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Pretty rational models for Poincaré duality pairs

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We prove that a large class of Poincaré duality pairs of spaces admit rational models (in the sense of Sullivan) of a convenient form associated to some Poincaré duality CDGA.

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1 Introduction

Sullivan theory [10] encodes the rational homotopy type of a simply connected space of finite type, X, into a commutative differential graded algebra (CDGA), (A, d_A), such that $\mathrm{H}(A, d_A) \cong \mathrm{H}^*(X; \mathbb{Q})$ and which is called a *CDGA model* of X (see Section 2.1 for a quick recapitulation on that theory). In [8] we proved that when X is a simply connected Poincaré duality space (the most important example being a closed manifold), we can construct a CDGA model whose underlying algebra satisfies Poincaré duality. These Poincaré duality CDGA models are often convenient and were used for example in Lambrechts and Stanley [6; 9], Idrissi [5] and Campos and Willwacher [1] to construct nice rational or real models for configuration spaces in closed manifolds, or in Félix and Thomas [4] to study the Chas–Sullivan product on the free loop space.

The aim of this paper is to exhibit convenient CDGA models for Poincaré duality *pairs* of spaces, like compact manifolds with boundary. Such a model should be a CDGA morphism between two CDGAs representing each element of the pair. Our main result is that many Poincaré duality pairs admit what we call *pretty models* (Definition 4.2). The main results of this paper are that the following Poincaré duality pairs admit such models:

- even-dimensional disk bundles over a simply connected closed manifold relative to their sphere bundles (Theorem 5.1);

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- Poincaré duality pairs $(W, \partial W)$, where ∂W retracts rationally on its half-skeleton (Definition 6.1 and Theorem 6.6), under some mild connectivity hypotheses;
- the complement of a subpolyhedron of high codimension in a closed manifold relative to its natural boundary (Proposition 4.5).

Let us describe roughly the form of these *pretty models* (see Section 4, and in particular Definition 4.2, for more details). A pretty model for $\partial W \hookrightarrow W$ is a CDGA morphism between mapping cones

(1) $$\varphi \oplus \mathrm{id}\colon P \oplus_{\varphi^!} \mathrm{ss}^{-n}\#Q \to Q \oplus_{\varphi\varphi^!} \mathrm{ss}^{-n}\#Q,$$

where

- P is a Poincaré duality CDGA in dimension $n = \dim W$, which roughly means that it is a CDGA whose underlying algebra satisfies Poincaré duality (see Definition 3.1 and Remark 3.2 for details);
- $\varphi\colon P \to Q$ is a CDGA morphism;
- $\mathrm{ss}^{-n}\#Q$ is the $(n-1)^{\mathrm{st}}$ suspension of the linear dual $\#Q = \hom_{\mathbb{Q}}(Q, \mathbb{Q})$;
- $\varphi^!\colon s^{-n}\#Q \to P$ is a P–dg-module morphism constructed out of φ and the Poincaré duality isomorphism on P;
- the CDGA structure on the mapping cones is the semitrivial one described at Section 2.2 (which requires that $\varphi\varphi^!$ is *balanced* in the sense of Definition 2.2).

In the special case when $\partial W = \varnothing$, we have $Q = 0$ and we recover a Poincaré duality CDGA model, P, for W as in [8].

Note also that the codomain of (1),

$$Q \oplus_{\varphi\varphi^!} \mathrm{ss}^{-n}\#Q,$$

is a Poincaré duality CDGA model in dimension $n-1$ for ∂W.

When $\partial W \neq \varnothing$, W is not a Poincaré duality space and thus does not admit a Poincaré duality CDGA model. However, often W has a model which is an explicit quotient of a Poincaré duality CDGA, as the following shows:

Proposition 1.1 (Corollary 4.4) *If $(W, \partial W)$ admits a pretty model (1) and if φ is surjective, then W has a CDGA model*

$$P/I,$$

where P is a Poincaré duality CDGA and $I = \varphi^!(s^{-n}\#Q)$ is a differential ideal.

These pretty models should be very convenient in many constructions in rational homotopy theory on Poincaré duality pairs. In particular, we use them in [2] to obtain explicit models for the complement of a subpolyhedron in a manifold with boundary, and in particular to the configuration space of two points in such a manifold. In a paper in preparation we will also use these pretty models for models of configurations spaces of any number of points in a manifold with boundary.

Here is the plan of the paper. In Section 2 we quickly review basic facts and terminology about rational homotopy theory, and we define the semitrivial CDGA structure on some mapping cones. In Section 3 we review the notion of a Poincaré duality CDGA modelling a given CDGA whose cohomology satisfies Poincaré duality and prove some existence results of such Poincaré duality CDGAs, refining the main result of [8]. In Section 4 we define (surjective) pretty models for Poincaré duality pairs of spaces and we motivate this definition by the example of the complement of a polyhedron in a closed manifold. In Section 5 we prove that even-dimensional disk bundles over simply connected Poincaré duality spaces admit surjective pretty models. We prove in Section 6 that any simply connected Poincaré duality pair whose boundary retracts rationally on its half-skeleton admits a surjective pretty model, under mild connectivity hypotheses. In the last section we discuss whether every Poincaré duality pair admits a pretty model.

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2 CDGAs, dg-modules and semitrivial CDGA structures on mapping cones

2.1 Rational homotopy theory

In this paper we will use the standard tools and results of rational homotopy theory, following the notation and terminology of [3]. Recall that A_{PL} is the Sullivan–de Rham contravariant functor and that for a simply connected space of finite type, X, $A_{\mathrm{PL}}(X)$ is a commutative differential graded algebra (CDGA for short, always nonnegatively graded). Any CDGA weakly equivalent to $A_{\mathrm{PL}}(X)$ is called a *CDGA model of X* and it completely encodes the rational homotopy type of X. Similarly, a CDGA model of a map of spaces $f\colon X \to Y$ is a CDGA morphism weakly equivalent to $A_{\mathrm{PL}}(f)\colon A_{\mathrm{PL}}(Y) \to A_{\mathrm{PL}}(X)$. All our dg-modules and CDGAs are over the field $\mathbb{Q}$. A CDGA, A, is *connected* if $A^0 = \mathbb{Q}$. It is k–connected if moreover $A^i = 0$ for $1 \le i \le k$. The unit is denoted by $1 \in A^0$ and the homology algebra of A by $\mathrm{H}(A)$. A *Poincaré duality CDGA* is, roughly speaking, a connected CDGA whose underlying algebra satisfies Poincaré duality (see Definition 3.1 for a precise definition).

2.2 Mapping cones of balanced morphisms and their semitrivial CDGA structures

Let A be a CDGA and let R be an A–dg-module. Since A is commutative, every left (or right) A–dg-module can be seen as a commutative A–dg-bimodule. We will denote by $s^k R$ the k^{th} suspension of R, ie $(s^k R)^p = R^{k+p}$, and for a map of A–dg-modules, $f\colon R \to Q$, we denote by $s^k f$ the k^{th} suspension of f. For example, $s^{-n}\mathbb{Q}$ is a dg-module concentrated in degree n. Furthermore, we will use # to denote the linear dual of a vector space, $\#V = \hom(V, \mathbb{Q})$, and $\#f$ to denote the linear dual of a map f. A dg-module is of *finite type* if it is of finite dimension in every degree. If M is a dg-module, we write $M^{>k} = 0$ to express that $M^i = 0$ for each $i > k$; similarly, we will write $M^{\ge k} = 0$, $M^{<k} = 0$, etc.

If $f\colon Q \to R$ is an A–dg-module morphism, *the mapping cone* of f is the A–dg-module

$$C(f) := (R \oplus sQ, \delta)$$

defined by $R \oplus sQ$ as an A–module and with a differential δ such that $\delta(r, sq) = (d_R(r) + f(q), -s d_Q(q))$. We also write $C(f) = R \oplus_f sQ$. When $f = 0$, we just write $C(0) = R \oplus sQ$.

When $R = A$, the mapping cone $C(f\colon Q \to A)$ can be equipped with a unique commutative graded algebra (CGA) structure that extends the algebra structure on A and the A–dg-module structure on sQ, and such that $(sq)(sq') = 0$, for $q, q' \in Q$.

Definition 2.1 [7, Section 4] We call the above CGA structure the *semitrivial structure* on the mapping cone $A \oplus_f sQ$.

This CGA $A \oplus_f sQ$ may not be a CDGA because the differential on the mapping cone could fail to satisfy the Leibniz rule for the multiplication coming from the semitrivial structure. The next definition and proposition characterize the mapping cones on which the semitrivial structure is that of a CGDA.

Definition 2.2 Let A be a CDGA and Q be an A–dg-module. An A–dg-module morphism $f\colon Q \to A$ is *balanced* if, for each $x, y \in Q$,

$$f(x)y = xf(y). \tag{2}$$

The importance of this notion comes from the following proposition:

Proposition 2.3 *Let Q be an A–dg-module and $f\colon Q \to A$ be an A–dg-module morphism. The mapping cone $C(f) = A \oplus_f sQ$ endowed with the semitrivial CGA structure is a CDGA if and only if f is balanced.*

Proof In one direction, assume that f is balanced. The only nontrivially verified condition for $C(f)$ being a CDGA is the Leibniz rule for the differential. Let $a, a' \in A$ and $q, q' \in Q$. For products of the form $(a, 0)(a', 0)$ and of the form $(a, 0)(0, sq)$, the Leibniz rule is verified because A is a CDGA and Q is an A–dg-module. For products of the form $(0, sq)(0, sq')$, by semitriviality of the CDGA structure of the mapping cone we have to verify that

$$(\delta(0, sq))(0, sq') + (-1)^{|q|+1}(0, sq)(\delta(0, sq')) = 0, \tag{3}$$

which is a direct consequence of the hypothesis that f is balanced and the formula for the differential on a mapping cone.

