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Nonarithmetic hyperbolic manifolds and trace rings

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We give a sufficient condition on the hyperplanes used in the Belolipetsky–Thomson inbreeding construction to obtain nonarithmetic manifolds. We explicitly construct infinitely many examples of such manifolds that are pairwise noncommensurable and estimate their volume.

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Let M be a finite-volume hyperbolic n–manifold, with $n \geq 2$. If M is complete, it can be written as a quotient $\Gamma\backslash\mathbb{H}^n$ for Γ a torsion-free lattice in the semisimple Lie group $\mathrm{PO}(n,1) \cong \mathrm{Isom}(\mathbb{H}^n)$, the group of isometries of the hyperbolic n–space $\mathbb{H}^n$. The lattice Γ is then uniform (ie cocompact) if and only if M is compact.

A standard way to construct hyperbolic manifolds in higher dimensions is via arithmetic lattices. In Lie groups with rank at least 2, this is actually the only possible construction (by Margulis' arithmeticity theorem); yet it is known that there are nonarithmetic lattices in $\mathrm{PO}(n,1)$ for every $n \geq 2$. Many examples in low dimensions were constructed using Coxeter groups, notably by Vinberg [16], but the first construction in arbitrary dimension was given by Gromov and Piatetski-Shapiro [7]. Roughly, their idea consists in constructing two pieces of noncommensurable arithmetic manifolds with isometric boundaries and gluing them together to form a nonarithmetic manifold. This construction has then been generalized by Raimbault [12] and Gelander and Levit [6] to produce many different commensurability classes of nonarithmetic manifolds.

A similar construction was introduced by Agol [1] in dimension 4 and generalized by Belolipetsky and Thomson [2] in arbitrary dimension to obtain manifolds with short systole. They start with two hyperplanes chosen at distance $\delta > 0$ apart and find a torsion-free arithmetic lattice Γ such that, in $M = \Gamma\backslash\mathbb{H}^n$, the hyperplanes project down to two disjoint hypersurfaces. Then they cut M open along the hypersurfaces and glue it back to a copy of itself along its boundary; as $\delta \to 0$, the systole of such a manifold then becomes arbitrarily small. Manifolds obtained via this construction will be referred to as *doubly cut gluings* and the two corresponding hyperplanes as the *cut hyperplanes* (see Section 1.2).

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An interesting consequence is that infinitely many of such doubly cut gluings are nonarithmetic and pairwise noncommensurable. Moreover, these are the first examples in arbitrary dimension of nonarithmetic manifolds that are quasiarithmetic (see Thomson [15]). However if one is only interested in constructing nonarithmetic manifolds, their proof is somehow nonexplicit in the sense that it relies on the systole argument for proving both nonarithmeticity and pairwise noncommensurability. Furthermore, it is hard to give an estimate on the volume of one particular nonarithmetic manifold.

In this paper we give a sufficient condition on the cut hyperplanes to obtain nonarithmetic doubly cut gluings. Recall that the group $\mathrm{PO}(n,1) \cong \mathrm{Isom}(\mathbb{H}^n)$ has a natural matrix representation in $\mathrm{O}_f(\mathbb{R}) \subset \mathrm{GL}_{n+1}(\mathbb{R})$ for $f = -x_0^2 + x_1^2 + \cdots + x_n^2$ the standard Lorentzian quadratic form (see Section 1.1).

Proposition 1 *Let M be a doubly cut gluing with cut hyperplanes R_1 and R_2. Let $\rho_1, \rho_2 \in \mathrm{O}_f(\mathbb{R})$ denote the reflections in R_1, R_2 respectively. If the trace of $g = \rho_1\rho_2$ is not an algebraic integer, then M is nonarithmetic.*

In order to study the commensurability classes of doubly cut gluings, we use an invariant called the *adjoint trace ring*. This invariant was introduced by Vinberg [17] as the minimal ring of definition of a lattice Γ (see Section 2.1). We first show that we can realize every finitely generated subring of $\mathbb{Q}$ as the adjoint trace ring of a doubly cut gluing.

Theorem 2 *Let $n \geq 4$. For every square-free integer $d > 1$, there exists a nonarithmetic lattice Γ_d in $\mathrm{PO}(n,1)$ with adjoint trace ring $\mathbb{Z}[1/d]$.*

These lattices are nonuniform by construction (see Remark 2.7). Since the adjoint trace ring is an invariant of the commensurability class, there is an immediate corollary:

Corollary 3 *The lattices Γ_d and $\Gamma_{d'}$ are noncommensurable whenever $d \neq d'$.*

Thus we are able to construct many noncommensurable nonarithmetic doubly cut gluings avoiding the nonexplicit systole argument. We will see in Section 3 that this allows us to give explicit estimates on the volumes of these constructions.

The proof mainly relies on an observation about manifolds which admit a mirror symmetry in two of their embedded hypersurfaces. In the first section we introduce the necessary background and prove Proposition 1. In Section 2 we explain how to proceed to obtain noncommensurable manifolds and prove Theorem 2. Section 3 is devoted to volume computations, and some generalizations of Theorem 2 are discussed in Section 4.

