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Logarithmic Hennings invariants for restricted quantum $\mathfrak{sl}(2)$

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We construct a Hennings-type logarithmic invariant for restricted quantum $\mathfrak{sl}(2)$ at a $2\mathrm{p}^{\mathrm{th}}$ root of unity. This quantum group U is not quasitriangular and hence not ribbon, but factorizable. The invariant is defined for a pair: a 3–manifold M and a colored link L inside M. The link L is split into two parts colored by central elements and by trace classes, or elements in the 0^{th} Hochschild homology of U, respectively. The two main ingredients of our construction are the universal invariant of a string link with values in tensor powers of U, and the *modified trace* introduced by the third author with his collaborators and computed on tensor powers of the regular representation. Our invariant is a colored extension of the logarithmic invariant constructed by Jun Murakami.

57M27; 17B37, 57M25

1 Introduction

In the 1990s, M Hennings [13] came up with a construction of 3–manifold invariants out of a factorizable ribbon Hopf algebra H. In his construction the right integral $\mu \in H^*$ satisfying

$$(\mu \otimes \mathrm{id})\Delta(x) = \mu(x)1_H \quad \text{for all } x \in H \tag{1}$$

plays the role of the Kirby color. If the category of H–modules is semisimple, Hennings recovers the Reshetikhin–Turaev invariant. However in the nonsemisimple case, his invariant vanishes for manifolds with positive first Betti number (see Kerler [15]). A TQFT based on the Hennings invariant was constructed by Kerler [14] and Lyubashenko [19]. It provides a representation of the mapping class group for closed surfaces with one boundary component, but in general, it has weak functoriality and monoidality properties.

More recently, a completely different nonsemisimple TQFT based on the unrolled quantum $\mathfrak{sl}(2)$ was defined by Blanchet, Costantino, Geer and Patureau-Mirand [3]. This construction uses the logarithmic 3–manifold invariant constructed previously

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by Costantino, Geer and Patureau-Mirand (CGP) in [5] by generalizing the Kashaev invariant. One of the most innovative ingredients of the CGP construction is the so-called *modified trace* which contrary to the usual quantum trace does not vanish on the projective modules. Motivated by extension of Kashaev's volume conjecture, Jun Murakami [20] has constructed a logarithmic invariant of knots in 3–manifolds with value in the center of the quantum group. This invariant is obtained from the universal invariant of links in the 3–sphere by evaluating surgery components with the right integral defined in (1).

In this paper we combine the Hennings approach with the modified trace methods of Geer, Kujawa and Patureau-Mirand [10] and Geer, Patureau-Mirand and Virelizier [11] and construct a family of invariants of 3–manifolds with colored links inside. In the case of a knot we recover Murakami's logarithmic invariant. Our construction for links is more general, and its formulation is well suited for TQFT extension (compare De Renzi, Geer and Patureau-Mirand [7]).

Another novelty of our approach is that we study the modified trace as a symmetric function on U rather than on its category of modules and compute this function explicitly. These ideas were further developed by Beliakova, Blanchet and Gainutdinov [2] and led to a conceptual understanding of a modified trace as a shifted integral in the context of finite-dimensional pivotal Hopf algebras. Finally, this paper shows a way to extend link and 3–manifold invariants outside of ribbon categories, giving an explicit example of constructions by Bruguières [4].

Results on modified trace

Let us denote by U the restricted quantum $\mathfrak{sl}(2)$ at a $2\mathsf{p}^{\text{th}}$ root of unity $q = e^{i\pi/\mathsf{p}}$, explicitly defined in the next section. The Hopf algebra U does not contain an R–matrix, but only monodromy or double braiding. However, U is a subalgebra of a ribbon Hopf algebra D obtained by adjoining a square root of the generator K. Since D is not factorizable and U is not quasitriangular, neither U nor D supports the Hennings–Kerler–Lyubashenko TQFT construction.

Let U–mod be the category of finite-dimensional U–modules. This is a finite pivotal tensor category. Hence, for any morphism $f\colon V \to V$ in U–mod there is a notion of a categorical (or quantum) left and right traces, denoted by $\mathrm{tr}_l^V(f)$ and $\mathrm{tr}_r^V(f)$, respectively.

Let U–pmod be its full subcategory of projective U–modules. An explicit description of U–pmod is given by Gainutdinov, Semikhatov, Tipunin and Feĭgin [9]. Let us denote by $\mathcal{P}_j^{\pm}$, with $j=1,\dots,\mathsf{p}$, the indecomposable projective modules. Here $\mathcal{P}_{\mathsf{p}}^{\pm}$ is a simple module with highest weight $\pm q^{\mathsf{p}-1}$. The module $\mathcal{P}_1^{+}$ is the projective cover of the trivial module. The space of endomorphisms $\mathrm{End}_U(\mathcal{P}_j^{\pm})$, where $1\le j\le \mathsf{p}-1$, is two-dimensional with basis given by the identity $\mathrm{id}_{\mathcal{P}_j^{\pm}}$ and a nilpotent endomorphism $x_j^{\pm}$, defined in Section 3 .

The subcategory U–pmod is an ideal of U–mod in the sense of Geer, Kujawa and Patureau-Mirand [10] and Geer, Patureau-Mirand and Virelizier [11]. A *modified trace* on U–pmod is a family of linear functions

$$\{\mathsf{t}_V\colon \mathrm{End}_U\, V\to\mathbb{C}\}_{V\in U\text{–pmod}}$$

such that the following two conditions hold:

Cyclicity If $X,V\in U$–pmod, then for any morphisms $f\colon V\to X$ and $g\colon X\to V$ in U–mod we have

$$\mathsf{t}_V(gf)=\mathsf{t}_X(fg).$$

Partial trace properties If $X\in U$–pmod and $W\in U$–mod, then for any morphisms $f\in\mathrm{End}_U(X\otimes W)$ and $g\in\mathrm{End}_U(W\otimes X)$ we have

$$\mathsf{t}_{X\otimes W}(f)=\mathsf{t}_X(\mathrm{tr}_r^W(f))\quad\text{and}\quad \mathsf{t}_{W\otimes X}(g)=\mathsf{t}_X(\mathrm{tr}_l^W(g)),$$

where tr_r^W and tr_l^W are the right and left partial categorical traces along W defined using the pivotal structure in (6).

Our first main result is the following.

Theorem 1 *There exists a unique family of linear functions*

$$\{\mathsf{t}_V\colon \mathrm{End}_U(V)\to\mathbb{C}\}_{V\in U\text{–pmod}},$$

satisfying cyclicity and the partial trace properties, normalized by

$$\mathsf{t}_{\mathcal{P}_{\mathsf{p}}^{+}}(\mathrm{id}_{\mathcal{P}_{\mathsf{p}}^{+}})=(-1)^{\mathsf{p}-1}.$$

Moreover, $\mathsf{t}_{\mathcal{P}_{\mathsf{p}}^{-}}(\mathrm{id}_{\mathcal{P}_{\mathsf{p}}^{-}})=1$ *and for* $1\le j\le\mathsf{p}-1$ *we have*

$$\mathsf{t}_{\mathcal{P}_j^{\pm}}(\mathrm{id}_{\mathcal{P}_j^{\pm}})=(\pm1)^{\mathsf{p}-1}(-1)^j(q^j+q^{-j})\quad\textit{and}\quad \mathsf{t}_{\mathcal{P}_j^{\pm}}(x_j^{\pm})=(\pm1)^{\mathsf{p}}(-1)^j[j]^2.$$

This family is called the modified trace on U–pmod. The proof uses the fact that U–mod is unimodular (ie the projective cover of the trivial module is self-dual) and it has a simple projective object. In this case, there exist unique left and right modified traces on U–pmod by [10, Corollary 3.2.1]. We actually compute these traces on the algebra of endomorphisms of indecomposable projectives and show that they are equal.

Observe that Theorem 1 applies to U, considered as a free left module over itself, called the regular representation, and its tensor powers. Recall from [9] that the regular representation decomposes as

$$U \cong \bigoplus_{j=1}^{\mathsf{p}} j\mathcal{P}_j^+ \oplus \bigoplus_{j=1}^{\mathsf{p}} j\mathcal{P}_j^- .$$

The algebra $\mathrm{End}_U(U)$ of the U–endomorphisms of U can be identified with U^{op} (ie U with the opposite multiplication). The isomorphism is given by sending an element x of U^{op} to the operator r_x of the right multiplication by x on the regular representation. By definition r_x commutes with the left action.

The space of characters (or symmetric functions; see Arike [1]) on U is defined as

$$\mathrm{Char}(U) := \{\phi \in U^* \mid \phi(xy) = \phi(yx) \text{ for any } x, y \in U\}.$$

This space is dual to the 0^{th} Hochschild homology $\mathrm{HH}_0(U)$, which is

$$\mathrm{HH}_0(U) := \frac{U}{[U,U]}, \quad \text{where } [U,U] = \mathrm{Span}\{xy - yx \mid x, y \in U\}.$$

There is an obvious action of the center $Z(U)$ on $\mathrm{Char}(U)$ by setting $z\phi(x) := \phi(zx)$ for any $z \in Z(U)$ and $x \in U$.

Modified trace provides us with symmetric linear maps

$$\mathsf{t}_{U^{\otimes m}} \colon \mathfrak{C}_m \to \mathbb{C},$$

where $\mathfrak{C}_m = \mathrm{End}_U(U^{\otimes m})$ for $m \geq 1$ are known as centralizer algebras. For $m = 1$, we obtain a special symmetric function $\mathsf{T} \in \mathrm{Char}(U)$ defined by $\mathsf{T}(x) = \mathsf{t}_U(r_x)$, which we call the *modified trace* of U.

Theorem 2 *The pairing* $\langle\ ,\ \rangle\colon Z(U) \times \mathrm{HH}_0(U) \to \mathbb{C}$ *defined from the modified trace by* $\langle z, u\rangle = \mathsf{T}(zu)$ *is nondegenerate.*

The proof is by direct computation of the pairing in the basis of the center and the basis of $\mathrm{HH}_0(U)$, defined in Sections 2 and 4, respectively.

Note that similarly to the space $\mathrm{Char}(U)$ we can define the space of quantum characters

$$\mathrm{qChar}(U) := \{\phi \in U^* \mid \phi(xy) = \phi(S^2(y)x) \text{ for any } x, y \in U\}.$$

Evaluation of quantum characters on central elements defines a pairing

$$\mathrm{qChar}(U) \times Z(U) \to \mathbb{C}.$$

This pairing is degenerate. Indeed, for any $1 \le j \le \mathsf{p}$ the quantum character $\mathrm{tr}_r^{\mathcal{P}_j^{\pm}}$ associated with an indecomposable projective $\mathcal{P}_j^{\pm}$ has zero evaluation on all central elements. However, its evaluation on the pivotal element gives the dimension of the module. This is why quantum characters cannot detect Murakami's invariant while the modified trace does.

Construction of invariants

For any ribbon Hopf algebra there exists a *universal* invariant associated to an oriented framed tangle T. This invariant is obtained by assigning the R–matrix to crossings and evaluation and coevaluation maps to maxima and minima (see Section 4). Although our restricted quantum group U is not ribbon, it has a ribbon extension D, which produces a universal invariant $J_T \in D^{\otimes m}$ for any tangle T with m components. If T is an upwards oriented string link, we argue that J_T actually belongs to the subspace $(U^{\otimes m})^U \subset U^{\otimes m}$ of invariants under left action.