In the other direction, if the Leibniz rule is satisfied for the semitrivial multiplication, then (3) holds for any $q, q' \in Q$, and, again by the formula for the differential on a mapping cone and the semitrivial multiplication, this implies that $f(q)q' = qf(q')$, hence f is balanced. □

Remark 2.4 In the rest of this paper, when a mapping cone is equipped with a CDGA structure it will be understood that is comes from the semitrivial structure.

We will also need the following:

Lemma 2.5 *Let $\varphi\colon P \to Q$ be a morphism of CDGA. Let D be a Q–dg-module, with its induced P–dg-module structure, and let $\psi\colon D \to P$ be a morphism of P–dg-modules. If $\varphi\psi\colon D \to Q$ is a balanced morphism of Q–dg-modules then ψ is a balanced morphism of P–dg-modules.*

Proof Let $x, y \in D$. Then we have, since the P–module structure on Q is induced through φ and since $\varphi\psi$ is balanced,

$$\psi(x)y = \varphi(\psi(x))y = x\varphi(\psi(y)) = x\psi(y).$$

In other words, ψ is balanced as a morphism of P–dg-modules. □

3 Poincaré duality CDGAs

In this section we define precisely Poincaré duality CDGAs and we review the main result of [8], which states that any 1–connected Poincaré duality space admits a 1–connected Poincaré duality CDGA-model. We prove some relative version of that result (Proposition 3.4). This will be used in the next sections to build pretty models.

A 1–connected Poincaré duality space in dimension n is a 1–connected space, M, such that there is an isomorphism

$$\mathrm{H}^*(M) \cong \#\,\mathrm{H}^{n-*}(M)$$

of $\mathrm{H}^*(M)$–modules, which is called a Poincaré duality isomorphism in cohomology. In [8] we proved that Poincaré duality holds not only in cohomology but also on some CDGA model of M. To make this precise, we review the following:

Definition 3.1 An *oriented Poincaré duality CDGA in dimension n*, or PDCDGA, is a connected CDGA of finite type, P, equipped with an isomorphism of P–dg-modules

$$\theta_P\colon P \xrightarrow{\cong} s^{-n}\,\#P. \tag{4}$$

Remark 3.2 Since P is a free P–dg-module generated by a single element, the isomorphism θ_P of (4) is unique up to a multiplication by a nonzero scalar. When this isomorphism is not specified, we talk of a *Poincaré duality CDGA* (dropping the adjective *oriented*).

Remark 3.3 It is easy to check that Definition 3.1 is equivalent to [8, Definition 2.2]. Indeed the map $\epsilon\colon P \to s^{-n}\mathbb{Q}$ required in [8, Definition 2.2-2.3] is obtained by $\epsilon := \theta_P(1)$, where 1 is the unit of P. Conversely, the isomorphism θ_P is obtained from such a map ϵ by $(\theta_P(x))(y) := \epsilon(xy)$ for $x, y \in P$.

The main result of [8] is that any CDGA whose cohomology is 1–connected and satisfies Poincaré duality is weakly equivalent to some 1–connected Poincaré duality CDGA. The aim of this section is to prove the following relative version of that result:

Proposition 3.4 *Let* $\psi\colon A \to B$ *be a morphism of CDGA such that* $\mathrm{H}(A)$ *is a* 1*–connected Poincaré duality algebra in dimension* n*,* $\mathrm{H}^{\geq \frac{n}{2}-1}(B) = 0$*,* $\mathrm{H}(B)$ *is* 1*–connected and of finite type, and* $\mathrm{H}^2(\psi)$ *is surjective.*

Then ψ *is weakly equivalent to some surjective CDGA morphism*

$$\varphi\colon P \twoheadrightarrow Q$$

such that P *is a* 1*–connected Poincaré duality CDGA in dimension* n*,* Q *is* 1*–connected and* $Q^{\geq \frac{n}{2}-1} = 0$*.*

Moreover, when A *is* 1*–connected of finite type,* $A^2 \subset \ker(d)$*,* $B^{\geq \frac{n}{2}-1} = 0$ *and* ψ *is surjective, then we can take* $Q = B$ *and* φ *such that there is a quasi-isomorphism* $\lambda\colon A \xrightarrow{\simeq} P$ *such that* $\psi = \varphi\lambda$*.*

In any case the morphism φ *can be constructed explicitly out of the morphism* ψ*.*

A key ingredient to prove this proposition is the following:

Proposition 3.5 *Let* A *be a CDGA such that* $\mathrm{H}(A)$ *is a Poincaré duality algebra in dimension* n*. Assume moreover that* $n \geq 7$*,* A *is* 1*–connected of finite type and* $A^2 \subset \ker d$*.*

Then one can construct a CDGA quasi-isomorphism

$$\lambda\colon A \xrightarrow{\simeq} P$$

such that P *is a Poincaré duality CDGA in dimension* n *and* λ *is an isomorphism in degrees* $< \frac{n}{2} - 1$*.*

This proposition is an improvement of the main result of [8] in the sense that the quasi-isomorphism λ to the Poincaré duality CDGA is an isomorphism below about half the dimension.

If we take for granted Proposition 3.5 then we can prove Proposition 3.4 as follows:

Proof of Proposition 3.4 If $n \leq 6$ then, since $\mathrm{H}^{\geq \frac{n}{2}-1}(B) = 0$ and $\mathrm{H}(B)$ is 1–connected, we have $\mathrm{H}(B) = \mathbb{Q}$ and the proposition is a consequence of the main result of [8] by taking $Q = \mathbb{Q}$. In fact when $n \leq 6$, A is formal because $\mathrm{H}(A)$ is a 1–connected Poincaré duality algebra of low dimension, and we can take $P = \mathrm{H}(A)$.

Assume now that $n \geq 7$. By passing to Sullivan models it is easy to see that ψ is weakly equivalent to a surjective morphism between 1–connected finite-type CDGAs. Thus, without loss of generality we assume that $\psi\colon A \to B$ is already like that. Moreover, since $\mathrm{H}^{\geq \frac{n}{2}-1}(B)=0$, by modding out B by a suitable acyclic ideal we get a surjective quasi-isomorphism

$$\pi\colon B \xrightarrow{\simeq} Q,$$

where $Q^{\geq \frac{n}{2}-1}=0$.

By Proposition 3.5, there is a quasi-isomorphism $\lambda\colon A \xrightarrow{\simeq} P$ which is an isomorphism in degrees $< \frac{n}{2}-1$ and such that P is a Poincaré duality CDGA. Since $(\ker\lambda)^{<\frac{n}{2}-1}=0$ and $Q^{\geq \frac{n}{2}-1}=0$, the morphism $\pi\psi$ extends along λ into the desired morphism $\varphi\colon P \twoheadrightarrow Q$. □

The rest of this section is devoted to the proof of Proposition 3.5. Since the techniques used here will not appear in the rest of this paper, the readers can safely jump to the next section if they wish. The proof is based on techniques of [8] and we assume that the reader is familiar with the notation and proofs of that paper. Let us quickly recall from that paper the two key notions of *orientation* and of *orphan*.

An *orientation* (in degree n) of a CDGA (A, d_A) is a chain map

$$\epsilon\colon A \to s^{-n}\,\mathbb{Q}$$

that is surjective in cohomology [8, Definition 2.3]. We then say that (A,d,ϵ) is an *oriented CDGA*. Of course, ϵ is completely determined by the linear map $\epsilon\colon A^n \to \mathbb{Q}$ since ϵ is zero in degrees other than n, and it is a chain map if and only if $\epsilon(d(A^{n-1}))=0$.

Given an oriented CDGA, (A,d,ϵ), its differential ideal of *orphans* is [8, Definition 3.1 and Proposition 3.2]

$$\mathcal{O}=\mathcal{O}(A,\epsilon) := \{a\in A \mid \epsilon(ab)=0 \text{ for all } b\in A\}.$$

The main interest of the notion of orphans is that if A is 1–connected of finite type and $\mathrm{H}(A)$ is a Poincaré duality algebra in degree n, then $P=A/\mathcal{O}$ is a Poincaré duality CDGA in degree n [8, Proposition 3.3]. Moreover, if $\mathcal{O}$ is acyclic then P is a Poincaré duality CDGA quasi-isomorphic to A.

The strategy of the proof of Proposition 3.5 is to build a Sullivan extension $(A,d) \rightarrowtail (\widehat{A} := A\otimes\wedge V, \widehat{d})$ with a suitable orientation $\widehat{\epsilon}$ such that $V^{<\frac{n}{2}-1}=0$ and such that the ideal of orphans $\widehat{\mathcal{O}}$ in $\widehat{A}$ is acyclic and without elements of degree $<\frac{n}{2}-1$. Then $\lambda\colon A \xrightarrow{\simeq} \widehat{A}/\widehat{\mathcal{O}}$ will be the desired quasi-isomorphism to a PDCDGA.

For the sake of the construction of this oriented extension we need the following definition:

Definition 3.6 Let (A, d, ϵ) be an oriented CDGA.

(i) The oriented CDGA (A, ϵ) has *no orphans in degrees* $\leq p$ if $(\mathcal{O}(A, \epsilon))^i = 0$ for $i \leq p$.

(ii) An *acyclic oriented Sullivan extension* is a Sullivan extension

$$(A, d) \overset{\simeq}{\rightarrowtail} (\widehat{A} := A \otimes \wedge V, \widehat{d})$$

that is a quasi-isomorphism and is equipped with an orientation $\widehat{\epsilon}\colon \widehat{A} \to s^{-n}\,\mathbb{Q}$ that extends $\epsilon\colon A \to s^{-n}\,\mathbb{Q}$.