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1 Background and nonarithmeticity

1.1 Background

Let $k \subset \mathbb{R}$ be a totally real number field with ring of integers $\mathcal{O}_k \subset k$. Let G be an absolutely simple adjoint algebraic k–group, and write $\mathrm{G}(\mathcal{O}_k)$ for $\mathrm{G}(k) \cap \mathrm{GL}_N(\mathcal{O}_k)$ in an arbitrary embedding $\mathrm{G} \subset \mathrm{GL}_N$ (this group is well defined up to commensurability). For $n \geq 4$, we say that G is *admissible* if $\mathrm{G}(\mathbb{R}) \cong \mathrm{PO}(n,1)$ and $\mathrm{G}^\sigma(\mathbb{R})$ is compact for any nontrivial embedding $\sigma\colon k \hookrightarrow \mathbb{R}$. In that case, $\mathrm{G}(\mathcal{O}_k)$ is a lattice in $\mathrm{G}(\mathbb{R})$.

A lattice $\Gamma \subset \mathrm{PO}(n,1)$ is called *arithmetic* if there exists an admissible algebraic k–group G and an isomorphism $\varphi\colon \mathrm{PO}(n,1) \to \mathrm{G}(\mathbb{R})$ such that

(1) $$\varphi(\Gamma) \text{ is commensurable to } \mathrm{G}(\mathcal{O}_k).$$

Since G is adjoint, we have $\varphi(\Gamma) \subset \mathrm{G}(k)$ (see Borel and Prasad [4, Proposition 1.2]). The lattice Γ is called *quasiarithmetic* if condition (1) is replaced by the weaker requirement that $\varphi(\Gamma) \subset \mathrm{G}(k)$. Thus arithmetic implies quasiarithmetic.

Let $n \geq 4$, and let f be a quadratic form of rank $n+1$ defined over k. Assume that f is *admissible*; that is, assume

(i) f has signature $(n,1)$ when seen over $\mathbb{R}$;

(ii) f^σ is positive definite for any nontrivial embedding $\sigma\colon k \hookrightarrow \mathbb{R}$.

Condition (i) implies the existence of an $\mathbb{R}$–isometry $f \cong -x_0^2 + x_1^2 + \cdots + x_n^2$. Therefore we may identify the hyperbolic space $\mathbb{H}^n$ with the "f–hyperbolic space" $\mathbb{H}_f$ which is obtained by choosing one of the two connected components of the space

$$\{x \in \mathbb{R}^{n+1} \mid f(x) = -1\}.$$

Let O_f denote the algebraic k–group of f–orthogonal matrices and $\mathrm{PO}_f = \mathrm{O}_f / Z(\mathrm{O}_f)$ the associated projective orthogonal group (in a matrix representation, we have $\mathrm{PO}_f = \mathrm{O}_f/\{\pm \mathrm{id}\}$). Conditions (i) and (ii) ensure that PO_f is an admissible algebraic group; the corresponding arithmetic groups are called *of the first type*.

Instead of directly working in $\mathrm{PO}_f(\mathbb{R})$, it is more convenient for our purposes to use the Lie group $\mathrm{O}'_f(\mathbb{R})$ consisting of the matrices in $\mathrm{GL}_{n+1}(\mathbb{R})$ which preserve $\mathbb{H}_f$. This group acts on $\mathbb{H}_f$ and may be identified with the group $\mathrm{Isom}(\mathbb{H}^n)$ of isometries of the hyperbolic n–space. Moreover there is an obvious isomorphism $\mathrm{O}'_f(\mathbb{R}) \cong \mathrm{PO}_f(\mathbb{R})$ which allows us to see elements of $\mathrm{PO}_f(\mathbb{R})$ as matrices. Since it is a matrix group, we can define $\mathrm{O}'_f(A)$ for any subring $A \subset \mathbb{R}$ as $\mathrm{O}'_f(\mathbb{R}) \cap \mathrm{GL}_{n+1}(A)$. In particular, the group $\mathrm{O}'_f(\mathcal{O}_k)$ is unambiguously defined and is an arithmetic subgroup of the first type.

With these identifications, a hyperplane of $\mathbb{H}_f$ is simply $\mathbb{H}_f$ intersected with a linear subspace of $\mathbb{R}^{n+1}$ of dimension n. We will say that such a hyperplane is *k–rational* (or more briefly *rational*) if the corresponding subspace in $\mathbb{R}^{n+1}$ is the f–orthogonal complement of a vector $v \in k^{n+1}$ (equivalently, $v \in \mathcal{O}_k{}^{n+1}$). Observe that the reflection in such a hyperplane is an isometry of $\mathbb{H}_f$ which lies in $\mathrm{O}'_f(k)$.

