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A simplicial James–Hopf map and decompositions of the unstable Adams spectral sequence for suspensions

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We use combinatorial group theory methods to extend the definition of the classical James–Hopf invariant to a simplicial group setting. This allows us to realize certain coalgebra idempotents at an sSet_* level and obtain a functorial decomposition of the spectral sequence, associated with the lower p–central series filtration on a free simplicial group.

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1 Introduction

One of the ultimate goals in unstable homotopy is to understand how the loops-over-suspension functor $\Omega\Sigma$ changes the homotopy type of spaces. One aspect of such understanding is the computation and investigation of homotopy groups of spaces and one of the most powerful methods for it is the unstable Adams spectral sequence, first introduced by Bousfield, Curtis, Kan, Quillen, Rector and Schlesinger [1]. Construction of this spectral sequence starts with a free simplicial group model for a loop space, equipped with the lower p–central series filtration, which produces the spectral sequence itself. So the framework for this machinery is the category of (pointed) simplicial sets sSet_*. Computations show that for a loop space over spheres, the mod-p unstable Adams spectral sequence is accelerated in a sense that nontrivial elements on its first page are concentrated in columns with numbers p^k (see [1]).

Another approach to investigating the loop-suspension functor $\mathrm{Top}_* \to \mathrm{Top}_*$ is given by the functorial direct product decompositions that were introduced by Selick and Wu [16]. This approach uses significantly a certain model for $\Omega\Sigma X$ in Top_* — the James construction JX — a reduced free monoid on points of X together with a word-length filtration J_kX on it. Among all (p–local) decompositions of the form

$$\Omega\Sigma X \simeq_p A \times B \tag{1}$$

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there is a minimal one, called $A^{\min}$, minimal in a sense that it is obtained from a minimal coalgebra retract of a tensor algebra functor T — which is a homological representation of $\Omega\Sigma$. It can be shown that the primitive elements of such a minimal coalgebra retract are concentrated in degrees p^k.

The aim of the present paper is to translate the above-mentioned decompositions to a category of simplicial sets and furthermore to extend them to a level of spectral sequences. From this point of view the primitive elements of $A^{\min}$ form a first page of a functorial spectral subsequence of the unstable Adams spectral sequence for suspensions and therefore results about the acceleration of the spectral sequence for ΩS^{n+1} become a particular case (since $A^{\min}(S^n) = S^n$) of a more general situation.

We will briefly recall the idea of the loop-suspension functor decompositions as in Selick and Wu [16]. Let $\boldsymbol{k} = \mathbb{Z}/p$ be a ground field. Then, by the Bott–Samelson theorem, the homology of $\Omega\Sigma X$ with coefficients in k is given by the tensor algebra $T(\overline{H}_*(X))$ on the reduced homology on X; ie functorially there is a map

$$[\Omega\Sigma, \Omega\Sigma] \xrightarrow{H_*} \mathrm{Hom}_{\mathrm{Coalg}}(T, T). \tag{2}$$

By the geometric realization theorem in [16] this map is surjective, and in fact all natural coalgebra maps $f\colon T \to T$ can be realized as elements in a certain subgroup H_∞ of $[J, \Omega\Sigma]$, called the *Cohen group*, which was first introduced in [2]. Now any coalgebra idempotent $f\colon T \to T$ can be lifted to a space level to obtain a map $\varphi\colon \Omega\Sigma \to \Omega\Sigma$, $\varphi_* = f$. The first piece of the decomposition (1) will be given by $A = \mathrm{hocolim}\,\varphi$, and the second piece is given by a homotopy mapping telescope of a complement map $[\mathrm{id}]\varphi^{-1}$.

To adapt this picture to sSet_*, one needs a suitable analogue of the Cohen group H_∞ which will realize coalgebra idempotents on the level of simplicial sets. In Top_* the Cohen group is defined as a subgroup of $[J, \Omega\Sigma]$, generated as a pro-group by compositions

$$JX \xrightarrow{H_k} JX^{\wedge k} \xrightarrow{J(\bar{\Delta})} JX^{\wedge l} \xrightarrow{J(\sigma)} J(X^{\wedge l}) \xrightarrow{W_l} \Omega\Sigma X, \tag{3}$$

where the first map is a combinatorial James–Hopf map (see Definition 2.1) and the last map is a Whitehead product. The most straightforward way to translate H_∞ to sSet_* is to push all objects through the geometric realization/singular chains functors. Unfortunately, the computationally heavy construction of $|-|-\mathrm{Sing}$ adjunction hides the combinatorial nature of H_∞, so preservation of the lower p–central series by its elements looks untraceable under this approach.

Instead, we will define generators of the Cohen group as certain natural transformations of functors on pointed sets and will extend this definition to sSet_* levelwise. The main object of interest here is Milnor's construction $F[-]$, a simplicial group model for the loop-suspension functor $\Omega\Sigma$. Our first goal is to extend the combinatorial James–Hopf map from a free monoid to a free group framework. This is achieved by translating the Hilton–Hopf map to the simplicial level and applying Hall's commutator collection process inside the simplicial version of the Hilton–Milnor theorem. The resulting definition, Definition 2.2, has all the desired properties of the classical James–Hopf map and moreover can be considered as a nonabelian version of the Magnus embedding; see Magnus, Karrass and Solitar [10]. Since everything happens levelwise, the Whitehead product can be considered simply as an iterated commutator map; permutations and iterated reduced diagonal maps stay the same as in (3).

With help from the Fox differential calculus it will be shown that elements of the Cohen group preserve the lower central series on the free simplicial group $F[-]$ and therefore induce maps between lower (p–)central towers of fibrations. Extending these maps to an exact couple level will equip the unstable Adams spectral sequence with an action of the Cohen group.

Such an action is used to obtain a functorial decomposition of this spectral sequence in the following way. As before, one starts with a functorial coalgebra idempotent $f\colon T \to T$, which is now extended to a natural self-transformation of the functor $T(\boldsymbol{k}[-])\colon \mathrm{sSet}_* \to \mathrm{sCoalg}_{\boldsymbol{k}}$. By the simplicial realization theorem, Theorem 4.3, this natural transformation can be lifted to a self-map of the lower central series tower and each level of the corresponding tower can be decomposed through a mapping telescope construction in the usual way. Such decompositions preserve the fibrations inside the lower central series tower, which can be therefore presented as a direct product of two towers, with layers weak equivalent to primitive elements of coalgebra retracts of $T(\boldsymbol{k}[-])$, corresponding to the idempotent f and its complement; see Theorem 4.6. The choice of the idempotent which corresponds to the minimal coalgebra retract ($A^{\min}$, a subfunctor of T) will produce a direct summand of the lower central series tower with nontrivial layers only at degrees p^k. Hence the corresponding spectral sequence will be accelerated in the same sense as in [1]. By choosing different idempotents (not necessarily the minimal one), different decompositions of the spectral sequence can be obtained. We provide one example of such a decomposition in (42) and show its consistency with the Λ–algebra description of the E^1–page given by Curtis [5].

The paper is organized as follows. Extension of the James–Hopf invariants to the simplicial group level (Definition 2.2) together with motivation for such extensions and connection with the Magnus embedding (Theorem 2.6) is given in Section 2. Preservation of the lower p–central series by elements of the Cohen group is discussed in Theorem 3.8 and the construction itself is given in Definition 3.1 of Section 3. Section 4 is devoted to applications of the action of the Cohen group on the spectral sequence (Theorem 3.10): decomposition of the spectral sequence (Theorem 4.6 and Corollary 4.7) together with the example (42), and acceleration of the part that corresponds to the minimal coalgebra decomposition (Corollary 4.8).

2 James–Hopf map for Milnor's construction

Set_* will denote the category of pointed sets (with distinct point $*$). In this category the *wedge sum* $X \vee Y = X \cup Y / *_X \sim *_Y \hookrightarrow X \times Y$ and the *smash product* $X \wedge Y = X \times Y / X \vee Y$ of pointed sets X and Y are defined. The category of groups Grp will also be considered pointed, with identity element of each object as a distinct point. Then there is a pair of adjoint functors

$$F\colon \mathrm{Set}_* \rightleftarrows \mathrm{Grp} : U,$$

where U is a forgetful functor and F is a *reduced* free group functor, that is, for any pointed set X, FX is a free group on X, quotiented out by the relation $* = 1$. The corresponding monad $UF\colon \mathrm{Set}_* \to \mathrm{Set}_*$ and its extension to the category of pointed simplicial sets sSet_* will be denoted by $F[-]$ and called *Milnor's construction*.

The similar monad, obtained from adjoint functors between Set_* and the category of monoids Mon will be denoted by J and called the *James construction*. J is a subfunctor of $F[-]$.

Remark Traditionally the James construction is defined for topological spaces as a free reduced topological monoid on points of the underlying topological space. In the present paper the James construction will be applied to a given simplicial set levelwise. Moreover, various functors and natural transformations between them will be considered for Set_* and sSet_* simultaneously, whenever it will not lead to confusion.

Definition 2.1 The k^{th} *(classical) combinatorial James–Hopf map* is a natural transformation of functors

$$H_k\colon J \to J(-)^{\wedge k}$$

such that for any pointed set X, $H_k\colon JX \to JX^{\wedge k}$ is defined by

$$H_k(x_1 \cdots x_n) = \prod_{i_1 < \cdots < i_k} x_{i_1} \wedge \cdots \wedge x_{i_k},$$

where the product is taken over all subsequences of $(1, \dots, n)$ (without repetitions of indices) arranged in the (right) lexicographic order. By functoriality the same map can be defined for simplicial sets levelwise.

The aim of this section is to extend Definition 2.1 in a natural way to the natural transformation $F[-] \to F[(-)^{\wedge n}]$ and provide a motivation for such a definition along with some basic properties. We will denote this extension by the same letter H_n.

Definition 2.2 The k^{th} *combinatorial James–Hopf* map is a natural transformation

$$H_k\colon F[-] \to F[(-)^{\wedge k}]$$

defined for any pointed (simplicial) set X on reduced words by

$$H_k(x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}) = \prod_{(i_1, \dots, i_k)} (x_{i_1} \wedge \cdots \wedge x_{i_k})^{\varepsilon_{i_1} \cdots \varepsilon_{i_k}},$$

where the product is taken over sequences of indices $(i_1, \dots, i_k)$ such that

$$i_j \le i_{j+1} - \tfrac{1}{2}(\varepsilon_{i_{j+1}} + 1).$$

In other words, the product is taken over all subsequences $(i_1, \dots, i_k)$ of $(1, \dots, n)$ with possible repetitions of indices and index i_j is a repetition only if the corresponding exponent ε_{i_j} is negative. The order in the product is (compare with 11.4 in [5]) $(i_1, \dots, i_k) < (i'_1, \dots, i'_k)$ if one of the following holds:

- $i_1 < i'_1$;
- $i_1 = i'_1$ and $i_2 < i'_2$ if $\varepsilon_{i_1} = 1$ or $i_2 > i'_2$ if $\varepsilon_{i_1} = -1$;
- $i_1 = i'_1$ and $i_2 = i'_2$ and $i_3 < i'_3$ if $\varepsilon_{i_1}\varepsilon_{i_2} = 1$ or $i_3 > i'_3$ if $\varepsilon_{i_1}\varepsilon_{i_2} = -1$;

$\vdots$

- $i_1 = i'_1$, $i_2 = i'_2, \dots, i_{k-1} = i'_{k-1}$ and $i_k < i'_k$ if $\varepsilon_{i_1} \cdots \varepsilon_{i_{k-1}} = 1$ or $i_k > i'_k$ if $\varepsilon_{i_1} \cdots \varepsilon_{i_{k-1}} = -1$.