We are now ready to state our main result. Let $\mu \in U^*$ be the right integral of U. It is unique up to a normalization, which we fix in Section 2. For $m \ge 1$, the function $\mathsf{t}_{U^{\otimes m}}$ defines a bilinear pairing $\langle\ ,\ \rangle\colon (U^{\otimes m})^U \times U^{\otimes m} \to \mathbb{C}$ by

$$\langle z, x\rangle = \mathsf{t}_{U^{\otimes m}}(l_z r_x).$$

Here l_z and r_x are the left and right multiplications, respectively. This bilinear pairing factorizes on the right through $\mathrm{HH}_0(U^{\otimes m})$.

Assume that a closed 3–manifold M with an (m_+, m_-)–component oriented framed link (L^+, L^-) inside is represented by surgery in S^3 along the m_0–component oriented framed link L^0. Let us color the components of L^+ (resp. L^-) by central elements $z_j \in Z(U)$ for $1 \le j \le m_+$ (resp. by trace classes $h_k \in \mathrm{HH}_0(U)$ for $1 \le k \le m_-$). Let $T = T^+ \cup T_0 \cup T^-$ be an upward oriented string link in S^3 obtained by opening all $m = m_+ + m_0 + m_-$ components. Let s be the signature of the linking matrix for L^0 and $\delta := \frac{1}{\sqrt{2}}(1-i)q^{\frac{1}{2}(3-\mathsf{p}^2)}$. We set $z^+ = \bigotimes_j z_j$ $\big($resp. $h^- = \bigotimes_k h_k\big)$, and denote by L the colored link $((L^+, z^+), (L^-, h^-))$. Note that to define z^+ and h^- as well as J_T, we need to fix an order on the components of L; change of this order will result in an obvious permutation of the entries. For each j, the central element z_j acting on the right integral μ defines the quantum character $\mu_{z_j}\colon x \mapsto \mu(z_j x)$. We denote the tensor product $\bigotimes_j \mu_{z_j}$ by $z^+\mu^{\otimes m_+}$. We define a number associated to the pair (M, L) as follows.

Theorem 3 *With the notation as above,*

$$\mathrm{H}^{\log}(M, L) := \delta^s \langle (z^+\mu^{\otimes m_+} \otimes \mu^{\otimes m_0} \otimes \mathrm{id})(J_T),\, h^- \rangle$$

is a topological invariant of the pair (M, L).

When L^- is empty, the colored Hennings invariants [13] are recovered. When L^- is a knot, the collection of our invariants for all colors of L^- is equivalent to the Murakami center-valued logarithmic invariant (see Proposition 11). Thus $\mathrm{H}^{\log}$ can be understood as a colored extension of the Murakami invariants. An action of the modular group $\mathrm{SL}(2,\mathbb{Z})$ on the center $Z(U)$ was studied by Feigin, Gainutdinov, Semikhatov and Tipunin [8]. We expect that $\mathrm{H}^{\log}$ can be used to extend these mapping class group representations in genus one to refined TQFTs with full functorial and monoidal properties.

The paper is organized as follows. In Section 2 we define the restricted quantum group U and its braided extension D. In Section 3 we discuss their categories of finite-dimensional modules. The universal tangle invariant is constructed in Section 4, where we also compute a basis for the space of trace classes $\mathrm{HH}_0(U)$. Our main theorems are proved in the two last sections.

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2 Restricted quantum $\mathfrak{sl}(2)$ and its braided extension

Definition of U

Fix an integer $\mathsf{p} \geq 2$ and let $q = e^{i\pi/\mathsf{p}}$ be a $2\mathsf{p}^{\text{th}}$ root of unity. Let $U = \overline{U}_q(\mathfrak{sl}(2))$ be the $\mathbb{C}$–algebra given by generators E, F, K, K^{-1} and relations

$$E^{\mathsf{p}} = F^{\mathsf{p}} = 0, \qquad K^{2\mathsf{p}} = 1, \qquad KK^{-1} = K^{-1}K = 1,$$

$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

The algebra U is a Hopf algebra where the coproduct, counit and antipode are defined by

$$\begin{aligned} \Delta(E) &= 1\otimes E + E\otimes K, & \varepsilon(E) &= 0, & S(E) &= -EK^{-1}, \\ \Delta(F) &= K^{-1}\otimes F + F\otimes 1, & \varepsilon(F) &= 0, & S(F) &= -KF, \\ \Delta(K) &= K\otimes K, & \varepsilon(K) &= 1, & S(K) &= K^{-1}, \\ \Delta(K^{-1}) &= K^{-1}\otimes K^{-1}, & \varepsilon(K^{-1}) &= 1, & S(K^{-1}) &= K. \end{aligned}$$

In what follows we will use Sweedler notation. For $x \in U$ we write

$$\Delta(x) = \sum x_{(1)}\otimes x_{(2)} \qquad \text{and} \qquad \Delta^{[n]}(x) = \sum x_{(1)}\otimes x_{(2)}\otimes\cdots\otimes x_{(n)} \quad \text{for } n\geq 1.$$

The center of U

The dimension of the center $Z(U)$ is $3\mathsf{p}-1$. A basis consists of $\mathsf{p}+1$ central idempotents $\boldsymbol{e}_j$ $(0\leq j\leq \mathsf{p})$ and $2\mathsf{p}-2$ elements $\boldsymbol{w}_j^{\pm}$ $(1\leq j\leq \mathsf{p}-1)$ in the radical [8]. These elements satisfy the following relations:

$$\begin{aligned} \boldsymbol{e}_s\boldsymbol{e}_t &= \delta_{s,t}\boldsymbol{e}_s & &\text{for } 0\leq s,t\leq \mathsf{p}, \\ \boldsymbol{e}_s\boldsymbol{w}_t^{\pm} &= \delta_{s,t}\boldsymbol{w}_t^{\pm} & &\text{for } 0\leq s\leq \mathsf{p} \text{ and } 1\leq t\leq \mathsf{p}-1, \\ \boldsymbol{w}_s^{\pm}\boldsymbol{w}_t^{\pm} &= \boldsymbol{w}_s^{\pm}\boldsymbol{w}_t^{\mp} = 0 & &\text{for } 1\leq s,t\leq \mathsf{p}-1. \end{aligned}$$

Braided extension

The Hopf algebra U is not braided; see [16]. However, it can be realized as a Hopf subalgebra of the following braided Hopf algebra. Let D be the Hopf algebra generated by e, ϕ, k and k^{-1} with the relations

$$e^{\mathsf{p}} = \phi^{\mathsf{p}} = 0, \qquad k^{4\mathsf{p}} = 1, \qquad kk^{-1} = k^{-1}k = 1,$$
$$kek^{-1} = qe, \quad k\phi k^{-1} = q^{-1}\phi, \quad [e,\phi] = \frac{k^2-k^{-2}}{q-q^{-1}},$$

and Hopf algebra structure

$$\begin{aligned} \Delta(e) &= 1\otimes e + e\otimes k^2, & \varepsilon(e) &= 0, & S(e) &= -ek^{-2}, \\ \Delta(\phi) &= k^{-2}\otimes \phi + \phi\otimes 1, & \varepsilon(\phi) &= 0, & S(\phi) &= -k^2\phi, \\ \Delta(k) &= k\otimes k, & \varepsilon(k) &= 1, & S(k) &= k^{-1}, \\ \Delta(k^{-1}) &= k^{-1}\otimes k^{-1}, & \varepsilon(k^{-1}) &= 1, & S(k^{-1}) &= k. \end{aligned}$$

The Hopf algebra D has two special invertible elements: the R–matrix

$$R = \frac{1}{4\mathsf{p}} \sum_{m=0}^{\mathsf{p}-1} \sum_{n,j=0}^{4\mathsf{p}-1} \frac{(q-q^{-1})^m}{[m]!} q^{m(m-1)/2+m(n-j)-nj/2} e^m k^n \otimes \phi^m k^j$$

and the ribbon element

$$r = \frac{1-i}{2\sqrt{\mathsf{p}}} \sum_{m=0}^{\mathsf{p}-1} \sum_{j=0}^{2\mathsf{p}-1} \frac{(q-q^{-1})^m}{[m]!} q^{-m/2+mj+(j+p+1)^2/2} \phi^m e^m k^{2j},$$

where $q^{\frac{1}{2}} = e^{\frac{1}{2}i\pi/\mathsf{p}}$, $[m] = (q^m - q^{-m})/(q-q^{-1})$ and $[m]! = [m][m-1]\cdots[1]$. The following theorem is well known; see [8].

Theorem 4 *The triple (D, R, r) is a ribbon Hopf algebra.*

Let us call $M = R_{21}R$ the double braiding or *monodromy*, where $R_{21} = \sum_i \beta_i \otimes \alpha_i$ with $R = \sum_i \alpha_i \otimes \beta_i$. A Hopf algebra A is called *factorizable* if its monodromy matrix can be written as

$$M = \sum_i m_i \otimes n_i,$$

where m_i and n_i are two bases of A. The Hopf algebra D is not factorizable. There is a Hopf algebra embedding $U \to D$ given by

$$E \mapsto e, \quad F \mapsto \phi \quad \text{and} \quad K \mapsto k^2.$$

It is easy to check that $r \in U$, and the monodromy $M = R_{21}R$ is in $U \otimes U$. Moreover, U is factorizable.

Ribbon and balancing elements of U

Let $u = \sum_i S(\beta_i)\alpha_i$ be the canonical element implementing the inner automorphism S^2, ie $S^2(x) = uxu^{-1}$ for any $x \in D$, and satisfying

$$\Delta(u) = M^{-1}(u \otimes u).$$

Using the formula for the R–matrix, it is easy to check that $u \in U$. The ribbon element $r \in U$ is central and invertible, so that

$$r^2 = uS(u), \quad S(r) = r, \quad \varepsilon(r) = 1, \quad \Delta(r) = M^{-1}(r \otimes r). \tag{2}$$

We then define

$$g := r^{-1}u = k^{2\mathsf{p}+2} = K^{\mathsf{p}+1} \in U,$$

the *balancing* element. This element is grouplike, ie

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1 \quad \text{and} \quad gxg^{-1} = S^2(x) \tag{3}$$

for any $x \in U$. The balancing element will be used to define the pivotal structure.

Remark As was shown by Drinfeld, the equations (3) determine g^2 only. Hence $g' = K$ is another balancing element, which will not be considered in this paper.

