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Equivariant cohomology Chern numbers determine equivariant unitary bordism for torus groups

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We show that the integral equivariant cohomology Chern numbers completely determine the equivariant geometric unitary bordism classes of closed unitary G–manifolds, which gives an affirmative answer to a conjecture posed by Guillemin, Ginzburg and Karshon (*Moment maps, cobordisms, and Hamiltonian group actions*, Remark H.5 in Appendix H.3), where G is a torus. As a further application, we also obtain a satisfactory solution of their Question (A) (Appendix H.1.1) on unitary Hamiltonian G–manifolds. Our key ingredients in the proof are the universal toric genus defined by Buchstaber, Panov and Ray and the Kronecker pairing of bordism and cobordism. Our approach heavily exploits Quillen's geometric interpretation of homotopic unitary cobordism theory. Moreover, this method can also be applied to the study of $(\mathbb{Z}_2)^k$–equivariant unoriented bordism and can still derive the classical result of tom Dieck.

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1 Introduction and main results

1.1 Background

In his seminal work [35], R Thom introduced the unoriented bordism theory, which corresponds to the infinite orthogonal group O. Since then, various other bordism theories, which correspond to subgroups $\mathbb{G}$ of the orthogonal group O as structure groups of stable tangent bundles or stable normal bundles of compact smooth manifolds, have been studied and established (eg see Milnor [29], Novikov [30] and Wall [36] and, for more details, see Landweber [24] and Stong [33]). When $\mathbb{G}$ is chosen to be SO (resp. U, SU, etc), the corresponding bordism theory is often called the oriented (resp. unitary, special unitary, etc) bordism theory. One of the main results in these bordism theories is that a closed manifold bounds if and only if certain characteristic numbers vanish. For example, when $\mathbb{G}$ is O or U, the corresponding characteristic numbers will be Stiefel–Whitney numbers or Chern numbers. Another main result is that the bordism ring $\Omega_*^{\mathbb{G}}$ associated to the given subgroup $\mathbb{G}$ can also be described quite

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completely, where $\Omega_*^{\mathbb{G}}$ consists of the bordism classes of all closed smooth manifolds with $\mathbb{G}$–structure (ie whose stable tangent bundles admit $\mathbb{G}$ as structure group). If $\mathbb{G}$ is O (resp. SO, U, etc), then a compact smooth manifold with $\mathbb{G}$–structure is often called an unoriented (resp. oriented, unitary, etc) manifold. Homotopy-theoretic bordism theories $M\mathbb{G}_*(X)$ (which is isomorphic to $\Omega_*^{\mathbb{G}}(X)$ by the Pontryagin–Thom construction) and cobordism theories $M\mathbb{G}^*(X)$ of a topological space X are due to M Atiyah [2], where $M\mathbb{G}$ is the corresponding Thom spectrum of $\mathbb{G}$ and $\Omega_*^{\mathbb{G}}(X)$ is the geometric bordism ring generated by the bordism classes of singular manifolds $f\colon M \to X$, where each M is a smooth closed manifold with $\mathbb{G}$–structure. Note that when X is a point, $\Omega_*^{\mathbb{G}}(X)$ is just $\Omega_*^{\mathbb{G}}$. These two kinds of theories are generalized homology and cohomology theories.

In the early 1960s, Conner and Floyd [12; 13; 14] began the study of geometric equivariant bordism theory by combining the ideas of bordism theory and transformation groups. They studied geometric equivariant unoriented, oriented and unitary bordism theories for smooth closed manifolds with periodic diffeomorphisms, and the subject has also continued to further develop by extending their ideas or combining with various ideas and theories from other research areas since then. For example, tom Dieck [15] introduced and studied the homotopy-theoretic equivariant bordisms $M\mathbb{G}_*^G(X)$ and cobordisms $M\mathbb{G}_G^*(X)$ for a G–space X, where G is a compact Lie group and, recently, Buchstaber, Panov and Ray [9] introduced and studied the universal toric genus in a geometric manner. The geometric equivariant bordism ring $\Omega_*^{\mathbb{G},G}(X)$ for a G–space X can also be defined in a natural way, and it is formed by singular G–manifolds $f\colon M \to X$, where each M is a smooth closed G–manifold with $\mathbb{G}$–structure such that the G–action preserves the $\mathbb{G}$–structure (in this case, we say that M admits a G–equivariant $\mathbb{G}$–structure). When X is a point, $\Omega_*^{\mathbb{G},G}(X)$ will be denoted by $\Omega_*^{\mathbb{G},G}$, and it is an equivariant analogue of $\Omega_*^{\mathbb{G}}$. It should be pointed out that in the above definition, the G–equivariant $\mathbb{G}$–structure of a smooth closed G–manifold with $\mathbb{G}$–structure is equipped on the stably G–equivariant tangent bundle of M rather than the stably G–equivariant normal bundle of the embedding of M in some G–representation because there is a substantial difference between both notions (see Hanke [19]). In this paper we will pay more attention to the case in which $\mathbb{G} = U$ (ie G–unitary bordism). A nice survey of the development of G–unitary bordism is contained in the introduction of Buchstaber, Panov and Ray [9].

Comparing with the nonequivariant case, a natural question arises and is stated as follows:

Question 1.1 *What kinds of equivariant characteristic numbers determine the equivariant geometric bordism class of a smooth closed G–manifold with $\mathbb{G}$–structure?*

As far as the authors know, the above question is far from solved. Most known works with respect to the above question have considered mainly the cases $(\mathbb{G}, G) = (O, (\mathbb{Z}_2)^k)$, $(U, T^k \times \mathbb{Z}_m)$, (SO, finite group) (eg see Bix and tom Dieck [6], Guillemin, Ginzburg and Karshon [18], Hattori [20], Khare [21], Lü and Tan [27], Lee and Wasserman [25], Sinha [32], Stong [34] and tom Dieck [16; 17]). In the following, we mainly give an investigation in the cases $(\mathbb{G}, G) = (U, T^k)$ and $(O, \mathbb{Z}_2^k)$.

