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On the asymptotic expansion of the quantum SU(2) invariant at $q = \exp(4\pi\sqrt{-1}/N)$ for closed hyperbolic 3–manifolds obtained by integral surgery along the figure-eight knot

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It is known that the quantum SU(2) invariant of a closed 3–manifold at $q = \exp(2\pi\sqrt{-1}/N)$ is of polynomial order as $N \to \infty$. Recently, Chen and Yang conjectured that the quantum SU(2) invariant of a closed hyperbolic 3–manifold at $q = \exp(4\pi\sqrt{-1}/N)$ is of order $\exp(N \cdot \varsigma(M))$, where $\varsigma(M)$ is a normalized complex volume of M. We can regard this conjecture as a kind of "volume conjecture", which is an important topic from the viewpoint that it relates quantum topology and hyperbolic geometry.

In this paper, we give a concrete presentation of the asymptotic expansion of the quantum SU(2) invariant at $q = \exp(4\pi\sqrt{-1}/N)$ for closed hyperbolic 3–manifolds obtained from the 3–sphere by integral surgery along the figure-eight knot. In particular, the leading term of the expansion is $\exp(N \cdot \varsigma(M))$, which gives a proof of the Chen–Yang conjecture for such 3–manifolds. Further, the semiclassical part of the expansion is a constant multiple of the square root of the Reidemeister torsion for such 3–manifolds. We expect that the higher-order coefficients of the expansion would be "new" invariants, which are related to "quantization" of the hyperbolic structure of a closed hyperbolic 3–manifold.

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1 Introduction

In the late 1980s, Witten [26] proposed topological invariants of a closed 3–manifold M for a simple compact Lie group G, what we call the quantum G invariant. This invariant is formally presented by a path integral whose Lagrangian is the Chern–Simons functional of G connections on M. There are two approaches to obtain mathematically rigorous information from a path integral: the operator formalism and the perturbative expansion. Motivated by the operator formalism of the Chern–Simons path integral, we obtain a rigorous mathematical construction of quantum invariants by using linear sums of quantum invariants of links. In particular, the quantum SU(2) invariant $\tau_N(M;q)$ of a closed 3–manifold M is defined to be a linear sum of the colored Jones polynomials $J_n(K;q)$ of a link L at q, where q is a primitive N^{th} root of unity, and L is a link such that M is obtained from S^3 by an integral surgery along L; for details, see eg a book of the author [17]. We note that, as mentioned in [17], when N is odd, $\tau_N(M;q)$ can be defined at $q = e^{4\pi\sqrt{-1}/N}$; we denote it by $\hat{\tau}_N(M)$.

The volume conjecture of Kashaev, Murakami and Murakami [9; 14] is an important topic, which relates quantum topology and hyperbolic geometry. A complexification of the volume conjecture of Murakami, Murakami, Okamoto, Takata and Yokota [15] states that, for a hyperbolic knot K, $(2\pi\sqrt{-1}/N)\log J_N(K;q^{2\pi\sqrt{-1}/N})$ goes to a (normalized) complex volume of K as $N \to \infty$. Further, it has been expected by Murakami [13] that the quantum invariant of a closed 3–manifold has a similar property, though it is known that $\tau_N(M;q)$ at $q = e^{2\pi\sqrt{-1}}$ is of polynomial order as $N \to \infty$. Recently, Chen and Yang [2] observed that $\hat{\tau}_N(M)$ is of exponential order as $N \to \infty$ for some hyperbolic 3–manifolds obtained by surgery along the figure-eight knot and the 5_2 knot, and conjectured that

$$4\pi\sqrt{-1}\cdot \lim_{\substack{N\to\infty \\ N \text{ odd}}} \frac{\log \hat{\tau}_N(M)}{N} = \mathrm{cs}(M) + \sqrt{-1}\,\mathrm{vol}(M)$$

for a closed hyperbolic 3–manifold M, where $\mathrm{cs}(M)$ and $\mathrm{vol}(M)$ denote the Chern–Simons invariant and the hyperbolic volume of M, respectively. We define a normalized *complex volume* of M by

$$\varsigma(M)=\frac{1}{4\pi\sqrt{-1}}(\mathrm{cs}(M)+\sqrt{-1}\,\mathrm{vol}(M)).$$

From the viewpoint of mathematical physics, we can regard this conjecture as a perturbative expansion of the Chern–Simons path integral; see Section 2, for details.

Let p be an integer, and let M_p be the 3–manifold obtained from S^3 by p surgery along the figure-eight knot. It is known that M_p is hyperbolic if and only if $|p|>4$. The aim of this paper is to show the following theorem, as a refinement of the above-mentioned conjecture:

Theorem 1.1 *Let N be an odd integer ≥ 3, and let p be an integer with $|p|>4$. Then the quantum invariant $\hat{\tau}_N(M_p)$ of M_p is expanded as $N\to\infty$ in the form*

$$\hat{\tau}_N(M_p)=(-1)^p e^{-(\pi\sqrt{-1}/4)pN}\sqrt{-1}^{\,\mathrm{sign}(p)(N-3)/2}e^{N\varsigma(M_p)}N^{3/2}\omega(M_p)\\ \times\left(1+\sum_{i=1}^{d}\kappa_i(M_p)\cdot\left(\frac{4\pi\sqrt{-1}}{N}\right)^i+O\left(\frac{1}{N^{d+1}}\right)\right)$$

for any integer $d\geq 1$, where $\omega(M_p)$ and $\kappa_i(M_p)$ are constants determined by M_p.

We conjecture that, similarly to in the theorem, $\hat{\tau}_N(M)$ of any closed hyperbolic 3–manifold can be expanded in the form

$$\hat{\tau}_N(M)=(\text{some root of unity})e^{N\varsigma(M)}N^{3/2}\omega(M)\\ \times\left(1+\sum_{i=1}^{d}\kappa_i(M)\cdot\left(\frac{4\pi\sqrt{-1}}{N}\right)^i+O\left(\frac{1}{N^{d+1}}\right)\right)$$

for any integer $d\geq 1$, with some constants $\omega(M)$ and $\kappa_i(M)$ determined by M.

We can numerically observe, for example, the behavior of $\hat{\tau}_N(M_8)$, as follows:

N	$\hat{\tau}_N(M_8)\sqrt{-1}^{(p/2)N-(N-3)/2}e^{-N\varsigma(M_8)}N^{-3/2}$
101	$0.0033167246\ldots-\sqrt{-1}\cdot 0.0219539338\ldots$
201	$0.0050414223\ldots-\sqrt{-1}\cdot 0.0215864601\ldots$
501	$0.0060677858\ldots-\sqrt{-1}\cdot 0.0213013492\ldots$
1001	$0.0064080099\ldots-\sqrt{-1}\cdot 0.0211954944\ldots$

Here, $\varsigma(M_8)$ and $\omega(M_8)$ are numerically given by

$$\begin{aligned}\varsigma(M_8) &= 0.1259843998\ldots - \sqrt{-1}\cdot 0.0858243597\ldots \\ &= \frac{1.0785007120\ldots + \sqrt{-1}\cdot 1.5831666606\ldots}{4\pi\sqrt{-1}}, \\ \omega(M_8) &= 0.0067471463\ldots - \sqrt{-1}\cdot 0.0210842217\ldots,\end{aligned}$$

and we can numerically observe that the complex numbers in the above table converge to $\omega(M_8)$ as $N \to \infty$.

The values of $\omega(M_p)$ are numerically given for some p, as follows:

p	$\omega(M_p)$
5	$0.0081594261\ldots - \sqrt{-1}\cdot 0.0558388944\ldots$
6	$0.0078610660\ldots - \sqrt{-1}\cdot 0.0356626288\ldots$
7	$0.0072993993\ldots - \sqrt{-1}\cdot 0.0265443774\ldots$
8	$0.0067471463\ldots - \sqrt{-1}\cdot 0.0210842217\ldots$
9	$0.0062386239\ldots - \sqrt{-1}\cdot 0.0173836407\ldots$

It is shown by the author and Takata in [21] that $\omega(M_p)^2$ is equal to a constant multiple of the twisted Reidemeister torsion of M_p. We note that a similar statement holds for the asymptotic expansion of the Kashaev invariant for the two-bridge knots, as shown by the author and Takata [20]. We also expect that $\kappa_i(M)$ are new invariants of a closed hyperbolic 3–manifold M.

Remark 1.2 There has been recent progress on the Chen–Yang conjectures. Chen and Yang [2] (whose first version was written in March 2015) gave two conjectures based on numerical observations: one is the "volume conjecture" for the quantum SU(2) invariant (the Reshetikhin–Turaev invariant) for closed 3–manifolds, and the other is the "volume conjecture" for the Turaev–Viro invariant for closed or cusped 3–manifolds. Since it is known (see eg a book of Turaev [25, Section VII.4]) that the Turaev–Viro invariant for closed 3–manifolds is determined from the Reshetikhin–Turaev invariant, the Chen–Yang conjecture for the Turaev–Viro invariant for closed 3–manifolds is a consequence of the Chen–Yang conjecture for the Reshetikhin–Turaev invariant. This paper (whose first version was written in August 2016) gives a rigorous proof for the Chen–Yang conjecture for the Reshetikhin–Turaev invariant for closed hyperbolic 3–manifolds obtained by integral surgery along the figure-eight knot. Further, there has been recent progress on the Chen–Yang conjecture for the Turaev–Viro invariant for some cusped 3–manifolds. We note that the Turaev–Viro invariant for a cusped 3–manifold is defined for an ideal triangulation of the cusped 3–manifold, and it might not be known so far whether there is a direct relation between this invariant and the Reshetikhin–Turaev invariant. Recently, Detcherry, Kalfagianni and Yang [3] have proved the Chen–Yang conjecture for the Turaev–Viro invariant for the complements of the figure-eight knot and the Borromean rings. Further, Belletti, Detcherry, Kalfagianni and Yang [1] have proved the Chen–Yang conjecture for the Turaev–Viro invariant for the complements of infinitely many hyperbolic links called "fundamental shadow links".

The paper is organized as follows. In Section 2, we explain a physical background of this topic. In Section 3, we explain preliminaries, which we need in the proof of Theorem 1.1. In Section 4, we review the definition of the quantum SU(2) invariant of closed 3–manifolds, and give a concrete presentation of the value of the quantum SU(2) invariant of M_p. In Sections 5 and 7, we give a proof of Theorem 1.1, when $|p| \ge 6$ and when $|p| = 5$, respectively. In Section 6, we show some propositions, which we need in the proof of Theorem 1.1. In the appendices, we show some lemmas, which we use in the proof of Theorem 1.1.

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2 Physical background

A physical background of quantum invariants of 3–manifolds is the Chern–Simons field theory of Witten [26]; for details, see eg a book of the author [17]. Further, we can regard the volume conjecture as a perturbative expansion of the Chern–Simons path integral. We explain these in this section.

Let M be an oriented closed 3–manifold. Let $\mathcal{A}$ denote the set of connections on the trivial SU(2) bundle $\mathrm{SU}(2) \times M \to M$. We identify $\mathcal{A}$ with the set $\Omega^1(M; \mathfrak{sl}_2)$ of $\mathfrak{sl}_2$–valued 1–forms on M. The Chern–Simons functional CS: $\mathcal{A} \to \mathbb{R}$ is defined by

$$\mathrm{CS}(A) = \frac{1}{8\pi^2} \int_M \mathrm{trace}\Big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \Big)$$

for a connection A. The gauge group $\mathcal{G}$ is the group of automorphisms of the bundle $\mathrm{SU}(2) \times M \to M$, and it is known that, when $g \in \mathcal{G}$ takes $A \in \mathcal{A}$ to g^*A, $\mathrm{CS}(g^*A)$ differs from $\mathrm{CS}(A)$ by an integer. Hence, the Chern–Simons functional induces the map CS: $\mathcal{A}/\mathcal{G} \to \mathbb{R}/\mathbb{Z}$. The *Chern–Simons path integral* is formally given by

$$Z_N(M) = \int_{\mathcal{A}/\mathcal{G}} \exp(2\pi\sqrt{-1}\, N\, \mathrm{CS}(A))\, \mathcal{D}A \tag{1}$$

for any positive integer N. We note that, since $\mathcal{A}/\mathcal{G}$ is infinite-dimensional, the path integral has not been defined mathematically, but the Chern–Simons path integral gives many interesting suggestions to mathematics. Witten [26] proposed that this formal integral $Z_N(M)$ gives a topological invariant of a closed 3–manifold M; this is a physical background of the quantum invariant of M. In physics, there are two approaches available to obtain observables of a path integral: the operator formalism or the perturbative expansion.

The operator formalism induces a formulation of the invariant in terms of the topological quantum field theory; that is, we can compute the invariant by cutting the 3–manifold along surfaces. In particular, we can formulate the invariant of a closed 3–manifold M as a linear sum of quantum invariants of a link L such that M is obtained from S^3 by an integral surgery along L. In Section 4, we review a mathematical definition of the quantum SU(2) invariant of closed 3–manifolds along this formulation.

The volume conjecture is a recent important topic, which relates quantum topology and hyperbolic geometry. We briefly review the volume conjecture, and explain the volume conjecture from the viewpoint of the perturbative expansion of the Chern–Simons path integral, in this paragraph. Kashaev [7; 8] defined the Kashaev invariant $\langle L \rangle_N \in \mathbb{C}$ of a link L for $N = 2, 3, \ldots$ by using the quantum dilogarithm. He conjectured [9] that, for any hyperbolic link L, $\frac{2\pi}{N}\log\langle L \rangle_N$ goes to the hyperbolic volume of $S^3 - L$ as $N \to \infty$. In 1999, H Murakami and J Murakami [14] proved that the Kashaev invariant $\langle L \rangle_N$ of any link L is equal to the colored Jones polynomial $J_N(L; e^{2\pi\sqrt{-1}/N})$ of L at $e^{2\pi\sqrt{-1}/N}$, and conjectured that, for any knot K, $\frac{2\pi}{N}\log|J_N(K; e^{2\pi\sqrt{-1}/N})|$ goes to a normalized simplicial volume of $S^3 - K$, as an extension of Kashaev's conjecture. This is called *the volume conjecture*. As a complexification of the volume conjecture, it is conjectured by Murakami, Murakami, Okamoto, Takata and Yokota [15] that, for a hyperbolic link L,

$$2\pi\sqrt{-1} \cdot \lim_{N\to\infty} \frac{J_N(L; e^{2\pi\sqrt{-1}/N})}{N} = \mathrm{cs}(S^3 - L) + \sqrt{-1}\,\mathrm{vol}(S^3 - L),$$

where "cs" and "vol" denote the Chern–Simons invariant and the hyperbolic volume, respectively. We define a normalized complex volume by

$$\varsigma(L) = \frac{1}{2\pi\sqrt{-1}}(\mathrm{cs}(S^3 - L) + \sqrt{-1}\,\mathrm{vol}(S^3 - L)).$$

Further, as a refinement of the above conjecture, it is shown by the author and Yokota [18; 22; 19] that, for hyperbolic knots K with up to seven crossings, the asymptotic expansion of the Kashaev invariant is presented by the form

$$\langle K \rangle_N = e^{N\varsigma(K)} N^{3/2} \omega(K) \cdot \Bigl(1 + \sum_{i=1}^{d} \kappa_i(K) \cdot \Bigl(\frac{2\pi\sqrt{-1}}{N}\Bigr)^i + O\Bigl(\frac{1}{N^{d+1}}\Bigr)\Bigr)$$

for any integer $d \geq 1$, where $\omega(K)$ and the $\kappa_i(K)$ are some scalars determined by K. As for the quantum invariant of closed 3–manifolds, it has been expected by Murakami [13] that the quantum invariant of a closed 3–manifold has a similar property, though it is known that $\tau_N(M;q)$ at $q = e^{2\pi\sqrt{-1}}$ is of polynomial order as $N \to \infty$. As we mentioned in the introduction, recently, Chen and Yang [2] observed that $\hat{\tau}_N(M)$ is of exponential order as $N \to \infty$ for some hyperbolic 3–manifolds, and conjectured that

$$4\pi\sqrt{-1} \cdot \lim_{\substack{N\to\infty \\ N \text{ odd}}} \frac{\log \hat{\tau}_N(M)}{N} = \mathrm{cs}(M) + \sqrt{-1}\,\mathrm{vol}(M)$$

for a closed hyperbolic 3–manifold M. We define a normalized complex volume of M by

$$\varsigma(M) = \frac{1}{4\pi\sqrt{-1}}(\mathrm{cs}(M) + \sqrt{-1}\,\mathrm{vol}(M)).$$

We can regard this conjecture as a perturbative expansion of the Chern–Simons path integral, as follows. Let $\mathcal{A}_{\mathbb{C}}$ be the set of connections on the trivial $\mathrm{SL}(2;\mathbb{C})$ bundle on a closed 3–manifold M. We identify $\mathcal{A}_{\mathbb{C}}$ with the set $\Omega^1(M;\mathfrak{sl}_2\mathbb{C})$ of $\mathfrak{sl}_2\mathbb{C}$–valued 1–forms on M. The gauge group $\mathcal{G}_{\mathbb{C}}$ is the group of automorphisms of the bundle $\mathrm{SL}(2;\mathbb{C}) \times M \to M$. For a closed hyperbolic 3–manifold M, we apply the saddle-point method formally to the integral (1), by moving the domain $\mathcal{A}/\mathcal{G}$ in $\mathcal{A}_{\mathbb{C}}/\mathcal{G}_{\mathbb{C}}$ in such a way that the new domain contains the $\mathrm{SL}(2;\mathbb{C})$ flat connection corresponding to the holonomy representation of the hyperbolic structure of M. By expanding the path integral at this flat connection, we obtain the complex volume in the leading term. Further, in the second term (the part of the semiclassical limit), we obtain the Reidemeister torsion as the determinant of the quadratic part of the Chern–Simons functional (see Witten [26]). Furthermore, in the higher-order terms, we obtain a power series in $\frac{1}{N}$ by coupling the quadratic part and the higher-order part of the Chern–Simons functional (see eg a book of the author [17]). Hence, we can expect that the quantum invariant is expanded in such a form, and this is a physical background of the expansion of Theorem 1.1. We note that this expansion is obtained from contributions of a neighborhood of the flat $\mathrm{SL}(2,\mathbb{C})$ connection corresponding to the holonomy representation of the hyperbolic structure of M, and we can regard them as perturbations of the hyperbolic structure of M. So we expect that the higher-order coefficients $\kappa_i(M)$ in the expansion of Theorem 1.1 might be related to "quantization" of the hyperbolic structure of M in some sense.

3 Preliminaries

3.1 Integral presentation of $(q)_n$

In this section, we review integral presentations of $(q)_n$ and some of their properties.

We put

$$(x)_n = (1-x)(1-x^2)\cdots(1-x^n)$$

for $n \geq 0$.

Let N be an integer ≥ 3. It is known [5; 27] that

$$\begin{aligned} (e^{2\pi\sqrt{-1}/N})_n &= \exp\Bigl(\varphi\Bigl(\frac{1}{2N}\Bigr) - \varphi\Bigl(\frac{2n+1}{2N}\Bigr)\Bigr), \\ (e^{-2\pi\sqrt{-1}/N})_n &= \exp\Bigl(\varphi\Bigl(1-\frac{2n+1}{2N}\Bigr) - \varphi\Bigl(1-\frac{1}{2N}\Bigr)\Bigr). \end{aligned} \tag{2}$$

Here, following Faddeev [4], we define a holomorphic function $\varphi(t)$ on the domain $\{t \in \mathbb{C} \mid 0 < \operatorname{Re} t < 1\}$ by

$$\varphi(t) = \int_{-\infty}^{\infty} \frac{e^{(2t-1)x}\,dx}{4x \sinh x \sinh \frac{x}{N}}.$$

We note that this integrand has poles at $n\pi\sqrt{-1}$ $(n \in \mathbb{Z})$, and, to avoid the pole at 0, we choose the contour of the integral

$$\gamma = (-\infty, -1] \cup \{z \in \mathbb{C} \mid |z| = 1,\ \operatorname{Im} z \geq 0\} \cup [1, \infty).$$

Further, it is known (due to Kashaev — see [18]) that

$$\varphi\Bigl(\frac{1}{2N}\Bigr) - \varphi\Bigl(1-\frac{1}{2N}\Bigr) = \log N. \tag{3}$$

Furthermore, it is known [18] that

$$\varphi(t)+\varphi(1-t)=2\pi\sqrt{-1}\Big(-\frac{N}{2}\Big(t^2-t+\frac{1}{6}\Big)+\frac{1}{24N}\Big) \tag{4}$$

for $0<\operatorname{Re}t<1$. Moreover,

$$\frac{1}{N}\varphi(t) \quad \text{uniformly converges to} \quad \frac{1}{2\pi\sqrt{-1}}\operatorname{Li}_2(e^{2\pi\sqrt{-1}t}) \tag{5}$$

in the domain

$$\{t\in\mathbb{C}\mid \delta\le\operatorname{Re}t\le 1-\delta,\ |\operatorname{Im}t|\le M\} \tag{6}$$

for any sufficiently small $\delta>0$ and any $M>0$.

Let N be an odd integer ≥ 3. We put $q=e^{4\pi\sqrt{-1}/N}$. It follows from (2), replacing N with $\frac{N}{2}$, that

$$\begin{aligned}(q)_n&=\exp\Big(\hat\varphi\Big(\frac{1}{N}\Big)-\hat\varphi\Big(\frac{2n+1}{N}\Big)\Big),\\ (\overline{q})_n&=\exp\Big(\hat\varphi\Big(1-\frac{2n+1}{N}\Big)-\hat\varphi\Big(1-\frac{1}{N}\Big)\Big)\end{aligned} \tag{7}$$

for $0\le n<\frac{N}{2}$, where we define $\hat\varphi(t)$ on $\{t\in\mathbb{C}\mid 0<\operatorname{Re}t<1\}$ by

$$\hat\varphi(t)=\int_{-\infty}^{\infty}\frac{e^{(2t-1)x}\,dx}{4x\sinh x\sinh\frac{2x}{N}}.$$

We note that (7) holds only for $0\le n<\frac{N}{2}$ since $\hat\varphi(t)$ is defined for $0<\operatorname{Re}t<1$, though $(q)_n$ and $(\overline{q})_n$ themselves are defined for $0\le n<N$. Further, it follows from (3) and (4) that

$$\hat\varphi\Big(\frac{1}{N}\Big)-\hat\varphi\Big(1-\frac{1}{N}\Big)=\log\frac{N}{2}, \tag{8}$$

$$\hat\varphi(t)+\hat\varphi(1-t)=2\pi\sqrt{-1}\Big(-\frac{N}{4}\Big(t^2-t+\frac{1}{6}\Big)+\frac{1}{12N}\Big) \tag{9}$$

for $0<\operatorname{Re}t<1$. Furthermore, it follows from (5) that

$$\frac{1}{N}\hat\varphi(t) \quad \text{uniformly converges to} \quad \frac{1}{4\pi\sqrt{-1}}\operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t}) \tag{10}$$

in the domain (6). Moreover, we have the following lemma, which is a modification of a formula of Murakami and Murakami [14]:

Lemma 3.1 (modification of a formula in [14]) *We have that*

$$(q)_n(\overline{q})_{N-n-1}=N \tag{11}$$

for $0\le n<N$.

Proof We have that

$$(q)_n(\overline{q})_{N-n-1} = (1-q)(1-q^2)\cdots(1-q^n)\times(1-\overline{q}^{N-n-1})(1-\overline{q}^{N-n-2})\cdots(1-\overline{q})$$
$$= (1-q)(1-q^2)\cdots(1-q^{N-1}) = (q)_{N-1}.$$

Hence, it is sufficient to show that $(q)_{N-1} = N$. We have that

$$\begin{aligned}(q)_{N-1} &= \prod_{\substack{j\in\mathbb{Z}/N\mathbb{Z}\\ j\neq 0}} (1-e^{(4\pi\sqrt{-1}/N)\cdot j}) = \prod_{\substack{k\in\mathbb{Z}/N\mathbb{Z}\\ k\neq 0}} (1-e^{(2\pi\sqrt{-1}/N)\cdot k})\\ &= \prod_{1\le k\le (N-1)/2} (1-e^{(2\pi\sqrt{-1}/N)\cdot k})(1-e^{-(2\pi\sqrt{-1}/N)\cdot k})\\ &= \prod_{1\le k\le (N-1)/2} |1-e^{(2\pi\sqrt{-1}/N)\cdot k}|\cdot|1-e^{-(2\pi\sqrt{-1}/N)\cdot k}|\\ &= \prod_{1\le k\le N-1} 2\sin\frac{k\pi}{N} = N,\end{aligned}$$

where we obtain the second equality by replacing k with $2j$, and obtain the last equality by a formula in the proof of Proposition 4.2 of [14]. Therefore, we obtain the lemma. □

3.2 The saddle-point method

In this section, we review a proposition obtained from the saddle-point method.

Proposition 3.2 (see [18]) *Let A be a nonsingular symmetric complex 2×2 matrix, and let $\psi(z_1,z_2)$ and $r(z_1,z_2)$ be holomorphic functions of the forms*

$$\begin{aligned}\psi(z_1,z_2) &= \boldsymbol{z}^T A\boldsymbol{z} + r(z_1,z_2),\\ r(z_1,z_2) &= \sum_{i,j,k} b_{ijk}z_iz_jz_k + \sum_{i,j,k,l} c_{ijkl}z_iz_jz_kz_l + \cdots,\end{aligned} \tag{12}$$

defined in a neighborhood of $\mathbf{0}\in\mathbb{C}^2$. Suppose that the restriction of the domain

$$\{(z_1,z_2)\in\mathbb{C}^2 \mid \operatorname{Re}\psi(z_1,z_2) < 0\} \tag{13}$$

to a neighborhood of $\mathbf{0}\in\mathbb{C}^2$ is homotopy equivalent to S^1. Let D be an oriented disk embedded in $\mathbb{C}^2$ such that ∂D is included in the domain (13) whose inclusion is *homotopic to a homotopy equivalence to the above* S^1 *in the domain* (13). *Then*

$$\int_D e^{N\psi(z_1,z_2)} dz_1\, dz_2 = \frac{\pi}{N\sqrt{\det(-A)}}\Big(1 + \sum_{i=1}^{d} \frac{\lambda_i}{N^i} + O\Big(\frac{1}{N^{d+1}}\Big)\Big)$$

for any d, *where we choose the sign of* $\sqrt{\det(-A)}$ *as explained in* [18], *and the* λ_i *are constants presented by using coefficients of the expansion of* $\psi(z_1,z_2)$; *such presentations are obtained by formally expanding the formula*

$$1 + \sum_{i=1}^{\infty} \frac{\lambda_i}{N^i} = \exp\Big(N r\Big(\frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2}\Big)\Big) \exp\Big(-\frac{1}{4N}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}^T A^{-1} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\Big)\Big|_{w_1=w_2=0}.$$

For a proof of the proposition, see [18].

Remark 3.3 As mentioned in [18, Remark 3.6], we can extend Proposition 3.2 to the case where $\psi(z_1,z_2)$ depends on N in such a way that $\psi(z_1,z_2)$ is of the form

$$\psi(z_1,z_2) = \psi_0(z_1,z_2) + \psi_1(z_1,z_2)\frac{1}{N} + \psi_2(z_1,z_2)\frac{1}{N^2} + \cdots + \psi_m(z_1,z_2)\frac{1}{N^m} + r_m(z_1,z_2)\frac{1}{N^{m+1}},$$

where the $\psi_i(z_1,z_2)$ are holomorphic functions independent of N, and we assume that $\psi_0(z_1,z_2)$ satisfies the assumption of the proposition and $|r_m(z_1,z_2)|$ is bounded by a constant which is independent of N.

3.3 The Poisson summation formula

In this section, we review the Poisson summation formula and a proposition obtained from it.

Recall (see eg the book of Stein and Weiss [23]) that the Poisson summation formula states that

$$\sum_{\boldsymbol{m}\in\mathbb{Z}^n} f(\boldsymbol{m}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^n} \hat{f}(\boldsymbol{m}) \tag{14}$$

for a continuous integrable function f on $\mathbb{R}^n$ which satisfies that

$$|f(\boldsymbol{z})| \le C(1+|\boldsymbol{z}|)^{-n-\delta}, \quad |\hat{f}(\boldsymbol{z})| \le C(1+|\boldsymbol{z}|)^{-n-\delta} \tag{15}$$

for some $C, \delta > 0$, where $\widehat{f}$ is the Fourier transform of f defined by

$$\widehat{f}(\boldsymbol{w}) = \int_{\mathbb{R}^n} f(\boldsymbol{z}) e^{-2\pi\sqrt{-1}\,\boldsymbol{w}^T\boldsymbol{z}} d\boldsymbol{z}.$$

The following proposition is obtained from the Poisson summation formula:

Proposition 3.4 (see [18]) *For $(c_1, c_2) \in \mathbb{C}^2$ and an oriented disk D' in $\mathbb{R}^2$, we put*

$$\Lambda = \left\{\left(\frac{i}{N} + c_1, \frac{j}{N} + c_2\right) \in \mathbb{C}^2 \;\middle|\; i, j \in \mathbb{Z}, \left(\frac{i}{N}, \frac{j}{N}\right) \in D'\right\},$$
$$D = \{(t + c_1, s + c_2) \in \mathbb{C}^2 \mid (t, s) \in D' \subset \mathbb{R}^2\}.$$

Let $\psi(t, s)$ be a holomorphic function defined in a neighborhood of $\mathbf{0} \in \mathbb{C}^2$ including D. We assume that ∂D is included in the domain

$$\{(t, s) \in \mathbb{C}^2 \mid \operatorname{Re}\psi(t, s) < -\varepsilon_0\}$$

for some $\varepsilon_0 > 0$. Further, we assume that ∂D is null-homotopic in each of the domains

$$\{(t + \delta\sqrt{-1}, s) \in \mathbb{C}^2 \mid (t, s) \in D',\ \delta \ge 0,\ \operatorname{Re}\psi(t + \delta\sqrt{-1}, s) < 2\pi\delta\}, \tag{16}$$

$$\{(t - \delta\sqrt{-1}, s) \in \mathbb{C}^2 \mid (t, s) \in D',\ \delta \ge 0,\ \operatorname{Re}\psi(t - \delta\sqrt{-1}, s) < 2\pi\delta\}, \tag{17}$$

$$\{(t, s + \delta\sqrt{-1}) \in \mathbb{C}^2 \mid (t, s) \in D',\ \delta \ge 0,\ \operatorname{Re}\psi(t, s + \delta\sqrt{-1}) < 2\pi\delta\}, \tag{18}$$

$$\{(t, s - \delta\sqrt{-1}) \in \mathbb{C}^2 \mid (t, s) \in D',\ \delta \ge 0,\ \operatorname{Re}\psi(t, s - \delta\sqrt{-1}) < 2\pi\delta\}. \tag{19}$$

Then

$$\frac{1}{N^2} \sum_{(t,s)\in\Lambda} e^{N\psi(t,s)} = \int_D e^{N\psi(t,s)} dt\, ds + O(e^{-N\varepsilon})$$

for some $\varepsilon > 0$.

Remark 3.5 In the assumption of the proposition, if we use the formula

$$\{(t + \delta\sqrt{-1}, s) \in \mathbb{C}^2 \mid (t, s) \in D',\ \delta \ge 0,\ \operatorname{Re}\psi(t + \delta\sqrt{-1}, s) < 4\pi\delta\} \tag{20}$$

instead of (16), then the following formula holds instead of the resulting formula of the proposition:

$$\frac{1}{N^2} \sum_{(t,s)\in\Lambda} e^{N\psi(t,s)} = \int_D e^{N\psi(t,s)} dt\, ds + \int_D e^{N(\psi(t,s) + 2\pi\sqrt{-1}t)} dt\, ds + O(e^{-N\varepsilon})$$

for some $\varepsilon > 0$.

Remark 3.6 In the assumption of the proposition, if we use (20) and the formula

$$\{(t - \delta\sqrt{-1}, s) \in \mathbb{C}^2 \mid (t, s) \in D',\ \delta \ge 0,\ \operatorname{Re}\psi(t - \delta\sqrt{-1}, s) < 4\pi\delta\}, \tag{21}$$

instead of (16) and (17), then the following formula holds instead of the resulting formula of the proposition:

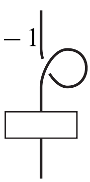
$$\frac{1}{N^2}\sum_{(t,s)\in\Lambda} e^{N\psi(t,s)} = \int_D e^{N(\psi(t,s)-2\pi\sqrt{-1}\,t)}dt\,ds + \int_D e^{N\psi(t,s)}dt\,ds + \int_D e^{N(\psi(t,s)+2\pi\sqrt{-1}\,t)}dt\,ds + O(e^{-N\varepsilon})$$

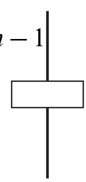
for some $\varepsilon > 0$.