Right integral

Recall that a *right integral* $\mu \in U^*$ is defined by the system of equations

$$(\mu \otimes \mathrm{id})\Delta(x) = \mu(x)1_U \quad \text{for all } x \in U.$$

For any finite-dimensional Hopf algebra over a field of characteristic zero, there is a unique solution to these equations up to a scalar. In our case in the PBW basis, it is given by the formula

$$\mu(E^m F^n K^l) = \zeta \delta_{m,\mathsf{p}-1} \delta_{n,\mathsf{p}-1} \delta_{l,\mathsf{p}+1}.$$

We fix a normalization as in [20] by setting

$$\zeta = -\sqrt{\frac{2}{\mathsf{p}}}([\mathsf{p}-1]!)^2.$$

The evaluation on the ribbon element and its inverse are given by

$$\mu(r) = \frac{1-i}{\sqrt{2}} q^{\frac{1}{2}(3-\mathsf{p}^2)} = \frac{1}{\mu(r^{-1})} = \delta.$$

Dually, a left (resp. right) *cointegral* is an element $\boldsymbol{c} \in U$ such that $x\boldsymbol{c} = \varepsilon(x)\boldsymbol{c}$ (resp. $\boldsymbol{c}x = \varepsilon(x)\boldsymbol{c}$) for all $x \in U$. Nontrivial left and right cointegrals exist and are unique up to scalar. A Hopf algebra is *unimodular* if its right cointegral is also left. Our Hopf algebra U is unimodular with cointegral

$$\boldsymbol{c} = F^{\mathsf{p}-1} E^{\mathsf{p}-1} \sum_{j=0}^{2\mathsf{p}-1} K^j.$$

The integral μ belongs to the space of quantum characters $\mathrm{qChar}(U)$. This is a general fact for a finite-dimensional unimodular Hopf algebra [22, Theorem 3]. Here it's enough to check the defining relation when x is in the PBW basis and y is a generator.

The center $Z(U)$ acts on qChar(U) by $z\phi(x) := \phi(zx)$ for any $z \in Z(U)$ and $x \in U$. Under this action qChar(U) is a free module of dimension one with basis given by the right integral μ. Hence, as a vector space qChar(U) has dimension $3\mathsf{p}-1$.

The space of quantum characters qChar(U) is naturally isomorphic to the space of characters. Indeed, we can define the map

$$Q\colon \mathrm{qChar}(U) \to \mathrm{Char}(U) \quad \text{by sending } \phi \mapsto \phi_g, \tag{4}$$

where $\phi_g(x) := \phi(gx)$ and g is the balancing element. Cyclicity can be verified as follows:

$$\phi_g(xy) = \phi(gxy) = \phi(S^2(y)gx) = \phi(gyx) = \phi_g(yx).$$

The inverse map is given by sending $\psi \in \mathrm{Char}(U)$ to $\psi_{g^{-1}} \in \mathrm{qChar}(U)$. Hence Q is an isomorphism. We get that $\mathrm{HH}_0(U)$ has dimension $3\mathsf{p}-1$ since its dual vector space is isomorphic to Char(U).

3 Categories of modules

In this section we will study the $\mathbb{C}$–linear categories of finite-dimensional modules over D and U, which we denote by D–mod and U–mod, respectively.

Category D–mod

The category D–mod is ribbon, with the usual braiding

$$c_{V,W}\colon V \otimes W \to W \otimes V, \quad \text{given by } u \otimes v \mapsto \tau R(u \otimes v),$$

where $\tau(x \otimes y) = y \otimes x$, with self-dual twist

$$\theta_V\colon V \to V, \quad \text{given by } v \mapsto r^{-1}v,$$

and with compatible duality

$$\begin{aligned}
\mathrm{coev}_V\colon \mathbb{C} &\to V \otimes V^*, && \text{given by } 1 \mapsto \textstyle\sum_i v_i \otimes v_i^*,\\
\mathrm{ev}_V\colon V^* \otimes V &\to \mathbb{C}, && \text{given by } f \otimes v \mapsto f(v),\\
\widetilde{\mathrm{coev}}_V\colon \mathbb{C} &\to V^* \otimes V, && \text{given by } 1 \mapsto \textstyle\sum_i v_i^* \otimes g^{-1}v_i,\\
\widetilde{\mathrm{ev}}_V\colon V \otimes V^* &\to \mathbb{C}, && \text{given by } v \otimes f \mapsto f(gv),
\end{aligned} \tag{5}$$

where g is the balancing element. The ribbon structure implies that the twist is *self-dual*, ie $\theta_{V^*} = \theta_V^*$.

The duality morphisms (5) define the pivotal structure on D–mod (see eg [11]). In particular, in the pivotal setting, one can define left and right (categorical) traces of any endomorphism $f\colon V \to V$ as

$$\mathrm{tr}_l(f) = \mathrm{ev}_V(\mathrm{id}_{V^*} \otimes f)\,\widetilde{\mathrm{coev}}_V \quad \text{and} \quad \mathrm{tr}_r(f) = \widetilde{\mathrm{ev}}_V(f \otimes \mathrm{id}_{V^*})\,\mathrm{coev}_V$$

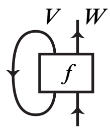
and dimensions of objects.

A *spherical* category is a pivotal category whose left and right traces are equal, ie $\mathrm{tr}_l(f) = \mathrm{tr}_r(f)$ for any endomorphism f. It is easy to see that any ribbon category is spherical. We call $\mathrm{tr}_l(\mathrm{id}_V) = \mathrm{tr}_r(\mathrm{id}_V)$ the *quantum* dimension of V.

We will use a standard graphical calculus to represent morphisms in D–mod by diagrams in the plane which are read from the bottom to the top, with the convention that an X–colored vertical arc oriented upwards represents the identity id_X.

In what follows, we will need *partial* categorical traces of endomorphisms. Given $V, W \in D$–mod and $f\colon V \otimes W \to V \otimes W$, let tr_l^V and tr_r^W be the left and right *partial* traces defined as follows:

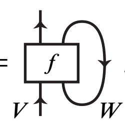
$$\mathrm{tr}_l^V(f) = (\mathrm{ev}_V \otimes \mathrm{id}_W)(\mathrm{id}_{V^*} \otimes f)(\widetilde{\mathrm{coev}}_V \otimes \mathrm{id}_W) =$$



,

(6)

$$\mathrm{tr}_r^W(f) = (\mathrm{id}_V \otimes \widetilde{\mathrm{ev}}_W)(f \otimes \mathrm{id}_{W^*})(\mathrm{id}_V \otimes \mathrm{coev}_W) =$$



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Category U–mod

Let us call a module *simple* if its endomorphism ring is one-dimensional. A module is *projective* if it is a direct summand of a free module.

The category U–mod includes the s–dimensional simple modules $\mathcal{X}_s^{\pm}$, and their projective covers $\mathcal{P}_s^{\pm}$ for $1 \le s \le \mathsf{p}$, which are $2\mathsf{p}$–dimensional for $1 \le s < \mathsf{p}$. The simple module $\mathcal{X}_s^{\pm}$ is determined by its highest-weight vector v with the action $Ev = 0$ and $Kv = \pm q^{s-1} v$. It is projective if and only if $s = p$. A category is called *unimodular* if the projective cover of the trivial module is self-dual. It is shown in [18, Section 2.5] that a finite-dimensional Hopf algebra is unimodular if and only if its category of finite-dimensional modules is unimodular, hence the category U–mod is unimodular.

The category U–mod inherits the pivotal structure, twist and double braiding from D–mod. The double braiding is

$$M_{V,W}\colon V\otimes W\to V\otimes W,\quad \text{given by } x\otimes y\mapsto M(x\otimes y),$$

where M is the monodromy matrix; the self-dual twist and duality are given by (5). One can check that U–mod is a *twisted* category with duality in the sense of Bruguières [4].

Let U–pmod be the full subcategory of U–mod consisting of projective modules. This category is nonabelian. To compute the modified trace on U–pmod, we will need an explicit structure of this category.

Structure of U–pmod

A module is *indecomposable* if it does not decompose as a direct sum of two modules. The indecomposable projective U–modules are classified up to isomorphism in [9]: they are precisely the projective covers $\mathcal{P}_j^{\pm}$ of the simple modules, where $j=1,\dots,\mathsf{p}$. In particular, $\mathcal{P}_{\mathsf{p}}^{\pm}$ is a simple module with highest weight $\pm q^{\mathsf{p}-1}$. The module $\mathcal{P}_1^+$ is the projective cover of the trivial one.

We will recall some facts about these projective modules. We reproduce in the appendix the defining relations for the modules $\mathcal{P}_j^+$ and $\mathcal{P}_{\mathsf{p}-j}^-$ for $1\le j\le \mathsf{p}-1$, in the form given in [8, Section C.2]. They have respective bases $\boldsymbol{B}^+$ and $\boldsymbol{B}^-$, given by

$$\boldsymbol{B}^{\pm}=\{\boldsymbol{x}_k^{\pm},\boldsymbol{y}_k^{\pm}\}_{0\le k\le \mathsf{p}-j-1}\cup\{\boldsymbol{a}_n^{\pm},\boldsymbol{b}_n^{\pm}\}_{0\le n\le j-1}. \tag{7}$$

Following [6] we call a weight vector v *dominant* if $(FE)^2v=0$. The vector $\boldsymbol{b}_0^+$ (resp. $\boldsymbol{y}_0^-$) is a generating dominant vector of $\mathcal{P}_j^+$ (resp. $\mathcal{P}_j^-$) with weight $\pm q^{j-1}$. Analyzing the possible images for this generating dominant weight vector of $\mathcal{P}_j^{\epsilon}$, where $\epsilon\in\{-,+\}$, we can completely determine the nontrivial morphisms between indecomposable projective modules. Let x_j^+ (resp. x_j^-) be the nilpotent endomorphism of $\mathcal{P}_j^+$ (resp. $\mathcal{P}_j^-$) determined by $\boldsymbol{b}_0^+\mapsto\boldsymbol{a}_0^+$ (resp. $\boldsymbol{y}_0^-\mapsto\boldsymbol{x}_0^-$), and let $a_j^+,b_j^+\colon\mathcal{P}_j^+\to\mathcal{P}_{\mathsf{p}-j}^-$ and $a_j^-,b_j^-\colon\mathcal{P}_j^-\to\mathcal{P}_{\mathsf{p}-j}^+$ be the morphisms defined by

$$a_j^+(\boldsymbol{b}_0^+)=\boldsymbol{a}_0^-,\quad b_j^+(\boldsymbol{b}_0^+)=\boldsymbol{b}_0^-,\quad a_j^-(\boldsymbol{y}_0^-)=\boldsymbol{x}_0^+\quad\text{and}\quad b_j^-(\boldsymbol{y}_0^-)=\boldsymbol{y}_0^+.$$

The nontrivial Hom–spaces between indecomposable projective modules are

- the endomorphism ring $\mathrm{End}_U(\mathcal{P}_j^{\pm})$, which is one-dimensional for $j=p$ and two-dimensional with basis $\{\mathrm{id}_{\mathcal{P}_j^{\pm}},x_j^{\pm}\}$ for $1\le j<\mathsf{p}$,
- the Hom–spaces $\mathrm{Hom}_U(\mathcal{P}_j^+,\mathcal{P}_{\mathsf{p}-j}^-)$ and $\mathrm{Hom}_U(\mathcal{P}_j^-,\mathcal{P}_{\mathsf{p}-j}^+)$, which are two-dimensional, with respective bases $\{a_j^+,b_j^+\}$ and $\{a_j^-,b_j^-\}$, for $1\le j<\mathsf{p}$.

