

Notes on open book decompositions for Engel structures

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We relate open book decompositions of a 4–manifold M with its Engel structures. Our main result is, given an open book decomposition of M whose binding is a collection of 2–tori and whose monodromy preserves a framing of a page, the construction of an Engel structure whose isotropic foliation is transverse to the interior of the pages and tangent to the binding.

In particular, the pages are contact manifolds and the monodromy is a compactly supported contactomorphism. As a consequence, on a parallelizable closed 4–manifold, every open book with toric binding carries in the previous sense an Engel structure. Moreover, we show that among the supported Engel structures we construct, there are loose Engel structures.

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1 Introduction

One particularly fruitful approach to the study of contact structures is the use of open book decompositions, that was introduced by E Giroux [13]. He showed that every cooriented contact structure on a closed $(2n-1)$ –manifold is supported by an open book decomposition of the underlying manifold, ie it can be defined by a 1–form α such that α gives a contact structure on the binding and α turns the pages into Weinstein domains. If $n = 2$, then there is a one-to-one correspondence between open book decompositions of M up to positive stabilization and isotopy classes of contact structures on M (see for example Etnyre [9]).

The simpler part of this correspondence—the construction of a contact structure starting from an open book decomposition—was established by W Thurston and H Winkelnkemper [23].

Engel structures form a class of plane fields on 4–manifolds which have attracted some interest recently, although our understanding of their properties is relatively underdeveloped. We refer to Montgomery [19] and Vogel [24] for explanations concerning

the motivation of their study. Here we note that Engel structures are closely related to contact structures in dimension 3. This fact was applied successfully in [24] to prove that every parallelizable 4–manifold admits an Engel structure.

The purpose of this note is to prove an analogue of the Thurston–Winkelnkemper theorem for Engel structures: under the assumptions that the binding is a collection of 2–tori and that the monodromy preserves a framing of a page, we will obtain Engel structures starting from an open book decomposition of the manifold that make the pages contact; see [Theorem 5.3](#). This follows from [Corollary 5.4](#) for every open book decomposition with toric binding of a parallelizable 4–manifold.

We also address stabilization constructions and the uniqueness question. In [Theorem 5.8](#) we show that amongst the supported Engel structures we construct there is a special class of loose ones which is invariant under stabilization. We also raise the problem of studying Engel structures supported by an open book with tight contact pages, a property that is invariant under stabilization by [Theorem 6.3](#).

Whether an Engel structure is homotopic — through Engel structures — to an Engel structure which is adapted to an open book decomposition is unknown. Notice that it is unknown whether nonloose Engel structures exist. Finding a supporting open book decomposition for a loose Engel structure up to Engel homotopy amounts, thanks to our construction, to finding an open book decomposition and a framing (e_1, e_2, e_3, e_4) such that the pages are transverse to a vector field e'_1 homotopic to e_1 and the binding tangent to e'_1 and linearly foliated by e'_1 .

Also, our definition of an Engel structure being adapted to an open book decomposition is not tightly suited since the requirements in the definition only use the associated even contact structure.

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2 Engel and even contact structures

We assume that the reader is familiar with the basics of contact topology in dimension 3; a lot of the material we use is covered in [11]. The Giroux correspondence for contact manifolds in dimension 3 is the subject of [9].

We now give definitions of Engel structures and associated distributions.

Definition 2.1 Let M be a $2n$ -dimensional manifold. A (cooriented) even contact structure on M is a hyperplane field \mathcal{E} defined as the kernel of a 1-form α such that $\alpha \wedge (d\alpha)^{n-1}$ never vanishes.

To each even contact structure one can associate a line field $\mathcal{W} \subset \mathcal{E}$, called its *isotropic foliation* or *kernel foliation*, which is the kernel of $d\alpha$ restricted to \mathcal{E} . When $n = 2$, the even contact condition is equivalent to $[\mathcal{E}, \mathcal{E}] = TM$.

Definition 2.2 An Engel structure \mathcal{D} is a smooth plane field on a 4-manifold M such that $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$ is an even contact structure.

Notice that if an even contact structure \mathcal{E} is induced by an Engel structure, ie $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$, then \mathcal{W} is tangent to \mathcal{D} . Thus, an Engel structure \mathcal{D} induces a flag of distributions

$$\mathcal{W} \subset \mathcal{D} \subset \mathcal{E} \subset TM.$$

Moreover, if \mathcal{E} comes from an Engel structure, then it is canonically oriented, and an orientation of TM induces an orientation of \mathcal{W} and vice versa. If we assume that both \mathcal{D} and M are oriented, then this shows that \mathcal{D} induces a framing of TM which is well defined up to homotopy. In particular, two Engel structures cannot be homotopic through Engel structures if the associated framings are not homotopic. Quite recently it was shown in [2] that every framing of a parallelizable 4-manifold is homotopic to the framing induced by an Engel structure on M .

Example 2.3 On \mathbb{R}^4 the 1-form $dz - y dx$ defines an even contact structure \mathcal{E} . The isotropic foliation is spanned by ∂_w (where w is the fourth coordinate). Locally, every even contact structure is diffeomorphic to the one given here.

The plane field $\mathcal{D} = \ker(dy - w dx) \cap \mathcal{E}$ is an Engel structure such that $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$. Every Engel structure is locally diffeomorphic to \mathcal{D} (this fact is due to F Engel [8]).

□

The fact that there is a line field associated to an even contact structure/Engel structure means that Gray's stability theorem cannot hold for these distributions: a varying family of even contact structures or Engel structures will induce a varying family of isotropic foliations. Hence, the even contact structures/Engel structures in the family are not diffeomorphic to each other since dynamical properties of the isotropic foliation can change. This is explored in [19].

If \mathcal{E} is an even contact structure on M and if Y is a hypersurface in M transverse to \mathcal{W} , then the plane field

$$\xi = TY \cap \mathcal{E}$$

is a contact structure on Y .

Recall also that even contact structures satisfy the h-principle: a formal (oriented) even contact structure on M is a pair $(\mathcal{E}, \mathcal{W})$, where \mathcal{E} is an oriented hyperplane field on M and $\mathcal{W} \subset \mathcal{E}$ is an oriented line field. According to [17], the space of even contact structures is homotopy equivalent to the space of formal even contact structures.

Remark 2.4 Once the even contact structure is fixed, the homotopy class of \mathcal{D} is also fixed as a plane field in \mathcal{E} . Indeed, let W be a vector field generating \mathcal{W} and N be a vector field orthogonal to \mathcal{D} in \mathcal{E} (assuming we have fixed a metric). The path of vector fields $\cos sW + \sin sN$ for $s \in [0, \frac{\pi}{2}]$ interpolates between W and N , and thus gives a path between their normal plane fields in \mathcal{E} .

However, for a fixed pair $\mathcal{W} \subset \mathcal{E}$, the homotopy class of \mathcal{D} as a distribution sitting on the sequence $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E}$ is not fixed and it corresponds to the homotopy classes of sections of the S^1 -bundle $S(\mathcal{E}/\mathcal{W}) \rightarrow M$.

The main motivating example of an even contact structure is the preimage of the contact structure under the projection $M = Y \times [0, 1] \rightarrow Y$ for a contact structure ξ on Y . The fibers of the projection are the leaves of the isotropic foliation. A slight modification of this example proves the following lemma:

Lemma 2.5 *Let (Y, ξ) be a contact manifold and φ the coordinate on $[0, 1]$. Then the hyperplane field $\mathcal{E} = \xi \oplus \mathbb{R}(\partial_\varphi + L)$ on $Y \times [0, 1]$ is an even contact structure whose isotropic foliation is spanned by $\partial_\varphi + L$ if and only if L is a contact vector field on (Y, ξ) .*

In order to find an Engel structure \mathcal{D} such that the even contact structure \mathcal{E} using Lemma 2.5 satisfies $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ we require, in addition, that ξ admits a trivialization

C_1, C_2 . The following lemma was certainly known to H Geiges [10] and we are not aware of an earlier reference.

Lemma 2.6 *Let L be a contact vector field on (Y, ξ) . For a sufficiently big positive integer k the plane field spanned by*

$$W = \partial_\varphi + L \quad \text{and} \quad X_k = \cos(k\varphi)C_1 + \sin(k\varphi)C_2$$

is an Engel structure \mathcal{D}_k such that $[\mathcal{D}_k, \mathcal{D}_k] = \mathcal{E}$. In particular, the isotropic foliation is spanned by W .

3 Open book decomposition of a 4–manifold

Open book decompositions — their definition can be found below — have been probably introduced into differential topology by J Alexander, who showed that every oriented 3–manifold has an open book decomposition. This led to the short proof of the fact that every oriented 3–manifold has a contact structure [23] that we will adapt to Engel structures.