(iii) The acyclic oriented Sullivan extension (ii) *adds no orphans in degree* $\leq q$ if

$$(\mathcal{O}(\widehat{A}, \widehat{\epsilon}))^i \subset (\mathcal{O}(A, \epsilon))^i \quad \text{for } i \leq q.$$

(iv) The acyclic oriented Sullivan extension (ii) *adds no generators in degree* $\leq m$ if

$$V^i = 0 \quad \text{for } i \leq m.$$

The construction of the desired oriented extension $(\widehat{A}, \widehat{\epsilon}, \widehat{d})$ is by a two-step induction based on two lemmas. The first lemma (Lemma 3.7) constructs an extension whose ideal of orphans is acyclic. The second lemma (Lemma 3.8) starts with an oriented CDGA whose ideal of orphans is acyclic and eliminates the orphans of the lowest dimension. In other words, the second lemma increases the connectivity of the ideal of orphans, assuming that it was acyclic. Moreover, in both lemmas, no generator is added below half the dimension. Repeating in succession these two lemmas, we eventually obtain an extension in which the ideal of orphans is acyclic and about $\frac{n}{2}$–connected.

Our first lemma, making the ideal of orphans acyclic without adding generators and orphans below half the dimension, is the following:

Lemma 3.7 *Let (A, d, ϵ) be an oriented CDGA of finite type that is 1–connected, such that $A^2 \subset \ker(d)$ and $\mathrm{H}(A, d)$ is a Poincaré duality algebra in degree $n \geq 7$.*

Then (A, d, ϵ) admits an acyclic oriented Sullivan extension that adds no orphans in degrees $\leq \frac{n}{2} - 1$ or generators in degrees $< \frac{n}{2} - 1$ and whose set of orphans is acyclic.

Proof The set of orphans of (A, d, ϵ) is $\frac{n}{2}$–half-acyclic (see [8, Definition 3.5] and the remark after). Since, by hypothesis, (A, d, ϵ) satisfies [8, (4.1)] we can apply [8, Proposition 5.1] iteratively for all integers k ranging from $\lceil \frac{n}{2} + 1 \rceil$ up to $n + 1$.

More precisely, at each step we construct the extension described in [8, Section 4] for the integer $k \geq \frac{n}{2}+1$. This is an acyclic Sullivan extension defined in the equation [8, (4.4)], which is oriented by [8, Lemma 4.5]. The set of orphans in this extension is k–half-acyclic by [8, Proposition 5.1].

The new generators of lowest degrees in the extension [8, (4.4)] are the w_i of degree $k-2 \geq \frac{n}{2}-1$. Thus, the extension adds no generator of degrees $< \frac{n}{2}-1$.

Since, by [8, Lemma 4.1], $d\gamma_i = \alpha_i$ and since, by [8, (4.2)], α_i is not the boundary of an orphan in A, there exists $\xi_i \in A$ such that $\epsilon(\gamma_i \xi_i) \neq 0$. By [8, (4.5)(ii)], $\widehat{\epsilon}(w_i d(\xi_i)) = \pm\epsilon(\gamma_i\xi_i) \neq 0$ and therefore w_i is not an orphan in $\widehat{A}$. Thus, the extension adds no orphans in degrees $\leq k-2$, hence in degrees $\leq \frac{n}{2}-1$.

When we reach $k = n+1$, the set of orphans is $(n+1)$–half-acyclic, and therefore is acyclic by [8, Proposition 3.6]. □

The second lemma, increasing the connectivity of the ideal of orphans, assuming that it is acyclic, is the following:

Lemma 3.8 *Let (A,d,ϵ) be as in Lemma 3.7. Assume that its set of orphans is acyclic and that there are no orphans in degrees $< p$ for some integer $1 \leq p < \frac{n}{2}-1$. Then (A,d,ϵ) admits an acyclic oriented Sullivan extension with no orphans in degrees $\leq p$ and which adds no generators in degrees $\leq \frac{n}{2}$.*

Proof Let $\mathcal{O}$ be the ideal of orphans in (A,d,ϵ). Since $\mathcal{O}$ is acyclic and $\mathcal{O}^{<p} = 0$, we have $\mathcal{O}^p \cap \ker(d) = 0$. Let $\{x_1,\ldots,x_r\}$ be a basis of $\mathcal{O}^p$. Consider the acyclic Sullivan extension

$$\widehat{A} := (A \otimes \wedge(u_1,\ldots,u_r,\overline{u}_1,\ldots,\overline{u}_r), \widehat{d})$$

with $\deg(u_i) = n-p-1$, $\deg(\overline{u}_i) = n-p$, $\widehat{d}(u_i) = \overline{u}_i$ and $\widehat{d}(\overline{u}_i) = 0$. We extend the orientation ϵ into an orientation $\widehat{\epsilon}$ of $\widehat{A}$ as follows. Let S be a supplement space of $\mathcal{O}^p \oplus (A^p \cap \ker(d))$ in A^p. Let T be a supplement space of $d(\mathcal{O}^p) \oplus d(S)$ in A^{p+1}. Since $n-p-1 > \frac{n}{2}$, we have

$$\widehat{A}^n = A^n \oplus \mathbb{Q}\{\overline{u}_1,\ldots,\overline{u}_r\} \otimes A^p \oplus \mathbb{Q}\{u_1,\ldots,u_r\} \otimes A^{p+1}.$$

Since

$$A^p = \mathcal{O}^p \oplus (A^p \cap \ker d) \oplus S \quad \text{and} \quad A^{p+1} = d(\mathcal{O}^p) \oplus d(S) \oplus T,$$

there is a unique degree 0 linear map

$$\widehat{\epsilon}\colon \widehat{A}^n \to \mathbb{Q}$$

extending $\epsilon\colon A^n \to \mathbb{Q}$ and such that, for each $1 \le i, j \le r$,

$$\begin{aligned} \widehat{\epsilon}(\overline{u}_i x_j) &= \delta_{ij}, \quad \text{where } \delta_{ij} \text{ is the Kronecker symbol,} \\ \widehat{\epsilon}(\overline{u}_i \ker(d)) &= 0, \\ \widehat{\epsilon}(\overline{u}_i S) &= 0, \\ \widehat{\epsilon}(u_i d(x_j)) &= (-1)^{n-p}\delta_{ij}, \\ \widehat{\epsilon}(u_i d(S)) &= 0, \\ \widehat{\epsilon}(u_i T) &= 0. \end{aligned}$$

Let us check that $\widehat{\epsilon}$ is an orientation. For this we only need to check that $\widehat{\epsilon}(d(\widehat{A}^{n-1}))=0$. We have

$$\widehat{A}^{n-1} = A^{n-1} \oplus \mathbb{Q}\{u_1, \dots, u_r\} \otimes \big(\mathcal{O}^p \oplus (A^p \cap \ker(d)) \oplus S\big) \oplus \mathbb{Q}\{\overline{u}_1, \dots, \overline{u}_r\} \otimes A^{p-1}. \tag{5}$$

For $z \in A^p \cap \ker(d)$, $s \in S$, $a \in A^{p-1}$ and $1 \le i, j \le r$, we compute that

$$\begin{aligned} \widehat{\epsilon}(\widehat{d}(u_i x_j)) &= \widehat{\epsilon}(\overline{u}_i x_j + (-1)^{|u_i|} u_i d(x_j)) = \delta_{ij} - \delta_{ij} = 0, \\ \widehat{\epsilon}(\widehat{d}(u_i z)) &= \widehat{\epsilon}(\overline{u}_i z) = 0, \\ \widehat{\epsilon}(\widehat{d}(u_i s)) &= \widehat{\epsilon}(\overline{u}_i s) \pm \widehat{\epsilon}(u_i d(s)) = 0 \pm 0 = 0, \\ \widehat{\epsilon}(\widehat{d}(\overline{u}_i a)) &= \pm\widehat{\epsilon}(\overline{u}_i d(a)) = 0. \end{aligned}$$

Also $\widehat{\epsilon}(\widehat{d}(A^{n-1})) = \epsilon(d(A^{n-1})) = 0$ since ϵ is an orientation. All of these equations and (5) imply that $\widehat{\epsilon}(\widehat{d}(\widehat{A}^{n-1})) = 0$, hence $\widehat{\epsilon}$ is an orientation on $(\widehat{A}, \widehat{d})$.

This acyclic extension $\widehat{A}$ of A adds no generators in degrees $< n-p-1$, and hence no generators in degrees $\le \frac{n}{2}$ $\big($because $p < \frac{n}{2} - 1\big)$. For the same reasons it adds no orphans in degrees $\le p$. Moreover, all the degree p orphans of A, which are linear combinations of $x_1, \dots, x_r$, are not orphans anymore in $\widehat{A}$ since $\widehat{\epsilon}(\overline{u}_i x_j) = \delta_{ij}$. Thus, $\widehat{A}$ has no orphans in degree $\le p$. □

Proof of Proposition 3.5 The idea of the proof is to apply inductively Lemmas 3.7 and 3.8. Indeed, Lemma 3.7 builds a quasi-isomorphic CDGA whose set of orphans is acyclic. On the other hand, when the set of orphans is acyclic, Lemma 3.8 eliminates the orphans of the lowest degrees $\big($up to degree $< \frac{n}{2} - 1\big)$. Moreover, both constructions add no generators in degree $\frac{n}{2} - 1$.