4 The quantum SU(2) invariant

In this section, we review the definition of the quantum SU(2) invariant, and calculate it for the 3–manifold M_p obtained from S^3 by p surgery along the figure-eight knot K_{4_1} for a positive integer p.

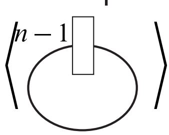
Let N be an odd integer ≥ 3. We review the definition of the quantum SU(2) invariant following the notation of Lickorish [12]. In this notation, we usually put A to be a $4N^{\text{th}}$ root of unity, but we note that, when N is odd, we can also put A to be a $2N^{\text{th}}$ root of unity; see [17]. We put $A = e^{\pi\sqrt{-1}/N}$ and $q = A^4 = e^{4\pi\sqrt{-1}/N}$. Let p be a positive integer, and let M_p be the 3–manifold obtained from S^3 by p surgery along the figure-eight knot K_{4_1}. We recall that



$= (-1)^{n-1}A^{n^2-1}$


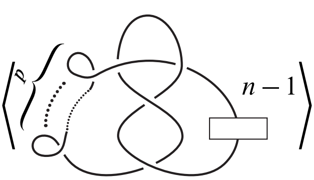
,

(22)



$= \Delta_{n-1} = (-1)^{n-1}\dfrac{A^{2n}-A^{-2n}}{A^2-A^{-2}}$

in the linear skein of [12]. Then, as in [12, Chapter 13], the quantum SU(2) invariant of M_p at $q = e^{4\pi\sqrt{-1}/N}$ is defined by

(23) $\widehat{\tau}_N(M_p) = \dfrac{1}{c_+}\displaystyle\sum_{n=1}^{N-1}\Delta_{n-1}$


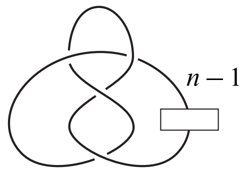
.

Here, c_+ is a constant given by

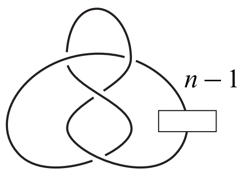
$$c_+ = \sum_{n=1}^{N-1} \Delta_{n-1} \Big\langle$$

$$\Big\rangle = \sum_{n=1}^{N-1} (-1)^{n-1} A^{n^2-1} \frac{(A^{2n} - A^{-2n})^2}{(A^2 - A^{-2})^2},$$

where we obtain the second equality by (22). Further, by (22), the formula (23) is rewritten as

$$\hat{\tau}_N(M_p) = \frac{1}{c_+} \sum_{n=1}^{N-1} ((-1)^{n-1} A^{n^2-1})^p \cdot (-1)^{n-1} \frac{A^{2n} - A^{-2n}}{A^2 - A^{-2}} \Big\langle$$

$$\Big\rangle.$$

We put

$$J_n(K_{4_1}) = \Big\langle$$

$$\Big\rangle.$$

Here, $J_n(K_{4_1})$ is the n^{th} colored Jones polynomial of the figure-eight knot K_{4_1}, where we normalize the colored Jones polynomial in such a way that

$$J_n(\text{trivial knot}) = (-1)^{n-1} \frac{A^{2n} - A^{-2n}}{A^2 - A^{-2}}.$$

Moreover, it is known [6; 11] that the colored Jones polynomial of the figure-eight knot can be presented by

$$J_n(K_{4_1}) = \frac{(-1)^{n-1}}{A^2 - A^{-2}} \sum_{j=0}^{n-1} (A^{2(n+j)} - A^{-2(n+j)})(A^{2(n+j-1)} - A^{-2(n+j-1)}) \cdots (A^{2(n-j)} - A^{-2(n-j)}).$$

We calculate c_+, as follows:

$$\begin{aligned} -(q^{1/2} - q^{-1/2})^2 c_+ &= \sum_{n=1}^{N-1} (-1)^n q^{(n^2-1)/4} (q^{n/2} - q^{-n/2})^2 \\ &= 2q^{-1/4}(q^{-1} - 1) \sum_{n \in \mathbb{Z}/n\mathbb{Z}} (-A)^{n^2} \\ &= 2q^{-1/4}(q^{-1} - 1)(-\sqrt{-1})^{(N-1)/2} \sqrt{N}, \end{aligned}$$

where we obtain the last equality by Lemma A.1. Further, we calculate $J_n(K_{4_1})$, as follows:

$$\begin{aligned} J_n(K_{4_1}) &= \frac{(-1)^n}{q^{1/2}-q^{-1/2}} \sum_{j=0}^{n-1} (1-q^{n+j})(1-q^{n+j-1}) \cdots (1-q^{n-j}) \cdot q^{-n(2j+1)/2} \\ &= \frac{(-1)^n}{q^{1/2}-q^{-1/2}} \sum_{j=0}^{n-1} q^{-n(2j+1)/2} \frac{(q)_{n+j}}{(q)_{n-j-1}}. \end{aligned}$$

Hence, we calculate $\widehat{\tau}_N(M_p)$, as follows:

$$\begin{aligned} \widehat{\tau}_N(M_p) &= \frac{-1}{c_+(q^{1/2}-q^{-1/2})} \sum_{j=0}^{n-1} ((-1)^{n-1} q^{(n^2-1)/4})^p (-1)^n (q^{n/2}-q^{-n/2}) J_n(K_{4_1}) \\ &= \frac{-1}{c_+(q^{1/2}-q^{-1/2})^2} \\ &\qquad \times \sum_{0 \le j < n < N} (-1)^{p(n-1)} q^{p(n^2-1)/4} q^{-n(2j+1)/2} (q^{n/2}-q^{-n/2}) \frac{(q)_{n+j}}{(q)_{n-j-1}} \\ &= \frac{(-1)^p q^{(5-p)/4}}{2(1-q)} \sqrt{-1}^{(N-1)/2} N^{-1/2} \\ &\qquad\qquad \times \sum_{\substack{0 \le j < n < N \\ n+j<N}} (-1)^{pn} q^{pn^2/4-nj} (1-q^{-n}) \frac{(q)_{n+j}}{(q)_{n-j-1}}, \end{aligned}$$

where we obtain the restriction "$n+j<N$" in the sum of the last line since $(q)_{n+j}=0$ for $n+j \ge N$.

5 Proof of Theorem 1.1 when $|p| \ge 6$

In this section, we give a proof of Theorem 1.1 when $|p| \ge 6$. (When $|p| = 5$, we give a proof of the theorem in Section 7, where there is the technical difficulty that Lemma 5.1 fails for $|p| = 5$, and we need an additional procedure there.)

Since the figure-eight know is isotopic to its mirror image, M_{-p} is homeomorphic to M_p with opposite orientation, and $\widehat{\tau}_N(M_{-p}) = \overline{\widehat{\tau}_N(M_p)}$. Hence, it is sufficient to show the theorem for $p > 4$, since the theorem for a negative p can be obtained from the theorem for a positive p. We assume that p is an integer ≥ 6 in this section.

As we calculated in the previous section, $\widehat{\tau}_N(M_p)$ is presented by

$$\widehat{\tau}_N(M_p) = \frac{(-1)^p q^{(5-p)/4}}{2(1-q)} \sqrt{-1}^{(N-1)/2} N^{-1/2} \times \sum_{\substack{0\le j<i<N \\ i+j<N}} (-1)^{pi} q^{pi^2/4-ij} (1-q^{-i}) \frac{(q)_{i+j}}{(q)_{i-j-1}}$$
$$= \frac{(-1)^p q^{(5-p)/4}}{2(1-q)} \sqrt{-1}^{(N-1)/2} N^{-1/2} \times \sum_{\substack{0\le j<i<N \\ i+j<N}} (-1)^{pi} q^{pi^2/4-ij} (1-q^{-i}) \frac{N}{(q)_{i-j-1}(\overline{q})_{N-i-j-1}},$$

where we obtain the second equality by (11). Further, by (7) and (8),

$$\widehat{\tau}_N(M_p) = \frac{(-1)^p q^{(5-p)/4}}{1-q} \sqrt{-1}^{(N-1)/2} N^{-1/2} \sum_{\substack{0\le j<i<N \\ i+j<N}} (-1)^{pi} (1-q^{-i}) \times \exp\Big(\frac{4\pi\sqrt{-1}}{N}\Big(\frac{p}{4}i^2-ij\Big)+\widehat{\varphi}\Big(\frac{2(i-j)-1}{N}\Big)-\widehat{\varphi}\Big(1-\frac{2(N-i-j)-1}{N}\Big)\Big).$$

Hence,

$$\widehat{\tau}_N(M_p) = \frac{q^{(5-p)/4}}{q-1} (-1)^p e^{-(\pi\sqrt{-1}/4)pN} \sqrt{-1}^{(N-1)/2} N^{-1/2} \times \sum_{\substack{0\le j<i<N \\ i+j<N}} (1-q^{-i}) \exp\Big(N\cdot \widetilde{V}\Big(\frac{1}{2}-\frac{i}{N}, \frac{1}{2}-\frac{j}{N}\Big)\Big),$$

where we put

$$\widetilde{V}(t,s) = \frac{1}{N}\Big(\widehat{\varphi}\Big(2s-2t-\frac{1}{N}\Big)-\widehat{\varphi}\Big(1-2t-2s+\frac{1}{N}\Big)\Big)+4\pi\sqrt{-1}\Big(\frac{p}{4}t^2-ts\Big),$$

since, putting $t=\frac{1}{2}-\frac{i}{N}$ and $s=\frac{1}{2}-\frac{j}{N}$,

$$\exp\Big(N\cdot 4\pi\sqrt{-1}\,\Big(\frac{p}{4}t^2-ts\Big)\Big)$$
$$= \exp\Big(N\cdot 4\pi\sqrt{-1}\Big(\frac{p}{4}\Big(\frac{i^2}{N^2}-\frac{i}{N}+\frac{1}{4}\Big)-\Big(\frac{ij}{N^2}-\frac{i}{2N}-\frac{j}{2N}+\frac{1}{4}\Big)\Big)\Big)$$
$$= \exp\Big(\frac{4\pi\sqrt{-1}}{N}\Big(\frac{p}{4}i^2-ij\Big)\Big)\exp\Big(4\pi\sqrt{-1}\,\Big(-\frac{p}{4}i+\frac{i}{2}+\frac{j}{2}\Big)\Big)\exp\Big(\pi\sqrt{-1}\,N\Big(\frac{p}{4}-1\Big)\Big)$$
$$= \exp\Big(\frac{4\pi\sqrt{-1}}{N}\Big(\frac{p}{4}i^2-ij\Big)\Big)(-1)^{pi} e^{(\pi\sqrt{-1}/4)pN}(-1).$$

Further, we replace s of $\widetilde{V}(t,s)$ with $s+\frac{1}{2N}$ in the following way:

$$\begin{aligned}\widetilde{V}\Big(t, s+\frac{1}{2N}\Big) &= \frac{1}{N}(\widehat{\varphi}(2s-2t)-\widehat{\varphi}(1-2t-2s)) + 4\pi\sqrt{-1}\Big(\frac{p}{4}t^2 - t\Big(s+\frac{1}{2N}\Big)\Big) \\ &= \frac{1}{N}(\widehat{\varphi}(2s-2t)-\widehat{\varphi}(1-2t-2s)) + 4\pi\sqrt{-1}\Big(\frac{p}{4}t^2 - ts\Big) - \frac{2\pi\sqrt{-1}}{N}t.\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}\hat{\tau}_N(M_p) = {}& \frac{q^{(5-p)/4}}{q-1}(-1)^p e^{-(\pi\sqrt{-1}/4)pN}\sqrt{-1}^{(N-1)/2} N^{-1/2} \\ &\times \sum_{\substack{0\le j<i<N \\ i+j<N}} (q^{i/2}-q^{-i/2}) \exp\Big(N\cdot V\Big(\frac{1}{2}-\frac{i}{N}, \frac{1}{2}-\frac{j}{N}-\frac{1}{2N}\Big)\Big),\end{aligned} \tag{24}$$

where we put

$$V(t,s) = \frac{1}{N}(\widehat{\varphi}(2s-2t)-\widehat{\varphi}(1-2t-2s)) + 4\pi\sqrt{-1}\Big(\frac{p}{4}t^2 - ts\Big).$$

By (10), $V(t,s)$ converges to

$$\widehat{V}(t,s) = \frac{1}{4\pi\sqrt{-1}}(\mathrm{Li}_2(e^{4\pi\sqrt{-1}(s-t)}) - \mathrm{Li}_2(e^{-4\pi\sqrt{-1}(t+s)})) + 4\pi\sqrt{-1}\Big(\frac{p}{4}t^2 - ts\Big).$$

We note that the summand of (24) contributes to the formula of Theorem 1.1 only when $\operatorname{Re} V(t,s) \ge \varsigma_R(M_p)$, where $\varsigma_R(M_p)$ is as we define in Section 5.1. Hence, in order to prove the theorem, it is sufficient to consider the domain $\{\operatorname{Re}\widehat{V}(t,s) \ge \varsigma_R(M_p)\}$, instead of the whole domain

$$\Delta = \big\{(t,s)\in\mathbb{R}^2 \mid s\ge t,\ s\ge -t,\ s\le \tfrac{1}{2}\big\}.$$

As shown in Figure 1, the domain $\{\operatorname{Re}\widehat{V}(t,s) \ge \varsigma_R(M_p)\}$ has three connected components. Corresponding to these three components, we decompose Δ into the three parts

$$\begin{aligned}\Delta_0 &= \big\{(t,s)\in\Delta \mid t+s\le\tfrac{1}{2},\ s-t\le\tfrac{1}{2}\big\} = \big\{(t,s)\in\mathbb{R}^2 \mid 0\le t+s\le\tfrac{1}{2},\ 0\le s-t\le\tfrac{1}{2}\big\}, \\ \Delta_1 &= \big\{(t,s)\in\Delta \mid s-t>\tfrac{1}{2}\big\}, \\ \Delta_2 &= \big\{(t,s)\in\Delta \mid t+s>\tfrac{1}{2}\big\}.\end{aligned}$$

In Section 6, we show that the contributions from Δ_1 and Δ_2 are sufficiently small, and we can ignore them. So it is sufficient to calculate the contribution from Δ_0.

In fact, we can further restrict Δ_0 to Δ_0' of the following lemma:

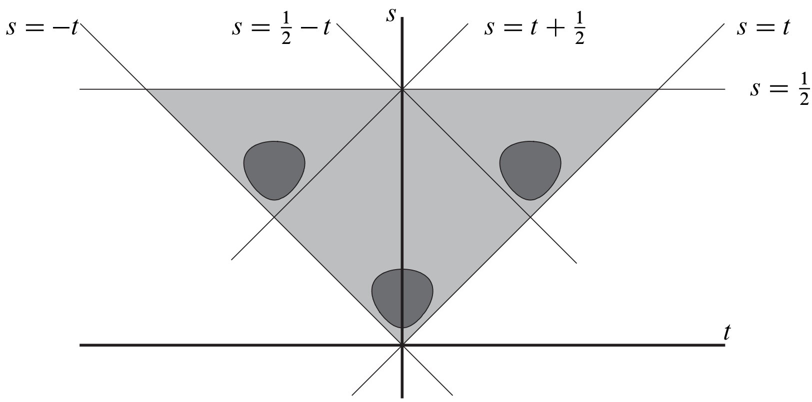


Figure 1: The light gray area is Δ and the dark gray area is $\{\operatorname{Re}\widehat{V}(t,s)\geq\varsigma_R(M_p)\}$ for $p=8$

Lemma 5.1 *We put*

$$\Delta_0'=\Big\{(t,s)\in\mathbb{R}^2\ \Big|\ 0.005\leq t+s\leq 0.24,\ 0.005\leq s-t\leq 0.24,\ |t|\leq\frac{0.74}{p}\Big\}.$$

Then the domain

$$\{(t,s)\in\Delta_0\mid\operatorname{Re}\widehat{V}(t,s)\geq\varsigma_R(M_p)-\varepsilon\}$$

is included in Δ_0' for sufficiently small $\varepsilon>0$.

We give a proof of the lemma in Appendix E. See Figure 2 for graphical representations of the inclusion of the lemma for $p=6,\ 12$.

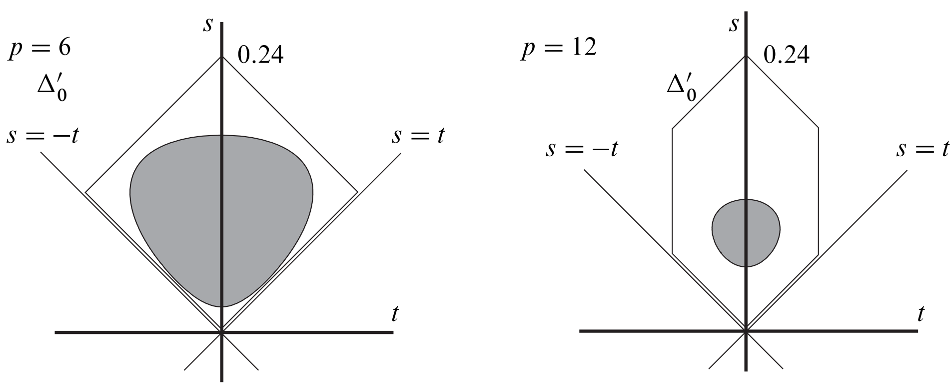


Figure 2: The domain $\{\operatorname{Re}\widehat{V}(t,s)\geq\varsigma_R(M_p)\}$ is included in Δ_0'

Proof of Theorem 1.1 We recall that $\hat{\tau}_N(M_p)$ is presented by the sum (24). Hence, by the above argument, we obtain that

$$\text{(25)}\quad \hat{\tau}_N(M_p) = \frac{q^{(5-p)/4}}{q-1}(-1)^p e^{-(\pi\sqrt{-1}/4)pN}\sqrt{-1}^{(N-1)/2} N^{-1/2} \times \sum_{(1/2-i/N,1/2-j/N)\in\Delta_0'} (q^{i/2}-q^{-i/2}) \times\exp\Big(N\cdot V\Big(\frac{1}{2}-\frac{i}{N},\frac{1}{2}-\frac{j}{N}-\frac{1}{2N}\Big)\Big) +O(e^{N(\varsigma_R(M_p)-\varepsilon)}).$$

By Proposition 3.4 (Poisson summation formula) (see also Remark 3.5), this sum is expressed by the integrals

$$\hat{\tau}_N(M_p) = \frac{q^{(5-p)/4}}{q-1}(-1)^p e^{-(\pi\sqrt{-1}/4)pN}\sqrt{-1}^{(N-1)/2} N^{3/2} \times\Big(\int_{\Delta_0'}(e^{-2\pi\sqrt{-1}\,t}-e^{2\pi\sqrt{-1}\,t})\exp(N\,V(t,s))\,dt\,ds+O(e^{N(\varsigma_R(M_p)-\varepsilon)}) + \int_{\Delta_0'}(e^{-2\pi\sqrt{-1}\,t}-e^{2\pi\sqrt{-1}\,t})\exp\big(N(V(t,s)+2\pi\sqrt{-1}\,t)\big)\,dt\,ds + O(e^{N(\varsigma_R(M_p)-\varepsilon)})\Big)$$

for some $\varepsilon > 0$, noting that we verify the assumptions of Proposition 3.4 in Section 5.3. We note that, by Lemma 5.2 below, the second integral of the above formula is equal to

$$\int_{\Delta_0'}(e^{-2\pi\sqrt{-1}\,t}-e^{2\pi\sqrt{-1}\,t})\exp(N\,V(-t,s))\,dt\,ds = \int_{\Delta_0'}(e^{-2\pi\sqrt{-1}\,t'}-e^{2\pi\sqrt{-1}\,t'})\exp(N\,V(t',s))\,dt'\,ds$$

putting $t' = -t$, which is equal to the first integral of the above formula of $\hat{\tau}_N(M_p)$. Therefore,

$$\text{(26)}\quad \hat{\tau}_N(M_p) = \frac{2q^{(5-p)/4}}{q-1}(-1)^p e^{-(\pi\sqrt{-1}/4)pN}\sqrt{-1}^{(N-1)/2} N^{3/2} \times\int_{\Delta_0'}(e^{-2\pi\sqrt{-1}\,t}-e^{2\pi\sqrt{-1}\,t})\exp(N\,V(t,s))\,dt\,ds + O(e^{N(\varsigma_R(M_p)-\varepsilon)}).$$

In order to apply the saddle-point method (Proposition 3.2), we consider a critical point of $\widehat{V}(t,s)$. The differentials of $\widehat{V}(t,s)$ are given by

$$\begin{aligned} \frac{\partial \widehat{V}}{\partial t} &= \log(1-e^{4\pi\sqrt{-1}(s-t)}) - \log(1-e^{-4\pi\sqrt{-1}(t+s)}) + 4\pi\sqrt{-1}\Big(\frac{p}{2}t - s\Big), \\ \frac{\partial \widehat{V}}{\partial s} &= -\log(1-e^{4\pi\sqrt{-1}(s-t)}) - \log(1-e^{-4\pi\sqrt{-1}(t+s)}) - 4\pi\sqrt{-1}\,t. \end{aligned} \tag{27}$$

Further, putting $z = e^{-4\pi\sqrt{-1}\,t}$ and $w = e^{-4\pi\sqrt{-1}\,s}$, their differentials are given by

$$\begin{aligned} \frac{\partial^2 \widehat{V}}{\partial t^2} &= 4\pi\sqrt{-1}\Big(\frac{e^{4\pi\sqrt{-1}(s-t)}}{1-e^{4\pi\sqrt{-1}(s-t)}} - \frac{e^{-4\pi\sqrt{-1}(t+s)}}{1-e^{-4\pi\sqrt{-1}(t+s)}} + \frac{p}{2}\Big) \\ &= 4\pi\sqrt{-1}\Big(\frac{z/w}{1-z/w} - \frac{zw}{1-zw} + \frac{p}{2}\Big) = 4\pi\sqrt{-1}\Big(\frac{z}{w-z} - \frac{zw}{1-zw} + \frac{p}{2}\Big), \\ \frac{\partial^2 \widehat{V}}{\partial t\,\partial s} &= 4\pi\sqrt{-1}\Big(-\frac{z}{w-z} - \frac{zw}{1-zw} - 1\Big), \\ \frac{\partial^2 \widehat{V}}{\partial s^2} &= 4\pi\sqrt{-1}\Big(\frac{z}{w-z} - \frac{zw}{1-zw}\Big). \end{aligned}$$

Let (t_0, s_0) be the critical point of $\widehat{V}(t_0, s_0)$ given in Section 5.1. We put

$$V_{tt} = \frac{\partial^2 \widehat{V}}{\partial t^2}(t_0,s_0), \qquad V_{ts} = \frac{\partial^2 \widehat{V}}{\partial t\,\partial s}(t_0,s_0), \qquad V_{ss} = \frac{\partial^2 \widehat{V}}{\partial s^2}(t_0,s_0).$$

Then, by applying Proposition 3.2 (see also Remark 3.3) to (26), we obtain that

$$\begin{aligned} \hat{\tau}_N(M_p) = \frac{2q^{(5-p)/4}}{q-1}(-1)^p e^{-(\pi\sqrt{-1}/4)pN}\sqrt{-1}^{(N-1)/2} N^{3/2} \\ \times (e^{-2\pi\sqrt{-1}t_0} - e^{2\pi\sqrt{-1}\,t_0}) \exp(N\widehat{V}(t_0,s_0)) \cdot \frac{2\pi}{N}(V_{tt}V_{ss} - V_{ts}^2)^{-1/2} \\ \times \Big(1 + O\Big(\frac{1}{N}\Big)\Big), \end{aligned}$$

noting that we verify the assumption of Proposition 3.2 in Section 5.2.

Further, noting that $q-1 = 4\pi\sqrt{-1}/N + O(1/N^2)$, we obtain that

$$\hat{\tau}_N(M_p) = (-1)^p e^{-(\pi\sqrt{-1}/4)pN}\sqrt{-1}^{(N-3)/2} \exp(N\widehat{V}(t_0,s_0)) N^{3/2}\omega(M_p) \times \Big(1 + O\Big(\frac{1}{N}\Big)\Big),$$

where

$$\omega(M_p) = (e^{-2\pi\sqrt{-1}\,t_0} - e^{2\pi\sqrt{-1}\,t_0})(V_{tt}V_{ss} - V_{ts}^2)^{-1/2}.$$

Hence, we obtain the required formula of the theorem. □

We used the following lemma in the above proof of the theorem:

Lemma 5.2 *We suppose that* $0<\mathrm{Re}(t+s)<\frac{1}{4}$ *and* $0<\mathrm{Re}(s-t)<\frac{1}{4}$. *Then*

$$V(-t,s)=V(t,s)+2\pi\sqrt{-1}\,t.$$

Proof From the definition of $V(t,s)$, we have that

$$V(-t,s)=\frac{1}{N}(\widehat{\varphi}(2s+2t)-\widehat{\varphi}(1-2s+2t))+4\pi\sqrt{-1}\Big(\frac{p}{4}t^2+ts\Big).$$

Further, by (9), we have that

$$\widehat{\varphi}(2s+2t)=-\widehat{\varphi}(1-2s-2t)+2\pi\sqrt{-1}\Big(-\frac{N}{4}\Big((2t+2s)^2-(2t+2s)+\frac{1}{6}\Big)+\frac{1}{12N}\Big),$$
$$\widehat{\varphi}(1-2s+2t)=-\widehat{\varphi}(2s-2t)+2\pi\sqrt{-1}\Big(-\frac{N}{4}\Big((2s-2t)^2-(2s-2t)+\frac{1}{6}\Big)+\frac{1}{12N}\Big).$$

Hence,

$$\begin{aligned}V(-t,s)&=\frac{1}{N}(\widehat{\varphi}(2s-2t)-\widehat{\varphi}(1-2t-2s))\\&\qquad+4\pi\sqrt{-1}\Big(\frac{p}{4}t^2+ts-\frac{1}{4}(2t+2s)^2+\frac{1}{4}(2s-2t)^2+\frac{1}{4}(2t+2s)\\&\qquad\qquad-\frac{1}{4}(2s-2t)\Big)\\&=\frac{1}{N}(\widehat{\varphi}(2s-2t)-\widehat{\varphi}(1-2t-2s))+4\pi\sqrt{-1}\Big(\frac{p}{4}t^2-ts+\frac{1}{2}t\Big)\\&=V(t,s)+2\pi\sqrt{-1}\,t,\end{aligned}$$

as required. □

5.1 A critical point of $\widehat{V}(t,s)$

In this section, we characterize a critical point (t_0,s_0) of $\widehat{V}(t,s)$, which we use in the proof of Theorem 1.1.

As shown in Appendix B, there exists a single critical point (t_0,s_0) of $\widehat{V}(t,s)$ in the domain

$$\big\{(t,s)\in\mathbb{C}^2\mid 0<\mathrm{Re}(t+s)<\tfrac{1}{4},\ 0<\mathrm{Re}(s-t)<\tfrac{1}{4},\ \mathrm{Re}\,t\ge 0\big\}. \tag{28}$$

We calculate this critical point concretely. By (27), a critical point of $\widehat{V}(t,s)$ is a solution of the equations

$$\begin{aligned}\log(1-e^{4\pi\sqrt{-1}(s-t)})-\log(1-e^{-4\pi\sqrt{-1}(t+s)})+4\pi\sqrt{-1}\Big(\frac{p}{2}t-s\Big)&=0,\\-\log(1-e^{4\pi\sqrt{-1}(s-t)})-\log(1-e^{-4\pi\sqrt{-1}(t+s)})-4\pi\sqrt{-1}\,t&=0,\end{aligned} \tag{29}$$

where we choose the branch of the log in the way that $-\pi < \operatorname{Im}\log(\cdot) < \pi$, noting that $\operatorname{Im}(1-e^{4\pi\sqrt{-1}(s-t)}) > 0$ and $\operatorname{Im}(1-e^{-4\pi\sqrt{-1}(t+s)}) < 0$. Putting $z = e^{-4\pi\sqrt{-1}\,t}$ and $w = e^{-4\pi\sqrt{-1}\,s}$, the above equations are rewritten as

$$z^{p/2}(1-zw) = w-z, \quad (1-zw)(w-z) = zw. \tag{30}$$

Further, as shown in [13], they are rewritten as

$$z^2 - \left(\frac{z+z^{p/2}}{z^{p/2}z+1} + 1 + \frac{z^{p/2}z+1}{z+z^{p/2}}\right)z + 1 = 0, \quad w = \frac{z+z^{p/2}}{z^{p/2}z+1}.$$

Furthermore, since

$$z - \left(\frac{z+z^{p/2}}{z^{p/2}z+1} + 1 + \frac{z^{p/2}z+1}{z+z^{p/2}}\right) + \frac{1}{z}$$
$$= -\frac{z^{p/2}z(z^{p/2}+z^{-p/2}-z^2-z^{-2}+z+z^{-1}+2)}{(z^{p/2}z+1)(z+z^{p/2})},$$

the above equations are rewritten as

$$z^{p/2}+z^{-p/2} = z^2+z^{-2}-z-z^{-1}-2, \quad w = \frac{z+z^{p/2}}{z^{p/2}z+1}. \tag{31}$$

By Lemma B.1, there exists a single solution of these equation which satisfies (28) and (29); we denote it by (t_0, s_0). For a concretely given p, we can obtain a numerical solution by calculating solutions of the above equations; we show some of concrete values of such numerical solutions below:

p	(t_0, s_0)
6	$(0.0743075\ldots - \sqrt{-1}\cdot 0.0382219\ldots, 0.1128050\ldots - \sqrt{-1}\cdot 0.0314723\ldots)$
7	$(0.0640105\ldots - \sqrt{-1}\cdot 0.0283809\ldots, 0.1065380\ldots - \sqrt{-1}\cdot 0.0212048\ldots)$
8	$(0.0566257\ldots - \sqrt{-1}\cdot 0.0221934\ldots, 0.1022661\ldots - \sqrt{-1}\cdot 0.0152090\ldots)$
9	$(0.0509104\ldots - \sqrt{-1}\cdot 0.0179265\ldots, 0.0991274\ldots - \sqrt{-1}\cdot 0.0113510\ldots)$
10	$(0.0462978\ldots - \sqrt{-1}\cdot 0.0148180\ldots, 0.0967225\ldots - \sqrt{-1}\cdot 0.0087183\ldots)$

Further, we put

$$\varsigma(M_p) = \widehat{V}(t_0, s_0), \quad \varsigma_R(M_p) = \operatorname{Re}\widehat{V}(t_0, s_0).$$

We note that $\varsigma(M_p)$ is a normalized complex volume of M_p; see [13; 16] for concrete formulas of the complex volume of M_p. In fact, as shown in Appendix B, we can give the hyperbolic structure of M_p by using the solution (t_0, s_0) of Lemma B.1, and a normalized hyperbolic volume of M_p is given by $\operatorname{Re}\widehat{V}(t_0, s_0)$. We show some numerical values of the complex volume of M_p, as follows:

p	$\varsigma(M_p)$	$4\pi\sqrt{-1}\varsigma(M_p) = \mathrm{cs}(M_p) + \sqrt{-1}\mathrm{vol}(M_p)$
6	$0.102216092\ldots - \sqrt{-1}\cdot 0.106706823\ldots$	$1.340917487\ldots + \sqrt{-1}\cdot 1.284485300\ldots$
7	$0.116483644\ldots - \sqrt{-1}\cdot 0.095216750\ldots$	$1.196528981\ldots + \sqrt{-1}\cdot 1.463776644\ldots$
8	$0.125984399\ldots - \sqrt{-1}\cdot 0.085824359\ldots$	$1.078500712\ldots + \sqrt{-1}\cdot 1.583166660\ldots$
9	$0.132719661\ldots - \sqrt{-1}\cdot 0.078024388\ldots$	$0.980483376\ldots + \sqrt{-1}\cdot 1.667804452\ldots$
10	$0.137695916\ldots - \sqrt{-1}\cdot 0.071457335\ldots$	$0.897959363\ldots + \sqrt{-1}\cdot 1.730337923\ldots$

5.2 Verifying the assumption of the saddle-point method for V

In this section, in Lemma 5.6, we verify the assumption of the saddle-point method (Proposition 3.2 and Remark 3.3) when we apply Proposition 3.2 and Remark 3.3 to (26).