Existence results for open book decompositions in dimension ≥ 6 were found by Winkelnkemper [25] and T Lawson [16]. Later, F Quinn [22] found necessary and sufficient conditions for the existence of an open book decomposition on closed manifolds of dimension ≥ 5 . To the best of our knowledge there is no reference dealing with open book decompositions of 4–manifolds specifically. The (simple) fact that if a closed 4–manifold admits an open book decomposition then its signature has to vanish was certainly known to Winkelnkemper [25].

For background on the relationship between contact structures and open book decompositions we refer to [9] when the underlying manifold has dimension 3, and to [13; 5] for the case of contact manifolds in higher dimension.

3.1 Open books with torus binding

In this section we summarize basic definitions concerning open book decompositions.

Definition 3.1 *Let M be a closed manifold. An open book decomposition of M is a pair (K, θ) where*

- $K \subset M$ is a nonempty, not necessarily connected, submanifold of codimension two with trivial normal bundle,

- $\theta: M \setminus K \rightarrow S^1$ is a fibration, and
 - one can choose coordinates on a neighborhood
- (1) $N(K) \simeq K \times D^2 = \{(x, (r, \phi)) \mid x \in K \text{ and } (r, \phi) \in D^2 \text{ are polar coordinates}\}$
of K such that $\theta(x, (r, \phi)) = \phi$ for $r \neq 0$.

The submanifold K is called the *binding* of the open book decomposition. For $t \in S^1$, the preimage $\theta^{-1}(t)$ will be called a *fiber* of (K, θ) . The compactification of any fiber obtained by addition of K is called a *page*. The natural orientation of S^1 lifts through θ to a natural coorientation of the pages. If the ambient manifold M is oriented, then we obtain a natural orientation on the pages, and K is naturally oriented as the boundary of a page.

The simplest example of a manifold with an open book decomposition is the 2–sphere. In this case, the binding consists of a pair of points, the complementary annulus fibers over the circle and the fibers are intervals.

Recall that every oriented 3–manifold admits an open book decomposition [1].

Definition 3.2 (Giroux [13]) Let (M, ξ) be a contact 3–manifold and (K, θ) an open book decomposition of M . Then (K, θ) *supports* ξ if there is a defining contact form α for ξ such that

- (i) $\alpha|_K$ is positive, and
- (ii) the restriction of $d\alpha$ to fibers of (K, θ) is a positive area form.

According to [13], in any odd dimension, every contact structure has a supporting open book decomposition. Details of the proof can be found in [21].

When M has dimension four, K is a disjoint union of embedded closed surfaces. From a vector field transverse to the fibers of $M \setminus K \rightarrow S^1$ and a vector field V_K with isolated zeroes on K one can obtain a vector field on M whose zeroes are precisely the zeroes of K such that the zeroes have the same index. Hence, the Euler characteristic of K equals the Euler characteristic of M by the Poincaré–Hopf index theorem.

For 4–manifolds we will require in addition that K is a union of tori. Then $\chi(M) = 0$, and this is a condition Engel manifolds satisfy.

Moreover, it follows from Novikov additivity [20] that the signature of a 4–manifold admitting an open book decomposition vanishes. However, 4–manifolds which admit

an open book decomposition do not always admit a Spin–structure. Consider for example the nontrivial S^2 –bundle $S^2 \hat{\times} S^2$ over S^2 . This manifold does not have a Spin–structure, but an open book decomposition can be obtained by pulling back the open book decomposition of S^2 .

It seems to be unknown whether all oriented 4–manifolds with vanishing signature admit an open book decomposition [22]. Below we describe simple examples of open book decompositions on 4–manifolds which fiber over 3–manifolds.

So far we have considered open book decomposition from an intrinsic point of view. From a more extrinsic point of view, an open book decomposition of an n –manifold M is a triple (Y, h, ϕ) where

- Y is a compact oriented $(n-1)$ –manifold with nonempty boundary,
- $h: Y \rightarrow Y$ is a diffeomorphism which is the identity on a neighborhood of ∂Y , and
- $\phi: \Sigma_0(Y, h) \rightarrow M$ is a homeomorphism, where $\Sigma_0(Y, h)$ is the relative mapping torus of (Y, h) .

This determines the manifold only up to homeomorphism. Recall that the relative mapping torus of (Y, h) is

$$\Sigma_0(Y, h) = Y \times [0, 1] / \sim_h,$$

where \sim_h is the equivalence relation

$$\begin{aligned} (h(x), 0) &\sim_h (x, 1) \quad \text{for all } x \in Y, \\ (x, t) &\sim_h (x, t') \quad \text{for all } x \in \partial Y, t, t' \in [0, 1]. \end{aligned}$$

The triple (Y, h, ϕ) gives rise to an open book decomposition of M whose binding is $\phi(\partial Y \times [0, 1] / \sim_h)$ and whose pages are $Y_t = \phi(Y \times \{t\})$. Conversely, any open book decomposition can be seen as the relative mapping torus of some (Y, h) (notice that we usually do not mention the identification ϕ).

Example 3.3 Let (K_0, θ_0) be an open book decomposition of a 3–manifold Y , also given by the relative mapping torus of (S_0, h_0) . Here S_0 is a compact oriented surface with nonempty boundary. Then $(S^1 \times K_0, \text{id} \times \theta_0)$ is an open book decomposition of $S^1 \times Y$, also described by the relative mapping torus of $(S^1 \times S_0, \text{id} \times h_0)$. \square

We close this section with an example of an open book decomposition of S^3 .

Example 3.4 On $S^3 \subset \mathbb{C}^2$ we consider the contact form $\alpha = r_1^2 d\varphi_1 + r_2^2 d\varphi_2$. Here r_1, φ_1 and r_2, φ_2 are polar coordinates on the first and second factors of \mathbb{C}^2 , respectively.

Then the Hopf link $K_0 = \{r_1 = 0\} \cup \{r_2 = 0\}$ is the binding of an open book decomposition which supports $\ker(\alpha)$. The complement of the Hopf link K_0 fibers over S^1 by

$$\theta_0: S^3 \setminus K \rightarrow S^1, \quad ((r_1, \varphi_1), (r_2, \varphi_2)) \mapsto \varphi_1 + \varphi_2.$$

One can check that (K_0, θ_0) is an open book decomposition of S^3 and that (K_0, θ_0) supports $\xi = \ker(\alpha)$. The page $A_0 = \theta_0^{-1}(0)$ of the open book is the annulus

$$\{(r, \varphi, \sqrt{1-r^2}, -\varphi) \in S^3 \mid 0 < r < 1 \text{ and } \varphi \in [0, 2\pi]\}.$$

The monodromy of this open book decomposition is a positive Dehn twist along the circle $\{r = \frac{1}{2}\} \subset \theta_0^{-1}(0)$. □

The following description of the stabilization of open book decompositions supporting a contact structure on a 3-manifold M can be found in [14]; see also [9]. Let (K, θ) be an open book which carries the contact 3-manifold (M, ξ) . On a page Σ we pick a properly embedded Legendrian arc β . Now choose a vector field on $M \setminus K$ which is transverse to the pages such that

- on $N(K) \setminus K \simeq K \times (D^2 \setminus \{0\})$ the vector field is tangent to the second factor,
- the time-1 flow is defined and it induces a self-diffeomorphism ψ_1 of Σ whose support lies in the interior of the page, and
- $\Sigma \cup \psi_{1/2}(\Sigma)$ is a surface whose interior is smooth.

Let $N(a) \subset M$ be a closed ball which intersects K in two arcs and contains $a = \psi_{1/2}(\beta)$ in its interior such that $\partial N(a) \setminus K$ has precisely two tangencies with pages of the open book decomposition. On a page of the open book decomposition of (S^3, ξ_{st}) described in Example 3.4 we choose an arc connecting the two components of the binding (a Hopf link) and a neighborhood N_{S^3} of that arc with properties analogous to those of $N(a)$.

By carefully performing the connected sum of (M, ξ) with (S^3, ξ_{st}) using the balls $N(a)$ and N_{S^3} such that the oriented singular foliations are induced by the open book decompositions on $\partial N(a)$ and ∂N_{S^3} , one obtains a new open book decomposition of $M \simeq M \# S^3$. This yields a contact structure supported by the resulting open book

decomposition which is isotopic to ξ by the classification of tight contact structures on the ball (see [7]).

Figure 1 shows the intersection of the pages with $\partial N(a)$. The thickened solid arcs K and K' represent the binding before and after stabilization, respectively. These arcs lie in the interior of the ball. Finally, the dashed arc represents pieces of β , a and γ .