It is clear that the James–Hopf invariant defined above is a natural extension of the combinatorial James–Hopf map for the James construction:

$$\begin{array}{ccc} F[X] & \xrightarrow{H_k} & F[X^{\wedge k}] \\ \uparrow & & \uparrow \\ J(X) & \xrightarrow{H_k} & J(X^{\wedge k}) \end{array} \tag{4}$$

For $x \in X$, let $x \wedge (-)\colon F[X^{\wedge k}] \to F[X^{\wedge k+1}]$ denote a homomorphic extension of

$$X^{\wedge k} \to X^{\wedge k+1}, \quad x_1 \wedge \cdots \wedge x_k \mapsto x \wedge x_1 \wedge \cdots \wedge x_k.$$

Using this notation one can compute H_k recursively:

Lemma 2.3 *For $x_i \in X$ and $\varepsilon_i = \pm 1$, the following formula holds:*

$$H_k(x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}) = \big(x_1 \wedge H_{k-1}(x_1^{(\varepsilon_1-1)/2} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n})\big)^{\varepsilon_1} H_k(x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}).$$

Proof Taking out the first letter x_{i_1} in Definition 2.2 gives

$$\begin{aligned} H_k(x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}) &= \prod_{i_1=1}^{n} \prod_{(i_2,\dots,i_k)} (x_{i_1} \wedge \cdots \wedge x_{i_k})^{\varepsilon_{i_1} \cdots \varepsilon_{i_k}} \\ &= \Bigg(x_1 \wedge \prod_{(i_2,\dots,i_k)} (x_{i_2} \wedge \cdots \wedge x_{i_k})^{\varepsilon_1 \varepsilon_{i_2} \cdots \varepsilon_{i_k}}\Bigg) H_k(x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}). \end{aligned}$$

Now if $\varepsilon_1 = 1$, then the first term is precisely $x_1 \wedge H_{k-1}(x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n})$. Indeed, only the order of subsequences in the product needs to be checked. By definition, $(i_2, \dots, i_k) < (i_2, \dots, i_k)$ if $i_2 < i_2'$; or if $i_2 = i_2'$ and $i_3 < i_3'$ if $\varepsilon_{i_2} = 1$ and $i_3 > i_3'$ if $\varepsilon_{i_2} = -1$; and so on. Similarly, if $\varepsilon_1 = -1$, the indices $i_2, \dots, i_k$ can take the value 1 and the order of elements is reversed, comparing to the order of terms in the product $H_{k-1}(x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n})$: $(i_2, \dots, i_k) < (i_2, \dots, i_k)$ if $i_2 > i_2'$; or if $i_2 = i_2'$ and $i_3 < i_3'$ if $-\varepsilon_{i_2} = 1$ and $i_3 > i_3'$ if $-\varepsilon_{i_2} = -1$; and so on. The assertion follows. □

We will use this recursive formula to show that H_k is well defined as a map from the free group $F[X]$.

Theorem 2.4 *The value of $H_k(g)$, where $g \in F[X]$, depends only on the element g itself and not on its presentation as a word in the set of generators X; ie for any words $w_1, w_2 \in F[X]$ and $x \in X$,*

$$H_k(w_1 x x^{-1} w_2) = H_k(w_1 x^{-1} x w_2) = H_k(w_1 w_2).$$

Proof We can assume that the word w_1 is reduced. The proof will be by induction on k and the length of w_1. It is clear that $H_1 = \mathrm{id}_{F[X]}$ is well defined. Assume that H_{k-1} is well defined. Then

$$\begin{aligned} H_k(xx^{-1}w_2) &= \big(x \wedge H_{k-1}(x^{-1}w_2)\big) H_k(x^{-1}w_2) \\ &= \big(x \wedge H_{k-1}(x^{-1}w_2)\big)\big(x \wedge H_{k-1}(x^{-1}w_2)\big)^{-1} H_k(w). \end{aligned}$$

Similarly,

$$\begin{aligned} H_k(x^{-1}xw) &= \big(x \wedge H_{k-1}(x^{-1}xw)\big)^{-1} H_k(xw) \\ &= \big(x \wedge H_{k-1}(w)\big)^{-1}\big(x \wedge H_{k-1}(w)\big) H_k(w). \end{aligned}$$

Now assume that assertion holds for words w_1 of length l. For $y \in X$ consider

$$\begin{aligned} H_k(y^{\varepsilon}w_1xx^{-1}w_2) &= \big(y \wedge H_{k-1}(y^{(\varepsilon-1)/2}w_1xx^{-1}w_2)\big)^{\varepsilon} H_k(w_1xx^{-1}w_2) \\ &= \big(y \wedge H_{k-1}(y^{(\varepsilon-1)/2}w_1w_2)\big)^{\varepsilon} H_k(w_1w_2) = H_k(y^{\varepsilon}w_1w_2). \end{aligned}$$

Proceed similarly for $H_k(y^{\varepsilon}w_1x^{-1}xw_2)$. □

To motivate our definition of the James–Hopf invariant we will follow the idea that Curtis apparently used in [5] to define the Hopf map H_2 in the simplicial EHP sequence.

In Top_* there is another map $h_k\colon \Omega\Sigma X \to \Omega\Sigma X^{\wedge k}$, which is called the *Hilton–Hopf invariant*. This map is very close to the classical James–Hopf invariant H_k (in a sense that both can be used in the EHP sequence for spheres), but defined only for co-H–spaces X; see [12]. We will use an extension of this definition to all of Top_* and afterwards will translate it to sSet_*. To define h_k first recall the statement of the Hilton–Milnor theorem as in [12]. For connected spaces X and Y the loop suspension over their wedge can be decomposed as a product

$$\Omega\Sigma(X \vee Y) \simeq \Omega\Sigma X \times \Omega\Sigma\big(\textstyle\bigvee_{n\geq 0} Y \wedge X^{\wedge n}\big). \tag{5}$$

The second summand can be decomposed further and iterated application of this theorem gives the decomposition

$$\Omega\Sigma(X \vee Y) \simeq \prod_{\omega\in\mathcal{B}(X,Y)} \Omega\Sigma(\omega(X,Y)), \tag{6}$$

where $\mathcal{B}(X,Y)$ denotes the Hall basis of the free Lie algebra on two letters X and Y and so each summand $\omega(X,Y)$ can be identified with a certain commutator bracket on two letters, written as a smash product.

Let X be a co-H–space with comultiplication μ'. For convenience, we will denote by A and B two identical copies of X. With this notation, comultiplication will be a map $\mu'\colon X \to A \vee B$ and the Hilton–Hopf map h_k is defined as a composition:

$$
\begin{array}{ccc}
\Omega\Sigma X & \xrightarrow{\quad h_k \quad} & \Omega\Sigma X^{\wedge k} \\
{\scriptstyle \Omega\Sigma\mu'}\big\downarrow & & \big\uparrow{\scriptstyle \pi_{B\wedge\cdots\wedge A}} \\
\Omega\Sigma(A\vee B) & \xrightarrow{\sim} & \prod_{\omega\in\mathcal{B}(A,B)} \Omega\Sigma(\omega(A,B))
\end{array}
\tag{7}
$$

Here the lower horizontal map is an inverse of the weak equivalence in the Hilton–Milnor theorem (6), the right vertical map is projection to a summand, which corresponds to a Lie bracket $[[[B,A]\cdots],A]$.

To extend h_k to all of Top_*, we will introduce another map $\Omega\Sigma X \to \Omega\Sigma(X\vee X)$, as in [5], which is homotopy equivalent to $\Omega\Sigma\mu'$ when X is a co-H–space. Let i_A and i_B be two inclusions of X in $\Omega\Sigma(A\vee B)$. Then $i_A * i_B$ is defined as a product of these inclusions in $\Omega\Sigma(A\vee B)$, ie as an extension of the composition

$$
X \xrightarrow{\Delta} X\times X \xrightarrow{i_A\times i_B} \Omega\Sigma(A\vee B)\times\Omega\Sigma(A\vee B) \xrightarrow{\mu} \Omega\Sigma(A\vee B) \tag{8}
$$

to an H–map $\Omega\Sigma X \to \Omega\Sigma A\vee B$.

Lemma 2.5 *For X a co-H–space,*

$$i_A * i_B \simeq \Omega\Sigma\mu'.$$

Proof Since the extension of (8) to the H–map is unique up to homotopy, this follows from the commutativity of the following diagram, which can be checked explicitly:

$$
\begin{array}{ccccccc}
X & \xrightarrow{\Delta} & X\times X & \xrightarrow{i_A\times i_B} & \Omega\Sigma(A\vee B)\times\Omega\Sigma(A\vee B) & \xrightarrow{\mu} & \Omega\Sigma(A\vee B) \\
& {\scriptstyle \mu'}\searrow & \big\uparrow & & & \nearrow & \\
& & X\vee X & & & &
\end{array}
$$

$\square$

Now we can translate the definition (7) to a simplicial group setting:

$$
\begin{array}{ccc}
F[X] & \xrightarrow{\quad H_k \quad} & F[X]^{\wedge k} \\
{\scriptstyle i_A * i_B}\big\downarrow & & \big\uparrow{\scriptstyle \pi_{B\wedge A\wedge\cdots\wedge A}} \\
F[A]*F[B] & \xrightarrow{\simeq} & \prod_{\omega\in\mathcal{B}(A,B)} F[\omega(A,B)]
\end{array}
\tag{9}
$$

Here $*$ denotes the free product $F[A]*F[B]\cong F[A\vee B]$ and the left vertical map sends the generator x_i to $a_i b_i$, the product of generators of $F[A]$ and $F[B]$.

In Top_*, the Hilton–Milnor decomposition given by a certain weak equivalence,

$$\Omega\Sigma X \times \Omega\Sigma\big(\textstyle\bigvee_{n\geq 0} Y \wedge X^{\wedge n}\big) \to \Omega\Sigma(X \vee Y),$$

is constructed from the iterated Whitehead products $[[[i_Y, i_X]\cdots], i_X]$, so its inverse (given by a lower horizontal arrow) is somewhat mysterious. The situation is opposite in sSet_*: the map from the free product, $F[A] * F[B] \to \prod_{\omega\in\mathcal{B}(A,B)} F[\omega(A,B)]$, can be seen as a collection process in the sense of Hall (see [7]), which allows us to describe the map purely combinatorially, and it turns out that such a map is a natural generalization of the combinatorial James–Hopf map from a free monoid to a free group. We will briefly recall the procedure of commutator collection applied to a free product $F[A] * F[B]$ from [5]. For simplicity we assume that A (with elements denoted by a_i) and B (with elements denoted by b_j) are just pointed sets. After that the process can be extended to simplicial sets by just applying it levelwise.