Tom Dieck first investigated the above question. In his series of papers [15; 16; 17], by combining the geometric approach of Conner and Floyd [13; 14] and the K–theory approach developed by Atiyah, Bott, Segal and Singer [3; 4; 5], tom Dieck introduced the bundling transformation which further develops ideas of Boardman [7] and Conner [11], and proved a series of integrality theorems in equivariant (co)bordism theory. Furthermore, he gave an answer to the above question in the cases $(\mathbb{G}, G) = (O, (\mathbb{Z}_2)^k)$, $(U, T^k \times \mathbb{Z}_m)$, which is stated as follows:

Theorem 1.2 (tom Dieck) *When $(\mathbb{G}, G) = (O, (\mathbb{Z}_2)^k)$, $\beta \in \Omega_*^{O,(\mathbb{Z}_2)^k}$ is zero if and only if all equivariant Stiefel–Whitney numbers of β vanish. When $(\mathbb{G}, G) = (U, T^k \times \mathbb{Z}_m)$, $\beta \in \Omega_*^{U,G}$ is zero if and only if all equivariant K–theoretic Chern numbers of β vanish.*

In their book [18, Appendix H], Guillemin, Ginzburg and Karshon discussed the problem of calculating the ring $\mathcal{H}_*^G$ of equivariant Hamiltonian bordism classes of all unitary Hamiltonian G–manifolds with integral equivariant cohomology classes $\frac{1}{2\pi}[\omega - \Phi]$, where G is a torus. With respect to the determination of the ring $\mathcal{H}_*^G$, they posed three series of questions, the first one of which is stated as follows:

Question 1.3 [18, Appendix H.1.1, Question (A)] *Do mixed equivariant characteristic numbers form a full system of invariants of Hamiltonian bordism?*

On this question, Guillemin, Ginzburg and Karshon constructed a monomorphism

$$\Sigma_G \colon \mathcal{H}_*^G \to \Omega_{*+2}^{U,G},$$

so that Question 1.3 is equivalent to asking if the integral equivariant cohomology Chern numbers determine the equivariant geometric unitary bordism classes for the ring $\Omega_*^{U,G}$. They showed that a closed unitary G–manifold M with only isolated fixed points represents the zero element in $\Omega_*^{U,G}$ if and only if all integral equivariant cohomology Chern numbers of M vanish, which gives a partial solution of Question 1.3. Furthermore, they posed the following conjecture without the restriction of isolated fixed points.

Conjecture 1.4 [18, Appendix H.3, Remark H.5] *Let G be a torus. Then $\beta \in \Omega_*^{U,G}$ is zero if and only if all integral equivariant cohomology Chern numbers of β vanish.*

For this detecting problem, it is natural to use the localization theorem to analyze the topological data of fixed-point sets, which has been used successfully in the case when the fixed point sets are isolated; see Guillemin, Ginzburg and Karshon [18]. For the general case, we found that the analysis of fixed data seems to be not easy to handle and we turned to consider the geometric presentation of the image $\Phi([M]) \in MU_*(BG)$ of *the universal toric genus* instead, which admits a natural geometric representation described by Buchstaber, Panov and Ray [9]. In particular, by using the Kronecker pairing, our argument can directly be reduced to a simpler calculation.

1.2 Main results

The motivation of this paper is mainly supplied by the work of Guillemin, Ginzburg and Karshon as mentioned above. The geometric description of the universal toric genus developed by Buchstaber, Panov and Ray [9] provides us much more insight, so that we can carry out our work by utilizing the Kronecker pairing of bordism and cobordism. This leads us to give an affirmative answer to Conjecture 1.4, which is stated as follows.

Theorem 1.5 *Conjecture 1.4 holds.*

As a further consequence, we obtain a satisfactory solution of Question 1.3.

Corollary 1.6 *Mixed equivariant characteristic numbers determine equivariant Hamiltonian bordism.*

It is interesting that our approach above can also be applied to the detecting problem of $(\mathbb{Z}_2)^k$–equivariant unoriented bordism, and, in particular, it can still derive the classical result of tom Dieck in the case $(\mathbb{G}, G) = (O, (\mathbb{Z}_2)^k)$ of Theorem 1.2 by replacing the Boardman map by Kronecker pairing in the original proof of tom Dieck [16].

This paper is organized as follows. In Section 2, we shall review Quillen's geometric interpretation of homotopic unitary cobordism theory first, which is of essential importance, so that the Kronecker pairing between bordism and cobordism can be calculated in a geometric way. We then review *the universal toric genus* defined by Buchstaber, Panov and Ray, which can be expressed in terms of Quillen's geometric interpretation. We also recollect some necessary facts about equivariant Chern classes and equivariant Chern numbers. Then we shall give the proof of Theorem 1.5 in Section 3, which is more geometric. In particular, our approach in the unitary case can also be carried out in the study of $(\mathbb{Z}_2)^k$–equivariant unoriented bordism, as we shall see in the final part of Section 3.

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2 Preliminaries

2.1 Geometric interpretation of elements in $MU^*(X)$ and Kronecker pairing

Given a topological space X, let $MU_*(X)$ and $MU^*(X)$ be the complex (homotopic) bordism and cobordism of X, which are defined as

$$MU_*(X) = \lim_{l\to\infty} [S^{2l+*}, X_+ \wedge MU(l)]$$

and

$$MU^*(X) = \lim_{l\to\infty} [S^{2l-*} \wedge X_+, MU(l)],$$

respectively, where X_+ denotes the union of X and a disjoint point, and $MU(l)$ denotes the Thom space of the universal complex l–dimensional vector bundle over $BU(l)$.