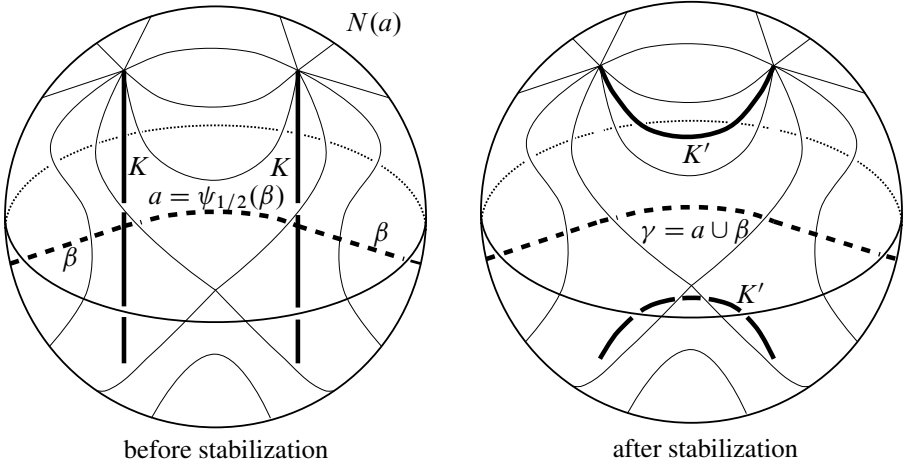


Figure 1: Stabilization of open book decompositions of 3-manifolds

3.2 Stabilization

As we have stated in the introduction, isotopy classes of contact structures on 3-manifolds are in one-to-one correspondence with open book decompositions up to positive stabilization. In this section we suggest a definition of stabilization for open book decompositions of 4-manifolds as a fiber connected sum along an annulus A in a page of the open book by taking the product of the stabilization in dimension 3 with S^1 , ie the annulus A will be of the form $a \times S^1$.

The stabilization of open books on 4-manifolds can be thought of as removing neighborhoods of annuli in M and $S^3 \times S^1$ and identifying the boundaries. In the following we also want to ensure that the binding is a union of tori after stabilization when this is true for the original open book decomposition.

Let M be a 4-manifold equipped with an open book decomposition (K, θ) obtained as the relative mapping torus of (Y, h) . We pick a properly embedded annulus $(A, \partial A) \subset (Y, \partial Y)$ such that no component of ∂A is contractible in ∂Y .

Consider the 3–manifold Y' obtained by gluing $S^1 \times [0, 1] \times [-1, 1]$ to Y , where the gluing map takes $S^1 \times \{0, 1\} \times [-1, 1]$ to ∂Y so that the two curves $S^1 \times \{0, 1\} \times \{0\}$ are identified with ∂A and $T = A \cup (S^1 \times [0, 1] \times \{0\})$ is a torus. The boundary of Y' is still a union of tori.

Let $\gamma \subset T$ be an embedded closed curve intersecting the circle $S^1 \times \{\frac{1}{2}\} \times \{0\}$ exactly once. We then take a small tubular neighborhood $N(T)$ of T in Y' and consider the fibered Dehn twist τ_γ along T in the direction of γ , supported in $N(T)$. Let $h' = \tau_\gamma \circ h \in \text{Diff}(Y')$.

Proposition 3.5 *The relative mapping torus of (Y', h') is diffeomorphic to M by a diffeomorphism which induces an injection from a page Y of $\Sigma_0(Y, h)$ to a page Y' of $\Sigma_0(Y', h')$.*

Proof Recall that an elementary positive stabilization of a 3–dimensional open book is obtained by performing the connected sum with S^3 on a suitable neighborhood $N(a)$ of a properly embedded arc a [13; 14].

We start from the open book decomposition (K_0, θ_0) of S^3 given by the relative mapping torus of a positive Dehn twist on an annulus as described in Example 3.4. Its binding is a positive Hopf link. Let A_0 be a page of this open book, and a a properly embedded arc on A_0 connecting the two boundary components.

Next, we consider the manifold $S^1 \times S^3$ together with the open book $(S^1 \times K_0, \text{id} \times \theta_0)$. It contains the annulus $S^1 \times a$ and the associated neighborhood $S^1 \times N(a)$, where $N(a)$ is a neighborhood of a as in [13]. The complement of $S^1 \times N(a)$ in $S^1 \times S^3$ is diffeomorphic to $S^1 \times B^3$.

Now we take coordinates $S^1 \times [0, 1]$ on $A \subset Y$ such that $\gamma = \{0\} \times [0, 1] \subset A$. As in [13], we can find a neighborhood $N(A)$ of A in M with coordinates $S^1 \times B^3$, so that the pages of (K, θ) define a “partial” open book decomposition on $S^1 \times B^3$. This open book is conjugate to $S^1 \times (N(a), K_0, \theta_0)$.

Since both restrictions $(S^1 \times B^3, K, \theta)$ and $S^1 \times (N(a), K_0, \theta_0)$ are S^1 –invariant, we can form an S^1 –invariant connected sum of open books, ie we glue

$$(S^1 \times S^3 \setminus \text{int}(S^1 \times N(a)), S^1 \times K_0, \text{id} \times \theta_0)$$

to $(M \setminus \text{int}N(A), K, \theta)$ in an S^1 –invariant way. In this manner we get a manifold diffeomorphic to M , with an open book of page Y' and monodromy h' . □

Another way to argue in the previous proof would be as follows: The stabilized manifold is obtained from the original manifold by removing a copy of $S^1 \times D^3$ and gluing it back in. According to [15] every diffeomorphism of $S^1 \times \partial D^3$ extends to $S^1 \times D^3$. Hence, the diffeomorphism type of the manifold does not change under this surgery operation.

4 Open book decompositions and even contact structures

Let M be an oriented closed 4–manifold together with an open book decomposition (K, θ) .

Definition 4.1 An even contact structure \mathcal{E} with kernel \mathcal{W} is *adapted to* (or *supported by*) (K, θ) if the following conditions are satisfied:

- K is a union of tori.
- \mathcal{W} is transverse to the fibers of θ . In particular, the interior of the pages Y_θ are naturally contact manifolds for the contact plane $\xi_\theta = \mathcal{E} \cap TY_\theta$.
- ξ_θ is a positive contact structure for the canonical orientation of Y_θ .
- \mathcal{E} is transverse to K .
- \mathcal{W} restricts to a linear vector field on each connected component of K .
- For each $\theta \in S^1$ there is a collar $U \simeq \partial Y_\theta \times (-1, 0]$ of ∂Y_θ in Y_θ such that $K = \partial Y_\theta \times \{0\}$ and the characteristic foliation on each connected component of $\partial Y_\theta \times \{r\}$ for $r \in (-1, 0]$ is linear.

Remark 4.2 Since all linear vector fields on T^2 are homotopic, two even contact structures supported by an open book are homotopic through even contact structures if the contact structures in the pages are homotopic as 1–parametric families of plane fields: the associated kernel foliations are homotopic through foliations which are transverse to the interiors of the pages and induce linear vector fields on the binding.

Two even contact structures on M supported by the same open book (K, θ) are not necessarily homotopic as formal even contact structures. A sufficient condition for two different even contact structures to be homotopic through formal even contact structures is that the associated gluing contactomorphisms are isotopic through formal contactomorphisms. In particular, if the return maps are contact isotopic, then the

even contact structures are formally homotopic and therefore homotopic through even contact structures according to [17].

Note that uniqueness holds in the 3–dimensional contact case, because the return map is an exact symplectomorphism of a surface, and two symplectomorphisms of a surface are isotopic through symplectomorphisms if they are isotopic through diffeomorphisms.

Example 4.3 Let ξ be a contact structure on Y carried by an open book (K, θ) , and α an adapted contact form. Its Reeb vector field X is transverse to the pages and tangent to the binding. We assume that ξ is trivial as a vector bundle.

Let $M = \widetilde{\mathbb{P}\xi}$ be the manifold consisting of 1–dimensional oriented subspaces of the contact planes and

$$\text{pr}: \widetilde{\mathbb{P}\xi} \rightarrow M$$

the S^1 –fibration which sends a line in ξ_p to p . Then $\text{pr}^*\alpha$ defines an orientable even contact structure \mathcal{E} whose isotropic foliation is tangent to the fibers of pr .

We want to modify \mathcal{E} so that the open book decomposition $(\text{pr}^{-1}(K), \theta \circ \text{pr})$ of $\widetilde{\mathbb{P}\xi}$ is adapted to the resulting even contact structure \mathcal{E}_ε . For this we identify $S^1 \times Y$ with $\widetilde{\mathbb{P}\xi}$, denote the coordinate in S^1 by t , and let \widetilde{X} be the lift of the Reeb field X to $S^1 \times Y$ using the connection $\{0\} \oplus TY$.