Proposition 5 [8, Proposition 4.4.4] *The action of the center on the indecomposable projective modules is as follows, where* $1 \le j < p$*:*

	$\mathcal{P}_p^-$	$\mathcal{P}_p^+$	$\mathcal{P}_j^+$	$\mathcal{P}_{p-j}^-$
$\boldsymbol{e}_0$	$\mathrm{id}_{\mathcal{P}_p^-}$	0	0	0
$\boldsymbol{e}_p$	0	$\mathrm{id}_{\mathcal{P}_p^+}$	0	0
$\boldsymbol{e}_j$	0	0	$\mathrm{id}_{\mathcal{P}_j^+}$	$\mathrm{id}_{\mathcal{P}_{p-j}^-}$
$\boldsymbol{w}_j^+$	0	0	x_j^+	0
$\boldsymbol{w}_j^-$	0	0	0	x_{p-j}^-

4 Tangle invariants and the trace of U

Our links and tangles are always assumed to be framed and oriented. The diagrams are going from bottom to top. A *string link* is a tangle without closed component whose arcs end at the same order as they start with upwards orientation. A pure braid is an example.

Reshetikhin–Turaev invariant

Given any ribbon category $\mathcal{C}$, there exists a unique ribbon functor

$$F_{\mathcal{C}} \colon \mathrm{Rib}_{\mathcal{C}} \to \mathcal{C},$$

where $\mathrm{Rib}_{\mathcal{C}}$ is the category of $\mathcal{C}$–colored ribbon graphs, which send an upwards oriented vertical string colored by $V \in \mathcal{C}$ to the identity id_V. This is shown in [24], with opposite orientation convention. Applying this construction to D–mod, for an m–component upwards string link T, colored with the regular representation D, we obtain $F_D(T) \in \mathrm{End}_D(D^{\otimes m})$. Here we use the shorthand F_D for $F_{D\text{–mod}}$.

The Reshetikhin–Turaev invariant of the colored link $(L, V_1, \dots, V_m)$ is obtained by evaluating the categorical right traces on $F_D(T)$, ie

$$\begin{aligned} J_L(V_1, \dots, V_m) &:= (\mathrm{tr}_r^{V_1} \otimes \cdots \otimes \mathrm{tr}_r^{V_m}) F_D(T) \\ &= (\mathrm{trace}^{V_1} \otimes \cdots \otimes \mathrm{trace}^{V_m})(g \otimes \cdots \otimes g) F_D(T), \end{aligned}$$

where T is a string link with braid closure isotopic to L. Here trace^V means the (usual) trace of a linear endomorphism. Recall also that $\mathrm{tr}_r^V = \mathrm{tr}_l^V$ since both closures are isotopic. Hence, we can replace g by g^{-1} in the last line. If one of the V_i is projective, this invariant vanishes.

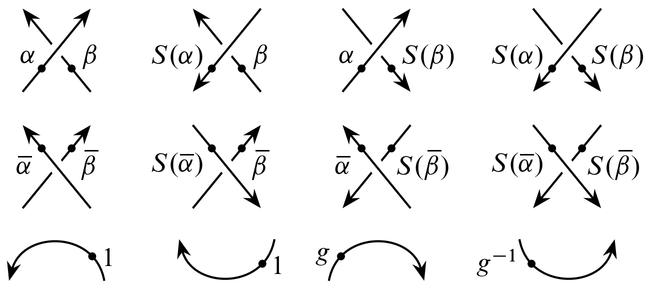

Figure 1: Local formulas for the universal invariant

Universal invariant

Associated with a ribbon Hopf algebra there is another powerful invariant: the *universal* invariant of links and tangles introduced by Lawrence [17] for some quantum groups and defined by Hennings [13] in the general case. For a string link T with m components, its universal invariant J_T is obtained by pasting together pieces shown in Figure 1. Here we write $R = \sum \alpha \otimes \beta$ and $R^{-1} = \sum \overline{\alpha} \otimes \overline{\beta}$.

More precisely, for each arc of T we obtain an element of D by writing a word from right to left with labels read using the order following the orientation. Thus we get $J_T \in D^{\otimes m}$. This element does not change by Reidemeister moves. The original proof of the invariance was stated for a link. The argument was extended to tangles in [12, Section 7.3].

Note that in [12] Habiro uses different conventions. His tangles are depicted from top to bottom and orientations are reversed. Hence, our model can be recovered from his after reflecting over a horizontal axis. The universal invariants coincide. In [21], Ohtsuki defines the universal invariant using orientation opposite to our convention, but the word is written there from left to right when following the orientation. Again the universal invariant is finally the same as ours.

Relation between them

The universal invariant is known to dominate Reshetikhin–Turaev invariants in the following sense (the proof is given in [21, Theorem 4.9]):

$$J_L(V_1, \dots, V_m) = (\mathrm{tr}_r^{V_1} \otimes \cdots \otimes \mathrm{tr}_r^{V_m})(J_T).$$

Let us denote by $(U^{\otimes m})^U \subset U^{\otimes m}$ the submodule centralizing the left action, ie

$$x \in (U^{\otimes m})^U \qquad \Longleftrightarrow \qquad \Delta^{[m]}(h)x = x\Delta^{[m]}(h) \quad \text{for all } h \in U.$$

The following lemma is folklore, but we are adding the proof for completeness.

Lemma 6 *The Reshetikhin–Turaev intertwiner $F_D(T)$ is equal to left multiplication by J_T. In addition, for an m–component string link T, its universal invariant J_T belongs to $(U^{\otimes m})^U$.*

Proof The fact that $F_D(T)$ is the left multiplication by J_T follows directly by comparing the definitions of these two invariants. More details are given in the proof of [21, Theorem 4.9]. Hence, multiplication by J_T has to commute with left action, and we conclude $J_T \in (D^{\otimes m})^D$.

Let us show that for a string link the universal invariant J_T actually belongs to $U^{\otimes m}$. This implies the claim, since $(U^{\otimes m})^D \subset (U^{\otimes m})^U$.

The linking matrix of a string link is diagonal mod 2. In [4, Section 1.3], Bruguières shows that *any* tangle with this property can be obtained as compositions and tensor products of evaluations, coevaluations and twists. Thus the universal invariant J_T is built up from the ribbon element r, the balancing element g, and their inverses by applying the Hopf algebra operations. We obtain $J_T \in U^{\otimes m}$. □

Evaluations of the universal invariant

Assume for simplicity that T is a $(1,1)$–tangle whose closure is the knot K. Then for any $\phi \in \mathrm{qChar}(U)$, the evaluation $\phi(J_T) \in \mathbb{C}$ is a knot invariant. To prove this fact, we need to show that this evaluation does not change by cyclic permutations of the word $J_K = g^{-1} J_T$ obtained by applying the algorithm described above to the left closure of T. Using (4), we get

$$\phi(J_T) = \phi_g(g^{-1} J_T) = \phi_g(J_K).$$

The last expression does not change by cyclic permutations since $\phi_g \in \mathrm{Char}(U)$. Applying this argument to the k leftmost components of a string link with m strands, we will get the following.

Lemma 7 *Let T be an m–component string link and $1 \le k \le m$. Let $\Phi = \bigotimes_{j=1}^{k} \phi_j$, with $\phi_j \in \mathrm{qChar}(U)$ a sequence of quantum characters. Then*

$$(\Phi \otimes \mathrm{id}_{U^{\otimes(m-k)}})(J_T) \in (U^{\otimes(m-k)})^U$$

is an invariant of the tangle obtained from T by closing the first k components.

Further observe that given $\phi \in \mathrm{qChar}(U)$ we can obtain another quantum character ϕ_z by twisting ϕ with a central element $z \in Z(U)$, where $\phi_z(x) := \phi(zx)$ for any $x \in U$.

In the next section we will define a family of linear functions

$$\{\mathsf{t}_V\colon \mathrm{End}_{\mathscr{C}}(V) \to \mathbb{C}\}_{V \in U\text{–pmod}}$$

satisfying cyclicity and the partial trace property. For any $m > 0$, we can use $\mathsf{t}_{U^{\otimes m}}$ to define a bilinear pairing

$$\langle\ ,\ \rangle\colon (U^{\otimes m})^U \otimes U^{\otimes m} \to \mathbb{C},$$

by the formula

$$\langle x, y\rangle = \mathsf{t}_{U^{\otimes m}}(l_x \circ r_y). \tag{8}$$

Here l_x and r_x are the operators of the left and right multiplications by x.

From the cyclicity of t, we obtain an induced pairing

$$\langle\ ,\ \rangle\colon (U^{\otimes m})^U \otimes \mathrm{HH}_0(U^{\otimes m}) \to \mathbb{C},$$

which we call the modified trace pairing. To achieve the full evaluation, besides the basis for the center, we will need a basis for $\mathrm{HH}_0(U)$.

The trace $\mathrm{HH}_0(U)$

Let us construct a basis of the trace of U.

Recall that 0^{th} Hochschild homology or trace of a linear category $\mathcal{C}$ is defined by

$$\mathrm{HH}_0(\mathcal{C}) := \frac{\bigoplus_{x\in\mathcal{C}} \mathrm{End}_{\mathcal{C}}(x)}{fg-gf} \quad \text{for any } f\colon x \to y \text{ and } g\colon y \to x.$$

The image of an endomorphism $f \in \mathrm{End}_{\mathcal{C}}(x)$ in $\mathrm{HH}_0(\mathcal{C})$ will be called its trace class and denoted by $[f]$.

For an algebra A (ie a category with one object) this reduces to

$$\mathrm{HH}_0(A) := \frac{A}{[A,A]} \quad \text{with } [A,A] = \mathrm{Span}\{xy - yx \mid x, y \in A\}.$$

This space supports a natural action of the center defined by $z[x] = [zx]$ for any $z \in Z(A)$. Recall that the dual of $\mathrm{HH}_0(A)$ is isomorphic to $\mathrm{Char}(A)$.

Proposition 8 *The trace $\mathrm{HH}_0(U)$ has dimension $3\mathsf{p}-1$, with basis consisting of*

- $\boldsymbol{h}_k^{\pm}$ *for $1 \le k \le \mathsf{p}$, represented by the minimal (noncentral) idempotent projecting onto a copy of the module $\mathcal{P}_k^{\pm}$, and*
- $\boldsymbol{h}_j = \boldsymbol{w}_j^{+}\boldsymbol{h}_j^{+} = \boldsymbol{w}_j^{-}\boldsymbol{h}_{\mathsf{p}-j}^{-}$ *for $1 \le j \le \mathsf{p}-1$.*

The proof will use the following well-known fact.

Proposition 9 *For any finite-dimensional algebra* A,

$$\mathrm{HH}_0(A\text{–pmod}) \simeq \mathrm{HH}_0(A).$$

Let us give a proof for completeness.