Geometrically, it is very well known that elements of $MU_n(X)$ are regarded as bordism classes of maps $M \to X$ of stably complex closed n–dimensional manifolds M to X since $MU_n(X) \cong \Omega_n^U(X)$.

On the geometric interpretation of elements in $MU^*(X)$, Quillen [31, Proposition 1.2] showed that for a manifold X, $MU^{\pm n}(X)$ is isomorphic to the group formed by cobordism classes of proper *complex-oriented maps* $f\colon M \to X$ of dimension $\mp n$, where $\dim X - \dim M = \pm n$. Recall that when n is even, a *complex-oriented map* $f\colon M \to X$ of dimension $\mp n$ is a composition of maps of manifolds

$$M \xrightarrow{i} \mathbb{E} \xrightarrow{\pi} X, \tag{1}$$

where $\pi\colon \mathbb{E} \to X$ is a complex vector bundle over X and $i\colon M \to \mathbb{E}$ is an embedding endowed with, a complex structure on its normal bundle. When n is odd, the complex orientation of f will be defined in a same way as above with $\mathbb{E}$ replaced by $\mathbb{E} \times \mathbb{R}$ in (1).

It is very well known that as a generalized cohomology and homology theory, $MU^*(X)$ and $MU_*(X)$ admit a canonical Kronecker pairing

$$\langle\, ,\rangle\colon MU^n(X) \otimes MU_m(X) \to MU_{m-n},$$

where $MU_{m-n} = MU_{m-n}(\mathrm{pt}) \cong \Omega^U_{m-n}$.

This Kronecker pairing can be calculated in a geometric way as follows. For example, if $\alpha \in MU^{-n}(X)$ is represented by a smooth fiber bundle of closed smooth stably complex manifolds $E \to X$ with $\dim E - \dim X = n$ and $\beta \in MU_m(X)$ can be represented by a smooth map $f\colon M \to X$, then the Kronecker pairing $\langle \alpha, \beta\rangle \in \Omega^U_{m-n}$ is the bordism class of the pullback $\tilde{f}^*(E) = E \times_X M$, as shown in the diagram

$$\begin{array}{ccc} E \times_X M & \xrightarrow{\tilde{f}} & E \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

(see [8, Construction D.3.4]).

2.2 The universal toric genus

Based upon the works of tom Dieck, Krichever and Löffler (see [15; 17; 23; 26]), Buchstaber, Panov and Ray [9] defined the *universal toric genus* Φ in a geometric manner, which is a ring homomorphism from the geometric unitary T^k–bordism ring to the complex cobordism ring of the classifying space BT^k

$$\Phi\colon \Omega_*^{U,T^k} \to MU^*(BT^k).$$

Note that, as an Ω_*^U–algebra, $MU^*(BT^k)$ is isomorphic to $\Omega_*^U[\![u_1,\dots,u_k]\!]$, where u_i is the cobordism Chern class $c_1^{MU}(\xi_i)$ of the conjugate Hopf bundle ξ_i over the i^{th} factor of BT^k for $i=1,\dots,k$, so $MU^*(BT^k)$ can be replaced by $\Omega_*^U[\![u_1,\dots,u_k]\!]$. Let $[M]_{T^k}\in\Omega_n^{U,T^k}$ be an element represented by a closed unitary T^k–manifold M. Buchstaber, Panov and Ray showed that $\Phi([M]_{T^k})$ can be defined to be the cobordism class of the complex oriented map $\pi\colon ET^k\times_{T^k}M\to BT^k$. More precisely, choose a T^k–equivariant embedding $i\colon M\hookrightarrow V$ into a unitary T^k–representation space V; then the Borelification of i gives a complex-oriented map

$$\pi_l\colon ET^k(l)\times_{T^k}M\hookrightarrow ET^k(l)\times_{T^k}V\to BT^k(l),$$

which determines a cobordism class α_l in $MU^{-n}(BT^k(l))$, where $BT^k(l)=(\mathbb{C}P^l)^k$ and $ET^k(l)=(S^{2l+1})^k$. Since $BT^k=\bigcup_l BT^k(l)$ and $ET^k=\bigcup_l ET^k(l)$, these cobordism classes α_l form an inverse system. This produces a class $\alpha=\varprojlim\alpha_l$ in $MU^{-n}(BT^k)$, which is represented geometrically by the complex-oriented map

$$\pi\colon ET^k\times_{T^k}M\hookrightarrow ET^k\times_{T^k}V\to BT^k.$$

Then $\Phi([M]_{T^k})$ is defined as this limit α.

The following result is essentially due to Löffler [26] and Comezaña [10], as noted by Hanke [19].

Proposition 2.1 *The ring homomorphism $\Phi\colon\Omega_*^{U,T^k}\to MU^*(BT^k)$ is injective.*

Corollary 2.2 *If M is a closed unitary T^k–manifold, then M is null-bordant in Ω_*^{U,T^k} if and only if the complex-oriented map $\pi\colon ET^k\times_{T^k}M\to BT^k$ represents the zero element in $MU^*(BT^k)$.*

2.3 Equivariant Chern classes and equivariant Chern numbers

Let M be a closed unitary T^k–manifold. Then applying the Borel construction to the stable tangent bundle TM of M gives a vector bundle $ET^k\times_{T^k}TM$ over $ET^k\times_{T^k}M$.