For $\varepsilon > 0$ small enough the 1–form

$$\alpha_\varepsilon := \text{pr}^*\alpha - \varepsilon dt$$

still defines an even contact structure \mathcal{E}_ε . Let ∂_t denote the vector field tangent to the fibers dual to dt . Then $\partial_t + \varepsilon\widetilde{X}$ is tangent to \mathcal{E}_ε and this vector field preserves \mathcal{E}_ε . The isotropic foliation of \mathcal{E}_ε is therefore spanned by $\partial_t + \varepsilon\widetilde{X}$, and the open book $(\text{pr}^{-1}(K), \theta \circ \text{pr})$ supports \mathcal{E}_ε provided that $\varepsilon > 0$. □

When one views an open book decomposition supporting an even contact structure as a mapping torus, then the monodromy is a contact diffeomorphism of a page. Using Eliashberg’s classification of overtwisted contact structures [6] it is easy to obtain contact diffeomorphisms from diffeomorphisms which preserve a given plane field up to homotopy.

Proposition 4.4 *Let Y be a closed 3–manifold, $e = (e_0, e_1, e_2)$ be a trivialization of TY and h a diffeomorphism of Y such that $h_*(e)$ is homotopic to e . Then there*

exists a contact structure ξ on Y , together with a contact diffeomorphism ψ of (Y, ξ) isotopic to h , such that ξ contains a nonvanishing vector field X with ψ_*X homotopic to X in ξ .

Proof Write $e_i^0 = e_i$ and $e_i^1 = h_*(e_i)$ for $i = 0, 1, 2$. By hypothesis, there exists a family (e_0^t, e_1^t, e_2^t) interpolating between these two framings. Let $\xi_t = \langle e_0^t, e_1^t \rangle$. Since the ball is contractible, we may assume that, after a homotopy of the framing e_i and an isotopy of the diffeomorphism h , there exists a ball $U \subset Y$ such that $\xi_t|_U = \xi_0|_U$ is contact and $\xi_0|_U$ contains an overtwisted disk. By [6], there exists a continuous family $\xi_{t,s}$ such that $\xi_{t,0} = \xi_t$ and $\xi_{t,1}$ are contact structures. We apply Eliashberg's classification theorem [6] as follows:

- (1) First, for $t = 0$, find $\xi_{0,s}$.
- (2) Then we have a solution for $t = 1$, given by $\xi_{1,s} = h_*\xi_{0,s}$.
- (3) The relative character with respect to the parameter of [6] allows us to extend it to $\xi_{t,s}$.

By contractibility of the interval, there is an extension $(e_0^{t,s}, e_1^{t,s})$ that coincides with (e_0^t, e_1^t) for $s = 0$. Moreover, $\xi_{t,s} = \langle e_0^{t,s}, e_1^{t,s} \rangle$, $h_*e_0^{0,1} = e_0^{1,1}$ and $h_*e_1^{0,1} = e_1^{1,1}$.

By Gray stability, there exists a flow $\varphi_t: Y \rightarrow Y$ such that $(\varphi_t)_*\xi_{0,1} = \xi_{t,1}$. Therefore, $(\varphi_1^{-1} \circ h)_*\xi_{0,1} = \xi_{0,1}$ shows that $(\varphi_1^{-1} \circ h)$ is the required contactomorphism. Finally, $X = e_0^{0,1}$ satisfies the stated hypothesis. \square

The same proof works for compact manifolds with boundary when the diffeomorphism h is the identity on the boundary and $h_*(e)$ is homotopic to e relative to the boundary. We then obtain a contact diffeomorphism ψ supported in the interior of Y and a nowhere-vanishing Legendrian vector field X such that $\psi_*(X)$ is homotopic (relative to the boundary) to X through nowhere-vanishing Legendrian vector fields.

The assumption in Proposition 4.4 that h preserves a framing up to homotopy turns out to be not too restrictive by the following observation of Geiges [10]:

Lemma 4.5 *Let M be a closed oriented 4-manifold with trivial tangent bundle and X a nowhere-vanishing vector field on M . Then there are vector fields e_1, e_2, e_3 such that X, e_1, e_2, e_3 is a framing.*

Proof We identify $TM \simeq M \times \mathbb{R}^4$ with $M \times \mathbb{H}$. Then X together with $e_1 = iX$, $e_2 = jX$ and $e_3 = kX$ is a framing of TM . \square

In particular, we can apply this lemma to the suspension vector field on the mapping torus of a diffeomorphism $h: N \rightarrow N$ when the mapping torus is parallelizable. Projecting e_1 , e_2 and e_3 to the fibers of a mapping torus, we obtain a framing of N with the desired properties.

Theorem 4.6 *Let Y be a compact 3–manifold bounded by a nonempty union of tori, and ξ a positive contact structure on Y which prints a linear characteristic linear foliation on ∂Y . If h is a contact diffeomorphism of (Y, ξ) which is the identity near the boundary, then the relative mapping torus of (Y, h) carries an even contact structure which induces ξ on (a small retraction of) the pages. This even contact structure is uniquely determined up to homotopy through even contact structures carried by the triple (Y, ξ, h) .*

Proof For simplicity, we assume that ∂Y is connected. Let $Y_s = Y \times [0, 1]/\sim$ be the mapping torus of (Y, h) . We denote the suspension coordinate on Y_s by φ and fix a coordinate system (x, y) on $T^2 \simeq \partial Y$ such that ∂Y is linear in terms of the coordinates. The suspension vector field is ∂_φ and $D_{r_0}^2 \subset \mathbb{R}^2$ is a disc of radius r_0 .

We view M as the mapping torus $Y_s = Y \times [0, 1]/\sim$ of (Y, h) , with a neighborhood $N(K) = \partial Y \times D_{r_0}^2$ of the binding K of the binding attached along the boundary such that $\{y\} \times \partial D_{r_0}^2$ is mapped to $\{y\} \times S^1$ in $\partial Y_s = \partial Y \times S^1$ for all $y \in \partial Y$.

Because h preserves ξ we get a well-defined 2–plane field ξ_s on Y_s which is tangent to the fibers of $\theta: Y_s \rightarrow S^1$. Together with the suspension vector field ∂_φ , this plane field spans an even contact structure \mathcal{E}_s on Y_s whose isotropic foliation is directed by ∂_φ .

Since h is the identity on a neighborhood of ∂Y and the characteristic foliation $\xi(\partial Y)$ is linear, we can choose a collar $U \simeq \partial Y \times [0, 2\varepsilon)$, with $\varepsilon > 0$, such that $h|_U = \text{id}_U$ and

$$\xi = \ker(\cos(r + a) dx - \sin(r + a) dy).$$

where the parameter $a \in \mathbb{R}$ is determined by the slope of the linear characteristic foliation $\xi(\partial Y)$. Since we are free to choose the coordinates (x, y) , we may assume that the slope is positive in terms of these coordinates. On $U \times S^1 \subset Y_s$ we consider the even contact structure $\mathcal{E}_0 = \xi \oplus \mathbb{R}\partial_\varphi$. The isotropic foliation is of course spanned by ∂_φ . We want to homotope \mathcal{E}_0 through even contact structures so that isotropic foliation of the resulting even contact structure \mathcal{E}_1 is $\partial_\varphi + \partial_y$.

Recall that the space of contact vector fields on a given contact manifold is in one-to-one correspondence to the space of smooth functions once a contact form is fixed (see [18, Section I.3.4]). This implies the existence of a contact vector field L with support in U and $L = \partial_y$ on a collar $V = \partial Y \times [0, \varepsilon] \subset U$ of ∂Y .

We use the same notation for L and its lift to the product $Y \times S^1$. The desired homotopy of even contact structures is $\mathcal{E}_s = \xi_s \oplus \mathbb{R}(\partial_\varphi + sL)$ with $s \in [0, 1]$.

Finally, we extend \mathcal{E}_s to an even contact structure over $V \cup N(K)$: let (r, φ) be polar coordinates on the second factor of $N(K) = \partial Y_s \times \mathbb{R}^2$ with $r \in [0, 1]$ and

$$\mathcal{E}_b = \ker(dx + r^2 d\varphi - r^2 dy).$$

One can check by a simple computation that \mathcal{E}_b is an even contact structure whose isotropic foliation is spanned by $\partial_\varphi + \partial_y$. In particular, \mathcal{E}_b induces a contact structure on $\{\varphi = \varphi_0, 0 < r \leq r_0\}$. The characteristic foliation on $T^2 = T^2 \times \{r = r_0, \varphi = \varphi_0\}$ is linear and the slope is $1/r^2 > 0$.

