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Dimensional reduction and the equivariant Chern character

AUGUSTO STOFFEL

We propose a dimensional reduction procedure for $1|1$–dimensional supersymmetric euclidean field theories (EFTs) in the sense of Stolz and Teichner. Our construction is well suited in the presence of a finite gauge group or, more generally, for field theories over an orbifold. As an illustration, we give a geometric interpretation of the Chern character for manifolds with an action by a finite group.

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1 Introduction

In the context of topological quantum field theory (that is, the study of symmetric monoidal functors d–Bord → Vect and variants thereof), dimensional reduction is the assignment of a $(d-1)$–dimensional theory to a d–dimensional theory induced by the functor of bordism categories $S^1 \times -\colon (d-1)\text{–Bord} \to d\text{–Bord}$. In the Stolz–Teichner framework of supersymmetric euclidean field theories (EFTs) [24; 23], dimensional reduction is a more subtle subject, but it can still be implemented and provides geometric interpretations of classical constructions in algebraic topology. To give the basic idea, we first recall that $0|1$–dimensional EFTs over a manifold X are in bijection, after passing to concordance classes, with de Rham cohomology classes of X; see Hohnhold, Kreck, Stolz and Teichner [15]. On the other hand, superparallel transport — see Dumitrescu [9] — allows us to associate a field theory $E_V \in 1|1\text{–EFT}(X)$ to any vector bundle with connection $V \in \mathrm{Vect}^\nabla(X)$, and a similar statement relating $1|1$–dimensional EFTs and topological K–theory is widely expected. Moreover, there is a dimensional reduction map red between (groupoids of) field theories over X that recovers the Chern character, in the sense that the diagram

$$\begin{tikzcd}
 & 1|1\text{–EFT}(X) \arrow[r, "\mathrm{red}"] \arrow[d, dotted] & 0|1\text{–EFT}(X) \arrow[d] \\
\mathrm{Vect}^\nabla(X) \arrow[ur, bend left, "E"] \arrow[r, bend right] & K^0(X) \arrow[r, "\mathrm{ch}"] & H^{\mathrm{ev}}(X;\mathbb{C})
\end{tikzcd}$$

commutes; see Dumitrescu [10] and Han [14].

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This paper is part of an ongoing project aiming to identify gauged supersymmetric field theories as geometric cocycles for equivariant cohomology theories; see Berwick-Evans [5], Berwick-Evans and Han [7] and Stolz [22]. Our main goal here is to extend the above dimensional reduction procedure for $1|1$–EFTs to the case where the manifold X is replaced by an orbifold $\mathfrak{X}$ (or, more generally, any stack on the site SM of supermanifolds). This will be based on a series of functors between variants of the euclidean bordism categories over $\mathfrak{X}$,

$$(1)\quad 0|1\text{–EBord}(\Lambda\mathfrak{X}) \xleftarrow{\mathcal{P}} 0|1\text{–EBord}^{\mathbb{T}}(\Lambda\mathfrak{X}) \xrightarrow{\mathcal{Q}} 0|1\text{–EBord}^{\mathbb{R}/\!/\mathbb{Z}}(\Lambda\mathfrak{X}) \xrightarrow{\mathcal{R}} 1|1\text{–EBord}(\mathfrak{X}).$$

Dimensional reduction of field theories (or twist functors) will then be realized as the pull–push operation induced by $\mathcal{R}$, $\mathcal{Q}$ and $\mathcal{P}$. The two middle objects in (1), which we call $\mathbb{T}$– and $\mathbb{R}/\!/\mathbb{Z}$–equivariant bordisms, respectively, over the inertia stack $\Lambda\mathfrak{X}$, as well as the maps involving them, are introduced in Section 3. Here, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ stands for the circle group and $\mathbb{R}/\!/\mathbb{Z}$ is the stack arising from the action of $\mathbb{Z}$ on $\mathbb{R}$. These are of course equivalent as group stacks, and our terminology just intends to indicate which model for the circle is directly involved in the definition of each bordism category. (The two equivariant bordism categories also turn out to be equivalent, though this requires proof; see Theorem 11.) We also remark that $\mathcal{R}$ takes values in the substack of closed bordisms. This allows us to avoid delving into the somewhat long definition of the full bordism category $1|1$–EBord$(\mathfrak{X})$, and focus on the stack $\mathfrak{K}(\mathfrak{X})$ of closed, connected bordisms, which we call euclidean supercircles.

As a simple but illustrative application, we specialize to the case where $\mathfrak{X} = X/\!/G$ is a global quotient orbifold and give a field-theoretic interpretation of the simplest instance of orbifold Chern character, namely the one concerning untwisted cohomology of global quotients; see Baum and Connes [2]. It is possible to extend the map E above for an orbifold $\mathfrak{X}$ in place of the manifold X; since the dimensional reduction only depends on the values of a field theory on closed bordisms, we will only describe the partition function Z_V of the field theory E_V in this paper. That is, we construct a map

$$Z\colon \mathrm{Vect}^{\nabla}(\mathfrak{X}) \to C^{\infty}(\mathfrak{K}(\mathfrak{X}))$$

(see Section 4.2). From the discussion of Section 2 it will follow that $0|1$–EFTs over the inertia orbifold $\Lambda\mathfrak{X}$ are geometric cocycles for the so-called delocalized cohomology $H_G^{\mathrm{ev}}(\widehat{X};\mathbb{C})$ — the codomain of the equivariant Chern character ch_G (see Section 4.1). Finally, in Section 4.3 we verify that the dimensional reduction of Z_V is a representative of $\mathrm{ch}_G(V)$.

Theorem 1 *Let $\mathfrak{X} = X/\!\!/G$ be the quotient stack arising from the action of a finite group on a manifold. Then the diagram*

$$\begin{tikzcd} & C^\infty(\mathfrak{K}(\mathfrak{X})) \arrow[r, "\mathrm{red}"] & 0|1\text{–}\mathrm{EFT}(\Lambda\mathfrak{X}) \arrow[dd] \\ \mathrm{Vect}^\nabla(\mathfrak{X}) \arrow[ur, bend left, "Z"] \arrow[dr, bend right] & & \\ & K^0_G(X) \arrow[r, "\mathrm{ch}_G"] & H^{\mathrm{ev}}_G(\widehat{X};\mathbb{C}) \end{tikzcd}$$

commutes, and moreover the vertical map induces a bijection after passing to concordance classes of field theories.

Remark 2 In a subsequent paper [21], we construct twists for $1|1$–EFTs over $\mathfrak{X}$ associated to classes in $H^3(\mathfrak{X},\mathbb{Z})$, using a representing gerbe with connection as input data, as well as twisted field theories from twisted vector bundles. We also employ the dimensional reduction procedure given here to relate these twists and twisted field theories with more general versions of the orbifold Chern character. In particular, when the twist is trivial, the field theories in question do indeed have Z as partition function. This allows us to replace $C^\infty(\mathfrak{K}(\mathfrak{X}))$ with a suitably defined groupoid $1|1$–EFT$(\mathfrak{X})$ of field theories over $\mathfrak{X}$ in the above theorem.

While this work was in preparation, closely related preprints by Daniel Berwick-Evans have appeared [5; 6]. His approach is heavily inspired by ideas from perturbative quantum field theory, while ours is more geometric, putting group actions on stacks at the forefront.

1.1 Terminology and background

For an extensive survey of the Stolz–Teichner program, see [24]. The facts more directly relevant to this paper, regarding $0|1$–dimensional field theories, can be found in [15]. Concerning supermanifolds, we generally follow the definitions and conventions of Deligne and Morgan [8], and in particular we routinely use the functor of points formalism. The necessary facts about euclidean structures are reviewed in Appendix B.

Vector bundles are always $\mathbb{Z}/2$–graded and over $\mathbb{C}$, and Vect^∇ denotes the stack of vector bundles with connection. C^∞ and Ω^* denote the sheaves of complex-valued functions and differential forms. In the category of supermanifolds, the notions of principal bundles and connections mimic the usual definitions; see Stavracou [19]. If G is a super-Lie group with super-Lie algebra $\mathfrak{g}$, a principal G–bundle over the base X is a manifold P with a free G–action and an invariant submersion $P \to X$ which is locally isomorphic to $X \times G \to X$. A connection is a real form $\omega \in \Omega^1(P; \mathfrak{g})$ of *even* parity satisfying the usual conditions (to be G–invariant and coincide with the Maurer–Cartan form of G on the fibers), and its curvature is $d\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2(X, P \times_{\mathrm{ad}} \mathfrak{g})$. More generally, if $X \to S$ is a submersion, then an S–family of differential forms, or fiberwise form, is a section of some exterior power of $\operatorname{Coker}(T^*S \to T^*X)$. Fiberwise connections and their curvature are families of forms defined in a similar fashion.

We treat stacks on the site SM of supermanifolds (where a covering is a collection of jointly surjective local diffeomorphisms) in a geometric way, meaning, for instance, that most of our diagrams involving manifolds must be interpreted as diagrams in stacks, where some of the objects happen to be representable sheaves. A differentiable stack is a stack $\mathfrak{X}$ that admits an atlas $X_0 \to \mathfrak{X}$, or, equivalently, can be presented by a Lie groupoid $X_1 \rightrightarrows X_0$. We recommend the appendix of Hohnhold et al [15] for a short introduction to stacks, and Behrend and Xu [4] as a more detailed reference, including the stacky perspective on orbifolds and cohomology of orbifolds. The less standard piece of descent theory needed in this paper concerns group actions on stacks. We offer a short overview (with further references) in Appendix A, where we also record a lemma that may be of independent interest (Proposition 18).

If $\mathfrak{X}$ is a stack, we define its *inertia* to be the mapping stack

$$\Lambda\mathfrak{X} = \underline{\mathrm{Fun}}_{\mathrm{SM}}(\mathrm{pt}/\!\!/\mathbb{Z}, \mathfrak{X}).$$

More concretely, $\Lambda\mathfrak{X}$ is the fibered category whose S–points are given by pairs (x, α) with $x \in \mathfrak{X}_S$ and α an automorphism of x. A morphism $(x, \alpha) \to (x', \alpha')$ is given by a morphism $\psi\colon x \to x'$ in $\mathfrak{X}$ such that $\alpha' \circ \psi = \psi \circ \alpha$. The stack $\mathrm{pt}/\!\!/\mathbb{Z}$ can be thought of as a categorical circle, and $\Lambda\mathfrak{X}$ is the stack of "hidden loops", ie those loops that are not seen by the coarse moduli space of $\mathfrak{X}$ (see eg Lupercio and Uribe [16] for more information). Notice that $\mathrm{pt}/\!\!/\mathbb{Z}$ is a group object in stacks, and it follows that $\Lambda\mathfrak{X}$ is acted upon by it. Concretely, such an action translates as an automorphism of $\mathrm{id}_{\Lambda\mathfrak{X}}$, namely the natural transformation assigning to (x, α) the automorphism α.

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2 Bordisms and field theories over an orbifold

A d–dimensional topological (quantum) field theory, in the usual definition of Atiyah and Segal, is a symmetric monoidal functor

$$E \in \mathrm{Fun}^{\otimes}(d\text{–Bord}, \mathrm{Vect})$$

between the category of d–dimensional bordisms and the category of vector spaces. The domain has as objects closed $(d-1)$–dimensional manifolds and as morphisms diffeomorphism classes of bordisms between them.

Stolz and Teichner [24] consider a refinement of the above, where each bordism is equipped with several additional geometric structures: supersymmetry, meaning that a bordism is now a supermanifold of dimension $d|\delta$; a euclidean structure in the sense of Appendix B; and finally a smooth map to a fixed manifold X. In order to make sense of the idea that field theories should depend smoothly on the input data, we are led to formulate the resulting bordism category $d|\delta\text{–EBord}(X)$ as a (weak) category internal to symmetric monoidal stacks. This also allows us to keep track of isometries between bordisms instead of just considering equivalence classes of bordisms modulo isometry.

Once this framework is in place, it is clear how to replace the manifold X by a "generalized manifold", or stack, $\mathfrak{X}$: an S–family of bordisms in $d|\delta\text{–EBord}(\mathfrak{X})$ is given by a submersion $\Sigma \to S$ of codimension $d|\delta$ with fiberwise euclidean structure, an object of $\mathfrak{X}_\Sigma$ (which, by the Yoneda lemma, corresponds to a map $\psi\colon \Sigma \to \mathfrak{X}$ in the realm of generalized manifolds) and lastly some boundary information we will not detail here (see [21, Section 2.8]). A morphism over $f\colon S' \to S$ in the stack of bordisms is determined by a fiberwise isometry $F\colon \Sigma' \to \Sigma$ covering f (and suitably compatible with the boundary information) together with a morphism ξ between objects of $\mathfrak{X}_{\Sigma'}$ as indicated in the diagram below:

$$\begin{tikzcd}
\Sigma' \arrow[dr, "F"] \arrow[drr, bend left, "\psi'"] \arrow[dd] & & \\
 & \Sigma \arrow[r, "\psi"] \arrow[dd] \arrow[u, phantom, "\Downarrow\xi"] & \mathfrak{X} \\
S' \arrow[dr, "f"] & & \\
 & S &
\end{tikzcd}$$

Finally, euclidean field theories of dimension $d|\delta$ over $\mathfrak{X}$ are functors of internal categories

$$d|\delta\text{–EFT}(\mathfrak{X}) = \mathrm{Fun}^{\otimes}_{\mathrm{SM}}(d|\delta\text{–EBord}(\mathfrak{X}), \mathrm{TV}),$$

where TV is an internal version of the category of topological vector spaces. These are contravariant objects on the variable $\mathfrak{X}$, and we call two EFTs $E_0, E_1 \in d|\delta$–EFT$(\mathfrak{X})$ concordant if there exist a field theory $E \in d|\delta$–EFT$(\mathfrak{X} \times \mathbb{R})$ such that $E \cong \mathrm{pr}_1^* E_0$ on $\mathfrak{X} \times (-\infty, 0)$ and $E \cong \mathrm{pr}_1^* E_1$ on $\mathfrak{X} \times (1, \infty)$.