There is a natural projection $F[A] * F[B] \to F[A]$ (forgetting all b_j) with a section induced by the inclusion $A \to A \vee B$, and by Schreier's lemma the kernel is given by $F[B] * F[B \wedge F[A]]$. Here the free group on the smash product $B \wedge F[A]$ is naturally identified with a subgroup of $F[A] * F[B]$ generated by commutators $[b, w]$, where $b \in B$ and $w \in F[A]$.The free summand $F[B \wedge F[A]]$ can be decomposed further using the Tietze transformations (m is any number)

$$\begin{aligned}(10)\quad F[B\wedge F[A]] &\cong F[B\wedge A] * F[B\wedge A\wedge F[A]]\\ &\cong \cdots \cong F[B\wedge A] * F[B\wedge A\wedge A] * \cdots * F[B\wedge A^{\wedge m}\wedge F[A]].\end{aligned}$$

This chain of isomorphisms can be seen as a process of expanding commutators of the form $[b, aw]$ as $[b, w][b, a][b, a, w]$, where $b \in B$, $a \in A$ and $w \in F[A]$. So, for any m, after forgetting the group structure, there is a map

$$\begin{aligned}(11)\quad & F[A] * F[B]\\ &\to F[A] \times F[B] * F[B\wedge A] * \cdots * F[B\wedge A^{\wedge m}] * F[B\wedge A^{\wedge m}\wedge F[A]]\\ &\to F[A] \times F\big[\textstyle\bigvee_{j=0}^{m} B\wedge A^{\wedge j}\big],\end{aligned}$$

given by the (set-theoretic) retraction $F[A] * F[B] \to F[B] * F[B \wedge F[A]]$, composed with a chain of isomorphisms (10) and projection away from $F[B \wedge A^{\wedge m} \wedge F[A]]$. The Hilton–Milnor theorem for simplicial sets states that this map becomes a weak equivalence as $m \to \infty$ since the connectivity of the pieces $F[B \wedge A^{\wedge m} \wedge F[A]]$ grows as m increases. The same procedure can be applied to $F[B] * F[B \wedge F[A]]$ to split off $F[B]$ as a direct summand and so on. A sequence of such procedures can be seen as Hall's commutator collection process, which is described in detail in [7] and [5]. Here we only sketch the basic idea.

Let $w = a_1^{\varepsilon_1} b_1^{\eta_1} \cdots a_n^{\varepsilon_n} b_n^{\eta_n}$, where $\varepsilon_i, \eta_j = \pm 1$, be a reduced word in $F[A] * F[B]$. For simplicity first assume that $\varepsilon_i = \eta_i = 1$ for every i. By pushing all the a_i to the beginning of the word w, starting with a_1 and using the identity

$$ca_i = a_i c[c, a_i], \quad c = b_j \quad \text{or} \quad c = [b_j, a_{i_1}, \dots, a_{i_k}],$$

the word w can be written as $w = w_A w'$, where $w_A = a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n}$ and w' consists of b_j and commutators of the form $[b_j, a_{i_1}, \dots, a_{i_k}]$, so $w' \in F[B \wedge F[A]]$ and the mapping $w \to w'$ gives a retraction $F[A] * F[B] \to F[B] * F[B \wedge F[A]]$. In the case of arbitrary exponents the following commutation rules should be applied (see [7, (11.1.6)] for details):

$$c^{-1}a = a[c, a]^{-1}c^{-1}, \tag{12}$$

$$ca^{-1} = a^{-1}c[c, a, a][c, a, a, a, a] \cdots [c, a, a, a]^{-1}[c, a]^{-1}, \tag{13}$$

$$c^{-1}a^{-1} = a^{-1}[c, a][c, a, a, a] \cdots [c, a, a, a, a]^{-1}[c, a, a]^{-1}c^{-1}. \tag{14}$$

Note that exponents of commutators that lie inside $F[B \wedge A]$ are products of exponents of c and a.

The described process collects (ie arranges in the (right) lexicographic order in the beginning of the word w) commutators of weight 0 (generators a_i). Since in (9) the last map is a projection to the summand $F[B \wedge A \wedge \cdots \wedge A]$ (taking commutators of the form $[b_j, a_{i_1}, \dots, a_{i_k}]$), to define H_n only one retraction $w \to w'$ is needed:

$$\begin{array}{ccccc}
F[X] & & \xrightarrow{\quad H_n \quad} & & F[X^{\wedge n}] \\
\downarrow{\scriptstyle i_1 * i_2} & & & & \uparrow{\scriptstyle \pi_{B\wedge A\wedge\cdots\wedge A}} \\
F[A \vee B] & \xrightarrow{\rho} & F\big[\bigvee_{j=0}^k B \wedge A^{\wedge j}\big] * F[B \wedge A^{\wedge k} \wedge F[A]] & \longrightarrow & \prod_{j=0}^k F[B \wedge A^{\wedge j}] \times F[B \wedge A^{\wedge k} \wedge F[A]]
\end{array} \tag{15}$$

Now we will describe how the combinatorics of the collection process described above leads to Definition 2.2. Fix $k > 0$ and consider $H_k(x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n})$ as in (9). The image of $x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n} \in F[X]$ under $i_1 * i_2$ is $(a_1 b_1)^{\varepsilon_1} \cdots (a_n b_n)^{\varepsilon_n} \in F[A] * F[B]$. During the collection process each $a_i^{\varepsilon_i}$ will give rise to commutators of the form $[b_j, a_{i_1}, \dots, a_{i_l}, a_i]^{\varepsilon_j \varepsilon_{i_1} \cdots \varepsilon_{i_l} \varepsilon_i}$, where $j \le i_1 \le \cdots \le i_l \le i$ and equality in this sequence appears only if the corresponding exponent is equal to -1 (rules (13) or (14) are applied).

These commutators appear in the following order: if $j < j'$, commutators $[b_j, \dots]$ are collected before $[b_{j'}, \dots]$, and when $j = j'$, if the exponent of b_j is positive, new commutators $[b_j, a_{i_1}, \dots]$ appear from the right side of b_j (meaning those with $i_1 > i'_1$ are collected before those with $i_1 < i'_1$) and if negative, from the left side (the opposite order is applied). Continuing this analysis one can see that the order of the commutators $[b_{i_1}, a_{i_2}, \dots, a_{i_k}]$ is very similar to that in Definition 2.2, with the opposite ordering applied when $i_s = i'_s$, for $s = 1, \dots, k$. This opposite ordering in Definition 2.2 is introduced to make the new definition coherent with the classical one for the James construction.

Now we will show that the product of all the H_k can be seen as a nonabelian version of the Magnus representation [10]. Let X be a pointed set and $\mathbb{Z}\langle\langle X\rangle\rangle$ be a ring of (noncommutative) formal power series on elements of X. The *Magnus representation* μ is a homomorphism from a free group on X to $\mathbb{Z}\langle\langle X\rangle\rangle$ defined on generators by

$$\mu\colon F[X] \to \mathbb{Z}\langle\langle X\rangle\rangle, \quad x \mapsto x + 1.$$

By the theorem of Magnus [10], this map is injective.

The ring $\mathbb{Z}\langle\langle X\rangle\rangle$ can be identified with an infinite product ($\overline{\mathbb{Z}}[-] = \mathbb{Z}[-]/\mathbb{Z}[*]$)

$$\prod_{n=0}^{\infty} \overline{\mathbb{Z}}[X]^{\otimes n} \cong \prod_{n=0}^{\infty} \overline{\mathbb{Z}}[X^{\wedge n}],$$

and for any k there are projections $\pi_k\colon \mathbb{Z}\langle\langle X\rangle\rangle \to \overline{\mathbb{Z}}[X^{\wedge k}]$ which map the formal power series to its elements of length k. By *length* of the formal power series we mean the minimum among all lengths of its elements.

Theorem 2.6 *There is a commutative diagram:*

$$\begin{array}{ccc} F[X] & \xhookrightarrow{\mu} & \mathbb{Z}\langle\langle X\rangle\rangle \\ {\scriptstyle H_k}\downarrow & & \downarrow{\scriptstyle \pi_k} \\ F[X^{\wedge k}] & \xrightarrow[\text{ab}]{} & \overline{\mathbb{Z}}[X^{\wedge k}] \end{array} \tag{16}$$

Proof Define $H_0\colon F[X] \to \mathbb{Z}$ by $x \mapsto 1$ and $H = \prod_{k=0}^{\infty} H_k : F[X] \to \prod_{k=0}^{\infty} F[X^{\wedge k}]$. Then the statement will follow from the commutativity of the diagram

$$\begin{array}{ccc} F[X] & \xhookrightarrow{\mu} & \mathbb{Z}\langle\langle X\rangle\rangle \\ {\scriptstyle H}\downarrow & \nearrow_{\text{ab}} & \\ \prod_{k=0}^{\infty} F[X^{\wedge k}] & & \end{array}$$

For any $w = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n} \in F[X]$ coefficients of the power series $\mu(w) \in \mathbb{Z}\langle\langle X \rangle\rangle$ can be described as augmentations of the Fox derivatives (see [6])

$$\mu(w) = \sum_{k=0}^{\infty} \sum_{(i_1,\dots,i_k)} \varepsilon\partial^k_{i_1\dots i_k}(w) x_{i_1} \cdots x_{i_k}. \tag{17}$$

Here we use a shortened notation $\partial^k_{i_1\dots i_k}(w)$ for the Fox derivative $\partial^k w/(\partial x_{i_1} \dots \partial x_{i_k})$. Each of these is given by the formula [6, (3.5)]

$$\varepsilon\partial^k_{i_1\dots i_k}(w) = \sum_{(\lambda_1,\dots,\lambda_k)} \varepsilon_{\lambda_1} \cdots \varepsilon_{\lambda_k},$$

with summation over all sequences of indices $(\lambda_1, \dots, \lambda_k)$ such that $i_{\lambda_j} = i_j$ for all $j = 1, \dots, k$ and

$$\lambda_j \le \lambda_{j+1} - \tfrac{1}{2}(\varepsilon_{\lambda_{j+1}} + 1).$$

So $\mu(w)$, after throwing away all terms $x_{i_1} \cdots x_{i_k}$ with zero coefficients before them, becomes an abelianized version of Definition 2.2:

$$H(w)_{\mathrm{ab}} = \sum_{k=0}^{\infty} \sum_{(i_1,\dots,i_k)} \varepsilon_{i_1} \cdots \varepsilon_{i_k} x_{i_1} \cdots x_{i_k},$$

with $i_j \le i_{j+1} - \frac{1}{2}(\varepsilon_{i_{j+1}} + 1)$ □

Corollary 2.7 (i) *$H\colon F[X] \to \prod_{k=0}^{\infty} F[X^{\wedge k}]$ is injective.*

(ii) *Let γ_n be the n^{th} term of the lower central series of $F[X]$ and $\mathcal{L}^n(X) \cong \gamma_n/\gamma_{n+1}$ be the n^{th} Lie power, viewed as the n^{th} homogeneous component of the associated graded object, corresponding to the lower central series filtration on $F[X]$. $\overline{H}_n$ will denote a composition of H_n with the abelianization map $F[X]^{\wedge n} \to \overline{\mathbb{Z}}[X]^{\otimes n}$. Then the following diagram commutes:*

$$\begin{array}{ccc} \gamma_n & \hookrightarrow & F[X] \\ \downarrow & & \downarrow \overline{H}_n \\ \mathcal{L}^n(X) & \hookrightarrow & \overline{\mathbb{Z}}[X]^{\otimes n} \end{array}$$

That is, $\overline{H}_n$ sends group commutators to ring commutators.