In more detail, let $p_{\max}$ be the largest integer $< \frac{n}{2} - 1$. Applying Lemmas 3.7, 3.8 and 3.7 again, successively for $p = 1, 2, \ldots, p_{\max}$ we obtain, by composition, an acyclic oriented Sullivan extension $\widehat{A}$ with an acyclic ideal of orphans, with no orphans in degrees $< \frac{n}{2} - 1$, and with no generators added in degree $< \frac{n}{2} - 1$. Therefore, the composite

$$\lambda\colon A \xrightarrow{\simeq} \widehat{A} \xrightarrow{\simeq} \frac{\widehat{A}}{\mathcal{O}(\widehat{A}, \widehat{\epsilon})}$$

is a quasi-isomorphism, and an isomorphism in degrees $< \frac{n}{2} - 1$, to a Poincaré duality CDGA. □

4 Pretty models

In this section we first describe precisely what we call *pretty models*, and next we motivate this definition by showing that these models arise naturally as models of complements of a high-codimension subpolyhedron in a closed manifold.

Suppose given

(i) a Poincaré duality CDGA, P, in dimension n (see Definition 3.1);

(ii) a connected CDGA, Q;

(iii) a CDGA morphism, $\varphi\colon P \to Q$.

By definition of a Poincaré duality CDGA, there exists an isomorphism of P–dg-modules

$$\theta_P\colon P \xrightarrow{\cong} s^{-n}\,\#P \tag{6}$$

and such an isomorphism is unique up to multiplication by a nonzero scalar because P is a free P–module generated by 1.

Let us recall the notion of a *shriek map*:

Definition 4.1 [7, Definition 5.1] Let P be a CDGA such that $H(P)$ is a Poincaré duality algebra in dimension n. A *shriek map* is a morphism of P–dg-modules, $\psi\colon D \to P$, such that $\mathrm{H}^n(\psi)$ is an isomorphism.

This notion was used in [7] to construct CDGA models of the complement of a polyhedron in a closed manifold. In our context, the composite $\varphi^!$ that we define now is a shriek map.

Consider the composite

$$\varphi^!\colon s^{-n}\#Q \xrightarrow{s^{-n}\#\varphi} s^{-n}\#P \xrightarrow{\theta_P^{-1}} P, \tag{7}$$

which is a morphism of P–dg-modules. Since $\mathrm{H}^0(\varphi)$ is an isomorphism, so is $\mathrm{H}^n(\varphi^!)$ and thus $\varphi^!$ is a shriek map.

Assume that

$$\varphi\varphi^!\colon s^{-n}\#Q \to Q$$

is a balanced morphism of Q–dg-modules (see Definition 2.2). By Lemma 2.5 (with $\psi = \varphi^!$ and $D = s^{-n}\#Q$), $\varphi^!$ is also balanced. By Proposition 2.3, the mapping cones

$$P \oplus_{\varphi^!} \mathrm{s}s^{-n}\#Q \quad \text{and} \quad Q \oplus_{\varphi\varphi^!} \mathrm{s}s^{-n}\#Q$$

are CDGAs and

$$\varphi \oplus \mathrm{id}\colon P \oplus_{\varphi^!} \mathrm{s}s^{-n}\#Q \to Q \oplus_{\varphi\varphi^!} \mathrm{s}s^{-n}\#Q \tag{8}$$

is a CDGA morphism.

Definition 4.2 Let $\varphi\colon P \to Q$ be a CDGA morphism with P a Poincaré duality CDGA in dimension n, consider $\varphi^!\colon s^{-n}\#Q \to P$ defined at (7) and assume that $\varphi\varphi^!$ is a balanced morphism of Q–dg-modules. Then the CDGA morphism

$$\varphi \oplus \mathrm{id}\colon P \oplus_{\varphi^!} \mathrm{s}s^{-n}\#Q \to Q \oplus_{\varphi\varphi^!} \mathrm{s}s^{-n}\#Q \tag{9}$$

is called the *pretty model associated to* φ. If moreover φ is surjective, we say that (9) is a *surjective pretty model*. It will be called a *(surjective) pretty model of the pair of spaces* $(W, \partial W)$ when (9) is a CDGA model of the map $A_{\mathrm{PL}}(W) \to A_{\mathrm{PL}}(\partial W)$.

Proposition 4.3 *If (9) is a surjective pretty model then the projection*

$$\pi\colon P \oplus_{\varphi^!} \mathrm{s}s^{-n}\#Q \xrightarrow{\simeq} P/I,$$

where $I := \varphi^!(s^{-n}\#Q)$, *is a quasi-isomorphism of CDGAs.*

Proof $I=\varphi^!(s^{-n}\#Q)$ is a differential ideal of the CDGA P because it is the image of a morphism of P–dg-modules. Since φ is surjective, by duality, $\varphi^!$ is injective and we have a short exact sequence

$$0\to s^{-n}\#Q\xrightarrow{\varphi^!}P\xrightarrow{\text{proj}}P/I\to 0.$$

Thus,

$$\pi:=(\text{proj},0)\colon P\oplus_{\varphi^!}ss^{-n}\#Q\to P/I$$

is a quasi-isomorphism of CDGAs. □

Corollary 4.4 *If a Poincaré duality pair $(W,\partial W)$ admits a surjective pretty model (9), then a CDGA model of W is given by P/I, where P is a Poincaré duality CDGA and $I=\varphi^!(s^{-n}\#Q)$ is a differential ideal.*

To motivate the above definition of pretty models, let us show how pretty models appear naturally as models of the complement of a high-codimension subpolyhedron in a closed manifold. Let V be a simply connected closed triangulated manifold and let $K\subset V$ be a subpolyhedron. Let T be a regular neighbourhood of K in V, which means that $T\subset V$ is a codimension 0 compact submanifold (with boundary) that retracts by deformation on K. Let $W=\overline{V\setminus T}$ be the closure of the complement of T in V. Then W is a compact manifold with boundary $\partial W=\partial T$. The next proposition shows that this complement $(W,\partial W)$ admits a pretty surjective model:

Proposition 4.5 *With the notation of the above paragraph, assume that V is of dimension n and that K is 2–connected of dimension $\dim(K)<\frac{n}{2}-1$. Then the inclusion $K\hookrightarrow V$ admits a CDGA model*

$$\varphi\colon P\twoheadrightarrow Q,$$

where P is a Poincaré duality CDGA in dimension n, Q is 2–connected, $Q^{\geq\frac{n}{2}-1}=0$, φ is surjective and the morphism (9) is a surjective pretty model of $(W,\partial W)$.

Proof By Proposition 3.4, $K\hookrightarrow V$ admits a surjective model φ with the properties in the statement.

We now rely on [7] to prove that (9) is a model of the complement W. Consider the morphism $\varphi^!\colon s^{-n}\#Q\to P$ defined as the composite (7). It is a *shriek map* in the sense of [7, Definition 5.1] or Definition 4.1 above. By the main result of that paper, their Theorem 1.2, the pretty model (9) is then a CDGA model of the inclusion $\partial W\hookrightarrow W$ (using the fact that $\varphi\varphi^!=0$, for degree reasons, and hence is balanced). □

Any compact manifold with boundary, W, arises as the complement of a subpolyhedron in a closed n–manifold. Indeed we can consider the double $V := W \cup_{\partial W} W$, which is a closed manifold. Then W is the complement of the second copy of W in V. If we assume moreover that W retracts by deformation on some high-codimensional subpolyhedron $K \subset W$ then we can look at the second copy of W in V as a thickening T of K. In other words, the first copy of W is the complement of a thickening of K in V. Of course, there does not always exists such a high-codimensional deformation retract K, which explains why Proposition 4.5 does not directly imply that any compact manifold with boundary admits a pretty model.

5 Disk bundles over Poincaré duality spaces

In this section we prove that we can construct explicit pretty models for the total space of an even-dimensional disk bundle over a closed manifold.

Theorem 5.1 *Let ξ be a real vector bundle of even rank over a simply connected Poincaré duality space. Then the pair $(D\xi, S\xi)$ of associated (disk, sphere) bundles admits a surjective pretty model.*

Moreover, this model can be explicitly constructed out of any CDGA model of the base and from the Euler class of the bundle.

Proof Assume that ξ is a vector bundle of rank $2k$ with base a Poincaré duality space in dimension $n-2k$ for some integer $n>2k$. Let Q be a Poincaré duality CDGA model of the base (which, by [8] or Proposition 3.5, exists and can be explicitly constructed out of any CDGA model of the base) and let $e \in Q^{2k} \cap \ker d$ be a representative of the Euler class of the bundle. Then a CDGA model of the sphere bundle is given by

$$Q \rightarrowtail (Q \otimes \wedge z, dz = e) \tag{10}$$

with $\deg(z) = 2k-1$, and this is also a model of the pair $(D\xi, S\xi)$.

We look for a CDGA model of $(D\xi, S\xi)$ of the form

$$P \oplus_{\varphi^!} \mathrm{ss}^{-n} \# Q \to Q \oplus_{\varphi\varphi^!} \mathrm{ss}^{-n} \# Q.$$

Notice that we already have a CDGA model for $S\xi$ of the form

$$(Q \otimes \wedge z, dz = e)$$

and, by Poincaré duality of Q, we have isomorphisms of Q–modules

$$Q \otimes \wedge z \cong Q \oplus zQ \cong Q \oplus \mathrm{s}s^{-n}\#Q,$$

which is the desired form for the second term of a pretty model of $(D\xi, S\xi)$ (not taking into account the differential). We now look for a suitable CDGA P and a morphism φ.