1.2 Nonarithmeticity

We briefly recall the construction of Belolipetsky and Thomson [2] in order to prove Proposition 1. Let f/k be an admissible quadratic form over a totally real number field k. Let R_1 and R_2 be two *disjoint* rational hyperplanes in $\mathbb{H}_f$; ie R_1 and R_2 do not meet in $\mathbb{H}_f$ or at infinity. In that case, there is a unique geodesic segment ν orthogonal to both R_1 and R_2, and its length is the distance between R_1 and R_2.

Let $\Lambda \subset \mathrm{O}'_f(\mathcal{O}_k)$ be a finite-index torsion-free subgroup such that

(i) for $i = 1, 2$ and each $\lambda \in \Lambda$, either λR_i is disjoint from R_i or they coincide;

(ii) λR_1 is disjoint from R_2 for any $\lambda \in \Lambda$.

In that case, for $i = 1, 2$, the orbit $\Lambda R_i = \{\lambda R_i \mid \lambda \in \Lambda\}$ forms a collection of disjoint hyperplanes, and $\Lambda R_1 \cap \Lambda R_2 = \varnothing$. It follows from [2, Lemma 3.1] that such a subgroup $\Lambda \subset \mathrm{O}'_f(\mathcal{O}_k)$ always exists.

Let $L = \Lambda \backslash \mathbb{H}_f$. By construction, the two hyperplanes R_1 and R_2 project down in L to two disjoint *hypersurfaces* (by which we mean finite-volume totally geodesic codimension 1 embedded submanifolds); we denote them by N_1 and N_2 respectively. The preimage of $L \setminus (N_1 \cup N_2)$ in $\mathbb{H}_f$ consists of disjoint convex open sets whose boundaries are hyperplane lifts of N_1 or N_2. Let $\widetilde{C}$ be the connected component among them having R_1 and R_2 as boundary components, and let C denote the image of $\widetilde{C}$ in L. Then, since $\widetilde{C}$ contains the geodesic segment ν, the set C is a connected component of $L \setminus (N_1 \cup N_2)$ such that the distance separating N_1 and N_2 in C equals the distance between R_1 and R_2 in $\mathbb{H}_f$. Finally let M denote the manifold obtained by gluing two copies of C to each other by identifying their boundaries (that is, M is the "double" of C). We will call M a *doubly cut gluing*, and R_1 and R_2 the *cut hyperplanes*. Observe that this manifold depends on the choice of the subgroup $\Lambda \subset \mathrm{O}'_f(\mathcal{O}_k)$.

Since the hypersurfaces N_1 and N_2 have finite volume, the manifold M is complete without boundary (even when N_1 and N_2 pass through cusps; see Gromov and Piatetski-Shapiro [7, Section 2.10.B]). Hence we may write it as $M = \Gamma\backslash\mathbb{H}_f$ with $\Gamma \subset \mathrm{O}'_f(\mathbb{R})$. Let D and D' denote the two copies of C, seen as submanifolds of M. A connected component $\widetilde{D}$ of the preimage of D under the covering map $\mathbb{H}_f \twoheadrightarrow M$ is then a universal cover of $D \cong C$. Thus upon conjugating Γ we can assume that $\widetilde{D} = \widetilde{C}$ and that

$$\mathrm{Stab}_\Gamma(\widetilde{C}) = \mathrm{Stab}_\Lambda(\widetilde{C}).$$

Write Λ^+ for this stabilizer; it corresponds to the image of $\pi_1(C)$ in Λ via the isomorphism $\pi_1(L) \cong \Lambda$.

The following lemma gives a generating set for Γ. A similar generating set appears in Thomson [15] in the proof of quasiarithmeticity. We include a proof of this specific version, which we will need in the sequel.

Lemma 1.1 *Let $\rho_1, \rho_2 \in \mathrm{O}'_f(k)$ denote the reflections in R_1, R_2 respectively. There exists $\lambda_1, \lambda_2 \in \Lambda$ such that*

$$\Gamma = \langle \Lambda^+,\ \rho_1\Lambda^+\rho_1,\ \rho_1\rho_2,\ \rho_1\lambda_1\rho_1\lambda_1^{-1},\ \rho_1\lambda_2\rho_2\lambda_2^{-1}\rangle.$$

In particular, $\Gamma \subset \langle \mathrm{O}'_f(\mathcal{O}_k), \rho_1, \rho_2\rangle$.

Proof Topologically, M consists of D and D' glued together along their boundaries. Now depending on whether N_1 (resp. N_2) separates L or not, ∂D consists of one or two copies of N_1 (resp. N_2). Write $\partial D = N_{1,1} \cup N_{1,2} \cup N_{2,1} \cup N_{2,2}$ with $N_{i,j} \cong N_i$ and possibly $N_{i,1} = N_{i,2}$. By our choice of Γ, we can assume (upon exchanging $N_{i,1}$ and $N_{i,2}$) that the hyperplane R_i is a lift of $N_{i,1}$. Moreover since the boundary components of C corresponding to $N_{i,1}$ and $N_{i,2}$ are identified in L, we can find a (possibly trivial) $\lambda_i \in \Lambda$ such that $\lambda_i R_i$ is a hyperplane lift of $N_{i,2}$ contained in the boundary of $\widetilde{C}$.