We note that, by (10), $V(t,s)$ uniformly converges to $\widehat{V}(t,s)$ on Δ'_0 as $N \to \infty$. So we verify the assumption of the saddle-point method for $\widehat{V}(t,s)$. To simplify the calculation of the behavior of $\widehat{V}(t,s)$, we change the variables (t,s) to (u,v) by putting $u = s+t$ and $v = s-t$. They are in the ranges that $0 < u < \frac{1}{4}$ and $0 < v < \frac{1}{4}$. Then $\widehat{V}(t,s)$ is rewritten as

$$\check{V}(u,v) = \frac{1}{4\pi\sqrt{-1}}(\mathrm{Li}_2(e^{4\pi\sqrt{-1}\,v}) - \mathrm{Li}_2(e^{-4\pi\sqrt{-1}\,u})) + 4\pi\sqrt{-1}\Big(\frac{p}{4}\cdot\frac{(u-v)^2}{4} + \frac{v^2-u^2}{4}\Big).$$

In order to show Lemmas 5.3 and 5.5 below, we calculate the behavior of the function

$$f_{u,v}(\delta_1,\delta_2) = \operatorname{Re}\check{V}(u+\delta_1\sqrt{-1}, v+\delta_2\sqrt{-1}).$$

The differentials of this function are given by

$$\begin{aligned}(32)\qquad \frac{\partial}{\partial\delta_1} f_{u,v}(\delta_1,\delta_2) &= \operatorname{Re}\Big(\sqrt{-1}\frac{\partial}{\partial u}\check{V}(u+\delta_1\sqrt{-1}, v+\delta_2\sqrt{-1})\Big)\\ &= \operatorname{Im}\Big(\log\Big(1-\frac{1}{x}\Big)\Big) + 4\pi\Big(-\frac{p}{4}\cdot\frac{u-v}{2} + \frac{u}{2}\Big)\\ &= \operatorname{Arg}\Big(1-\frac{1}{x}\Big) + 4\pi\Big(-\frac{p}{4}\cdot\frac{u-v}{2} + \frac{u}{2}\Big),\end{aligned}$$

$$\begin{aligned}(33)\qquad \frac{\partial}{\partial\delta_2} f_{u,v}(\delta_1,\delta_2) &= \operatorname{Re}\Big(\sqrt{-1}\frac{\partial}{\partial v}\check{V}(u+\delta_1\sqrt{-1}, v+\delta_2\sqrt{-1})\Big)\\ &= \operatorname{Im}(\log(1-y)) + 4\pi\Big(\frac{p}{4}\cdot\frac{u-v}{2} - \frac{v}{2}\Big)\\ &= \operatorname{Arg}(1-y) + 4\pi\Big(\frac{p}{4}\cdot\frac{u-v}{2} - \frac{v}{2}\Big),\end{aligned}$$

where $x = e^{4\pi\sqrt{-1}(u+\delta_1\sqrt{-1})}$ and $y = e^{4\pi\sqrt{-1}(v+\delta_2\sqrt{-1})}$.

Lemma 5.3 *Fixing $(u,v) \in \Delta'_0$ and $\delta_2 \in \mathbb{R}$, we regard $f_{u,v}(X,\delta_2)$ as a function of $X \in \mathbb{R}$.*

(1) *If $v \geq \bigl(1-\frac{4}{p}\bigr)u$, then $f_{u,v}(X,\delta_2)$ is monotonically increasing for $X \in \mathbb{R}$.*

(2) *If $\bigl(1+\frac{4}{p}\bigr)u - \frac{2}{p} < v < \bigl(1-\frac{4}{p}\bigr)u$, then $f_{u,v}(X,\delta_2)$ has a unique minimal point at $X = g_1(u,v)$, where*

$$g_1(u,v) = \frac{1}{4\pi} \log \frac{\sin 4\pi\bigl(\frac{p}{4}\cdot\frac{u-v}{2} - \frac{u}{2}\bigr)}{\sin 4\pi\bigl(\frac{1}{4} - \frac{p}{4}\cdot\frac{u-v}{2} - \frac{u}{2}\bigr)},$$

ie $f_{u,v}(X,\delta_2)$ is monotonically decreasing for $X < g_1(u,v)$, and is monotonically increasing for $X > g_1(u,v)$.

(3) *If $v \leq \bigl(1+\frac{4}{p}\bigr)u - \frac{2}{p}$, then $f_{u,v}(X,\delta_2)$ is monotonically decreasing for $X \in \mathbb{R}$.*

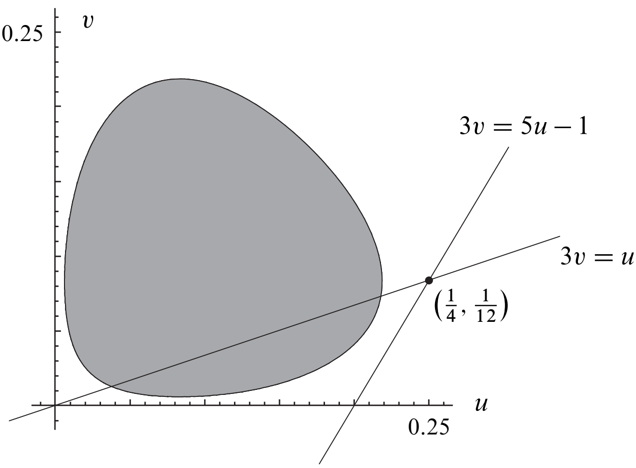


Figure 3: The domain $\{\mathrm{Re}\,\check{V}(u,v) \geq \varsigma_R(M_6)\}$ and the lines $3v = u$ and $3v = 5u-1$

Remark 5.4 When $p = 6$ (for example), the domain $\{\mathrm{Re}\,\check{V}(u,v) \geq \varsigma_R(M_6)\}$ and the lines $3v = u$ and $3v = 5u-1$ (which appear in the statement of the lemma) are located as shown in Figure 3. We note that the crossing point $(u,v) = \bigl(\frac{1}{4}, \frac{1}{12}\bigr)$ of these two lines does not belong to Δ'_0, since it does not satisfy that $u = t+s \leq 0.24$; this is an important point, because the gradient flow of $-\mathrm{Re}\,\check{V}$ does not behave well only at this point. We also note that some cases in the statement of the lemma might not be realized if we were to choose Δ'_0 sufficiently close to the domain $\{\mathrm{Re}\,\check{V}(u,v) \geq \varsigma_R(M_p)\}$.

Proof of Lemma 5.3 We put $x=e^{4\pi\sqrt{-1}(u+X\sqrt{-1})}$. Then $1/x=e^{4\pi X}e^{-4\pi\sqrt{-1}\,u}$. We put $\theta=\operatorname{Arg}\bigl(1-\frac{1}{x}\bigr)$ in this proof. Since $0<u<\frac{1}{4}$, θ is in the range

$$0<\theta<4\pi\Bigl(\frac{1}{4}-u\Bigr).$$

When $v\ge\bigl(1-\frac{4}{p}\bigr)u$, we show the lemma, as follows. By (32),

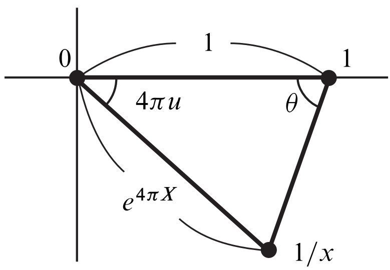
$$\frac{\partial}{\partial X}f_{u,v}(X,\delta_2)=\theta+4\pi\Bigl(-\frac{p}{4}\cdot\frac{u-v}{2}+\frac{u}{2}\Bigr)>0.$$

Therefore, $f_{u,v}(X,\delta_2)$ is monotonically increasing, and (1) holds.

When $\bigl(1+\frac{4}{p}\bigr)u-\frac{2}{p}<v<\bigl(1-\frac{4}{p}\bigr)u$, we show the lemma, as follows. In this case, by (32),

$$\frac{\partial}{\partial X}f_{u,v}(X,\delta_2)\begin{cases}>0 & \text{if }\ \theta>4\pi\bigl(\frac{p}{4}\cdot\frac{u-v}{2}-\frac{u}{2}\bigr),\\ =0 & \text{if }\ \theta=4\pi\bigl(\frac{p}{4}\cdot\frac{u-v}{2}-\frac{u}{2}\bigr),\\ <0 & \text{if }\ \theta<4\pi\bigl(\frac{p}{4}\cdot\frac{u-v}{2}-\frac{u}{2}\bigr).\end{cases}$$

Further, θ and X are related as shown in the following picture:



Hence, X is monotonically increasing as a function of θ, and they satisfy that

$$\frac{e^{4\pi X}}{\sin\theta}=\frac{1}{\sin(\pi-4\pi u-\theta)}.$$

This is rewritten as

$$X=\frac{1}{4\pi}\log\frac{\sin\theta}{\sin(\pi-4\pi u-\theta)}.$$

Therefore,

$$\frac{\partial}{\partial X}f_{u,v}(X,\delta_2)\begin{cases}>0 & \text{if }\ X>g_1(u,v),\\ =0 & \text{if }\ X=g_1(u,v),\\ <0 & \text{if }\ X<g_1(u,v),\end{cases}$$

where we put

$$g_1(u,v)=\frac{1}{4\pi}\log\frac{\sin 4\pi\bigl(\frac{p}{4}\cdot\frac{u-v}{2}-\frac{u}{2}\bigr)}{\sin 4\pi\bigl(\frac{1}{4}-\frac{p}{4}\cdot\frac{u-v}{2}-\frac{u}{2}\bigr)}.$$

Hence, (2) holds.

When $v \le \left(1+\frac{4}{p}\right)u - \frac{2}{p}$, we show the lemma, as follows. By (32),

$$\frac{\partial}{\partial X} f_{u,v}(X,\delta_2) = \theta + 4\pi\left(-\frac{p}{4}\cdot\frac{u-v}{2}+\frac{u}{2}\right) < 4\pi\left(\frac{1}{4}-u\right) + 4\pi\left(-\frac{p}{4}\cdot\frac{u-v}{2}+\frac{u}{2}\right) < 0.$$

Therefore, $f_{u,v}(X,\delta_2)$ is monotonically decreasing, and (3) holds. □

Lemma 5.5 *Fixing $(u,v)\in\Delta_0'$ and $\delta_1\in\mathbb{R}$, we regard $f_{u,v}(\delta_1,Y)$ as a function of $Y\in\mathbb{R}$.*

(1) *If $u \le \left(1+\frac{4}{p}\right)v$, then $f_{u,v}(\delta_1,Y)$ is monotonically decreasing for $Y\in\mathbb{R}$.*

(2) *If $\left(1+\frac{4}{p}\right)v < u < \left(1-\frac{4}{p}\right)v + \frac{2}{p}$, then $f_{u,v}(\delta_1,Y)$ has a unique minimal point at $Y = g_2(u,v)$, where*

$$g_2(u,v) = \frac{1}{4\pi}\log\frac{\sin 4\pi\left(\frac{1}{4}-\frac{p}{4}\cdot\frac{u-v}{2}-\frac{v}{2}\right)}{\sin 4\pi\left(\frac{p}{4}\cdot\frac{u-v}{2}-\frac{v}{2}\right)},$$

ie $f_{u,v}(\delta_1,Y)$ is monotonically decreasing for $Y < g_2(u,v)$, and is monotonically increasing for $Y > g_2(u,v)$.

(3) *If $u \ge \left(1-\frac{4}{p}\right)v + \frac{2}{p}$, then $f_{u,v}(\delta_1,Y)$ is monotonically increasing for $Y\in\mathbb{R}$.*

Proof We put $y = e^{4\pi\sqrt{-1}(v+Y\sqrt{-1})}$. We put $\theta = -\mathrm{Arg}(1-y)$ in this proof. Since $0 < v < \frac{1}{4}$, θ is in the range

$$0 < \theta < 4\pi\left(\tfrac{1}{4}-v\right).$$

When $u \le \left(1+\frac{4}{p}\right)v$, we show the lemma, as follows. By (33),

$$\frac{\partial}{\partial Y} f_{u,v}(\delta_1,Y) = -\theta + 4\pi\left(\frac{p}{4}\cdot\frac{u-v}{2}-\frac{v}{2}\right) < 0.$$

Therefore, $f_{u,v}(\delta_1,Y)$ is monotonically decreasing, and (1) holds.

When $\left(1+\frac{4}{p}\right)v < u < \left(1-\frac{4}{p}\right)v+\frac{2}{p}$, we show the lemma, as follows. In this case, by (33),

$$\frac{\partial}{\partial Y} f_{u,v}(\delta_1,Y) \begin{cases} >0 & \text{if } \theta < 4\pi\left(\frac{p}{4}\cdot\frac{u-v}{2}-\frac{v}{2}\right), \\ =0 & \text{if } \theta = 4\pi\left(\frac{p}{4}\cdot\frac{u-v}{2}-\frac{v}{2}\right), \\ <0 & \text{if } \theta > 4\pi\left(\frac{p}{4}\cdot\frac{u-v}{2}-\frac{v}{2}\right). \end{cases}$$

Further, Y is monotonically decreasing as a function of θ, and they satisfy that

$$\frac{e^{-4\pi Y}}{\sin\theta} = \frac{1}{\sin(\pi-4\pi v-\theta)}.$$

This is rewritten as

$$Y=\frac{1}{4\pi}\log\frac{\sin(\pi-4\pi v-\theta)}{\sin\theta}.$$

Therefore,

$$\frac{\partial}{\partial Y}f_{u,v}(\delta_1,Y)\begin{cases}>0 & \text{if } Y>g_2(u,v),\\ =0 & \text{if } Y=g_2(u,v),\\ <0 & \text{if } Y<g_2(u,v),\end{cases}$$

where we put

$$g_2(u,v)=\frac{1}{4\pi}\log\frac{\sin 4\pi\left(\frac{1}{4}-\frac{p}{4}\cdot\frac{u-v}{2}-\frac{v}{2}\right)}{\sin 4\pi\left(\frac{p}{4}\cdot\frac{u-v}{2}-\frac{v}{2}\right)}.$$

Hence, (2) holds.

When $u\ge\left(1-\frac{4}{p}\right)v+\frac{2}{p}$, we show the lemma, as follows. By (33),

$$\frac{\partial}{\partial Y}f_{u,v}(\delta_1,Y)=-\theta+4\pi\left(\frac{p}{4}\cdot\frac{u-v}{2}-\frac{v}{2}\right)>-4\pi\left(\frac{1}{4}-v\right)+4\pi\left(\frac{p}{4}\cdot\frac{u-v}{2}-\frac{v}{2}\right)>0.$$

Therefore, $f_{u,v}(\delta_1,Y)$ is monotonically increasing, and (3) holds. □

The argument of the proof of the following lemma is due to Yokota [28].

Lemma 5.6 *When we apply Proposition 3.2 (saddle-point method) to (26), the assumption of Proposition 3.2 holds.*

Proof We show that there exists a homotopy $\Delta'_{(\delta)}$ $(0\le\delta\le1)$ between $\Delta'_{(0)}=\Delta'_0$ and $\Delta'_{(1)}$ such that

(34) $$(t_c,s_c)\in\Delta'_{(1)},$$

(35) $$\Delta'_{(1)}-\{(t_c,s_c)\}\subset\{(t,s)\in\mathbb{C}^2\mid \operatorname{Re}\widehat{V}(t,s)<\varsigma_R\},$$

(36) $$\partial\Delta'_{(\delta)}\subset\{(t,s)\in\mathbb{C}^2\mid \operatorname{Re}\widehat{V}(t,s)<\varsigma_R\}.$$

For a sufficiently large $R>0$, we put

$$\hat{g}_1(t,s)=\begin{cases}-R & \text{if } v\ge\left(1-\frac{4}{p}\right)u,\\ \max\{-R,g_1(t,s)\} & \text{if } \left(1+\frac{4}{p}\right)u-\frac{2}{p}<v<\left(1-\frac{4}{p}\right)u,\\ -R & \text{if } v\le\left(1+\frac{4}{p}\right)u-\frac{2}{p},\end{cases}$$

$$\hat{g}_2(t,s)=\begin{cases}-R & \text{if } u\le\left(1+\frac{4}{p}\right)v,\\ \max\{-R,g_2(t,s)\} & \text{if } \left(1+\frac{4}{p}\right)v<u<\left(1-\frac{4}{p}\right)v+\frac{2}{p},\\ -R & \text{if } u\ge\left(1-\frac{4}{p}\right)v+\frac{2}{p}.\end{cases}$$

We note that, since $g_1(t,s) \to -\infty$ as $v \to \big(1+\frac{4}{p}\big)u-\frac{2}{p}, \big(1-\frac{4}{p}\big)u$, the function $\hat{g}_1(t,s)$ is continuous, and similarly, since $g_2(t,s) \to -\infty$ as $u \to \big(1+\frac{4}{p}\big)v, \big(1-\frac{4}{p}\big)v+\frac{2}{p}$, the function $\hat{g}_2(t,s)$ is continuous. We put

$$\Delta'_{(\delta)} = \{(t+\delta\cdot\hat{g}_1(t,s)\sqrt{-1},\ s+\delta\cdot\hat{g}_2(t,s)\sqrt{-1}) \in \mathbb{C}^2 \mid (t,s) \in \Delta'_0\}.$$

We show (36), as follows. From the definition of Δ'_0,

$$\partial\Delta'_0 \subset \{(t,s) \in \mathbb{C}^2 \mid \operatorname{Re}\hat{V}(t,s) < \varsigma_R\}.$$

Further, by Lemmas 5.3 and 5.5,

$$\operatorname{Re}\hat{V}(t+\delta\cdot\hat{g}_1(t,s)\sqrt{-1}, s+\delta\cdot\hat{g}_2(t,s)\sqrt{-1}) \le \operatorname{Re}\hat{V}(t,s)$$

for any $\delta \in [0,1]$ and any $(t,s) \in \Delta'_0$. Hence, (36) holds.

We show (34) and (35), as follows. Consider the functions

$$F(t,s,X,Y) = \operatorname{Re}\hat{V}(t+X\sqrt{-1}, s+Y\sqrt{-1}),$$
$$h(t,s) = F(t,s,\hat{g}_1(t,s),\hat{g}_2(t,s)).$$

When $v \ge \big(1-\frac{4}{p}\big)u$ or $v \le \big(1+\frac{4}{p}\big)u-\frac{2}{p}$ or $u \le \big(1+\frac{4}{p}\big)v$ or $u \ge \big(1-\frac{4}{p}\big)v+\frac{2}{p}$, $-h(t,s)$ is sufficiently large (because we let R be sufficiently large), and (35) holds in this case. When $\big(1+\frac{4}{p}\big)u-\frac{2}{p} < v < \big(1-\frac{4}{p}\big)u$ and $\big(1+\frac{4}{p}\big)v < u < \big(1-\frac{4}{p}\big)v+\frac{2}{p}$, it follows from the definitions of $g_1(t,s)$ and $g_2(t,s)$ that $\frac{\partial F}{\partial X} = 0$ at $X = g_1(t,s)$ and $\frac{\partial F}{\partial Y} = 0$ at $Y = g_2(t,s)$. Hence,

$$\operatorname{Im}\frac{\partial\hat{V}}{\partial t} = \operatorname{Im}\frac{\partial\hat{V}}{\partial s} = 0$$

at $(t+g_1(t,s)\sqrt{-1}, s+g_2(t,s)\sqrt{-1})$. Further, $\frac{\partial h}{\partial t} = \operatorname{Re}\frac{\partial\hat{V}}{\partial t}$ and $\frac{\partial h}{\partial s} = \operatorname{Re}\frac{\partial\hat{V}}{\partial s}$ at $(t+g_1(t,s)\sqrt{-1}, s+g_2(t,s)\sqrt{-1})$. Therefore, when (t,s) is a critical point of $h(t,s)$, $(t+g_1(t,s)\sqrt{-1}, s+g_2(t,s)\sqrt{-1})$ is a critical point of $\hat{V}$. It follows that $h(t,s)$ has a unique maximal point at $(t,s) = (\operatorname{Re}t_c, \operatorname{Re}s_c)$. Therefore, (34) and (35) hold. □

5.3 Verifying the assumption of the Poisson summation formula for V

In this section, in Lemma 5.7, we verify the assumption of the Poisson summation formula (Proposition 3.4 and Remark 3.5), when we apply them to (25). As in the previous section, we consider $\hat{V}(t,s)$ instead of $V(t,s)$. We assume that $0 < t+s < \frac{1}{4}$ and $0 < s-t < \frac{1}{4}$ in this section.

We calculate an upper bound of $\widehat{V}(t,s) - \varsigma_R(M_p)$ for $(t,s) \in \Delta'_0$, as follows. From the definition of $\widehat{V}(t,s)$, we have that

$$\begin{aligned}\operatorname{Re} \widehat{V}(t,s) &= \operatorname{Re}\Big(\frac{1}{4\pi\sqrt{-1}}(\mathrm{Li}_2(e^{4\pi\sqrt{-1}(s-t)}) - \mathrm{Li}_2(e^{-4\pi\sqrt{-1}(t+s)}))\Big)\\ &= \frac{1}{2}\Lambda(2s-2t) + \frac{1}{2}\Lambda(2t+2s) \le \Lambda\Big(\frac{1}{6}\Big).\end{aligned}$$

Hence, by Lemma D.1,

$$\operatorname{Re} \widehat{V}(t,s) - \varsigma_R(M_p) \le \Lambda\Big(\frac{1}{6}\Big) - \varsigma_R(M_p) \le \frac{3}{p^2} \le \frac{1}{12} = 0.0833\ldots. \tag{37}$$

Further, the following inequalities hold for $\delta \in \mathbb{R}$, $x = e^{4\pi\sqrt{-1}(t+s+\delta\sqrt{-1})}$ and $y = e^{4\pi\sqrt{-1}(s-t-\delta\sqrt{-1})}$, which we use in the proofs of lemmas in this section:

$$\begin{aligned}0 < \operatorname{Arg}\Big(1 - \frac{1}{x}\Big) &< 4\pi\Big(\frac{1}{4} - t - s\Big),\\ -4\pi\Big(\frac{1}{4} - s + t\Big) &< \operatorname{Arg}(1-y) < 0.\end{aligned} \tag{38}$$

Lemma 5.7 *When we apply Proposition 3.4 and Remark 3.5 to (25), their assumptions hold.*

Proof We verify (20), (17), (18) and (19) in Lemmas 5.8, 5.9, 5.10 and 5.11, respectively. The other assumptions of Proposition 3.4 can be verified easily. □

Lemma 5.8 *The assumption (20) holds for* $\widehat{V}(t,s) - \varsigma_R(M_p)$.

Proof As for the assumption (20), we show that $\partial\Delta'_0$ is null-homotopic in

$$\{(t+\delta\sqrt{-1}, s) \in \mathbb{C}^2 \mid (t,s) \in \Delta'_0,\ \delta \ge 0,\ \operatorname{Re} \widehat{V}(t+\delta\sqrt{-1}, s) < \varsigma_R(M_p) + 4\pi\delta\}.$$

To show it, we show that the following disk bounds $\partial\Delta'_0$ in the above domain:

$$\{(t+\delta_0\sqrt{-1}, s) \in \mathbb{C}^2 \mid (t,s) \in \Delta'_0\} \cup \{(t+\delta\sqrt{-1}, s) \in \mathbb{C}^2 \mid (t,s) \in \partial\Delta'_0,\ \delta \in [0,\delta_0]\}.$$

We put

$$F_{t,s}(\delta) = \operatorname{Re} \widehat{V}(t+\delta\sqrt{-1}, s) - \varsigma_R(M_p) - 4\pi\delta$$

in this proof. Then it is sufficient to show that

$$F_{t,s}(\delta_0) < 0 \quad \text{for any } (t,s) \in \Delta'_0 \tag{39}$$

and

$$F_{t,s}(\delta) < 0 \quad \text{for any } (t,s) \in \partial\Delta'_0 \text{ and } \delta \in [0,\delta_0] \tag{40}$$

for some $\delta_0 > 0$.

To show these, we estimate the differential of $F_{t,s}(\delta)$, as follows. The differential of $F_{t,s}(\delta)$ is given by

$$\begin{aligned}\frac{d}{d\delta}F_{t,s}(\delta) &= \mathrm{Re}\Big(\sqrt{-1}\frac{\partial}{\partial t}\widehat{V}(t+\delta\sqrt{-1},s)\Big)-4\pi\\ &= \mathrm{Im}\Big(-\log(1-y)+\log\Big(1-\frac{1}{x}\Big)+4\pi\sqrt{-1}\Big(-\frac{p}{2}t+s-1\Big)\Big)\\ &= -\mathrm{Arg}(1-y)+\mathrm{Arg}\Big(1-\frac{1}{x}\Big)+4\pi\Big(-\frac{p}{2}t+s-1\Big),\end{aligned}$$

where we put $x=e^{4\pi\sqrt{-1}(t+s+\delta\sqrt{-1})}$ and $y=e^{4\pi\sqrt{-1}(s-t-\delta\sqrt{-1})}$. By (38), we can estimate it as follows:

$$\begin{aligned}\frac{d}{d\delta}F_{t,s}(\delta) &< 4\pi\Big(\frac{1}{4}-s+t\Big)+4\pi\Big(\frac{1}{4}-t-s\Big)+4\pi\Big(-\frac{p}{2}t+s-1\Big)\\ &= 4\pi\Big(-\frac{p}{2}t-s-\frac{1}{2}\Big)<4\pi\Big(-\frac{p}{2}t-\frac{1}{2}\Big)<-4\pi\cdot 0.13,\end{aligned}$$

where we obtain the last inequality since $|t|\le 0.74/p$ by Lemma 5.1.

We show (39), as follows. We have that

$$F_{t,s}(\delta_0)=F_{t,s}(0)+\int_0^{\delta_0}\frac{d}{d\delta}F_{t,s}(\delta)\,d\delta<F_{t,s}(0)-4\pi\cdot\frac{0.1}{2}\cdot\delta_0.$$

Further, by (37),

$$F_{t,s}(0)=\mathrm{Re}\,\widehat{V}(t,s)-\varsigma_R\le\frac{1}{12}=0.0833\dots.$$

Hence, (39) is satisfied for a sufficiently large δ_0.

We show (40), as follows. From the definition of Δ', we have that $F_{t,s}(0)<0$ for any $(t,s)\in\partial\Delta'$. Since $\frac{d}{d\delta}F_{t,s}(\delta)<0$ as shown above, it is shown similarly as above that $F_{t,s}(\delta)<0$ for any $\delta\ge 0$. Hence, (40) is satisfied. □

Lemma 5.9 *The assumption (17) holds for $\widehat{V}(t,s)-\varsigma_R(M_p)$.*

Proof We put

$$F_{t,s}(\delta)=\mathrm{Re}\,\widehat{V}(t-\delta\sqrt{-1},s)-\varsigma_R(M_p)-2\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon>0$ such that

$$\frac{d}{d\delta}F_{t,s}(\delta)<-\varepsilon \tag{41}$$

for any $(t,s)\in\Delta'_0$.

To show this, we estimate the differential of $F_{t,s}(\delta)$, as follows. The differential of $F_{t,s}(\delta)$ is given by

$$\begin{aligned}\frac{d}{d\delta}F_{t,s}(\delta) &= \mathrm{Re}\Big(-\sqrt{-1}\,\frac{\partial}{\partial t}\hat{V}(t-\delta\sqrt{-1},s)\Big)-2\pi\\ &= \mathrm{Im}\Big(\log(1-y)-\log\Big(1-\frac{1}{x}\Big)+4\pi\sqrt{-1}\Big(\frac{p}{2}t-s-\frac{1}{2}\Big)\Big)\\ &= \mathrm{Arg}(1-y)-\mathrm{Arg}\Big(1-\frac{1}{x}\Big)+4\pi\Big(\frac{p}{2}t-s-\frac{1}{2}\Big),\end{aligned}$$

where we put $x = e^{4\pi\sqrt{-1}(t+s-\delta\sqrt{-1})}$ and $y = e^{4\pi\sqrt{-1}(s-t+\delta\sqrt{-1})}$. By (38), we can estimate it as follows:

$$\frac{d}{d\delta}F_{t,s}(\delta) < 0+0+4\pi\Big(\frac{p}{2}t-s-\frac{1}{2}\Big) < 4\pi\Big(\frac{p}{2}t-\frac{1}{2}\Big) < -4\pi\cdot 0.13,$$

where we obtain the last inequality since $|t| \le 0.74/p$ by Lemma 5.1. Hence, (41) holds, as required. □

Lemma 5.10 *The assumption (18) holds for* $\hat{V}(t,s)-\varsigma_R(M_p)$*.*

Proof We put

$$F_{t,s}(\delta) = \mathrm{Re}\,\hat{V}(t,s+\delta\sqrt{-1})-\varsigma_R(M_p)-2\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon > 0$ such that

$$\frac{d}{d\delta}F_{t,s}(\delta) < -\varepsilon \tag{42}$$

for any $(t,s) \in \Delta_0'$.

To show these, we estimate the differential of $F_{t,s}(\delta)$, as follows. The differential of $F_{t,s}(\delta)$ is given by

$$\begin{aligned}\frac{d}{d\delta}F_{t,s}(\delta) &= \mathrm{Re}\Big(\sqrt{-1}\,\frac{\partial}{\partial s}\hat{V}(t,s+\delta\sqrt{-1})\Big)-2\pi\\ &= \mathrm{Im}\Big(\log(1-y)+\log\Big(1-\frac{1}{x}\Big)+4\pi\sqrt{-1}\,t\Big)-2\pi\\ &= \mathrm{Arg}(1-y)+\mathrm{Arg}\Big(1-\frac{1}{x}\Big)+4\pi\Big(t-\frac{1}{2}\Big),\end{aligned}$$

where we put $x = e^{4\pi\sqrt{-1}(t+s+\delta\sqrt{-1})}$ and $y = e^{4\pi\sqrt{-1}(s-t+\delta\sqrt{-1})}$. By (38), we can estimate it as follows:

$$\frac{d}{d\delta}F_{t,s}(\delta) < 0+4\pi\Big(\frac{1}{4}-t-s\Big)+4\pi\Big(t-\frac{1}{2}\Big) = 4\pi\Big(-s-\frac{1}{4}\Big) < -\pi.$$

Hence, (42) holds, as required. □

Lemma 5.11 *The assumption (19) holds for $\widehat{V}(t,s)-\varsigma_R(M_p)$.*

Proof We put

$$F_{t,s}(\delta)=\operatorname{Re}\widehat{V}(t,s-\delta\sqrt{-1})-\varsigma_R(M_p)-2\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon>0$ such that

$$\frac{d}{d\delta}F_{t,s}(\delta)<-\varepsilon \tag{43}$$

for any $(t,s)\in\Delta_0'$.

To show this, we estimate the differential of $F_{t,s}(\delta)$, as follows. The differential of $F_{t,s}(\delta)$ is given by

$$\begin{aligned}\frac{d}{d\delta}F_{t,s}(\delta)&=\operatorname{Re}\Big(-\sqrt{-1}\,\frac{\partial}{\partial s}\widehat{V}(t,s-\delta\sqrt{-1})\Big)-2\pi\\&=\operatorname{Im}\Big(-\log(1-y)-\log\Big(1-\frac{1}{x}\Big)-4\pi\sqrt{-1}\,t\Big)-2\pi\\&=-\operatorname{Arg}(1-y)-\operatorname{Arg}\Big(1-\frac{1}{x}\Big)+4\pi\Big(-t-\frac{1}{2}\Big),\end{aligned}$$

where we put $x=e^{4\pi\sqrt{-1}(t+s-\delta\sqrt{-1})}$ and $y=e^{4\pi\sqrt{-1}(s-t-\delta\sqrt{-1})}$. By (38), we can estimate it as follows:

$$\frac{d}{d\delta}F_{t,s}(\delta)<4\pi\Big(\frac{1}{4}-s+t\Big)+0+4\pi\Big(-t-\frac{1}{2}\Big)=4\pi\Big(-s-\frac{1}{4}\Big)<-\pi.$$

Hence, (43) holds, as required. □

6 Contributions from Δ_1 and Δ_2

In this section, in the following two propositions, we show that, when we restrict the range of the sum (24) to Δ_1 and Δ_2, the values of the restricted sums are of sufficiently small order; this fact is used in the proof of Theorem 1.1 in Section 5.

The aim of this section is to show the following two propositions; the second proposition is immediately obtained from the first proposition, as we show below.

Proposition 6.1 *For any integer p with $|p|\ge 6$,*

$$\sum_{\substack{0\le j<i<N\\ i+j<N/2}}(-1)^{pi}q^{pi^2/4-ij}(1-q^{-i})\frac{(q)_{i+j}}{(q)_{i-j-1}}=O(\varepsilon^{N(\varsigma_R(M_p)-\varepsilon)})$$

for some $\varepsilon>0$.

Proposition 6.2 *For any integer* p *with* $|p|\geq 6$,

$$\sum_{\substack{0\leq j<i-N/2\\ i+j<N}} (-1)^{pi} q^{pi^2/4-ij}(1-q^{-i})\frac{(q)_{i+j}}{(q)_{i-j-1}} = O(\varepsilon^{N(\varsigma_R(M_p)-\varepsilon)})$$

for some $\varepsilon>0$.