Proof Part (i) is immediate since μ is injective.

For part (ii), because $\overline{H}_n = \pi_n \circ \mu$, it is sufficient to show that

$$\pi_n(\mu[x_1, \dots, x_n]) = [x_1, \dots, x_n]_{\mathrm{Rings}}.$$

Following [10], for an element $u \in \mu(F[X])$, denote by $\delta(u)$ its *deviation*, which is the nonvanishing homogeneous component of lowest positive degree in the power series u. Then formula (7) from Lemma (5.4) in [10] computes the deviation of group commutator:

$$\delta(\mu([x, y])) = [\delta(\mu(x)), \delta(\mu(y))]_{\mathrm{Rings}} \quad \text{for } x, y \in F[X].$$

Using induction on n we see that

$$\delta(\mu([x_1, \dots, x_n])) = [x_1, \dots, x_n]_{\mathrm{Rings}},$$

which has length precisely n, so it coincides with $\pi_n(\mu[x_1, \dots, x_n])$. □

3 Cohen group for the lower p–central series tower

In this section we will define the Cohen group as a subgroup of a group of natural transformations of functors $F[-] \to F^{\wedge}[-]$ which acts naturally on the lower p–central series tower, and therefore on the corresponding spectral sequence.

Let X be a (pointed) simplicial set and p be a prime. Recall that the lower central series γ_n and the lower p–central series $\gamma_n^{[p]}$ of $F[X]$ are defined as simplicial subgroups of $F[X]$ such that in each dimension k they are the usual lower (p–)central series of the group $F[X]_k$:

$$(\gamma_n F[X])_k = \langle \{[g_1, \dots, g_n] \mid g_l \in F[X_k]\} \rangle, \tag{18}$$

$$(\gamma_n^{[p]} F[X])_k = \langle \{[g_1, \dots, g_i]^{p^j} \mid g_l \in F[X_k] \text{ and } i p^j \geq n\} \rangle. \tag{19}$$

The lower (p–)central series γ_n and $\gamma_n^{[p]}$ will be considered as subfunctors of $F[-]$.

Both series lead to the towers (20) of fibrations of simplicial sets (denoted by $\Gamma(X)$ and $\Gamma_p(X)$ respectively) which are functorial with respect to X:

$$
\begin{array}{ccccccc}
F^\wedge[X] = \varprojlim F[X]/\gamma_n & & & & F^\wedge_p[X] = \varprojlim F[X]/\gamma_n^{[p]} & & \\
\downarrow & & & & \downarrow & & \\
\vdots & & & & \vdots & & \\
\downarrow & & & & \downarrow & & \\
F[X]/\gamma_{n+1} & \hookleftarrow & \mathcal{L}^n(\overline{\mathbb{Z}}[X]) & & F[X]/\gamma_{n+1}^{[p]} & \hookleftarrow & \mathcal{L}^n_{\mathrm{res}}(\overline{\mathbb{Z}}/p[X]) \\
\twoheaddownarrow & & & & \twoheaddownarrow & & \\
F[X]/\gamma_n & & & & F[X]/\gamma_n^{[p]} & & \\
\twoheaddownarrow & & & & \twoheaddownarrow & & \\
\vdots & & & & \vdots & & \\
\twoheaddownarrow & & & & \twoheaddownarrow & & \\
F[X]/\gamma_2 = \overline{\mathbb{Z}}[X] & & & & F[X]/\gamma_2^{[p]} = \overline{\mathbb{Z}}/p[X] & &
\end{array}
\tag{20}
$$

Fibers of (20), denoted by $E^0\Gamma$ and $E^0\Gamma_p$, can be identified with Lie powers and restricted Lie powers functors on X and both towers lead to spectral sequences with horizontal convergence (which is due to Curtis [4]):

$$
\begin{aligned}
E^1_{p,q} &= \pi_p\mathcal{L}^q(\overline{\mathbb{Z}}[X]) \Rightarrow \pi_p F[X], \\
E^1_{p,q} &= \pi_p\mathcal{L}^q(\overline{\mathbb{Z}}/p[X]) \Rightarrow \pi_p F[X] \otimes \mathbb{Z}/p.
\end{aligned}
\tag{21}
$$

Remark Traditionally the spectral sequence constructed from the lower p–central series filtration on the Kan construction GX is called the *unstable Adams spectral sequence* and was introduced in [1] (accelerated version) and in [15]. The spectral sequence (21) is isomorphic to the unstable Adams spectral sequence for $G(\Sigma X)$ by a natural isomorphism of simplicial groups $F[X] \to G(\Sigma X)$.

The (levelwise) *Whitehead product* W_n for Milnor's construction is a homomorphism (a natural transformation, in fact) $W_n\colon F[X^{\wedge n}] \to F[X]$ defined on generators by

$$W_n(x_1 \wedge \cdots \wedge x_n) = [[[x_1, x_2], \cdots], x_n].$$

Now we can construct an analogue of the Cohen group (3) for simplicial sets as a subgroup of $\mathrm{Hom}_{\mathrm{sSet}_*}(F[-], F^\wedge[-])$, the (topological) group (under pointwise multiplication) of all simplicial natural transformations between Milnor's construction and its pro-nilpotent completion. The reason for choosing the pro-nilpotent completion here is that the original Cohen group H_∞ was constructed as a pro-group and therefore contains certain infinite products, which can be crucial for realizing coalgebra idempotents.

Definition 3.1 The *Cohen group* H_∞ is the closure of a subgroup of

$$\mathrm{Hom}_{\mathrm{sSet}_*}(F[-], F^\wedge[-])$$

(in the topology of $\mathrm{Hom}_{\mathrm{sSet}_*}(F[-], F^\wedge[-])$) generated by compositions (evaluated at arbitrary X)

$$F[X] \xrightarrow{H_k} F[X^{\wedge k}] \xrightarrow{F[\bar{\Delta}]} F[X^{\wedge l}] \xrightarrow{F[\sigma]} F[X^{\wedge l}] \xrightarrow{W_l} F[X] \to F^\wedge[X] \tag{22}$$

for all positive $k \le l$ and permutations $\sigma \in \Sigma_l$. Here H_k is the James–Hopf map (Definition 2.2), $\bar{\Delta}\colon X^{\wedge k} \to X^{\wedge l}$ is the iterated reduced diagonal, $\sigma\colon X^{\wedge l} \to X^{\wedge l}$ is permutation of factors in the smash product and W_l is the Whitehead product.

To check that elements of H_∞ induce maps between towers $\Gamma \to \Gamma$ and $\Gamma_p \to \Gamma_p$, it is sufficient to check that the generators (22) preserve γ_n and $\gamma_n^{[p]}$, and since $F(\bar{\Delta})$, $F(\sigma)$ and W_l are homomorphisms, only preservation of the lower (p–)central series by H_k needs to be checked. Unfortunately, this is not the case, as the simplest example shows:

$$H_2([x,y]) = [(x\wedge y),(x\wedge x)][(y\wedge y),(y\wedge x)](y\wedge x)^{-1}(x\wedge y).$$

To overcome this difficulty we will define the weighted lower central series filtration on $F[X^{\wedge m}]$ and show that the James–Hopf map preserves the lower (p–)central series in this weighted sense, which is sufficient to induce a map of towers Γ and Γ_p.

Definition 3.2 The *weighted lower (p–)central series*

$$\gamma_n^w(F[X^{\wedge m}]) \quad \text{and} \quad \gamma_n^{[p],w}(F[X^{\wedge m}])$$

are sequences of simplicial subgroups of $F[X]$ defined as follows:

(i) If $n < m$ then $\gamma_n^w(F[X^{\wedge m}]) = F[X^{\wedge m}]$.

(ii) If $n \ge m$, ie $n = qm + s$, where $s < m$, then

$$\gamma_n^w(F[X^{\wedge m}]) = \gamma_{\lceil n/m\rceil}(F[X^{\wedge m}]) = \begin{cases} \gamma_q(F[X^{\wedge m}]) & \text{if } s = 0, \\ \gamma_{q+1}(F[X^{\wedge m}]) & \text{if } s > 0, \end{cases}$$

and similarly for $\gamma_n^{[p],w}(F[X^{\wedge m}])$.