Definition 2.3 The *total equivariant cohomology Chern class* of M is defined to be the total Chern class of the vector bundle $ET^k\times_{T^k}TM$ over $ET^k\times_{T^k}M$, ie

$$\begin{aligned}c^{T^k}(M)&:=c(ET^k\times_{T^k}TM)\\&=1+c_1(ET^k\times_{T^k}TM)+c_2(ET^k\times_{T^k}TM)+\cdots\\&=1+c_1^{T^k}(M)+c_2^{T^k}(M)+\cdots\in H^*(ET^k\times_{T^k}M)=H^*_{T^k}(M).\end{aligned}$$

Let $p\colon M \to \mathrm{pt}$ be the constant map, and let $p_!^{T^k}\colon H^*(ET^k \times_{T^k} M) \to H^*(BT^k)$ be *the equivariant Gysin map* induced by p. Then, the *integral equivariant cohomology Chern numbers* of M are defined to be

$$c_\omega^{T^k}[M]_{T^k} := p_!^{T^k}(c_\omega^{T^k}(M)),$$

each of which is a homogeneous polynomial of degree $2|\omega| - \dim M$ in the polynomial ring $H^*(BT^k) = \mathbb{Z}[x_1, \dots, x_k]$ with $\deg x_i = 2$, where $\omega = (i_1, i_2, \dots, i_s)$ is a partition of weight $|\omega| = i_1 + i_2 + \cdots + i_s$ and $c_\omega^{T^k}(M)$ denotes the product $c_{i_1}^{T^k}(M) c_{i_2}^{T^k}(M) \cdots c_{i_s}^{T^k}(M)$.

In particular, by definition, when $\dim M$ *is odd*, one has $c_\omega^{T^k}[M]_{T^k} = 0$ for all ω. Hence, in what follows, it is safe to assume $\dim M = 2n$ most of the time.

Note that the map $\pi\colon ET^k \times_{T^k} M \to BT^k$ is just the Borelification of $p\colon M \to \mathrm{pt}$, so $p_!^{T^k}$ is often replaced by the Gysin map $\pi_!$ in this paper.

3 Proof of the main theorem and equivariant unoriented bordism

3.1 Determining equivariant cohomology Chern numbers by ordinary Chern numbers

Let M be a $2n$–dimensional closed unitary T^k–manifold; we consider the equivariant cohomology Chern numbers $c_\omega^{T^k}[M]_{T^k} \in H^*(BT^k; \mathbb{Z})$ for the partition $\omega = (i_1, i_2, \dots, i_s)$ of weight $|\omega| = i_1 + i_2 + \cdots + i_s$. Since $H^*(BT^k; \mathbb{Z}) \cong \mathbb{Z}[x_1, x_2, \dots, x_k]$ with $\deg x_i = 2$, the equivariant Chern number $c_\omega^{T^k}[M]_{T^k}$ admits the form

$$(*) \qquad c_\omega^{T^k}[M]_{T^k} = \sum_J n_J^\omega x^J,$$

where the k–tuples of natural numbers $J = (j_1, j_2, \dots, j_k)$ range over all k–partitions of weight $|\omega| - n$ and x^J means $x_1^{j_1} x_2^{j_2} \cdots\ x_k^{j_k}$, with the coefficients $n_J^\omega \in \mathbb{Z}$. It follows that to determine the equivariant cohomology Chern numbers $c_\omega^{T^k}[M]_{T^k}$ it is equivalent to determine the coefficient n_J^ω for each $J = (j_1, j_2, \dots, j_k)$.

The coefficient n_J^ω is closely related to the ordinary Chern numbers of certain manifolds. Indeed, for $J = (j_1, j_2, \dots, j_k)$, let $\left(\prod_{i=1}^k S^{2j_i+1}\right) \times_{T^k} M$ be the pullback of the natural inclusion $\prod_{i=1}^{k} \mathbb{C}P^{j_i} \hookrightarrow BT^k$:

$$\begin{array}{ccc} \left(\prod_{i=1}^{k} S^{2j_i+1}\right) \times_{T^k} M & \xrightarrow{\tilde{f}_J} & ET^k \times_{T^k} M \\ \pi_J \Big\downarrow & & \pi \Big\downarrow \\ \prod_{i=1}^{k} \mathbb{C}P^{j_i} & \xrightarrow{\quad f_J \quad} & BT^k \end{array}$$

First, we find that:

Lemma 3.1 *For fixed ω and J,*

$$n_J^{\omega} x^J = (\pi_J)_! \biggl(c_\omega \biggl(\biggl(\prod_{i=1}^{k} S^{2j_i+1} \biggr) \times_{T^k} TM \biggr) \biggr),$$

where $(\pi_J)_!$ is the Gysin map induced by π_J.

Proof By the definition of equivariant Chern numbers,

$$c_\omega^{T^k}[M]_{T^k} = \pi_!(c_\omega(ET^k \times_{T^k} TM)) = \sum_J n_J^{\omega} x^J \in H^*(BT^k; \mathbb{Z}).$$

Since $\left(\prod_{i=1}^{k} S^{2j_i+1}\right) \times_{T^k} TM = \tilde{f}_J^*(ET^k \times_{T^k} TM)$ and Gysin maps are commutative in the pullback square, ie $f_J^* \pi_! = (\pi_J)_! \tilde{f}_J^*$, we have

$$\begin{aligned} n_J^{\omega} x^J &= f_J^* \pi_!(c_\omega(ET^k \times_{T^k} TM)) \\ &= (\pi_J)_! \tilde{f}_J^* c_\omega(ET^k \times_{T^k} TM) \\ &= (\pi_J)_! \biggl(c_\omega \biggl(\biggl(\prod_{i=1}^{k} S^{2j_i+1} \biggr) \times_{T^k} TM \biggr) \biggr). \end{aligned}$$ □

Let $\omega = (i_1, i_2, \dots, i_s)$ and $\omega' = (i'_1, i'_2, \dots, i'_s)$ be two partitions; we say $\omega' \leqslant \omega$ provided $i'_j \leqslant i_j$ for each j and we say $\omega' < \omega$ if $\omega' \leqslant \omega$ and $\omega' \neq \omega$. We denote by $\omega \pm \omega'$ the partition $(i_1 \pm i'_1, i_2 \pm i'_2, \dots, i_s \pm i'_s)$. Let $J = (i_1, \dots, i_k)$ and $J' = (i'_1, \dots, i'_k)$ be two k–partitions; we also define $J' \leqslant J$, $J' < J$ and $J \pm J'$ in the same way.