Proof By putting $i'=N-i$ and $p'=-p$, the left-hand side of the formula of the proposition is calculated as follows:

$$\begin{aligned}
&\sum_{\substack{0\leq j<i'<N\\ i'+j<N/2}} (-1)^{p'(N-i')} q^{-(p'/4)(N-i')^2-(N-i')j}(1-q^{i'})\frac{(q)_{N-i'+j}}{(q)_{N-i'-j-1}}\\
&\qquad = (-1)^{p'N} q^{-(p'/4)N^2} \sum_{\substack{0\leq j<i'<N\\ i'+j<N/2}} (-1)^{p'i'} q^{-(p'/4)i'^2+i'j}(1-q^{i'})\frac{(\overline{q})_{i'+j}}{(\overline{q})_{i'-j-1}}\\
&\qquad = \sum_{\substack{0\leq j<i'<N\\ i'+j<N/2}} (-1)^{p'i'} q^{-(p'/4)i'^2+i'j}(1-q^{i'})\frac{(\overline{q})_{i'+j}}{(\overline{q})_{i'-j-1}}.
\end{aligned}$$

Since this is equal to the complex conjugate of the left-hand side of the formula of Proposition 6.1, we obtain the required formula of the proposition. □

The rest of this section is devoted to the proof of Proposition 6.1.

Proof of Proposition 6.1 In a similar way as in Section 5, putting $t=\frac{1}{4}-\frac{i}{N}$ and $s=\frac{1}{4}-\frac{j}{N}$, the sum of Proposition 6.1 can be rewritten as

$$\sum_{\substack{0\leq j<i<N\\ i+j<N/2}} \exp\Big(N\widetilde{U}\Big(\frac{1}{4}-\frac{i}{N},\frac{1}{4}-\frac{j}{N}\Big)\Big) = O(\varepsilon^{N(\varsigma_R(M_p)-\varepsilon)}), \tag{44}$$

where we put

$$\widetilde{U}(t,s)=\frac{1}{N}\Big(\hat{\varphi}\Big(2s-2t-\frac{1}{N}\Big)-\hat{\varphi}\Big(1-2t-2s+\frac{1}{N}\Big)\Big)+4\pi\sqrt{-1}\,\Big(\frac{p}{4}t^2-ts+\alpha t+\frac{1}{4}s\Big),$$

and we put

$$\alpha=\begin{cases} -\frac{1}{8} & \text{if } p\equiv 1 \mod 4,\\ 0 & \text{if } p\equiv 2 \mod 4,\\ \frac{1}{8} & \text{if } p\equiv 3 \mod 4,\\ \frac{1}{4} & \text{if } p\equiv 0 \mod 4,\end{cases}$$

To be precise, there should be a factor $(1-q^{-i})$ in the sum of (44), but such a factor does not contribute the resulting statement of Proposition 6.1, and so we omit such a factor in the following argument. Further, modifying $\widetilde{U}(t,s)$ to

$$U(t,s)=\frac{1}{N}(\hat{\varphi}(2s-2t)-\hat{\varphi}(1-2t-2s))+4\pi\sqrt{-1}\Big(\frac{p}{4}t^2-ts+\alpha t+\frac{1}{4}s\Big),$$

it is sufficient to show that

$$\sum_{\substack{0\le j<i<N\\ i+j<N/2}}\exp\Big(NU\Big(\frac{1}{4}-\frac{i}{N},\frac{1}{4}-\frac{j}{N}\Big)\Big)=O(\varepsilon^{N(\varsigma_R(M_p)-\varepsilon)}). \tag{45}$$

We apply the Poisson summation formula (Proposition 3.4 and Remarks 3.5 and 3.6) to (45), noting that we verify the assumptions of Proposition 3.4 in Section 6.1. Then it follows that the resulting formula is a linear sum of the following formulas. Hence, it is sufficient to show that

$$\int_{\Delta_0'}\exp(NU(t,s))\,dt\,ds=O(\varepsilon^{N(\varsigma_R(M_p)-\varepsilon)}),$$
$$\int_{\Delta_0'}\exp\big(N(U(t,s)+2\pi\sqrt{-1}\,t)\big)\,dt\,ds=O(\varepsilon^{N(\varsigma_R(M_p)-\varepsilon)}),$$
$$\int_{\Delta_0'}\exp\big(N(U(t,s)-2\pi\sqrt{-1}\,t)\big)\,dt\,ds=O(\varepsilon^{N(\varsigma_R(M_p)-\varepsilon)})$$

for some $\varepsilon>0$, noting that we need the third formula only when $\alpha=\frac{1}{4}$. Therefore, from the definition of $U(t,s)$, putting

$$U'(t,s)=\frac{1}{N}(\hat{\varphi}(2s-2t)-\hat{\varphi}(1-2t-2s))+4\pi\sqrt{-1}\Big(\frac{p}{4}t^2-ts+\alpha' t+\frac{1}{4}s\Big),$$

it is sufficient to show the following formula for $\alpha'=-\frac{2}{8},-\frac{1}{8},0,\dots,\frac{5}{8},\frac{6}{8}$:

$$\int_{\Delta_0'}\exp(NU'(t,s))\,dt\,ds=O(\varepsilon^{N(\varsigma_R(M_p)-\varepsilon)}) \tag{46}$$

for some $\varepsilon>0$. Further, we note that, in the same way as Lemma 5.2, we can obtain that

$$U'(-t,s)=\frac{1}{N}(\hat{\varphi}(2s-2t)-\hat{\varphi}(1-2t-2s))+4\pi\sqrt{-1}\Big(\frac{p}{4}t^2-ts+\Big(\frac{1}{2}-\alpha'\Big)t+\frac{1}{4}s\Big).$$

By using this formula, we can show that it is sufficient to show (46) only for $\alpha'=-\frac{1}{4},-\frac{1}{8},0,\frac{1}{8},\frac{1}{4}$. This is shown in Lemma 6.10, in a similar way as the procedure of the saddle-point method. Hence, we obtain the proposition. □

6.1 Verifying the assumptions of the Poisson summation formula for U

In this section, in Lemma 6.3, we verify the assumption of the Poisson summation formula (Proposition 3.4 and Remarks 3.5 and 3.6), when we apply them to (45).

We note that $U(t,s)$ uniformly converges on Δ_0' as $N \to \infty$ to

$$\widehat{U}(t,s) = \frac{1}{4\pi\sqrt{-1}}(\mathrm{Li}_2(e^{4\pi\sqrt{-1}(s-t)}) - \mathrm{Li}_2(e^{-4\pi\sqrt{-1}(t+s)})) + 4\pi\sqrt{-1}\Big(\frac{p}{4}t^2 - ts + \alpha t + \frac{1}{4}s\Big).$$

As in Section 5.3, we consider $\widehat{U}(t,s)$ instead of $U(t,s)$.

The differentials of $\widehat{U}(t,s)$ are given by

$$\frac{\partial \widehat{U}}{\partial t} = \log(1 - e^{4\pi\sqrt{-1}(s-t)}) - \log(1 - e^{-4\pi\sqrt{-1}(t+s)}) + 4\pi\sqrt{-1}\Big(\frac{p}{2}t - s + \alpha\Big),$$
$$\frac{\partial \widehat{U}}{\partial s} = -\log(1 - e^{4\pi\sqrt{-1}(s-t)}) - \log(1 - e^{-4\pi\sqrt{-1}(t+s)}) + 4\pi\sqrt{-1}\Big(-t + \frac{1}{4}\Big).$$

We note that similar estimates as (38) hold also in this section.

Lemma 6.3 *When we apply Proposition 3.4 and Remarks 3.5 and 3.6 to (45), their assumptions hold.*

Proof We verify (20), (17), (21), (18) and (19) in Lemmas 6.4, 6.5(1), 6.5(2), 6.6 and 6.7, respectively. The other assumptions of Proposition 3.4 can be verified easily. □

Lemma 6.4 *The assumption (20) holds for $\widehat{U}(t,s) - \varsigma_R(M_p)$.*

Proof We put

$$F_{t,s}(\delta) = \mathrm{Re}\,\widehat{U}(t + \delta\sqrt{-1}, s) - \varsigma_R(M_p) - 4\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon > 0$ such that

$$\frac{d}{d\delta}F_{t,s}(\delta) < -\varepsilon \tag{47}$$

for any $(t,s) \in \Delta_0'$.

To show this, we estimate the differential of $F_{t,s}(\delta)$, as follows. The differential of $F_{t,s}(\delta)$ is given by

$$\begin{aligned}\frac{d}{d\delta}F_{t,s}(\delta) &= \operatorname{Re}\Big(\sqrt{-1}\,\frac{\partial}{\partial t}\widehat{U}(t+\delta\sqrt{-1},s)\Big) - 4\pi \\ &= \operatorname{Im}\Big(-\log(1-y)+\log\Big(1-\frac{1}{x}\Big)+4\pi\sqrt{-1}\Big(-\frac{p}{2}t+s-\alpha\Big)\Big) - 4\pi \\ &= -\operatorname{Arg}(1-y)+\operatorname{Arg}\Big(1-\frac{1}{x}\Big)+4\pi\Big(-\frac{p}{2}t+s-\alpha-1\Big),\end{aligned}$$

where we put $x = e^{4\pi\sqrt{-1}(t+s+\delta\sqrt{-1})}$ and $y = e^{4\pi\sqrt{-1}(s-t-\delta\sqrt{-1})}$. By (38), we can estimate it as follows:

$$\begin{aligned}\frac{d}{d\delta}F_{t,s}(\delta) &< 4\pi\Big(\frac{1}{4}-s+t\Big)+4\pi\Big(\frac{1}{4}-t-s\Big)+4\pi\Big(-\frac{p}{2}t+s-\alpha-1\Big) \\ &= 4\pi\Big(-\frac{p}{2}t-s-\alpha-\frac{1}{2}\Big) < 4\pi\Big(-\frac{p}{2}t-\alpha-\frac{1}{2}\Big) < 4\pi\Big(-\frac{p}{2}t-\frac{3}{8}\Big) \\ &< 4\pi\Big(\frac{0.74}{2}-\frac{3}{8}\Big) = -0.02\pi.\end{aligned}$$

Hence, (47) holds, as required. □

Lemma 6.5 (1) *When $\alpha = \frac{1}{4}$, the assumption (21) holds for $\widehat{U}(t,s) - \varsigma_R(M_p)$.*

(2) *When $\alpha = 0, \pm\frac{1}{8}$, the assumption (17) holds for $\widehat{U}(t,s) - \varsigma_R(M_p)$.*

Proof We put

$$F_{t,s}(\delta) = \operatorname{Re}\widehat{U}(t-\delta\sqrt{-1},s) - \varsigma_R(M_p) - 2k\pi\delta$$

in this proof, where we put $k = 1$ if $\alpha = \frac{1}{4}$, and $k = 2$ if $\alpha = 0, \pm\frac{1}{8}$. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon > 0$ such that

$$\frac{d}{d\delta}F_{t,s}(\delta) < -\varepsilon \tag{48}$$

for any $(t,s) \in \Delta_0'$.

To show this, we estimate the differential of $F_{t,s}(\delta)$, as follows. The differential of $F_{t,s}(\delta)$ is given by

$$\begin{aligned}\frac{d}{d\delta}F_{t,s}(\delta) &= \operatorname{Re}\Big(-\sqrt{-1}\,\frac{\partial}{\partial t}\widehat{U}(t-\delta\sqrt{-1},s)\Big) - 2k\pi \\ &= \operatorname{Im}\Big(\log(1-y)-\log\Big(1-\frac{1}{x}\Big)+4\pi\sqrt{-1}\Big(\frac{p}{2}t-s+\alpha\Big)\Big) - 2k\pi \\ &= \operatorname{Arg}(1-y)-\operatorname{Arg}\Big(1-\frac{1}{x}\Big)+4\pi\Big(\frac{p}{2}t-s+\alpha-\frac{k}{2}\Big),\end{aligned}$$

where we put $x = e^{4\pi\sqrt{-1}(t+s-\delta\sqrt{-1})}$ and $y = e^{4\pi\sqrt{-1}(s-t+\delta\sqrt{-1})}$. By (38), we can estimate it as follows:

$$\frac{d}{d\delta}F_{t,s}(\delta) < 0+0+4\pi\Big(\frac{p}{2}t - s + \alpha - \frac{k}{2}\Big) < 4\pi\Big(\frac{p}{2}t + \alpha - \frac{k}{2}\Big) \le 4\pi\Big(\frac{p}{2}t - \frac{3}{8}\Big)$$
$$< 4\pi\Big(\frac{0.74}{2} - \frac{3}{8}\Big) = -0.02\pi.$$

Hence, (48) holds, as required. □

Lemma 6.6 *The assumption (18) holds for* $\hat{U}(t,s) - \varsigma_R(M_p)$.

Proof We put

$$F_{t,s}(\delta) = \operatorname{Re}\hat{U}(t, s+\delta\sqrt{-1}) - \varsigma_R(M_p) - 2\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon > 0$ such that

$$\frac{d}{d\delta}F_{t,s}(\delta) < -\varepsilon \tag{49}$$

for any $(t,s) \in \Delta'_0$.

To show these, we estimate the differential of $F_{t,s}(\delta)$, as follows. The differential of $F_{t,s}(\delta)$ is given by

$$\begin{aligned}\frac{d}{d\delta}F_{t,s}(\delta) &= \operatorname{Re}\Big(\sqrt{-1}\,\frac{\partial}{\partial s}\hat{U}(t, s+\delta\sqrt{-1})\Big) - 2\pi \\ &= \operatorname{Im}\Big(\log(1-y) + \log\Big(1-\frac{1}{x}\Big) + 4\pi\sqrt{-1}\Big(t-\frac{1}{4}\Big)\Big) - 2\pi \\ &= \operatorname{Arg}(1-y) + \operatorname{Arg}\Big(1-\frac{1}{x}\Big) + 4\pi\Big(t-\frac{3}{4}\Big),\end{aligned}$$

where we put $x = e^{4\pi\sqrt{-1}(t+s+\delta\sqrt{-1})}$ and $y = e^{4\pi\sqrt{-1}(s-t+\delta\sqrt{-1})}$. By (38), we can estimate it as follows:

$$\frac{d}{d\delta}F_{t,s}(\delta) < 0 + 4\pi\Big(\frac{1}{4} - t - s\Big) + 4\pi\Big(t - \frac{3}{4}\Big) = 4\pi\Big(-s-\frac{1}{2}\Big) < -2\pi.$$

Hence, (49) holds, as required. □

Lemma 6.7 *The assumption (19) holds for* $\hat{U}(t,s) - \varsigma_R(M_p)$.

Proof We put

$$F_{t,s}(\delta) = \operatorname{Re}\hat{U}(t, s-\delta\sqrt{-1}) - \varsigma_R(M_p) - 2\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon > 0$ such that

$$\frac{d}{d\delta} F_{t,s}(\delta) < -\varepsilon \tag{50}$$

for any $(t,s) \in \Delta_0'$.

To show this, we estimate the differential of $F_{t,s}(\delta)$, as follows. The differential of $F_{t,s}(\delta)$ is given by

$$\begin{aligned}\frac{d}{d\delta} F_{t,s}(\delta) &= \mathrm{Re}\Big(-\sqrt{-1}\,\frac{\partial}{\partial s}\hat{U}(t, s-\delta\sqrt{-1})\Big) - 2\pi \\ &= \mathrm{Im}\Big(-\log(1-y) - \log\Big(1-\frac{1}{x}\Big) + 4\pi\sqrt{-1}\Big(-t+\frac{1}{4}\Big)\Big) - 2\pi \\ &= -\mathrm{Arg}(1-y) - \mathrm{Arg}\Big(1-\frac{1}{x}\Big) + 4\pi\Big(-t-\frac{1}{4}\Big),\end{aligned}$$

where we put $x = e^{4\pi\sqrt{-1}(t+s-\delta\sqrt{-1})}$ and $y = e^{4\pi\sqrt{-1}(s-t-\delta\sqrt{-1})}$. By (38), we can estimate it as follows:

$$\frac{d}{d\delta} F_{t,s}(\delta) < 4\pi\Big(\frac{1}{4} - s + t\Big) + 0 + 4\pi\Big(-t-\frac{1}{4}\Big) = -4\pi s < -4\pi\cdot 0.005,$$

since $s \ge 0.005$ by Lemma 5.1. Hence, (50) holds, as required. □

6.2 The integral (46) is sufficiently small

In this section, in Lemma 6.10, we show that the integral (46) is of sufficiently small order. We use a similar procedure as the saddle-point method of Section 5.2, but we have no critical point in the domain of the integral in this case. So, as a consequence, we can show that the value of the integral is of sufficiently small order.

We note that $U'(t,s)$ uniformly converges on Δ_0' as $N \to \infty$ to

$$\hat{U}'(t,s) = \frac{1}{4\pi\sqrt{-1}}\big(\mathrm{Li}_2(e^{4\pi\sqrt{-1}(s-t)}) - \mathrm{Li}_2(e^{-4\pi\sqrt{-1}(t+s)})\big) + 4\pi\sqrt{-1}\Big(\frac{p}{4}t^2 - ts + \alpha' t + \frac{1}{4}s\Big).$$

To simplify the calculation of the behavior of $\hat{U}(t,s)$, we change the variables (t,s) to (u,v) by $u = s+t$ and $v = s-t$. They are in the ranges that $0 < u < \frac{1}{4}$ and $0 < v < \frac{1}{4}$. Then $\hat{U}(t,s)$ is rewritten as

$$\check{U}(u,v) = \frac{1}{4\pi\sqrt{-1}}\big(\mathrm{Li}_2(e^{4\pi\sqrt{-1}\,v}) - \mathrm{Li}_2(e^{-4\pi\sqrt{-1}\,u})\big) + 4\pi\sqrt{-1}\Big(\frac{p}{4}\cdot\frac{(u-v)^2}{4} + \frac{v^2-u^2}{4} + \alpha'\frac{u-v}{2} + \frac{u+v}{8}\Big).$$

Its differentials are given by

$$\frac{\partial \check{U}}{\partial u} = -\log(1 - e^{-4\pi\sqrt{-1}\,u}) + 4\pi\sqrt{-1}\Big(\frac{p}{4}\cdot\frac{u-v}{2} - \frac{u}{2} + \frac{\alpha'}{2} + \frac{1}{8}\Big),$$
$$\frac{\partial \check{U}}{\partial v} = -\log(1 - e^{4\pi\sqrt{-1}\,v}) + 4\pi\sqrt{-1}\Big(\frac{p}{4}\cdot\frac{v-u}{2} + \frac{v}{2} - \frac{\alpha'}{2} + \frac{1}{8}\Big).$$

In order to show Lemmas 6.8 and 6.9 below, we calculate the behavior of the function

$$f_{u,v}(\delta_1,\delta_2) = \operatorname{Re}\check{U}(u+\delta_1\sqrt{-1}, v+\delta_2\sqrt{-1}).$$

The differentials of this function are given by

$$\begin{aligned}
\frac{\partial}{\partial\delta_1} f_{u,v}(\delta_1,\delta_2) &= \operatorname{Re}\Big(\sqrt{-1}\,\frac{\partial}{\partial u}\check{U}(u+\delta_1\sqrt{-1}, v+\delta_2\sqrt{-1})\Big) \\
&= \operatorname{Arg}\Big(1-\frac{1}{x}\Big) - 4\pi\Big(\frac{p}{4}\cdot\frac{u-v}{2} - \frac{u}{2} + \frac{\alpha'}{2} + \frac{1}{8}\Big),
\end{aligned} \tag{51}$$

$$\begin{aligned}
\frac{\partial}{\partial\delta_2} f_{u,v}(\delta_1,\delta_2) &= \operatorname{Re}\Big(\sqrt{-1}\,\frac{\partial}{\partial v}\check{U}(u+\delta_1\sqrt{-1}, v+\delta_2\sqrt{-1})\Big) \\
&= \operatorname{Arg}(1-y) - 4\pi\Big(\frac{p}{4}\cdot\frac{v-u}{2} + \frac{v}{2} - \frac{\alpha'}{2} + \frac{1}{8}\Big),
\end{aligned} \tag{52}$$

where $x = e^{4\pi\sqrt{-1}(u+\delta_1\sqrt{-1})}$ and $y = e^{4\pi\sqrt{-1}(v+\delta_2\sqrt{-1})}$.

Lemma 6.8 *Fixing $(u,v)\in\Delta_0'$ and $\delta_2\in\mathbb{R}$, we regard $f_{u,v}(X,\delta_2)$ as a function of $X\in\mathbb{R}$. If $\frac{p}{2}t+\alpha'\geq\frac{1}{4}$, then $f_{u,v}(X,\delta_2)$ is monotonically decreasing for $X\in\mathbb{R}$, and $f_{u,v}(X,\delta_2)\to-\infty$ as $X\to\infty$.*

Proof Since $0<u<\frac{1}{4}$, we have that

$$0 < \operatorname{Arg}\Big(1-\frac{1}{x}\Big) < 4\pi\Big(\frac{1}{4}-u\Big).$$

Hence,

$$\begin{aligned}
\frac{d}{dX} f_{u,v}(X,\delta_2) &< 4\pi\Big(\frac{1}{4}-u\Big) - 4\pi\Big(\frac{p}{4}\cdot\frac{u-v}{2} - \frac{u}{2} + \frac{\alpha'}{2} + \frac{1}{8}\Big) \\
&= -4\pi\Big(\frac{pt}{4} + \frac{u}{2} - \frac{1}{8} + \frac{\alpha'}{2}\Big) = -2\pi\Big(\frac{pt}{2} + \alpha' - \frac{1}{4} + u\Big) \\
&\leq -2\pi u \leq -2\pi\cdot 0.005,
\end{aligned}$$

since $u\geq 0.005$ by Lemma 5.1. Therefore, $f_{u,v}(X,\delta_2)$ is monotonically decreasing, as required. □

Lemma 6.9 *Fixing $(u,v) \in \Delta'_0$ and $\delta_1 \in \mathbb{R}$, we regard $f_{u,v}(\delta_1, Y)$ as a function of $Y \in \mathbb{R}$. If $\frac{p}{2}t + \alpha' \le \frac{1}{4}$, then $f_{u,v}(\delta_1, Y)$ is monotonically decreasing for $Y \in \mathbb{R}$, and $f_{u,v}(\delta_1, Y) \to -\infty$ as $Y \to \infty$.*

Proof Since $0 < v < \frac{1}{4}$, we have that

$$-4\pi\Big(\frac{1}{4} - v\Big) < \operatorname{Arg}(1-y) < 0.$$

Hence,

$$\begin{aligned}\frac{d}{dY} f_{u,v}(\delta_1, Y) &< -4\pi\Big(\frac{p}{4}\cdot\frac{v-u}{2} + \frac{v}{2} - \frac{\alpha'}{2} + \frac{1}{8}\Big)\\ &= 4\pi\Big(\frac{pt}{4} - \frac{v}{2} - \frac{1}{8} + \frac{\alpha'}{2}\Big) = 2\pi\Big(\frac{pt}{2} + \alpha' - \frac{1}{4} - v\Big)\\ &\le -2\pi v \le -2\pi\cdot 0.005,\end{aligned}$$

since $v \ge 0.005$ by Lemma 5.1. Therefore, $f_{u,v}(\delta_1, Y)$ is monotonically decreasing, as required. □

Lemma 6.10 *For $\alpha' = 0, \pm\frac{1}{8}, \pm\frac{1}{4}$,*

$$\int_{\Delta'_0} \exp(NU'(t,s))\, dt\, ds = O(e^{N(\varsigma_R(M_p)-\varepsilon)})$$

for some $\varepsilon > 0$.

Proof We show that there exists a homotopy $\Delta'_{(\delta)}$ $(0 \le \delta \le \delta_0)$ between $\Delta'_{(0)} = \Delta'_0$ and $\Delta'_{(\delta_0)}$ such that

$$\Delta'_{(\delta_0)} \subset \{(t,s) \in \mathbb{C}^2 \mid \operatorname{Re} U'(t,s) < \varsigma_R(M_p) - \varepsilon\}, \tag{53}$$

$$\partial\Delta'_{(\delta)} \subset \{(t,s) \in \mathbb{C}^2 \mid \operatorname{Re} U'(t,s) < \varsigma_R(M_p) - \varepsilon\}. \tag{54}$$

We note that $U'(t,s)$ uniformly converges to $\widehat{U}'(t,s)$ on Δ'_0 and the error term is of order $O(1/N^2)$. So we show the existence of such a homotopy for $\widehat{U}'(t,s)$, instead of $U'(t,s)$. We recall that we defined $\check{U}(u,v)$ to be $\widehat{U}'(t,s)$ changing variables by putting $u = s+t$ and $v = s-t$. We construct such a homotopy by using $\check{U}(u,v)$.

For each fixed (u,v), we move (X,Y) from $(0,0)$ along the gradient flow of the function $-\operatorname{Re}\check{U}(u+X\sqrt{-1}, v+Y\sqrt{-1})$. Then the value of $\operatorname{Re}\check{U}(u+X\sqrt{-1}, v+Y\sqrt{-1})$ monotonically decreases, and in particular, by Lemmas 6.8 and 6.9, it goes to $-\infty$. As for (54), since $\partial\Delta'_0$ is originally included in this domain and the value of $\operatorname{Re}\check{U}$ monotonically decreases, (54) holds. As for (53), since the value of $\operatorname{Re}\check{U}$ uniformly goes to $-\infty$, (53) is satisfied for sufficiently large δ_0. Hence, such a homotopy exists, as required. □

7 Proof of Theorem 1.1 when $|p| = 5$

We gave a proof of Theorem 1.1 when $|p| \geq 6$ in Section 5. Unlike that case, we need additional procedure when $|p| = 5$ in the proof of Theorem 1.1. In this case, the domain $\{\mathrm{Re}\,\widehat{V}(t,s) \geq \varsigma_{\boldsymbol{R}}(M_5)\}$ slightly intersects the lines $s = t$ and $s = -t$ (see Figure 4), and we must remove neighborhoods of these lines from this domain before we apply the Poisson summation formula and the saddle-point method. We show this procedure and give a proof of Theorem 1.1 when $|p| = 5$ in this section. Since M_{-5} is homeomorphic to M_5 with opposite orientation as mentioned at the beginning of Section 5, it is sufficient to show the proof only for $p = 5$.

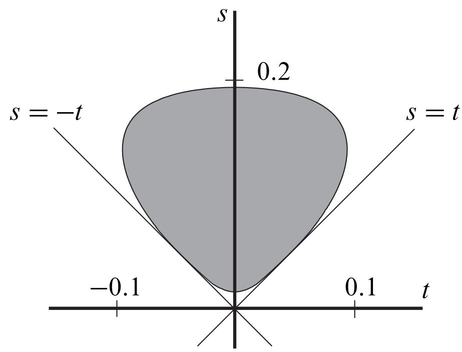


Figure 4: The domain $\{\mathrm{Re}\,\widehat{V}(t,s) \geq \varsigma_{\boldsymbol{R}}(M_5)\}$

We verify that the domain $\{\mathrm{Re}\,\widehat{V}(t,s) \geq \varsigma_{\boldsymbol{R}}(M_5)\}$ intersects the lines $s = t$ and $s = -t$, as follows. As shown in Appendix E, $\mathrm{Re}\,\widehat{V}(t,s)$ is presented by

$$\mathrm{Re}\,\widehat{V}(t,s) = \tfrac{1}{2}\Lambda(2s-2t) + \tfrac{1}{2}\Lambda(2s+2t) \geq \varsigma_{\boldsymbol{R}}(M_5),$$

where $\Lambda(\cdot)$ is as defined in Appendix E. As mentioned in Appendix E, the maximal value of $\Lambda(\cdot)$ is given by $\Lambda\big(\frac{1}{6}\big)$, and $\frac{1}{2}\Lambda\big(\frac{1}{6}\big) = 0.0807665\ldots$, while $\varsigma_{\boldsymbol{R}}(M_5)$ is given by

$$\varsigma_{\boldsymbol{R}}(M_5) = 0.07809485\ldots,$$

which is smaller than $\frac{1}{2}\Lambda\big(\frac{1}{6}\big)$. Hence, since the behavior of $\Lambda(\cdot)$ is as shown in Appendix E, the values of $s-t$ and $s+t$ can be zero in the domain $\{\mathrm{Re}\,\widehat{V}(t,s) \geq \varsigma_{\boldsymbol{R}}(M_5)\}$. (On the other hand, we note that $\varsigma_{\boldsymbol{R}}(M_p) > \frac{1}{2}\Lambda\big(\frac{1}{6}\big)$ for $p \geq 6$.) Therefore, the domain $\{\mathrm{Re}\,\widehat{V}(t,s) \geq \varsigma_{\boldsymbol{R}}(M_5)\}$ intersects the lines $s = t$ and $s = -t$, unlike the case of $p \geq 6$.

We recall that $\hat{\tau}_N(M_5)$ is presented by the following sum, by (24) putting $p=5$:

$$\begin{aligned}(55)\quad \hat{\tau}_N(M_5) = \frac{1}{1-q} e^{-(5\pi\sqrt{-1}/4)N} \sqrt{-1}^{(N-1)/2} N^{-1/2} \\ \times \sum_{(1/2-i/N,1/2-j/N)\in\Delta_0} (q^{i/2}-q^{-i/2}) \\ \times \exp\Big(N\cdot V\Big(\frac{1}{2}-\frac{i}{N},\frac{1}{2}-\frac{j}{N}-\frac{1}{2N}\Big)\Big) \\ + O(e^{N(\varsigma_R(M_p)-\varepsilon)}),\end{aligned}$$

where we recall that Δ_0 is given by

$$\Delta_0 = \big\{(t,s)\in\mathbb{R}^2 \mid 0\le t+s\le \tfrac{1}{2},\ 0\le s-t\le \tfrac{1}{2}\big\}.$$

We restrict Δ_0 to

$$\begin{aligned}\Delta_0'' &= \{(t,s)\in\Delta_0 \mid 0.001\le t+s,\ 0.001\le s-t\} \\ &= \big\{(t,s)\in\mathbb{R}^2 \mid 0.001\le t+s\le \tfrac{1}{2},\ 0.001\le s-t\le \tfrac{1}{2}\big\}.\end{aligned}$$

In fact, in the sum (55), the summand itself is too large for our purpose, but the values of the restricted sums along the lines $s=t+\rho$ and $s=-t+\rho$ are of sufficiently small order for any fixed ρ with $0\le\rho\le 0.001$; we show this in Lemma F.1. Hence, we obtain that

$$\begin{aligned}(56)\quad \hat{\tau}_N(M_5) = \frac{1}{1-q} e^{-(5\pi\sqrt{-1}/4)N} \sqrt{-1}^{(N-1)/2} N^{-1/2} \\ \times \sum_{(1/2-i/N,1/2-j/N)\in\Delta_0''} (q^{i/2}-q^{-i/2}) \\ \times \exp\Big(N\cdot V\Big(\frac{1}{2}-\frac{i}{N},\frac{1}{2}-\frac{j}{N}-\frac{1}{2N}\Big)\Big) \\ + O(e^{N(\varsigma_R(M_p)-\varepsilon)}).\end{aligned}$$

Further, in order to apply the Poisson summation formula and the saddle-point method later, we consider to restrict Δ_0'' to Δ_0' of the following lemma; we will use the defining inequalities of Δ_0' later when we verify the assumption of the Poisson summation formula and the saddle-point method in Sections 7.2 and 7.3.

Lemma 7.1 *We put*

$$\Delta_0' = \{(t,s)\in\mathbb{R}^2 \mid 0.001\le t+s\le 0.26,\ 0.001\le s-t\le 0.26,\ |t|\le 0.099,\ s\le 0.2\}.$$

Then the domain

$$\{(t,s)\in\Delta_0'' \mid \operatorname{Re}\hat{V}(t,s)\ge\varsigma_R(M_5)-\varepsilon\}$$

is included in Δ_0' for sufficiently small $\varepsilon>0$.

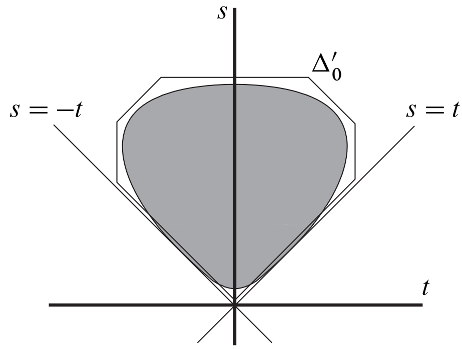


Figure 5: The domain $\{\mathrm{Re}\,\widehat{V}(t,s) \geq \varsigma_R(M_5)\}$ (the gray area) and the domain Δ'_0

We give a proof of the lemma in Appendix G. See Figure 5 for a graphical representation of the inclusion of the lemma.