Lemma 3.3 *Whitehead products send each weighted lower* (p–)*central series to the corresponding usual one:*

$$W_m(\gamma_n^w(F[X^{\wedge m}])) \subset \gamma_n \quad \text{and} \quad W_m(\gamma_n^{[p],w}(F[X^{\wedge m}])) \subset \gamma_n^{[p]}.$$

Proof The left-hand inclusion follows from the well-known fact that $[\gamma_n, \gamma_m] \subset \gamma_{n+m}$. The mod-$p$ version follows from the integral one:

$$\begin{aligned} W_m\Bigg(\prod_{(i,j):ip^j \geq \lceil n/m \rceil} (\gamma_i)^{p^j}\Bigg) &= \prod_{(i,j):ip^j \geq \lceil n/m \rceil} W_m(\gamma_i)^{p^j} \\ &\subset \prod_{(i,j):ip^j \geq \lceil n/m \rceil} (\gamma_{im})^{p^j} \subset \prod_{(i,j):i'p^j \geq n} (\gamma_{i'})^{p^j}, \end{aligned}$$

where $i' = im$ in the last product. □

Since we are working with free (simplicial) groups, it is convenient to switch our attention from the lower (p–)central series γ_n and $\gamma_n^{[p]}$ to the powers of (mod-p) augmentation ideals Δ^n and Δ_p^n. Recall that the *augmentation ideal* and the *(mod-p) augmentation ideal* are kernels of the augmentation homomorphisms

$$\Delta \hookrightarrow \mathbb{Z}[F[X]] \xrightarrow{\varepsilon} \mathbb{Z} \quad \text{and} \quad \Delta_p \hookrightarrow \mathbb{Z}/p[F[X]] \xrightarrow{\varepsilon} \mathbb{Z}/p.$$

According to the Magnus–Witt theorem, γ_n and $\gamma_n^{[p]}$ can be expressed through Δ^n and Δ_p^n:

$$\gamma_n = F[X] \cap (1 + \Delta^n) \quad \text{and} \quad \gamma_n^{[p]} = F[X] \cap (1 + \Delta_p^n).$$

We will use the following lemma (stated simultaneously for integral and mod-p cases):

Lemma 3.4 *Let* $f\colon A \to B$ *be a pointed* (ie $f(1) = 1$) *map between free groups* A *and* B *such that* $\mathbb{Z}f(\Delta^n) \subset \Delta^m$ *(resp.* $\mathbb{Z}/pf(\Delta_p^n) \subset \Delta_p^m$*). Then* $f(\gamma_n) \subset \gamma_m$ *(resp.* $f(\gamma_n^{[p]}) \subset \gamma_m^{[p]}$*) and there is a well-defined map* $\bar{f}\colon A/\gamma_n \to B/\gamma_m$ *(resp.* $\bar{f}\colon A/\gamma_n^{[p]} \to B/\gamma_m^{[p]}$*).*

Proof We will prove only the integral case. We will use the identifications

$$\gamma_n A = A \cap (1 + \Delta^n) \quad \text{and} \quad \gamma_n B = B \cap (1 + \Delta^n) \tag{23}$$

of the terms of the lower central series of A and B with their dimension subgroups (see for example [9]). Note that here it is essential that A and B are free groups: for a general group G, the central series $\{\gamma_n G\}_n$ and $\{G \cap (1 + \Delta^n)\}_n$ are different.

By (23), for every $g \in \gamma_n$, $g = 1 + a$ for some $a \in \Delta^n$. So $f(g) = \mathbb{Z}f(1+a) = 1 + \mathbb{Z}f(a) \in (1+\Delta^m) \cap B = \gamma_m$. For the second statement it is sufficient to show that for every $g \in A$ and $x \in \gamma_n$, $f(gx) \in f(g)\gamma_m$. Using the second identification, $x = 1+b$ for some $b \in \Delta^n$, which implies $f(gx) = f(g) + \mathbb{Z}f(gb)$, so $f(gx) - f(g) \in \Delta^m$, $f(g)^{-1}f(gx) - 1 \in \Delta^m$, and finally $f(g)^{-1}f(gx) \in \gamma_m$ □

The standard machinery for working with the powers of the augmentation ideal is the Fox free differential calculus, first described in [6]. Here we will briefly sketch the definition of the Fox derivatives and state the theorem of Fox that connects these derivatives and the powers of the augmentation ideal.

Definition 3.5 Let FX be a free group on the set of generators X, and let $x_i \in X$. The *Fox derivative with respect to* x_i is a linear map

$$\partial_i\colon \mathbb{Z}[FX] \to \mathbb{Z}[FX]$$

uniquely determined by the following properties:

$$\partial_i(x_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \qquad \text{and} \qquad \partial_i(ab) = \partial(a)\varepsilon(b) + a\partial(b) \quad \text{for } a, b \in \mathbb{Z}[FX].$$

For a sequence $x_{i_1}, \dots, x_{i_k}$ of generators, $\partial^k_{i_1 \dots i_k}$ will denote the higher Fox derivative $\partial_{i_1} \circ \cdots \circ \partial_{i_k}$ of order k. The Fox derivative of order 0 is defined to be the augmentation homomorphism ε.

Theorem 3.6 (Fox [6]) *Let $a \in \mathbb{Z}[FX]$. Then $a \in \Delta^n$ if and only if for all $0 \le k < n$ and for all sequences $(i_1, \dots, i_k)$ of indices, $\varepsilon\partial^k_{i_1 \dots i_k}(a) = 0$; ie an element of a group ring lies in the n^{th} power of the augmentation ideal if and only if the augmentations of all its Fox derivatives up to order $n-1$ vanish.*

Remark This theorem can be seen as a corollary of formula (17) above, since the image of Δ^n under the Magnus embedding (extended linearly to a group ring) is $P_n \subset \mathbb{Z}\langle\langle X \rangle\rangle$, formal power series of length $\ge n$.

Corollary 3.7 *If $a \in \mathbb{Z}/p[FX]$, then $a \in \Delta^n_p$ if and only if for every k and every $(i_1, \dots, i_k)$,*

$$\varepsilon\partial^k_{i_1 \dots i_k}(a) \equiv 0 \mod p.$$

Proof The corollary follows from the mod-p version of the Magnus embedding

$$\mu_p\colon FX \to \mathbb{Z}/p\langle\langle X \rangle\rangle, \quad x \mapsto x + 1,$$

which is the composition of μ and the quotient map $\mathbb{Z}\langle\langle X\rangle\rangle \to \mathbb{Z}/p\langle\langle X\rangle\rangle$. Coefficients of elements in the image of μ_p are given by augmentations of the Fox derivatives modulo the prime p. And, as before, after extending μ_p to the group ring $\mathbb{Z}/p[FX]$, the powers of the augmentation ideals Δ_p^n will map to elements of length $\leq n$. These two facts give the result. □

Theorem 3.6, together with Lemma 3.4, provides a convenient way to check when H_k preserves the weighted lower (p–)central series filtration.

Theorem 3.8 *The simplicial James–Hopf map $H_n\colon F[X] \to F[X^{\wedge n}]$ sends the lower central series to the weighted one:*

$$H_n(\gamma_m) \subset \gamma_m^w.$$

Proof By Lemma 3.4 it is sufficient to prove the preservation of the weighted augmentation ideal filtration, ie that $H_n((g_1-1)\cdots(g_m-1)) \in \Delta_w^m$ for every $g_1, \dots, g_m \in F[X]$. After writing each g_i in normal form, the statement can be reduced to the case when all g_i are generators of $F[X]$, possibly with negative exponents. Since H_n is natural with respect to maps between sets of generators $Y \to X$,

$$\begin{array}{ccc} \mathbb{Z}F[Y] & \xrightarrow{\mathbb{Z}H_n} & \mathbb{Z}F[Y^{\wedge n}] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\wedge\cdots\wedge f} \\ \mathbb{Z}F[X] & \xrightarrow[\mathbb{Z}H_n]{} & \mathbb{Z}F[X^{\wedge n}] \end{array}$$

all these generators can be taken distinct from each other and ordered lexicographically. So

$$\mathbb{Z}H_n((x_{i_1}^{\varepsilon_{i_1}}-1)\cdots(x_{i_m}^{\varepsilon_{i_m}}-1)) = (f\wedge\cdots\wedge f)\mathbb{Z}H_n((y_1^{\varepsilon_1}-1)\cdots(y_m^{\varepsilon_m}-1)).$$

According to Theorem 3.6, it is sufficient to show that the augmentations of all derivatives $\partial^i_{J_i\dots J_1}$ of $\mathbb{Z}H_n((y_1^{\varepsilon_1}-1)\cdots(y_m^{\varepsilon_m}-1))$ of orders up to q (or $q-1$) vanish for all possible combinations of multi-indices $J_k = y_{k_1}\wedge\cdots\wedge y_{k_n}$.

Let $Y = y_1^{\varepsilon_1}\cdots y_m^{\varepsilon_m}$, and let $|v|$ be the length of the word v. We write $s \subset v$ to denote that s is a subword (without repetitions) of the word v and $s \subseteq v$ to denote that s is a subword (with possible repetitions) of the word v. With this notation,

$$\varepsilon\partial^i_{J_i\dots J_1}\mathbb{Z}H_n((y_1^{\varepsilon_1}-1)\cdots(y_m^{\varepsilon_m}-1)) = \sum_{j=0}^{m}(-1)^{m-j}\sum_{\substack{v\subset Y\\ |v|=j}}\varepsilon(\partial^i_{J_i\dots J_1}H_n(v)). \tag{24}$$

By formula (3.5) of [6], in which higher derivatives of arbitrary reduced words in a free group are computed, and using the formula $H_n(v) = \prod_{d \subseteq v} d^{\varepsilon_d}$, we see that

$$\varepsilon\partial^i_{J_i \dots J_1}(H_n(v)) = \sum \varepsilon_{d_1} \cdots \varepsilon_{d_i}, \tag{25}$$

where the sum is over all sequences of subwords (with possible repetitions) d_k of v such that for each subword d_k in the sequence, $J_k = d_k$ and the subword d_k can repeat if the corresponding exponent ε_{d_k} is negative.

This means that the augmentation of the derivative is not zero if the corresponding v contains all subwords J_k (again, repetitions are allowed if the corresponding exponents are negative). First, all derivatives with respect to tuples $(J_q, \dots, J_1)$ which are not in the order defined in Definition 2.2 are zero. Second, since all elements in Y are different, all subwords $d \subseteq v$ are different from each other, so all J_k appear in $H_n(v)$ only once, with the fixed exponent ε_{J_k} and (24) will take the form

$$\varepsilon\partial^i_{J_i \dots J_1}\mathbb{Z}H_n((y_1^{\varepsilon_1}-1)\cdots(y_m^{\varepsilon_m}-1)) = \varepsilon_{J_i}\cdots\varepsilon_{J_1}\sum_{j=0}^{m}(-1)^{m-j}\sum_{v \subset Y,\ |v|=j,\ J_i \cdots J_1 \subseteq v} 1.$$

Next we will count the number of subwords $v \subset Y$ of length j such that $J_i \cdots J_1 \subseteq v$, for the fixed tuple $(J_i, \dots, J_1)$. If the generator y_l has a positive exponent in Y then the derivative with respect to a tuple which contains the multi-index $J_p = y_{p_1} \wedge \cdots \wedge y_l \wedge y_l \wedge \cdots \wedge y_{p_n}$ with repetition of y_l will be zero. So we can assume that each J_k has repetition of indices inside only if the exponent of the corresponding generator is negative in Y. Same goes for the case when $J_k = J_{k+1}$ — such derivatives are equal to zero if the product of the exponents of the generators in Y which also form J_k is positive (this does not depend on v). With these restrictions, the number of words v of length j such that $J_i \cdots J_1 \subseteq v$ is precisely

$$\binom{m-n_1-\cdots-n_r}{j-n_1-\cdots-n_r}.$$

Here r is the total number ($1 \le r \le i$) of different J_k, and n_k is the total number ($1 \le n_k \le n$) of different generators in J_k. Writing $N = n_1 + \cdots + n_r$ and substituting this number in the formula we have

$$\begin{aligned}\varepsilon\partial^i_{J_i \dots J_1}\mathbb{Z}H_n((y_1^{\varepsilon_1}-1)\cdots(y_m^{\varepsilon_m}-1)) &= \varepsilon_{J_i}\cdots\varepsilon_{J_1}\sum_{j=N}^{m}(-1)^{m-j}\binom{m-N}{j-N} \\ &= \begin{cases} 0 & \text{if } m-N \neq 0, \\ \varepsilon_{J_i}\cdots\varepsilon_{J_1} & \text{if } m-N = 0.\end{cases}\end{aligned}$$

Since $N \le in$, $s < n$ and $i \le q$, the last case can only happen if $s = 0$ and $N = qn$, which is exactly the first case in Definition 3.2. □

Remark For the lower powers of the augmentation ideal and the lower order of the James–Hopf map the preservation of the weighted filtration can be seen explicitly, without referring to the Fox calculus. We will illustrate such preservation in the first nontrivial case, $\mathbb{Z}H_2(\Delta^3 F[X]) \subset \Delta^2 F[X \wedge X]$, combinatorially and compare it with a computation using Fox derivatives. This will serve as a demonstration of the combinatorics of the James–Hopf map on a group ring level.