Proof Assume M is projective. Then there exists another projective module N such that $M \oplus N = A^{\oplus n}$ for some n. Hence in A–pmod there are morphisms

$$i\colon M \to A^{\oplus n} \quad \text{and} \quad p\colon A^{\oplus n} \to M$$

with $p \circ i = \mathrm{id}_M$ and $i \circ p$ an idempotent. For $1 \le k \le n$, let us denote by $a_k\colon M \to A$ the composition of i with the k^{th} projection, and by $b_k\colon A \to M$ the composition of the k^{th} inclusion with p. We have

$$\mathrm{id}_M = \sum_{k=1}^{n} b_k \, \mathrm{id}_A \, a_k. \tag{9}$$

This can be used to define a map $\mathrm{HH}_0(A\text{–pmod}) \to \mathrm{HH}_0(A)$ in the following manner: Given $f \in \mathrm{End}_A(M)$ and $M \in A$–pmod, we then associate to f the trace class $\left[\sum_k (a_k f b_k)(1)\right] \in \mathrm{HH}_0(A)$. We can check that this trace class depends neither on the choice giving identity (9) nor on the representative of $[f] \in \mathrm{HH}_0(A\text{–pmod})$. Hence we have a map

$$\Phi\colon \mathrm{HH}_0(A\text{–pmod}) \to \mathrm{HH}_0(A)$$

defined by $\Phi([f]) = \left[\sum_k (a_k f b_k)(1)\right]$. The inverse of this map sends $[x] \in \mathrm{HH}_0(A)$ to the trace class $[r_x] \in \mathrm{HH}_0(A\text{–pmod})$ of the right multiplication $r_x \in \mathrm{End}_A(A)$. Hence, we have an isomorphism. □

Let us recall that U as a free left U–module decomposes into a direct sum of indecomposable projectives as follows:

$$U \cong \bigoplus_{j=1}^{\mathsf{p}} j\mathcal{P}_j^+ \oplus \bigoplus_{j=1}^{\mathsf{p}} j\mathcal{P}_j^-.$$

The module $\mathbb{P} = \bigoplus_{j,\pm} \mathcal{P}_j^{\pm}$ is called the projective generator of U and $B = \mathrm{End}(\mathbb{P})$ the *basic* algebra. Using decomposition of any projective module M into indecomposable ones, we show that the map $B \to \mathrm{HH}_0(U\text{–pmod})$ sending f to its trace class induces a bijection

$$\mathrm{HH}_0(B) \cong \mathrm{HH}_0(U\text{–pmod}).$$

Let us use the notation $\mathrm{id}_j^{\pm} = \mathrm{id}_{\mathcal{P}_j^{\pm}}$ for simplicity.

Lemma 10 *A basis for* $\mathrm{HH}_0(B)$ *is represented by* $[\mathrm{id}_k^{\pm}]$ *for* $1 \le k \le \mathsf{p}$, *along with* $[x_j^+] = [x_{\mathsf{p}-j}^-]$ *for* $1 \le j < \mathsf{p}$.

Proof Recall that

$$B = \bigoplus_{j,\epsilon,j',\epsilon'} \mathrm{Hom}(\mathcal{P}_j^{\epsilon}, \mathcal{P}_{j'}^{\epsilon'}).$$

A linear basis for B, as defined in Section 3, consists of

$$\mathrm{id}_k^{\pm}, \quad x_j^{\epsilon}, \quad a_j^{\epsilon}\colon \mathcal{P}_j^{\epsilon} \to \mathcal{P}_{\mathsf{p}-j}^{-\epsilon} \quad \text{and} \quad b_j^{\epsilon}\colon \mathcal{P}_j^{\epsilon} \to \mathcal{P}_{\mathsf{p}-j}^{-\epsilon},$$

where $1 \le k \le \mathsf{p}$, $1 \le j < \mathsf{p}$ and $\epsilon \in \{-,+\}$. For $1 \le j < \mathsf{p}$, we have $a_j^{\epsilon}\,\mathrm{id}_j^{\epsilon} = a_j^{\epsilon}$, while $\mathrm{id}_j^{\epsilon}\,a_j^{\epsilon} = 0$, so that a_j^{ϵ} and similarly b_j^{ϵ} vanish in $\mathrm{HH}_0(B)$. We also have $b_{\mathsf{p}-j}^{-\epsilon} a_j^{\epsilon} = x_j^{\epsilon}$, while $a_j^{\epsilon} b_{\mathsf{p}-j}^{-\epsilon} = x_{\mathsf{p}-j}^{-\epsilon}$. We get the relation $[x_j^{\epsilon}] = [x_{\mathsf{p}-j}^{-\epsilon}] \in \mathrm{HH}_0(B)$. Since the resulting set of generators has cardinality $3\mathsf{p}-1 = \dim(\mathrm{HH}_0(U))$, this completes the proof of the lemma. □

Proof of Proposition 8 Combining Proposition 9 with Lemma 10, we conclude that $\mathrm{HH}_0(U)$ has dimension $3\mathsf{p}-1$, with basis consisting of

- $\boldsymbol{h}_k^{\pm}$ for $1 \le k \le \mathsf{p}$, represented by the minimal (noncentral) idempotent projecting onto a copy of the module $\mathcal{P}_k^{\pm}$, and
- $\boldsymbol{h}_j = \boldsymbol{w}_j^+ \boldsymbol{h}_j^+ = \boldsymbol{w}_j^- \boldsymbol{h}_{p-j}^-$ for $1 \le j \le \mathsf{p}-1$.

On the regular representation in U–pmod these elements act by the right multiplication. □

5 Proofs of Theorems 1 and 2

In this section we construct the modified trace on U–pmod and compute it for the regular representation U and its tensor powers. This will provide the main tool to prove Theorems 1 and 2.

Modified trace

The subcategory U–pmod is an ideal of U–mod in the sense of [10; 11], that is:

(1) If $V \in U$–pmod and $W \in U$–mod, then $V \otimes W \in U$–pmod and $V \otimes W \in U$–pmod.

(2) If $V \in U$–pmod and $W \in U$–mod and there exist morphisms $f\colon W \to V$ and $g\colon V \to W$ such that $gf = \mathrm{id}_W$, then $W \in U$–pmod.

Let us recall that a *modified trace* on U–pmod is a family of linear functions

$$\{\mathsf{t}_V\colon \mathrm{End}_{\mathscr{C}}(V) \to \mathbb{C}\}_{V \in U\text{–pmod}}$$

such that the following two conditions hold:

Cyclicity If $X, V \in U$–pmod, then for any morphisms $f\colon V \to X$ and $g\colon X \to V$ in U–mod we have

$$\mathsf{t}_V(gf) = \mathsf{t}_X(fg).$$

Partial trace properties If $X \in U$–pmod and $W \in U$–mod then for any morphisms $f \in \mathrm{End}_U(X \otimes W)$ and $g \in \mathrm{End}_U(W \otimes X)$ we have

$$\mathsf{t}_{X\otimes W}(f) = \mathsf{t}_X(\mathrm{tr}_r^W(f)) \quad \text{and} \quad \mathsf{t}_{W\otimes X}(g) = \mathsf{t}_X(\mathrm{tr}_l^W(g)),$$

where tr_r^W and tr_l^W are the right and left partial categorical traces along W defined by (6).

If only the first (resp. the second) of the two partial trace properties is satisfied, we call the modified trace right (resp. left).

Proof of Theorem 1

Corollary 3.2.1 of [10] implies the existence of a unique (up to global scalar) right modified trace in any unimodular pivotal category with enough projectives and a simple projective object L such that $\widetilde{\mathrm{ev}}_L$ is surjective. The category U–pmod does satisfy all these assumptions. Hence there exists a unique right modified trace

$$\{\mathsf{t}^R_V\colon \mathrm{End}(V) \to \mathbb{C}\}_{V \in U\text{–pmod}}$$

normalized by

$$\mathsf{t}^R_{\mathcal{P}^+_{\mathsf{p}}}(\mathrm{id}_{\mathcal{P}^+_{\mathsf{p}}}) = (-1)^{\mathsf{p}-1}.$$

Analogous arguments imply the existence of the unique left trace

$$\{\mathsf{t}^L_V\colon \mathrm{End}(V) \to \mathbb{C}\}_{V \in U\text{–pmod}}$$

normalized by

$$\mathsf{t}^L_{\mathcal{P}^+_{\mathsf{p}}}(\mathrm{id}_{\mathcal{P}^+_{\mathsf{p}}}) = (-1)^{\mathsf{p}-1}.$$

We will compute both and show that they coincide. For this, we will use the additivity of trace functions for direct sums, which follows from the cyclicity. Hence, it is enough to compute modified traces on the endomorphisms of the indecomposable projectives.

Identity endomorphisms Recall $\mathcal{X}_1^+$ is the one-dimensional trivial U–module whose action on any vector v is given by $Ev = Fv = 0$ and $K^{\pm 1}v = v$. There is another one-dimensional module $\mathcal{X}_1^-$ whose action on any vector v is given by $Ev = Fv = 0$ and $K^{\pm 1}v = -v$. Hence, $\mathrm{tr}_r(\mathrm{id}_{\mathcal{X}_1^-}) = \mathrm{tr}^{\mathcal{X}_1^-}(K^{p+1}) = (-1)^{p+1}$. Using $\mathcal{P}_j^- \cong \mathcal{P}_j^+ \otimes \mathcal{X}_1^-$, we can compute the trace on $\mathcal{P}_{\mathsf{p}}^-$:

$$\mathsf{t}^{\boldsymbol{R}}_{\mathcal{P}_{\mathsf{p}}^-}(\mathrm{id}_{\mathcal{P}_{\mathsf{p}}^-}) = \mathsf{t}^{\boldsymbol{R}}_{\mathcal{P}_{\mathsf{p}}^+\otimes\mathcal{X}_1^-}(\mathrm{id}_{\mathcal{P}_{\mathsf{p}}^+\otimes\mathcal{X}_1^-}) = \mathsf{t}^{\boldsymbol{R}}_{\mathcal{P}_{\mathsf{p}}^+}(\mathrm{tr}_{\boldsymbol{r}}^{\mathcal{X}_1^-}(\mathrm{id}_{\mathcal{P}_{\mathsf{p}}^+\otimes\mathcal{X}_1^-})) = 1.$$

Similarly, $\mathcal{P}_{\mathsf{p}}^- \cong \mathcal{X}_1^- \otimes \mathcal{P}_p^+$ implies that $\mathsf{t}^{\boldsymbol{L}}_{\mathcal{P}_{\mathsf{p}}^-}(\mathrm{id}_{\mathcal{P}_{\mathsf{p}}^-}) = 1$.