Proof of Theorem 1.1 when $p = 5$ We recall that $\hat{\tau}_N(M_5)$ is presented by (56). Hence, by Lemma 7.1, we obtain that

$$\begin{aligned}
(57)\quad \hat{\tau}_N(M_5) = \frac{1}{1-q} e^{-(5\pi\sqrt{-1}/4)N} \sqrt{-1}^{(N-1)/2} N^{-1/2} \\
\times \sum_{(1/2-i/N,1/2-j/N)\in\Delta'_0} (q^{i/2}-q^{-i/2}) \\
\times \exp\Big(N\cdot V\Big(\frac{1}{2}-\frac{i}{N},\frac{1}{2}-\frac{j}{N}-\frac{1}{2N}\Big)\Big) \\
+ O(e^{N(\varsigma_R(M_p)-\varepsilon)}).
\end{aligned}$$

By the Poisson summation formula (Proposition 7.2), this sum is expressed by the integrals

$$\begin{aligned}
\hat{\tau}_N(M_5) = \frac{1}{1-q} e^{-(5\pi\sqrt{-1}/4)N} \sqrt{-1}^{(N-1)/2} N^{3/2} \\
\times \Big(\int_{\Delta'_0} (e^{-2\pi\sqrt{-1}\,t} - e^{2\pi\sqrt{-1}\,t}) \exp(N\,V(t,s))\,dt\,ds + O(e^{N(\varsigma_R(M_p)-\varepsilon)}) \\
+ \int_{\Delta'_0} (e^{-2\pi\sqrt{-1}\,t} - e^{2\pi\sqrt{-1}\,t}) \exp\big(N(V(t,s)+2\pi\sqrt{-1}\,t)\big)\,dt\,ds \\
+ O(e^{N(\varsigma_R(M_p)-\varepsilon)}) \Big)
\end{aligned}$$

for some $\varepsilon > 0$. Further, similarly as in Section 5, by Lemma 5.2, the second integral is equal to the first integral, and we can rewrite the above formula as

$$\hat{\tau}_N(M_5) = \frac{2}{1-q} e^{-(5\pi\sqrt{-1}/4)N} \sqrt{-1}^{(N-1)/2} N^{3/2} \times \int_{\Delta_0'} (e^{-2\pi\sqrt{-1}\,t} - e^{2\pi\sqrt{-1}\,t}) \exp(N V(t,s))\, dt\, ds + O(e^{N(\varsigma_R(M_p)-\varepsilon)}).$$

Let (t_0, s_0) be the critical point of $\widehat{V}(t,s)$ given in Section 7.1. Then, by applying the saddle-point method (Proposition 7.8), we obtain that

$$\hat{\tau}_N(M_5) = \frac{2}{1-q} e^{-(5\pi\sqrt{-1}/4)N} \sqrt{-1}^{(N-1)/2} N^{3/2} \times (e^{-2\pi\sqrt{-1}\,t_0} - e^{2\pi\sqrt{-1}\,t_0}) \exp(N \widehat{V}(t_0,s_0)) \cdot \frac{2\pi}{N} (V_{tt}V_{ss} - V_{ts}^2)^{-1/2} \times \Bigl(1 + O\Bigl(\frac{1}{N}\Bigr)\Bigr),$$

in a similar way as in Section 5. Hence, we obtain the required formula of the theorem. □

7.1 A critical point of $\widehat{V}(t,s)$

In this section, we characterize a critical point (t_0, s_0) of $\widehat{V}(t,s)$, which we use in the proof of Theorem 1.1.

In the same way as in Section 5.1, we show that there is a single critical point in the domain

$$\bigl\{(t,s) \in \mathbb{C}^2 \mid 0 < \operatorname{Re}(t+s) < \tfrac{1}{4},\ 0 < \operatorname{Re}(s-t) < \tfrac{1}{4},\ \operatorname{Re} t \ge 0\bigr\}. \tag{58}$$

As in Section 5.1, putting $z = e^{-4\pi\sqrt{-1}\,t}$ and $w = e^{-4\pi\sqrt{-1}\,s}$, such a critical point is obtained from a solution of

$$z^2 - \left(\frac{z + z^{5/2}}{z^{5/2}z + 1} + 1 + \frac{z^{5/2}z + 1}{z + z^{5/2}}\right) z + 1 = 0, \qquad w = \frac{z + z^{5/2}}{z^{5/2}z + 1}.$$

We can verify that there is a single solution which satisfies (58); we denote it by (t_0, s_0). It is numerically given by

$$(t_0, s_0) = (0.0916106\ldots - \sqrt{-1}\cdot 0.0574205\ldots,\ 0.1238288\ldots - \sqrt{-1}\cdot 0.0530897\ldots).$$

Further, we put

$$\varsigma(M_5) = \widehat{V}(t_0, s_0), \quad \varsigma_R(M_5) = \operatorname{Re} \widehat{V}(t_0, s_0) = 0.07809485\ldots.$$

7.2 The Poisson summation formula for V

The aim of this section is to show the following proposition, which is obtained from the Poisson summation formula (Proposition 3.4 and Remark 3.5).

Proposition 7.2 *We have*

$$\sum_{(1/2-i/N,1/2-j/N)\in\Delta_0'} (q^{i/2}-q^{-i/2})\exp\Bigl(N\cdot V\Bigl(\frac{1}{2}-\frac{i}{N},\frac{1}{2}-\frac{j}{N}-\frac{1}{2N}\Bigr)\Bigr)$$
$$=\int_{\Delta_0'}(e^{-2\pi\sqrt{-1}\,t}-e^{2\pi\sqrt{-1}\,t})\exp(N\,V(t,s))\,dt\,ds$$
$$+\int_{\Delta_0'}(e^{-2\pi\sqrt{-1}\,t}-e^{2\pi\sqrt{-1}\,t})\exp\bigl(N(V(t,s)+2\pi\sqrt{-1}\,t)\bigr)\,dt\,ds$$
$$+O(e^{N(\varsigma_R(M_5)-\varepsilon)}).$$

Proof We put a smooth function $g\colon\mathbb{R}\to\mathbb{R}$ by

$$g(t)=\begin{cases}1 & \text{if } t\ge 0.001,\\ 0 & \text{if } t\le 0.0005,\end{cases}\qquad 0\le g(t)\le 1\quad\text{if } 0.0005<t<0.001$$

for $t\in\mathbb{R}$. Further, by extending Δ_0', we put

$$\Delta_0'''=\{(t,s)\in\mathbb{R}^2\mid 0.0005\le t+s\le 0.26,\ 0.0005\le s-t\le 0.26,\ |t|\le 0.1,\ s\le 0.2\}.$$

Then, by applying the Poisson summation formula (Proposition 3.4 and Remark 3.5) to $g(s+t)g(s-t)(e^{-2\pi\sqrt{-1}t}-e^{2\pi\sqrt{-1}t})V(t,s)$, we obtain that

$$\sum_{(1/2-i/N,1/2-j/N)\in\Delta_0'''} g\Bigl(1-\frac{i}{N}-\frac{j}{N}\Bigr)g\Bigl(\frac{i}{N}-\frac{j}{N}\Bigr)(q^{i/2}-q^{-i/2})$$
$$\times\exp\Bigl(N\cdot V\Bigl(\frac{1}{2}-\frac{i}{N},\frac{1}{2}-\frac{j}{N}-\frac{1}{2N}\Bigr)\Bigr)$$
$$=\int_{\Delta_0'''}g(s+t)g(s-t)(e^{-2\pi\sqrt{-1}\,t}-e^{2\pi\sqrt{-1}\,t})\exp(N\,V(t,s))\,dt\,ds$$
$$+\int_{\Delta_0'''}g(s+t)g(s-t)(e^{-2\pi\sqrt{-1}\,t}-e^{2\pi\sqrt{-1}\,t})\exp\bigl(N(V(t,s)+2\pi\sqrt{-1}\,t)\bigr)\,dt\,ds$$
$$+O(e^{N(\varsigma_R(M_5)-\varepsilon)}),$$

noting that we verify the assumption of the Poisson summation formula in Lemma 7.3 below. As we show in Appendix F, we can ignore the error term, which is derived from the function g. Hence, from the above formula, we obtain the required formula of the proposition. □

Lemma 7.3 *When we apply Proposition 3.4 and Remark 3.5 in the proof of Proposition 7.2, their assumptions hold.*

Proof It is sufficient to show the assumptions for $\widehat{V}(t,s)$. We verify (20), (17), (18) and (19) in Lemmas 7.4, 7.5, 7.6 and 7.7, respectively. The other assumptions of Proposition 3.4 can be verified easily. □

Before we show Lemmas 7.4, 7.5, 7.6 and 7.7, we note some inequalities, which we use in the proofs of the lemmas. Let $(t,s) \in \Delta'_0$. The following inequalities hold for $\delta \in \mathbb{R}$ and $x = e^{4\pi\sqrt{-1}(t+s+\delta\sqrt{-1})}$:

$$\operatorname{Arg}\Big(1-\frac{1}{x}\Big) \begin{cases} < 4\pi\big(\frac{1}{4}-t-s\big) < \pi & \text{if } 0 < t+s < \frac{1}{4}, \\ \leq 0 & \text{if } \frac{1}{4} \leq t+s \leq 0.26, \end{cases}$$

$$-\operatorname{Arg}\Big(1-\frac{1}{x}\Big) \begin{cases} \leq 0 & \text{if } 0 < t+s \leq \frac{1}{4}, \\ < 4\pi\big(t+s-\frac{1}{4}\big) \leq 4\pi \cdot 0.01 & \text{if } \frac{1}{4} < t+s \leq 0.26. \end{cases}$$

Hence,

$$\operatorname{Arg}\Big(1-\frac{1}{x}\Big) < \pi, \quad -\operatorname{Arg}\Big(1-\frac{1}{x}\Big) < 4\pi \cdot 0.01. \tag{59}$$

Further, the following inequalities hold for $\delta \in \mathbb{R}$ and $y = e^{4\pi\sqrt{-1}(s-t-\delta\sqrt{-1})}$:

$$\operatorname{Arg}(1-y) \begin{cases} \leq 0 & \text{if } 0 < s-t \leq \frac{1}{4}, \\ < 4\pi\big(s-t-\frac{1}{4}\big) < 4\pi \cdot 0.01 & \text{if } \frac{1}{4} < s-t \leq 0.26, \end{cases}$$

$$-\operatorname{Arg}(1-y) \begin{cases} < 4\pi\big(\frac{1}{4}-s+t\big) < \pi & \text{if } 0 < s-t < \frac{1}{4}, \\ \leq 0 & \text{if } \frac{1}{4} \leq s-t \leq 0.26. \end{cases}$$

Hence,

$$\operatorname{Arg}(1-y) < 4\pi \cdot 0.01, \quad -\operatorname{Arg}(1-y) < \pi. \tag{60}$$

Lemma 7.4 *The assumption (20) holds for $\widehat{V}(t,s) - \varsigma_R(M_5)$.*

Proof We put

$$F_{t,s}(\delta) = \operatorname{Re}\widehat{V}(t+\delta\sqrt{-1}, s) - \varsigma_R(M_5) - 4\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon > 0$ such that

$$\frac{d}{d\delta} F_{t,s}(\delta) < -\varepsilon \tag{61}$$

for any $(t,s) \in \Delta'_0$.

To show this, we estimate the differential of $F_{t,s}(\delta)$, as follows. As shown in the proof of Lemma 5.8, the differential of $F_{t,s}(\delta)$ is given by

$$\frac{d}{d\delta}F_{t,s}(\delta) = -\mathrm{Arg}(1-y) + \mathrm{Arg}\Big(1-\frac{1}{x}\Big) + 4\pi\Big(-\frac{5}{2}t+s-1\Big),$$

where we put $x = e^{4\pi\sqrt{-1}\,(t+s+\delta\sqrt{-1})}$ and $y = e^{4\pi\sqrt{-1}(s-t-\delta\sqrt{-1})}$. By (59) and (60), we can estimate it as follows:

$$\begin{aligned}\frac{d}{d\delta}F_{t,s}(\delta) &< \pi + \pi + 4\pi\Big(-\frac{5}{2}t+s-1\Big)\\ &= 4\pi\Big(-\frac{5}{2}t+s-\frac{1}{2}\Big) < 4\pi\Big(\frac{5}{2}\cdot 0.1 + 0.2 - \frac{1}{2}\Big) = -4\pi\cdot 0.05,\end{aligned}$$

where we obtain the second inequality since $|t| \le 0.1$ and $s \le 0.2$ by Lemma 7.1. Hence, (61) holds, as required. □

Lemma 7.5 *The assumption* (17) *holds for* $\widehat{V}(t,s) - \varsigma_R(M_5)$.

Proof We put

$$F_{t,s}(\delta) = \mathrm{Re}\,\widehat{V}(t-\delta\sqrt{-1},s) - \varsigma_R(M_p) - 2\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon > 0$ such that

$$\frac{d}{d\delta}F_{t,s}(\delta) < -\varepsilon \tag{62}$$

for any $(t,s) \in \Delta'_0$.

To show this, we estimate the differential of $F_{t,s}(\delta)$, as follows. As shown in the proof of Lemma 5.9, the differential of $F_{t,s}(\delta)$ is given by

$$\frac{d}{d\delta}F_{t,s}(\delta) = \mathrm{Arg}(1-y) - \mathrm{Arg}\Big(1-\frac{1}{x}\Big) + 4\pi\Big(\frac{5}{2}t-s-\frac{1}{2}\Big),$$

where we put $x = e^{4\pi\sqrt{-1}(t+s-\delta\sqrt{-1})}$ and $y = e^{4\pi\sqrt{-1}(s-t+\delta\sqrt{-1})}$. By (59) and (60), we can estimate it as follows:

$$\frac{d}{d\delta}F_{t,s}(\delta) < 4\pi\cdot 0.01 + 4\pi\cdot 0.01 + 4\pi\Big(\frac{5}{2}t-s-\frac{1}{2}\Big) \le 4\pi\Big(0.02+\frac{5}{2}\cdot 0.1-\frac{1}{2}\Big) = -4\pi\cdot 0.23,$$

where we obtain the second inequality since $|t| \le 0.1$ and $s > 0$ by Lemma 7.1. Hence, (62) holds, as required. □

Lemma 7.6 *The assumption* (18) *holds for* $\widehat{V}(t,s) - \varsigma_R(M_5)$.

Proof We put

$$F_{t,s}(\delta) = \mathrm{Re}\,\widehat{V}(t, s+\delta\sqrt{-1}) - \varsigma_{R}(M_5) - 2\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon > 0$ such that

$$\frac{d}{d\delta} F_{t,s}(\delta) < -\varepsilon \tag{63}$$

for any $(t, s) \in \Delta'_0$.

To show these, we estimate the differential of $F_{t,s}(\delta)$, as follows. As shown in the proof of Lemma 5.10, the differential of $F_{t,s}(\delta)$ is given by

$$\frac{d}{d\delta} F_{t,s}(\delta) = \mathrm{Arg}(1-y) + \mathrm{Arg}\Big(1-\frac{1}{x}\Big) + 4\pi\Big(t-\frac{1}{2}\Big),$$

where we put $x = e^{4\pi\sqrt{-1}(t+s+\delta\sqrt{-1})}$ and $y = e^{4\pi\sqrt{-1}(s-t+\delta\sqrt{-1})}$. By (59) and (60), we can estimate it as follows:

$$\frac{d}{d\delta} F_{t,s}(\delta) < 4\pi\cdot 0.01 + \pi + 4\pi\Big(t-\frac{1}{2}\Big) = 4\pi\Big(0.01+\frac{1}{4}+t-\frac{1}{2}\Big) \le -4\pi\cdot 0.14,$$

where we obtain the last inequality since $|t| \le 0.1$ by Lemma 7.1. Hence, (63) holds, as required. □

Lemma 7.7 *The assumption (19) holds for* $\widehat{V}(t,s) - \varsigma_{R}(M_5)$.

Proof We put

$$F_{t,s}(\delta) = \mathrm{Re}\,\widehat{V}(t, s-\delta\sqrt{-1}) - \varsigma_{R}(M_5) - 2\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon > 0$ such that

$$\frac{d}{d\delta} F_{t,s}(\delta) < -\varepsilon \tag{64}$$

for any $(t, s) \in \Delta'_0$.

To show this, we estimate the differential of $F_{t,s}(\delta)$, as follows. As shown in the proof of Lemma 5.11, the differential of $F_{t,s}(\delta)$ is given by

$$\frac{d}{d\delta} F_{t,s}(\delta) = -\mathrm{Arg}(1-y) - \mathrm{Arg}\Big(1-\frac{1}{x}\Big) + 4\pi\Big(-t-\frac{1}{2}\Big),$$

where we put $x = e^{4\pi\sqrt{-1}(t+s-\delta\sqrt{-1})}$ and $y = e^{4\pi\sqrt{-1}(s-t-\delta\sqrt{-1})}$. By (59) and (60), we can estimate it as follows:

$$\frac{d}{d\delta}F_{t,s}(\delta) < \pi + 4\pi\cdot 0.01 + 4\pi\Big(-t-\frac{1}{2}\Big) = 4\pi\Big(\frac{1}{4}+0.01-t-\frac{1}{2}\Big) \le -4\pi\cdot 0.14,$$

where we obtain the last inequality since $|t| \le 0.1$ by Lemma 7.1. Hence, (64) holds, as required. □

7.3 Verifying the assumption of the saddle-point method for V

The aim of this section is to show the following proposition, which is obtained from the saddle-point method (Proposition 3.2 and Remark 3.3).

Proposition 7.8 *We have*

$$\int_{\Delta_0'} (e^{-2\pi\sqrt{-1}\,t} - e^{2\pi\sqrt{-1}\,t}) \exp(N V(t,s))\, dt\, ds$$
$$= (e^{-2\pi\sqrt{-1}\,t_0} - e^{2\pi\sqrt{-1}\,t_0}) \exp(N\hat{V}(t_0,s_0))\cdot\frac{2\pi}{N}(V_{tt}V_{ss}-V_{ts}^2)^{-1/2}\Big(1+O\Big(\frac{1}{N}\Big)\Big).$$

Proof When we apply the saddle-point method, unlike the case of Section 5, it is a problem that the boundary of Δ_0' is not included in the domain $\{\mathrm{Re}\,\hat{V}(t,s) < \varsigma_R(M_5)\}$. To make an appropriate boundary, we consider to extend Δ_0' by adding some additional parts. We note that Δ_0' intersects $\{\mathrm{Re}\,\hat{V}(t,s) \ge \varsigma_R(M_5)\}$ along the lines $s = t+\rho$ and $s = -t+\rho$, where we put $\rho = 0.001$. We put $\partial_1\Delta_0'$ to be the edge of Δ_0' on $\{s = t+\rho\}$, and put $\partial_2\Delta_0'$ to be the edge of Δ_0' on $\{s = -t+\rho\}$. As in the proof of Lemma 7.12, we move $\partial_1\Delta_0'$ and $\partial_2\Delta_0'$ along the gradient flow of $-\mathrm{Re}\,\hat{V}$, and denote the resulting segments by $\partial_1^{(\delta)}\Delta_0'$ and $\partial_2^{(\delta)}\Delta_0'$ for $\delta \ge 0$. As shown in the proof of Lemma 7.12, $\partial_1^{(\delta_0)}\Delta_0'$ and $\partial_2^{(\delta_0)}\Delta_0'$ are included in

$$\{(t,s)\in\mathbb{C}^2 \mid \mathrm{Re}\,\hat{V}(t,s) < \varsigma_R(M_5)-\varepsilon\} \tag{65}$$

for a sufficiently large δ_0 and a sufficiently small $\varepsilon > 0$. We define a new domain $\hat{\Delta}_0'$ by

$$\hat{\Delta}_0' = \Delta_0' \cup \bigcup_{0\le\delta\le\delta_0} (\partial_1^{(\delta)}\Delta_0' \cup \partial_2^{(\delta)}\Delta_0').$$

Then the boundary of $\hat{\Delta}_0'$ is included in (65) for a sufficiently small $\varepsilon > 0$. When we extend Δ_0' to $\hat{\Delta}_0'$, we have the error term of the value of the integral of the proposition. We note that, as in Appendix F, we can show that this error term is of sufficiently small order.

Hence, in a similar way as in Section 5.2, we can apply the saddle-point method (Proposition 3.2 and Remark 3.3) for $\widehat{\Delta}'_0$, noting that we verify the assumption of the saddle-point method in Lemma 7.12. Therefore, we obtain the required formula of the proposition. □

As in Section 5.2, putting $u = t + s$ and $v = s - t$, we put $\check{V}(u, v)$ to be $\widehat{V}(t, s)$, in order to simplify the calculation.

Lemma 7.9 *Fixing $(u, v) \in \Delta'_0$ and $\delta_2 \in \mathbb{R}$, we regard $f_{u,v}(X, \delta_2)$ as a function of $X \in \mathbb{R}$.*

(1) *If $5v \geq u$ and $5v \geq 9u - 2$, then $f_{u,v}(X, \delta_2)$ is monotonically increasing for $X \in \mathbb{R}$.*

(2) *If $9u - 2 < 5v < u$ or $u < 5v < 9u - 2$, then $f_{u,v}(X, \delta_2)$ has a unique minimal point at $X = g_1(u, v)$, where*

$$g_1(u, v) = \frac{1}{4\pi} \log \frac{\sin \frac{\pi}{2}(u - 5v)}{\sin \frac{\pi}{2}(2 - 9u + 5v)},$$

ie $f_{u,v}(X, \delta_2)$ is monotonically decreasing for $X < g_1(u, v)$, and is monotonically increasing for $X > g_1(u, v)$.

(3) *If $5v \leq u$ and $5v \leq 9u - 2$, then $f_{u,v}(X, \delta_2)$ is monotonically decreasing for $X \in \mathbb{R}$.*

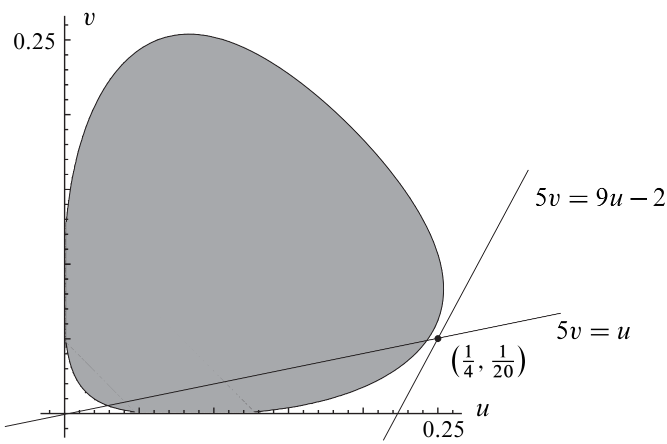


Figure 6: The domain $\{\mathrm{Re}\, \check{V}(u, v) \geq \varsigma_R(M_5)\}$ and the lines $5v = u$ and $5v = 9u - 2$

Remark 7.10 The point $\left(\frac{1}{4}, \frac{1}{20}\right)$ is the intersection of the two lines $5v = u$ and $5v = 9u - 2$, which appear in the statement of the lemma. This point is in the exterior of the domain $\{\operatorname{Re} \check{V}(u, v) \geq \varsigma_R(M_5)\}$, since

$$\begin{aligned}\check{V}\left(\tfrac{1}{4}, \tfrac{1}{20}\right) - \varsigma_R(M_5) &= \tfrac{1}{2}\Lambda\left(\tfrac{1}{2}\right) + \tfrac{1}{2}\Lambda\left(\tfrac{1}{10}\right) - \varsigma_R(M_5)\\ &= 0 + 0.0735101\ldots - 0.0780948\ldots = -0.0045847\ldots < 0.\end{aligned}$$

In fact, the domain $\{\operatorname{Re} \check{V}(u, v) \geq \varsigma_R(M_5)\}$, and the two lines are located as shown in Figure 6. Hence, some cases in the statement of the lemma might not be realized if we were to choose Δ_0' sufficiently close to the domain $\{\operatorname{Re} \check{V}(u, v) \geq \varsigma_R(M_5)\}$.

Proof of Lemma 7.9 We can show the lemma in a similar way as the proof of Lemma 5.3. A different point is that "which cases are actually realized"; we note that there are 4 cases depending on the signs of $5v - u$ and $5v - 9u + 2$. It is also an important point that the point $(u, v) = \left(\frac{1}{4}, \frac{1}{20}\right)$ $\left(\text{ie } (t, s) = \left(\frac{1}{10}, \frac{3}{20}\right)\right)$ does not belong to Δ_0', because the gradient flow of $-\operatorname{Re} \check{V}$ does not behave well only at this point. Hence, we can show the lemma in a similar way as the proof of Lemma 5.3. □

Lemma 7.11 *Fixing $(u, v) \in \Delta_0'$ and $\delta_1 \in \mathbb{R}$, we regard $f_{u,v}(\delta_1, Y)$ as a function of $Y \in \mathbb{R}$.*

(1) *If $5u \leq 9v$, then $f_{u,v}(\delta_1, Y)$ is monotonically decreasing for $Y \in \mathbb{R}$.*

(2) *If $5u > 9v$, then $f_{u,v}(\delta_1, Y)$ has a unique minimal point at $Y = g_2(u, v)$, where*

$$g_2(u, v) = \frac{1}{4\pi} \log \frac{\sin \frac{\pi}{2}(v - 5u + 2)}{\sin \frac{\pi}{2}(5u - 9v)},$$

ie $f_{u,v}(\delta_1, Y)$ is monotonically decreasing for $Y < g_2(u, v)$, and is monotonically increasing for $Y > g_2(u, v)$.

Proof We can show the lemma in a similar way as the proof of Lemma 5.5. The main difference is that (3) of Lemma 5.5 does not happen in this lemma. □

Lemma 7.12 *When we apply Proposition 3.2 (saddle-point method) in the proof of Proposition 7.8, the assumption of Proposition 3.2 holds.*

Proof We can show the lemma in a similar way as the proof of Lemma 5.6. A difference is that we use $\widehat{\Delta}_0'$ instead of Δ_0' in this proof. When we make a homotopy of $\widehat{\Delta}_0'$, we move $\widehat{\Delta}_0'$ along the gradient flow of $-\operatorname{Re} \check{V}$. Then the resulting domain after moving by the homotopy is the same as that of $\widehat{\Delta}_0'$. Hence, we can show the lemma in a similar way as the proof of Lemma 5.6. □

Appendix A Gauss sum

The aim of this section is to show Lemma A.1 below. For a textbook of number theory, see eg [10].

Lemma A.1 *Let N be a positive odd integer, and let A be $e^{\pi\sqrt{-1}/N}$. Then*

$$\sum_{n\in\mathbb{Z}/N\mathbb{Z}} (-A)^{n^2} = (-\sqrt{-1})^{(N-1)/2}\sqrt{N}.$$

We note that

$$-A = e^{\pi\sqrt{-1}+\pi\sqrt{-1}/N} = e^{(2\pi\sqrt{-1}/N)\cdot(N+1)/2} = e^{(2\pi\sqrt{-1}/N)\cdot\bar{2}},$$

where we denote by $\bar{2}$ the inverse of 2 in $\mathbb{Z}/N\mathbb{Z}$. Hence, the sum of the lemma is a Gauss sum.

It is known (see eg [10]) that we can calculate the value of a Gauss sum concretely. We review this procedure in the remainder of this section to show Lemma A.1.

Let b be a positive odd integer, and let a to an integer coprime to b. We put

$$G(a,b) = \sum_{n\in\mathbb{Z}/b\mathbb{Z}} e^{(2\pi\sqrt{-1}/b)\cdot an^2},$$

and call it a *Gauss sum*. It is known (see [10]) that

$$G(a,b) = \Big(\frac{a}{b}\Big) G(1,b),$$

where $\big(\frac{a}{b}\big)$ is a natural generalization of the Legendre symbol. Our aim is to calculate $G(\bar{2},N)$ for any positive odd integer N.

The simplest case of Lemma A.1 is the case where N is an odd prime. We show this case in the following lemma:

Lemma A.2 *For an odd prime ℓ,*

$$G(\bar{2},\ell) = (-\sqrt{-1})^{(\ell-1)/2}\sqrt{\ell}.$$

Proof We have that

$$G(\bar{2},\ell) = G(2,\ell) = \Big(\frac{2}{\ell}\Big) G(1,\ell).$$

Further, it is known (see [10]) that

$$\left(\frac{2}{\ell}\right) = (-1)^{(\ell^2-1)/8}, \qquad G(1,\ell) = \begin{cases} \sqrt{\ell} & \text{if } \ell \equiv 1 \mod 4, \\ \sqrt{-1}\sqrt{\ell} & \text{if } \ell \equiv 3 \mod 4. \end{cases}$$

Hence, we can verify the lemma for each $\ell \equiv 0, 1, \dots, 7 \mod 8$. □

The second simplest case of Lemma A.1 is the case where N is an odd prime power. We show this case in the following lemma:

Lemma A.3 *For an odd prime ℓ and a positive integer r,*

$$G(\bar{2}, \ell^r) = (-\sqrt{-1})^{(\ell^r-1)/2}\sqrt{\ell^r}.$$

Proof It is known (see [10]) that

$$G(a, \ell^r) = \ell G(a, \ell^{r-2}).$$

By Lemma A.2, it is sufficient to show the lemma recursively, assuming the lemma for $r-2$. We have that

$$G(\bar{2}, \ell^r) = \ell G(\bar{2}, \ell^{r-2}) = (-\sqrt{-1})^{(\ell^{r-2}-1)/2}\sqrt{\ell^r}.$$

Hence, it is sufficient to show that

$$(-\sqrt{-1})^{(\ell^r-1)/2} = (-\sqrt{-1})^{(\ell^{r-2}-1)/2}. \tag{66}$$

We have that

$$\frac{\ell^r-1}{2} - \frac{\ell^{r-2}-1}{2} = \frac{\ell^r-\ell^{r-2}}{2} = \frac{\ell^{r-2}(\ell-1)(\ell+1)}{2}.$$

This is divisible by 4, since $(\ell-1)(\ell+1)$ is divisible by 8. Hence, (66) holds, as required. □

Let b_1 and b_2 be coprime odd positive integers, and let a be an integer coprime to $b_1 b_2$. Then it is known (see [10]) that

$$G(a, b_1 b_2) = G(ab_2, b_1)G(ab_1, b_2).$$

Further, it is known (see [10]) that, for coprime odd positive integers b_1 and b_2,

$$\left(\frac{b_1}{b_2}\right)\left(\frac{b_2}{b_1}\right) = (-1)^{((b_1-1)/2)((b_2-1)/2)},$$

as a natural generalization of the reciprocity law.

Proof of Lemma A.1 We show the lemma by induction on N. It is sufficient to show that, for coprime odd positive integers N_1 and N_2, the lemma holds for $N_1 N_2$, assuming the lemma for N_1 and for N_2. Hence, it is sufficient to show that

$$G(\bar{2}, N_1 N_2) = (-\sqrt{-1})^{(N_1 N_2 - 1)/2} \sqrt{N_1 N_2},$$

assuming that

$$G(\bar{2}, N_1) = (-\sqrt{-1})^{(N_1 - 1)/2} \sqrt{N_1} \quad \text{and} \quad G(\bar{2}, N_2) = (-\sqrt{-1})^{(N_2 - 1)/2} \sqrt{N_2}.$$

We have that

$$\begin{aligned} G(\bar{2}, N_1 N_2) &= G(\bar{2} N_2, N_1) G(\bar{2} N_1, N_2) \\ &= \left(\frac{N_2}{N_1}\right) G(\bar{2}, N_1) \left(\frac{N_1}{N_2}\right) G(\bar{2}, N_2) \\ &= (-1)^{((N_1-1)/2)((N_2-1)/2)} G(\bar{2}, N_1) G(\bar{2}, N_2) \\ &= (-1)^{((N_1-1)/2)((N_2-1)/2)} (-\sqrt{-1})^{(N_1-1)/2} (-\sqrt{-1})^{(N_2-1)/2} \sqrt{N_1 N_2}. \end{aligned}$$

Hence, it is sufficient to show that

$$(-1)^{((N_1-1)/2)((N_2-1)/2)} (-\sqrt{-1})^{(N_1-1)/2} (-\sqrt{-1})^{(N_2-1)/2} \sqrt{N_1 N_2} \\ = (-\sqrt{-1})^{(N_1 N_2 - 1)/2}.$$

This formula holds, since

$$\begin{aligned} &2 \cdot \frac{N_1 - 1}{2} \cdot \frac{N_2 - 1}{2} + \frac{N_1 - 1}{2} + \frac{N_2 - 1}{2} - \frac{N_1 N_2 - 1}{2} \\ &= \frac{1}{2}((N_1 - 1)(N_2 - 1) + N_1 - 1 + N_2 - 1 - N_1 N_2 + 1) = 0. \end{aligned}$$

Therefore, the lemma holds. □

Appendix B Critical points of $\widehat{V}(t, s)$

The aim of this section is to show the following lemma, which implies that there exists a single critical point of $\widehat{V}(t, s)$ in the domain (28).