Therefore, we can choose r_0 so that the even contact structures \mathcal{E}_b and \mathcal{E}_1 on Y_s can be glued, and we obtain an even contact structure supported by the resulting open book on $Y_s \cup_\partial (\partial Y \times D_{r_0}^2)$. The pages of this open book are formed by fibers of $Y_s \rightarrow S^1$ with $\{\varphi = \varphi_0, 0 < r \leq r_0\}$ attached along ∂Y .

We now prove the uniqueness part of the theorem. Let \mathcal{E} and \mathcal{E}' be two even contact structures carried by the open book decomposition (Y, ξ, h) . Their isotropic foliations \mathcal{W} and \mathcal{W}' are spanned by vector fields of the form $\partial_\varphi + L$ and $\partial_\varphi + L'$ on both $N(K)$ and Y_s , where L and L' are tangent to the pages and to K .

The path of hyperplanes fields $\mathcal{E}_s = \xi \oplus \mathbb{R}(\partial_\varphi + (1-s)L + sL')$ for $s \in [0, 1]$ interpolates between \mathcal{E} and \mathcal{E}' , satisfies the even contact condition and is carried by (Y, ξ, h) . \square

Remark 4.7 We are able to prove that the built even contact structure is unique (see Remark 4.2) because we start with an open book given by a triple (Y, ξ, h) , and not by a pair (K, θ) .

Using the flexibility properties of overtwisted contact structures [6], we can show the existence of a contact structure satisfying the assumptions of Theorem 4.6 for a given open book decomposition (Y, h) such that ∂Y is a union of tori and h preserves a plane field up to homotopy relative to the boundary.

Theorem 4.8 *Let Y be a compact 3–manifold bounded by a nonempty union of tori, and h a diffeomorphism of Y which is the identity near the boundary such that there exists a plane field P of Y such that h_*P is homotopic to P relative to ∂Y .*

Then the relative mapping torus of h carries an even contact structure.

The proof of [Theorem 4.8](#) follows from a combination of [Theorem 4.6](#) and a variant of [Proposition 4.4](#). Under the hypothesis that the monodromy h preserves a plane field \mathcal{P} up to homotopy relative to the boundary, a slightly modified version of [Proposition 4.4](#) gives an overtwisted contact structure homotopic to \mathcal{P} and preserved by h . [Theorem 4.6](#) then gives the conclusion.

The following theorem shows that one can always arrange that the page of an open book supporting an even contact structure is overtwisted if one allows homotopies through even contact structures which are not necessarily supported by the open book decomposition. In particular, ξ' can be taken overtwisted even if ξ is tight.

Theorem 4.9 *If \mathcal{E} is carried by (Y, h) and \mathcal{E} induces a contact structure ξ on Y . Then, for every overtwisted contact structure ξ' on Y which is homotopic to ξ as oriented plane field relative to the boundary, the even contact structures \mathcal{E} and \mathcal{E}' associated to ξ and ξ' are homotopic through even contact structures.*

In particular, ξ' can be taken overtwisted even if ξ is tight.

Proof It is enough to prove the result when ξ' is overtwisted. In this situation, $h_*\xi'$ is overtwisted and homotopic to ξ' rel ∂Y . By Eliashberg’s classification of overtwisted contact structures [6], ξ' is isotopic to $h_*\xi'$.

We can thus deform h to h' to make ξ' an h' –invariant contact structure. Then we construct an even contact structure \mathcal{E}' carried by (Y, ξ', h') . The pair $(\mathcal{E}', \mathcal{W}')$ is homotopic to $(\mathcal{E}, \mathcal{W})$ by construction. The h-principle [17] implies that \mathcal{E} and \mathcal{E}' are homotopic through even contact structures: On the interior of the suspension of (Y, h) , the homotopy of plane fields ζ_t for $t \in [0, 1]$ interpolating between ξ and ξ' induces a homotopy of formal even contact structures $(\zeta_t \oplus \mathbb{R}\partial_\varphi, \mathbb{R}\partial_\varphi)$, and both \mathcal{W} and \mathcal{W}' are homotopic to $\mathbb{R}\partial_\varphi$ as line fields transverse to the pages. \square

Still, rigidity could come from tightness of the pages.

Question 4.10 Which even contact structures are carried by at least one open book decomposition with a tight page?

5 Open book decompositions and Engel structures

Let M be a closed oriented 4–manifold and (K, θ) an open book decomposition of M .

Definition 5.1 An Engel structure \mathcal{D} is *adapted to* (or *supported by*) (K, θ) if the associated even contact structure $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ is adapted to (K, θ) .

As we have stated in the introduction, this is a somewhat unsatisfactory definition. For example, it is possible that different Engel structures are adapted to the same open book even if the associated framings are not homotopic.

Example 5.2 Let ξ be a contact structure on a compact 3–manifold Y which is trivial as a plane field and is supported by an open book (K, θ) . We fix an adapted contact form α together with the corresponding Reeb vector field R and a trivialization C_1, C_2 of ξ . Let $\text{pr}: M = S^1 \times Y \rightarrow Y$ be the projection. We denote the horizontal lifts of the trivialization C_1, C_2 and of the Reeb vector field R to $S^1 \times Y$ with the same letters, and the coordinate on S^1 will be t .

For $\varepsilon > 0$ small enough, the plane field \mathcal{D}_k , with $k \geq 1$, spanned by

$$(2) \quad W = \partial_t + \varepsilon R, \quad X = \cos(2\pi kt)C_1 + \sin(2\pi kt)C_2$$

is an Engel structure supported by $(S^1 \times K, \theta \circ \text{pr})$.

This can be applied to the contact structure on $S(T^*T^2)$ for different k . The homotopy class of the induced framing of $S(T^*T^2) \times S^1$ depends on the parity of k . Thus, an open book can generate Engel structures which cannot be homotopic through Engel structures. \square

5.1 Invariants of Engel structures supported by open books

Fix \mathcal{D} an Engel structure carried by (Y, h) ; associate an integer k to \mathcal{D} in the following way: Pick a page Y_0 and a collar neighborhood $\partial Y_0 \times (-1, 0]$ of ∂Y_0 in Y_0 such that the characteristic foliation $\xi_0(\partial Y_0 \times \{s\})$ for $s \in (-1, 0]$ is linear and where the monodromy h is the identity. Let $x_0 \in \partial Y_0 \times \{s_0\}$ be a point in that neighborhood.

Along the path $x_0 \times S^1 \subset \Sigma_0(Y, h)$, we have a trivialization of $\xi_t = \mathcal{E} \cap TY_t$ at the point (x_0, t) for $t \in S^1$ by the line field $C = \xi_t \cap T(\partial Y_t \times \{s_0\})$. There is another

line field $C' = \mathcal{D} \cap \xi_t$. We define k to be the rotation number of C' with respect to C in ξ_t when we go around $x_0 \times S^1$.

Now, if \mathcal{D} and \mathcal{D}' are two Engel structures which are carried by the same open book and induce the same contact structure ξ_0 on one page Y_0 , then there are two Legendrian line fields $L = \xi_0 \cap \mathcal{D}$ and $L' = \xi_0 \cap \mathcal{D}'$. The degree of L' with respect to L induces a map which defines a homomorphism

$$\delta(\mathcal{D}, \mathcal{D}'): H_1(Y_0, \mathbb{Z}) \rightarrow \mathbb{Z}.$$

If \mathcal{W} and \mathcal{W}' are the respective kernels of \mathcal{D} and \mathcal{D}' and are equal near K , their first return maps ϕ and ϕ' of the kernel foliations are contactomorphisms on the interior of (Y_0, ξ_0) which coincide near ∂Y_0 . The even contact structures \mathcal{E} and \mathcal{E}' are homotopic through even contact structures supported by the open book if and only if there is a family (ϕ_t, ξ_t) such that ϕ_t is a contactomorphism of the contact structure ξ_t on Y_0 such that $\phi_0 = \phi$, $\phi_1 = \phi'$, ϕ_t has support in the interior of Y_0 and ξ_t is constant on a neighborhood of ∂Y_0 .

If we moreover assume $\mathcal{W} = \mathcal{W}'$, then $\xi_t = \xi'_t$ for all $t \in S^1$, and $\delta(\mathcal{D}, \mathcal{D}') \circ \phi_* = \delta(\mathcal{D}, \mathcal{D}')$.

We now define *twisting numbers*, which will be used to keep track of the homotopy class of plane fields in even contact structures such that the plane field is tangent to the isotropic foliation of the even contact structure.