These observations are the foundation of an equivariant extension of Stolz–Teichner program. In this paper, we are only interested in the cases $d|\delta = 0|1$ or $1|1$, so we can work with simplified definitions, which we discuss in the remainder of this section.

2.1 Dimension 0|1

Since every $0|1$–dimensional bordism is closed, a $0|1$–EFT is nothing but the assignment of a complex number to each euclidean $0|1$–manifold, in a way that is invariant under isometries and such that disjoint unions map to products. In particular, a $0|1$–EFT is determined by its values on connected bordisms. Thus, we can just define

$$0|1\text{–EFT}(\mathfrak{X}) = \mathrm{Fun}_{\mathrm{SM}}(\mathfrak{B}(\mathfrak{X}), \mathbb{C}) = C^\infty(\mathfrak{B}(\mathfrak{X})),$$

where $\mathfrak{B}(\mathfrak{X})$ is some model for the full substack comprising fiberwise connected bordisms in $0|1$–EBord$(\mathfrak{X})$. Concretely, we take it to be

$$\mathfrak{B}(\mathfrak{X}) = \Pi T\mathfrak{X} /\!\!/ \mathrm{Isom}(\mathbb{R}^{0|1}), \quad \text{where } \Pi T\mathfrak{X} = \underline{\mathrm{Fun}}_{\mathrm{SM}}(\mathbb{R}^{0|1}, \mathfrak{X}).$$

Here, $\mathrm{Fun}_{\mathrm{SM}}$ denotes the groupoid of fibered functors and natural transformations over SM, while $\underline{\mathrm{Fun}}_{\mathrm{SM}}$ denotes the mapping stack. Thus, $\mathfrak{B}(\mathfrak{X})$ is the quotient stack arising from a group action on a stack; see Appendix A. The notation $\Pi T\mathfrak{X}$ is motivated by the fact that when $\mathfrak{X}$ is a manifold, the internal hom in question is in fact representable by the parity-reversed tangent bundle.

Theorem 3 *For any differentiable stack $\mathfrak{X}$, there is a natural bijection*

$$0|1\text{–EFT}(\mathfrak{X}) \cong \Omega^{\mathrm{ev}}_{\mathrm{cl}}(\mathfrak{X})$$

between $0|1$–EFTs over $\mathfrak{X}$ and closed differential forms of even parity. If $\mathfrak{X}$ is an orbifold, passing to concordance classes gives an isomorphism with even de Rham cohomology,

$$0|1\text{–EFT}(\mathfrak{X})/\text{concordance} \cong H^{\mathrm{ev}}_{\mathrm{dR}}(\mathfrak{X}).$$

Here, $\Omega^{\mathrm{ev}}_{\mathrm{cl}}(\mathfrak{X}) = \mathrm{Fun}_{\mathrm{SM}}(\mathfrak{X}, \Omega^{\mathrm{ev}}_{\mathrm{cl}})$, where $\Omega^{\mathrm{ev}}_{\mathrm{cl}}$ is the sheaf on SM of even, closed differential forms.

Proof For $\mathfrak{X}$ a manifold, this is Theorem 1 in Hohnhold et al [15], and the main ingredient of the proof is to identify the action of $\mathrm{Isom}(\mathbb{R}^{0|1})=\mathbb{R}^{0|1}\rtimes\mathbb{Z}/2$ on ΠTX. It turns out that on $C^\infty(\Pi TX)=\Omega^*(X)$, $\mathbb{Z}/2$ acts as the mod 2 grading involution, and the odd vector field generating the $\mathbb{R}^{0|1}$–action is precisely the de Rham differential.

Now, let $X_1 \rightrightarrows X_0$ be a groupoid presentation of $\mathfrak{X}$. Then $\Pi TX_1 \rightrightarrows \Pi TX_0$ is a groupoid presentation of $\Pi T\mathfrak{X}$, since both stacks assign a groupoid equivalent to

$$\mathrm{SM}(S\times\mathbb{R}^{0|1},X_1)\rightrightarrows \mathrm{SM}(S\times\mathbb{R}^{0|1},X_0)$$

to any contractible S. It follows [15, Proposition 7.13] that

$$C^\infty(\Pi T\mathfrak{X})\cong \lim(\Omega^*(X_0)\rightrightarrows\Omega^*(X_1))=\Omega^*(\mathfrak{X}).$$

The $\mathrm{Isom}(\mathbb{R}^{0|1})$–action on $\Omega^*(\mathfrak{X})\subset\Omega^*(X_0)$ is, again, generated by the de Rham differential and the $\mathbb{Z}/2$–grading operator.

Now,

$$0|1\text{–EFT}(\mathfrak{X})=\mathrm{Fun}_{\mathrm{SM}}(\Pi T\mathfrak{X}/\!\!/\,\mathrm{Isom}(\mathbb{R}^{0|1}),\mathbb{C})$$

can be calculated from Proposition 18, and is given by

$$\lim\bigl(C^\infty(\Pi T\mathfrak{X})\rightrightarrows C^\infty(\Pi T\mathfrak{X}\times\mathrm{Isom}(\mathbb{R}^{0|1}))\bigr)=\Omega^*(\mathfrak{X})^{\mathrm{Isom}(\mathbb{R}^{0|1})}=\Omega^{\mathrm{ev}}_{\mathrm{cl}}(\mathfrak{X}).$$

By the Stokes theorem, forms in $\Omega^{\mathrm{ev}}_{\mathrm{cl}}(X_0)$ are concordant if and only if they are cohomologous. The same type of argument shows that concordance through closed, X_1–invariant forms is the same relation as being cohomologous in the chain complex $\Omega^*(\mathfrak{X})$. Thus,

$$0|1\text{–EFT}(\mathfrak{X})/\text{concordance}\cong H^{\mathrm{ev}}(\Omega^*(\mathfrak{X}),d).$$

For orbifolds, the right-hand side can be taken as the definition of de Rham cohomology [3, Corollary 25]. □

Remark 4 Differential forms and cohomology classes of odd degree are similarly related to field theories twisted by the basic twist $\mathcal{T}_1$ of [15, Definition 6.2]. This statement is proven similarly, using, as in [15], the fact that sections of $\mathcal{T}_1$ correspond to closed, odd differential forms (see also [21, Section 2], where more general twists are considered).

2.2 Dimension 1|1

In order to construct the functor of internal categories $\mathcal{R}$ of diagram (1), all details about the stack of objects of $1|1$–EBord($\mathfrak{X}$) and nonclosed bordisms are entirely irrelevant; this is, again, due to the fact that the domain of $\mathcal{R}$ has trivial object stack. Thus, it suffices to work with the moduli stack of closed and connected bordisms in $1|1$–EBord$(\mathfrak{X})$, which we will also call the stack of *euclidean supercircles over* $\mathfrak{X}$ and denote by $\mathfrak{K}(\mathfrak{X})$.

The moduli stack $\mathfrak{K}(\mathfrak{X})$ of euclidean supercircles over $\mathfrak{X}$ is defined as follows. An object (K, ψ) of $\mathfrak{K}(\mathfrak{X})$ over S is given by an S–family K of closed, connected euclidean $1|1$–manifolds together with a map $\psi\colon K \to \mathfrak{X}$. A morphism $(K', \psi') \to (K, \psi)$ over a map $f\colon S' \to S$ is given by a fiberwise isometry $F\colon K' \to K$ covering f together with a 2–morphism $\psi' \to \psi \circ F$; compositions are performed in the obvious way. The data of a morphism can be summarized by the following diagram:

$$\begin{tikzcd}
K' \arrow[dr, "F"] \arrow[drr, bend left, "\psi'"] \arrow[dd] \arrow[from=1-1, to=2-3, phantom, "\Downarrow \xi"] & & \\
& K \arrow[r, "\psi"] \arrow[dd] & \mathfrak{X} \\
S' \arrow[dr, "f"] & & \\
& S &
\end{tikzcd}$$

Remark 5 A complete definition of the bordism category $1|1$–EBord$(\mathfrak{X})$ is given in [21]. It is easy to see that $\mathfrak{K}(\mathfrak{X})$, as given here, is indeed the substack of closed and connected bordisms there. Alternatively, the reader may prefer to think of the present description of $\mathfrak{K}(\mathfrak{X})$ as being sufficiently reasonable, and thus a sanity check for the more general construction.

A detailed study of the stack $\mathfrak{K} = \mathfrak{K}(\mathrm{pt})$ is given in Section B.2. Examples of (families of) supercircles can be obtained by choosing a "length" parameter $l\colon S \to \mathbb{R}^{1|1}_{>0}$, and then letting

$$K_l = (S \times \mathbb{R}^{1|1})/\mathbb{Z}l$$

be given by the orbit space of the translation by l. Proposition 23 shows that, at least locally in S, every supercircle is of this form (but not canonically). Moreover, any morphism $K_{l'} \to K_l$ is determined by a smooth map $S' \to \mathbb{R}^{1|1} \rtimes \mathbb{Z}/2$, which fixes a certain relation between l' and l; see Section B.2 for more details.

In order to define the functor $\mathcal{R}$ in Section 3.1, we will use an alternative method to construct supercircles, provided by the following theorem:

Theorem 6 *Let $\Sigma \to S$ be an S–family of euclidean $0|1$–manifolds and $P \to \Sigma$ a principal $\mathbb{T}$–bundle. Then a fiberwise (in S) connection form ω on P whose curvature agrees with the tautological 2–form ζ on Σ canonically determines a euclidean structure on P. Isometries of P correspond bijectively to connection-preserving bundle maps covering an isometry of Σ.*

This is just a restatement of Theorem 25, proven at the end of the paper.

Remark 7 To see why the data of ω is essential here, notice that the short exact sequence of super-Lie groups

$$1 \to \mathbb{R} \to \mathbb{R}^{1|1} \to \mathbb{R}^{0|1} \to 1$$

is not split. As a consequence, the cartesian product of euclidean manifolds of dimensions 1 and $0|1$ is not endowed with a canonical euclidean structure. This makes dimensional reduction in our setting quite subtle, since "crossing with S^1" is not a well-defined operation in the euclidean category, and therefore there is no direct functor $S^1 \times -\colon 0|1\text{–EBord} \to 1|1\text{–EBord}$.

Finally, we remark that every $1|1$–EFT over $\mathfrak{X}$ determines a smooth function on $\mathfrak{K}(\mathfrak{X})$, the *partition function* of the theory. Again, this is an immediate consequence of the fact that the empty manifold, being the monoidal unit in the bordism category, is required to map to the vector space $\mathbb{C}$.

3 Dimensional reduction

The upshot of Section 2 is that it suffices to discuss the functors (1) of internal categories in terms of the corresponding substacks of (fiberwise) closed and connected families of bordisms; we reserve the term *moduli stack* for these objects. We have already discussed $\mathfrak{B}(\mathfrak{X})$ and $\mathfrak{K}(\mathfrak{X})$ in Section 2. The two middle moduli stacks, as well as the maps

$$\mathfrak{B}(\Lambda\mathfrak{X}) \xleftarrow{\mathcal{P}} \mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X}) \xrightarrow{\mathcal{Q}} \mathfrak{B}^{\mathbb{R}/\!\!/\mathbb{Z}}(\Lambda\mathfrak{X}) \xrightarrow{\mathcal{R}} \mathfrak{K}(\mathfrak{X})$$

relating them, will be defined in the ensuing subsections. We will refer to the two middle stacks as the $\mathbb{T}$–equivariant and $\mathbb{R}/\!\!/\mathbb{Z}$–equivariant moduli stacks of euclidean $0|1$–manifolds over $\Lambda\mathfrak{X}$.

The lack of a direct map from left to right in the above span of moduli stacks is due to a subtlety of supereuclidean geometry: if Σ is a euclidean $0|1$–manifold, the product

$S^1 \times \Sigma$ does not come with a canonical euclidean structure; to choose one essentially amounts to the choice of preimage along $\mathcal{P}$ (see Theorem 6 and Remark 7). This is not a serious issue for us, since $\mathcal{P}$ induces a bijection between the set of functions on each moduli stack (or, equivalently, between field theories based on each variant of the bordism category; see Proposition 10).

Following the physical (and, by now, mathematical) jargon, restriction of $1|1$–EFTs (or just functions on $\mathfrak{K}(\mathfrak{X})$) to $0|1$–EFTs via the above maps of bordism stacks will be referred to as dimensional reduction. Our motivation for doing this is that the stack $\mathfrak{K}(\mathfrak{X})$ of euclidean supercircles over $\mathfrak{X}$ is "infinite-dimensional", and therefore unwieldy to analysis; dimensional reduction allows us to probe its geometry by means of $0|1$–dimensional gadgets over $\mathfrak{X}$.