Denote by $\overline{z}$ a generator of degree $2k$ and define the CDGA

$$P := \left(\frac{Q \otimes \wedge \overline{z}}{(\overline{z}^2 - e\overline{z})}, D\overline{z} = 0 \right),$$

where $(\overline{z}^2 - e\overline{z})$ is the ideal in $Q \otimes \wedge \overline{z}$ generated by this difference. Then P is a Poincaré duality CDGA in dimension n. As vector spaces we have $P \cong Q \oplus Q\overline{z}$.

Define

$$\varphi\colon\ P \to Q$$

by $\varphi(q_1 + q_2\overline{z}) = q_1 + eq_2$ for $q_1, q_2 \in Q$, which is a surjective CDGA morphism.

We will show that the pretty model associated to φ is weakly equivalent to the CDGA morphism (10), which will establish the theorem. Consider the diagram of P–dg-modules

$$\begin{array}{ccccc}
s^{-n}\#Q & \xrightarrow{\ s^{-n}\#\varphi\ } & s^{-n}\#P & \xrightarrow[\cong]{\ \theta_P^{-1}\ } & P \\
\cong \big\uparrow s^{-2k}\,\theta_Q & & & \nearrow_{\Phi^!} & \\
s^{-2k}\,Q & & & &
\end{array}$$

where θ_Q and θ_P are Poincaré duality isomorphisms for Q and P, and set

$$\varphi^! := \theta_P^{-1} \circ (s^{-n}\#\varphi) \quad \text{and} \quad \Phi^! := \varphi^! \circ (s^{-2k}\,\theta_Q).$$

We now prove that we can assume that $\Phi^!$ is given by

$$\Phi^!(s^{-2k}\,q) = q\overline{z}. \tag{11}$$

Indeed $\Phi^!(s^{-2k}\,1) = \alpha + \lambda\overline{z}$ for some $\alpha \in Q^{2k}$ and some $\lambda \in Q^0 = \mathbb{Q}$. Since $\Phi^!$ is a morphism of P–dg-modules,

$$\overline{z}\Phi^!(s^{-2k}\,1) = \Phi^!(\overline{z}\,s^{-2k}\,1),$$

which implies, using that

$$\overline{z}\,s^{-2k}\,1 = \varphi(\overline{z})\,s^{-2k}\,1 = e\,s^{-2k}\,1,$$

that

$$\alpha\bar{z} + \lambda\bar{z}^2 = e\alpha + \lambda e\bar{z}.$$

Therefore, since $\bar{z}^2 = e\bar{z}$, $\alpha = 0$. Also $\lambda \neq 0$ because $\Phi^!$ induces an isomorphism in $\mathrm{H}^n(-)$, since $s^{-n}\#\varphi$ does. We can replace the Poincaré duality isomorphism θ_P by the isomorphism $\lambda\theta_P$ and we get

$$\Phi^!(s^{-2k}\,1) = \bar{z},$$

which implies (11).

A direct computation shows that $\varphi\Phi^!$ is balanced, and hence also $\varphi\varphi^!$ is. The pretty model associated to φ is isomorphic to

$$\varphi \oplus \mathrm{id}\colon\ P \oplus_{\Phi^!} \mathrm{ss}^{-2k}\, Q \to Q \oplus_{\varphi\Phi^!} \mathrm{ss}^{-2k}\, Q.$$

The codomain of $\varphi \oplus \mathrm{id}$ is isomorphic to $(Q \otimes \wedge z, dz = e)$ because $\varphi\Phi^!(s^{-2k}\,1) = e$. The inclusion of Q in the domain of $\varphi \oplus \mathrm{id}$,

$$Q \hookrightarrow P \hookrightarrow P \oplus_{\Phi^!} \mathrm{ss}^{-2k}\, Q,$$

is clearly a quasi-isomorphism. Thus, $\varphi \oplus \mathrm{id}$ is weakly equivalent to (10), which is a CDGA model of $(D\xi, S\xi)$, and the theorem is proved. □

6 Poincaré duality spaces that retract rationally on their half-skeleton

In this section we exhibit in Theorem 6.6 a quite large class of Poincaré duality pairs $(W, \partial W)$ that admit a surjective pretty model. The main hypothesis (in addition to some connectivity hypotheses) is on the boundary ∂W, which should *retract rationally on its half-skeleton* in the sense of the following definition:

Definition 6.1 Let M be a simply connected Poincaré duality space in dimension $n-1$. We say that M *retract rationally on its half-skeleton* if M admits a CDGA model A together with a morphism of connected CDGAs

$$\rho\colon Q \to A$$

such that

(i) $\mathrm{H}^{\geq \frac{n}{2}-1}(Q) = 0$, and

(ii) $\mathrm{H}^k(\rho)$ is an isomorphism for $k \leq \frac{n}{2}$.

Remark 6.2 The terminology comes from the fact that the conditions of the definition imply that the realization of ρ can be thought of as a retraction of M on a skeleton of half the dimension, as is clear from diagram (12) in the proof of Proposition 6.5.

Remark 6.3 Poincaré duality of M and (i)–(ii) in the previous definition imply that M has no cohomology about the middle dimension. More precisely, if n is even then $\mathrm{H}^{n/2-1}(M) = \mathrm{H}^{n/2}(M) = 0$, and if n is odd then $\mathrm{H}^{(n-1)/2}(M) = 0$.

Example 6.4 (1) Consider the total space W of a d–dimensional disk bundle over a closed manifold of dimension $< d-1$. Then ∂W retracts rationally on its half-skeleton, as one checks by building a model of the sphere bundle.

(2) Our next example is very much related to Proposition 4.5 and its setting. Let K be a compact polyhedron embedded in a closed triangulated manifold V of dimension n. Assume that K and V are 1–connected and $\dim K < \frac{n}{2} - 1$. Let T be a regular neighbourhood of K in V. Then $M := \partial T$ retracts rationally on its half-skeleton. Indeed, Theorem 1.2 of [7] gives a model of ∂T of the form $Q \oplus sD$, where Q is a CDGA model of K and $D \simeq s^{-n} \# Q$, and the conclusion follows.

(3) As a special case of the previous example consider a 1–connected polyhedron K embedded in $S^n = \mathbb{R}^n \cup \{\infty\}$ with $n \geq 2\dim K + 3$. Then the boundary of a thickening of K in S^n retracts rationally on its half-skeleton.

Our next proposition is a characterization of Poincaré duality spaces that retract rationally on their half-skeleton in terms of a nice CDGA model of the space.

Proposition 6.5 *Let n be an integer > 2 and M be a simply connected Poincaré duality space in dimension $n-1$. Then M retracts rationally on its half-skeleton if and only if there exists a connected CDGA, Q, such that*

(a) $Q^{\geq \frac{n}{2}-1} = 0$, *and*

(b) $Q \oplus \mathrm{s}s^{-n} \# Q$ *is a CDGA model of M,*

where $Q \oplus \mathrm{s}s^{-n} \# Q$ is the mapping cone of the zero map $0\colon s^{-n} \# Q \to Q$, which is balanced, equipped with the semitrivial CDGA structure.

Proof It is clear that (a) and (b) imply that M retracts rationally on its half-skeleton.

Let us prove the converse. Let

$$\rho'\colon Q' \to A'$$

be a morphism between connected CDGAs that satisfies (i)–(ii) of Definition 6.1 (with the added decoration "prime") and A' is a CDGA model of M. Consider a minimal Sullivan extension $\widehat{\rho}$,

$$Q' \xrightarrow{\widehat{\rho}} (Q' \otimes \wedge V, D') \xrightarrow{\simeq} A',$$

that factors ρ'. Let h be the integer such that $n = 2h$ or $n = 2h+1$. Since $\mathrm{H}^{\leq h}(\rho)$ is an isomorphism and $\mathrm{H}^{>h}(Q') = 0$, by minimality we have $V^{\leq h} = 0$. Since Q' is connected and $\mathrm{H}^{\geq \frac{n}{2}-1}(Q') = 0$, there exists an acyclic ideal $J \subset Q'$ such that $Q'^{\geq \frac{n}{2}-1} \subset J$. Set $Q := Q'/J$ and consider the pushout of CDGAs

$$\begin{array}{ccc} Q' & \xrightarrow{\rho'} & (Q' \otimes \Lambda V, D') \\ \simeq\downarrow & \text{pushout} & \downarrow\simeq \\ Q & \xrightarrow[\rho]{} & (Q \otimes \Lambda V, D) \end{array}$$

and set $A := (Q \otimes \wedge V, D)$. Note that ρ is an isomorphism in degrees $\leq h$. Also it endows A with the structure of a Q–dg-module.

Let S be a complement of $A^h \cap \ker d$ in A^h and set

$$I := S \oplus A^{>h},$$

which is an ideal since A is connected. For degree reasons and since $\mathrm{H}^{\leq h}(\rho)$ is an isomorphism, the composite

$$Q \xrightarrow{\rho} A \xrightarrow{\text{proj}} A/I$$

is a quasi-isomorphism.

By the lifting lemma [3, Proposition 14.6], in the diagram

$$\begin{array}{ccc} Q & = & Q \\ \rho\downarrow & \overset{\pi}{\nearrow} & \downarrow\simeq \\ A & \xrightarrow[\text{proj}]{} & A/I \end{array} \tag{12}$$

we get a CDGA morphism π that makes the upper-left triangle commute and the lower-right triangle commute up to homotopy; in other words, Q is a retract of A.