The restriction ξ_φ of \mathcal{E} to each page is a contact structure. We assume that there is a section X of \mathcal{E} which is tangent to the pages. In the situations we are going to study, X will be $-\partial_r$ on collars of the boundary of $T^2 \times D^2$ and of $Y_s = Y \times [0, 1]/\sim$.

Let C_1, C_2 be an oriented framing of ξ_φ . For a closed oriented curve γ there are functions g_1 and g_2 such that $X(\gamma(t)) = g_1(t)C_1(\gamma(t)) + g_2(t)C_2(\gamma(t))$. We define the twisting number $\text{tw}(C_1, \gamma)$ to be the degree of the map

$$S^1 \rightarrow S^1, \quad t \mapsto \frac{(g_1(t), g_2(t))}{\|(g_1(t), g_2(t))\|}.$$

Obviously, $\text{tw}(C_1, \gamma)$ depends only on the homology class of γ and on the homotopy class of C_1 as a nowhere-vanishing section of ξ_φ .

5.2 Construction of Engel structures

In this section we adapt the construction from [Section 4](#) to obtain Engel structures and prove the following theorem:

Theorem 5.3 *Let Y be a compact 3–manifold bounded by a nonempty union of tori, and h a diffeomorphism of Y which is the identity near the boundary and for which there exists a framing e of Y such that h_*e is homotopic to e relatively to ∂Y . Then the relative mapping torus of h carries an Engel structure.*

We will prove this theorem in [Section 5.2](#). Using [Lemma 4.5](#) we obtain the following corollary:

Corollary 5.4 *If M is an oriented closed parallelizable 4–manifold, every open book decomposition with toric binding (K, θ) of M carries an Engel structure.*

Proposition 5.5 *Let \mathcal{D} be an Engel structure carried by (Y, h) and $\delta: H_1(Y, \mathbb{Z}) \rightarrow \mathbb{Z}$ be a morphism. If $\delta \circ h_* = \delta$, then there exists an Engel structure \mathcal{D}' supported by (Y, h) which induces the same contact structure as \mathcal{D} on $\text{int}(Y_0)$ (with $\mathcal{W}' = \mathcal{W}$ and $\mathcal{E}' = \mathcal{E}$ outside a neighborhood of K) such that $\delta(\mathcal{D}, \mathcal{D}') = \delta$.*

Question 5.6 *Is every Engel structure homotopic to an Engel structure carried by an open book?*

Question 5.7 *Which Engel structures are carried by an open book decomposition with tight pages?*

Let \mathcal{E} be an even contact structure on the total space of a fibration over S^1 such that the isotropic foliation is transverse to the fibers of θ . (We are going to consider two cases, $\theta: Y_s \rightarrow S^1$ and $\theta: T^2 \times (D^2 \setminus \{0\}) \rightarrow S^1$.) We denote the vertical tangent vectors in \mathcal{E} by \mathcal{E}^v . The analogous notation \mathcal{D}^v will be used later for Engel structures \mathcal{D} whose isotropic foliation is transverse to a fibration over S^1 .

In what follows, γ_x , γ_y and γ_φ are simple closed curves in $\partial Y \times S^1$ (or, equivalently, in $T^2 \times \partial D^2 \subset T^2 \times D^2$) which correspond to the coordinates x , y and φ .

5.2.1 Engel structure on Y_s Let S_ξ be vector field directing the characteristic foliation on $T^2 \times \{r\} \times \{\varphi\}$ such that ∂_r , S_ξ is an oriented framing of ξ on a neighborhood of ∂Y and ∂_r points into Y_s along ∂Y_s . We choose coordinates x , y on $T^2 = \partial Y$ such that

$$(3) \quad \text{tw}(C_1, \gamma_x) = 0, \quad \text{tw}(C_1, \gamma_y) = \lambda,$$

where λ is an integer. This determines the homotopy class of C_1 as a nowhere-vanishing section of ξ , and we assume

$$(4) \quad \begin{aligned} C_1 &= \cos(\lambda y) \partial_r + \sin(\lambda y) S_\xi, \\ C_2 &= -\sin(\lambda y) \partial_r + \cos(\lambda y) S_\xi \end{aligned}$$

on a collar of $T^2 \subset \partial Y$. A simple computation shows that C_1 has the right twisting numbers (see (3)). Note in particular that, along $\{y = 0\}$, $C_1 = -\partial_r$ and $C_2 = S_\xi$.

On $Y \times [0, 2]$ with coordinates (p, φ) consider the plane field \mathcal{D}_k spanned by the isotropic foliation of the even contact structure defined as in Section 4 using the contact structure ξ , and

$$(5) \quad X_k := \cos(2\pi k \varphi) C_1 + \sin(2\pi k \varphi) C_2$$

with $k \in \mathbb{N}$. According to Lemma 2.6, \mathcal{D}_k is an Engel structure when $k > 0$ is large enough and $[\mathcal{D}_k, \mathcal{D}_k] = \mathcal{E}$ is independent of k . The orientation of \mathcal{E} induced by \mathcal{D}_k is the one determined by the framing C_1, C_2 . Moreover, assuming again that k is large enough, there is a smooth function $f: Y \rightarrow (0, 2)$ with $f \equiv 1$ on U (where $h = \text{id}$) such that

$$(6) \quad h: Y \simeq \{(p, f(p)) \mid p \in Y\} \rightarrow Y = Y \times \{0\}$$

preserves the line field spanned by X_k . At this point we use the assumption that the framing $(h_*(C_1), h_*(C_2))$ is homotopic to (C_1, C_2) together with the fact that the holonomy of the characteristic foliation of \mathcal{D}_k is the identity on U . Thus, we obtain an Engel structure \mathcal{D}_k on Y_s . Notice that near ∂Y_s the Engel structure \mathcal{D}_k is $\langle \partial_\varphi + \partial_y \rangle \oplus \langle X_k \rangle$.

By construction, $\text{tw}(X_k, \gamma_\varphi) = k$. In order to make subsequent constructions possible, we require that k is odd.

5.2.2 Even contact structure on $T^2 \times D^2$ We need to extend the Engel structure \mathcal{D}_k to $T^2 \times D^2$. For that we will first extend the associated even contact structure \mathcal{E} in such a way that the plane field \mathcal{D}_k also extends to a plane field in \mathcal{E} containing \mathcal{W} . This extra condition requires a modification of the construction in Section 4.

We will choose the even contact structure on $T^2 \times D^2 = S_x^1 \times S_y^1 \times D^2$ so that it has a section C'_1 extending X_k over $(0, 0) \times D^2$ which is never tangent to \mathcal{W} . Note that the homotopy type of X_k as a section of \mathcal{E}^v along $(0, 0) \times \partial D^2$ depends only on k (and not on the choice of C_1 in Y since it is φ -invariant). This is done as follows.

On $S_x^1 \times D^2$ consider an x -invariant contact structure ξ' with the following properties:

- $\xi' = \ker(dx + r^2 d\varphi)$ near the boundary of $S_x^1 \times D^2$. In particular, $\xi' = \xi$ along $\partial Y_s = \partial N(K)$.
- The characteristic foliation on $\{x = 0\}$ has only nondegenerate singularities. Let e_{\pm} and h_{\pm} denote the number of elliptic and hyperbolic singularities, respectively. The subscript refers to the sign of the singularities. We require

$$(7) \quad \begin{aligned} e_+ &= \frac{1}{2}(k + 1), & e_- &= 0, \\ h_+ &= 0, & h_- &= \frac{1}{2}(k - 1). \end{aligned}$$

Since k is odd, these numbers are all integers.

Such a foliation exists since $e_+ + e_- - h_+ - h_- = 1 = \chi(D^2)$ and standard results from the theory of convex surfaces imply that the singular foliation on $\{x = 0\}$ determines a x -invariant contact structure ξ' on $S_x^1 \times D^2$. Let \mathcal{E}' be the extension of \mathcal{E} given by $\mathcal{E}' = \xi' \oplus \langle \partial_y + \partial_\varphi \rangle$.

We let X'_k be a vector field along ∂Y_s generating $\mathcal{D}_k \cap \xi'$. Note that

$$e(\xi')[D^2] = e_+ - e_- - h_+ + h_- = k = \text{tw}(X'_k, \partial D^2).$$

By the Poincaré–Hopf theorem, the Legendrian vector field X'_k of $\xi'|_{\partial D^2}$ extends to a nowhere-vanishing section C'_1 of \mathcal{E}' on $T^2 \times (0, 0)$ such that C'_1 is nowhere tangent to $\mathcal{W}' = \partial_y + \partial_\varphi$.

This extension is unique up to homotopy and there is a Legendrian vector field C'_2 such that (C'_1, C'_2) is a framing of ξ' on $(0, 0) \times D^2$.