To further motivate our dimensional reduction procedure, note that $\mathcal{Q}$ is an equivalence of stacks (Theorem 11), even though its inverse does not admit a nice geometric description. Thus, $\mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X})$ and $\mathfrak{B}^{\mathbb{R}/\!\!/\mathbb{Z}}(\Lambda\mathfrak{X})$ can be seen as different presentations of the same entity; the former presentation has a direct relationship with $\mathfrak{B}(\Lambda\mathfrak{X})$, while the latter leads us to a suitable definition of a map to $\mathfrak{K}(\mathfrak{X})$.

Remark 8 To understand the relevance of $\mathbb{R}/\!\!/\mathbb{Z}$–actions for dimensional reduction, we can consider a naive replacement for the composition $\mathcal{R} \circ \mathcal{Q}$: instead of performing the descent constructions of Section 3.3, we could simply perform a pullback along $P \to \Sigma$. Then it is easy to see that, with these modifications, Theorem 1 would recover the naive Chern character

$$K_G^0(X) \xrightarrow{\alpha} K^0(EG \times_G X) \xrightarrow{\mathrm{ch}} H^{\mathrm{ev}}(EG \times_G X) = H_G^{\mathrm{ev}}(X)$$

(or, more precisely, its pullback to $\widehat{X}$, as defined in (6)). Here, the map α is given, at the level of vector bundles, by the homotopy quotient construction. Thus, as explained at the end of Section 4.1, this alternative construction forgets too much information.

At the end of this section, to illustrate the ideas, we specialize these constructions to the case where $\mathfrak{X} = X /\!\!/ G$ is a global quotient by a finite group.

3.1 The $\mathbb{R}/\!\!/\mathbb{Z}$–equivariant moduli stack and the map $\mathcal{R}$

We define a stack $\mathfrak{B}^{\mathbb{R}/\!\!/\mathbb{Z}}(\Lambda\mathfrak{X})$ where an object over S is given by the following data:

(1) a family $\Sigma \to S$ of connected euclidean $0|1$–manifolds;

(2) a principal $\mathbb{T}$–bundle $P \to \Sigma$ with a fiberwise connection ω whose curvature agrees with the tautological (fiberwise) 2–form ζ on Σ (see Appendix B); and

(3) an $\mathbb{R}/\!/\mathbb{Z}$–equivariant map $\psi\colon P \to \Lambda\mathfrak{X}$ with equivariance datum ρ, where $\mathbb{R}/\!/\mathbb{Z}$ acts on P and $\Lambda\mathfrak{X}$ via the usual homomorphisms $\mathbb{R}/\!/\mathbb{Z} \to \mathbb{T}$ and $\mathbb{R}/\!/\mathbb{Z} \to \mathrm{pt}/\!/\mathbb{Z}$, respectively.

(Recall that $\mathbb{R}/\!/\mathbb{Z}$–equivariance is not just a condition on ψ, but rather extra data encoded by the 2–morphism ρ; see Appendix A). We will usually denote this object by (Σ, P, ψ, ρ) or, diagrammatically,

$$\begin{tikzcd} P \arrow[r, "\psi", "\mathbb{R}/\!/\mathbb{Z}"'] \arrow[d] & \Lambda\mathfrak{X} \\ \Sigma & \end{tikzcd}$$

A morphism $(\Sigma', P', \psi', \rho') \to (\Sigma, P, \psi, \rho)$ covering a map of supermanifolds $S' \to S$ is given by

(1) a fiberwise isometry $F\colon \Sigma' \to \Sigma$ covering $S' \to S$,

(2) a connection-preserving bundle map $\Phi\colon P' \to P$ covering F, and

(3) an equivariant 2–morphism $\xi\colon \psi' \to \psi \circ \Phi$.

Compositions are performed as suggested by the geometry.

Now we discuss the map $\mathcal{R}\colon \mathfrak{B}^{\mathbb{R}/\!/\mathbb{Z}}(\Lambda\mathfrak{X}) \to \mathfrak{K}(\mathfrak{X})$. An object (Σ, P, ψ, ρ) over S is mapped to the supercircle over $\mathfrak{X}$ consisting of

(1) the family of $1|1$–dimensional manifolds P endowed with the fiberwise euclidean structure determined by ω (see Theorem 6), and

(2) the map $P \to \mathfrak{X}$ obtained by composing ψ with the forgetful map $\Lambda\mathfrak{X} \to \mathfrak{X}$.

Notice that this construction forgets the $\mathbb{T}$–action on P as well as the equivariance datum ρ. To define $\mathcal{R}$ at the level of morphisms, recall, again by Theorem 6, that a connection-preserving bundle map $P' \to P$ covering a fiberwise isometry $\Sigma' \to \Sigma$ is a fiberwise (over S) isometry with respect to the euclidean structures on P' and P.

3.2 The $\mathbb{T}$–equivariant moduli stack and the map $\mathcal{P}$

For any stack $\mathfrak{X}$, we define $\mathfrak{B}^{\mathbb{T}}(\mathfrak{X})$ to be the stack whose S–points are given by an S–family of connected euclidean $0|1$–manifolds $\Sigma \to S$ together with two pieces of data:

(1) a principal $\mathbb{T}$–bundle $P \to \Sigma$ with a fiberwise connection ω whose curvature agrees with the tautological 2–form ζ on Σ, and

(2) a map $\psi\colon \Sigma \to \mathfrak{X}$.

Morphisms between two objects (Σ', P', ψ') and (Σ, P, ψ) over $f\colon S' \to S$ consist of a fiberwise isometry $F\colon \Sigma' \to \Sigma$ covering f, a connection-preserving bundle map $\Phi\colon P' \to P$ covering F and a 2–morphism $\xi\colon \psi' \to \psi \circ F$. Compositions are performed as suggested by the geometry.

The data (1) and (2) above are completely unrelated in the sense that

$$\mathfrak{B}^{\mathbb{T}}(\mathfrak{X}) \cong \mathfrak{B}^{\mathbb{T}} \times_{\mathfrak{B}} \mathfrak{B}(\mathfrak{X}),$$

and our map $\mathcal{P}\colon \mathfrak{B}^{\mathbb{T}}(\mathfrak{X}) \to \mathfrak{B}(\mathfrak{X})$ is simply the projection onto the second component. Our interest in $\mathfrak{B}^{\mathbb{T}}(\mathfrak{X})$ is due to the fact that it admits a straightforward quotient stack presentation. Write $\mathbb{T}^{1|1} = \mathbb{R}^{1|1}/\mathbb{Z}$ for the (length 1) supercircle group.

Proposition 9 *There is an equivalence of stacks*

$$\Pi T\mathfrak{X} /\!\!/ \operatorname{Isom}(\mathbb{T}^{1|1}) \to \mathfrak{B}^{\mathbb{T}}(\mathfrak{X}),$$

where the action of $\operatorname{Isom}(\mathbb{T}^{1|1})$ *on* $\Pi T\mathfrak{X}$ *is through the quotient*

$$\pi\colon \operatorname{Isom}(\mathbb{T}^{1|1}) = \mathbb{T}^{1|1} \rtimes \mathbb{Z}/2 \to \mathbb{R}^{0|1} \rtimes \mathbb{Z}/2 = \operatorname{Isom}(\mathbb{R}^{0|1}).$$

Proof For $\mathfrak{X} = \mathrm{pt}$, this follows from (the proof of) Theorem 25. Therefore, in the general case we have

$$\mathfrak{B}^{\mathbb{T}}(\mathfrak{X}) \cong \Pi T\mathfrak{X} /\!\!/ \operatorname{Isom}(\mathbb{R}^{0|1}) \times_{\mathrm{pt}/\!\!/ \operatorname{Isom}(\mathbb{R}^{0|1})} \mathrm{pt} /\!\!/ \operatorname{Isom}(\mathbb{T}^{1|1})$$

and the result follows from Proposition 19. □

Proposition 10 *For any sheaf* $\mathfrak{F}$*, the map* $\mathcal{P}\colon \mathfrak{B}^{\mathbb{T}}(\mathfrak{X}) \to \mathfrak{B}(\mathfrak{X})$ *induces a bijection*

$$\mathrm{Fun}_{\mathrm{SM}}(\mathfrak{B}(\mathfrak{X}), \mathfrak{F}) \to \mathrm{Fun}_{\mathrm{SM}}(\mathfrak{B}^{\mathbb{T}}(\mathfrak{X}), \mathfrak{F}).$$

Proof Under the identification of the previous proposition, $\mathcal{P}\colon \mathfrak{B}^{\mathbb{T}}(\mathfrak{X}) \to \mathfrak{B}(\mathfrak{X})$ becomes the natural map

$$\Pi T\mathfrak{X} /\!\!/ \operatorname{Isom}(\mathbb{T}^{1|1}) \to \Pi T\mathfrak{X} /\!\!/ \operatorname{Isom}(\mathbb{R}^{0|1})$$

induced by the surjection

$$\pi\colon \operatorname{Isom}(\mathbb{T}^{1|1}) \to \operatorname{Isom}(\mathbb{R}^{0|1}).$$

Thus, Proposition 18 identifies the set $\mathrm{Fun}_{\mathrm{SM}}(\mathfrak{B}(\mathfrak{X}), \mathfrak{F})$ with the subset of $\operatorname{Isom}(\mathbb{R}^{0|1})$–invariants in $\mathrm{Fun}_{\mathrm{SM}}(\Pi T\mathfrak{X}, \mathfrak{F})$, and similarly for $\mathrm{Fun}_{\mathrm{SM}}(\mathfrak{B}^{\mathbb{T}}(\mathfrak{X}), \mathfrak{F})$. This proves the claim. □

Taking $\mathfrak{F} = \mathbb{C}$, we get a bijection between $0|1$–EFT$(\mathfrak{X})$ and $C^\infty(\mathfrak{B}^{\mathbb{T}}(\mathfrak{X}))$. This shows that the last step of our dimensional reduction procedure, pushing forward along $\mathcal{P}$, is well defined.

3.3 The map $\mathcal{Q}\colon \mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X}) \to \mathfrak{B}^{\mathbb{R}/\!\!/\mathbb{Z}}(\Lambda\mathfrak{X})$

We denote by α the canonical automorphism of the identity of $\Lambda\mathfrak{X}$. It suffices to describe the restriction of the desired map $\mathcal{Q}\colon \mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X}) \to \mathfrak{B}^{\mathbb{R}/\!\!/\mathbb{Z}}(\Lambda\mathfrak{X})$ to the full prestack of objects where all bundles involved are trivial. To $(\Sigma, P, \psi) \in \mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X})_S$ with

$$\Sigma = S \times \mathbb{R}^{0|1}, \quad P = S \times \mathbb{T}^{1|1}, \quad \psi\colon \Sigma \to \Lambda\mathfrak{X},$$

and the standard euclidean structure and connection form, we want to assign an object $(\Sigma, P, \psi_!\colon P \to \Lambda\mathfrak{X}, \rho) \in \mathfrak{B}^{\mathbb{R}/\!\!/\mathbb{Z}}(\mathfrak{X})_S$. Consider the covering $U = S \times \mathbb{R}^{1|1} \to P$. Our goal is to descend $\tilde{\psi}\colon U \to \Lambda\mathfrak{X}$, the pullback of ψ via $U \to \Sigma$, to a map $\psi_!\colon P \to \Lambda\mathfrak{X}$:

$$\begin{tikzcd} U \arrow[r] \arrow[rrr, bend left, "\tilde{\psi}"] & P \arrow[r] \arrow[rr, bend right, dashed, "\psi_!"'] & \Sigma \arrow[r, "\psi"] & \Lambda\mathfrak{X} \end{tikzcd}$$

In order to do that, we need to provide certain isomorphisms over double overlaps and then check a coherence condition on triple overlaps. Denote by $\mathrm{pr}_1, \mathrm{pr}_2\colon U \times_P U \rightrightarrows U$ the projections. Then we are looking for a 2–morphism $\tilde{\alpha}\colon \tilde{\psi} \circ \mathrm{pr}_1 \to \tilde{\psi} \circ \mathrm{pr}_2$ (whose domain and codomain happen to be the same map, henceforth denoted by $\psi \circ \mathrm{pr}$). Note that $U \times_P U$ breaks up as a disjoint union indexed by $\mathbb{Z}$, where the n^{th} component comprises pairs of the form $(x, n \cdot x)$. On that component, we set $\tilde{\alpha}$ to be the horizontal composition (whiskering)

$$\tilde{\alpha} = \alpha^n \circ (\psi \circ \mathrm{pr}).$$

Regarding the coherence condition, we need to check that

$$\mathrm{pr}_{13}^* \tilde{\alpha} = \mathrm{pr}_{23}^* \tilde{\alpha} \circ \mathrm{pr}_{12}^* \tilde{\alpha}, \tag{2}$$

where pr_{ij} denotes the projection $U \times_P U \times_P U \to U \times_p U$ forgetting the third index. The threefold fiber product breaks up as a disjoint union indexed by $\mathbb{Z} \times \mathbb{Z}$, where the component (n, m) and its image through the pr_{ij} are as follows:

$$\begin{tikzcd} & (x, n \cdot x, (n+m) \cdot x) \arrow[dl, mapsto, "\mathrm{pr}_{12}"'] \arrow[d, mapsto, "\mathrm{pr}_{13}"] \arrow[dr, mapsto, "\mathrm{pr}_{23}"] & \\ (x, n \cdot x) & (x, (n+m) \cdot x) & (n \cdot x, (n+m) \cdot x) \end{tikzcd}$$

Therefore, on that component,

$$\mathrm{pr}_{23}^* \widetilde{\alpha} = \alpha^m \circ (\psi \circ \mathrm{pr}), \quad \mathrm{pr}_{12}^* \widetilde{\alpha} = \alpha^n \circ (\psi \circ \mathrm{pr}), \quad \mathrm{pr}_{13}^* \widetilde{\alpha} = \alpha^{n+m} \circ (\psi \circ \mathrm{pr}),$$

and their vertical compositions are as required by (2). We thus obtain the desired $\psi_!\colon P \to \Lambda\mathfrak{X}$.