Next we compute some partial traces involving $\mathcal{X}_2^+$. The module $\mathcal{X}_2^+$ has a basis $\{w_0, w_1\}$ with action

$$\begin{aligned} Ew_0 &= 0, & Fw_0 &= w_1, & Kw_0 &= qw_0, \\ Ew_1 &= w_0, & Fw_1 &= w_0, & Kw_1 &= q^{-1}w_1. \end{aligned}$$

Recall that an endomorphism of $\mathcal{P}_j^+$ is determined by the image of the dominant vector $\boldsymbol{b}_0^+$. We can use this fact to compute the following partial trace:

$$(\mathrm{tr}_{\boldsymbol{l}}^{\mathcal{X}_2^+}(\mathrm{id}_{\mathcal{X}_2^+\otimes\mathcal{P}_j^+}))(\boldsymbol{b}_0^+) = w_0^*(K^{\mathsf{p}+1}w_0)\boldsymbol{b}_0^+ + w_1^*(K^{\mathsf{p}+1}w_1)\boldsymbol{b}_0^+ = (q^{\mathsf{p}+1}+q^{-\mathsf{p}-1})\boldsymbol{b}_0^+.$$

Thus, $\mathrm{tr}_{\boldsymbol{l}}^{\mathcal{X}_2^+}(\mathrm{id}_{\mathcal{X}_2^+\otimes\mathcal{P}_j^+}) = -(q+q^{-1})\,\mathrm{id}_{\mathcal{P}_j^+}$ for $j = 1, \dots, \mathsf{p}-1$. Similarly, for the right partial trace of the identity of $\mathcal{P}_j^+ \otimes \mathcal{X}_2^+$ we get

$$\mathrm{tr}_{\boldsymbol{r}}^{\mathcal{X}_2^+}(\mathrm{id}_{\mathcal{P}_j^+\otimes\mathcal{X}_2^+}) = (-q-q^{-1})\,\mathrm{id}_{\mathcal{P}_j^+}.$$

The decomposition of tensor products of a simple module with a projective indecomposable module is given in [23, Proposition 4.1] (also see [16, Theorems 3.1.5 and 3.2.1]). In particular,

$$\begin{aligned}
&(10) & \mathcal{P}_{\mathsf{p}}^\pm \otimes \mathcal{X}_2^+ &\cong \mathcal{P}_{p-1}^\pm \cong \mathcal{X}_2^+ \otimes \mathcal{P}_{\mathsf{p}}^\pm, \\
&(11) & \mathcal{P}_{\mathsf{p}-1}^\pm \otimes \mathcal{X}_2^+ &\cong \mathcal{P}_{\mathsf{p}-2}^\pm \oplus 2\mathcal{P}_{\mathsf{p}}^\pm \cong \mathcal{X}_2^+ \otimes \mathcal{P}_{\mathsf{p}-1}^\pm, \\
&(12) & \mathcal{P}_j^\pm \otimes \mathcal{X}_2^+ &\cong \mathcal{P}_{j-1}^\pm \oplus \mathcal{P}_{j+1}^\pm \cong \mathcal{X}_2^+ \otimes \mathcal{P}_j^\pm \quad \text{for } j \in \{2, \dots, \mathsf{p}-2\}.
\end{aligned}$$

Combining these formulas with properties of the modified trace we have

$$\mathsf{t}^{\boldsymbol{R}}_{\mathcal{P}_j^\pm\otimes\mathcal{X}_2^+}(\mathrm{id}_{\mathcal{P}_j^+\otimes\mathcal{X}_2^+}) = \mathsf{t}^{\boldsymbol{R}}_{\mathcal{P}_j^\pm}(\mathrm{tr}_{\boldsymbol{r}}^{\mathcal{X}_2^+}(\mathrm{id}_{\mathcal{P}_j^+\otimes\mathcal{X}_2^+})) = (-q-q^{-1})\mathsf{t}^{\boldsymbol{R}}_{\mathcal{P}_j^\pm}(\mathrm{id}_{\mathcal{P}_j^+}),$$

where $j \in \{2, \dots, \mathsf{p}\}$. This equality together with the isomorphism on the left-hand side of (10) for $j = \mathsf{p}$ gives

$$\mathsf{t}^{\boldsymbol{R}}_{\mathcal{P}_{\mathsf{p}-1}^\pm}(\mathrm{id}_{\mathcal{P}_{\mathsf{p}-1}^\pm}) = (\mp 1)^{\mathsf{p}-1}(-q-q^{-1}).$$

Then the isomorphism of the left-hand side of (11) implies

$$\mathrm{t}^{R}_{\mathcal{P}^{\pm}_{\mathsf{p}-2}}(\mathrm{id}_{\mathcal{P}^{\pm}_{\mathsf{p}-2}}) = (\mp 1)^{\mathsf{p}-1}((-q-q^{-1})^2-2) = (\mp 1)^{\mathsf{p}-1}(q^2+q^{-2}).$$

Finally, for $j \in \{2,\dots,\mathsf{p}-2\}$, the isomorphism of the left-hand side of (12) implies

$$-(q+q^{-1})\mathrm{t}^{R}_{\mathcal{P}^{\pm}_{j}}(\mathrm{id}_{\mathcal{P}^{\pm}_{j}}) = \mathrm{t}^{R}_{\mathcal{P}^{\pm}_{j-1}}(\mathrm{id}_{\mathcal{P}^{\pm}_{j-1}}) + \mathrm{t}^{R}_{\mathcal{P}^{\pm}_{j+1}}(\mathrm{id}_{\mathcal{P}^{\pm}_{j+1}}). \tag{13}$$

Recursively the last equality implies, for $j \in \{2,\dots,\mathsf{p}-1\}$,

$$\mathrm{t}^{R}_{\mathcal{P}^{\pm}_{j}}(\mathrm{id}_{\mathcal{P}^{\pm}_{j}}) = (\pm 1)^{p-1}(-1)^j(q^j+q^{-j}). \tag{14}$$

Using the right-hand side of the tensor product in each of (10), (11) and (12) we compute similarly the left trace of the identities and get

$$\mathrm{t}^{L}_{\mathcal{P}^{\pm}_{j}}(\mathrm{id}_{\mathcal{P}^{\pm}_{j}}) = \mathrm{t}^{R}_{\mathcal{P}^{\pm}_{j}}(\mathrm{id}_{\mathcal{P}^{\pm}_{j}}) \quad \text{for } 1 \le j \le \mathsf{p}.$$

Endomorphisms $\boldsymbol{x}^{\pm}_{\boldsymbol{j}}$ For $x \in U$, let us denote by l^V_x the operator of the left multiplication by x on V. Sometimes, we will omit V for simplicity. To compute $\mathrm{t}^{R}_{\mathcal{P}^{\pm}_{j}}(x^{\pm}_{j})$ we will use the action of the Casimir element

$$C = FE + \frac{qK+q^{-1}K^{-1}}{(q-q^{-1})^2}.$$

Since C is central, the action of C commutes with the left U–action, and hence defines an intertwiner $l^{\mathcal{P}}_C \in \mathrm{End}_U(\mathcal{P})$.

On simple modules, C acts by a scalar, hence

$$\mathrm{t}^{R}_{\mathcal{P}^{\pm}_{\mathsf{p}}}(l_C^{\mathcal{P}^{\pm}_{\mathsf{p}}}) = \frac{2(\mp 1)^{\mathsf{p}}}{(q-q^{-1})^2}. \tag{15}$$

For $j \in \{1,\dots,\mathsf{p}-1\}$, the dominant vector $\boldsymbol{b}^{+}_{0}$ (resp. $\boldsymbol{y}^{-}_{0}$) of $\mathcal{P}^{+}_{j}$ (resp. $\mathcal{P}^{-}_{j}$) has weight $\pm q^{j-1}$. The action of C on this vector is

$$C\boldsymbol{b}^{+}_{0} = \boldsymbol{a}^{+}_{0} + \frac{q^j+q^{-j}}{(q-q^{-1})^2}\boldsymbol{b}^{+}_{0} \quad \left(\text{resp. } C\boldsymbol{y}^{-}_{0} = \boldsymbol{x}^{-}_{0} - \frac{q^j+q^{-j}}{(q-q^{-1})^2}\boldsymbol{y}^{-}_{0}\right).$$

Thus, for $j \in \{1,\dots,\mathsf{p}-1\}$ we have

$$l_C^{\mathcal{P}^{\pm}_{j}} = x^{\pm}_{j} \pm \frac{q^j+q^{-j}}{(q-q^{-1})^2}\,\mathrm{id}_{\mathcal{P}^{\pm}_{j}}. \tag{16}$$

To compute the action of C on tensor products we need the formula

$$\Delta(C) = K^{-1}\otimes FE + K^{-1}E\otimes FK + F\otimes E + FE\otimes K + \frac{qK\otimes K+q^{-1}K^{-1}\otimes K^{-1}}{(q-q^{-1})^2}.$$

The second and third terms of $\Delta(C)$ have no diagonal contribution, and hence vanish when computing the partial trace $\operatorname{tr}_r^{\mathcal{X}_2^+}\big(l_{\Delta(C)}^{\mathcal{P}_i^{\pm}\otimes\mathcal{X}_2^+}\big)$. In particular, for $v\in\mathcal{P}_i^{\pm}$ we have

$$\big[\operatorname{tr}_r^{\mathcal{X}_2^+}\big(l_{\Delta(C)}^{\mathcal{P}_i^{\pm}\otimes\mathcal{X}_2^+}\big)\big](v)\\ =-q^{-1}K^{-1}v-(q^2+q^{-2})FEv-\frac{(q^2+q^{-2})q}{(q-q^{-1})^2}Kv+\frac{-2q^{-1}}{(q-q^{-1})^2}K^{-1}v.$$

When $i\in\{1,\dots,\mathsf{p}-1\}$ and $v=\boldsymbol{b}_0^+\in\mathcal{P}_i^+$ or $v=\boldsymbol{y}_0^-\in\mathcal{P}_i^-$ is a generating dominant vector of $\mathcal{P}_i^{\pm}$ this equality implies

$$\operatorname{tr}_r^{\mathcal{X}_2^+}\big(l_{\Delta(C)}^{\mathcal{P}_i^{\pm}\otimes\mathcal{X}_2^+}\big)=-(q^2+q^{-2})x_i^{\pm}\mp\frac{(q^2+q^{-2})}{(q-q^{-1})^2}(q^i+q^{-i})\operatorname{id}_{\mathcal{P}_i^{\pm}}.$$

Similarly, if v is the highest weight of $\mathcal{P}_{\mathsf{p}}^{\pm}$, we obtain

$$\operatorname{tr}_r^{\mathcal{X}_2^+}\big(l_{\Delta(C)}^{\mathcal{P}_{\mathsf{p}}^{\pm}\otimes\mathcal{X}_2^+}\big)=\pm2\frac{(q^2+q^{-2})}{(q-q^{-1})^2}\operatorname{id}_{\mathcal{P}_{\mathsf{p}}^{\pm}}. \tag{17}$$

Now we can compute $\mathsf{t}^{\boldsymbol{R}}_{\mathcal{P}_{\mathsf{p}-1}^{\pm}}\big(x_{\mathsf{p}-1}^{\pm}\big)$. From the isomorphism in (10), we have

$$\mathsf{t}^{\boldsymbol{R}}_{\mathcal{P}_{\mathsf{p}}^{\pm}\otimes\mathcal{X}_2^+}\big(l_{\Delta(C)}^{\mathcal{P}_{\mathsf{p}}^{\pm}\otimes\mathcal{X}_2^+}\big)=\mathsf{t}^{\boldsymbol{R}}_{\mathcal{P}_{\mathsf{p}-1}^{\pm}}\big(l_C^{\mathcal{P}_{\mathsf{p}-1}^{\pm}}\big).$$

Using (16) and (17) we can simplify the last equality as follows:

$$\pm2\frac{q^2+q^{-2}}{(q-q^{-1})^2}\mathsf{t}^{\boldsymbol{R}}_{\mathcal{P}_{\mathsf{p}}^{\pm}}\big(\operatorname{id}^{\pm}_{\mathcal{P}_{\mathsf{p}}^{\pm}}\big)=\mathsf{t}^{\boldsymbol{R}}_{\mathcal{P}_{\mathsf{p}-1}^{\pm}}\big(x_{\mathsf{p}-1}^{\pm}\big)\pm\frac{q^{\mathsf{p}-1}+q^{-\mathsf{p}+1}}{(q-q^{-1})^2}\mathsf{t}^{\boldsymbol{R}}_{\mathcal{P}_{\mathsf{p}-1}^{\pm}}\big(\operatorname{id}^{\pm}_{\mathcal{P}_{\mathsf{p}-1}^{\pm}}\big),$$

or

$$\mathsf{t}^{\boldsymbol{R}}_{\mathcal{P}_{\mathsf{p}-1}^{\pm}}\big(x_{\mathsf{p}-1}^{\pm}\big)=\pm(\mp)^{\mathsf{p}-1}\frac{(q^{\mathsf{p}-1}-q^{-\mathsf{p}+1})^2}{(q-q^{-1})^2}.$$