For each partition ω and each k–partition J, the corresponding coefficient n_J^{ω} is determined by the ordinary Chern number $c_\omega\bigl[\left(\prod_{i=1}^{k} S^{2j_i+1}\right) \times_{T^k} M\bigr]$ and the coefficients $n_{J'}^{\omega'}$ with $J' < J$ and $\omega' < \omega$. More precisely, we have:

Proposition 3.2 *When* $\dim M = 2n > 0$*, one has*

$$c_\omega\Big[\Big(\prod_{i=1}^{k} S^{2j_i+1}\Big)\times_{T^k} M\Big] = n_J^\omega + \sum_{\substack{\omega'+\omega''=\omega\\ \omega'<\omega,\,\omega''<\omega,\,J'<J}} n_{J'}^{\omega'} m_{(J-J')}^{\omega''},$$

where $m_{J-J'}$ *is the coefficient of* $x^{J-J'}$ *in the Chern class*

$$c_{\omega''}\big(\prod_{i=1}^{k}\mathbb{C}P^{j_i}\big)\in H^*\Big(\prod_{i=1}^{k}\mathbb{C}P^{j_i};\mathbb{Z}\Big) = \mathbb{Z}[x_1,x_2,\dots,x_k]/(x_1^{j_1+1},\dots,x_k^{j_k+1}).$$

Proof The stable tangent bundle of $\big(\prod_{i=1}^{k} S^{2j_i+1}\big)\times_{T^k} M$ admits a decomposition

$$T\Big(\Big(\prod_{i=1}^{k} S^{2j_i+1}\Big)\times_{T^k} M\Big) = \Big(\Big(\prod_{i=1}^{k} S^{2j_i+1}\Big)\times_{T^k} TM\Big)\oplus \pi_J^* T\Big(\prod_{i=1}^{k}\mathbb{C}P^{j_i}\Big).$$

By the Cartan formula, we have

$$\begin{aligned}
c_\omega\Big(\Big(\prod_{i=1}^{k} S^{2j_i+1}\Big)\times_{T^k} M\Big)
&= c_\omega\Big(\Big(\prod_{i=1}^{k} S^{2j_i+1}\Big)\times_{T^k} TM\Big) + \pi_J^* c_\omega\Big(\prod_{i=1}^{k}\mathbb{C}P^{j_i}\Big)\\
&\quad + \sum_{\substack{\omega'+\omega''=\omega\\ \omega'<\omega,\omega''<\omega}} c_{\omega'}\Big(\Big(\prod_{i=1}^{k} S^{2j_i+1}\Big)\times_{T^k} TM\Big)\pi_J^* c_{\omega''}\Big(\prod_{i=1}^{k}\mathbb{C}P^{j_i}\Big).
\end{aligned}$$

Applying $(\pi_J)_!$ to both sides, we obtain

$$\begin{aligned}
&c_\omega\Big[\Big(\prod_{i=1}^{k} S^{2j_i+1}\Big)\times_{T^k} M\Big]x^J\\
&= (\pi_J)_! c_\omega\Big(\Big(\prod_{i=1}^{k} S^{2j_i+1}\Big)\times_{T^k} M\Big)\\
&= (\pi_J)_!\Big(c_\omega\Big(\Big(\prod_{i=1}^{k} S^{2j_i+1}\Big)\times_{T^k} TM\Big)\Big) + (\pi_J)_!\Big(\pi_J^* c_\omega\Big(\prod_{i=1}^{k}\mathbb{C}P^{j_i}\Big)\Big)\\
&\quad + \sum_{\substack{\omega'+\omega''=\omega\\ \omega'<\omega,\omega''<\omega}} (\pi_J)_!\Big(c_{\omega'}\Big(\Big(\prod_{i=1}^{k} S^{2j_i+1}\Big)\times_{T^k} TM\Big)\pi_J^* c_{\omega''}\Big(\prod_{i=1}^{k}\mathbb{C}P^{j_i}\Big)\Big)
\end{aligned}$$

$$= n_J^{\omega} x^J + (\pi_J)_!(1) c_\omega\Big(\prod_{i=1}^k \mathbb{C}P^{j_i}\Big) + \sum_{\substack{\omega'+\omega''=\omega \\ \omega'<\omega, \omega''<\omega, J'<J}} (\pi_J)_!\Big(c_{\omega'}\Big(\Big(\prod_{i=1}^k S^{2j_i+1}\Big) \times_{T^k} TM\Big)\Big) c_{\omega''}\Big(\prod_{i=1}^k \mathbb{C}P^{j_i}\Big),$$

where the first equality uses that $(\pi_J)_!$ is isomorphic on the top cohomology and the last uses that $(\pi_J)_!(1) = 0$ when $\dim M > 0$.