Next, we need to provide the $\mathbb{R}/\!\!/\mathbb{Z}$–equivariance datum ρ for $\psi_!$. To analyze the putative square

$$\begin{tikzcd} P \times \mathbb{R}/\!\!/\mathbb{Z} \arrow[r, "\mu"] \arrow[d, "\psi_! \times \mathrm{id}"'] & P \arrow[d, "\psi_!"] \\ \Lambda\mathfrak{X} \times \mathbb{R}/\!\!/\mathbb{Z} \arrow[r] \arrow[ur, Rightarrow, "\rho"'] & \Lambda\mathfrak{X} \end{tikzcd} \tag{3}$$

we notice that, after a suitable base change, any S–point of $P \times \mathbb{R}/\!\!/\mathbb{Z}$ can be pulled back from the atlas $i_0\colon P \times \mathbb{R} \to P \times \mathbb{R}/\!\!/\mathbb{Z}$, or, for that matter, from any of the atlases $i_n\colon (p,t) \mapsto i_0(p, t+n)$, where $n \in \mathbb{Z}$; moreover, any morphism of S–points can be pulled back from $m\colon i_n \to i_{n+m}$. Thus, we can extract all information encoded by ρ by evaluating the above diagram on each i_n and m. The top-right composition factors through $P \times \mathbb{T}$, so every i_n maps to the same $\mu^* \psi_! \in \Lambda\mathfrak{X}_{P\times\mathbb{R}}$, and m maps to the identity. The left-bottom composition factors through $\Lambda\mathfrak{X} \times \mathrm{pt}/\!\!/\mathbb{Z}$, so, for any n, i_n maps to $\mathrm{pr}_1^* \psi_! \in \Lambda\mathfrak{X}_{P\times\mathbb{R}}$, and $m\colon i_n \to i_{n+m}$ maps to $\mathrm{pr}_1^* \alpha^m\colon \mathrm{pr}_1^* \psi_! \to \mathrm{pr}_1^* \psi_!$. For each i_n, the fibered natural transformation ρ should give a morphism $\rho(i_n)\colon \mathrm{pr}_1^* \psi_! \to \mu^* \psi_!$ fitting in the diagram below:

$$\begin{tikzcd} \mathrm{pr}_1^* \psi_! \arrow[r, "\rho(i_n)"] \arrow[d, "\mathrm{pr}_1^* \alpha^m"'] & \mu^* \psi_! \arrow[d, equal] \\ \mathrm{pr}_1^* \psi_! \arrow[r, "\rho(i_{n+m})"] & \mu^* \psi_! \end{tikzcd}$$

This means ρ is completely specified by $\rho(i_0)$, and naturality imposes no further restrictions on the latter. To provide $\rho(i_0)$, it suffices to give a morphism $\mathrm{pr}_1^* \widetilde{\psi} \to \mu^* \widetilde{\psi}$, where the latter is the composition

$$U \times \mathbb{R} \to P \times \mathbb{R} \xrightarrow{\mu} P \to \Sigma \xrightarrow{\psi} \Lambda\mathfrak{X},$$

satisfying appropriate coherence conditions on $U \times_P U \times \mathbb{R}$. Since $\mu^* \widetilde{\psi} = \mathrm{pr}_1 \widetilde{\psi}$, we can take that to be the identity. One can check that ρ satisfies the coherence conditions required of the equivariance datum.

The effect of $\mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X}) \to \mathfrak{B}^{\mathbb{R}/\!/\mathbb{Z}}(\Lambda\mathfrak{X})$ on morphisms is also given by descent. Given a morphism in $\mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X})$

$$\begin{tikzcd}
P \arrow[r, "\Phi"] \arrow[dd] & P' \arrow[dd] & \\
 & & \Lambda\mathfrak{X} \\
\Sigma \arrow[r, "F"] \arrow[rru, bend left=40, "\psi", ""{name=A, below}] & \Sigma' \arrow[ru, bend right, "\psi'"', ""{name=B, above}] &
\arrow[Rightarrow, from=A, to=B, "\xi"]
\end{tikzcd}$$

where $\Sigma' = S' \times \mathbb{R}^{0|1}$, $P' = S' \times \mathbb{T}^{1|1}$ are also trivial families, consider the fiberwise universal cover $U' = S \times \mathbb{R}^{1|1} \to P'$ and choose a lift $\widetilde{\Phi}\colon U \to U'$. We can then lift ψ, ψ' and ξ by composing and whiskering, respectively, with $U \to \Sigma$ or $U' \to \Sigma'$:

$$\begin{tikzcd}
U \arrow[rr, "\widetilde{\Phi}"] \arrow[dr, "\widetilde{\psi}"', ""{name=A, right}] & & U' \arrow[dl, "\widetilde{\psi}'", ""{name=B, left}] \\
 & \Lambda\mathfrak{X} &
\arrow[Rightarrow, from=A, to=B, "\widetilde{\xi}"]
\end{tikzcd}$$

and descend $\widetilde{\xi}$ to a morphism $\xi_!\colon \psi_! \to \Phi^*\psi'_!$. To justify that, we need to show that on the n^{th} component of $U \times_P U$ the diagram

$$\begin{tikzcd}
\mathrm{pr}_1^* \widetilde{\psi} \arrow[r, "\mathrm{pr}_1^* \widetilde{\xi}"] \arrow[d, "\alpha^n"'] & \mathrm{pr}_1^*(\widetilde{\psi}' \circ \widetilde{\Phi}) \arrow[d, "\alpha^n"] \\
\mathrm{pr}_2^* \widetilde{\psi} \arrow[r, "\mathrm{pr}_2^* \widetilde{\xi}"] & \mathrm{pr}_2^*(\widetilde{\psi}' \circ \widetilde{\Phi})
\end{tikzcd}$$

commutes. (To be precise, α^n above stands, respectively, for $\alpha^n \circ (\psi \circ \mathrm{pr})$, the gluing isomorphism used to build $\psi_!$, and its counterpart for $\Phi^*\psi'_!$.) This follows immediately from the compatibility condition between ξ and α, namely $\xi \circ \alpha_\psi = \alpha_{\Phi^*\psi'} \circ \xi$. The morphism $\xi_!$ thus obtained is independent of the choice of lift $\widetilde{\Phi}$, since it only depends on the composition $\widetilde{\psi}' \circ \widetilde{\Phi}$. We omit the verification that $\xi_!$ is compatible with the equivariance data.

Finally, we assign to the morphism in $\mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X})$ prescribed by the data (F, Φ, ξ) the morphism in $\mathfrak{B}^{\mathbb{R}/\!/\mathbb{Z}}(\Lambda\mathfrak{X})$ prescribed by the data $(F, \Phi, \xi_!)$. That this assignment respects compositions follows from uniqueness for descent of morphisms.

This finishes the construction of $\mathcal{Q}$. The next result is not used directly in the remainder of the paper, and is rather meant as a motivation for introducing the stacks $\mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X})$ and $\mathfrak{B}^{\mathbb{R}/\!/\mathbb{Z}}(\Lambda\mathfrak{X})$, which turn out to be just different presentations of the same object. In fact, they are presentations adapted to establishing a relationship with $\mathfrak{B}(\Lambda\mathfrak{X})$ and $\mathfrak{K}(\mathfrak{X})$, respectively, as witnessed by the relatively easy definition of the maps $\mathcal{P}$

and $\mathcal{R}$ above. Note also that the proof of Theorem 11 is indirect, and does not explicitly provide an inverse to $\mathcal{Q}$; thus, it does not seem possible to simplify our presentation of the dimensional reduction procedure by removing any mention of $\mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X})$.

Theorem 11 *The fibered functor* $\mathcal{Q}\colon \mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X}) \to \mathfrak{B}^{\mathbb{R}/\!\!/\mathbb{Z}}(\Lambda\mathfrak{X})$ *is an equivalence.*

Proof At the morphism level, the effect of the functor in question was described in two steps: $\xi \mapsto \widetilde{\xi} \mapsto \xi_!$. This is a one-to-one procedure because the first step is injective (since $U \to \Sigma$ has local sections) and the second step (descent) is in fact bijective. Thus, it remains to show that the fibered functor $\mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X}) \to \mathfrak{B}^{\mathbb{R}/\!\!/\mathbb{Z}}(\Lambda\mathfrak{X})$ is full and essentially surjective. In order to do that, we will build a prestack $\mathfrak{B}^{\mathrm{triv}}$ and a factorization

$$\begin{tikzcd}
& \mathfrak{B}^{\mathrm{triv}} \arrow[ld, "v"'] \arrow[rd, "u"] & \\
\mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X}) \arrow[rr, "\mathcal{Q}"] & & \mathfrak{B}^{\mathbb{R}/\!\!/\mathbb{Z}}(\Lambda\mathfrak{X})
\end{tikzcd} \tag{4}$$

where u is full and essentially surjective on the groupoid of S–point for any contractible S.

The prestack $\mathfrak{B}^{\mathrm{triv}}$ is defined as follows:

(1) an object consists of an object $(\Sigma, P, \psi, \rho) \in \mathfrak{B}^{\mathbb{R}/\!\!/\mathbb{Z}}(\Lambda\mathfrak{X})$ together with a section $s\colon \Sigma \to P$, and

(2) a morphism $(\Sigma', P', \psi', \rho', s') \to (\Sigma, P, \psi, \rho, s)$ is a pair consisting of a morphism (F, Φ, ξ) of the underlying objects in $\mathfrak{B}^{\mathbb{R}/\!\!/\mathbb{Z}}(\mathfrak{X})$ together with a map $r\colon \Sigma' \to \mathbb{R}$ relating s and s' in the sense that $\Phi \circ s' = (s \circ F)e^{2\pi i r}$.

With a little poetic license, a morphism can be depicted as follows (the square containing r would literally make sense, as a 2–commutative diagram, if we replaced P with $P/\!\!/\mathbb{R}$):

$$\begin{tikzcd}
P' \arrow[rrd, bend left, "\psi'"] \arrow[rd, "\Phi"] \arrow[dd, shift left] & {} \arrow[d, Rightarrow, shorten >=10pt, "\xi"] & \\
& P \arrow[r, "\psi"] \arrow[dd, shift right] & \Lambda\mathfrak{X} \\
\Sigma' \arrow[uu, dashed, shift left, "s'"] \arrow[ru, Rightarrow, "r"] \arrow[rd, "F"] & & \\
& \Sigma \arrow[uu, dashed, shift right, "s"'] &
\end{tikzcd} \tag{5}$$

We define $u\colon \mathfrak{B}^{\mathrm{triv}} \to \mathfrak{B}^{\mathbb{R}/\!/\mathbb{Z}}(\Lambda\mathfrak{X})$ to be the forgetful functor, which simply discards s and r, so it is clearly full and essentially surjective over contractible S, as claimed.

Next, we construct $v\colon \mathfrak{B}^{\mathrm{triv}} \to \mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X})$. To an object $(\Sigma, P, \psi, \rho, s) \in \mathfrak{B}^{\mathrm{triv}}$, we assign the object $(\Sigma, P, s^*\psi)$ in $\mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X})$. Now fix a morphism as in (5). To define its image in $\mathfrak{B}^{\mathbb{T}}(\Lambda\mathfrak{X})$, the only new data we need to provide is a morphism $(s')^*\psi' \to (s \circ F)^*\psi$, which we take to be the composition

$$(s')^*\psi' \xrightarrow{(s')^*\xi} (s')^*\Phi^*\psi \cong (\Phi \circ s')^*\psi = ((s \circ F)e^{2\pi i r})^*\psi \xrightarrow{\rho^{-1}_{s\circ F, r}} (s \circ F)^*\psi.$$

We omit the verification of functoriality.

To finish the proof, we just need to show that (4) commutes (up to 2–isomorphism). It suffices to look at $(\Sigma, P, \psi, \rho, s) \in \mathfrak{B}^{\mathrm{triv}}$ where P and Σ are trivial families, and pick s to be the unit section; our goal is to produce an isomorphism between $(s^*\psi)_!$ and ψ, natural in the input data $(\Sigma, P, \psi, \rho, s)$ and compatible with the respective equivariance data. From the discussion leading to the construction of the ρ in (3), we see that the data of the present (arbitrarily given) ρ is essentially an isomorphism $\rho_0\colon \mathrm{pr}_1^*\psi \to \mu^*\psi$ in $\Lambda\mathfrak{X}_{P\times\mathbb{R}}$. Now, let $\pi^*\psi$ be the pullback through $\pi\colon U \to P$ and recall that $\widetilde{s^*\psi}$ is the U–point of $\Lambda\mathfrak{X}$ used to put together $(s^*\psi)_!$. Note that each half of the diagram

$$\begin{tikzcd}
U = \Sigma \times \mathbb{R} \arrow[r, "s\times\mathrm{id}"] \arrow[rrr, bend left, "\widetilde{s^*\psi}"] \arrow[rrr, bend right, "\pi^*\psi"'] & P \times \mathbb{R} \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mu"'] & P \arrow[r, "\psi"] & \Lambda\mathfrak{X}
\end{tikzcd}$$

commutes, so ρ_0 gives a morphism $\widetilde{s^*\psi} \to \pi^*\psi$ and, by descent, a morphism $(s^*\psi)_! \to \psi$. We omit the naturality and compatibility checks. □

3.4 Global quotients

Let us illustrate the above constructions when $\mathfrak{X} = X/\!/G$ is the quotient orbifold associated to the action of a finite group G on a manifold X.