Using the isomorphism in (12), for $j\in\{2,\dots,\mathsf{p}-2\}$ we obtain the recursive relation

$$\mathsf{t}^{\boldsymbol{R}}_{\mathcal{P}_{j-1}^{\pm}}\big(x_{j-1}^{\pm}\big)+\mathsf{t}^{\boldsymbol{R}}_{\mathcal{P}_{j+1}^{\pm}}\big(x_{j+1}^{\pm}\big)+(q^2+q^{-2})\mathsf{t}^{\boldsymbol{R}}_{\mathcal{P}_j^{\pm}}\big(x_j^{\pm}\big)=-2(\pm1)^{\mathsf{p}}(-1)^j. \tag{18}$$

Using (11) we can show that this formula also holds for $j=\mathsf{p}-1$ by setting $x_{\mathsf{p}}^{\pm}=0$. We deduce the general formula for $j\in\{1,\dots,\mathsf{p}-1\}$,

$$\mathsf{t}^{\boldsymbol{R}}_{\mathcal{P}_j^{\pm}}\big(x_j^{\pm}\big)=(\pm1)^{\mathsf{p}}(-1)^j[j]^2,$$

which is compatible with the computation at $j=\mathsf{p}-2$ or $j=\mathsf{p}-1$ and satisfies the recursive relation for $j\in\{2,\dots,\mathsf{p}-2\}$.

With a similar computation we get the same value for $\mathsf{t}^{L}_{\mathcal{P}_j^{\pm}}(x_j^{\pm})$. Thus, we have proved that the left and right modified traces are equal on U–pmod. Let us summarize our computations of the modified trace $(0 < j < \mathsf{p})$:

$\mathrm{id}_{\mathcal{P}_{\mathsf{p}}^-}$	$\mathrm{id}_{\mathcal{P}_{\mathsf{p}}^+}$	$\mathrm{id}_{\mathcal{P}_j^-}$	$\mathrm{id}_{\mathcal{P}_j^+}$	x_j^-	x_j^+
1	$(-1)^{\mathsf{p}-1}$	$(-1)^{\mathsf{p}+j-1}(q^j+q^{-j})$	$(-1)^j(q^j+q^{-j})$	$(-1)^{\mathsf{p}+j}[j]^2$	$(-1)^j[j]^2$

Proof of Theorem 2

Let us apply the modified trace construction to the regular representation $U \in U$–pmod. Using the isomorphism of algebras

$$r\colon U^{\mathrm{op}} \cong \mathrm{End}_U(U), \quad x \mapsto r_x,$$

where $r_x(y) = yx$ is the right multiplication, we define

$$\mathsf{T}\colon U \to \mathbb{C} \quad \text{by} \ \ \mathsf{T}(x) = \mathsf{t}_U(r_x).$$

By Theorem 1, the linear map T is a character and satisfies the partial trace property.

We can now show by direct computation that the pairing

$$Z(U) \times \mathrm{HH}_0(U) \to \mathbb{C} \quad \text{given by} \ \ (z, x) \mapsto \mathsf{T}(zx)$$

is nondegenerate. Using Proposition 5 and Theorem 1 we can explicitly compute this pairing in the basis of the center and the trace. For example

$$\mathsf{T}(\boldsymbol{w}_j^+\boldsymbol{h}_j^+) = \mathsf{t}^{R}_{\mathcal{P}_j^+}(x_j^+) = (-1)^j[j]^2 \quad \text{or} \quad \mathsf{T}(\boldsymbol{e}_0\boldsymbol{h}_{\mathsf{p}}^-) = \mathsf{t}^{R}_{\mathcal{P}_{\mathsf{p}}^-}(\mathrm{id}_{\mathcal{P}_{\mathsf{p}}^-}) = 1.$$

Completing the computation, we obtain the pairing shown in Table 1. From the table it is easy to see the pairing is nondegenerate.

	$\boldsymbol{h}_{\mathsf{p}}^+$	$\boldsymbol{h}_{\mathsf{p}}^-$	$\boldsymbol{h}_s$	$\boldsymbol{h}_s^+$	$\boldsymbol{h}_{\mathsf{p}-s}^-$
$\boldsymbol{e}_{\mathsf{p}}$	$(-1)^{\mathsf{p}-1}$	0	0	0	0
$\boldsymbol{e}_0$	0	1	0	0	0
$\boldsymbol{e}_j$	0	0	$(-1)^j[j]^2$	$(-1)^j(q^j+q^{-j})$	$(-1)^j(q^j+q^{-j})$
$\boldsymbol{w}_j^+$	0	0	0	$(-1)^j[j]^2$	0
$\boldsymbol{w}_j^-$	0	0	0	0	$(-1)^j[j]^2$

Table 1: Values of the pairing on a basis, where $1 \le j < \mathsf{p}$

6 Proof of Theorem 3

In this section we will define our logarithmic 3–manifold invariant $\mathrm{H}^{\log}(M,L)$, prove Theorem 3 and compare $\mathrm{H}^{\log}(M,L)$ with the invariant defined by Murakami in [20].

Logarithmic invariant

Assume we are given an oriented framed link (L^+,L^-) with (m_+,m_-) components inside a 3–manifold $M=S^3(L^0)$, where $L^0\subset S^3$ is a surgery oriented framed link for M with m_0 components. Here m_+ or m_- can be zero. We suppose that (L^+,L^-) is in $S^3\setminus L^0$, and choose an upwards oriented string link $T=(T^+,T^0,T^-)$ whose closure is (L^+,L^0,L^-). By Lemma 6, the universal invariant satisfies

$$J_T\in (U^{\otimes(m_++m_0+m_-)})^U.$$

Let us color the components of L^+ by central elements $z_j\in Z(U)$, where $1\le j\le m_+$, and L^- by trace classes $h_k\in \mathrm{HH}_0(U)$, where $1\le k\le m_-$, and write $z^+=\bigotimes_j z_j$ and $h^-=\bigotimes_k h_k$. Let us denote by L the resulting colored link $((L^+,z^+),(L^-,h^-))$. For each j, the central element z_j acting on the right integral μ defines the quantum character $\mu_{z_j}\colon x\mapsto \mu(z_jx)$. We denote the tensor product $\bigotimes_j\mu_{z_j}$ by $z^+\mu^{\otimes m_+}$. Using the map $\mathrm{HH}_0(U)^{\otimes m_-}\to \mathrm{HH}_0(U^{\otimes m_-})$ induced by $\mathrm{id}_{U^{\otimes m_-}}$, the tensor product h^- gives an element in $\mathrm{HH}_0(U^{\otimes m_-})$ which we also denote by h^-. We obtain

$$\mathrm{H}^{\log}(M,L):=\delta^s\langle (z^+\mu^{\otimes m_+}\otimes\mu^{\otimes m_0}\otimes \mathrm{id})(J_T),\, h^-\rangle$$

by evaluating components of J_T corresponding to (L^+,L^0) with the right integral μ twisted with central elements and by applying to the result the modified trace pairing. Here s is the signature of the linking matrix for L^0. Theorem 3 claims that $\mathrm{H}^{\log}(M,L)$ is a topological invariant of the pair (M,L).

Proof of Theorem 3 When L^- is empty the formula recovers the Hennings invariant. We further assume $m_->0$. We first show that $\mathrm{H}^{\log}(M,L)$ is an invariant of the colored link L, ie it does not depend on the choice of T. By applying Lemma 7 we see that the partial closure $(z^+\mu^{\otimes m_+}\otimes\mu^{\otimes m_0}\otimes\mathrm{id})(J_T)$ is invariant of the tangle T_1 obtained by closing the first m_++m_0 components of T.

We may continue the procedure as follows. To an element $h\in U$ we associate the quantum character χ_h defined by

$$\chi_h(x)=\mathrm{tr}_l^H(l_xr_h)\quad\text{for any } x\in U.$$

This character χ_h only depends on the trace class of h in $\mathrm{HH}_0(U)$. Hence to each trace class h_k we associate a quantum character χ_{h_k}. If $m_- > 1$, using the partial trace property of the modified trace, we have the formula

$$\mathrm{H}^{\log}(M,L) = \delta^s \left\langle \left(z^+ \mu^{\otimes m_+} \otimes \mu^{\otimes m_0} \otimes \left(\bigotimes_{k=1}^{m_- -1} \chi_{h_k} \right) \otimes \mathrm{id} \right)(J_T),\, h_{m_-} \right\rangle \tag{19}$$

Now from Lemma 7 we get that the central element

$$J_K = \left(z^+ \mu^{\otimes m_+} \otimes \mu^{\otimes m_0} \otimes \left(\bigotimes_{k=1}^{m_- -1} \chi_{h_k} \right) \otimes \mathrm{id} \right)(J_T)$$

is an invariant of the $(1,1)$–tangle K obtained by closing all components but the last one. The main point is that it does not matter which $m_- - 1$ components we choose! Hence $\mathrm{H}^{\log}(M,L)$ is invariant of the $(1,1)$–tangle K obtained from T_1 by closing all but one trace-class-colored component.

But the resulting central element J_K does not depend on the point where we cut L into K. The proof mimics the argument showing that long knots and knots in S^3 are equivalent.

Indeed, let us think about our $(1,1)$–tangle K as being a long knot. We need to show that our central element does not change if we move an arc through "infinity" (or the cutting disc). This move can be alternatively realized by moving the arc in the opposite way through the long knot, which is just a sequence of Reidemeister moves, under which we know J_K to be stable.

It remains to show invariance under Kirby moves: sliding along a component of L^0 and stabilization with a ± 1 framed unknot. The defining property of the right integral ensures the sliding invariance (see eg [15]). Note that if we change the orientation on one of the T^0 components, this will change J_T by applying S at the corresponding position, but now $\mu \circ S$ is a left integral, hence the sliding property holds after rearranging the components.

Adding to L^0 a ± 1 framed unknot multiplies $\mathrm{H}^{\log}(M,L)$ by $\mu(v^{\mp}) = \delta^{\mp}$, and changes the signature s by ± 1, so $\mathrm{H}^{\log}(M,L)$ remains the same. □

Relation with Murakami invariants

Here we show that the logarithmic invariant in [20] is a special case of $\mathrm{H}^{\log}(M,L)$ where L^- has precisely one component.

Murakami's invariant is defined for a knot and a colored link in a 3–manifold. We will adapt our notation to his setting. Let us consider a link (L^+, L^-) in a 3–manifold $M = S^3(L_0)$ as before, but with $L_- = K$ a knot and $m_- = 1$. The link L^+ is colored by z^+ as before. The logarithmic knot invariant defined by Murakami is then

$$\mathrm{J}^{\log}(M, (L^+, z^+), K) = \delta^s(z^+\mu^{\otimes m_+} \otimes \mu^{\otimes m_0} \otimes \mathrm{id})(J_T) \in Z(U).$$

Due to different conventions in the definition of J_T, Murakami's original invariant corresponds to the mirror image of the link in our notation.