Let $a, b, c \in X$ be three generators of $F[X]$ and consider some element defined by them in $\Delta^3 F[X]$, for example, $\alpha = (a^{-1} - 1)(b - 1)(c - 1)$. We will show that $\mathbb{Z}H_2(\alpha) \in \Delta^2 F[X \wedge X]$. After opening up the brackets in α, we will get

$$\begin{aligned}&\mathbb{Z}H_2(\alpha)\\ &= H_2(a^{-1}bc) - H_2(a^{-1}b) - H_2(a^{-1}c) - H_2(bc) + H_2(a^{-1}) + H_2(b) + H_2(c) - 1.\end{aligned}$$

Using Definition 2.2, we get

$$\begin{aligned}\mathbb{Z}H_2(\alpha) = (a \wedge c)^{-1}(a \wedge b)^{-1}(a \wedge a)(b \wedge c) - (a \wedge b)^{-1}(a \wedge a)\\ - (a \wedge c)^{-1}(a \wedge a) - (b \wedge c) + (a \wedge a) + 1 + 1 - 1.\end{aligned} \tag{26}$$

For convenience we will denote the elements $(a \wedge a)$, $(a \wedge b)^{-1}$, $(a \wedge c)^{-1}$ and $(b \wedge c)$ by w, x, y and z respectively. Then we can decompose $\mathbb{Z}H_2(\alpha)$ as a sum of elements of $\Delta^2 F[X]$, for example, as follows:

$$\begin{aligned}&yxwz - xw - yw - z + w + 1\\ &= (yx-1)(wz-1) + (w-1)(z-1) + (y-1)(x-1) - (x-1)(w-1) - (y-1)(w-1).\end{aligned}$$

The same computation with the Fox derivatives is much more straightforward. To show that $\mathbb{Z}H_2(\alpha)$ lies in the square of the augmentation ideal we need to check that the augmentations of all the Fox derivatives of first order vanish. These derivatives are organized in Table 1.

Note that expanding the element of the form $(x_1^{\varepsilon_1} - 1) \cdots (x_n^{\varepsilon_n} - 1) \in \Delta^n F[X]$ as a linear combination of elements of the group ring can be understood as computing all possible classical James–Hopf invariants (Definition 2.1) on an element $x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n} \in J(X \cup X^{-1})$. This is due to the relation between the James–Hopf invariants and the Magnus embedding (Theorem 2.6). Therefore, understanding the augmentation of the

$\partial_\bullet$	$\partial_\bullet \mathbb{Z} H_2(\alpha)$	$\varepsilon\partial_\bullet \mathbb{Z} H_2(\alpha)$
$\partial_{(a\wedge a)}$	$((a\wedge c)^{-1}-1)((a\wedge b)^{-1}-1)$	0
$\partial_{(a\wedge b)}$	$(1-(a\wedge c)^{-1})(a\wedge b)^{-1}$	0
$\partial_{(a\wedge c)}$	0	0
$\partial_{(b\wedge a)}$	0	0
$\partial_{(b\wedge b)}$	0	0
$\partial_{(b\wedge c)}$	$(a\wedge c)^{-1}(a\wedge b)^{-1}(a\wedge a)-1$	0
$\partial_{(c\wedge a)}$	0	0
$\partial_{(c\wedge b)}$	0	0
$\partial_{(c\wedge c)}$	0	0

Table 1

Fox derivatives of elements such as $\mathbb{Z} H_2(\alpha)$ in some sense can be seen as studying coefficients in a composition of two James–Hopf maps.

Remark Passi (see [13; 14]) has defined a map f between a group G and an R–module M to be a *polynomial of degree* $\leq n$ if its extension to a group algebra $R[G]$ vanishes on Δ_R^{n+1}. In this terminology, Theorem 3.8 states that the maps $\varepsilon\partial^j_{J_j\dots J_1}\mathbb{Z} H_k\colon \mathbb{Z} F[X]\to\mathbb{Z}$ are $\mathbb{Z}$–polynomials of degree $\leq jk$.

Corollary 3.9 *The James–Hopf invariants H_k each send the lower p–central series to the weighted one:*

$$H_k(\gamma_n^{[p]})\subset\gamma_n^{[p],w}.$$

Proof As before, we switch our attention to the powers of the augmentation ideal Δ_p. By Corollary 3.7 we need to show that augmentations of the corresponding Fox derivatives of H_k vanish mod p. This follows from Proposition (1.11) in [14], which states that the composition of a $\mathbb{Z}$–polynomial map $f\colon G\to A$ and a homomorphism $\theta\colon A\to M$ for some R–module M is an R–polynomial map. An application of this proposition to $\varepsilon\partial^j_{J_j\dots J_1}H_k$ and the quotient map $\mathbb{Z}\to\mathbb{Z}/p$ gives the result. □

Theorem 3.10 *Elements of the Cohen group $H_\infty(X)$ induce maps of towers $\Gamma\to\Gamma$ and $\Gamma_p\to\Gamma_p$, and therefore they equip the spectral sequences* (21) *with an action $H_\infty\times E^r_{p,q}\to E^r_{p,q}$ which preserves the differential.*

Proof Combining Lemma 3.3 with Theorem 3.8 we see that generators of the Cohen group (and hence all elements) preserve the lower (p–)central series filtration, and it is immediate that the inclusion $F[-] \to F^\wedge[-]$ induces isomorphisms

$$F[-]/\gamma_n \cong F^\wedge[-]/\gamma_n \quad \text{and} \quad F[-]/\gamma_n^{[p]} \cong F^\wedge[-]/\gamma_n^{[p]}, \tag{27}$$

so there are well-defined maps

$$\begin{aligned} H_\infty &\longrightarrow \mathrm{Hom}(\Gamma, \Gamma), \\ f\colon F[-] \to F^\wedge[-] &\longmapsto \{f_n\}_n\colon \{F[-]/\gamma_n\}_n \to \{F[-]/\gamma_n\}_n, \end{aligned} \tag{28}$$

$$\begin{aligned} H_\infty &\longrightarrow \mathrm{Hom}(\Gamma_p, \Gamma_p), \\ f\colon F[-] \to F^\wedge[-] &\longmapsto \{f_n\}_n\colon \{F[-]/\gamma_n^{[p]}\}_n \to \{F[-]/\gamma_n^{[p]}\}_n. \end{aligned} \tag{29}$$

Note that $\mathrm{Hom}(\Gamma, \Gamma)$ and $\mathrm{Hom}(\Gamma_p, \Gamma_p)$ are groups with respect to a pointwise multiplication on each layer of the tower, and hence (28) and (29) are homomorphisms. Here f_n denotes an induced map on the quotients $F[-]/\gamma_n$ (or $F[-]/\gamma_n^{[p]}$). The images of these maps will be called the *Cohen groups for the towers* Γ *and* Γ_p and denoted by $H\Gamma_\infty$ and $H\Gamma_{p,\infty}$.

The action of H_∞ on spectral sequences comes from the corresponding action on towers through the induced action on exact couples in a standard way. Each $f \in H_\infty$ gives a map $E^1_{p,q} \to E^1_{p,q}$ which is an induced map on homotopy groups of fibers. Preservation of the differential follows from the fact that f induces a map of towers. □

4 Functorial decomposition of the lower p–central series spectral sequence

Throughout this section, $\boldsymbol{k} = \mathbb{Z}/p$ will be the ground field with p elements, and V will be a $\boldsymbol{k}$–module. First we will review a couple of facts about natural coalgebra transformations $T \to T$. The tensor algebra $T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$ can be considered as a (connected) Hopf algebra (see [11]) with the standard multiplication m and comultiplication ψ defined by

$$\psi(v) = v \otimes 1 + 1 \otimes v, \quad v \in V,$$

on generators (making them primitive) and extended to all $T(V)$. The conjugation s is defined by

$$s(x_1 \cdots x_k) = (-1)^k x_k \cdots x_1.$$

Using this comultiplication and conjugation, one can define a group structure on the set of all coalgebra maps $\mathrm{Hom}_{\mathrm{Coalg}_{\boldsymbol{k}}}(T(V), T(V))$, using the so-called *convolution product*: for $f, g\colon T(V) \to T(V)$, the convolution product $f * g$ is the composition

$$T(V) \xrightarrow{\psi} T(V) \otimes T(V) \xrightarrow{f \otimes g} T(V) \otimes T(V) \xrightarrow{m} T(V). \tag{30}$$

Note that this product is a complete analogue of the pointwise multiplication on the set of all maps $G \to G$ for a group G:

$$G \xrightarrow{\Delta} G \times G \xrightarrow{f \times g} G \times G \xrightarrow{\mu} G.$$

As a coalgebra, T is naturally filtered by the *James filtration*

$$J_k = \bigoplus_{n=0}^{k} T_n,$$

where T_n stands for the functor $T_n(V) = V^{\otimes n}$. This filtration is complete and gives $\mathrm{Hom}_{\mathrm{Coalg}_{\boldsymbol{k}}}(T(V), T(V))$ the structure of a pro-group.