We start noticing that a quotient stack presentation for $\Lambda(X/\!/G)$ can be given as follows. Consider the product $X \times G$ with diagonal G–action, where G acts on itself by conjugation. There is an invariant submanifold

$$\hat{X} = \{(x, g) \in X \times G \mid x \in X^g\}, \tag{6}$$

and an object over S in the quotient stack $\widehat{X}/\!\!/G$ consists of a pair $(Q,(f,A))$, where $Q\to S$ is a principal G–bundle and $(f,A)\colon Q\to \widehat{X}\subset X\times G$ is a G–equivariant smooth map. Denote by $\alpha\colon Q\to Q$ the bundle automorphism determined by A; on T–points, it is given by

$$\alpha(q)=qA(q),\quad q\in Q_T.$$

Notice that this automorphism preserves f, and therefore (Q,f,α) determines an S–point of $\Lambda(X/\!\!/G)$. Conversely, given an S–point (Q,f,α) of $\Lambda(X/\!\!/G)$, we can specify a G–equivariant map $A\colon Q\to G$ by requiring that the above equation holds, and compatibility between f and α implies that the resulting map $(f,A)\colon Q\to X\times G$ factors through $\widehat{X}$, thus determining an object of $\widehat{X}/\!\!/G$ over S.

The translation back and forth between A and α provides a $\mathrm{pt}/\!\!/\mathbb{Z}$–equivariant equivalence between $\Lambda(X/\!\!/G)$ and $\widehat{X}/\!\!/G$, compatible with the maps $\Lambda(X/\!\!/G)\to X/\!\!/G$ forgetting the prescribed automorphism and $\widehat{X}/\!\!/G\to X/\!\!/G$ induced the projection $\mathrm{pr}_1\colon X\times G\to X$. We will shift freely between these two formulations.

The geometric content of an S–family in $\mathfrak{B}^{\mathbb{R}/\!\!/\mathbb{Z}}(\widehat{X}/\!\!/G)$ is the following:

(1) a family $\Sigma\to S$ of connected euclidean $0|1$–manifolds;

(2) a principal $\mathbb{T}$–bundle $P\to\Sigma$ with a fiberwise connection ω whose curvature agrees with the tautological 2–form on Σ;

(3) a principal G–bundle $Q\to P$;

(4) a G–equivariant map $(f,A)\colon Q\to\widehat{X}\subset X\times G$, or, equivalently, a bundle automorphism $\alpha\colon Q\to Q$ and a G–equivariant map $f\colon Q\to X$ such that $f\circ\alpha=f$; and, finally,

(5) a collection of natural isomorphisms of G–torsors

$$\rho_{p,t}\colon Q_p\to Q_{pe^{2\pi it}}$$

for each pair of T–points $p\colon T\to P$, $t\colon T\to\mathbb{R}$, intertwining the maps

$$f_p\colon Q_p\to X,\quad f_{pe^{2\pi it}}\colon Q_{pe^{2\pi it}}\to X$$

and subject to the condition that for any $n\colon T\to\mathbb{Z}$ the diagram

(7)
$$\begin{tikzcd} Q_p \arrow[r, "\rho_{p,t}"] \arrow[d, "\alpha_p^n"'] & Q_{pe^{2\pi it}} \arrow[d, equal] \\ Q_p \arrow[r, "\rho_{p,t+n}"] & Q_{pe^{2\pi i(t+n)}} \end{tikzcd}$$

commutes.

The last condition means that α agrees with the holonomy of Q around the fibers of P. A morphism in $\mathfrak{B}^{\mathbb{R}/\!\!/\mathbb{Z}}(\widehat{X}/\!\!/G)$ is given by a fiberwise isometry $F\colon \Sigma' \to \Sigma$, a connection-preserving bundle map $\Phi\colon P' \to P$ covering F and a bundle map $Q' \to Q$ covering Φ which is required to be compatible in the obvious way with the data in (4) and (5) above.

The geometric content of an S–family in $\mathfrak{B}^{\mathbb{T}}(\widehat{X}/\!\!/G)$ is the following:

(1) a family $\Sigma \to S$ of connected euclidean $0|1$–manifolds,

(2) a principal $\mathbb{T}$–bundle $P \to \Sigma$ with a connection ω whose curvature agrees with the tautological 2–form on Σ,

(3) a principal G–bundle $Q \to \Sigma$, and

(4) a G–equivariant map $f\colon Q \to \widehat{X}$.

A morphism

$$(\Sigma', P', Q', f') \to (\Sigma, P, Q, f)$$

consists of a fiberwise isometry $F\colon \Sigma' \to \Sigma$, a connection-preserving bundle map $\Phi\colon P' \to P$ covering F and a bundle map $Q' \to Q$ covering F and intertwining the maps $f\colon Q \to \widehat{X}$ and $f'\colon Q' \to \widehat{X}$. From Proposition 9, it follows that $\mathfrak{B}^{\mathbb{T}}(\widehat{X}/\!\!/G)$ admits the presentation

$$(\Pi T(\widehat{X}/\!\!/G))/\!\!/\operatorname{Isom}(\mathbb{T}^{1|1}) \cong \Pi T\widehat{X}/\!\!/(\operatorname{Isom}(\mathbb{T}^{1|1}) \times G).$$

Finally, let us describe the map $\mathcal{Q}$ relating the $\mathbb{T}$–equivariant and $\mathbb{R}/\!\!/\mathbb{Z}$–equivariant moduli stacks of euclidean $0|1$–manifolds in this special situation. Fix $(\Sigma, P, Q, f) \in \mathfrak{B}^{\mathbb{T}}(\widehat{X}/\!\!/G)_S$ and let $(\Sigma, P, Q_!, f_!, \rho) \in \mathfrak{B}^{\mathbb{R}/\!\!/\mathbb{Z}}(\widehat{X}/\!\!/G)_S$ be its image. Locally in S, f determines a conjugacy class of G and $Q_! \to P$ is the G–bundle with that holonomy around the fibers of $P \to \Sigma$. More specifically, let us assume P and Q are trivial; if S is connected, then f determines an element $g \in G$, namely the one corresponding to the connected component of $\widehat{X} = \coprod_{g\in G} X^g$ in which $f|_{\Sigma\times\{e\}}$ takes values. Then $Q_! \to P$ is the G–bundle built as a quotient

$$Q_! = (\Sigma \times \mathbb{R} \times G)/\mathbb{Z} \to P = \Sigma \times \mathbb{T},$$

where the $\mathbb{Z}$–action is generated by the diffeomorphism prescribed, on T–points, by $(s,t,h) \mapsto (s,t+1,gh)$. The map $f_!\colon Q_! \to \widehat{X}$ is induced by the $\mathbb{Z}$–invariant map $(s,t,h) \mapsto f(s,e)\cdot h$. The automorphism of $Q_!$ determined by the G–component of $f_!$ can be expressed as $(s,t,h) \mapsto (s,t,gh)$.

4 The Chern character for global quotients

In this section, we show how to recover, in terms of dimensional reduction of field theories, the delocalized Chern character of Baum and Connes [2] (and, before them, Słomińska [18]), concerning the case of a finite group G acting on a manifold X.

We start by briefly recalling the classical construction of ch_G in Section 4.1. On the field theory side, we can associate to each vector bundle with connection V on an orbifold $\mathfrak{X}$ a field theory $E_V \in 1|1\text{–EFT}(\mathfrak{X})$. For the sake of brevity, we will only describe, in Section 4.2, the partition function of this theory, denoted by $Z_V \in C^\infty(\mathfrak{K}(\mathfrak{X}))$. Finally, in Section 4.3 we prove Theorem 1.

4.1 The Baum–Connes Chern character

As before, we write $\widehat{X} = \{(x,g) \in X \times G \mid xg = x\} = \coprod_{g\in G} X^g$. The equivariant Chern character is a ring homomorphism

$$\mathrm{ch}_G\colon K^i_G(X) \to H^i_G(\widehat{X};\mathbb{C}) \cong \Bigl[\bigoplus_{g\in G} H^i(X^g;\mathbb{C})\Bigr]^G. \tag{8}$$

Here, $i \in \mathbb{Z}/2$ and ordinary cohomology is $\mathbb{Z}/2$–graded. We recall that the equivariant ordinary cohomology of $\widehat{X}$ with coefficients in $\mathbb{C}$ can be identified with the invariants in its nonequivariant cohomology; this can be deduced from the Serre spectral sequence for the fibration $EG \times_G X \to BG$ using the fact that the integral reduced cohomology of a finite group is torsion.

For each $g \in G$, we define the homomorphism $\mathrm{ch}_g\colon K^i_G(X) \to H^i(X^g;\mathbb{C})$ as the composition

$$K^i_G(X) \to K^i_{\langle g\rangle}(X^g) \cong K^i(X^g) \otimes R(\langle g\rangle) \xrightarrow{\mathrm{ch}\otimes\mathrm{tr}_g} H^i(X^g;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

(The middle isomorphism is due to the fact that the action of the cyclic group $\langle g\rangle$ generated by g on X^g is trivial; ch denotes the usual, nonequivariant Chern character; and tr_g assigns to any representation of $\langle g\rangle$ the trace of the operator g.) Finally, we let $\mathrm{ch}_G\colon K^i_G(X) \to H^i_G(\widehat{X};\mathbb{C})$ be the direct sum of all ch_g via the identification (8).

Concretely, the effect of ch_g on the K–theory class represented by a G–equivariant vector bundle $V \to X$ is the following. For each $x \in X^g$, the fiber V_x is a representation of the cyclic group generated by g. Let $\lambda_1, \dots \lambda_r$ be distinct eigenvalues and

$V_x^1, \dots, V_x^r$ the corresponding eigenspaces. Each λ_i is a $|g|$–root of unity, so $V|_{X^g}$ can be written as direct sum of vector bundles

$$V|_{X^g} = V^1 \oplus \cdots \oplus V^r.$$

Then

$$\mathrm{ch}_g(V) = \sum \lambda_i \, \mathrm{ch}(V^i) \in H^{\mathrm{ev}}(X^g; \mathbb{C})$$

and

$$\mathrm{ch}_G(V) = \bigoplus_{g \in G} \mathrm{ch}_g(V) \in \Bigl[\bigoplus_{g \in G} H^{\mathrm{ev}}(X^g; \mathbb{C}) \Bigr]^G . \tag{9}$$

This is the correct equivariant extension of the Chern character in the sense that, for compact X, ch_G induces an isomorphism after tensoring with $\mathbb{C}$. Note that, in light of the Atiyah–Segal completion theorem [1], the so-called delocalized cohomology ring $H_G^*(\hat{X}; \mathbb{C})$ is a stronger invariant than ordinary equivariant cohomology of X. For instance, taking $X = \mathrm{pt}$, ch_G provides an identification between the complexified representation ring $K_G^0(\mathrm{pt}) \otimes \mathbb{C}$ and the ring of characters of G. On the other hand, $\widetilde{H}_G^*(\mathrm{pt}; \mathbb{C}) = 0$.

4.2 Parallel transport and field theories

Let $\mathfrak{X}$ be a stack on SM and $V\colon \mathfrak{X} \to \mathrm{Vect}^\nabla$ a vector bundle with connection. (If $\mathfrak{X}$ is representable by a manifold, then, by the Yoneda lemma, a fibered functor $\mathfrak{X} \to \mathrm{Vect}^\nabla$ is just a vector bundle with connection on $\mathfrak{X}$. In general, V provides a natural assignment, to each S–point $x\colon S \to \mathfrak{X}$, of a vector bundle with connection V_x on S.) Then we would like to construct a field theory $E_V \in 1|1\text{–EFT}(\mathfrak{X})$ using parallel transport along superpaths in $\mathfrak{X}$. Roughly speaking, this EFT assigns to a superpoint $x\colon \mathrm{spt} \to \mathfrak{X}$ the fiber V_x, and to a bordism between those the superparallel transport map constructed by Dumitrescu [9]. It is then part of the conjecture of Stolz and Teichner on the relation between $1|1$–EFTs and K–theory that, for reasonable $\mathfrak{X}$, the field theory above corresponds to the K–theory class represented by V.