Note that $f_J^*\colon H^*(BT^k;\mathbb{Z}) \to H^*\big(\prod_{i=1}^k \mathbb{C}P^{j_i};\mathbb{Z}\big)$ is an isomorphism up to dimension $2|J|$, so we have

$$\begin{aligned}(\pi_J)_!\Big(c_{\omega'}\Big(\Big(\prod_{i=1}^k S^{2j_i+1}\Big) \times_{T^k} TM\Big)\Big) &= f_J^*(\pi_! c_{\omega'}(ET^k \times_{T^k} TM)) \\ &= f_J^*\Big(\Big(\sum n_{J'}^{\omega'}\Big) x^{J'}\Big) \qquad \text{(by } (*)\text{)} \\ &= \sum n_{J'}^{\omega'} x^{J'},\end{aligned}$$

because $f_J^*(x^{J'}) = x^{J'}$ when $J' \leqslant J$. Write $c_{\omega''}\big(\prod_{i=1}^k \mathbb{C}P^{j_i}\big) = \sum_{J''} m_{J''}^{\omega''} x^{J''}$ and we have

$$\begin{aligned}c_\omega\Big[\Big(\prod_{i=1}^k S^{2j_i+1}\Big) \times_{T^k} M\Big] x^J &= n_J^\omega x^J + \sum_{\substack{\omega'+\omega''=\omega \\ \omega'<\omega, \omega''<\omega, J'<J}} \Big(\sum n_{J'}^{\omega'} x^{J'}\Big)\Big(\sum_{J''} m_{J''}^{\omega''} x^{J''}\Big) \\ &= \Big(n_J^\omega + \sum_{\substack{\omega'+\omega''=\omega \\ \omega'<\omega, \omega''<\omega, J'<J}} n_J^{w'} m_{J-J'}^{\omega''}\Big) x^J.\end{aligned}$$

Therefore, by comparing the coefficients of both sides, one has the equation

$$c_\omega\Big[\Big(\prod_{i=1}^k S^{2j_i+1}\Big) \times_{T^k} M\Big] = n_J^\omega + \sum_{\substack{\omega'+\omega''=\omega \\ \omega'<\omega, \omega''<\omega, J'<J}} n_{J'}^{\omega'} m_{(J-J')}^{\omega''}. \qquad \square$$

Remark When the dimension of M is odd, all Chern numbers and equivariant Chern numbers are zero for dimensional reasons.

Clearly, one has:

Corollary 3.3 *$c_\omega\big[\big(\prod_{i=1}^k S^{2j_i+1}\big)\times_{T^k} M\big] \in (n^{\omega'}_{J'}) \subseteq \mathbb{Z}$ with $\omega' \leqslant \omega$ and $J' = (j_1',\dots,j_k') \leqslant J$, where $(n^{\omega'}_{J'})$ is the ideal in $\mathbb{Z}$ generated by the integers $n^{\omega'}_{J'}$.*

On the other hand, when $\dim M = 2n > 0$, the coefficient n^ω_J of the equivariant Chern number $c^{T^k}_\omega[M]_{T^k}$ is determined by the ordinary Chern numbers

$$c_{\omega'}\bigg[\bigg(\prod_{i=1}^k S^{2j_i'+1}\bigg)\times_{T^k} M\bigg]$$

of the manifolds $\big(\prod_{i=1}^k S^{2j_i'+1}\big)\times_{T^k} M$ with partition $\omega' \leqslant \omega$ and $J' \leqslant J$. Indeed, we have:

Proposition 3.4 *$n^\omega_J \in \big(c_{\widetilde\omega}\big[\big(\prod_{i=1}^k S^{2\tilde j_i+1}\big)\times_{T^k} M\big]\big) \subset \mathbb{Z}$ with $\widetilde\omega \leqslant \omega$ and $\widetilde J \leqslant J$, where the ideal $\big(c_{\widetilde\omega}\big[\big(\prod_{i=1}^k S^{2\tilde j_i+1}\big)\times_{T^k} M\big]\big)$ in $\mathbb{Z}$ is generated by the integers $c_{\widetilde\omega}\big[\big(\prod_{i=1}^k S^{2\tilde j_i+1}\big)\times_{T^k} M\big]$.*

Proof Note that when the weight satisfies $|\omega| < n$, we have $n^\omega_J = 0$ for all J. When $|\omega| = n$, the corresponding k–partition is $J_0 = (0,\dots,0)$ and we have $n^\omega_{J_0} = c_\omega\big[\big(\prod_{i=1}^k S^{2j_i+1}\big)\times_{T^k} M\big]$.

When $|\omega| > n$, by Proposition 3.2 we see

$$n^\omega_J - c_\omega\bigg[\bigg(\prod_{i=1}^k S^{2j_i+1}\bigg)\times_{T^k} M\bigg] \in (n^{\omega'}_{J'}) \subseteq \mathbb{Z}$$

with $\omega' < \omega$ and $J' < J$. When $|\omega'| < |\omega|$, by induction on the weight $|\omega|$ we obtain that

$$(n^{\omega'}_{J'}) \subset \bigg(c_{\omega''}\bigg[\bigg(\prod_{i=1}^k S^{2j_i''+1}\bigg)\times_{T^k} M\bigg]\bigg) \subseteq \mathbb{Z},$$

with $\omega'' \leqslant \omega'$ and $J'' \leqslant J'$. Hence, we have

$$n^\omega_J \in c_\omega\bigg[\bigg(\prod_{i=1}^k S^{2j_i+1}\bigg)\times_{T^k} M\bigg] + (n^{\omega'}_{J'}) \subset \bigg(c_{\widetilde\omega}\bigg[\bigg(\prod_{i=1}^k S^{2\tilde j_i+1}\bigg)\times_{T^k} M\bigg]\bigg) \subseteq \mathbb{Z}$$

with $\widetilde\omega \leqslant \omega$ and $\widetilde J \leqslant J$. □

Theorem 3.5 *The equivariant Chern number $c^{T^k}_\omega[M]_{T^k}$ equals 0 for all partitions ω if and only if the ordinary Chern number $c_\omega\big[\big(\prod_{i=1}^k S^{2j_i+1}\big)\times_{T^k} M\big]$ equals 0 for all ω and all $J = (j_1, j_2, \dots, j_k)$.*

Proof When $\dim M = 0$, M is the disjoint union of some isolated points with trivial torus action, and the statement certainly holds. When $\dim M$ is odd, clearly all $c_\omega^{T^k}[M]_{T^k}$ are 0 and all $c_\omega\big[\big(\prod_{i=1}^k S^{2j_i+1}\big) \times_{T^k} M\big]$ are 0 for dimensional reasons.