Lemma B.1 *The system of equations (29) has a single solution in the domain (28).*

Before showing Lemma B.1, we show the following lemma:

Lemma B.2 *Let p be an integer > 4. Then the equation*

$$z^{p/2} + z^{-p/2} = z^2 + z^{-2} - z - z^{-1} - 2$$

has exactly two solutions in the domain

$$D = \Big\{ e^{-4\pi\sqrt{-1}\,t} \in \mathbb{C} \;\Big|\; 0 < \operatorname{Re} t < \frac{1}{p} \Big\}.$$

Proof Since the equation of the lemma is a polynomial equation of degree $\le 2p$, we can verify the lemma by calculating all solutions concretely for a concrete p. So we assume that $p \ge 100$ in the remainder of this proof.

We put $z = e^{-4\pi\sqrt{-1}\,t}$. Then the equation of the lemma is rewritten as

$$(67)\quad e^{p\cdot 2\pi\sqrt{-1}\,t} + e^{-p\cdot 2\pi\sqrt{-1}\,t} = e^{8\pi\sqrt{-1}\,t} + e^{-8\pi\sqrt{-1}\,t} - e^{4\pi\sqrt{-1}\,t} - e^{-4\pi\sqrt{-1}\,t} - 2.$$

We note that, when t_0 is a solution of this equation, $\overline{t_0}$ is also a solution. Hence, it is sufficient to show that there is a single solution of (67) in the domain

$$D' = \Big\{ t \in \mathbb{C} \;\Big|\; 0 < \operatorname{Re} t < \frac{1}{p},\ 0 \le \operatorname{Im} t \Big\}.$$

We put $t = x + y\sqrt{-1}$. Then $z = e^{-4\pi\sqrt{-1}\,x} e^{4\pi y}$ and $z^{p/2} = e^{-p\cdot 2\pi\sqrt{-1}\,x} e^{p\cdot 2\pi y}$. In the remainder of this proof, we show that there is a single solution of (67) in D' by decomposing D' into seven subdomains in the following seven cases, respectively; there is no solution in Cases 1–6 and a single solution in Case 7.

Case 1 We show that, in $\{x + y\sqrt{-1} \in D' \mid y \ge \frac{0.3}{p}\}$, there is no solution of (67), as follows.

Putting $u = z^{p/2}$, since

$$|u + u^{-1}|^2 = (u + u^{-1})(\overline{u} + \overline{u}^{-1}) = |u|^2 + |u|^{-2} + \frac{u}{\overline{u}} + \frac{\overline{u}}{u} \ge |u|^2 + |u|^{-2} - 2,$$

we have that

$$|z^{p/2} + z^{-p/2}| \ge \sqrt{|z^{p/2}|^2 + |z^{-p/2}|^2 - 2} \ge \sqrt{e^{p\cdot 4\pi y} + e^{-p\cdot 4\pi y} - 2} = e^{p\cdot 2\pi y} - e^{-p\cdot 2\pi y}.$$

Further,

$$|z^2 + z^{-2} - z - z^{-1} - 2| \le |z|^2 + |z|^{-2} + |z| + |z|^{-1} + 2 = e^{8\pi y} + e^{-8\pi y} + e^{4\pi y} + e^{-4\pi y} + 2.$$

Hence, putting

$$f(y) = e^{p\cdot 2\pi y} - e^{-p\cdot 2\pi y} - (e^{8\pi y} + e^{-8\pi y} + e^{4\pi y} + e^{-4\pi y} + 2),$$

it is sufficient to show that $f(y) > 0$ for $y \geq \frac{0.3}{p}$. Since

$$\begin{aligned} f'(y) &= p \cdot 2\pi e^{p\cdot 2\pi y} - 8\pi e^{8\pi y} - 4\pi e^{4\pi y} + p \cdot 2\pi e^{-p\cdot 2\pi y} + 8\pi e^{-8\pi y} + 4\pi e^{-4\pi y} \\ &> (p-6)\cdot 2\pi e^{p\cdot 2\pi y} + 8\pi(e^{p\cdot 2\pi y} - e^{8\pi y}) - 4\pi(e^{p\cdot 2\pi y} - e^{4\pi y}) > 0, \end{aligned}$$

$f(y)$ is monotonically increasing. Therefore, it is sufficient to show that $f\big(\frac{0.3}{p}\big) > 0$. In fact,

$$\begin{aligned} f\Big(\frac{0.3}{p}\Big) &= e^{2\pi\cdot 0.3} - e^{-2\pi\cdot 0.3} - (e^{8\pi\cdot 0.3/p} + e^{-8\pi\cdot 0.3/p} + e^{4\pi\cdot 0.3/p} + e^{-4\pi\cdot 0.3/p} + 2) \\ &> e^{2\pi\cdot 0.3} - e^{-2\pi\cdot 0.3} - (e^{8\pi\cdot 0.003} + e^{-8\pi\cdot 0.003} + e^{4\pi\cdot 0.003} + e^{-4\pi\cdot 0.003} + 2) \\ &= 0.427117\ldots > 0, \end{aligned}$$

since $p \geq 100$. Hence, there is no solution in the domain of this case, as required.

Case 2 We show that, in $\big\{x + y\sqrt{-1} \in D' \mid \frac{0.01}{p} \leq x \leq \frac{0.49}{p},\ 0 < y < \frac{0.3}{p}\big\}$, there is no solution of (67), as follows.

Since $z = e^{-4\pi\sqrt{-1}\,x} e^{p\cdot 2\pi y}$, we have that

$$\begin{aligned} \operatorname{Im}(z^{p/2} + z^{-p/2}) &= -(e^{p\cdot 2\pi y} - e^{-p\cdot 2\pi y}) \sin(p \cdot 2\pi x), \\ \operatorname{Im}(z^2 + z^{-2} - z - z^{-1} - 2) &= -(e^{8\pi y} - e^{-8\pi y}) \sin 8\pi x + (e^{4\pi y} - e^{-4\pi y}) \sin 4\pi x. \end{aligned}$$

Hence,

$$\begin{aligned} |\operatorname{Im}(z^{p/2} + z^{-p/2})| &= (e^{p\cdot 2\pi y} - e^{-p\cdot 2\pi y}) \sin(p \cdot 2\pi x), \\ |\operatorname{Im}(z^2 + z^{-2} - z - z^{-1} - 2)| &\leq (e^{8\pi y} - e^{-8\pi y}) \sin 8\pi x. \end{aligned}$$

Therefore,

$$\sin(p \cdot 2\pi x) \leq \frac{e^{8\pi y} - e^{-8\pi y}}{e^{p\cdot 2\pi y} - e^{-p\cdot 2\pi y}} \cdot \sin 8\pi x.$$

Further, since

$$\begin{aligned} \frac{e^{8\pi y} - e^{-8\pi y}}{16\pi y} &\leq \frac{e^{8\pi\cdot 0.003} - e^{-8\pi\cdot 0.003}}{16\pi\cdot 0.003} = 1.00094775\ldots < 1.01, \\ e^{p\cdot 2\pi y} - e^{-p\cdot 2\pi y} &\geq 2p \cdot 2\pi y, \end{aligned}$$

we have that

$$\begin{aligned} \sin(p \cdot 2\pi x) &\leq \frac{16\pi y \cdot 1.01}{2p \cdot 2\pi y} \cdot \sin 8\pi \frac{1}{2p} \leq \frac{4 \cdot 1.01}{p} \cdot \frac{4\pi}{p} \leq \frac{4 \cdot 1.01 \cdot 4\pi}{100^2} \\ &= 0.005076\ldots < 0.01. \end{aligned}$$

On the other hand, since $0.01 \le px \le \frac{1}{2} - 0.01$, we have that

$$\sin(p \cdot 2\pi x) \ge \sin(2\pi \cdot 0.01) = 0.0627905\ldots > 0.05.$$

Hence, we have no solution in the domain of this case, as required.

Case 3 We can show that, in $\{x + y\sqrt{-1} \in D' \mid \frac{0.51}{p} \le x \le \frac{0.99}{p},\ 0 < y < \frac{0.3}{p}\}$, there is no solution of (67), in a similar way as in Case 2.

Case 4 We show that, in $\{x + y\sqrt{-1} \in D' \mid 0 < x < \frac{0.01}{p},\ 0 \le y < \frac{0.3}{p}\}$, there is no solution of (67), as follows.

Since $z = e^{4\pi y}(\cos 4\pi x - \sqrt{-1}\sin 4\pi x)$, we have that

$$\begin{aligned} |z^2 - z| &= ((e^{8\pi y}\cos 8\pi x - e^{4\pi y}\cos 4\pi x)^2 + (e^{8\pi y}\sin 8\pi x - e^{4\pi y}\sin 4\pi x)^2)^{1/2} \\ &\le \big((e^{8\pi\cdot 0.003} - e^{-4\pi\cdot 0.003}\cos(4\pi\cdot 0.0001))^2 + (e^{8\pi\cdot 0.003}\sin(8\pi\cdot 0.0001))^2\big)^{1/2} \\ &= 0.1153434\ldots < 0.12. \end{aligned}$$

Similarly, we can obtain that

$$|z^{-2} - z^{-1}| < 0.12.$$

Further, we have that

$$\begin{aligned} |z^{p/2} - 1| &= \big((e^{p\cdot 2\pi y}\cos(p \cdot 2\pi x) - 1)^2 + (e^{p\cdot 2\pi y}\sin(p\cdot 2\pi x))^2\big)^{1/2} \\ &\le \big((e^{-2\pi\cdot 0.3}\cos(2\pi \cdot 0.01) - 1)^2 + (e^{2\pi\cdot 0.3}\sin(2\pi\cdot 0.01))^2\big)^{1/2} \\ &= 0.9438792\ldots < 1. \end{aligned}$$

Similarly, we can obtain that

$$|z^{-p/2} - 1| < 1.$$

Hence,

$$4 = |(z^{p/2} - 1) + (z^{-p/2} - 1) - (z^2 - z) - (z^{-2} - z^{-1})| \le 2 + 2 \cdot 0.12 = 2.24,$$

and this is a contradiction. Therefore, there is no solution in the domain of this case, as required.

Case 5 We can show that, in $\{x + y\sqrt{-1} \in D' \mid \frac{0.99}{p} < x < \frac{1}{p},\ 0 \le y < \frac{0.3}{p}\}$, there is no solution of (67), in a similar way as in Case 4.

Case 6 We show that, in $\{x \in D' \mid \frac{0.01}{p} < x < \frac{0.99}{p}\}$, there is no solution of (67), as follows.

The equation of the lemma is rewritten as

$$\cos(p \cdot 2\pi x) + 1 = \cos 8\pi x - \cos 4\pi x.$$

Since $\cos(p \cdot 2\pi x) + 1 \geq 0$ and $\cos 8\pi x - \cos 4\pi x < 0$, there is no solution in the domain of this case, as required.

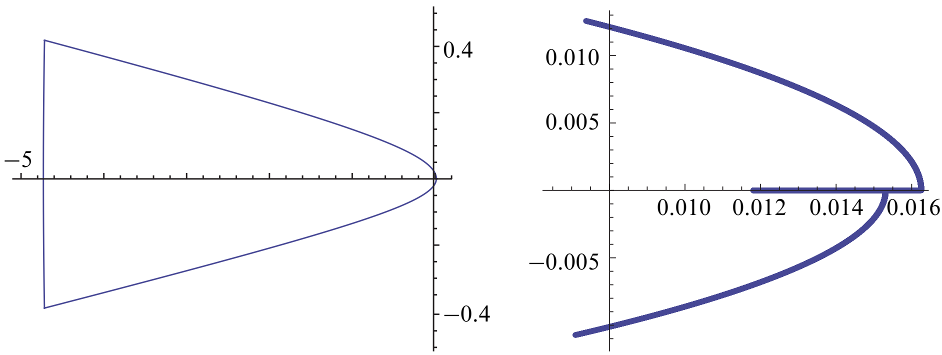


Figure 7: The image of the domain of Case 7 under the map ψ (left) and a local enlargement around the origin (right) for $p = 100$

Case 7 We show that, in $\{x + y\sqrt{-1} \in D' \mid \frac{0.49}{p} < x < \frac{0.51}{p},\ 0 < y < \frac{0.3}{p}\}$, there is a single solution of (67), as follows.

A solution of (67) is a zero of the holomorphic function

$$\psi(t) = e^{p \cdot 2\pi\sqrt{-1}\,t} + e^{-p \cdot 2\pi\sqrt{-1}\,t} - e^{8\pi\sqrt{-1}\,t} - e^{-8\pi\sqrt{-1}\,t} + e^{4\pi\sqrt{-1}\,t} + e^{-4\pi\sqrt{-1}\,t} + 2.$$

This function maps the four edges of the domain of this case in the following way:

$$\begin{aligned}
\Big\{x \in D' \;\Big|\; \frac{0.49}{p} \leq x \leq \frac{0.51}{p}\Big\} &\to \{u \in \mathbb{C} \mid \operatorname{Re} u > 0\},\\
\Big\{\frac{0.51}{p} + y\sqrt{-1} \in D' \;\Big|\; 0 < y \leq \frac{0.3}{p}\Big\} &\to \{u \in \mathbb{C} \mid \operatorname{Im} u > 0\},\\
\Big\{x + \frac{0.3}{p}\sqrt{-1} \in D' \;\Big|\; \frac{0.49}{p} \leq x \leq \frac{0.51}{p}\Big\} &\to \{u \in \mathbb{C} \mid \operatorname{Re} u < 0\},\\
\Big\{\frac{0.49}{p} + y\sqrt{-1} \in D' \;\Big|\; 0 < y \leq \frac{0.3}{p}\Big\} &\to \{u \in \mathbb{C} \mid \operatorname{Im} u < 0\}.
\end{aligned}$$

See Figure 7 for a graphical representation of this image. Hence, the boundary of the domain is taken to a closed path, which goes around the origin once. Therefore, since ψ is holomorphic, ψ has a single zero in the domain. Hence, there is a single solution of (67) in the domain of this case, as required. □

Proof of Lemma B.1 As mentioned at the beginning of the proof of Lemma B.2, we can verify the lemma for a concrete p, and so we assume that $p \ge 100$ in the remainder of this proof.

We consider a solution of (29) in the domain (28). Since $0 < \mathrm{Re}(t+s) < \frac{1}{4}$ and $0 < \mathrm{Re}(s-t) < \frac{1}{4}$, we have that

$$-4\pi\Big(\frac{1}{4} - s_R + t_R\Big) < \mathrm{Arg}(1 - e^{4\pi\sqrt{-1}(s-t)}) < 0,$$
$$0 < \mathrm{Arg}(1 - e^{-4\pi\sqrt{-1}(t+s)}) < 4\pi\Big(\frac{1}{4} - t_R - s_R\Big),$$

where we put $t_R = \mathrm{Re}\, t$ and $s_R = \mathrm{Re}\, s$. Further, as the imaginary part of the first equation of (29),

$$\mathrm{Arg}(1 - e^{4\pi\sqrt{-1}(s-t)}) - \mathrm{Arg}(1 - e^{-4\pi\sqrt{-1}(t+s)}) + 4\pi\Big(\frac{p}{2}t_R - s_R\Big) = 0.$$

Hence,

$$0 < 4\pi\Big(\frac{p}{2}t_R - s_R\Big) < 4\pi\Big(\frac{1}{4} - s_R + t_R\Big) + 4\pi\Big(\frac{1}{4} - t_R - s_R\Big) = 4\pi\Big(\frac{1}{2} - 2s_R\Big).$$

Therefore,

$$s_R < \frac{p}{2}t_R < \frac{1}{2} - s_R.$$

Hence, since $s_R = \mathrm{Re}\, s > 0$ by the condition of (28), we obtain that

$$0 < \mathrm{Re}\, t < \frac{1}{p}.$$

Further, putting $z = e^{-4\pi\sqrt{-1}\,t}$, a solution (29) satisfies (31). Hence, by Lemma B.2, the first equation of (31) has exactly two solutions. As shown in the proof of Lemma B.2, putting $t = x + y\sqrt{-1}$, these two solutions belong to the following domains, respectively:

$$\Big\{x + y\sqrt{-1} \in \mathbb{C} \;\Big|\; \frac{0.49}{p} < x < \frac{0.51}{p},\ 0 < y < \frac{0.3}{p}\Big\}, \tag{68}$$

$$\Big\{x + y\sqrt{-1} \in \mathbb{C} \;\Big|\; \frac{0.49}{p} < x < \frac{0.51}{p},\ -\frac{0.3}{p} < y < 0\Big\}. \tag{69}$$

We let t_0 be the solution in (69). Then the other solution $\overline{t_0}$ belongs to (68).

We show that the solution $\overline{t_0}$ does not induce a solution of (29) in the domain (28), as follows. By (31), putting $z = e^{-4\pi\sqrt{-1}\,\overline{t_0}}$, we have that

$$w = \frac{z^{p/2} + z}{z^{p/2}z + 1} = \frac{(z^{p/2} + z)(\overline{z}^{p/2}\overline{z} + 1)}{|z^{p/2}z + 1|^2} = \frac{|z|^p\overline{z} + z^{p/2} + \overline{z}^{p/2} + z}{|z^{p/2}z + 1|^2}.$$

Further, putting $x = \frac{1}{2p} + \frac{\varepsilon}{p}$ for $-0.01 < \varepsilon < 0.01$, the numerator of the last term is calculated as

$$e^{4\pi\sqrt{-1}\,x} e^{p\cdot 4\pi y} - e^{2\pi\sqrt{-1}\,\varepsilon} e^{p\cdot 2\pi y} - e^{-2\pi\sqrt{-1}\,\varepsilon} e^{p\cdot 2\pi y} + e^{-4\pi\sqrt{-1}\,x} e^{4\pi y}$$
$$= (e^{(p+1)4\pi y} + e^{4\pi y}) \cos 4\pi x - 2e^{p\cdot 2\pi y} \cos 2\pi\varepsilon + \sqrt{-1}(e^{(p+1)4\pi y} - e^{4\pi y}) \sin 4\pi x.$$

We note that $0 < x < \frac{0.51}{100}$. Since $e^{(p+1)4\pi y} - e^{4\pi y} > 0$ for $y > 0$, we have that $\operatorname{Im} w > 0$. However, since $0 < \operatorname{Re} s < \frac{1}{4}$ by the condition of (28), the imaginary part of $w = e^{-4\pi\sqrt{-1}\,s}$ must be negative, and this is a contradiction. Therefore, $\overline{t}_0$ does not induce a solution of (29) in the domain (28).

Hence, in order to show the lemma, it is sufficient to show that t_0 induces a single solution of (29) in the domain (28). We show that a solution of (29) induced by t_0 belongs to the domain (28), as follows. We put $z_0 = e^{-4\pi\sqrt{-1}\,t_0}$ and

$$w_0 = \frac{z_0^{p/2} + z_0}{z_0^{p/2} z_0 + 1}.$$

Then, in a similar way as above, we can show that $\operatorname{Im} w_0 < 0$, since $y < 0$ in this case. We put s_0 to be the value satisfying that $w_0 = e^{-4\pi\sqrt{-1}\,s_0}$ and $0 < \operatorname{Re} s_0 < \frac{1}{4}$. It is sufficient to show that this (t_0, s_0) satisfies (28) and (29). We recall that $\frac{0.49}{p} < \operatorname{Re} t_0 < \frac{0.51}{p}$; in particular, $\operatorname{Re} t_0 > 0$. Further, in order to show that $0 < \operatorname{Re}(t_0 + s_0) < \frac{1}{4}$ and $0 < \operatorname{Re}(s_0 - t_0) < \frac{1}{4}$, we estimate the value of $\operatorname{Re} s_0$. By the second equation of (30), we have that $(1 - z_0 w_0)\bigl(\frac{1}{z_0} - \frac{1}{w_0}\bigr) = 1$. Hence,

$$w_0 + \frac{1}{w_0} = z_0 + \frac{1}{z_0} - 1.$$

We put $t_0 = x + y\sqrt{-1}$, noting that $\frac{0.49}{p} < x < \frac{0.51}{p}$ and $-\frac{0.3}{p} < y < 0$. Further, we put

$$c = z_0 + \frac{1}{z_0} - 1.$$

Then we have that

$$\begin{aligned}
|c - 1| &= |z_0^{1/2} - z_0^{-1/2}|^2 = (z_0^{1/2} - z_0^{-1/2})(\overline{z}_0^{1/2} - \overline{z}_0^{-1/2}) \\
&= |z_0| + |z_0|^{-1} - \frac{z_0^{1/2}}{\overline{z}_0^{1/2}} - \frac{\overline{z}_0^{1/2}}{z_0^{1/2}} = e^{4\pi y} + e^{-4\pi y} - e^{-4\pi\sqrt{-1}\,x} - e^{4\pi\sqrt{-1}\,x} \\
&= e^{4\pi y} + e^{-4\pi y} - 2\cos 4\pi x \le e^{4\pi\cdot 0.003} + e^{-4\pi\cdot 0.003} - 2\cos(4\pi \cdot 0.0051) \\
&= 0.00552732\ldots < 0.006.
\end{aligned}$$

Since $w_0 + \frac{1}{w_0} = c$, we have that

$$w_0 = \frac{1}{2}(c - \sqrt{-1}\,\sqrt{4-c^2}),$$

where we choose the sign of $\sqrt{4-c^2}$ in the way that $\operatorname{Re}\sqrt{4-c^2} > 0$, noting that c is close to 1, as shown above. This term is calculated as

$$\sqrt{4-c^2} = \sqrt{3+(1-c^2)} = \sqrt{3}\,\sqrt{1+\frac{1-c^2}{3}} = \sqrt{3}\,\sqrt{1+\delta},$$

where we put $\delta = \frac{1-c^2}{3}$. Further, the last term is estimated by

$$|\sqrt{1+\delta}-1| = \frac{|\delta|}{|\sqrt{1+\delta}+1|} \le \frac{\delta_0}{\sqrt{1-\delta_0}+1} = 0.00200802\ldots < 0.003,$$

where, noting that $\delta = \frac{1-c^2}{3}$, we define δ_0 by

$$|\delta| = \frac{1}{3}|c-1|\cdot|c+1| < \frac{1}{3}\cdot 0.006\cdot 2.006 = \delta_0.$$

Since $|\sqrt{4-c^2}-\sqrt{3}| = \sqrt{3}\cdot|\sqrt{1+\delta}-1|$, we have that

$$\Big|w_0 - \frac{1-\sqrt{-1}\sqrt{3}}{2}\Big| \le \frac{1}{2}|c-1| + \frac{1}{2}|\sqrt{c^2-1}-\sqrt{3}| \le \frac{1}{2}\cdot 0.006 + \frac{\sqrt{3}}{2}\cdot 0.003,$$

which means that w_0 is close to $e^{-\pi\sqrt{-1}/3}$. Hence,

$$\Big|\operatorname{Arg}(w_0) + \frac{\pi}{3}\Big| \le \arcsin\Big(\frac{1}{2}\cdot 0.006 + \frac{\sqrt{3}}{2}\cdot 0.003\Big).$$

Therefore, noting that $w_0 = e^{-4\pi\sqrt{-1}\,s_0}$, we obtain that

$$\Big|\operatorname{Re} s_0 - \frac{1}{12}\Big| \le \frac{1}{4\pi}\arcsin\Big(\frac{1}{2}\cdot 0.006 + \frac{\sqrt{3}}{2}\cdot 0.003\Big) = 0.000445483\ldots < 0.001.$$

Hence,

$$\frac{1}{12} - 0.001 < \operatorname{Re} s_0 < \frac{1}{12} + 0.001. \tag{70}$$

Further, since $0 < \operatorname{Re} t_0 < \frac{0.51}{p} \le 0.0051$, we obtain that

$$0 < \operatorname{Re}(t_0+s_0) < \frac{1}{4}, \qquad 0 < \operatorname{Re}(s_0-t_0) < \frac{1}{4}, \tag{71}$$

and (t_0, s_0) satisfies (28), as required. In order to complete the proof of the lemma, it is sufficient to show that (t_0, s_0) satisfies (29). Since (z_0, w_0) satisfies (30), we have that

$$\log(1-e^{4\pi\sqrt{-1}\,(s_0-t_0)})-\log(1-e^{-4\pi\sqrt{-1}\,(t_0+s_0)})+4\pi\sqrt{-1}\Big(\frac{p}{2}\,t_0-s_0\Big)\in 2\pi\sqrt{-1}\,\mathbb{Z},$$
$$-\log(1-e^{4\pi\sqrt{-1}\,(s_0-t_0)})-\log(1-e^{-4\pi\sqrt{-1}\,(t_0+s_0)})-4\pi\sqrt{-1}\,t_0\in 2\pi\sqrt{-1}\,\mathbb{Z}.$$

Hence, it is sufficient to show that their imaginary part is in the range $-2\pi<\cdot<2\pi$. That is, it is sufficient to show that

$$-2\pi<\operatorname{Arg}(1-e^{4\pi\sqrt{-1}\,(s_0-t_0)})-\operatorname{Arg}(1-e^{-4\pi\sqrt{-1}(t_0+s_0)})+4\pi\ \operatorname{Re}\Big(\frac{p}{2}t_0-s_0\Big)<2\pi,$$
$$-2\pi<\operatorname{Arg}(1-e^{4\pi\sqrt{-1}(s_0-t_0)})+\operatorname{Arg}(1-e^{-4\pi\sqrt{-1}(t_0+s_0)})+4\pi\ \operatorname{Re}t_0<2\pi.$$

Hence, putting $t_R=\operatorname{Re}t_0$, $s_R=\operatorname{Re}s_0$ and

$$c_1=\frac{1}{4\pi}\operatorname{Arg}(1-e^{4\pi\sqrt{-1}\,(s_0-t_0)})-\frac{1}{4\pi}\operatorname{Arg}(1-e^{-4\pi\sqrt{-1}(t_0+s_0)})+\Big(\frac{p}{2}t_R-s_R\Big),$$
$$c_2=\frac{1}{4\pi}\operatorname{Arg}(1-e^{4\pi\sqrt{-1}(s_0-t_0)})+\frac{1}{4\pi}\operatorname{Arg}(1-e^{-4\pi\sqrt{-1}(t_0+s_0)})+t_R,$$

it is sufficient to show that

$$-\frac{1}{2}<c_1<\frac{1}{2},\quad -\frac{1}{2}<c_2<\frac{1}{2}. \tag{72}$$

Since (71) holds, we have that

$$-4\pi\Big(\frac{1}{4}-s_R+t_R\Big)<\operatorname{Arg}(1-e^{4\pi\sqrt{-1}(s_0-t_0)})<0,$$
$$0<\operatorname{Arg}(1-e^{-4\pi\sqrt{-1}(t_0+s_0)})<4\pi\Big(\frac{1}{4}-t_R-s_R\Big).$$

Further, we recall that $\frac{0.49}{p}<t_R<\frac{0.51}{p}$ and $\frac{1}{12}-0.001<s_R<\frac{1}{12}+0.001$. Hence, we obtain that

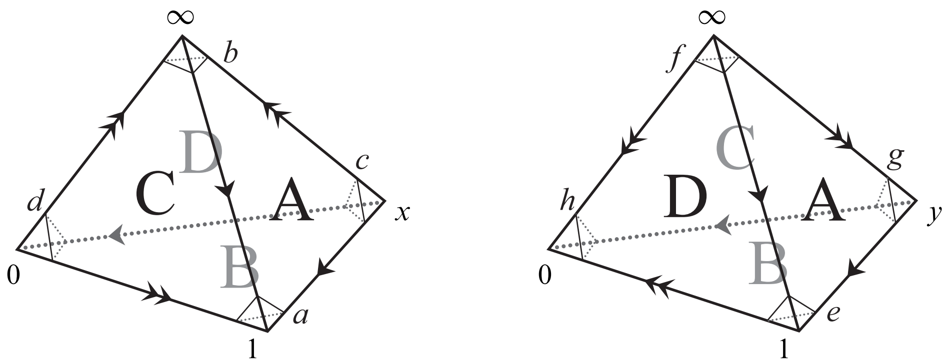
$$c_1>-\Big(\frac{1}{4}-s_R+t_R\Big)-\Big(\frac{1}{4}-t_R-s_R\Big)+\frac{p}{2}t_R-s_R=-\frac{1}{2}+\frac{p}{2}t_R+s_R$$
$$>-\frac{1}{2}+\frac{0.49}{2}+\frac{1}{12}-0.001=-0.172667\ldots>-0.2,$$
$$c_1<\frac{p}{2}t_R-s_R<\frac{0.51}{2}-\frac{1}{12}+0.001=0.172667\ldots<0.2,$$
$$c_2>-\Big(\frac{1}{4}-s_R+t_R\Big)+t_R=-\frac{1}{4}+s_R>-\frac{1}{4}+\frac{1}{12}-0.001=-0.167667\ldots>-0.2,$$
$$c_2<\Big(\frac{1}{4}-t_R-s_R\Big)+t_R=\frac{1}{4}-s_R<\frac{1}{4}-\frac{1}{12}-0.001=0.165667\ldots<0.2.$$

Therefore, (72) holds, as required. This completes the proof of the lemma. □

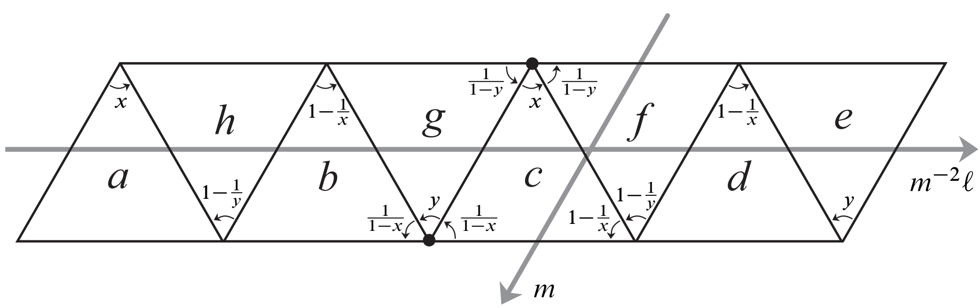
Appendix C The hyperbolic structure of M_p

In this section, we review the hyperbolic structure of M_p, and, in Lemma C.1, we explain how it is related to the critical point (t_0, s_0) of $\widehat{V}(t,s)$ given in Appendix B. For a textbook of hyperbolic geometry, see [24].

We review the hyperbolic structure of M_p given in [24], where we recall that M_p is the 3–manifold obtained from S^3 by p surgery along the figure-eight knot. As shown in [24], the complement of the figure-eight knot can be expressed as the union of the following two ideal tetrahedra:



Here, the four faces "A", "B", "C" and "D" are glued, respectively. Then the resulting space has two edges: the edge of a single arrow and the edge of double arrows. The shapes of the tetrahedra are given by parameters written at the vertices. Further, the boundary torus of a tubular neighborhood of the figure-eight knot is expressed as the union of eight triangles "a", "b", …, "h", which appear in neighborhoods of the vertices of the above ideal tetrahedra:



As shown in [24], the holonomy m of the meridian and the holonomy ℓ of the longitude are given by

$$m = \frac{1-\frac{1}{x}}{\frac{1}{1-y}} = -\frac{(1-x)(1-y)}{x}, \quad m^{-2}\ell = \frac{x^2\left(1-\frac{1}{x}\right)^2}{y^2\left(1-\frac{1}{y}\right)^2} = \frac{(1-x)^2}{(1-y)^2}.$$

We have some requirements to obtain the hyperbolic structure from these tetrahedra. We require that

$$\operatorname{Im} x > 0, \quad \operatorname{Im} y > 0, \tag{73}$$

which implies that the two tetrahedra have positive orientations. Further, we require the gluing equation,

$$xy\left(\frac{1}{1-x}\right)^2\left(\frac{1}{1-y}\right)^2 = 1,$$

$$\operatorname{Arg}(x) + \operatorname{Arg}(y) + 2\operatorname{Arg}\left(\frac{1}{1-x}\right) + 2\operatorname{Arg}\left(\frac{1}{1-y}\right) = 2\pi, \tag{74}$$

which we obtain as the product of moduli around the vertex of the dot. The first equation is rewritten as

$$(1-x)^2(1-y)^2 = xy. \tag{75}$$

We note that, for the gluing equations around the other vertices, we obtain equivalent conditions. Furthermore, we require that

$$m^p\ell = 1, \tag{76}$$

$$(p+2)\operatorname{Arg}(m) + \operatorname{Arg}(m^{-2}\ell) = 2\pi, \tag{77}$$

which is necessary in order that M_p be obtained as the closure of the union of the two tetrahedra; see Figure 8 for a graphical representation of this condition.