Murakami further expands his invariant in the basis of the center as follows:

$$\mathrm{J}^{\log}(M, (L^+, z^+), K) = \sum_{j=0}^{p} a_j \boldsymbol{e}_j + \sum_{j=1}^{p-1} b_j^+ \boldsymbol{w}_j^+ + \sum_{j=1}^{p-1} b_j^- \boldsymbol{w}_j^-.$$

The coefficients are clearly topological invariants of the triple $(M, (L^+, z^+), K)$.

Proposition 11 *With the above notation, we have, for* $1 \le j < \mathsf{p}$,

$$\begin{aligned}
a_0 &= \mathrm{H}^{\log}(M, (L^+, z^+), (L^-, \boldsymbol{h}_{\mathsf{p}}^-)),\\
a_{\mathsf{p}} &= (-1)^{\mathsf{p}-1}\mathrm{H}^{\log}(M, (L^+, z^+), (L^-, \boldsymbol{h}_{\mathsf{p}}^+)),\\
a_j &= \frac{(-1)^j}{[j]^2}\mathrm{H}^{\log}(M, (L^+, z^+), (L^-, \boldsymbol{h}_j)),\\
b_j^+ &= \frac{(-1)^j}{[j]^4}\mathrm{H}^{\log}\big(M, (L^+, z^+), (L^-, [j]^2\boldsymbol{h}_j^+ - (q^j + q^{-j})\boldsymbol{h}_j)\big),\\
b_j^- &= \frac{(-1)^j}{[j]^4}\mathrm{H}^{\log}\big(M, (L^+, z^+), (L^-, [j]^2\boldsymbol{h}_{\mathsf{p}-j}^- - (q^j + q^{-j})\boldsymbol{h}_j)\big).
\end{aligned}$$

Proof For any trace class $h \in \mathrm{HH}_0(U)$ we have

$$\mathrm{H}^{\log}(M, (L^+, z^+), (L^-, h)) = \langle \mathrm{J}^{\log}(M, (L^+, z^+), L^-), h\rangle.$$

From Table 1 we get

$$\begin{aligned}
\mathrm{H}^{\log}(M, (L^+, z^+), (L^-, \boldsymbol{h}_{\mathsf{p}}^-)) &= a_0,\\
\mathrm{H}^{\log}(M, (L^+, z^+), (L^-, \boldsymbol{h}_{\mathsf{p}}^+)) &= (-1)^{\mathsf{p}-1}a_{\mathsf{p}},\\
\mathrm{H}^{\log}(M, (L^+, z^+), (L^-, \boldsymbol{h}_j)) &= (-1)^j[j]^2 a_j,\\
\mathrm{H}^{\log}(M, (L^+, z^+), (L^-, \boldsymbol{h}_j^+)) &= (-1)^j(q^j + q^{-j})a_j + (-1)^j[j]^2 b_j^+,\\
\mathrm{H}^{\log}(M, (L^+, z^+), (L^-, \boldsymbol{h}_{\mathsf{p}-j}^-)) &= (-1)^j(q^j + q^{-j})a_j + (-1)^j[j]^2 b_j^-.
\end{aligned}$$

The claim follows. □

To summarize, Murakami's invariant is an invariant of a link in a 3–manifold together with a choice of one *preferred* component of the link (the one which is not evaluated). Our approach provides a tool to see why this choice is actually irrelevant.

Indeed, to each trace class h we have an associated quantum character χ_h, defined by

$$\chi_h(x) = \mathrm{tr}_l^U(l_x r_h) \quad \text{for any } x \in U.$$

On the other hand, using the Radford isomorphism between $\mathrm{qChar}(U)$ and $Z(U)$ we obtain $z^+ \in Z(U)$ such that $z^+\mu = \chi_h$.

From the proof of Theorem 3 we get the following statement, which allows us to move all components but an *arbitrary* one from the negative to the positive part of the link.

Proposition 12 *Let $L = (L^+, L^-) \subset M$ be a colored link as above, with more than one component in L^-. Assume that L' is obtained from L by moving an h^-–colored component from L^- to L^+ and assigning the color z^+ to it so that $z^+\mu = \chi_{h^-}$. Then*

$$\mathrm{H}^{\log}(M, L) = \mathrm{H}^{\log}(M, L').$$

Appendix Indecomposable projective modules

We reproduce here the defining relations for the modules $\mathcal{P}_j^+$ and $\mathcal{P}_{\mathsf{p}-j}^-$, where $1 \le j \le \mathsf{p}-1$, as given in [8, Section C.2]. The module $\mathcal{P}_j^\pm$, where $1 \le j \le \mathsf{p}-1$, is the indecomposable projective module whose irreducible quotient is given by $\mathscr{X}_j^\pm$.

Let j be an integer such that $1 \le j \le \mathsf{p}-1$. The indecomposable projective module $\mathcal{P}_j^+$ has the basis

$$\{\boldsymbol{x}_k^+, \boldsymbol{y}_k^+\}_{0 \le k \le \mathsf{p}-j-1} \cup \{\boldsymbol{a}_n^+, \boldsymbol{b}_n^+\}_{0 \le n \le j-1},$$

with the U–action given by

$$K\boldsymbol{x}_k^+ = -q^{\mathsf{p}-j-1-2k}\boldsymbol{x}_k^+, \quad K\boldsymbol{y}_k^+ = -q^{\mathsf{p}-j-1-2k}\boldsymbol{y}_k^+ \qquad \text{for } 0 \le k \le \mathsf{p}-j-1,$$

$$K\boldsymbol{a}_n^+ = q^{j-1-2n}\boldsymbol{a}_n^+, \qquad K\boldsymbol{b}_n^+ = q^{j-1-2n}\boldsymbol{b}_n^+ \qquad \text{for } 0 \le n \le j-1,$$

$$E\boldsymbol{x}_k^+ = -[k][\mathsf{p}-j-k]\boldsymbol{x}_{k-1}^+ \quad \text{for } 0 \le k \le \mathsf{p}-j-1 \quad (\text{where } \boldsymbol{x}_{-1}^+ = 0),$$

$$E\boldsymbol{y}_k^+ = \begin{cases} -[k][\mathsf{p}-j-k]\boldsymbol{y}_{k-1}^+ & \text{for } 1 \le k \le \mathsf{p}-j-1, \\ \boldsymbol{a}_{j-1}^+ & \text{for } k = 0, \end{cases}$$

$$E\boldsymbol{a}_n^+ = [n][j-n]\boldsymbol{a}_{n-1}^+ \quad \text{for } 0 \le n \le j-1 \quad (\text{where } \boldsymbol{a}_{-1}^+ = 0),$$

$$E\boldsymbol{b}_n^+ = \begin{cases} [n][j-n]\boldsymbol{b}_{n-1}^+ + \boldsymbol{a}_{n-1}^+ & \text{for } 1 \le n \le j-1, \\ \boldsymbol{x}_{\mathsf{p}-j-1}^+ & \text{for } n = 0, \end{cases}$$

$$F\boldsymbol{x}_k^+ = \begin{cases} \boldsymbol{x}_{k+1}^+ & \text{for } 0 \le k \le \mathsf{p} - j - 2, \\ \boldsymbol{a}_0^+ & \text{for } k = \mathsf{p} - j - 1, \end{cases}$$
$$F\boldsymbol{y}_k^+ = \boldsymbol{a}_{k+1}^+ \quad \text{for } 0 \le k \le \mathsf{p} - j - 1 \quad (\text{where } \boldsymbol{y}_{\mathsf{p}-j}^+ = 0),$$
$$F\boldsymbol{a}_n^+ = \boldsymbol{a}_{n+1}^+ \quad \text{for } 0 \le n \le j - 1 \quad (\text{where } \boldsymbol{a}_j^+ = 0),$$
$$F\boldsymbol{b}_n^+ = \begin{cases} \boldsymbol{b}_{n+1}^+ & \text{for } 0 \le n \le j - 2, \\ \boldsymbol{y}_0^+ & \text{for } n = j - 1. \end{cases}$$

Let j be an integer such that $1 \le j \le \mathsf{p}-1$. The indecomposable projective module $\mathcal{P}_{\mathsf{p}-j}^-$ has the basis

$$\{\boldsymbol{x}_k^-, \boldsymbol{y}_k^-\}_{0 \le k \le \mathsf{p}-j-1} \cup \{\boldsymbol{a}_n^-, \boldsymbol{b}_n^-\}_{0 \le n \le j-1},$$

with the U–action given by

$$K\boldsymbol{x}_k^- = -q^{\mathsf{p}-j-1-2k}\boldsymbol{x}_k^-, \quad K\boldsymbol{y}_k^- = -q^{\mathsf{p}-j-1-2k}\boldsymbol{y}_k^-, \qquad \text{for } 0 \le k \le \mathsf{p} - j - 1,$$
$$K\boldsymbol{a}_n^- = q^{j-1-2n}\boldsymbol{a}_n^-, \qquad K\boldsymbol{b}_n^- = q^{j-1-2n}\boldsymbol{b}_n^-, \qquad \text{for } 0 \le n \le j - 1,$$
$$E\boldsymbol{x}_k^- = -[k][\mathsf{p} - j - k]\boldsymbol{x}_{k-1}^- \quad \text{for } 0 \le k \le \mathsf{p} - j - 1 \quad (\text{where } \boldsymbol{x}_{-1}^- = 0),$$
$$E\boldsymbol{y}_k^- = \begin{cases} -[k][\mathsf{p} - j - k]\boldsymbol{y}_{k-1}^- + \boldsymbol{x}_{k-1}^- & \text{for } 1 \le k \le \mathsf{p} - j - 1, \\ \boldsymbol{a}_{j-1}^- & \text{for } k = 0, \end{cases}$$
$$E\boldsymbol{a}_n^- = [n][j - n]\boldsymbol{a}_{n-1}^- \quad \text{for } 0 \le n \le j - 1 \quad (\text{where } \boldsymbol{a}_{-1}^- = 0),$$
$$E\boldsymbol{b}_n^- = \begin{cases} [n][j - n]\boldsymbol{b}_{n-1}^- & \text{for } 1 \le n \le j - 1, \\ \boldsymbol{x}_{\mathsf{p}-j-1}^- & \text{for } n = 0, \end{cases}$$
$$F\boldsymbol{x}_k^- = \boldsymbol{x}_{k+1}^- \quad \text{for } 0 \le k \le \mathsf{p} - j - 1 \quad (\text{where } \boldsymbol{x}_{\mathsf{p}-j}^- = 0),$$
$$F\boldsymbol{y}_k^- = \begin{cases} \boldsymbol{y}_{k+1}^- & \text{for } 0 \le k \le \mathsf{p} - j - 2, \\ \boldsymbol{b}_0^- & \text{for } k = \mathsf{p} - j - 1, \end{cases}$$
$$F\boldsymbol{a}_n^- = \begin{cases} \boldsymbol{a}_{n+1}^- & \text{for } 0 \le n \le j - 2, \\ \boldsymbol{x}_0^- & \text{for } n = j - 1, \end{cases}$$
$$F\boldsymbol{b}_n^- = \boldsymbol{b}_{n+1}^- \quad \text{for } 0 \le n \le j - 1 \quad (\text{where } \boldsymbol{b}_j^- = 0).$$

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