Near $T^2 \times \partial D^2$ we require C_i and C'_i have the same image in \mathcal{E}/\mathcal{W} for $i = 1, 2$. Then there is a homotopy between C_i and C'_i in $\mathcal{D}_k \setminus \mathcal{W}$ such that C'_i can be viewed as an extension of C_i . Near $T^2 \times \partial D^2$, we have

$$(8) \quad \begin{aligned} C'_1 &= \cos(k\varphi) \partial_r + \sin(k\varphi) S_\xi, \\ C'_2 &= -\sin(k\varphi) \partial_r + \cos(k\varphi) S_\xi. \end{aligned}$$

This is consistent with the choice of X_k near ∂Y_s , ie the C_i together with C'_i form a smooth vector field on a neighborhood of $T^2 \times \partial D^2$ in $M = Y_s \cup_\partial (T^2 \times D^2)$ for $i = 1, 2$.

5.2.3 Engel structure on $T^2 \times D^2$ Now we define \widehat{D}_l on $T^2 \times D^2$ as the span of ∂_y and $\widetilde{X}_l = \cos(l\varphi)C'_1 + \sin(l\varphi)C'_2$. This is an Engel structure for all $l \in \mathbb{N}$. The

isotropic foliation is spanned by ∂_y , so it is not transverse to the fibers of θ . However, the Engel structure \mathcal{D}'_l obtained by pushing forward $\widehat{\mathcal{D}}_l$ with the diffeomorphism

$$f: S^1 \times (S^1 \times D^2) \rightarrow S^1 \times (S^1 \times D^2), \quad (x, y, r, \varphi) \mapsto (x, y, r, \varphi + y),$$

has the desired property since its isotropic foliation is spanned by $\partial_y + \partial_\varphi$. Let

$$(9) \quad X'_l = f_*(\tilde{X}_l) = \cos(l y) f_*(C'_1) + \sin(l y) f_*(C'_2).$$

By construction, the gluing map g maps the isotropic foliation (which is tangent to ∂Y_s) to the isotropic foliation of \mathcal{D}'_l (which is also tangent to $T^2 \times \partial D^2$). As in Section 4 we can arrange that the gluing map preserves the singular foliation on the fibers of θ .

5.2.4 Gluing the pieces together The even contact structures induced by \mathcal{D}_k and \mathcal{D}'_l coincide with the even contact structures near $\partial Y \times S^1$ and $T^2 \times \partial D^2$, respectively, which arose in Section 4. Therefore, the only remaining problem is to ensure that the gluing maps the line field $\mathcal{D}'_l{}^v$ to $\mathcal{D}_k{}^v$. From (5) and (4) we obtain

$$X_k = \cos(k\varphi + \lambda y) \partial_r + \sin(k\varphi + \lambda y) S_\xi$$

and, according to (8) and (9), X'_l is given by

$$X'_l = \cos((l - k)y + k\varphi) \partial_r + \sin((l - k)y + k\varphi) S_\xi$$

near $T^2 \times \partial D^2$. The conditions on the parameters l and k are

- k is a positive odd integer,
- l is a positive integer.

Thus, we obtain a smooth Engel structure on $M = (T^2 \times D^2) \cup_\partial Y_s$ if $l - k = \lambda$ when k is large enough.

5.2.5 Uniqueness Using recent results about flexibility of Engel structures [3] we can prove the following:

Theorem 5.8 *For a fixed choice of framing e , the Engel structures constructed in Theorem 5.3 are unique up to homotopy through Engel structures for k large enough.*

According to [3], we only need to check that

- any pair of them are homotopic through formal Engel structures, ie the induced framings of TM are homotopic, and
- the Engel structure is loose in the sense of [3] when k is big enough.

Recall that k is odd, therefore the Engel structures on the mapping torus associated to k and $k + 2$ are formally homotopic, ie the associated framings are homotopic. Near the boundary, we can be more specific using the coordinates x , y and φ on the boundary of the mapping torus: The homotopy of framings can be chosen to be:

- Constant on the neighborhood $D_{1/2} \times T^2$ of the binding, and we may assume that the characteristic foliation on $D_{1/2}$ contains all singular points introduced during the construction (7) for a fixed, odd value of k at the beginning of the homotopy. At the end of the homotopy, $D_1 \setminus \mathring{D}_{1/2}$ contains an additional pair of singularities, so that, at the end of the homotopy, the characteristic foliation on D_1 has the number of singular points prescribed by (7) for $k + 2$ instead of k .
- The hyperplane field on $D_1 \setminus D_{1/2}$ spanned by the first three components of the framing is invariant under rotations in the (x, y) -direction throughout the homotopy.
- Throughout the homotopy, the framing is invariant under translations in the x -direction; the first and the last component of the framing are also y -invariant.
- The second and the third components are not y -invariant in general, but they are twisted and the twisting is determined by λ as in (4).

A homotopy with all these properties can be found by first fixing it on the annulus $D_1 \setminus \mathring{D}_{1/2}$ and then extending the homotopy using the desired invariance properties.

Since $\pi_2(\mathrm{SO}(4)) = \{0\}$, this homotopy is unique, and we may assume that at the end of the homotopy, the characteristic foliation on $D_1 \setminus \mathring{D}_{1/2} \times \{(0, 0)\} \subset D^2 \times T^2$ has precisely one elliptic and one hyperbolic singularity (as in (4) for $k + 2$ instead of k).

The end result of the homotopy of framings coincides (up to homotopy) with the Engel framing obtained by our construction for $k + 2$. In particular, the twisting number along γ_y is fixed.

We use the following definition for looseness of Engel structures. A chart in (M, \mathcal{D}) is an N -Darboux chart for $N \in \mathbb{N}$ if there is an Engel embedding of the ball $D^3 \times (0, N)$ with coordinates (x, y, z, θ) and with Engel structure

$$\ker(dy - z dx) \cap \ker(\cos(2\pi(\theta - \theta_0)) dz - \sin(2\pi(\theta - \theta_0)) dx)$$

for some $\theta_0 \in \mathbb{R}$.

In other words, the chart is a Darboux chart of an even contact structure in which the vector transverse to \mathcal{W} rotates N times. We say that \mathcal{D} is N -loose if any point of the manifold can be placed in the interior (not necessarily in the middle) of an N -Darboux chart. The main result of [3] implies that there is an integer N_0 with the following property: if \mathcal{D}_0 and \mathcal{D}_1 are N -loose with $N > N_0$ and formally homotopic, then they are homotopic through Engel structures.

The Engel structure on Y_s is $(k-L)$ -loose, where L depends only on the gluing map φ and the framing (see (6)). The Engel structure \mathcal{D}'_l on $T^2 \times D^2$ is l -loose. Finally, recall that $l = k + \lambda$. Since both λ and L are independent of k , this implies [Theorem 5.8](#).

6 Stabilization

Let (Y, ξ) be a contact manifold whose boundary is a nonempty union of tori such that the characteristic foliation of ξ on each component of ∂Y is linear.

Let A be a properly embedded annulus in $(Y, \partial Y)$ such that

- (i) ∂A is not contractible in ∂Y ,
- (ii) the characteristic foliation $\xi|_A$ is nonsingular and made of intervals going from one boundary component to the other, and
- (iii) the characteristic foliations on the two 2-tori formed by A and the two annuli cut by ∂A in ∂Y are linear.

Like in the topological case explained in [Section 3.2](#), we glue a handle $S^1 \times [0, 1] \times [-1, 1]$ to ∂Y and consider the torus $T = A \cup S^1 \times [0, 1] \times \{0\}$. Recall that the stabilization procedure requires the choice of a curve γ in T intersecting the cocore $S^1 \times \{\frac{1}{2}\} \times \{0\}$ exactly once.

We extend ξ to the handle $S^1 \times [0, 1] \times [-1, 1]$ as follows. First, we extend the characteristic foliation $\xi|_A$ of A to a linear foliation of T such that γ is an integral curve. This determines the germ of the contact structure ξ' along T . After reducing the thickness of the handle, we obtain a contact structure ξ' on $Y' = Y \cup S^1 \times [0, 1] \times [-1, 1]$. Because of condition (iii), the characteristic foliations of the two tori formed by gluing $S^1 \times [0, 1] \times \{0\}$ and the two annuli cut by $\partial A = S^1 \times \{0, 1\} \times \{0\}$ in ∂Y are linear. This implies that we can choose the boundary of Y' so that the characteristic foliation of ξ' is linear.