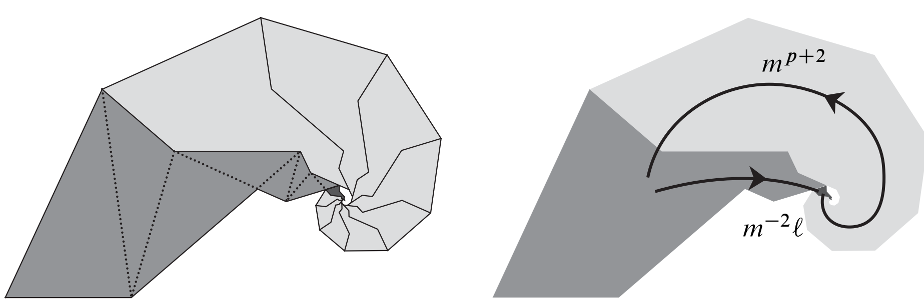


Figure 8: A fundamental domain of the boundary torus (the dark gray area) and its image by the action of $m^{-2}\ell$ (the black area) and its images by the actions of m^k for $k = 1, 2, \dots, p+1$ (the light gray area) for $p = 8$

The following lemma implies that the critical point of $\widehat{V}(t,s)$ given in Appendix B is related to the hyperbolic structure of M_p.

Lemma C.1 *Let* (t_0, s_0) *be the critical point of* $\widehat{V}(t,s)$ *given in Lemma B.1. We put* $x_0 = e^{4\pi\sqrt{-1}\,(s_0 - t_0)}$ *and* $y_0 = e^{4\pi\sqrt{-1}\,(t_0+s_0)}$*. Then* (x_0, y_0) *gives the hyperbolic structure of* M_p *in the above-mentioned way.*

Proof It is sufficient to show that (x_0, y_0) satisfies (73), (74), (75), (76) and (77). We put $z_0 = e^{-4\pi\sqrt{-1}\,t_0}$ and $w_0 = e^{-4\pi\sqrt{-1}\,s_0}$ as in the proof of Lemma B.1. Then $x_0 = z_0/w_0$ and $y_0 = 1/(z_0 w_0)$.

We obtain (73) from the definitions of x_0 and y_0, since $0 < \mathrm{Re}(s_0 - t_0) < \frac{1}{4}$ and $0 < \mathrm{Re}(t_0 + s_0) < \frac{1}{4}$ by (28).

We show (75), as follows. We have that

$$\begin{aligned}\frac{(1-x_0)^2(1-y_0)^2}{x_0 y_0} &= \frac{(1-z_0/w_0)^2(1-1/(z_0w_0))^2}{(z_0/w_0)\cdot 1/(z_0 w_0)}\\ &= \frac{1}{z_0^2 w_0^2}(w_0 - z_0)^2(1 - z_0 w_0)^2 = 1,\end{aligned}$$

where we obtain the last equality by (30). Hence, (x_0, y_0) satisfies (75), as required.

We show (74), as follows. By (75), we have that

$$\mathrm{Arg}(x_0) + \mathrm{Arg}(y_0) + 2\,\mathrm{Arg}\Big(\frac{1}{1-x_0}\Big) + 2\,\mathrm{Arg}\Big(\frac{1}{1-y_0}\Big) \in 2\pi\mathbb{Z}.$$

Hence, putting

$$\begin{aligned}c_1 &= \frac{1}{2\pi}\Big(\mathrm{Arg}(x_0) + \mathrm{Arg}(y_0) + 2\,\mathrm{Arg}\Big(\frac{1}{1-x_0}\Big) + 2\,\mathrm{Arg}\Big(\frac{1}{1-y_0}\Big)\Big)\\ &= \frac{1}{2\pi}\big(\mathrm{Arg}(x_0) + \mathrm{Arg}(y_0) - 2\,\mathrm{Arg}(1-x_0) - 2\,\mathrm{Arg}(1-y_0)\big),\end{aligned}$$

it is sufficient to show that

$$0 < c_1 < 2.$$

Since $x_0 = e^{4\pi\sqrt{-1}(s_0 - t_0)}$ and $y_0 = e^{4\pi\sqrt{-1}(t_0+s_0)}$,

$$\begin{aligned}&\mathrm{Arg}(x_0) = 4\pi(s_R - t_R), \quad -4\pi\big(\tfrac{1}{4} - s_R + t_R\big) < \mathrm{Arg}(1-x_0) < 0,\\ &\mathrm{Arg}(y_0) = 4\pi(t_R + s_R), \quad -4\pi\big(\tfrac{1}{4} - t_R - s_R\big) < \mathrm{Arg}(1-y_0) < 0,\end{aligned}$$

where we put $t_R = \operatorname{Re} t_0$ and $s_R = \operatorname{Re} s_0$. It follows that

$$c_1 > 2(s_R - t_R) + 2(t_R + s_R) = 4s_R,$$
$$c_1 < 2(s_R - t_R) + 2(t_R + s_R) + 2\big(\tfrac{1}{4} - s_R + t_R\big) + 2\big(\tfrac{1}{4} - t_R - s_R\big) = 2 - 8s_R.$$

Since $\frac{1}{12} - 0.001 < s_R < \frac{1}{12} + 0.001$ by (70), we obtain that $0 < c_1 < 2$, as required.

We show (76), as follow. We have that

$$\begin{aligned} m &= -\frac{(1-x_0)(1-y_0)}{x_0} = -\frac{(1-z_0/w_0)(1-1/(z_0w_0))}{z_0/w_0} \\ &= \frac{1}{z_0}(w_0 - z_0)\frac{1-z_0w_0}{z_0w_0} = \frac{1}{z_0}, \end{aligned}$$

where we obtain the last equality by (30). Further,

$$m^{-2}\ell = \frac{(1-x_0)^2}{(1-y_0)^2} = \frac{(1-z_0/w_0)^2}{(1-1/(z_0w_0))^2} = \frac{z_0^2(w_0-z_0)^2}{1-z_0w_0}^{2} = z_0^p \cdot z_0^2,$$

where we obtain the last equality by (30). Hence,

$$m^p\ell = \frac{z_0^p \cdot z_0^2}{z_0^p \cdot z_0^2} = 1.$$

Therefore, (x_0, y_0) satisfies (76), as required.

We show (77), as follows. By (76), we have that

$$(p+2)\operatorname{Arg}(m) + \operatorname{Arg}(m^{-2}\ell) \in 2\pi\mathbb{Z}.$$

Hence, putting

$$\begin{aligned} c_2 &= \frac{1}{2\pi}((p+2)\operatorname{Arg}(m) + \operatorname{Arg}(m^{-2}\ell)) \\ &= \frac{1}{2\pi}\left(-(p+2)\operatorname{Arg}(z_0) + \operatorname{Arg}\left(\frac{(1-x_0)^2}{(1-y_0)^2}\right)\right) \\ &= (p+2)\cdot 2t_R + \frac{1}{2\pi}(2\operatorname{Arg}(1-x_0) - 2\operatorname{Arg}(1-y_0)), \end{aligned}$$

it is sufficient to show that

$$0 < c_2 < 2.$$

Since $\frac{0.49}{p}<t_R<\frac{0.51}{p}$ and $\frac{1}{12}-0.001<s_R<\frac{1}{12}+0.001$, as shown in the proof of Lemma B.1 (for $p\geq 100$), it follows that

$$c_2>(p+2)\cdot 2t_R-4\left(\tfrac{1}{4}-s_R+t_R\right)=2pt_R-1+4s_R>0.98-1+4\left(\tfrac{1}{12}-0.001\right)>0,$$
$$c_2<(p+2)\cdot 2t_R+4\left(\tfrac{1}{4}-t_R-s_R\right)=2pt_R+1-4s_R<1.02+1-4\left(\tfrac{1}{12}+0.001\right)<2.$$

Hence (since we can verify the inequalities for $p<100$ by calculating (t_0,s_0) concretely), we obtain that $0<c_2<2$, as required. □

It is known (see [13; 16]) that the critical value $\widehat{V}(t_0,s_0)$ is equal to a normalized complex volume. We verify the real part of this fact, as follows. It is known that the hyperbolic volume of an ideal tetrahedron of modulus z is presented by the Bloch–Wigner function,

$$D(z)=\operatorname{Im}\operatorname{Li}_2(z)+\log|z|\cdot\operatorname{Arg}(1-z).$$

Hence,

$$\begin{aligned}
\operatorname{vol}(M_p)&=D(x_0)+D(y_0)=D\left(\frac{z_0}{w_0}\right)+D\left(\frac{1}{z_0w_0}\right)=D\left(\frac{z_0}{w_0}\right)-D(z_0w_0)\\
&=\operatorname{Im}\left(\operatorname{Li}_2\left(\frac{z_0}{w_0}\right)-\operatorname{Li}_2(z_0w_0)\right)+\log\left|\frac{z_0}{w_0}\right|\cdot\operatorname{Arg}\left(1-\frac{z_0}{w_0}\right)\\
&\qquad\qquad-\log|z_0w_0|\cdot\operatorname{Arg}(1-z_0w_0)\\
&=\operatorname{Im}\left(\operatorname{Li}_2\left(\frac{z_0}{w_0}\right)-\operatorname{Li}_2(z_0w_0)\right)+\log|z_0|\left(\operatorname{Arg}\left(1-\frac{z_0}{w_0}\right)-\operatorname{Arg}(1-z_0w_0)\right)\\
&\qquad\qquad+\log|w_0|\left(-\operatorname{Arg}\left(1-\frac{z_0}{w_0}\right)-\operatorname{Arg}(1-z_0w_0)\right).
\end{aligned}$$

Further, by using the imaginary part of (29), the last line is calculated as

$$\begin{aligned}
\operatorname{Re}(-4\pi\sqrt{-1}\,t_0)\left(-4\pi\ \operatorname{Re}\left(\frac{p}{2}t_0-s_0\right)\right)&+\operatorname{Re}(-4\pi\sqrt{-1}\,s_0)(4\pi\operatorname{Re}t_0)\\
&=(4\pi)^2\left(-\operatorname{Im}t_0\ \operatorname{Re}\left(\frac{p}{2}t_0-s_0\right)+\operatorname{Im}s_0\ \operatorname{Re}t_0\right)\\
&=-(4\pi)^2\ \operatorname{Re}\left(\frac{p}{2}t_0^2-t_0s_0\right).
\end{aligned}$$

Therefore,

$$\frac{1}{4\pi}\operatorname{vol}(M_p)=\frac{1}{4\pi}\ \operatorname{Im}\left(\operatorname{Li}_2\left(\frac{z_0}{w_0}\right)-\operatorname{Li}_2(z_0w_0)\right)-4\pi\ \operatorname{Re}\left(\frac{p}{2}t_0^2-t_0s_0\right)=\operatorname{Re}\widehat{V}(t_0,s_0).$$

Hence, $\operatorname{Re}\widehat{V}(t_0,s_0)$ is equal to a normalized hyperbolic volume of M_p.

Appendix D Estimate of the hyperbolic volume of M_p

In this section, in Lemma D.1, we give a lower bound of $\varsigma_R(M_p)$, which is 4π times the hyperbolic volume of M_p. We use this lemma to show Lemma 5.1 in Appendix E. The aim of this section is to show Lemma D.1.

Lemma D.1 *For an integer $p \geq 6$,*

$$\varsigma_R(M_p) \geq \Lambda\Big(\frac{1}{6}\Big) - \frac{3}{p^2}.$$

We give a proof of this lemma later in this section.

We recall the definition of $\varsigma_R(M_p)$, as follows. We recall that the potential function is given by

$$\widehat{V}(t,s) = \frac{1}{4\pi\sqrt{-1}}(\mathrm{Li}_2(e^{4\pi\sqrt{-1}(s-t)}) - \mathrm{Li}_2(e^{-4\pi\sqrt{-1}(t+s)})) + 4\pi\sqrt{-1}\Big(\frac{p}{4}t^2 - ts\Big),$$

and its differentials are given by

$$\frac{\partial\widehat{V}}{\partial t} = \log(1-e^{4\pi\sqrt{-1}(s-t)}) - \log(1-e^{-4\pi\sqrt{-1}(t+s)}) + 4\pi\sqrt{-1}\Big(\frac{p}{2}t - s\Big) = 0,$$
$$\frac{\partial\widehat{V}}{\partial s} = -\log(1-e^{4\pi\sqrt{-1}(s-t)}) - \log(1-e^{-4\pi\sqrt{-1}(t+s)}) - 4\pi\sqrt{-1}\,t = 0.$$

As mentioned in Section 5.1, we have a single solution of these equations in the domain (28). Putting $\gamma = \frac{1}{p}$, we regard it as a function of γ, and denote it by $(t(\gamma), s(\gamma))$. We recall that $\varsigma_R(M_p)$ is defined by

$$\varsigma_R(M_p) = \mathrm{Re}\,\widehat{V}(t(\gamma), s(\gamma)).$$

By expanding the above equations, we can obtain the expansions of $t(\gamma)$ and $s(\gamma)$ as

$$t(\gamma) = \frac{1}{2}\gamma - \sqrt{3}\sqrt{-1}\,\gamma^2 - 6\gamma^3 + O(\gamma^4),$$
$$s(\gamma) = \frac{1}{12} + \frac{\pi}{\sqrt{3}}\gamma^2 - 4\pi\sqrt{-1}\gamma^3 + O(\gamma^4).$$

Hence, we can obtain that

$$\widehat{V}(t(\gamma), s(\gamma)) = \frac{1}{4\pi\sqrt{-1}}(\mathrm{Li}_2(e^{\pi\sqrt{-1}/3}) - \mathrm{Li}_2(e^{-\pi\sqrt{-1}/3})) - \frac{\pi\sqrt{-1}}{4}\gamma - \frac{\sqrt{3}\pi}{2}\gamma^2 + O(\gamma^3).$$

Therefore,

$$\varsigma_R(M_p) = \operatorname{Re} \widehat{V}(t(\gamma), s(\gamma)) = \Lambda\Big(\frac{1}{6}\Big) - \frac{\sqrt{3}\pi}{2}\gamma^2 + O(\gamma^3).$$

Since $\frac{\sqrt{3}\pi}{2} = 2.72069904\dots$, we can verify that Lemma D.1 holds for sufficiently large p. (In fact, it probably holds that $\varsigma_R(M_p) \ge \Lambda\big(\frac{1}{6}\big) - \frac{\sqrt{3}\pi}{2}\gamma^2$.)

Proof of Lemma D.1 We put

$$\begin{aligned} f(\gamma) &= \widehat{V}(t(\gamma), s(\gamma), \gamma) \\ &= \frac{1}{4\pi\sqrt{-1}}(\operatorname{Li}_2(e^{4\pi\sqrt{-1}(s-t)}) - \operatorname{Li}_2(e^{-4\pi\sqrt{-1}(t+s)})) + 4\pi\sqrt{-1}\Big(\frac{1}{4\gamma}t^2 - ts\Big), \end{aligned}$$

regarding $\widehat{V}$ as a function of t, s and γ. Then

$$\begin{aligned} f'(\gamma) &= \frac{\partial \widehat{V}}{\partial t}(t(\gamma), s(\gamma), \gamma)\cdot t'(\gamma) + \frac{\partial \widehat{V}}{\partial s}(t(\gamma), s(\gamma), \gamma)\cdot s'(\gamma) - 4\pi\sqrt{-1}\,\frac{1}{4\gamma^2}t(\gamma)^2 \\ &= -\pi\sqrt{-1}\,\hat{t}(\gamma)^2, \end{aligned}$$

where we put

$$\hat{t}(\gamma) = \frac{t(\gamma)}{\gamma} = \frac{1}{2} - \sqrt{3}\sqrt{-1}\,\gamma - 6\gamma^2 + O(\gamma^3).$$

Hence,

$$f''(\gamma) = -2\pi\sqrt{-1}\,\hat{t}(\gamma)\hat{t}'(\gamma).$$

By the Taylor expansion, we have that

$$\operatorname{Re} f(\gamma) = f(0) + \operatorname{Re} f'(0)\gamma + \operatorname{Re} f''(c)\frac{\gamma^2}{2} = \Lambda\Big(\frac{1}{6}\Big) + \gamma^2\pi\ \operatorname{Im}(\hat{t}(c)\hat{t}'(c))$$

for some c with $0 \le c \le \gamma$. Therefore, it is sufficient to show that

$$\operatorname{Im}(\hat{t}(\gamma)\hat{t}'(\gamma)) \ge -\frac{\sqrt{3}}{2}. \tag{78}$$

Since $\operatorname{Im}(\hat{t}(\gamma)\hat{t}'(\gamma)) = \operatorname{Re}\hat{t}(\gamma)\cdot\operatorname{Im}\hat{t}'(\gamma) + \operatorname{Im}\hat{t}(\gamma)\cdot\operatorname{Re}\hat{t}'(\gamma)$, it is sufficient to show that

$$0 \le \operatorname{Re}\hat{t}(\gamma) \le \frac{1}{2}, \quad -\sqrt{3} \le \operatorname{Im}\hat{t}'(\gamma) \le 0, \quad \operatorname{Re}\hat{t}'(\gamma) \le 0, \quad \operatorname{Im}\hat{t}(\gamma) \le 0. \tag{79}$$

We can verify that, for sufficiently large p, the inequalities (79) hold, since

$$\operatorname{Re}\hat{t}(\gamma) = \frac{1}{2} - 6\gamma^2 + O(\gamma^4), \quad \operatorname{Im}\hat{t}(\gamma) = -\sqrt{3}\,\gamma + \frac{4}{\sqrt{3}}(\pi^2 + 9)\gamma^3 + O(\gamma^5).$$

Further, we can numerically verify (79) for $p \le 100$; see Remark D.2 below.

We explain how we can give a rigorous proof of (79), as follows. As shown in Section 5.1, putting $z = e^{-4\pi\sqrt{-1}\,t}$, z satisfies that

$$z^{p/2} + z^{-p/2} - z^2 - z^{-2} + z + z^{-1} + 2 = 0.$$

Hence,

$$e^{2\pi\sqrt{-1}\,pt} + e^{-2\pi\sqrt{-1}\,pt} - e^{8\pi\sqrt{-1}\,t} - e^{-8\pi\sqrt{-1}\,t} + e^{4\pi\sqrt{-1}\,t} + e^{-4\pi\sqrt{-1}\,t} + 2 = 0.$$

Further, putting $\hat{t} = tp = \frac{t}{\gamma}$, we have that

$$e^{2\pi\sqrt{-1}\,\hat{t}} + e^{-2\pi\sqrt{-1}\,\hat{t}} - e^{8\pi\sqrt{-1}\,\gamma\hat{t}} - e^{-8\pi\sqrt{-1}\,\gamma\hat{t}} + e^{4\pi\sqrt{-1}\,\gamma\hat{t}} + e^{-4\pi\sqrt{-1}\,\gamma\hat{t}} + 2 = 0.$$

We put

$$\hat{t}(\gamma) = t_R(\gamma) + \sqrt{-1}\,\gamma t_I(\gamma),$$

where

$$t_R(\gamma) = \frac{1}{2} - 6\gamma^2 + O(\gamma^4), \quad t_I(\gamma) = -\sqrt{3} + \frac{4}{\sqrt{3}}(\pi^2 + 9)\gamma^2 + O(\gamma^4).$$

We note that $\hat{t}(-\gamma) = \overline{\hat{t}(\gamma)}$, since M_{-p} is homeomorphic to M_p with the opposite orientation, and we choose $\bar{z}^{-1}$ instead of z when we change the sign of p. Hence, $t_R(\gamma)$ and $t_I(\gamma)$ are real-valued even functions. From the above equation, we have that

$$\begin{aligned}(80)\quad & e^{2\pi\sqrt{-1}\,t_R - 2\pi\gamma t_I} + e^{-2\pi\sqrt{-1}\,t_R + 2\pi\gamma t_I} - e^{8\pi\sqrt{-1}\,\gamma t_R - 8\pi\sqrt{-1}\,\gamma^2 t_I} \\ & - e^{-8\pi\sqrt{-1}\,\gamma t_R + 8\pi\sqrt{-1}\,\gamma^2 t_I} + e^{4\pi\sqrt{-1}\,\gamma t_R - 4\pi\gamma^2 t_I} \\ & + e^{-4\pi\sqrt{-1}\,\gamma t_R + 4\pi\gamma^2 t_I} + 2 = 0.\end{aligned}$$

By changing the sign of γ, we have that

$$\begin{aligned}(81)\quad & e^{2\pi\sqrt{-1}\,t_R + 2\pi\gamma t_I} + e^{-2\pi\sqrt{-1}\,t_R - 2\pi\gamma t_I} - e^{-8\pi\sqrt{-1}\,\gamma t_R - 8\pi\sqrt{-1}\,\gamma^2 t_I} \\ & - e^{8\pi\sqrt{-1}\,\gamma t_R + 8\pi\sqrt{-1}\,\gamma^2 t_I} + e^{-4\pi\sqrt{-1}\,\gamma t_R - 4\pi\gamma^2 t_I} \\ & + e^{4\pi\sqrt{-1}\,\gamma t_R + 4\pi\gamma^2 t_I} + 2 = 0.\end{aligned}$$

By adding the above two equations, we obtain that

$$\begin{aligned}&(e^{2\pi\sqrt{-1}\,t_R} + e^{-2\pi\sqrt{-1}\,t_R})(e^{2\pi\gamma t_I} + e^{-2\pi\gamma t_I}) \\ &\quad - (e^{8\pi\sqrt{-1}\,\gamma t_R} + e^{-8\pi\sqrt{-1}\,\gamma t_R})(e^{8\pi\sqrt{-1}\,\gamma^2 t_I} + e^{-8\pi\sqrt{-1}\,\gamma^2 t_I}) \\ &\quad + (e^{4\pi\sqrt{-1}\,\gamma t_R} + e^{-4\pi\sqrt{-1}\,\gamma t_R})(e^{4\pi\gamma^2 t_I} + e^{-4\pi\gamma^2 t_I}) + 4 = 0.\end{aligned}$$

Hence,

$$\cos 2\pi t_R \cosh 2\pi\gamma t_I - \cos 8\pi\gamma t_R \cosh 2\pi\gamma^2 t_I + \cos 4\pi\gamma t_R \cosh 4\pi\gamma^2 t_I + 1 = 0.$$

Similarly, by subtracting (80) from (81), we can obtain that

$$\sin 2\pi t_R \sinh 2\pi\gamma t_I - \sin 8\pi\gamma t_R \sinh 2\pi\gamma^2 t_I + \sin 4\pi\gamma t_R \sinh 4\pi\gamma^2 t_I = 0.$$

The above two equations are real-valued equations for real-valued functions $t_R(\gamma)$ and $t_I(\gamma)$, and we can estimate the functions sin, cos, sinh and cosh as precisely as we need by the Taylor expansion. Hence, we can estimate $t_R(\gamma)$ and $t_I(\gamma)$ as precisely as we need, and we can (in principle) show (79). □

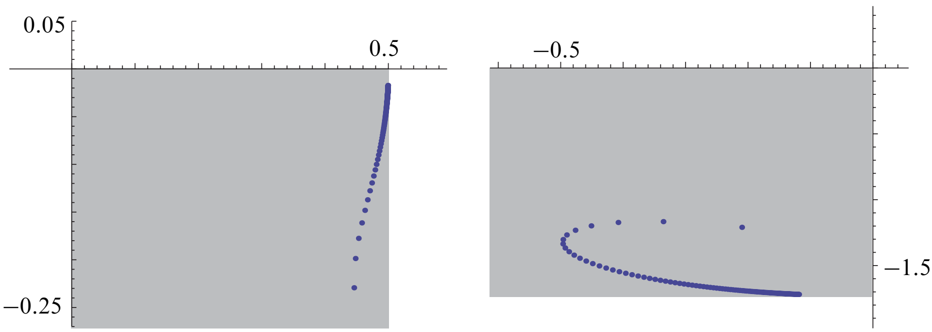


Figure 9: $\hat{t} \in \mathbb{C}$ for $p = 6, 7, \dots, 100$ (left) and $\hat{t}' \in \mathbb{C}$ for $p = 7, 8, \dots, 100$ (right). The gray areas are the areas of (79).

Remark D.2 We can numerically verify that $\hat{t}$ satisfies (79) for $p = 6, 7, \dots, 100$ and $\hat{t}'$ satisfies (79) for $p = 7, 8, \dots, 100$; see Figure 9 for their graphical representations. When $p = 6$, $\operatorname{Re}\hat{t}' > 0$ and it does not satisfy (79), but we can directly verify that (78) holds in this case.

Appendix E Proof of Lemma 5.1

The aim of this section is to give a proof of Lemma 5.1.

Proof of Lemma 5.1 We recall that

$$\Delta_0 = \left\{(t,s) \in \mathbb{R}^2 \mid 0 \le t+s \le \tfrac{1}{2},\ 0 \le s-t \le \tfrac{1}{2}\right\}.$$

In order to prove Lemma 5.1, it is sufficient to show that the domain

$$\{(t,s) \in \Delta_0 \mid \operatorname{Re} \widehat{V}(t,s) \geq \varsigma_R(M_p)\} \tag{82}$$

is included in the interior of Δ_0', since the domain (82) is a compact domain. Supposing that $\operatorname{Re} \widehat{V}(t,s) \geq \varsigma_R(M_p)$ for $(t,s) \in \Delta_0$, it is sufficient to show that

$$0.005 < t+s < 0.24, \quad 0.005 < s-t < 0.24, \quad |t| < \frac{0.74}{p}.$$

We show the first and second formulas in Lemma E.1 below, and show the third formula in Lemma E.3 below. Hence, we obtain the lemma. □

Before showing Lemmas E.1 and E.3, we review some behavior of the dilogarithm function. We put

$$\Lambda(t) = \operatorname{Re}\left(\frac{1}{2\pi\sqrt{-1}} \operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t})\right).$$

Since

$$\Lambda'(t) = -\log 2\sin \pi t, \quad \Lambda''(t) = -\pi \cot \pi t,$$

the behavior of $\Lambda(t)$ is as follows:

t	0	$\cdots$	$\frac{1}{6}$	$\cdots$	$\frac{1}{2}$	$\cdots$	$\frac{5}{6}$	$\cdots$	1
$\Lambda(t)$	0	$\nearrow$	$\Lambda\left(\frac{1}{6}\right)$	$\searrow$	0	$\searrow$	$-\Lambda\left(\frac{1}{6}\right)$	$\nearrow$	0
$\Lambda'(t)$		$+$	0	$-$	$-$	$-$	0	$+$	
$\Lambda''(t)$		$-$	$-$	$-$	0	$+$	$+$	$+$	

Here, $\Lambda\left(\frac{1}{6}\right) = 0.161533\ldots$. For the graph of $\Lambda(t)$, see Figure 10.

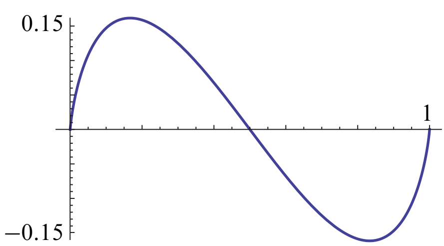


Figure 10: The graph of $\Lambda(t)$ for $0 \leq t \leq 1$

We note that, from the definition of $\widehat{V}$,

$$\operatorname{Re} \widehat{V}(t,s) = \tfrac{1}{2}\Lambda(2s-2t) + \tfrac{1}{2}\Lambda(2s+2t)$$

for $(t,s) \in \Delta_0$.

Lemma E.1 *Let* p *be an integer* ≥ 6. *We suppose that* $\operatorname{Re}\widehat{V}(t,s) \geq \varsigma_R(M_p)$ *for* $(t,s) \in \Delta_0$. *Then*

$$0.005 < t+s < 0.24, \quad 0.005 < s-t < 0.24.$$

Proof We put $u = t+s$ and $v = s-t$. Then

$$\operatorname{Re}\widehat{V} = \tfrac{1}{2}\Lambda(u) + \tfrac{1}{2}\Lambda(v).$$

It is sufficient to show that

$$0.005 < u < 0.24, \quad 0.005 < v < 0.24.$$

We have that

$$\tfrac{1}{2}\Lambda(2u) + \tfrac{1}{2}\Lambda(2v) = \operatorname{Re}\widehat{V} \geq \varsigma_R(M_p) \geq \varsigma_R(M_6).$$

Hence,

$$\begin{aligned}\Lambda(2u) &\geq 2\,\varsigma_R(M_6) - \Lambda(2v) \geq 2\,\varsigma_R(M_6) - \Lambda\left(\tfrac{1}{6}\right)\\ &= 2\cdot 0.102216\ldots - 0.161533\ldots = 0.042899\ldots > 0.\end{aligned}$$

Therefore, from the above-mentioned behavior of $\Lambda(t)$, we have that

$$0 < 2u < 0.5.$$

More precisely, since

$$\begin{aligned}\Lambda(0.01) - \left(2\,\varsigma_R(M_6) - \Lambda\left(\tfrac{1}{6}\right)\right) &= -0.00522573\ldots < 0,\\ \Lambda(0.48) - \left(2\,\varsigma_R(M_6) - \Lambda\left(\tfrac{1}{6}\right)\right) &= -0.0290494\ldots < 0,\end{aligned}$$

we obtain that

$$0.01 < 2u < 0.48.$$

Hence, $0.005 < u < 0.24$, as required. We can obtain that $0.005 < v < 0.24$ in the same way. □

Before we show Lemma E.3, we show the following lemma:

Lemma E.2 *For* $0 \leq \frac{1}{6} + t \leq 0.5$,

$$\Lambda\left(\tfrac{1}{6} + t\right) \leq \Lambda\left(\tfrac{1}{6}\right) - 1.4t^2.$$

Proof We put

$$f(t)=\Lambda\bigl(\tfrac{1}{6}\bigr)-1.4t^2-\Lambda\bigl(\tfrac{1}{6}+t\bigr).$$

Then its differentials are given by

$$f'(t)=-2.8t+\log 2\sin\pi\bigl(\tfrac{1}{6}+t\bigr),\qquad f''(t)=-2.8+\pi\cot\pi\bigl(\tfrac{1}{6}+t\bigr).$$

For $-\frac{1}{6}\le t\le\frac{1}{3}$, the solution of $f''(t)=0$ is given by

$$t=t_1=-\frac{1}{6}+\frac{1}{\pi}\arctan\Bigl(\frac{\pi}{2.8}\Bigr)=0.101613\ldots.$$

Further, for $-\frac{1}{6}\le t\le\frac{1}{3}$, the solutions of $f'(t)=0$ are given by

$$t=0,t_2,\quad \text{where } t_2=0.227398\ldots.$$

Hence, the behavior of $f(t)$ is given as follows:

t	$-\frac{1}{6}$	$\cdots$	0	$\cdots$	t_1	$\cdots$	t_2	$\cdots$	$\frac{1}{3}$
$f(t)$	$f\bigl(-\frac{1}{6}\bigr)$	$\searrow$	0	$\nearrow$	$f(t_1)$	$\nearrow$	$f(t_2)$	$\searrow$	$f\bigl(\frac{1}{3}\bigr)$
$f'(t)$		$-$	0	$+$	$+$	$+$	0	$-$	
$f''(t)$		$+$	$+$	$+$	0	$-$	$-$	$-$	

Here,

$$f\bigl(\tfrac{1}{3}\bigr)=\Lambda\bigl(\tfrac{1}{6}\bigr)-\tfrac{1.4}{9}=0.00597742\ldots>0.$$

Therefore, $f(t)\ge 0$, and we obtain the lemma. □

Lemma E.3 *Let p be an integer ≥ 6. We suppose that $\operatorname{Re}\widehat{V}(t,s)\ge\varsigma_R(M_p)$ for $(t,s)\in\Delta_0$. Then*

$$|t|<\frac{0.74}{p}.$$

Proof By Lemma D.1, we have that

$$\operatorname{Re}\widehat{V}(t,s)\ge\varsigma_R(M_p)\ge\Lambda\Bigl(\frac{1}{6}\Bigr)-\frac{3}{p^2}.$$

Further, putting $s=\frac{1}{12}+\hat{s}$,

$$\begin{aligned}\operatorname{Re}\widehat{V}(t,s)&=\frac{1}{2}\Lambda(2s-2t)+\frac{1}{2}\Lambda(2s+2t)\le\frac{1}{2}\Lambda\Bigl(\frac{1}{6}+2\hat{s}-2t\Bigr)+\frac{1}{2}\Lambda\Bigl(\frac{1}{6}+2\hat{s}+2t\Bigr)\\&\le\Lambda\Bigl(\frac{1}{6}\Bigr)-\frac{1.4}{2}(2\hat{s}-2t)^2-\frac{1.4}{2}(2\hat{s}+2t)^2\le\Lambda\Bigl(\frac{1}{6}\Bigr)-5.6(\hat{s}^2+t^2),\end{aligned}$$

where we obtain the second last inequality by Lemma E.2. Hence,

$$\Lambda\Big(\frac{1}{6}\Big) - 5.6(\hat{s}^2 + t^2) \geq \Lambda\Big(\frac{1}{6}\Big) - \frac{3}{p^2}.$$

Therefore,

$$\frac{3}{5.6p^2} \geq \hat{s}^2 + t^2 \geq t^2.$$

It follows that

$$|t| \leq \frac{\sqrt{\frac{3}{5.6}}}{p} < \frac{0.74}{p},$$

since $\sqrt{\frac{3}{5.6}} = 0.731925\ldots$. Hence, we obtain the lemma. □

Appendix F Restriction of Δ_0 to Δ_0''

In Section 7, we restrict Δ_0 to Δ_0'', and it is necessary to show that the following error term is sufficiently small:

$$\sum_{(1/2-i/N,1/2-j/N)\in\Delta_0-\Delta_0''} (q^{i/2} - q^{-i/2}) \exp\Big(N \cdot V\Big(\frac{1}{2} - \frac{i}{N}, \frac{1}{2} - \frac{j}{N} - \frac{1}{2N}\Big)\Big) = O(e^{N(\varsigma_R(M_5)-\varepsilon)}).$$

We can reduce this formula to the following lemma; the aim of this section is to show this lemma.