Since $\mathrm{H}(A)$ satisfies Poincaré duality in dimension $n-1$, there is a quasi-isomorphism of A–dg-modules, hence of Q–dg-modules,

$$\theta\colon A \xrightarrow{\simeq} \mathrm{ss}^{-n}\#A,$$

and we have the diagram

$$\begin{array}{ccc} A & \xrightarrow[\theta]{\simeq} & \mathrm{ss}^{-n}\#A \\ \rho\uparrow & & \downarrow \mathrm{ss}^{-n}\#\rho \\ Q & & \mathrm{ss}^{-n}\#Q \end{array}$$

Set $\lambda = (\mathrm{ss}^{-n}\#\rho)\circ\theta$, which is a morphism of Q–dg-modules. Since ρ induces an isomorphism in homology in degrees $\leq \frac{n}{2}$, we get that $\mathrm{ss}^{-n}\#\rho$, and hence λ, induces an isomorphism in homology in degrees $\geq \frac{n}{2}-1$.

Consider the Q–dg-module morphism

$$\gamma = (\pi,\lambda)\colon A \to Q \oplus \mathrm{ss}^{-n}\#Q.$$

For degree reasons and since π (respectively λ) induces isomorphism in homology below (respectively above) degree $\frac{n}{2}$, we get that γ is a quasi-isomorphism.

We prove that γ is a morphism of algebras. Let a, $a' \in A$. If $\deg(a) \leq h$, then, since ρ is an isomorphism in that degree, the multiplication by a is determined by the Q–module structure, and since γ is of Q–module we get that $\gamma(aa') = \gamma(a)\gamma(a')$. The same arguments work if $\deg(a') \leq h$. If both a and a' are of degrees $\geq h+1$, then $\deg(aa') > n$ and then $\gamma(aa') = 0 = \gamma(a)\gamma(a')$ for degree reasons.

Thus, γ is a CDGA quasi-isomorphism and the proposition is proved. □

Theorem 6.6 *Let $(W,\partial W)$ be a Poincaré duality pair of spaces with W simply connected, ∂W 2–connected and $\mathrm{H}^3(W) \to \mathrm{H}^3(\partial W)$ surjective. Assume that ∂W retracts rationally on its half-skeleton. Then $(W,\partial W)$ admits a surjective pretty model.*

Remark 6.7 Interestingly enough, the hypothesis in Theorem 6.6 is mainly on the boundary ∂W. More precisely, when ∂W is 3–connected and retracts rationally on its half-skeleton, any other Poincaré duality pair $(W',\partial W')$ with $\partial W' = \partial W$ and such that W' is also simply connected will also admit a pretty model. One can get many such manifolds W' by performing a connected sum in the *interior* of W with a simply connected closed n–manifold N. Or, more generally, we can modify W by surgeries on its interior, which do not change its boundary. We can thus perform such surgeries on the manifolds from Example 6.4 to get many other examples.

The rest of this section is devoted to the proof of Theorem 6.6. For this we need first to introduce the notion of homotopy kernel and also to discuss some results around a notion in CDGAs analogous to the notion of Poincaré duality spaces.

Definition 6.8 Let $f\colon M \to N$ be a morphism of A–dg-modules. The *homotopy kernel of* f is the A–dg-module mapping cone

$$\mathrm{hoker}(f) := s^{-1}\, N \oplus_{s^{-1} f} M,$$

which comes with a canonical map

$$\mathrm{hoker}(f) \to M, \quad (s^{-1}\, n, m) \mapsto m.$$

The following result is a direct consequence of the five lemma and justifies the terminology "homotopy kernel":

Proposition 6.9 *Let $f\colon M \to N$ be a morphism of A–dg-modules. If f is surjective then the morphism*

$$\varphi\colon \ker f \to \mathrm{hoker}\, f, \quad m \mapsto (0, m),$$

is an A–dg-module quasi-isomorphism.

This notion of homotopy kernel is useful to define the analog of a Poincaré duality pair of spaces at the level of rational models, as we explain now. Recall that a pair of simply connected spaces (X, Y) is a Poincaré duality pair in dimension n if there is an isomorphism of $\mathrm{H}^*(X)$–modules between $\mathrm{H}^*(X)$ and $\mathrm{H}_{n-*}(X, Y)$. For (co)homology over the field of rationals $\mathbb{Q}$, this is equivalent to an isomorphism between $\mathrm{H}^*(X)$ and $s^{-n}\, \#\, \mathrm{H}^*(X, Y)$. Moreover, because of the short exact sequence

$$0 \to A_{\mathrm{PL}}(X, Y) \to A_{\mathrm{PL}}(X) \to A_{\mathrm{PL}}(Y) \to 0,$$

$\mathrm{H}^*(X, Y)$ is the homology of the homotopy kernel of $A_{\mathrm{PL}}(X) \to A_{\mathrm{PL}}(Y)$. This motivates the following:

Definition 6.10 We say that a morphism of CDGA $p\colon A \to B$ has *relative Poincaré duality in dimension n* if there is an isomorphism of $\mathrm{H}(A)$–modules between $\mathrm{H}(A)$ and $\mathrm{H}(s^{-n}\, \#\, \mathrm{hoker}(p))$.

For example, if (X, Y) is a Poincaré duality pair of spaces in dimension n (eg if $(X, Y) = (W, \partial W)$ where W is a compact manifold of dimension n), then $A_{\mathrm{PL}}(X) \to A_{\mathrm{PL}}(Y)$ has relative Poincaré duality in dimension n. Also, when $B = 0$ in the previous definition, $\mathrm{H}(A)$ is a Poincaré duality algebra in dimension n.

Our next lemma lifts relative Poincaré duality from homology to the level of dg-modules, and states a useful "zero-or-quasi-isomorphism" alternative.

Lemma 6.11 *Let $p\colon A \to B$ be a CDGA morphism with relative Poincaré duality in dimension n. Then there exists a quasi-isomorphism of A–dg-modules*

$$A \xrightarrow{\simeq} s^{-n}\#\mathrm{hoker}(p).$$

If, moreover, $\mathrm{H}^0(A) \cong \mathbb{Q}$, then any A–dg-module morphism $A \to s^{-n}\#\mathrm{hoker}(p)$ is either a quasi-isomorphism or homotopic to the zero map.

Proof Since A is a free A–dg-module generated by 1, any A–dg-module morphism $\Phi\colon A \to s^{-n}\#\mathrm{hoker}(p)$ is determined by the cocycle $\Phi(1)$, and for any cocycle $\omega \in (s^{-n}\#\mathrm{hoker}(p))^0$ there exists a unique A–dg-module morphism Φ such that $\Phi(1) = \omega$. Similarly, any $\mathrm{H}(A)$–module morphism $\varphi\colon \mathrm{H}(A) \to \mathrm{H}(s^{-n}\#\mathrm{hoker}(p))$ is determined by $\varphi([1])$.

In particular, if we pick an isomorphism $\varphi_{\mathrm{iso}}\colon \mathrm{H}(A) \xrightarrow{\cong} \mathrm{H}(s^{-n}\#\mathrm{hoker}(p))$ (which exists by the relative Poincaré duality hypothesis) and if ω is a cocycle representing $\varphi_{\mathrm{iso}}([1])$, then the A–dg-module morphism $\Phi\colon A \to s^{-n}\#\mathrm{hoker}(p)$ defined by $\Phi(1) = \omega$ is such that $\mathrm{H}(\Phi) = \varphi_{\mathrm{iso}}$, and hence Φ is a quasi-isomorphism.

When $\mathrm{H}^0(A) \cong \mathbb{Q}$, given any morphism $\Phi\colon A \to s^{-n}\#\mathrm{hoker}(p)$, either $[\Phi(1)] = 0$ in $\mathrm{H}^0(s^{-n}\#\mathrm{hoker}(p))$, in which case Φ is nullhomotopic, or $[\Phi(1)]$ is a nonzero multiple of $\varphi_{\mathrm{iso}}[1]$ and then $\mathrm{H}(\Phi)$ is a nonzero multiple of the isomorphism φ_{iso} and hence Φ is a quasi-isomorphism. □

When W_1 and W_2 are two compact manifolds of dimension n with the same boundary, $\partial W_1 = \partial W_2$, the homotopy pushout $W_1 \cup_{\partial W_1} W_2$ is a closed manifold of dimension n. A homotopical version of that result is given by [11, Theorem 2.1], which states that if $(W_1, \partial W_1)$ and $(W_2, \partial W_2)$ are Poincaré duality pairs in dimension n with $\partial W_1 = \partial W_2$, then the homotopy pushout $W_1 \cup_{\partial W_1} W_2$ is a Poincaré duality space in dimension n. The following proposition is a rational version of that result:

Proposition 6.12 *Let*

$$\begin{array}{ccc} P & \xrightarrow{q_1} & A_1 \\ {\scriptstyle q_2}\downarrow & h.p.b & \downarrow{\scriptstyle p_1} \\ A_2 & \xrightarrow[p_2]{} & B \end{array}$$

be a homotopy pullback of CDGAs of finite type. Assume that, for $i = 1, 2$,

- $\mathrm{H}^{\geq n-1}(A_i) = 0$, *and*
- $A_i \to B$ *has relative Poincaré duality in dimension* n.

Then $\mathrm{H}(P)$ *is a Poincaré duality algebra in dimension* n.

Proof Factorize each map $p_i\colon A_i \to B$ as a weak equivalence followed by a surjection,

$$A_i \overset{\simeq}{\rightarrowtail} A_i' \overset{p_i'}{\twoheadrightarrow} B.$$

Take the pullback P' of the diagram

$$A_1' \overset{p_1'}{\twoheadrightarrow} B \overset{p_2'}{\twoheadleftarrow} A_2'.$$

Since the maps p_i' are surjective, this pullback is a homotopy pullback. By the universal property of the pullback there is an induced map $P \to P'$ and it is a quasi-isomorphism because the square in the statement is a homotopy pullback. It is then enough to show that $\mathrm{H}(P')$ is a Poincaré duality algebra. Therefore, using this construction, we can assume without loss of generality that the morphisms p_i are surjective and that the square in the statement is a genuine pullback. This implies that q_i are also surjections.