Observe that by construction the reflections in the $N_{i,j}$ are isometries of M, and that all these reflections have the same effect on M: they exchange D and D'. Since $\widetilde{C}$

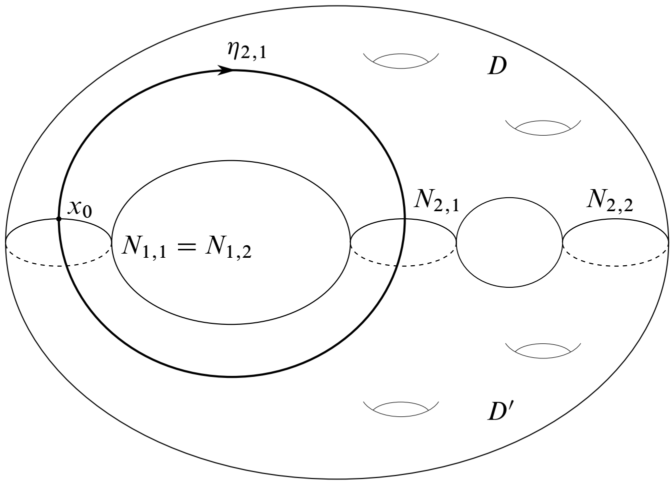

Figure 1: A doubly cut gluing M

contains all R_i and $\lambda_i R_i$ as boundary, it follows that the reflection about $N_{i,j}$ is induced by ρ_i if $j = 1$ and $\lambda_i \rho_i$ if $j = 2$. Thus for $i = 1, 2$ and $\lambda \in \{1, \lambda_i\}$, the elements ρ_1 and $\lambda \rho_i \lambda^{-1}$ induce the same isometry on M. Therefore

$$\rho_1 \lambda \rho_i \lambda^{-1} \in \Gamma.$$

Now pick a basepoint x_0 in $N_{1,1}$, and let $\tilde{x}_0 \in R_1$ be a preimage of x_0. The geodesic between $\tilde{x}_0$ and $\rho_2 \circ \rho_1(\tilde{x}_0)$ is a segment which crosses R_2 exactly once and does not meet any other hyperplane lifts of the $N_{i,j}$ except at its endpoints. Therefore, $\rho_1 \rho_2 = (\rho_2 \rho_1)^{-1} \in \Gamma$ corresponds to the homotopy class $[\eta_{2,1}] \in \pi_1(M, x_0)$ of a loop $\eta_{2,1}$ at x_0 which crosses $N_{2,1}$ exactly once and is contained in D' before this crossing and in D after this crossing.

Similarly, for $i = 1, 2$, the element $\rho_1 \lambda_i \rho_i \lambda_i^{-1}$ corresponds to the class of a loop $\eta_{i,2}$ crossing $N_{i,2}$ exactly once. Consequently, the group $\pi_1(M, x_0)$ is generated by

$$\pi_1(D) \cup \pi_1(D') \cup \{[\eta_{2,1}], [\eta_{1,2}], [\eta_{2,2}]\}.$$

Since $\pi_1(D')$ corresponds to $\rho_1 \Lambda^+ \rho_1$ in Γ, the lemma follows. □

We turn towards the proof of Proposition 1.

Proof of Proposition 1 We prove the contrapositive. Assume M is arithmetic. Since Γ and Λ share the Zariski-dense subgroup Λ^+, it follows from Gromov and Piatetski-Shapiro [7, Section 1.6] that $\Gamma \cap \mathrm{O}'_f(\mathcal{O}_k)$ is a finite-index subgroup of Γ. Therefore the element $g = \rho_1\rho_2 \in \Gamma$ must have a power g^N in $\mathrm{O}'_f(\mathcal{O}_k)$. Let α be an eigenvalue of g. Then α^N is an eigenvalue of g^N, and is thus an algebraic integer (since it lies in an integral extension of $\mathcal{O}_k$). It is easy to see that the same is true about α. Since α is arbitrary, we get that $\mathrm{tr}(g) = \sum(\text{eigenvalues})$ is also an algebraic integer. □

Remark 1.2 The proof actually only uses the fact that $\rho_1\rho_2$ has an eigenvalue which is not an algebraic integer. Therefore the conclusion of the proposition still holds if the trace condition is replaced by this requirement.

2 Noncommensurability and examples over $\mathbb{Q}$

2.1 The adjoint trace ring

Proposition 1 gives a way to control the nonarithmeticity of doubly cut gluings, but does not say anything about their commensurability. To that end, we will use an invariant introduced by Vinberg called the adjoint trace ring [17].