A construction of the field theory E_V necessitates a longer discussion on the bordism category $1|1\text{–EBord}(\mathfrak{X})$, and, for $\mathfrak{X}$ an orbifold, is given in a subsequent paper [21]. In any case, we are presently only interested in its partition function, that is, the function $Z_V \in C^\infty(\mathfrak{K}(\mathfrak{X}))$ obtained by restricting E_V to closed, connected bordisms. (Note that the reduced field theory $\mathrm{red}(E_V)$ relevant for Theorem 1 only depends on Z_V.) The partition function admits a straightforward description independent of the details of the construction of the full EFT; the goal of this subsection is to provide a detailed construction of the functor

$$V \in \mathrm{Vect}^\nabla(\mathfrak{X}) \mapsto Z_V \in C^\infty(\mathfrak{K}(\mathfrak{X})).$$

We start recalling Dumitrescu's superversion of parallel transport, modified to better fit our conventions and perspective. Fix, as above, a vector bundle with connection $V\colon \mathfrak{X} \to \mathrm{Vect}^\nabla$ and a map $\psi\colon S \times \mathbb{R}^{1|1} \to \mathfrak{X}$, which we think of as an S–family of superpaths. Fix also sections $a, b\colon S \to S \times \mathbb{R}^{1|1}$, which we think of as specifying endpoints of the superinterval $[a, b] \subset S \times \mathbb{R}^{1|1}$. The composition

$$S \times \mathbb{R}^{1|1} \xrightarrow{\psi} \mathfrak{X} \xrightarrow{V} \mathrm{Vect}^\nabla$$

determines a vector bundle V_ψ over $S \times \mathbb{R}^{1|1}$ with connection

$$\nabla\colon C^\infty(S \times \mathbb{R}^{1|1}; V_\psi) \to \Omega^1(S \times \mathbb{R}^{1|1}; V_\psi).$$

Further restricting via a and b gives us vector bundles $V_{\psi(a)}$ and $V_{\psi(b)}$ over S. Now, we define a section s of V_ψ to be *parallel* if $\nabla_D s = 0$. (Here, as in Appendix B, $D = \partial_\theta - \theta\partial_t$ is the standard left-invariant vector field on $\mathbb{R}^{1|1}$. Since $D^2 = -\partial_t$, this can be though of as a “half-order” differential equation.) It can be shown—see [9, Proposition 3.1]—that any section s_a of $V_{\psi(a)}$ determines a unique parallel section s of V_ψ. Parallel transport is then the linear map

$$\mathrm{SP}(\psi, a, b)\colon V_{\psi(a)} \to V_{\psi(b)}$$

obtained by restricting to $V_{\psi(b)}$ the parallel section with given value on $V_{\psi(a)}$.

The main properties of superparallel transport are established in [9, Theorem 3.5]. We recall them here for the convenience of the reader.

(SP1) If a map $F\colon S' \times \mathbb{R}^{1|1} \to S \times \mathbb{R}^{1|1}$ covering $f\colon S' \to S$ is conformal, that is, preserves the distribution generated by D, and, moreover, $F(a') = a \circ f$ and $F(b') = b \circ f$, then

$$\mathrm{SP}(F^*\psi, a', b') = f^*\mathrm{SP}(\psi, a, b).$$

(SP2) Given $a, b, c\colon S \to \mathbb{R}^{1|1}$, we have

$$\mathrm{SP}(\psi, a, c) = \mathrm{SP}(\psi, b, c) \circ \mathrm{SP}(\psi, a, b).$$

Property (SP1) encapsulates both the fact that parallel transport depends smoothly on the input data ψ, a and b, and that it is invariant under conformal reparametrization of superpaths (and in particular under the euclidean reparametrizations we are concerned with in this paper). Note that it is *not* invariant under arbitrary reparametrizations. Property (SP2) is the expected compatibility with gluing of superintervals.

Remark 12 Dumitrescu describes parallel transport with respect to both D and its right-invariant counterpart $\partial_\theta + \theta\partial_t$, emphasizing the latter [9, Remark 3.3]. Because of the way we set up supereuclidean structures, we must work with D–parallel transport. This leads to different sign conventions, for instance in the proof of Proposition 16. The second half of Dumitrescu's paper is concerned with the more subtle notion of a parallel transport operation for Quillen superconnections; we will not deal with this case here.

We now return to our task of constructing, out of $V \in \mathrm{Vect}^\nabla(\mathfrak{X})$, a function Z_V on $\mathfrak{K}(\mathfrak{X})$. The idea is to associate, to each S–point $(K, \psi\colon K \to \mathfrak{X})$ of $\mathfrak{K}(\mathfrak{X})$, the supertrace of the holonomy along K. To make this precise, note that, by Proposition 23, it suffices to restrict to K of the form $K_l = (S \times \mathbb{R}^{1|1})/\mathbb{Z}l$ for some length parameter $l\colon S \to \mathbb{R}^{1|1}_{>0}$. With a slight abuse of notation, we still write ψ for the induced periodic map $S \times \mathbb{R}^{1|1} \to \mathfrak{X}$, so that $V_{\psi(0)} = V_{\psi(l)}$. Finally, we set

$$Z_V(K_l, \psi) = \mathrm{str}(\mathrm{SP}(\psi, 0, l)) \in C^\infty(S). \tag{10}$$

Proposition 13 *For any morphism $F\colon (K_{l'}, \psi') \to (K_l, \psi)$ covering $f\colon S' \to S$ in $\mathfrak{K}(\mathfrak{X})$, we have $Z_V(K_{l'}, \psi') = f^* Z_V(K_l, \psi)$. Therefore,* (10) *indeed defines a function $Z_V \in C^\infty(\mathfrak{K}(\mathfrak{X}))$.*

Proof Again by Proposition 23, any isometry $F\colon K_{l'} \to K_l$ lifts to a fiberwise isometry $\widetilde{F}\colon S' \times \mathbb{R}^{1|1} \to S \times \mathbb{R}^{1|1}$ over the same $f\colon S' \to S$. If $\widetilde{F}(0) = 0$ and $\widetilde{F}(l') = l$, then the proposition follows immediately from (SP1). Thus, replacing K_l by its pullback to S' if needed, we may assume that $S' = S$ and f is the identity.

Now, $\widetilde{F}$ is determined by a map $S' \to \mathbb{Z}/2 \ltimes \mathbb{R}^{1|1}$; simple calculations, done separately for the case of flips and translations, show that $\mathrm{SP}(\psi, 0, \widetilde{F}(0)) = \mathrm{SP}(\psi, l, \widetilde{F}(l'))$. Thus, using (SP2) and the vanishing of supertrace on commutators,

$$\begin{aligned} Z_V(K_l, \psi) &= \mathrm{str}\big(\mathrm{SP}(\psi, \widetilde{F}(0), l) \circ \mathrm{SP}(\psi, 0, \widetilde{F}(0))\big) \\ &= \mathrm{str}\big(\mathrm{SP}(\psi, 0, \widetilde{F}(0)) \circ \mathrm{SP}(\psi, \widetilde{F}(0), l)\big) \\ &= \mathrm{str}\big(\mathrm{SP}(\psi, l, \widetilde{F}(l')) \circ \mathrm{SP}(\psi, \widetilde{F}(0), l)\big) \\ &= \mathrm{str}\big(\mathrm{SP}(\psi, \widetilde{F}(0), \widetilde{F}(l'))\big). \end{aligned}$$

Finally, using (SP1), we get

$$Z_V(K_{l'}, \psi') = Z_V(K_{l'}, F^*\psi) = \operatorname{str}\big(\mathrm{SP}(\psi, \tilde{F}(0), \tilde{F}(l'))\big) = Z_V(K_l, \psi),$$

which finishes the proof. □

The construction of Z_V is clearly natural in V, and thus defines a functor

$$Z\colon \mathrm{Vect}^\nabla(\mathfrak{X}) \to C^\infty(\mathfrak{K}(\mathfrak{X})).$$

4.3 Proof of Theorem 1

As before, fix $V\colon X/\!\!/G \to \mathrm{Vect}^\nabla$. This map classifies a G–equivariant vector bundle over X, which we still call V, with a G–invariant connection ∇. To get started, we need to describe the pullback of V to a supercircle over $X/\!\!/G$.

Proposition 14 *Fix a supercircle $\psi\colon K \to X/\!\!/G$ and denote by $\pi\colon Q \to K$ and $f\colon Q \to X$ the principal G–bundle and G–equivariant map classified by ψ. Then there is a natural connection-preserving isomorphism of vector bundles*

$$\begin{tikzcd}
(f^*V)/G \arrow[r] \arrow[d] & \psi^*V \arrow[d] \\
Q/G \arrow[r, equal] & K
\end{tikzcd}$$

Proof Consider the diagram

$$\begin{tikzcd}
Q\times_K Q \arrow[r, shift left] \arrow[r, shift right] & Q \arrow[r, "f"] \arrow[d, "\pi"] & X \arrow[d, "x"] & \\
& K \arrow[r, "\psi"] & X/\!\!/G \arrow[r, "V"] & \mathrm{Vect}^\nabla
\end{tikzcd}$$

Here, $x\colon X \to X/\!\!/G$ is the standard atlas and hence $V \circ x$ classifies the vector bundle with connection $V \to X$. Notice that the square 2–commutes. In fact, the top-right composition $Q \to X/\!\!/G$ classifies the trivial G–bundle $Q \times G \to Q$, while the left-bottom composition classifies the G–bundle $\pi^*Q \to Q$ (together with the corresponding equivariant maps into X induced by f), and these two Q–points of $X/\!\!/G$ are isomorphic.

Now, the composition $V \circ x \circ f$ classifies the vector bundle $f^*V \to Q$, and the G–equivariance information provides descent data for the covering $Q \times_K Q \cong Q \times G \rightrightarrows Q \to K$. The descended vector bundle with connection can be described explicitly as $(f^*V)/G \to K$. Thus, 2–commutativity of the square above and the uniqueness property of descent provide a canonical isomorphism $\psi^*V \cong (f^*V)/G$. □

Our goal now is to identify $\mathrm{red}(Z_V) \in C^\infty(\mathfrak{B}(\widehat{X} /\!\!/ G))$, the dimensional reduction of $Z_V \in C^\infty(\mathfrak{K}(\mathfrak{X}))$, with a ($G$–invariant, even, closed) differential form on $\widehat{X}$, following the identifications of Theorem 3. To do so, we need to consider the versal $\Pi T\widehat{X}$–family $\Sigma_{\mathrm{versal}} \in \mathfrak{B}(\widehat{X} /\!\!/ G)$,

$$\begin{tikzcd} \Pi T\widehat{X} \times \mathbb{R}^{0|1} \arrow[r, "\mathrm{ev}"] \arrow[d] & \widehat{X} \arrow[r, "\hat{x}"] & \widehat{X} /\!\!/ G \\ \Pi T\widehat{X} & & \end{tikzcd} \tag{11}$$

and calculate the smooth function on the parameter manifold $\Pi T\widehat{X}$ assigned to it via $\mathrm{red}(Z_V)$. Here, $\hat{x}$ denotes the usual atlas.

Proposition 15 *In $C^\infty(\Pi T\widehat{X})$ we have the identity*

$$\mathrm{red}(Z_V)(\Sigma_{\mathrm{versal}}) = Z_V(K, Q, f),$$

where $(K, Q, f) \in \mathfrak{K}(\mathfrak{X})$ is as defined below.

Proof This is an exercise in chasing through the definition of dimensional reduction. The first step is to pick a preimage of Σ_{versal} via $\mathcal{P}$. Such a preimage is obtained by adding to (11) the trivial principal $\mathbb{T}$–bundle with standard connection over $\Pi T\widehat{X} \times \mathbb{R}^{0|1}$. The second step is to map that gadget to $\mathfrak{B}^{\mathbb{R}/\!\!/\mathbb{Z}}(\widehat{X} /\!\!/ G)$ via $\mathcal{Q}$. From the considerations at the end of Section 3.4, it follows that the resulting $\Pi T\widehat{X}$–family, once restricted to $\Pi TX^g \subset \Pi T\widehat{X}$, comprises the following data:

1. the family of euclidean $0|1$–manifolds $\Sigma = \Pi TX^g \times \mathbb{R}^{0|1} \to \Pi TX^g$;
2. the trivial $\mathbb{T}$–bundle $P_g = \Pi TX^g \times \mathbb{R}^{0|1} \times \mathbb{T} \to \Sigma$, with the standard connection form $\omega = dt - \theta\, d\theta$;
3. the principal G–bundle $Q_g = (\Pi TX^g \times \mathbb{R}^{1|1} \times G)/\mathbb{Z} \to P_g$, where the $\mathbb{Z}$–action is generated by the map described on S–points by
$$(x, t, h) \in (\Pi TX^g \times \mathbb{R}^{1|1} \times G)_S \mapsto (x, 1 \cdot t, gh);$$
4. the map $f_g \colon Q_g \to \widehat{X} \subset X \times G$ given by $(x, t, h) \mapsto (\mathrm{ev}(t_1, x) \cdot h, h^{-1} g h)$, which is well defined since $\mathrm{ev}(t_1, x)$ lies in X^g.

Finally, mapping to $\mathfrak{K}(\mathfrak{X})$ via $\mathcal{R}$ results in the $\Pi T\widehat{X}$–family $(K, Q, f) \in \mathfrak{K}(\mathfrak{X})$ determined, over ΠTX^g, by the data of P_g, seen as a supercircle, and the map $P_g \to \mathfrak{X}$ determined by Q_g and $\mathrm{pr}_1 \circ f_g$. By construction, the equation in the statement of the proposition holds true. □

Our next task is to compute $Z_V(K, Q, f)$; this is, by definition, the supertrace of the holonomy (around K) of the pullback of V by the map $\psi\colon K \to X/\!\!/G$ determined by (Q, f). Proposition 14 identifies that pullback of V with the vector bundle with connection $W = (f^*V)/G \to K$.