This pullback square of CDGA is also a pullback of P–dg-modules, and therefore we have isomorphisms of P–dg-modules

$$\ker(q_2) \cong \ker(p_1) \quad \text{and} \quad \ker(q_1) \cong \ker(p_2).$$

We then have the following two short exact sequences of P–dg-modules (where γ_2 in the first sequence is the composition of the inclusion of $\ker(q_1)$ in P with the isomorphism $\ker(p_2) \cong \ker(q_1)$, and the second sequence is obtained from the first by switching the roles of the indices 1 and 2 and applying the contravariant functor $s^{-n}\#$):

$$0 \to \ker(p_2) \xrightarrow{\gamma_2} P \xrightarrow{q_1} A_1 \to 0,$$

$$0 \to s^{-n}\#A_2 \xrightarrow{s^{-n}\#q_2} s^{-n}\#P \xrightarrow{s^{-n}\#\gamma_1} s^{-n}\#\ker(p_1) \to 0.$$

Pick a quasi-isomorphism of A_1–dg-modules

$$\alpha_1\colon A_1 \xrightarrow{\simeq} s^{-n}\#\ker(p_1),$$

which exists by relative Poincaré duality of $p_1\colon A_1 \twoheadrightarrow B$ and Lemma 6.11.

Since $\mathrm{H}^{\leq 1}(s^{-n}\#A_2) \cong \mathrm{H}^{\geq n-1}(A_2) = 0$, we get that $\mathrm{H}^0(s^{-n}\#\gamma_1)$ is an isomorphism. Since, moreover, $s^{-n}\#\gamma_1$ is surjective, there exists a cocycle $\omega \in (s^{-n}\#P)^0$ whose image by $s^{-n}\#\gamma_1$ is $\alpha_1(1) \in s^{-n}\#\ker(p_1)$. Let

$$\lambda\colon P \to s^{-n}\#P$$

be the unique P–dg-module morphism such that $\lambda(1) = \omega$. Then $\alpha_1 q_1$ and $(s^{-n}\#\gamma_1)\lambda$ are two P–dg-module morphisms that send $1 \in P^0$ to the same element $\alpha_1(1) \in s^{-n}\#\ker(p_1)$. Since P is generated by 1 and all the morphisms are of P–dg-modules, this implies that $\alpha_1 q_1 = (s^{-n}\#\gamma_1)\lambda$. We therefore have the commutative ladder

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(p_2) & \xrightarrow{\gamma_2} & P & \xrightarrow{q_1} & A_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda'} & & \downarrow{\scriptstyle \lambda} & & {\scriptstyle\simeq}\downarrow{\scriptstyle \alpha_1} & & \\
0 & \longrightarrow & s^{-n}\#A_2 & \xrightarrow{s^{-n}\#q_2} & s^{-n}\#P & \xrightarrow{s^{-n}\#\gamma_1} & s^{-n}\#\ker(p_1) & \longrightarrow & 0
\end{array}$$

where λ' is the restriction of λ to the kernels. We will show that λ' is a quasi-isomorphism, which implies by the five lemma that λ is a quasi-isomorphism and hence that $\mathrm{H}(P)$ is a Poincaré duality algebra.

Apply the contravariant involutive functor $s^{-n}\#$ to the left square of that ladder to obtain the commutative diagram of P–dg-modules

$$\begin{array}{ccc}
s^{-n}\#\ker(p_2) & \xleftarrow{s^{-n}\#\gamma_2} & s^{-n}\#P \\
{\scriptstyle s^{-n}\#\lambda'}\uparrow & & \uparrow{\scriptstyle s^{-n}\#\lambda} \\
A_2 & \xleftarrow[q_2]{} & P
\end{array}$$

Since $\mathrm{H}^0(\lambda) \neq 0$, we have $\mathrm{H}(\lambda) \neq 0$ and hence $\mathrm{H}(s^{-n}\#\lambda) \neq 0$. Since $\mathrm{H}(P)$ is generated by $[1] \in \mathrm{H}^0(P)$, this implies that $\mathrm{H}^0(s^{-n}\#\lambda) \neq 0$. Since $\mathrm{H}^{\geq n-1}(A_1) = 0$, $\mathrm{H}^0(s^{-n}\#\gamma_2)$ is an isomorphism, which implies that the composite

$$\mathrm{H}^0((s^{-n}\#\gamma_2)(s^{-n}\#\lambda))\colon \mathrm{H}^0(P) \to \mathrm{H}^0(s^{-n}\#\ker(p_2))$$

is nontrivial, and hence the morphism $\mathrm{H}(s^{-n}\#\lambda')$ is not the zero map. Since the CDGA map $q_2\colon P \to A_2$ is a surjection, the P–dg-module morphism $s^{-n}\#\lambda'$ is also a morphism of A_2–dg-modules. By Lemma 6.11, since $s^{-n}\#\lambda'$ is not nullhomotopic, it is a quasi-isomorphism. Therefore, λ' is also a quasi-isomorphism and the proposition is proved. $\square$

We are now ready for the proof of Theorem 6.6.

Proof of Theorem 6.6 Since ∂W retracts rationally on its half-skeleton, by Proposition 6.5 there exists a connected CDGA, Q, of finite type such that $Q^{\geq\frac{n}{2}-1}=0$ and $A_{\mathrm{PL}}(\partial W)\simeq Q\oplus \mathrm{ss}^{-n}\#Q$. Since ∂W is 2–connected, we can assume that Q is 2–connected and $Q^3\subset\ker(d)$. Since W is simply connected and $\mathrm{H}^3(W)\to\mathrm{H}^3(\partial W)$ is surjective, there exists a 1–connected CDGA model R of $A_{\mathrm{PL}}(W)$ such that R is of finite type and $R^2\subset\ker(d)$, and a surjective morphism

$$\psi\colon R\twoheadrightarrow Q\oplus\mathrm{ss}^{-n}\#Q$$

that is a CDGA model of $A_{\mathrm{PL}}(W)\to A_{\mathrm{PL}}(\partial W)$.

Consider the pullback diagram in CDGA

$$\begin{array}{ccc} P' & \xrightarrow{\bar{\psi}} & Q \\ {\scriptstyle i}\downarrow & \text{pullback} & \downarrow{\scriptstyle \iota} \\ R & \xrightarrow{\psi} & Q\oplus\mathrm{ss}^{-n}\#Q \end{array} \tag{13}$$

where ι is the obvious inclusion. This pullback is a homotopy pullback because ψ is surjective.

The strategy of the proof is as follows: Using Proposition 6.12 we will show that $\mathrm{H}(P')$ is a Poincaré duality algebra, and by the results of Section 3 we can replace P' by a Poincaré duality CDGA, P. The bottom morphism ψ in the above square is a CDGA model of $(W,\partial W)$. Consider the mapping cone of the natural map from the homotopy kernel of i (respectively of ι) to P' (respectively to Q). These mapping cones are weakly equivalent, as P'–dg-modules, to R and $Q\oplus\mathrm{ss}^{-n}\,Q$, respectively. Moreover, the natural map induced between these mapping cones is weakly equivalent to the morphism ψ, as a morphism of P'–dg-modules. Since the square is a homotopy pullback, the homotopy kernels of i and ι are weakly equivalent to each other, and hence to $s^{-n}\#Q$, as is clear from the right vertical map. The map induced between those mapping cones is therefore weakly equivalent, as a P'–dg-module map, to the pretty model (9), and it is also weakly equivalent to the model ψ of $(W,\partial W)$. A connectivity argument will prove that these maps are weakly equivalent as CDGA morphisms, which gives our pretty model for the Poincaré duality pair. The rest of the proof develops the details.

The morphism ψ satisfies relative Poincaré duality because it is a model of the inclusion $\partial W \hookrightarrow W$. Moreover, $\mathrm{hoker}(\iota)$ is weakly equivalent to $s^{-n}\#Q$ as a Q–dg-module, hence ι also satisfies relative Poincaré duality in dimension n. Moreover, $\mathrm{H}^{\geq n-1}(R) \cong \mathrm{H}^{\geq n-1}(W) \cong \#\mathrm{H}^{\leq 1}(W,\partial W) = 0$ because W and ∂W are simply connected, and $\mathrm{H}^{\geq n-1}(Q) = 0$. Proposition 6.12 implies that $\mathrm{H}(P')$ is a Poincaré duality algebra in degree n.

Since R is 1–connected of finite type and $R^2 \subset \ker(d)$, the same is true for P' because $P' \subset R$. The morphism $\bar{\psi}$ is surjective and $Q^{\geq \frac{n}{2}-1} = 0$. Therefore, by Proposition 3.4, we can factorize $\bar{\psi}$ in CDGA as

$$\begin{array}{ccc} P' & \xrightarrow{\quad\bar{\psi}\quad} & Q \\ & {\scriptstyle\lambda}\searrow^{\simeq} \quad \nearrow\!\!\!\!\nearrow{\scriptstyle\varphi} & \\ & P & \end{array}$$

where P is a 1–connected Poincaré duality CDGA and φ is surjective.