Let Γ be a Zariski-dense subgroup of a semisimple algebraic group G. The *adjoint trace field* of Γ is the field

$$K(\Gamma) = \mathbb{Q}(\{\mathrm{tr}\,\mathrm{Ad}(\gamma) \mid \gamma \in \Gamma\}),$$

where Ad denotes the adjoint representation. Similarly, the *adjoint trace ring* $A(\Gamma)$ of Γ is the integral closure of the ring $\mathbb{Z}[\{\mathrm{tr}\,\mathrm{Ad}(\gamma) \mid \gamma \in \Gamma\}]$. If $k = K(\Gamma)$ is a number field, we simply have (see Davis [5])

$$A(\Gamma) = \mathcal{O}_k[\{\mathrm{tr}\,\mathrm{Ad}(\gamma) \mid \gamma \in \Gamma\}].$$

In his paper, Vinberg defines the minimal field (resp. ring) of definition of $\mathrm{Ad}\,\Gamma$, and shows that it coincides with $K(\Gamma)$ (resp. $A(\Gamma)$) [17, Corollary of Theorem 1]. The adjoint trace field is an invariant of the commensurability class of Γ, and the same is true for the adjoint trace ring when the adjoint trace field is a number field [17, Theorem 3].

Let us assume that G is defined over $\mathbb{R}$. Suppose further that $\mathrm{G}(\mathbb{R}) \cong \mathrm{PO}(n,1)$ as Lie groups (with $n \geq 4$) and that $\Gamma \subset \mathrm{G}(\mathbb{R})$. For example, one can take $\mathrm{G} = \mathrm{PO}_f$ for f a signature $(n,1)$ quadratic form. In that case, the algebraic adjoint representation and the one coming from the Lie group structure coincide on $\mathrm{G}(\mathbb{R})$. Thus the adjoint trace field (resp. ring) of Γ does not depend on G, and it makes sense to speak of the adjoint trace field $K(M)$ (resp. of the adjoint trace ring $A(M)$) of a hyperbolic manifold M.

Remark 2.1 For $n = 3$ the group $\mathrm{Isom}(\mathbb{H}^3) \cong \mathrm{PSL}_2(\mathbb{C})$ has a structure of a complex Lie group. The adjoint trace field — for the complex adjoint representation — of a Zariski-dense $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ coincides with the invariant trace field in the sense of Maclachlan and Reid [8, Definition 3.3.6]. If one uses an algebraic group G such that $\mathrm{G}(\mathbb{R}) \cong \mathrm{Isom}(\mathbb{H}^3)$, one gets a different adjoint trace field. For $n \geq 4$ this ambiguity does not appear, as $\mathrm{PO}(n,1)$ then does not possess a complex algebraic structure.

By Borel's density theorem, any lattice Γ is Zariski-dense in $\mathrm{G}(\mathbb{R})$ or $\mathrm{G}(\mathbb{R})^0$. Furthermore, it follows from local rigidity that the adjoint trace field of Γ is a number field (even if Γ is nonarithmetic; see Vinberg, Gorbatsevich and Shvartsman [18, Chapter 1, Section 6] or Bergeron and Gelander [3, page 124]).

Remark 2.2 When Γ is arithmetic and commensurable to $\mathrm{G}(\mathcal{O}_k)$, it follows from Prasad and Rapinchuk [11, Lemma 2.6] that $K(\Gamma) = k$ and $A(\Gamma) = \mathcal{O}_k$. In particular, the adjoint trace field of Λ^+ (defined in Section 1.2) is k since it contains the stabilizer of a rational hyperplane which is an arithmetic group of adjoint trace field k.

The following easy lemma is useful to compute the adjoint trace ring.

Lemma 2.3 *Let M be a doubly cut gluing with associated quadratic form f over the field k. Write $M = \Gamma\backslash\mathbb{H}_f$ with $\Gamma \subset \mathrm{O}'_f(\mathbb{R})$. Then the adjoint trace field $K(\Gamma)$ of Γ is k and its adjoint trace ring is*

$$A(\Gamma) = \mathcal{O}_k[\{\mathrm{tr}\,\gamma \mid \gamma \in \Gamma\}].$$

Proof We first look at the adjoint trace field. Choose Γ as in Lemma 1.1. Since Γ contains the Zariski-dense subgroup Λ^+ we get $K(\Gamma) \supset k$ (see Remark 2.2). From Lemma 1.1 we see that $\Gamma \subset \mathrm{O}'_f(k)$, thus establishing the other inclusion.

For the adjoint trace ring, we first apply Theorem 2 of Vinberg [17] to the representation $\Gamma \hookrightarrow \mathrm{O}'_f(k)$ given by the inclusion. We obtain that a conjugate of Γ lies in $\mathrm{O}'_f(A(\Gamma))$. By [17, Lemma 1 and 2] this happens exactly when $\{\mathrm{tr}(\gamma) \mid \gamma \in \Gamma\} \subset A(\Gamma)$ and the proof is complete. □

Remark 2.4 It follows that a doubly cut gluing is arithmetic if and only if the trace of all its elements are algebraic integers (that is, contained in $\mathcal{O}_k$). With the help of Lemma 1.1, Proposition 1 already implied one direction of this equivalence.