Lemma F.1 *For any fixed k with $0 < \frac{k}{N} \leq 0.001$, we have that*

$$\sum_{\substack{(1/2-i/N,1/2-j/N)\in\Delta_0 \\ j-i=k}} (q^{i/2} - q^{-i/2}) \exp\Big(N \cdot V\Big(\frac{1}{2} - \frac{i}{N}, \frac{1}{2} - \frac{j}{N} - \frac{1}{2N}\Big)\Big) = O(e^{N(\varsigma_R(M_5)-\varepsilon)}),$$

$$\sum_{\substack{(1/2-i/N,1/2-j/N)\in\Delta_0 \\ i+j=N-k}} (q^{i/2} - q^{-i/2}) \exp\Big(N \cdot V\Big(\frac{1}{2} - \frac{i}{N}, \frac{1}{2} - \frac{j}{N} - \frac{1}{2N}\Big)\Big) = O(e^{N(\varsigma_R(M_5)-\varepsilon)})$$

for sufficiently small $\varepsilon > 0$.

We let $\rho = \frac{k}{N}$ in the remainder of this section. Then ρ is a fixed real number with $0 \leq \rho \leq 0.001$. We note that we can ignore a factor such as $(q^{i/2} - q^{-i/2})$ in the formulas of the lemma, because the aim of this section is to estimate the exponential order of the formulas of the lemma and such a factor does not contribute to the exponential order, as we see in the calculation in Section 5, while the behavior of V does contribute to the exponential order. Hence, in order to show the lemma, it is sufficient to show that

$$\sum_{1/2-i/N\in\Delta_{0,1}} \exp\Big(N\cdot W_1\Big(\frac{1}{2}-\frac{i}{N}\Big)\Big) = O(e^{N(\varsigma_R(M_5)-\varepsilon)}), \tag{83}$$

$$\sum_{1/2-i/N\in\Delta_{0,2}} \exp\Big(N\cdot W_2\Big(\frac{1}{2}-\frac{i}{N}\Big)\Big) = O(e^{N(\varsigma_R(M_5)-\varepsilon)}), \tag{84}$$

where we put

$$W_1(t) = \frac{1}{N}(\hat\varphi(2\rho)-\hat\varphi(1-4t-2\rho)) + 4\pi\sqrt{-1}\Big(\frac{1}{4}t^2-\rho t\Big),$$
$$W_2(t) = \frac{1}{N}(\hat\varphi(-4t+2\rho)-\hat\varphi(1-2\rho)) + 4\pi\sqrt{-1}\Big(\frac{9}{4}t^2-\rho t\Big),$$

and

$$\Delta_{0,1} = \big\{t\in\mathbb{R} \mid 0\le t\le \tfrac{1}{8}\big\},$$
$$\Delta_{0,2} = \big\{t\in\mathbb{R} \mid -\tfrac{1}{8}\le t\le 0\big\}.$$

We note that, by Lemma 5.2, we can obtain that

$$W_2(-t) = W_1(t) + 2\pi\sqrt{-1}t. \tag{85}$$

The functions W_i converge to the following functions as $N\to\infty$:

$$\widehat W_1(t) = \frac{1}{4\pi\sqrt{-1}}(\mathrm{Li}_2(e^{4\pi\sqrt{-1}\rho}) - \mathrm{Li}_2(e^{-4\pi\sqrt{-1}(2t+\rho)})) + 4\pi\sqrt{-1}\Big(\frac{1}{4}t^2-\rho t\Big),$$
$$\widehat W_2(t) = \frac{1}{4\pi\sqrt{-1}}(\mathrm{Li}_2(e^{-4\pi\sqrt{-1}(2t-\rho)}) - \mathrm{Li}_2(e^{-4\pi\sqrt{-1}\rho})) + 4\pi\sqrt{-1}\Big(\frac{9}{4}t^2-\rho t\Big).$$

We note again that, by (85), we obtain that

$$\widehat W_2(-t) = \widehat W_1(t) + 2\pi\sqrt{-1}\,t. \tag{86}$$

Similarly as the calculation of Section 5, we can restrict $\Delta_{0,1}$ to $\Delta'_{0,1}$ of the following lemma:

Lemma F.2 *We recall that ρ is a fixed real number with $0\le\rho\le 0.001$. We put*

$$\Delta'_{0,1} = \{t\in\mathbb{R} \mid 0.02\le t\le 0.07\}.$$

Then the domain

$$\{t\in\Delta_{0,1} \mid \operatorname{Re}\widehat W_1(t) \ge \varsigma_R(M_5)-\varepsilon\}$$

is included in Δ'_0 for sufficiently small $\varepsilon>0$.

Proof Since the domain $\{\operatorname{Re}\widehat{W}_1(t) \geq \varsigma_R(M_5)\}$ is compact, assuming that $\operatorname{Re}\widehat{W}_1(t) \geq \varsigma_R(M_5)$, it is sufficient to show that

$$0.02 < t < 0.07.$$

From the definition of $\widehat{W}_1(t)$, we have that

$$\operatorname{Re}\widehat{W}_1(t) = \tfrac{1}{2}\Lambda(4t+2\rho) + \tfrac{1}{2}\Lambda(2\rho) \geq \varsigma_R(M_5).$$

Hence,

$$\Lambda(4t+2\rho) \geq 2\varsigma_R(M_5) - \Lambda(2\rho) \geq 2\varsigma_R(M_5) = 0.1561897\ldots,$$

and $\Lambda(0.09) = 0.141707\ldots$ and $\Lambda(0.26) = 0.142161\ldots$, which do not satisfy the above condition. Therefore, since the behavior of $\Lambda(\cdot)$ is as shown in Appendix E, we obtain that

$$0.09 \leq 4t + 2\rho \leq 0.26.$$

Further, since $0 \leq \rho \leq 0.001$, we obtain that

$$0.02 < t < 0.07,$$

as required. $\square$

Proof of Lemma F.1 We recall that it is sufficient to show (83) and (84). For simplicity, we show a proof of (83). (We can show (84) similarly.) By Lemma F.2, we can restrict the range of the sum of (83) to $\Delta'_{0,1}$. Hence, it is sufficient to show that

$$\sum_{1/2-i/N\in\Delta'_{0,1}} \exp\Bigl(N\cdot W_1\Bigl(\frac{1}{2}-\frac{i}{N}\Bigr)\Bigr) = O(e^{N(\varsigma_R(M_5)-\varepsilon)}).$$

Further, by Proposition F.4 (Poisson summation formula), it is sufficient to show that

$$\int_{\Delta'_{0,1}} \exp(N\cdot W_1(t))\,dt = O(e^{N(\varsigma_R(M_5)-\varepsilon)}). \tag{87}$$

We show this formula in Proposition F.7 by a similar procedure to the saddle-point method. Therefore, we obtain the lemma. $\square$

F.1 Critical points of $\widehat{W}_i$

In this section, we calculate critical points of $\widehat{W}_1$ and $\widehat{W}_2$, which we use in Section F.3 later.

From the definitions of $\widehat{W}_1$ and $\widehat{W}_2$, their differentials are given by

$$\frac{d\widehat{W}_1}{dt} = -2\log(1-e^{-4\pi\sqrt{-1}\,(2t+\rho)}) + 4\pi\sqrt{-1}\,\Big(\frac{1}{2}t-\rho\Big), \tag{88}$$

$$\frac{d\widehat{W}_2}{dt} = 2\log(1-e^{-4\pi\sqrt{-1}(2t-\rho)}) + 4\pi\sqrt{-1}\Big(\frac{9}{2}t-\rho\Big). \tag{89}$$

We consider a critical point of $\widehat{W}_1$ in the domain

$$\{t \in \mathbb{C} \mid 0.02 \le \operatorname{Re} t \le 0.08\}. \tag{90}$$

We note that this range is more extended than the range of Lemma F.2, because there is a critical point in the extended part and we use it in Section F.3. The equation $d\widehat{W}_1/dt = 0$ is rewritten as

$$\log(1-e^{-4\pi\sqrt{-1}(2t+\rho)}) = 2\pi\sqrt{-1}\Big(\frac{t}{2}-\rho\Big).$$

Further, it is rewritten as

$$1-e^{-4\pi\sqrt{-1}(2t+\rho)} = e^{2\pi\sqrt{-1}(t/2-\rho)},$$

and

$$1-x^8 e^{-4\pi\sqrt{-1}\,\rho} = \frac{1}{x}e^{-2\pi\sqrt{-1}\,\rho},$$

where we put $x = e^{-\pi\sqrt{-1}\,t}$. We can verify that there is a single solution of this equation such that the corresponding t is in (90); we put this solution t_0. We show some numerical values of t_0, as follows:

ρ	t_0
0	$0.0787594\ldots - \sqrt{-1}\cdot 0.0467182\ldots$
0.0002	$0.0788007\ldots - \sqrt{-1}\cdot 0.0468217\ldots$
0.0004	$0.0788424\ldots - \sqrt{-1}\cdot 0.0469251\ldots$
0.0006	$0.0788846\ldots - \sqrt{-1}\cdot 0.0470283\ldots$
0.0008	$0.0789272\ldots - \sqrt{-1}\cdot 0.0471313\ldots$
0.001	$0.0789703\ldots - \sqrt{-1}\cdot 0.0472342\ldots$

We consider a critical point of $\widehat{W}_2$ in the domain

$$\{t \in \mathbb{C} \mid -0.08 \le \operatorname{Re} t \le -0.02\}.$$

Then, noting (86), we can show that $-t_0$ is a single critical point of $\widehat{W}_2$ in the above domain.

F.2 The Poisson summation formula for W_i

The aim of this section is to show Proposition F.4, which we use in the proof of Lemma F.2.

In order to show Proposition F.4, we review the following proposition, which corresponds to the 1–dimensional case of Proposition 3.4:

Proposition F.3 (see [18]) *For $c \in \mathbb{C}$ and a closed interval D' in $\mathbb{R}$, we put*

$$\Lambda = \left\{ \frac{i}{N} + c_1 \in \mathbb{C} \;\middle|\; i \in \mathbb{Z},\ \frac{i}{N} \in D' \right\},$$
$$D = \{ t + c_1 \in \mathbb{C} \mid t \in D' \subset \mathbb{R} \}.$$

Let $\psi(t)$ be a holomorphic function defined in a neighborhood of $0 \in \mathbb{C}$ including D. We assume that ∂D is included in the domain

$$\{ t \in \mathbb{C} \mid \operatorname{Re} \psi(t) < -\varepsilon_0 \}$$

for some $\varepsilon_0 > 0$. Further, we assume that the two points of ∂D can be connected by a path in each of the domains

$$\{ t + \delta\sqrt{-1} \in \mathbb{C} \mid t \in D',\ \delta \geq 0,\ \operatorname{Re} \psi(t + \delta\sqrt{-1}) < 2\pi\delta \}, \tag{91}$$
$$\{ t - \delta\sqrt{-1} \in \mathbb{C} \mid t \in D',\ \delta \geq 0,\ \operatorname{Re} \psi(t - \delta\sqrt{-1}) < 2\pi\delta \}. \tag{92}$$

Then

$$\frac{1}{N} \sum_{t \in \Lambda} e^{N\psi(t)} = \int_D e^{N\psi(t)} dt + O(e^{-N\varepsilon})$$

for some $\varepsilon > 0$.

The aim of this section is to show the following proposition, by using the above proposition:

Proposition F.4 *We have*

$$\frac{1}{N} \sum_{1/2 - i/N \in \Delta'_{0,1}} \exp\Bigl(N \cdot W_1 \Bigl(\frac{1}{2} - \frac{i}{N} \Bigr) \Bigr) = \int_{\Delta'_{0,1}} \exp(N \cdot W_1(t))\, dt + O(e^{N(\varsigma_R(M_5) - \varepsilon)}).$$

Proof By applying Proposition F.3 to $\widehat{W}_1(t) - \varsigma_R(M_5)$, we obtain the proposition. We verify the assumptions (91) and (92) in Lemmas F.5 and F.6 below. The other assumptions of Proposition F.3 can be verified easily. $\square$

We note that the following inequalities hold for $x = e^{4\pi\sqrt{-1}\,(2t+\rho+2\delta\sqrt{-1})}$ and $\delta \in \mathbb{R}$:

$$0 < \operatorname{Arg}\Big(1-\frac{1}{x}\Big) < 4\pi\Big(\frac{1}{4}-2t-\rho\Big); \tag{93}$$

we use these in the proofs of the following two lemmas:

Lemma F.5 *The assumption (91) holds in the proof of Proposition F.4.*

Proof We let

$$F_t(\delta) = \operatorname{Re}\widehat{W}_1(t+\delta\sqrt{-1}) - \varsigma_R(M_5) - 2\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon > 0$ such that

$$\frac{d}{d\delta}F_t(\delta) < -\varepsilon, \tag{94}$$

for any $t \in \Delta'_{0,1}$.

To show this, we estimate the differential of $F_t(\delta)$, as follows. By (88), the differential of $F_t(\delta)$ is given by

$$\frac{dF_t}{d\delta} = \operatorname{Re}\Big(\sqrt{-1}\,\frac{d}{dt}\widehat{W}_1(t+\delta\sqrt{-1})\Big) - 2\pi = 2\operatorname{Arg}\Big(1-\frac{1}{x}\Big) - 4\pi\Big(\frac{1}{2}t-\rho\Big) - 2\pi,$$

where we put $x = e^{4\pi\sqrt{-1}(2t+\rho+2\delta\sqrt{-1})}$. By (93), we can estimate it as follows:

$$\frac{dF_t}{d\delta} < 2\cdot 4\pi\Big(\frac{1}{4}-2t-2\rho\Big) - 4\pi\Big(\frac{1}{2}t-\rho\Big) - 2\pi = -4\pi\Big(\frac{9}{2}t+\rho\Big) \le -4\pi\cdot 0.09,$$

where we obtain the last inequality since $t \ge 0.02$ by Lemma F.2. Hence, (94) holds, as required. □

Lemma F.6 *The assumption (92) holds in the proof of Proposition F.4.*

Proof We let

$$F_t(\delta) = \operatorname{Re}\widehat{W}_1(t-\delta\sqrt{-1}) - \varsigma_R(M_5) - 2\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon > 0$ such that

$$\frac{d}{d\delta}F_t(\delta) < -\varepsilon \tag{95}$$

for any $t \in \Delta'_{0,1}$.

To show this, we estimate the differential of $F_t(\delta)$, as follows. By (88), the differential of $F_t(\delta)$ is given by

$$\frac{dF_t}{d\delta} = \mathrm{Re}\Bigl(-\sqrt{-1}\frac{d}{dt}\widehat{W}_1(t+\delta\sqrt{-1})\Bigr) - 2\pi = -2\,\mathrm{Arg}\Bigl(1-\frac{1}{x}\Bigr) + 4\pi\Bigl(\frac{1}{2}t-\rho\Bigr) - 2\pi,$$

where we put $x = e^{4\pi\sqrt{-1}(2t+\rho-2\delta\sqrt{-1})}$. By (93), we can estimate it as follows:

$$\frac{dF_t}{d\delta} < 0 + 4\pi\Bigl(\frac{1}{2}t-\rho\Bigr) - 2\pi = 4\pi\Bigl(\frac{1}{2}t-\rho-\frac{1}{2}\Bigr) \le -4\pi\cdot\frac{0.93}{2},$$

where we obtain the last inequality since $t \le 0.07$ by Lemma F.2. Hence, (95) holds, as required. □

F.3 The integral (87) is sufficiently small

In this section, we show that the integral (87) is of sufficiently small order in the following proposition. The aim of this section is to show this proposition.

Proposition F.7 *We have*

$$\int_{\Delta'_{0,1}} \exp(N\cdot W_1(t))\,dt = O(e^{N(\varsigma_R(M_5)-\varepsilon)}).$$

We give a proof of the proposition later in this section.

We use a similar procedure as the saddle-point method, but we have no critical point whose real part belongs to $\Delta'_{0,1}$ in this case. Hence, the conclusion is that the integral of the proposition is of sufficiently small order. We put

$$f_t(X) = \mathrm{Re}\,\widehat{W}_1(t+X\sqrt{-1}).$$

Then, in the same way as in the previous section, we have that

$$\frac{df_t}{dX} = 2\,\mathrm{Arg}\Bigl(1-\frac{1}{x}\Bigr) - 4\pi\Bigl(\frac{1}{2}t-\rho\Bigr), \tag{96}$$

where we put $x = e^{4\pi\sqrt{-1}(2t+\rho+2X\sqrt{-1})}$.

Lemma F.8 *We fix a real number t with $0.02 \le t \le 0.08$. We recall that ρ is a fixed real number with $0 \le \rho \le 0.001$. Then $f_t(X)$ has a unique minimal point at $X = g(t)$, where*

$$g(t) = \frac{1}{8\pi}\log\frac{\sin 2\pi\bigl(\frac{1}{2}t-\rho\bigr)}{\sin 2\pi\bigl(\frac{1}{2}-\frac{9}{2}t-\rho\bigr)},$$

ie $f_t(X)$ is monotonically decreasing for $X < g(t)$ and is monotonically increasing for $X > g(t)$.

Proof We put $x = e^{4\pi\sqrt{-1}(2t+\rho+2X\sqrt{-1})}$. Then $\frac{1}{x} = e^{8\pi X} e^{-4\pi\sqrt{-1}(2t+\rho)}$. We put $\theta = \mathrm{Arg}\bigl(1-\frac{1}{x}\bigr)$ in this proof. Since $0.02 \le t \le 0.08$, θ is in the range

$$0 < \theta < 4\pi\bigl(\tfrac{1}{4} - 2t - \rho\bigr).$$

By (96),

$$\frac{\partial}{\partial X} f_t(X) \begin{cases} > 0 & \text{if } \theta > 2\pi\bigl(\frac{1}{2}t - \rho\bigr), \\ = 0 & \text{if } \theta = 2\pi\bigl(\frac{1}{2}t - \rho\bigr), \\ < 0 & \text{if } \theta < 2\pi\bigl(\frac{1}{2}t - \rho\bigr). \end{cases}$$

Further, in the same way as in the proof of Lemma 5.3, we can show that X is monotonically increasing as a function of θ, and they satisfy that

$$\frac{e^{8\pi X}}{\sin\theta} = \frac{1}{\sin\bigl(4\pi\bigl(\frac{1}{4} - 2t - \rho\bigr) - \theta\bigr)}.$$

This is rewritten as

$$X = \frac{1}{8\pi} \log \frac{\sin\theta}{\sin\bigl(4\pi\bigl(\frac{1}{4} - 2t - \rho\bigr) - \theta\bigr)}.$$

Therefore,

$$\frac{\partial}{\partial X} f_t(X) \begin{cases} > 0 & \text{if } X > g(t), \\ = 0 & \text{if } X = g(t), \\ < 0 & \text{if } X < g(t), \end{cases}$$

where we put

$$g(t) = \frac{1}{8\pi} \log \frac{\sin 2\pi\bigl(\frac{1}{2}t - \rho\bigr)}{\sin 2\pi\bigl(\frac{1}{2} - \frac{9}{2}t - \rho\bigr)}.$$

Hence, we obtain the lemma. □

Proof of Proposition F.7 We put

$$\Delta''_{0,1} = \{t \in \mathbb{R} \mid 0.02 \le t \le 0.08\}.$$

We show that there exists a homotopy $\Delta''_{(\delta)}$ $(0 \le \delta \le 1)$ between $\Delta''_{(0)} = \Delta''_{0,1}$ and $\Delta''_{(1)}$ such that

$$\Delta''_{(1)} \subset \{t \in \mathbb{C} \mid \mathrm{Re}\, W_1(t) < \varsigma_R(M_5) - \varepsilon\}, \tag{97}$$

$$\partial\Delta''_{(\delta)} \subset \{t \in \mathbb{C} \mid \mathrm{Re}\, W_1(t) < \varsigma_R(M_5) - \varepsilon\}. \tag{98}$$

We note that $W_1(t)$ uniformly converges to $\widehat{W}_1(t)$ on $\Delta''_{0,1}$ and the error term is of order $O(1/N^2)$. So we show the existence of such a homotopy for $\widehat{W}_1(t)$, instead of $W_1(t)$.

For each fixed t, we move X from 0 along the gradient flow of $-\mathrm{Re}\,\widehat{W}_1(t + X\sqrt{-1})$. Then the value of $\mathrm{Re}\,\widehat{W}_1(t + X\sqrt{-1})$ monotonically decreases. By Lemma F.8, X arrives at $g(t)$. We put $\Delta''_{(\delta)}$ to be

$$\Delta''_{(\delta)} = \{t + \delta \cdot g(t)\sqrt{-1} \in \mathbb{C} \mid 0.02 \le t \le 0.08\}.$$

It is sufficient to show that this homotopy satisfies (98) and (97).

We show (98), as follows. Since $\partial\Delta''_{0,1}$ is originally included in this domain and the value of $\mathrm{Re}\,\widehat{W}_1$ monotonically decreases, $\partial\Delta''_{(\delta)}$ is also included in this domain. Hence, (98) holds.

We show (97), as follows. We consider the functions

$$F(t, X) = \mathrm{Re}\,\widehat{W}_1(t + X\sqrt{-1}), \quad h(t) = F(t, g(t)) = \mathrm{Re}\,\widehat{W}_1(t + g(t)\sqrt{-1}).$$

It follows from the definition of $g(t)$ that $\frac{\partial F}{\partial X} = 0$ at $X = g(t)$. Hence, $\mathrm{Im}\, d\widehat{W}_1/dt = 0$ at $(t + g(t)\sqrt{-1})$. Further, $dh/dt = \mathrm{Re}\, d\widehat{W}_1/dt$ at $(t + g(t)\sqrt{-1})$. Therefore, when t is a critical point of $h(t)$, $(t + g(t)\sqrt{-1})$ is a critical point of $\widehat{W}_1$. Further, as shown in Section F.1, $\widehat{W}_1$ has a single critical point at t_0 in (90). Hence, putting $t_R = \mathrm{Re}\, t_0$, $h(t)$ has a unique maximal point at t_R and no other critical points. Originally, t_R belongs to the domain of (97). Further, since $\mathrm{Re}\,\widehat{W}_1$ monotonically decreases by the above-mentioned gradient flow, t_0 belongs to the domain of (97). Furthermore, since $h(t)$ has a unique maximal point at t_R, $(t + g(t)\sqrt{-1})$ also belongs to the domain of (97). Therefore, (97) holds, as required. □

Appendix G Proof of Lemma 7.1

The aim of this section is to give a proof of Lemma 7.1.

Proof of Lemma 7.1 Since the domain $\{\mathrm{Re}\,\widehat{V}(t,s) \ge \varsigma_R(M_5)\}$ is compact, assuming that $\mathrm{Re}\,\widehat{V}(t,s) \ge \varsigma_R(M_5)$, it is sufficient to show that

$$t + s < 0.26, \quad s - t < 0.26, \quad |t| < 0.099, \quad s < 0.2.$$

We show these inequalities in this proof. We recall that

$$\mathrm{Re}\,\widehat{V}(t,s) = \tfrac{1}{2}\Lambda(2t + 2s) + \tfrac{1}{2}\Lambda(2s - 2t),$$

as shown in Appendix E. We put $u = t + s$ and $v = s - t$.

We show that $t+s<0.26$ and $s-t<0.26$, as follows. Assuming that

$$\tfrac{1}{2}\Lambda(2u)+\tfrac{1}{2}\Lambda(2v)\ge \varsigma_{\boldsymbol{R}}(M_5),$$

it is sufficient to show that $u<0.26$ and $v<0.26$. We have that

$$\begin{aligned}\Lambda(2u)&\ge 2\,\varsigma_{\boldsymbol{R}}(M_5)-\Lambda(2v)\ge 2\,\varsigma_{\boldsymbol{R}}(M_5)-\Lambda\big(\tfrac{1}{6}\big)\\ &=2\cdot 0.078094\ldots-0.161533\ldots=-0.005343\ldots,\end{aligned}$$

and $\Lambda(2\cdot 0.26)=-0.0138498$, which does not satisfy the above condition. Hence, since the behavior of $\Lambda(\cdot)$ is as shown in Appendix E, we obtain that $u<0.26$. We also obtain that $v<0.26$ in the same way.

We show that $s<0.2$, as follows. By Lemma G.1 below, $\{\operatorname{Re}\widehat{V}(t,s)\ge\varsigma_{\boldsymbol{R}}(M_5)\}$ is a convex domain. Further, this domain is symmetric with respect to the change of the sign of t. Hence, s has a maximal value when $t=0$. Putting $t=0$, we have that

$$\operatorname{Re}\widehat{V}(0,s)=\Lambda(2s)\ge\varsigma_{\boldsymbol{R}}(M_5)=0.078094\ldots,$$

and $\Lambda(2\cdot 0.2)=0.0676532\ldots$, which does not satisfy the above condition. Hence, since the behavior of $\Lambda(\cdot)$ is as shown in Appendix E, we obtain that $s<0.2$.

We show that $|t|<0.099$, as follows. Noting that the domain $\{\operatorname{Re}\widehat{V}(t,s)\ge\varsigma_{\boldsymbol{R}}(M_5)\}$ is symmetric with respect to the change of the sign of t, it is sufficient to show that $t<0.099$. We calculate the maximal value of $t_{\max}$ in this domain. It satisfies the equations

$$\begin{cases}\tfrac{1}{2}\Lambda(2t+2s)+\tfrac{1}{2}\Lambda(2s-2t)=\varsigma_{\boldsymbol{R}}(M_5),\\ \tfrac{\partial}{\partial s}\big(\tfrac{1}{2}\Lambda(2t+2s)+\tfrac{1}{2}\Lambda(2s-2t)\big)=0.\end{cases}$$

They are rewritten as

$$\begin{cases}\Lambda(2t+2s)+\Lambda(2s-2t)=2\varsigma_{\boldsymbol{R}}(M_5),\\ \Lambda'(2t+2s)+\Lambda'(2s-2t)=0.\end{cases}$$

We note that this system of equations has exactly two solutions, corresponding to the maximal and minimal values of t, since the domain $\{\operatorname{Re}\widehat{V}(t,s)\ge\varsigma_{\boldsymbol{R}}(M_5)\}$ is a convex domain whose boundary curve is smooth by Lemma G.1 below. By calculating a solution of them by Newton's method from $(t,s)=(0.1,0.2)$, we obtain $t_{\max}=0.097454002299\ldots$. Hence, $t<0.099$, as required. $\square$

Lemma G.1 *The domain*

$$\big\{(t,s)\in\mathbb{R}^2 \mid \operatorname{Re}\widehat{V}(t,s)\ge\varsigma_{\boldsymbol{R}}(M_5),\ 0\le t+s<\tfrac{1}{2},\ 0\le s-t<\tfrac{1}{2}\big\}$$

is a convex domain.

See Figure 6 for a graphical representation of this lemma.

Proof Putting $u=t+s$ and $v=s-t$, it is sufficient to show that the domain

$$\big\{(u,v)\in\mathbb{R}^2 \mid \Lambda(2u)+\Lambda(2v)\geq 2\varsigma_{\boldsymbol{R}}(M_5),\ 0\leq u<\tfrac{1}{2},\ 0\leq v<\tfrac{1}{2}\big\} \tag{99}$$

is a convex domain. Hence, it is sufficient to show that the curvature of the boundary curve of this domain with respect to the outward normal vector is negative everywhere except for the lines $\{u=0\}$ and $\{v=0\}$.

When $0\leq u\leq\frac{1}{4}$ and $0\leq v\leq\frac{1}{4}$, $\Lambda(2u)$ and $\Lambda(2v)$ are concave functions, and it follows that the domain of the lemma is convex, and the curvature of the boundary curve with respect to the outward normal vector is negative in this area.

When $0\leq u\leq\frac{1}{4}$ and $\frac{1}{4}<v<\frac{1}{2}$, we show the lemma, as follows. As we can observe in Figure 6, this area is relatively small. We can obtain that

$$0.062<u<0.107,$$

since $\Lambda(2u)\geq 2\varsigma_{\boldsymbol{R}}(M_5)$. Further, we can obtain that

$$\tfrac{1}{4}<v<0.254,$$

since $\Lambda(2v)\geq 2\varsigma_{\boldsymbol{R}}(M_5)-\Lambda(2u)\geq 2\varsigma_{\boldsymbol{R}}(M_5)-\Lambda\big(\frac{1}{6}\big)$. Hence, noting the behavior of $\Lambda(\,\cdot\,)$ shown in Appendix E, we can obtain that

$$-0.22<\Lambda'(2u)<-0.28,$$

since $\Lambda'(2\cdot 0.062)=0.275018\ldots$ and $\Lambda'(2\cdot 0.107)=-0.219598\ldots$, and can obtain that

$$-0.7<\Lambda'(2v)<-0.69,$$

since $\Lambda'\big(2\cdot\frac{1}{4}\big)=-0.693147\ldots$ and $\Lambda'(2\cdot 0.254)=-0.692831\ldots$, and can obtain that

$$-8<\Lambda''(2u)<-2,$$

since $\Lambda''(2\cdot 0.062)=-7.65238\ldots$ and $\Lambda''(2\cdot 0.107)=-3.94669\ldots$, and can obtain that

$$0<\Lambda''(2v)<0.1,$$

since $\Lambda'\big(2\cdot\frac{1}{4}\big)=0$ and $\Lambda'(2\cdot 0.254)=0.0789735\ldots$. We consider the boundary curve of the domain (99). It is given by $F(u,v)=0$, where we put

$$F(u,v)=\Lambda(2u)+\Lambda(2v)-2\varsigma_{\boldsymbol{R}}(M_5).$$

When $0.062 < u < 0.107$ and $\frac{1}{4} < v < 0.254$, this curve can be presented by $v = f(u)$ with some function f satisfying that $F(u, f(u)) = 0$. Its differentials are given by

$$\frac{d}{du}F(u, f(u)) = 2\big(\Lambda'(2u) + \Lambda'(2f(u))f'(u)\big) = 0,$$
$$\frac{d^2}{du^2}F(u, f(u)) = 2\big(2\Lambda''(2u) + 2\Lambda''(2f(u))f'(u)^2 + \Lambda'(2f(u))f''(u)\big) = 0.$$

Since

$$f'(u) = -\frac{\Lambda'(2u)}{\Lambda'(2v)},$$

we obtain that $|f'(u)| < 1$ from the above inequalities. Further, since

$$f''(u) = -\frac{1}{\Lambda'(2v)}(2\Lambda''(2u) + 2\Lambda''(2v)f'(u)^2),$$

we obtain that $f''(u) < 0$ from the above inequalities. Therefore, the curvature of the boundary curve with respect to the outward normal vector is negative in the area that $0.062 < u < 0.107$ and $\frac{1}{4} < v < 0.254$.

When $0 \le v \le \frac{1}{4}$ and $\frac{1}{4} < u < \frac{1}{2}$, we can show that the above-mentioned curvature is negative in the area that $0.062 < v < 0.107$ and $\frac{1}{4} < u < 0.254$, in the same way as above, exchanging u and v.

Therefore, the curvature of the boundary curve of the domain (99) with respect to the outward normal vector is negative everywhere except for the lines $\{u = 0\}$ and $\{v = 0\}$, as required. □

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