Since (13) is a homotopy pullback diagram, $\mathrm{hoker}(i)$ is weakly equivalent as a P'–dg-module to $\mathrm{hoker}(\iota) \simeq s^{-n}\#Q$. Therefore, there exists a cofibrant P'–dg-module D with weak equivalences

$$\mathrm{hoker}(i) \xleftarrow[\gamma']{\simeq} D \xrightarrow[\gamma]{\simeq} s^{-n}\#Q.$$

Set

$$\varphi^! := \theta_P^{-1} \circ s^{-n}\#\varphi\colon s^{-n}\#Q \to s^{-n}\#P \xrightarrow{\cong} P,$$

consider the canonical map $l\colon \mathrm{hoker}(i) \to P'$ and recall that $\lambda\colon P' \to P$ is the quasi-isomorphism used to factorize the morphism $\bar{\psi}$ through P.

Since $\mathrm{H}^0(\varphi)$ is an isomorphism, we get that $\mathrm{H}^n(s^{-n}\#\varphi)$ and hence $\mathrm{H}^n(\varphi^!)$ are also isomorphisms. In other words, $\varphi^!$ is a *shriek map* or *top-degree map* in the sense of [7, Definition 5.1]. Also, $\mathrm{H}^{\geq n-1}(R) \cong \mathrm{H}^{\geq n-1}(W) \cong \mathrm{H}_{\leq 1}(W,\partial W) = 0$ implies that $\mathrm{H}^n(l)$ is an isomorphism, and hence $\lambda \circ l$ is also a shriek map. Therefore, $\varphi^! \circ \gamma$ and $\lambda \circ l \circ \gamma'$ are both shriek maps $D \to P$. Since $\mathrm{H}^n(D) \cong \mathrm{H}^n(s^{-n}\#Q) \cong \mathbb{Q}$ and $\mathrm{H}(P)$ is a Poincaré duality algebra, Proposition 5.6 of [7] (which asserts that the homotopy class of a shriek map is unique up to a multiplicative nonzero constant) implies that, after maybe replacing γ by some multiple by a nonzero scalar, the left pentagon of the following diagram of P'–dg-modules commutes up to homotopy and the right square commutes:

(14)
$$\begin{array}{ccccc} \operatorname{hoker} i & \xrightarrow{\ l\ } & P' & \xrightarrow{\ \bar\psi\ } & Q \\ {\scriptstyle\simeq}\big\uparrow{\scriptstyle\gamma'} & & \Big| & & \Big\| \\ D & \sim & {\scriptstyle\lambda}\Big\downarrow{\scriptstyle\simeq} & & \Big\| \\ {\scriptstyle\simeq}\big\downarrow{\scriptstyle\gamma} & & \Big| & & \Big\| \\ s^{-n}\#Q & \xrightarrow{\ \varphi^!\ } & P & \xrightarrow{\ \varphi\ } & Q \end{array}$$

Composition induces a P'–dg-modules morphism between the mapping cone of the morphism l and the mapping cone of the morphism $\bar\psi \circ l$,

(15)
$$\bar\psi \oplus \mathrm{id}\colon\ P' \oplus_l s\operatorname{hoker}(i) \to Q \oplus_{\bar\psi l} s\operatorname{hoker}(i).$$

The homotopy commutative diagram (14) implies that this last morphism is weakly equivalent as a P'–dg-module to the morphism of P'–dg-modules

(16)
$$\varphi \oplus \mathrm{id}\colon\ P \oplus_{\varphi^!} \mathrm{ss}^{-n}\#Q \to Q \oplus_{\varphi\varphi^!} \mathrm{ss}^{-n}\#Q.$$

Since diagram (13) is a pullback, the map induced by the top horizontal map $\bar\psi$ between the mapping cones of the inclusions of the vertical homotopy kernels,

$$P' \oplus_l s\operatorname{hoker}(i) \to Q \oplus_{l'} s\operatorname{hoker}(\iota),$$

is a P'–dg-module model of the bottom horizontal map ψ. Because of the pullback, these homotopy kernels $\operatorname{hoker}(i)$ and $\operatorname{hoker}(\iota)$ are weakly equivalent, therefore we get that the morphism (15) is also weakly equivalent, as a morphism of P'–dg-modules, to $\psi\colon R \to Q \oplus \mathrm{ss}^{-n}\#Q$, which is a CDGA model of $A_{\mathrm{PL}}(W) \to A_{\mathrm{PL}}(\partial W)$.

All of this implies that the pretty model $\varphi \oplus \mathrm{id}$ from (16) is weakly equivalent, as a morphism of P'–dg-modules, to the CDGA model ψ of the Poincaré pair $(W, \partial W)$. It remains to prove that this weak equivalence is actually of CDGA morphisms. First note that for degree reasons $\varphi\varphi^! = 0$, thus the codomains of (16) and of ψ are the same. Moreover, $\varphi\varphi^!$ is balanced, as well as $\varphi^!$ by Lemma 2.5, thus the morphism (15) is of CDGAs. The weak equivalence between the morphisms implies that there is cofibrant P'–dg-module, of the form $(P' \otimes X, D)$, weakly equivalent to both R and $P \oplus_{\varphi^!} \mathrm{ss}^{-n}\#Q$, and a commutative diagram of P'–dg-modules

$$\begin{array}{ccccc} & & R & & \\ & {\scriptstyle\simeq}\nearrow{\scriptstyle r} & & \searrow{\scriptstyle\psi} & \\ P' \otimes X & & & & Q \oplus \mathrm{ss}^{-n}\#Q \\ & {\scriptstyle\simeq}\searrow{\scriptstyle r'} & & \nearrow{\scriptstyle\varphi\oplus\mathrm{id}} & \\ & & P \oplus_{\varphi^!} \mathrm{ss}^{-n}\#Q & & \end{array}$$

Moreover, since $(\mathrm{ss}^{-n}\,\#Q)^{\leq\frac{n}{2}}=0$, $\mathrm{incl}\circ\lambda\colon P'\to P\oplus_{\varphi^!}\mathrm{ss}^{-n}\,\#Q$ induces an isomorphism in homology in degrees $\leq\frac{n}{2}$, therefore, by choosing the cofibrant dg-module $P'\otimes X$ minimal, we can assume that $X=\mathbb{Q}\oplus V$ with $V=V^{\geq\frac{n}{2}}$. We can canonically embed this cofibrant P'–dg-module into a relative Sullivan algebra as

$$u\colon (P'\otimes(\mathbb{Q}\oplus V),D)\hookrightarrow(P'\otimes\wedge V,d),$$

where the differential d is the extension of D as a derivation. Since $V^{<\frac{n}{2}}=0$, the morphism u induces an isomorphism in homology in degrees $<n$. We can then add generators of degrees $\geq n-1$ to kill all the homology of $(P'\otimes\wedge V,d)$ in degrees $\geq n$; in other words, there is a CDGA cofibration

$$v\colon (P'\otimes\wedge V,d)\to(P'\otimes\wedge V\otimes\wedge W,\tilde{d})$$

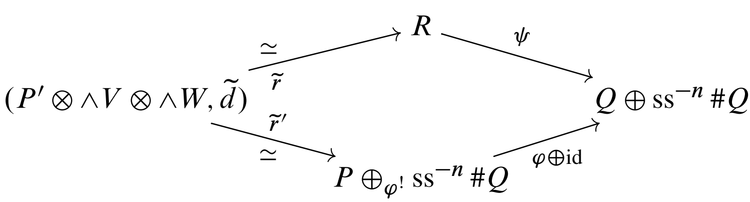
with $W=W^{\geq n-1}$, which induces an isomorphism in homology in degrees $<n$, and such that the homology of the codomain of v vanishes in degrees $\geq n$. Since $\mathrm{H}(P'\otimes(\mathbb{Q}\oplus V),D)\cong\mathrm{H}(W)$ vanishes in degrees $\geq n-1$, we deduce that the composite

$$vu\colon (P'\otimes(\mathbb{Q}\oplus V),D)\to(P'\otimes\wedge V\otimes\wedge W,\tilde{d})$$

is a quasi-isomorphism. Since $W^{<n-1}=0$ and the homologies of R and $P\oplus_{\varphi^!}\mathrm{ss}^{-n}\,\#Q$ vanish in degrees $\geq n-1$, there are no obstructions to extend the P'–dg-module quasi-isomorphisms r and r' to CDGA morphisms

$$\tilde{r}\colon(P'\otimes\wedge V\otimes\wedge W,\tilde{d})\to R\quad\text{and}\quad\tilde{r}'\colon(P'\otimes\wedge V\otimes\wedge W,\tilde{d})\to P\oplus_{\varphi^!}\mathrm{ss}^{-n}\,\#Q.$$

These morphisms $\tilde{r}$ and $\tilde{r}'$ are quasi-isomorphisms because r, r' and vu are. Thus, we get a commutative diagram of CDGAs



which proves that ψ and $\varphi\oplus\mathrm{id}$ are weakly equivalent as CDGA morphisms. □

7 An open question

We finish this article by asking whether every Poincaré duality pair admits a pretty model. This would imply that every boundary manifold ∂W admits a model of the form $Q \oplus_{\varphi\varphi^!} \mathrm{ss}^{-n}\#Q$, which is a very special form of a Poincaré duality CDGA. Thus, a preliminary algebraic question might be the following:

Question 7.1 *Let (A,d) be a CDGA whose homology satisfies Poincaré duality in dimension $n-1$ and whose signature is 0. Does there always exist a CDGA, Q, and a balanced Q–dg-module morphism*

$$\Psi\colon s^{-n}\#Q \to Q$$

such that (A,d) is quasi-isomorphic to the CDGA

$$Q \oplus_{\Psi} \mathrm{ss}^{-n}\#Q\,?$$

A positive answer to this question would be an interesting refinement of the main result of [8]. Note that we cannot drop the hypothesis of having zero signature, since this is clearly the case for the CDGA $Q \oplus_{\Psi} \mathrm{ss}^{-n}\#Q$. Of course, a boundary manifold such as ∂W is always of zero signature.

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