Remark 2.5 In the complex orthogonal group, the following formula relates the trace of an element and that of its adjoint:

$$\operatorname{tr}\operatorname{Ad}(g)=\frac{(\operatorname{tr}g)^2-\operatorname{tr}(g^2)}{2}\quad\text{for any } g\in \mathrm{O}_n(\mathbb{C}).$$

This can be proven by direct computation using a basis of the Lie algebra. Since the formula holds over $\mathbb{C}$, it remains valid in any orthogonal group O_f for an arbitrary quadratic form f, and can therefore be used to compute the adjoint trace ring of hyperbolic manifolds.

2.2 Examples over $\mathbb{Q}$

To prove Theorem 2, we will apply the results of the previous section to the quadratic form $f=-x_0^2+x_1^2+\cdots+x_n^2$ defined over $\mathbb{Q}$. In this section, norms, scalar products and orthogonal complements are to be understood with respect to the quadratic form f.

Fix the hyperplane $R_1=\{x_1=0\}=v^\perp$, where $v=(0,1,0,\dots,0)$. Let $w=(w_0,w_1,\dots,w_n)\in\mathbb{Z}^{n+1}$ be such that

$$w_1^2\geq\langle w,w\rangle>0.$$

Then the hyperplanes R_1 and $R_2=w^\perp$ do not intersect. Indeed we have $|w_1|=|\langle v,w\rangle|\geq\|v\|\|w\|=\sqrt{\langle w,w\rangle}$, and by [13, Theorem 3.2.7] it follows that $R_1\cap R_2=\varnothing$.

Let M_w be a doubly cut gluing with cut hyperplanes R_1 and R_2. Write $M_w=\Gamma_w\backslash\mathbb{H}_f$ with $\Gamma_w\subset\mathrm{O}'_f(\mathbb{R})$. Let ρ_1 and ρ_2 denote the reflections in R_1 and R_2 respectively.

Lemma 2.6 *The adjoint trace ring $A(M_w)$ of the manifold $M_w=\Gamma_w\backslash\mathbb{H}_f$ satisfies*

$$\mathbb{Z}\left[\frac{4w_1^2}{\langle w,w\rangle}\right]\subset A(M_w)\subset\mathbb{Z}\left[\frac{2}{\langle w,w\rangle}\right].$$

Proof By elementary computations, the matrix of ρ_2 is

$$\rho_2=I-\frac{2}{\langle w,w\rangle}ww^tJ,\quad\text{where } J=\operatorname{diag}(-1,1,\dots,1).$$

Multiplying by ρ_1 and using linearity and cyclic invariance of the trace, one gets

$$\operatorname{tr}(\rho_1\rho_2)=(n-1)-2\frac{\langle w,\rho_1w\rangle}{\langle w,w\rangle},\tag{2}$$

which in our case equals $n-3+4w_1^2/\langle w,w\rangle$ by an easy computation. With Lemma 2.3 this establishes the left inclusion. The right inclusion follows from Lemma 1.1 since $\mathrm{O}'_f(\mathbb{Z})\cup\{\rho_1,\rho_2\}\subset\mathrm{O}'_f(\mathbb{Z}[2/\langle w,w\rangle])$. □

We now dive into the proof Theorem 2.

Proof of Theorem 2 Set $b=d$ if 2 does not divide d, and $b=4d$ otherwise. Pick $w_1\in\mathbb{Z}$ such that $w_1^2>b$ and $\gcd(w_1,b)=1$. Choose integers $w_0,w_2,\dots,w_n$ such that $-w_0^2+w_2^2+\cdots+w_n^2=(b-w_1^2)$. The latter is possible since the quadratic form $-x_0^2+x_2^2+\cdots+x_n^2$ represents any arbitrary integer ($-x_0^2+x_2^2$ represents every odd integer, and we have at least one more variable to represent 1). If $w=(w_0,w_1,\dots,w_n)$, we have $\langle w,w\rangle=b$ whence $w_1^2>\langle w,w\rangle>0$. Thus we can construct $M_w=\Gamma_w\backslash\mathbb{H}_f$ as in the beginning of this section. Since $\gcd(w_1^2,b)=1$ we have, using standard properties of subrings of $\mathbb{Q}$,

$$\mathbb{Z}\left[\frac{4w_1^2}{\langle w,w\rangle}\right]=\mathbb{Z}[4/b]=\mathbb{Z}[1/d]=\mathbb{Z}\left[\frac{2}{\langle w,w\rangle}\right].$$

Therefore the theorem follows from Lemma 2.6, with $\Gamma_d=\Gamma_w$. □

Remark 2.7 It follows from the proof that the lattices Γ_d are all contained in $\mathrm{O}'_f(\mathbb{Q})$ for the same quadratic form f. Since its restriction to R_1 is simply $-x_0^2+x_2^2+\cdots+x_n^2$, it follows (this form being isotropic) that N_1 and thus also M_w are noncompact for any (admissible) w.