Proposition 16 *On ΠTX^g, the supertrace of the holonomy of $W = (f^*V)/G$ around K is a differential form representative of $\mathrm{ch}_g(V)$.*

Here, of course, we employ the usual identification $C^\infty(\Pi TX^g) \cong \Omega^*(X^g)$.

Proof Consider the standard superpath $c\colon \Pi TX^g \times \mathbb{R}^{1|1} \to \Pi TX^g \times \mathbb{T}^{1|1} \subset K$ with endpoints $i_t\colon \Pi TX^g \to \Pi TX^g \times \mathbb{R}^{1|1}$, $x \mapsto (x, t)$, for $t = 0, 1$, and denote by SP: $c_0^*W \to c_1^*W$ the parallel transport operator along that superinterval. There is a slight subtlety to notice here. Since the maps $c_0 = c \circ i_0$ and $c_1 = c \circ i_1$ are equal, c_0^*W and c_1^*W are the same vector bundle, but the correct way to identify them (for the purposes of computing the holonomy) is via the action of g. Indeed, let us form the pullback of principal bundles

$$\begin{tikzcd}
\widetilde{Q} \arrow[r] \arrow[d] \arrow[rr, bend left, "\widetilde{f}"] & Q \arrow[r, "f"'] \arrow[d] & X \\
\Pi TX^g \times \mathbb{R}^{1|1} \arrow[r, "c"] & K &
\end{tikzcd}$$

where $\widetilde{Q} = \Pi TX^g \times \mathbb{R}^{1|1} \times G$. Then the pullback c^*W can be identified with the restriction of the pullback of V to the identity section of $\widetilde{Q}$,

$$c^*W \cong (\widetilde{f}^*V)/G \cong (\widetilde{f}^*V)|_{\Pi TX^g \times \mathbb{R}^{1|1} \times \{e\}},$$

so we identify

$$c_0^*W_x = \widetilde{f}^*V_{(x,0,e)} = \widetilde{f}^*V_{(x,1,g)} \xrightarrow{g^{-1}} \widetilde{f}^*V_{(x,1,e)} = c_1^*W_x.$$

We write, as before, $V|_{X^g} = V^1 \oplus \cdots \oplus V^r$ as a direct sum of eigenspaces for eigenvalues $\lambda_1, \dots, \lambda_r$, with connection ∇_i on each component. Since $\widetilde{f}|_{\Pi TX^g \times \mathbb{R}^{1|1} \times \{e\}}$ takes values in X^g, this induces a similar decomposition of c^*W into a sum of vector bundles $\widetilde{W}^i$ with connection. We are finally ready to invoke the calculations of Dumitrescu recovering the usual (nonequivariant) Chern character in terms of parallel transport. Denoting by $\mathrm{SP}_i\colon \widetilde{W}^i|_{\Pi TX^g \times 0} \to \widetilde{W}^i|_{\Pi TX^g \times 1}$ the superparallel transport for one unit of time on each $\widetilde{W}^i$, the main theorem of [10] states that $\mathrm{SP}_i = \exp(\nabla_i^2)$, so that

$$\mathrm{ch}(\nabla_i) = \mathrm{str}(\exp(\nabla_i^2)) = \mathrm{str}(\mathrm{SP}_i).$$

The holonomy endomorphism $H\colon c_0^* W \to c_0^* W$ can be expressed as the composition

$$c_0^* W = \bigoplus_i \widetilde{W}^i|_{\Pi T X^g \times 0} \xrightarrow{\bigoplus_i \mathrm{SP}_i} \bigoplus_i \widetilde{W}^i|_{\Pi T X^g \times 1} = c_1^* W \xrightarrow{g} c_0^* W$$

and we conclude that

$$\mathrm{str}(H) = \sum_{1 \le i \le r} \mathrm{str}(g\,\mathrm{SP}_i) = \sum_{1 \le i \le r} \lambda_i\, \mathrm{str}(\mathrm{SP}_i) = \sum_{1 \le i \le r} \lambda_i\, \mathrm{ch}(\nabla_i)$$

is a differential form representative of $\mathrm{ch}_g(V)$. □

Finally, recall that the differential form on $\widehat{X} = \coprod_{g \in G} X^g$ associated to the field theory $\mathrm{red}(Z_V)$ is the form corresponding to the function $\mathrm{red}(Z_V)(\Sigma_{\mathrm{versal}})$ on $\Pi T \widehat{X}$. In particular, by Theorem 3, this form is G–invariant and represents an element of $H_G^{\mathrm{ev}}(\widehat{X}; \mathbb{C})$. By the above proposition and (9), this element is $\mathrm{ch}_G(V)$. This finishes the proof that the diagram of Theorem 1 commutes. The claim that the vertical arrow in that diagram induces a bijection after passing to concordance classes is the content of Theorem 3.

Appendix A Group actions on stacks

We briefly review the definitions of group action on a stack and quotient of a stack, following Romagny [17] and Ginot and Noohi [13], and then prove a useful lemma (Proposition 18). Note that limits and colimits here are always taken in the sense of bicategories. These are often called 2–(co)limits, bi(co)limits or homotopy (co)limits.

A.1 Basic definitions

Let $\mathfrak{X}$ be a groupoid fibration over a site $\mathfrak{S}$ and G a strict monoid object in the 2–category of fibrations over $\mathfrak{S}$. We denote by $m\colon G \times G \to G$ and $1\colon \mathrm{pt} \to G$ the multiplication law and unit map of G. Then we define a *(left) action of G on $\mathfrak{X}$* to be a map of groupoid fibrations $\mu\colon G \times \mathfrak{X} \to \mathfrak{X}$ together with (necessarily invertible)

2–morphisms α and $\mathfrak{a}$ as in the diagram below:

$$\begin{tikzcd}
G\times G\times\mathfrak{X} \arrow[r, "m\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\mu"'] & G\times\mathfrak{X} \arrow[d, "\mu"] \\
G\times\mathfrak{X} \arrow[r, "\mu"] \arrow[ur, Rightarrow, "\alpha", shorten=4mm] & \mathfrak{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
G\times\mathfrak{X} \arrow[r, "\mu"] \arrow[dr, Rightarrow, "\mathfrak{a}", shorten >=8mm] & \mathfrak{X} \\
\mathfrak{X} \arrow[u, "1\times\mathrm{id}"] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

In formulas, given an object $x\in\mathfrak{X}_S$ and $g,h\in G_S$, and using a dot to denote the group action, we are given natural isomorphisms

$$\alpha^x_{g,h}\colon g\cdot(h\cdot x)\to(gh)\cdot x,\quad \mathfrak{a}^x\colon 1\cdot x\to x.$$

This data is required to satisfy compatibility conditions that bear some resemblance to the axioms of a monoidal category. Firstly, a kind of pentagon identity relating the different ways in which the action of three group elements $g,h,k\in G_S$ can be associated:

$$\alpha^x_{g,hk}\circ g\cdot\alpha^x_{h,k}=\alpha^x_{gh,k}\circ\alpha^{g\cdot x}_{g,h}.$$

Second, a condition on the two ways of associating the action of the unit and another group element:

$$g\cdot\mathfrak{a}^x=\alpha^x_{g,1}\quad\text{and}\quad\mathfrak{a}^{g\cdot x}=\alpha^x_{1,g}.$$

It seems appropriate to call α and $\mathfrak{a}$ the associator and unitor for the action, in analogy to the terminology used in the theory of monoidal categories. We say the action is strict if α and $\mathfrak{a}$ are both the identity.

Now, suppose we are given fibrations with G–action $(\mathfrak{X},\mu,\alpha,\mathfrak{a})$ and $(\mathfrak{Y},\nu,\beta,\mathfrak{b})$. Then a *$G$–equivariant map* between them is a morphism of fibrations $f\colon\mathfrak{X}\to\mathfrak{Y}$ together with a 2–morphism

$$\begin{tikzcd}
G\times\mathfrak{X} \arrow[r, "\mu"] \arrow[d, "\mathrm{id}\times f"'] & \mathfrak{X} \arrow[d, "f"] \\
G\times\mathfrak{Y} \arrow[r, "\nu"] \arrow[ur, Rightarrow, "\rho", shorten=4mm] & \mathfrak{Y}
\end{tikzcd}$$

satisfying the following compatibility condition: for each $x\in\mathfrak{X}_S$ and $g,h\in G_S$, we have

$$f(\alpha^x_{g,h})\circ\rho^{h\cdot x}_g\circ g\cdot\rho^x_h=\rho^x_{gh}\circ\beta^{f(x)}_{g,h}\quad\text{and}\quad f(\mathfrak{a}^x)\circ\rho^x_1=\mathfrak{b}^{f(x)}.$$

We will call ρ the equivariance datum. Finally, a *G–equivariant 2–morphism* between morphisms (f,ρ) and (f',ρ') as above is defined to be a 2–morphism $\xi\colon f\to f'$

between the underlying fibered functors which is compatible with ρ and ρ' in the sense that

$$\rho'^{x}_{g} \circ g \cdot \xi^{x} = \xi^{g \cdot x} \circ \rho^{x}_{g}$$

for any $x \in \mathfrak{X}_S$, $g \in G_S$.

In terms of pasting diagrams, the conditions on ρ are expressed by the commutativity of the cube whose two halves are depicted below:

$$\begin{tikzcd}
G \times G \times \mathfrak{X} \arrow[dd] \arrow[dr, "m\times\mathrm{id}"] & & & \\
& G \times \mathfrak{X} \arrow[rr, "\mu"] \arrow[dd] & & \mathfrak{X} \arrow[dd] \\
G \times G \times \mathfrak{Y} \arrow[rr, "\mathrm{id}\times\nu"] \arrow[dr, "m\times\mathrm{id}"'] & & G \times \mathfrak{Y} \arrow[dr, "\nu"] \arrow[dl, Rightarrow, "\beta"'] & \\
& G \times \mathfrak{Y} \arrow[rr, "\nu"] & & \mathfrak{Y} \arrow[ul, Rightarrow, "\rho"']
\end{tikzcd}
\qquad
\begin{tikzcd}
G \times G \times \mathfrak{X} \arrow[rr, "\mathrm{id}\times\mu"] \arrow[dd] \arrow[dr, "m\times\mathrm{id}"'] & & G \times \mathfrak{X} \arrow[dd] \arrow[dr, "\mu"] \arrow[dl, Rightarrow, "\alpha"'] & \\
& G \times \mathfrak{X} \arrow[rr, "\mu"] & & \mathfrak{X} \arrow[dd] \\
G \times G \times \mathfrak{Y} \arrow[rr, "\mathrm{id}\times\nu"] \arrow[uur, Rightarrow, "\mathrm{id}\times\rho"'] & & G \times \mathfrak{Y} \arrow[dr, "\nu"] \arrow[ur, Rightarrow, "\rho"] & \\
& & & \mathfrak{Y}
\end{tikzcd}$$

and commutativity of the prism

$$\begin{tikzcd}
& G \times \mathfrak{X} \arrow[dr, "\mu"] \arrow[dd] \arrow[d, phantom, "\mathfrak{a}\Downarrow"] & \\
\mathfrak{X} \arrow[ur, "1\times\mathrm{id}"] \arrow[rr, "\mathrm{id}"'] \arrow[dd] & {} & \mathfrak{X} \arrow[dd] \\
& G \times \mathfrak{Y} \arrow[dr, "\nu"] \arrow[ur, Rightarrow, "\rho"'] \arrow[d, phantom, "\mathfrak{b}\Downarrow"] & \\
\mathfrak{Y} \arrow[ur, "1\times\mathrm{id}"] \arrow[rr, "\mathrm{id}"'] & {} & \mathfrak{Y}
\end{tikzcd}$$

Here, all vertical maps are products of f and the identity of G. The condition on ξ is the commutativity of the following diagram:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
G \times \mathfrak{X} \arrow[r, "\mu"] \arrow[d, bend right=50, "\mathrm{id}\times f"'] \arrow[d, bend left=50, "\mathrm{id}\times f'"] \arrow[d, phantom, "\overset{\mathrm{id}\times\xi}{\Longrightarrow}"] & \mathfrak{X} \arrow[d, bend right=50, "f"'] \arrow[d, bend left=50, "f'"] \arrow[d, phantom, "\overset{\xi}{\Longrightarrow}"] \arrow[dl, phantom, "\rho' \Nearrow \quad \rho \Nearrow"] \\
G \times \mathfrak{Y} \arrow[r, "\nu"'] & \mathfrak{Y}
\end{tikzcd}$$

We are mostly interested in the case where G is a (representable) sheaf of groups, but we will also consider the group stack $\mathrm{pt}/\!\!/\mathbb{Z}$. Note that a strict action of $\mathrm{pt}/\!\!/\mathbb{Z}$ on a stack $\mathfrak{X}$ is precisely the data of an automorphism of $\mathrm{id}_{\mathfrak{X}}$, ie a natural choice of automorphism for each object of $\mathfrak{X}$. For instance, the inertia stack $\Lambda\mathfrak{X}$ comes with a canonical $\mathrm{pt}/\!\!/\mathbb{Z}$–action. We will also make use of a 2–categorical model for the circle group, to be denoted by $\mathbb{R}/\!\!/\mathbb{Z}$. It is presented by the Lie 2–group $\mathbb{Z}\times\mathbb{R}\rightrightarrows\mathbb{R}$ (the transport groupoid of the $\mathbb{Z}$–action on $\mathbb{R}$) endowed with the multiplication map determined by the group structures on the spaces of objects and morphisms, and unit $0\in\mathbb{R}$. At the Lie 2–group level, there are evident strict homomorphisms

$$\mathbb{T}\leftarrow\mathbb{R}/\!\!/\mathbb{Z}\rightarrow\mathrm{pt}/\!\!/\mathbb{Z}.$$

The left map gives us an equivalence of group stacks, but in concrete situations it may be more convenient to consider one model or the other.