Lemma 4.1 [16, Proposition 2.4] *There is a short exact sequence of groups*

$$\mathrm{Hom}_{\boldsymbol{k}}(T_n, \mathcal{L}^n) \hookrightarrow \mathrm{Hom}_{\mathrm{Coalg}_{\boldsymbol{k}}}(J_n, T) \twoheadrightarrow \mathrm{Hom}_{\mathrm{Coalg}_{\boldsymbol{k}}}(J_{n-1}, T). \tag{31}$$

For the simplicial realization theorem below we will use a specific set of topological generators of the pro-group $\mathrm{Hom}_{\mathrm{Coalg}_{\boldsymbol{k}}}(T, T)$, which apparently first appeared in [3]:

Theorem 4.2 *Any natural coalgebra self-map $f_V\colon T(V) \to T(V)$ (evaluated at V) can be written as a (possibly infinite) convolution product $\prod_{k=1}^{\infty} f_k$ of maps of the form*

$$T(V) \xrightarrow{H_k^{\mathrm{alg}}} T(V^{\otimes k}) \xrightarrow{T(\sigma)} T(V^{\otimes k}) \xrightarrow{\beta_k^T} T(V), \tag{32}$$

where H_k^{alg} is the ***algebraic James–Hopf map*** [3, Proposition (5.3)], *$\sigma \in \Sigma_k$ and β_k^T is the extension of the map $\beta_k\colon V^{\otimes k} \to T(V)$, $\beta_k(x_1 \cdots x_k) = [x_1 \cdots x_k]$ to $T(V^{\otimes k})$.*

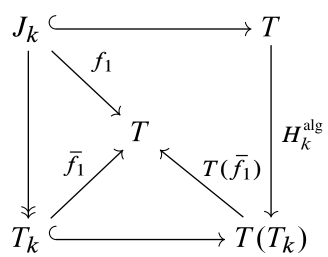
Proof It is sufficient to show the statement for each element in the inverse system. The statement is trivial for maps in $\mathrm{Hom}_{\mathrm{Coalg}_{\boldsymbol{k}}}(J_1, T)$. Suppose that any element $f \in \mathrm{Hom}(J_{k-1}, T)$ can be expressed as

$$f = \prod_{j=1}^{k-1} f_j.$$

A given $f\colon J_k \to T$ can be decomposed into a convolution product $f_1 * f_2$, where $f_1\colon J_k \to T$ is a coalgebra map which is the identity map on $(J_k)_0 = \boldsymbol{k}$, is trivial on $(J_k)_i$, for $0 < i < k$, and satisfies $(f_1)_k = (f)_k$, and f_2 is the restriction of f to J_{k-1}. Therefore it is sufficient to express f_1 as a convolution product of maps of the form (32). We will use the universal property of the algebraic James–Hopf map, which is a direct analogue of the universal property of the classical combinatorial James–Hopf map:

$$\text{(33)}\qquad \mathrm{Top}_*\colon\ \begin{array}{ccc} J_kX & \hookrightarrow & JX \\ \downarrow\!\!\downarrow & & \downarrow H_k \\ X^{\wedge k} & \hookrightarrow & JX^{\wedge k} \end{array} \qquad \mathrm{Coalg}\colon\ \begin{array}{ccc} J_kV & \hookrightarrow & T(V) \\ \downarrow\!\!\downarrow & & \downarrow H_k^{\mathrm{alg}} \\ V^{\otimes k} & \hookrightarrow & T(V^{\otimes k}) \end{array}$$

Applying it to f_1 (which factors through T_k by construction), we see that



Since $\mathrm{Hom}_{\boldsymbol{k}}(T_k, T_l) = \boldsymbol{k}[\Sigma_k]$ if $k = l$ and 0 otherwise, the map $\bar{f_1}\colon T_k \to T$ is completely determined by a certain element $\sum_i a_i\sigma_i$ of a group algebra of the symmetric group Σ_k. Moreover, since $\bar{f_1}$ came initially from a coalgebra map, its image lies in $\mathcal{L}^k_{\mathrm{res}}$ (see Proposition 2.4 in [16]) and therefore it factors through β_k. Finally, under the isomorphism in (31) the sum of permutations maps to a product and we have

$$f_1 = \prod_i \beta_k^T \circ \sigma_i^{a_1} \circ H_k. \qquad \square$$

Remark In contrast with Definition 3.1, there are no reduced diagonals in generators of $\mathrm{Hom}_{\mathrm{Coalg}_{\boldsymbol{k}}}(T, T)$ — these maps are trivial on the tensor algebra.

Similarly to the construction of the lower central series tower, using the powers of the augmentation ideal filtration on a group algebra $\boldsymbol{k}[F[-]]$, one can form a tower of simplicial abelian groups (denoted by $\boldsymbol{k}\Gamma_p$)

$$
\begin{array}{ccc}
\boldsymbol{k}[F]^\wedge = \varprojlim \boldsymbol{k}[F]/\Delta_p^n & & \\
\downarrow & & \\
\vdots & & \\
\downarrow & & \\
\boldsymbol{k}[F]/\Delta_p^{n+1} & \hookleftarrow & T_n(\bar{\boldsymbol{k}}[-]) \\
\downarrow\!\!\!\downarrow & & \\
\boldsymbol{k}[F]/\Delta_p^n & & \\
\downarrow\!\!\!\downarrow & & \\
\vdots & & \\
\downarrow\!\!\!\downarrow & & \\
\boldsymbol{k}[F]/\Delta_p^2 = \boldsymbol{k} \oplus \bar{\boldsymbol{k}}[-] & &
\end{array}
\tag{34}
$$

with fibers identified with the homogeneous components of the tensor algebra functor $T(\bar{\boldsymbol{k}}[-])\colon \mathrm{sSet}_* \to \mathrm{sMod}_{\boldsymbol{k}}$ (analogue of Magnus–Witt theorem; see [5])

$$T_n(\bar{\boldsymbol{k}}[-]) \xrightarrow{\cong} \Delta_p^n/\Delta_p^{n+1}, \quad x_1 \cdots x_n \mapsto (x_1 - 1)\cdots(x_n - 1) + \Delta_p^{n+1}.$$

The natural inclusion $F[-] \to \boldsymbol{k}[F[-]]$ of Milnor's construction to its group algebra can be extended to the level of towers,

$$F[-]/\gamma_n^{[p]} \to \boldsymbol{k}[F[-]]/\Delta_p^n,$$

in a way that on the level of fibers one gets the inclusion $\mathcal{L}_{\mathrm{res}}^n \to T_n$ of homogeneous components of the free restricted Lie algebra to homogeneous components of the tensor algebra as primitive elements in degree n (see also [11]).

As before, E^0 denotes the functor of the fibers of the tower. In the case of the towers (20) and (34), it takes values in graded objects of $\mathrm{sMod}_{\boldsymbol{k}}$. Hence there is a map

$$\mathrm{Hom}_{\mathrm{sSet}_*}(\Gamma_p, \Gamma_p) \xrightarrow{E^0 \circ \boldsymbol{k}[-]} \mathrm{Hom}_{\mathrm{sCoalg}_{\boldsymbol{k}}}(T(\bar{\boldsymbol{k}}[-]), T(\bar{\boldsymbol{k}}[-])). \tag{35}$$

Theorem 4.3 *The map* (35) *is a well-defined epimorphism of groups; that is, every coalgebra natural transformation of the functor* $T(\bar{\boldsymbol{k}}[-])$ *to itself can be lifted to a natural self-transformation of the tower* $\Gamma_p \to \Gamma_p$*.*

Proof First we will show that $E^0 \circ \boldsymbol{k}$ is a well-defined homomorphism; ie for any $f\colon \Gamma_p \to \Gamma_p$, its image $E^0(\boldsymbol{k}[f])$ is a coalgebra map and products are preserved. To see this we pass to the limits of the towers:

$$\begin{array}{ccccc} \mathrm{Hom}(\Gamma_p, \Gamma_p) & \xrightarrow{\ \boldsymbol{k}\ } & \mathrm{Hom}(\boldsymbol{k}\Gamma_p, \boldsymbol{k}\Gamma_p) & \xrightarrow{\ E^0\ } & \mathrm{Hom}(T, T) \\ \Big\downarrow{\scriptstyle \varprojlim} & & \Big\downarrow{\scriptstyle \varprojlim} & \nearrow{\scriptstyle E^0} & \\ \mathrm{Hom}(F_p^\wedge[-], F_p^\wedge[-]) & \xrightarrow[\varprojlim \boldsymbol{k}]{} & \mathrm{Hom}(\boldsymbol{k}[F]^\wedge, \boldsymbol{k}[F]^\wedge) & & \end{array}$$

Here we consider $\boldsymbol{k}[F]^\wedge$ as a complete coalgebra with a filtration, induced from the Δ_p–adic filtration on $\boldsymbol{k}[F]$, and in this context E^0 is the functor of passing to the associated graded coalgebra. E^0 respects (completed) tensor products:

$$E^0(- \mathbin{\hat{\otimes}} -) = E^0(- \otimes -) = E^0(-) \otimes E^0(-). \tag{36}$$

$\varprojlim \boldsymbol{k}$ is an extension of the group algebra functor $\boldsymbol{k}[-]\colon \mathrm{Grp} \to \mathrm{Hopf}_{\boldsymbol{k}}$ to completions, and therefore $\varprojlim \boldsymbol{k}(g)$ is a map of complete coalgebras for any map $g\colon F_p^\wedge \to F_p^\wedge$. This shows that the map (35) is well defined.

By (36) and since $\varprojlim$ commutes with direct products and

$$\varprojlim \boldsymbol{k}(F_p^\wedge \times F_p^\wedge) = \boldsymbol{k}F^\wedge \mathbin{\hat{\otimes}} \boldsymbol{k}F^\wedge,$$

$E^0 \circ \boldsymbol{k}$ sends the pointwise multiplication

$$F/\gamma_n^{[p]} \xrightarrow{\ \Delta\ } F/\gamma_n^{[p]} \times F/\gamma_n^{[p]} \xrightarrow{\ f \times g\ } F/\gamma_n^{[p]} \times F/\gamma_n^{[p]} \xrightarrow{\ \mu\ } F/\gamma_n^{[p]}$$

to the convolution product (30).

To prove that $E^0 \circ \boldsymbol{k}[-]$ is surjective we will show that generators of the Cohen group for towers map to generators of $\mathrm{Hom}_{\mathrm{sCoalg}_{\boldsymbol{k}}}(T(\bar{\boldsymbol{k}}[-]), T(\bar{\boldsymbol{k}}[-]))$, which are described in Theorem 4.2. It is clear that $E^0(\boldsymbol{k}[F[\sigma]]) = T(\sigma)$ and $E^0(\boldsymbol{k}[W_n]) = \beta_n^T$. Only the universal property (33) of H_k^{alg} is used in Theorem 4.2, and $E^0(\boldsymbol{k}[H_k])$ satisfies it since $E^0(\boldsymbol{k}[J]) \equiv E^0(\boldsymbol{k}[F])$ — the associated graded of the free group ring and the free monoid ring are naturally isomorphic (the isomorphism is induced by the inclusion $J \to F$). The result follows from diagrams (33) and (4). □

We proceed to a decomposition of the spectral sequence (21). First we translate the well-known result about homotopy idempotents of H–spaces to sSet_*.

Lemma 4.4 *Let G be a connected simplicial group and $f\colon G \to G$ be a simplicial self-map such that*

$$f_*\colon \pi_* G \to \pi_* G$$

is an idempotent. Then, as a simplicial set, G is weak equivalent to a product $A \times B$,

$$A = \operatorname{colim} f, \quad B = \operatorname{colim} g,$$

where $g = [\mathrm{id}] f^{-1}$ denotes a complement to f in the group $\mathrm{Hom}_{\mathrm{sSet}_*}(G, G)$.