More generally, for $n\geq 5$ any doubly cut gluing M with adjoint trace field $K(M)=\mathbb{Q}$ is noncompact. Indeed, the adjoint trace field of the gluing hypersurface must then also be $\mathbb{Q}$, and since it is arithmetic (of the first type) it corresponds to a quadratic form over $\mathbb{Q}$ with at least 5 variables. Such a quadratic form is always isotropic, and thus, by Godement's compactness criterion, the hypersurface, and hence M, are noncompact.

Remark 2.8 Theorem 2 still holds for $n=3$, but the notion of adjoint trace ring/field in the statement differs from its usual meaning if the lattices are considered in $\mathrm{PSL}_2(\mathbb{C})$ (see Remark 2.1). Thus Theorem 2 has been stated for $n\geq 4$ to avoid confusion.

3 Volume bound

The doubly cut gluings M_w from previous section depend on the choice of a finite-index subgroup $\Lambda\subset\mathrm{O}'_f(\mathbb{Z})$; in this section we will write M_w^Λ for the M_w obtained using the subgroup Λ. Note that it is not clear if two commensurable subgroups $\Lambda,\Lambda'\subset\mathrm{O}'_f(\mathbb{Z})$ give rise to commensurable manifolds M_w^Λ and $M_w^{\Lambda'}$.

Define V_w to be the minimal volume of M_w^Λ for any choice of $\Lambda \subset \mathrm{O}'_f(\mathbb{Z})$ (satisfying the requirements given in the beginning of Section 1.2). Our goal is to give an upper-bound on V_w by constructing explicitly such a Λ. The arguments of this section are inspired by the proof of [9, Lemma 10] of Margulis and Vinberg.

We use the notation of the previous section. The subgroup $\Lambda \subset \mathrm{O}'_f(\mathbb{Z})$ must be torsion-free and satisfy, for all $\lambda \in \Lambda$,

(i) $\lambda R_1 \cap R_1 = \varnothing$,

(ii) $\lambda R_2 \cap R_2 = \varnothing$,

(iii) $\lambda R_1 \cap R_2 = \varnothing$.

We focus on principal congruence subgroups of the form

$$\Lambda_m = \{\lambda \in \mathrm{O}'_f(\mathbb{Z}) \mid \lambda \equiv \mathrm{id} \bmod m\} \quad \text{(for } m \in \mathbb{Z} \text{ not necessarily prime).}$$

The goal is to find $m \geq 2$ such that Λ_m fulfills (i)–(iii).

To obtain a torsion-free subgroup, it is enough to take a congruence subgroup Λ_m with $m > 2$ (see [10, Theorem IX.7]). Furthermore, it follows from Gromov and Piatetski-Shapiro [7, Section 2.8] that any principal congruence subgroup satisfies (i).

For (ii), we need to ensure that for any $\lambda \in \Lambda_m$ we have (using [13, Theorem 3.2.7], as above)

$$|\langle \lambda w, w\rangle| \geq \langle w, w\rangle. \tag{3}$$

Now if we choose $m \geq 2\langle w, w\rangle$, the fact that

$$\langle \lambda w, w\rangle = \langle w, w\rangle \quad \bmod m$$

implies inequality (3).

Finally for (iii) the situation is similar: we just need to ensure that

$$|\langle \lambda v, w\rangle| \geq \|w\|. \tag{4}$$

If $m \geq 2w_1^2$, the fact that $w_1^2 \geq \langle w, w\rangle$ and the equation

$$\langle \lambda v, w\rangle^2 = \langle v, w\rangle^2 = w_1^2 \quad \bmod m$$

imply that inequalities (3) and (4) are fulfilled.

To sum up, we define $\Lambda = \Lambda_m$, where $m = \max(3, 2w_1^2)$. This congruence subgroup is then such that $L = \Lambda \backslash \mathbb{H}_f$ is a hyperbolic manifold in which R_1 and R_2 project to disjoint hypersurfaces, as desired. Now the doubly cut gluing M_w^{Λ} constructed from Λ is obtained as the double of a piece of L. Thus its volume cannot exceed twice the volume of L, and we have proven:

Proposition 3.1 *The minimal volume V_w of a doubly cut gluing M_w constructed as in Section 2.2 satisfies*

$$V_w \leq 2 \cdot |\mathrm{O}_f(\mathbb{Z}/m\mathbb{Z})| \cdot \operatorname{covol} \mathrm{O}_f(\mathbb{Z}),$$

where $m = \max(3, 2w_1^2)$.