There exists a neighborhood $T \times [-\varepsilon, \varepsilon]$ of $T \simeq T \times \{0\}$ such that each torus $T \times \{s\}$ for $s \in [-\varepsilon, \varepsilon]$ has a linear characteristic foliation for ξ' . We now choose a representative of τ_γ which preserves each torus together with its foliation. Following Giroux [13; 9], we then obtain another representative τ_γ which preserves ξ' . Let $h' = \tau_\gamma \circ h$.

Continuing Example 4.3, we give an example of a situation where the above conditions are satisfied.

Example 6.1 Let (K, θ) be an open book decomposition of a closed 3-manifold Y supporting a contact structure ξ which is trivial as a vector bundle. In Example 4.3 we showed that for $\varepsilon > 0$ small enough there is an even contact structure \mathcal{E}_ε on $S^1 \times Y$ which is homotopic to $\text{pr}_*^{-1}(\xi)$ through even contact structures. Recall that pr denotes the projection onto the second factor of $S^1 \times Y$.

We now consider a stabilization of (K, θ) . For this we choose a properly embedded arc a inside a page S . By Giroux's realization lemma [12], it is always possible to isotope ξ through contact structures carried by (K, θ) so that a is Legendrian. Then the open book decomposition $(\text{pr}^{-1}(K), \theta \circ \text{pr})$ is adapted to \mathcal{E}_ε and the annulus $S^1 \times a$ verifies the conditions stated at the beginning of this section. \square

Theorem 6.2 *The even contact structures \mathcal{E} and \mathcal{E}' determined by the mapping tori of (Y, ξ, h) and (Y', ξ', h') are homotopic.*

Proof Recall first that if an open book decomposition supporting a contact structure ξ on a closed 3-manifold Z is stabilized along an arc a properly embedded in a page, then the stabilized open book decomposition also carries a contact structure ξ' and that ξ and ξ' are isotopic. More precisely, after Giroux and Goodman [14] (see also [9]), one chooses a to be Legendrian. Then

$$(Z, \xi') = (Z, \xi) \# (S^3, \xi_{\text{st}}),$$

where the sphere separating the two summands bounds a small neighborhood of a in Z . The connected sum with S^3 does of course not change the manifold and the contact structure ξ' is isotopic to ξ since one Darboux ball (the neighborhood $N(a)$ of a) is removed and another Darboux ball is glued in. But then ξ' is homotopic to ξ relative to the complement of $N(a)$ since all tight contact structures on a ball which induce the same characteristic foliation on the boundary are isotopic by Eliashberg [7] (in this case it can be checked directly).

Now let $A \simeq S^1 \times [0, 1]$ be an annulus satisfying the conditions (i), (ii) and (iii) stated at the beginning of this section. Then one can find a neighborhood $N_0(A) \simeq A \times (-\delta, \delta)$ of A in the page Y such that the plane field ξ is defined by $dt - f(y) dx$, where (t, x) are product coordinates on $A = S^1 \times [0, 1]$ and $y \in (-\delta, \delta)$. Remember that ξ is contact in the interior of Y , so $f'(y) > 0$ when $x \in (0, 1)$ and $f'(y) = 0$ on the binding $\partial Y \cap N_0(A) = \{x = 0, 1\}$.

In order to describe the even contact structure \mathcal{E}' one can now refer to [Example 6.1](#) when we view $A = S^1 \times a$, where a is a leaf of the characteristic foliation of a fixed annular page of the open book of S^3 described in [Example 3.4](#), joining the two components of the Hopf link binding. In particular, there is a neighborhood $N(A)$ of $N_0(A)$ in M where \mathcal{E} is, possibly after homotopy, conjugate to $(S^1 \times N(a), \mathcal{E}_\varepsilon) \subset (S^1 \times S^3, \mathcal{E}_\varepsilon)$.

Notice that the contact structure ξ_{st} on S^3 supported by this open book is not trivial as a vector bundle, but it is over trivial $N(a)$, where we can apply the construction of [Example 4.3](#). The even contact structure \mathcal{E}' is obtained from \mathcal{E} by first removing $(S^1 \times N(a), \mathcal{E})$ and then attaching $(S^1 \times (S^3 \setminus N(a)), \mathcal{E}_\varepsilon)$. Here, again, ξ_{st} is trivial over $S^3 \setminus N(a)$, so the construction makes sense, and there is an orientation-reversing identification sending the boundary of $(S^1 \times N(a), \mathcal{E}_\varepsilon)$ to the boundary of $(S^1 \times (S^3 \setminus N(a)), \mathcal{E}_\varepsilon)$.

It now follows that $(\mathcal{E}', \mathcal{W}')$ and $(\mathcal{E}, \mathcal{W})$ are homotopic as formal even contact structures, and hence as even contact structures [\[17\]](#). □

In view of [Questions 4.10](#) and [5.7](#), a nice feature of the stabilization operation is the following:

Theorem 6.3 *If (Y, ξ) is universally tight then so is (Y', ξ') .*

Proof The contact manifold (Y', ξ') is obtained by gluing a universally tight contact structure on $T^2 \times [0, 1]$ to (Y, ξ) along the prelagrangian torus $\partial Y \subset (Y, \xi)$. The result is universally tight by [\[4\]](#). □

Finally, we describe how to obtain an Engel structure \mathcal{D}' through a stabilization procedure on an open book decomposition supporting an Engel structure \mathcal{D} on a 4-manifold M . In many situations, as in the case of even contact structures, if an open book (Y, ξ, h) supports an Engel structure \mathcal{D} and if $A \subset (Y, \xi)$ is an annulus satisfying conditions (i), (ii) and (iii) on page [4298](#), then using [Theorem 5.3](#) one can construct an

Engel structure \mathcal{D}' carried by the stabilization (Y', ξ', h') of (Y, ξ, h) along A . We can check that they are formally homotopic and by [3] they are homotopic for k large enough, since we have shown that they are loose. However, we lose control of the geometry.

We present a more intrinsic operation below. Unfortunately, this construction involves a specific rigid normal form for \mathcal{D} near the annulus A along which we stabilize. We do not know how restrictive it is.

More explicitly, in [Example 5.2](#), where ξ_v is a contact structure on some closed 3-manifold Y supported by an open book (K, θ) , we pick a Legendrian arc a properly embedded in a page S of (K, θ) and consider the annulus $S^1 \times a = \text{pr}^{-1}(a)$ together with an S^1 -invariant neighborhood of it $S^1 \times N(a)$. We assume that the Engel structure \mathcal{D} on a neighborhood $N(A)$ of A is conjugated with some Engel structure \mathcal{D}_k from [Example 5.2](#) on $S^1 \times N(a)$. Recall the construction of \mathcal{D}_k involves the choice of a trivialization and of a (K, θ) -adapted Reeb vector field of ξ_v on $N(a)$.

In order to obtain an Engel structure carried by the open book of M stabilized along A , we replace the Engel structure $\mathcal{D} \simeq \mathcal{D}_k$ on $N(A) \simeq S^1 \times N(a)$ with an Engel structure \mathcal{D}' obtained in the following way.

We stabilize the open book (K, θ) of Y along a using Giroux and Goodman's stabilization construction [14]. This is given, as mentioned in the proof of [Theorem 6.2](#), by replacing ξ_v and the partial open book induced by (K, θ) on $N(a)$ by a structure ξ'_v — contactomorphic to ξ_v — and a stabilized partial open book (K', θ') supporting ξ'_v on $N(a)$. The trivialization and the Reeb vector field of ξ_v along $\partial N(a)$ extend to a trivialization and a (K', θ') -adapted Reeb vector field of $\xi'_v \simeq \xi_v$ on $N(a)$.

Then we again apply the construction of [Example 5.2](#) to ξ'_v and (K', θ') to get \mathcal{D}' as an Engel structure \mathcal{D}'_k on $N(A) \simeq S^1 \times N(a)$. It is supported by the original open book of M stabilized along A .

This concludes the description of \mathcal{D}' . As in the case of even contact structures it follows that \mathcal{D}' is homotopic to \mathcal{D} through Engel structures: the space of framings of ξ'_v which coincide with a fixed framing on $\partial N(a)$ is contractible. Moreover, notice that ξ_v and ξ'_v are both tight near $\partial N(a)$. According to a theorem of Eliashberg [7], any two contact structures on the ball which coincide near $\partial N(a)$ are isotopic relative to the boundary of the ball. Finally, any two contact vector fields which are transverse to a fixed contact structure are homotopic through contact vector fields which are transverse

to that contact structure (this can be seen using convex combinations of two such contact vector fields).

Remark 6.4 Given an annulus A in a page (Y, ξ) satisfying conditions (i), (ii) and (iii), it is not clear whether one can find a neighborhood $N(A)$ of A and a homotopy of \mathcal{D} such that the resulting Engel structure is conjugate to one of our models.

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