Now we assume $\dim M = 2n > 0$, and suppose $c_\omega^{T^k}[M]_{T^k} = 0$ for all ω; then $n_{\widetilde{J}}^{\widetilde{\omega}} = 0$ for all $\widetilde{\omega}$ and $\widetilde{J}$. By Proposition 3.4, $c_\omega\big[\big(\prod_{i=1}^k S^{2j_i+1}\big) \times_{T^k} M\big] \in (n_{\widetilde{J}}^{\widetilde{\omega}}) = (0) \subseteq \mathbb{Z}$. The inverse direction is similar. □

3.2 Proof of Theorem 1.5

Let M be a closed unitary T^k–manifold. Since the universal toric genus

$$\Phi\colon \Omega_*^{U,T^k} \to MU^*(BT^k)$$

is injective, it is sufficient to show that the image $\Phi([M]_{T^k}) = 0$ if and only if all integral equivariant cohomology Chern numbers of M vanish.

Geometrically, the image $\Phi([M]_{T^k}) \in MU^*(BT^k)$ is represented by the map

$$\pi\colon ET^k \times_{T^k} M \to BT^k.$$

The Kronecker pairing

$$MU^*(BT^k) \otimes MU_*(BT^k) \to MU_*$$

defines an isomorphism

$$MU^*(BT^k) \cong \mathrm{Hom}_{MU_*}(MU(BT^k), MU_*),$$

because $MU^*(BT^k)$ is a free MU_*–module in even-dimensional generators.

Therefore, the image $\Phi([M]_{T^k})$ is zero if and only if its evaluation with all the MU_* generators of $MU_*(BT^k)$ is zero.

According to [1; 22], the natural inclusion maps $\mathbb{C}P^n \hookrightarrow BT^1$ are the MU_*–generators of $MU_*(BT^1)$, so one can take the natural inclusion maps $\prod_{i=1}^k \mathbb{C}P^{j_i} \hookrightarrow BT^k$ as MU_*–generators of $MU_*(BT^k)$.

By Quillen's construction, the Kronecker pairing of

$$\Phi([M]_{T^k}) \quad \text{and} \quad \Big[\prod_{i=1}^k \mathbb{C}P^{j_i} \to BT^k\Big]$$

defines the bordism class of the manifold

$$\Big\langle \Phi([M]_{T^k}), \Big[\prod_{i=1}^{k} \mathbb{C}P^{j_i} \to BT^k\Big]\Big\rangle = \Big[\Big(\prod_{i=1}^{k} S^{2j_i+1}\Big) \times_{T^k} M\Big] \in MU_*.$$

Therefore, $\Phi([M]_{T^k})$ is zero if and only if $\big[\big(\prod_{i=1}^{k} S^{2j_i-1}\big) \times_{T^k} M\big] = 0 \in MU_*$ for all the inclusions $\prod_{i=1}^{k} \mathbb{C}P^{j_i} \hookrightarrow BT^k$ for all $J = (j_1, j_2, \dots, j_k)$.

On the other hand, by Theorem 3.5, we see the equivariant Chern numbers $c_\omega^{T^k}[M]_{T^k}$ are 0 if and only if the ordinary Chern numbers $c_\omega\big[\big(\prod_{i=1}^{k} S^{2j_i-1}\big) \times_{T^k} M\big]$ are 0 for all ω and all $J = (j_1, j_2, \dots, j_k)$.

Thus, we conclude that the equivariant Chern number $c_\omega^{T^k}[M]$ is 0 for all ω if and only if $\Phi([M]) = 0$. Hence, Theorem 1.5 has been proved.

Together with the above arguments, we conclude:

Theorem 3.6 *Let M be a closed unitary T^k–manifold. Then the following statements are equivalent:*

(1) *M is null-bordant in Ω_*^{U,T^k}.*

(2) *All integral equivariant cohomology Chern numbers of M vanish.*

(3) *For any partition $J = (j_1, \dots, j_k)$ with each $j_i \geq 0$, $\big(\prod_{i=1}^{k} S^{2j_i+1}\big) \times_{T^k} M$ is null-bordant in Ω_*^U.*

(4) *The fibration $\pi\colon ET^k \times_{T^k} M \to BT^k$ is null-cobordant in $MU^*(BT^k)$.*

3.3 Equivariant unoriented bordism

Our approach above can also be carried out in the case of equivariant unoriented bordism. Let $\mathfrak{N}_*^{(\mathbb{Z}_2)^k}$ (or $\Omega_*^{O,(\mathbb{Z}_2)^k}$) be the ring formed by the equivariant bordism classes of all unoriented closed smooth $(\mathbb{Z}_2)^k$–manifolds, and let $MO^*(X)$ be the unoriented (homotopic) cobordism ring of a topological space X, ie

$$MO^*(X) = \lim_{l\to\infty} [S^{l-*} \wedge X_+, MO(l)].$$

Then Quillen's geometric approach on $MU^*(X)$ can be carried out in the case of $MO^*(X)$, so that the Kronecker pairing

$$\langle\ ,\ \rangle\colon MO^*(X) \otimes MO_*(X) \to MO_* \cong \mathfrak{N}_*$$

can be calculated in a geometric way similar to the unitary case as in Section 2.1 (see [8, Constructions D.2.8 and D.3.4]), where $\mathfrak{N}_*$ is the nonequivariant Thom unoriented bordism ring. In [16], tom Dieck showed that there is also a monomorphism

$$\Phi_{\mathbb{R}}\colon \mathfrak{N}_*^{(\mathbb{Z}_2)^k} \to MO^*(B(\mathbb{Z}_2)^k).$$