As a corollary we obtain the following examples of relatively small volume:

Corollary 3.2 *For any $n \geq 4$, there exists an n–dimensional nonarithmetic doubly cut gluing M with*

$$\operatorname{vol}(M) \leq 2 \cdot |\mathrm{O}_f(\mathbb{Z}/8\mathbb{Z})| \cdot \operatorname{covol} \mathrm{O}_f(\mathbb{Z}).$$

Proof We pick $w = (1, 2, 0, \dots, 0)$. Observe that $w_1^2 = 4 > \langle w, w \rangle = 3 > 0$. Therefore, we can construct doubly cut gluings M_w^{Λ} as before. Let $M = M_w^{\Lambda}$ be a doubly cut gluing realizing the volume V_w. This manifold has adjoint trace ring $\mathbb{Z}\left[\frac{1}{3}\right]$ (by Lemma 2.6) and is therefore nonarithmetic. The volume bound is a consequence of Proposition 3.1. □

This volume bound is larger than — but still comparable with — the volume bound $|\mathrm{O}_f(\mathbb{Z}/3\mathbb{Z})| \cdot \operatorname{covol}(\mathrm{O}_f(\mathbb{Z}))$ in the arithmetic case (which is obtained using the principal congruence subgroup mod 3). For exact computations of the volumes we refer to the formulas of Ratcliffe and Tschantz [14] and the references therein.

4 About generalizations

There are two easy-to-state generalizations of the doubly cut gluing construction of nonarithmetic manifolds. The first one is to increase the number of cut hyperplanes. Indeed, if one finds n disjoint rational hyperplanes, then it is possible (using again Belolipetsky and Thomson [2, Lemma 3.1]) to find an arithmetic lattice $\Gamma \subset \mathrm{PO}_f$ such that they project down to disjoint hypersurfaces. If one chooses the hyperplanes in such a way that their reflections have interesting rational properties, one might get better results regarding, for example, the minimal volume estimate of a manifold with prescribed adjoint trace ring.

The second possible generalization is to give an analog of Theorem 2 for number fields. One way to proceed would be to replace $\mathbb{Z}[1/d]$ with the ring $\mathcal{O}_S$ of S–integers of a totally real number field k, where S is a finite set of nonarchimedean places. For specific examples of S this is feasible (as suggested in the toy example of the next proposition). However in order to get a general result one has to face the problem that an admissible quadratic form over $k \neq \mathbb{Q}$ is not isotropic, and thus cannot represent any element of k. Moreover, even when we restrict to quadratic extensions, it is likely that more hyperplanes will be needed to generate $\mathcal{O}_S$, thereby making the construction more complicated.

Proposition 4.1 *Let $k = \mathbb{Q}(\alpha)$ be a totally real number field, with α an algebraic integer which is positive at all but one embedding $k \hookrightarrow \mathbb{R}$. Let $d \in \mathbb{Z}$ be a square-free integer. Then there exists $c \in \mathbb{Z}$ having the same prime factors as d and a doubly cut gluing M of arbitrary dimension $n \geq 4$ with adjoint trace ring satisfying*

$$\mathcal{O}_k\left[\frac{1}{d}\right] \subset A(M) \subset \mathcal{O}_k\left[\frac{1}{d}, \frac{1}{c-\alpha}\right].$$

Proof Start with $c = d$. By eventually increasing the powers of the primes occurring in it, we can ensure that $c > \max_\sigma |\sigma(\alpha)|$, where σ ranges over all embeddings $k \hookrightarrow \mathbb{R}$. Consider the quadratic form $f = \alpha x_0^2 + (c-\alpha)x_1^2 + x_2^2 + \cdots + x_n^2$. The conditions on α and c imply that f is admissible. Fix an embedding $k \subset \mathbb{R}$ such that $\alpha < 0$, and set $v = (0, 1, 0, \ldots, 0)$ and $w = (1, 1, 0, \ldots, 0)$. We have

$$f(v) = c - \alpha \quad \text{and} \quad f(w) = c,$$

hence both have positive f–norm, and the hyperplanes they define do not intersect since $\langle v, w\rangle_f^2 = (c-\alpha)^2 > (c-\alpha)c = f(v)f(w)$. Now if ρ_1 and ρ_2 denote the reflections at the hyperplanes defined by v and w respectively, a computation shows that formula (2) in the proof of Lemma 2.6 still holds when one uses the scalar product induced by our f instead. We get

$$\operatorname{tr}(\rho_1\rho_2) = (n-1) - 2 \cdot \frac{2\alpha - c}{c} \in -\frac{4\alpha}{c} + \mathbb{Z}.$$

Let $p(x)$ denote the minimal polynomial of α, and let $N = p(0)$. Then $(p(\alpha)-N)/\alpha$ is in $\mathcal{O}_k$, and thus

$$\mathcal{O}_k[1/c] \supset \mathcal{O}_k[\operatorname{tr}(\rho_1\rho_2)] \supset \mathcal{O}_k\left[\frac{p(\alpha)-N}{\alpha} \cdot \frac{4\alpha}{c}\right] \supset \mathcal{O}_k[4N/c].$$

Now by possibly increasing further the powers of the primes in c that divide $4N$, we get that the rings on both sides of the chain of inclusions coincide and are equal to $\mathcal{O}_k[1/d]$.

This proves the first inclusion of the proposition. The second inclusion follows from the same argument as in Lemma 2.6, observing that $\rho_1 \in \mathrm{O}'_f(\mathcal{O}_k[1/c-\alpha])$ and $\rho_2 \in \mathrm{O}'_f(\mathcal{O}_k[1/d])$. □

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