Proof Since f_* is a (graded) idempotent element of $\mathrm{End}\big(\bigoplus_i \pi_i G\big)$, there is an isomorphism

$$\bigoplus_i \pi_i G \equiv \bigoplus_i \operatorname{im} f_i \oplus \operatorname{im}(\mathrm{id} - f_i), \tag{37}$$

where $\operatorname{im} f_i = \operatorname{colim} f_i = \pi_i \operatorname{colim} f$, $\operatorname{im}(\mathrm{id}-f_i) = \operatorname{colim}(\mathrm{id}-f_i) = \pi_i \operatorname{colim}[\mathrm{id}][f^{-1}]$ and the isomorphism (37) can be realized by

$$G \xrightarrow{\Delta} G \times G \to A \times B. \qquad \square$$

Well-known results about filtered colimits and fibrations in sSet_* will be organized in the next lemma for simplicity.

Lemma 4.5 *Let $X_0 \to X_1 \to \cdots \to X$ be a sequence of simplicial sets and simplicial maps between them with colimit X, and let*

$$\begin{array}{ccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots & \longrightarrow & X \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & & & \downarrow{\scriptstyle f} \\ Y_0 & \longrightarrow & Y_1 & \longrightarrow & \cdots & \longrightarrow & Y \end{array}$$

be a map of such sequences. Then:

(i) $\pi_* X = \operatorname{colim}\{\pi_*(X_0) \to \pi_*(X_1) \to \cdots\}$.

(ii) *If f_n is a Kan fibration for every n, then f is also a Kan fibration.*

(iii) *The fiber of f is a colimit of fibers of f_n.*

Given an arbitrary natural coalgebra decomposition $T \simeq A \otimes B$ of the tensor algebra, it can be extended to sSet_* by switching from the functor $T\colon \mathrm{Mod}_{\boldsymbol{k}} \to \mathrm{Coalg}_{\boldsymbol{k}}$

to $T(\bar{\boldsymbol{k}}[-])\colon \mathrm{sSet}_* \to \mathrm{sCoalg}_{\boldsymbol{k}}$. For any $X \in \mathrm{sSet}_*$ the composition of retraction and inclusion

$$f\colon T(\bar{\boldsymbol{k}}[X]) \xrightarrow{r_A} A(\bar{\boldsymbol{k}}[X]) \xrightarrow{i_A} T(\bar{\boldsymbol{k}}[X])$$

is an idempotent in $\mathrm{Hom}_{\mathrm{sCoalg}_{\boldsymbol{k}}}(T(\bar{\boldsymbol{k}}[X]), T(\bar{\boldsymbol{k}}[X]))$ and $A(\bar{\boldsymbol{k}}[X]) \simeq \operatorname{colim} f$. This idempotent gives rise to a map of towers $\varphi\colon \Gamma_p \to \Gamma_p$ by the simplicial realization theorem (Theorem 4.3) on each level n denoted by $\varphi_n\colon F[X]/\gamma_n^{[p]} \to F[X]/\gamma_n^{[p]}$.

Theorem 4.6 *For any natural (simplicial) coalgebra decomposition $T \cong A \otimes B$ there exist towers of fibrations $\mathcal{A}$ and $\mathcal{B}$ such that*

$$\Gamma_p \simeq \mathcal{A} \times \mathcal{B}$$

and

$$E^0(\mathcal{A}) \simeq PA, \quad E^0(\mathcal{B}) \simeq PB,$$

where P denotes the primitive elements functor.

Proof Given an $X \in \mathrm{sSet}_*$ and a decomposition $T(\boldsymbol{k}[X]) \simeq A(\boldsymbol{k}[X]) \otimes B(\boldsymbol{k}[X])$, let $f_X\colon T(\boldsymbol{k}[X]) \to T(\boldsymbol{k}[X])$ be an idempotent, corresponding to $A(\boldsymbol{k}[X])$. As before, its realization on the level of towers is $\varphi_X\colon \Gamma_p(X) \to \Gamma_p(X)$. Since, in each dimension k, X_k is a finite set, the groups $(F[X]/\gamma_n^{[p]})_k$ are finite for every n. So, after sufficiently many iterations, $\varphi_{X,n}\colon F[X]/\gamma_n^{[p]} \to F[X]/\gamma_n^{[p]}$ will become an idempotent; ie there exists a number $N(n,k)$ such that

$$(\varphi_{X,n})_k^{N(n,k)}\colon (F[X]/\gamma_n^{[p]})_k \to (F[X]/\gamma_n^{[p]})_k \tag{38}$$

are idempotents. Since φ_X was initially a map of towers, φ_X^N is also one, which follows from commutativity of the following diagrams:

$$\begin{array}{ccc} (F[X]/\gamma_n^{[p]})_k & \overset{\longrightarrow}{\underset{\longleftarrow}{\cdots}} & (F[X]/\gamma_n^{[p]})_{k-1} \\ {\scriptstyle (\varphi_{X,n})_k^{N(n,k)}}\downarrow & & \downarrow{\scriptstyle (\varphi_{X,n})_k^{N(n,k-1)}} \\ (F[X]/\gamma_n^{[p]})_k & \overset{\longrightarrow}{\underset{\longleftarrow}{\cdots}} & (F[X]/\gamma_n^{[p]})_{k-1} \end{array} \qquad \begin{array}{ccc} F[X]/\gamma_n^{[p]} & \xrightarrow{(\varphi_{X,n})_k^{N(n)}} & F[X]/\gamma_n^{[p]} \\ \downarrow & & \downarrow \\ F[X]/\gamma_{n-1}^{[p]} & \xrightarrow[(\varphi_{X,n-1})_k^{N(n-1)}]{} & F[X]/\gamma_{n-1}^{[p]} \end{array}$$

Now Lemma 4.4 is applied to the groups $F[X]/\gamma_n^{[p]}$ and the idempotents (38) to obtain decompositions

$$F[X]/\gamma_n^{[p]} \simeq \mathcal{A}_n(X) \times \mathcal{B}_n(X), \quad \text{where } \mathcal{A}_n(X) = \operatorname{colim} \varphi_{X,n}^{N(n)} = \operatorname{colim} \varphi_{X,n}. \tag{39}$$

This weak equivalence is consistent with the maps $F[X]/\gamma_n^{[p]} \twoheadrightarrow F[X]/\gamma_{n-1}^{[p]}$ and by Lemma 4.5 the natural maps $\mathcal{A}_n(X) \to \mathcal{A}_{n-1}(X)$ are fibrations, similarly for $\mathcal{B}_n(X)$, so (39) indeed gives the decomposition of the whole tower $\Gamma_p(X)$, and this decomposition is functorial with respect to X.

By commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Hom}(\Gamma_p, \Gamma_p) & \xrightarrow{\boldsymbol{k}} & \mathrm{Hom}(\boldsymbol{k}\Gamma_p, \boldsymbol{k}\Gamma_p) \\ {\scriptstyle E^0}\downarrow & & \downarrow{\scriptstyle E^0} \\ \mathrm{Hom}(\mathcal{L}, \mathcal{L}) & \xleftarrow[P]{} & \mathrm{Hom}(T, T) \end{array}$$

and Lemma 4.5,

$$E^0\mathcal{A} = E^0 \operatorname{colim} \varphi \simeq \operatorname{colim} E^0\varphi = \operatorname{colim} Pf = P(\operatorname{colim} f) = PA. \qquad \square$$

Remark After passing to the limits of towers one has a functorial decomposition of the pro-p completion of Milnor's construction $F[-]$,

$$F_p^\wedge \simeq \varprojlim \mathcal{A} \times \varprojlim \mathcal{B}, \tag{40}$$

and since $|F_p^\wedge[X]| \simeq_p \Omega\Sigma|X|$, (40) can be seen as a translation of the classical decomposition (1) to the simplicial setting.

Corollary 4.7 *Any natural coalgebra decomposition $T \simeq A \otimes B$ induces a decomposition of the spectral sequence*

$$E^1_{p,q} = \pi_p \mathcal{L}^q_{\mathrm{res}} \Rightarrow \pi_p F, \quad E^r_{p,q} = E^r_{p,q}(A) \oplus E^r_{p,q}(B), \tag{41}$$

as a functor on sSet_*, *with the first pages of $E^r_{p,q}(A)$ and $E^r_{p,q}(B)$ given by the homotopy groups of primitive elements of the simplicial coalgebras A and B:*

$$E^1_{p,q}(A) = \pi_p(PA)_q \quad \text{and} \quad E^1_{p,q}(B) = \pi_p(PB)_q.$$

Remark As a sample application of Corollary 4.7 we will use the *block decomposition*

$$T = \bigotimes_{i=0}^{\infty} C^{m_i} \tag{42}$$

constructed in [8]. Here $\{m_i\}_i$ is the ordered set of all positive integers coprime with p (the characteristic of the base field), with $m_0 = 1$. The C^{m_i} are natural coalgebra retracts of T such that their primitive elements are

$$P_n C^{m_i} = \begin{cases} \mathcal{L}^n_{\text{res}} & \text{if } n = m_i p^r \text{ for some } r \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now (42) leads to a decomposition of the unstable Adams spectral sequence for suspensions,

$$E^r_{p,q} = \bigoplus_{i=0}^{\infty} E^r_{p,q}(C^{m_i}).$$

In particular, on the first page $E^1(C^{m_i})$ nonzero cells are concentrated in columns with numbers $m_i p^r$ for some r. This means that in the case $p = 2$, on the main spectral sequence the only nontrivial differential on the first page is acting between the first and second columns. This is consistent with the completely different description of the E_1–term as a module over a Λ–algebra given in [5] (see also [1] for the definition of the Λ–algebra). According to this description, for a simplicial set X, the unstable Adams spectral sequence converging to ΩX has first page $E_1 \cong \mathcal{L}_{\text{res}}(H_* X) \widehat{\otimes} \Lambda$, with differential d^1 acting on elements $x \in \mathcal{L}_{\text{res}}(H_* X)$ as (Theorem (8.10) in [5])

$$d^1(x) = \begin{cases} \partial^{\Delta}(x) + \sum_{i=1}^{[n/2]} (x).\mathrm{Sq}^i_* \otimes \lambda_{i-1} & \text{for } x \in H_n X, \\ \partial^{\Delta}(x) & \text{for } x \in \mathcal{L}^k_{\text{res}}(H_* X), \text{ where } k \geq 2. \end{cases}$$

The term $\partial^{\Delta}(x)$ is given by a dual to a cup product on X and vanishes for suspensions $X = \Sigma X'$, hence d^1 acts nontrivially only on elements of $H_* \Sigma X'$, which are concentrated in the first column.

In conclusion we will formulate the result about accelerated functorial spectral subsequence of the unstable Adams spectral sequence for suspensions, which is a direct corollary of Corollary 4.7.

Corollary 4.8 *Let $A^{\min}$ be a minimal functorial coalgebra retract of T, and let $T = A^{\min} \otimes B^{\max}$ be a corresponding decomposition. Then the spectral subsequence $E^r_{p,q}(\mathcal{A}^{\min})$ of $E^r_{p,q}$ has nontrivial cells only in columns with numbers p^k.*

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