It is well known that $MO^*(B(\mathbb{Z}_2)^k) = \varprojlim MO^*(B(\mathbb{Z}_2)^k(n))$, where $B(\mathbb{Z}_2)^k(n) = (\mathbb{R}P^n)^k$ and $B(\mathbb{Z}_2)^k = \bigcup_n B(\mathbb{Z}_2)^k(n)$. Then, applying the finite approximation method yields that for a class $[M]_{(\mathbb{Z}_2)^k} \in \mathfrak{N}_*^{(\mathbb{Z}_2)^k}$, the image $\Phi_{\mathbb{R}}([M]_{(\mathbb{Z}_2)^k})$ is geometrically represented by the fibration $\pi\colon E(\mathbb{Z}_2)^k \times_{(\mathbb{Z}_2)^k} M \to B(\mathbb{Z}_2)^k$.

Theorem 3.7 *Let M be a closed smooth $(\mathbb{Z}_2)^k$–manifold. Then the following statements are equivalent:*

(1) *M is null-bordant in $\mathfrak{N}_*^{(\mathbb{Z}_2)^k}$.*

(2) *All equivariant Stiefel–Whitney numbers of M vanish.*

(3) *For any partition $I = (j_1, \dots, j_k)$ with each $j_i \geq 0$, $\left(\prod_{i=1}^k S^{j_i+1}\right) \times_{(\mathbb{Z}_2)^k} M$ is null-bordant in $\mathfrak{N}_*$.*

(4) *The fibration*

$$\pi\colon E(\mathbb{Z}_2)^k \times_{(\mathbb{Z}_2)^k} M \to B(\mathbb{Z}_2)^k$$

is null-cobordant in $MO^*(B(\mathbb{Z}_2)^k)$.

Proof The proof follows closely that of Theorem 3.6 in a very similar way. We leave it as an exercise to the reader. □

Remark Theorem 3.7 tells us that $(\mathbb{Z}_2)^k$–equivariant unoriented bordism is determined by the $(\mathbb{Z}_2)^k$–equivariant Stiefel–Whitney numbers. This result is essentially due to tom Dieck with an argument involving the Boardman map

$$B\colon MO^*(B(\mathbb{Z}_2)^k) \to H^*(B(\mathbb{Z}_2)^k; \mathbb{Z}_2) \mathbin{\widehat{\otimes}} H_*(BO; \mathbb{Z}_2)$$

(see [16]). Here we exactly replace the use of the Boardman map by the Kronecker pairing.

It is well known that as an $\mathfrak{N}_*$–algebra, $MO^*(B(\mathbb{Z}_2)^k)$ is isomorphic to the ring $\mathfrak{N}_*[\![v_1, \dots, v_k]\!]$, where v_i denotes the first cobordism Stiefel–Whitney class of the canonical line bundle over the i^{th} factor of $B(\mathbb{Z}_2)^k$ for $i = 1, \dots, k$. Thus, in the sense of Buchstaber, Panov and Ray [9], we can also define the $(\mathbb{Z}_2)^k$*–equivariant universal genus*, which is an $\mathfrak{N}_*$–algebra homomorphism,

$$\Phi_{\mathbb{R}}\colon \mathfrak{N}_*^{(\mathbb{Z}_2)^k} \to \mathfrak{N}_*[\![v_1, \dots, v_k]\!],$$

by $\Phi_{\mathbb{R}}([M]_{(\mathbb{Z}_2)^k}) = \sum_{\omega=(j_1,\dots,j_k)} g_\omega^{\mathbb{R}}(M) v^\omega$, where $v^\omega = v_1^{j_1} \cdots v_k^{j_k}$ for $\omega = (j_1, \dots, j_k)$ with $j_i \geq 0$, and $g_\omega^{\mathbb{R}}(M) \in \mathfrak{N}_*$ with $\dim g_\omega^{\mathbb{R}}(M) = |\omega| + \dim M$.

The determination of the coefficients $g_\omega^{\mathbb{R}}(M)$ depends upon the choice of a dual basis in $MO_*(B(\mathbb{Z}_2)^k)$ to the basis $\{v^\omega\}$ in $MO^*(B(\mathbb{Z}_2)^k)$. Buchstaber, Panov and Ray's approach in the unitary case provides us much more insight on the determination of coefficients $g_\omega^{\mathbb{R}}(M)$. In our case, since it was shown in [28, Corollary 6.4] that any real Bott manifold is null-bordant in $\mathfrak{N}_*$, we may employ the real Bott manifolds to realize a dual basis in $MO_*(B(\mathbb{Z}_2)^k)$ of the basis $\{v^\omega\}$ in $MO^*(B(\mathbb{Z}_2)^k)$, so that the coefficients $g_\omega^{\mathbb{R}}(M)$ can be represented by manifolds $G_\omega^{\mathbb{R}}(M) = (S^1)^\omega \times_{(\mathbb{Z}_2)^\omega} M$, where the action of $(\mathbb{Z}_2)^\omega = (\mathbb{Z}_2)^{j_1} \times \cdots \times (\mathbb{Z}_2)^{j_k}$ on $(S^1)^\omega = (S^1)^{j_1} \times \cdots \times (S^1)^{j_k}$ is coordinatewise, and on M via the representation $(g_{1,1}, \dots, g_{1,j_1}; \dots; g_{k,1}, \dots, g_{k,j_k}) \mapsto (g_{1,j_1}, \dots, g_{k,j_k})$. The argument is close to that of the unitary case in [8, Section 9.2].

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