A.2 Quotient stacks of G–stacks

Let $\mathfrak{X}$ be a stack endowed with a left action of a sheaf of groups G. Then we define a new stack $G\backslash\!\!\backslash\mathfrak{X}$ whose S–points are given by a left G–torsor $P\to S$ together with a G–equivariant map $\psi\colon P\to\mathfrak{X}$; a morphism $(P',\psi')\to(P,\psi)$ covering $f\colon S'\to S$ is given by a diagram

$$\begin{tikzcd}
P' \arrow[dr, "\Phi"] \arrow[drr, bend left, "\psi'"] \arrow[dd] & \Downarrow\xi & \\
& P \arrow[r, "\psi"] \arrow[dd] & \mathfrak{X} \\
S' \arrow[dr, "f"] & & \\
& S &
\end{tikzcd}$$

where Φ is a map of G–torsors and ξ an equivariant 2–morphism.

There is a faithful functor $i\colon\mathfrak{X}\to G\backslash\!\!\backslash\mathfrak{X}$ sending $x\colon S\to\mathfrak{X}$ to the S–point of $G\backslash\!\!\backslash\mathfrak{X}$ consisting of the trivial G–torsor $G\times S\to S$ together with the G–equivariant map

$$\psi\colon G\times S\xrightarrow{\mathrm{id}\times x}G\times\mathfrak{X}\xrightarrow{\mu}\mathfrak{X}.$$

This makes the diagram below 2–cartesian:

$$\begin{tikzcd}
G\times\mathfrak{X} \arrow[r, "\mu"] \arrow[d, "\mathrm{pr}_2"'] & \mathfrak{X} \arrow[d, "i"] \\
\mathfrak{X} \arrow[r, "i"] & G\backslash\!\!\backslash\mathfrak{X}
\end{tikzcd}$$

Now, we can attempt to perform the construction of a transport groupoid $G\times\mathfrak{X}\rightrightarrows\mathfrak{X}$ internally in the 2–category of stacks. For this to work, we need to define internal categories with the appropriate degree of weakness (eg if the action is not strictly unital, the same must be allowed of our internal categories). In any case, it is clear that we get a "nerve", that is, an augmented (weak) simplicial object

$$G\backslash\!\!\backslash\mathfrak{X} \xleftarrow{i} \mathfrak{X} \leftleftarrows G\times\mathfrak{X} \mathrel{\substack{\leftarrow\\ \leftarrow\\ \leftarrow}} G\times G\times\mathfrak{X} \mathrel{\substack{\leftarrow\\ \leftarrow\\ \leftarrow\\ \leftarrow}} \cdots . \tag{12}$$

Since the various compositions $G^n \times \mathfrak{X} \to G\backslash\!\!\backslash\mathfrak{X}$ are not equal, just isomorphic (with a specified isomorphism), the augmentation depends, strictly speaking, on a choice. For definiteness, we take that to be the composition of i with the projection $\mathrm{pr}_{n+1}\colon G^n \times \mathfrak{X} \to \mathfrak{X}$.

Proposition 17 *The above induces an equivalence of stacks*

$$G\backslash\!\!\backslash\mathfrak{X} \xleftarrow{j} \operatorname{colim}\Big(\mathfrak{X} \leftleftarrows G\times\mathfrak{X} \mathrel{\substack{\leftarrow\\ \leftarrow\\ \leftarrow}} G\times G\times\mathfrak{X} \mathrel{\substack{\leftarrow\\ \leftarrow\\ \leftarrow\\ \leftarrow}} \cdots\Big).$$

The reader well versed on colimits of categories may be able to interpret the discussion in Sections 3.2 and 4.2 of Ginot and Noohi [13] as a proof, even though it does not use the language of colimits. In any case, we will provide our own argument. Before getting there, we give some background on (homotopy) colimits in Cat. Given a diagram of small categories $F\colon D \to \mathrm{Cat}$ indexed by a small 1–category (with no strictness requirements on F), we denote by $D \ltimes F$ the Grothendieck construction. It is the oplax colimit of F, meaning that for each $C \in \mathrm{Cat}$, there is an equivalence between the category of functors $D \ltimes F \to C$ and the category of lax natural transformations $F \to \mathrm{const}_C$ and modifications between them. The colimit of F is obtained by localizing $D \ltimes F$ at the class of opcartesian morphisms.

Spelling out the above, the colimit can be described in terms of generators and relations as follows. We write i, j, etc, for objects of D, and A_i, A_j for their images via F; also, we use the same notation both for a morphism $f\colon i \to j$ in D and its image $f\colon A_i \to A_j$. To build $A = \operatorname{colim}_D A_i$, we start with the disjoint union $\bigsqcup_{i\in D} A_i$ and then freely adjoin inverse morphisms

$$f_x\colon x \to f(x), \quad f_x^{-1}\colon f(x) \to x$$

for each $f\colon i \to j$ in D and $x \in A_i$; finally, we impose a number of natural relations, most notably

$$(x \xrightarrow{\phi} y \xrightarrow{f_y} f(y)) = (x \xrightarrow{f_x} f(x) \xrightarrow{f(\phi)} f(y)),$$

where ϕ is a morphism in A_i, as well as its counterpart involving f_x^{-1} and f_y^{-1}. This process can be made precise using the free category generated by a directed graph and congruences. For more details, including the proof that this has the desired universal property, see Fiore [11, Chapter 4].

Proof of Proposition 17 Colimits of stacks are obtained by taking colimits objectwise in $\mathfrak{S}$ and then stackifying. Thus, it suffices to show that, for each $S \in \mathfrak{S}$,

$$(G\backslash\!\backslash\mathfrak{X})_S \xleftarrow{j_S} \operatorname{colim}\Big(\mathfrak{X}_S \leftleftarrows (G\times\mathfrak{X})_S \mathrel{\substack{\leftarrow\\[-0.5ex]\leftarrow\\[-0.5ex]\leftarrow}} (G\times G\times\mathfrak{X})_S \mathrel{\substack{\leftarrow\\[-0.5ex]\leftarrow\\[-0.5ex]\leftarrow\\[-0.5ex]\leftarrow}} \cdots\Big)$$

gives an equivalence of the right-hand side with the full subgroupoid $(G\backslash\!\backslash\mathfrak{X})_S^{\mathrm{triv}}$ of the left-hand side involving only trivial G–torsors. To simplify the argument, we assume, without loss of generality, that the G_S–action on $\mathfrak{X}_S$ is strict [17, Proposition 1.5].

Consider the functor $l\colon (G\backslash\!\backslash\mathfrak{X})_S^{\mathrm{triv}} \to \operatorname{colim}_n(G^n\times\mathfrak{X})_S$ prescribed by the following conditions. First, on $\mathfrak{X}_S$, seen as a subgroupoid of both the domain (via $i\colon \mathfrak{X}_S \hookrightarrow (G\backslash\!\backslash\mathfrak{X})_S^{\mathrm{triv}}$) and codomain, l is just the identity. Second, to the morphism $x \to g\cdot x$ in $(G\backslash\!\backslash\mathfrak{X})_S^{\mathrm{triv}}$ determined by $g \in G_S$, l associates the morphism

$$\mu_g^x\colon x \xrightarrow{\mathrm{pr}_2^{-1}} (g,x) \xrightarrow{\mu} g\cdot x$$

in the colimit groupoid. To see that this is well defined and respects compositions, it suffices to check that the outer square of the following diagram in the colimit groupoid commutes for any $g,h\in G_S$ and $\xi\colon g\cdot x \to y$ in $\mathfrak{X}_S$:

$$\begin{tikzcd}
g\cdot x \arrow[rrr, "\xi"] \arrow[ddd, "\mu_h^{g\cdot x}"'] & & & y \arrow[ddd, "\mu_h^y"] \\
& (h,g\cdot x) \arrow[lu, "\mathrm{pr}_2"'] \arrow[r, "\mathrm{id}\times\xi"] \arrow[ldd, "\mu"] & (h,y) \arrow[ru, "\mathrm{pr}_2"] \arrow[rdd, "\mu"'] & \\
& & & \\
hg\cdot x \arrow[rrr, "h\cdot\xi"] & & & h\cdot y
\end{tikzcd}$$

This follows from the fact that each circuit traveling inside the square commutes.

Now, the composition $j_S \circ l$ is equal to the identity, and we claim that the reverse composition is isomorphic to the identity. In fact, $l\circ j_S(g_1,\dots,g_n,x) = x$, and we define a natural transformation $u\colon \mathrm{id} \to l\circ j_S$ by

$$u_{(g_1,\dots,g_n,x)} = \mathrm{pr}_{n+1}\colon (g_1,\dots,g_n,x) \to x.$$

Naturality with respect to those morphisms in the colimit groupoid which arise from morphisms in $(G^n\times\mathfrak{X})_S$ is obvious. A general morphism arising from the indexing category Δ^{op} is as in the left vertical arrow of the diagram below:

$$\begin{tikzcd}
(g_1,\dots,g_n,x) \arrow[rr, "\mathrm{pr}_{n+1}"] \arrow[dr] \arrow[dd] & & x \arrow[dd, "\mu^x_{g_J}"] \\
& (g_J,x) \arrow[ur, "\mathrm{pr}_2"] \arrow[dr, "\mu"'] & \\
(g_{I_1},\dots,g_{I_k},g_J\cdot x) \arrow[rr, "\mathrm{pr}_{k+1}"] & & g_J\cdot x
\end{tikzcd}$$

where $I_1,\dots,I_k,J\subset[n]$ are (possibly empty) disjoint and adjacent subsets whose union contains n and $g_{\{i_1,\dots,i_j\}}=g_{i_1}\dots g_{i_j}$. Its image through $l\circ j_S$ is the right vertical arrow, and naturality of u, that is, the claim that the outer square commutes, follows from commutativity of the circuits involving (g_J,x). This finishes the proof that j_S is an equivalence onto $(G\backslash\backslash\mathfrak{X})^{\mathrm{triv}}_S$. □

Now, given a stack $\mathfrak{C}$, applying $\mathrm{Fun}_{\mathfrak{S}}(-,\mathfrak{C})$ to diagram (12) produces a (weak) cosimplicial groupoid. The following descent calculation for G–stacks is then a corollary of Proposition 17:

Proposition 18 *For any stack $\mathfrak{C}$ and G–stack $\mathfrak{X}$, diagram* (12) *induces an equivalence of groupoids*

$$\mathrm{Fun}_{\mathfrak{S}}(G\backslash\backslash\mathfrak{X},\mathfrak{C})\cong\lim\Big(\mathrm{Fun}_{\mathfrak{S}}(\mathfrak{X},\mathfrak{C})\rightrightarrows\mathrm{Fun}_{\mathfrak{S}}(G\times\mathfrak{X},\mathfrak{C})\mathrel{\substack{\rightarrow\\ \rightarrow\\ \rightarrow}}\cdots\Big).$$

Again, a concrete description of 2–limits in the 2–category of small categories can be found in Fiore [11, Chapter 5]. For the convenience of the reader, we give a quick summary here. We fix the same notation as in the discussion of colimits above; in particular, we have a diagram $F\colon D\to\mathrm{Cat}$. Then (a model for) the limit of F is the category whose objects are (pseudo)natural transformations $\Delta_{\mathrm{pt}}\to F$ with domain the constant functor with value the discrete category with one object, and whose morphisms are modifications between them. In concrete terms, an object consists of a collection of objects $a_i\in A_i$ for each $i\in D$ together with isomorphisms $\tau_f\colon f(a_i)\to a_j$ for each morphism $f\colon i\to j$ in D; these data are required to satisfy certain coherence conditions. A morphism $(a'_i,\tau'_f)\to(a_i,\tau_f)$ consists of a collection of morphisms $a'_i\to a_i$ in A_i for each $i\in D$, subject to appropriate conditions.

Proposition 19 *Given a homomorphism of sheaves of groups $h\colon G\to H$ and an H–stack $\mathfrak{X}$, we have an equivalence*

$$G\backslash\backslash\mathfrak{X}\xrightarrow{\cong}G\backslash\backslash\mathrm{pt}\times_{H\backslash\backslash\mathrm{pt}}H\backslash\backslash\mathfrak{X}.$$

Proof The various maps of stacks involved in the statement of the proposition are induced by the obvious maps between the simplicial diagrams of which they are a colimit (see Proposition 17), as well as the universal property of the fiber product.

It follows from Proposition 17 that $G\backslash\!\backslash\mathfrak{X}$ is obtained by stackifying the prestack that assigns to $S \in \mathfrak{S}$ the groupoid whose objects are objects $x, y, \ldots \in \mathfrak{X}$, and morphisms $x \to y$ are pairs (g, ξ), where $g \in G_S$ and $\xi\colon g \cdot x \to y$ is a morphism in $\mathfrak{X}_S$. The formation of fiber products commutes with stackification (since the latter is built using limits and filtered colimits), so the codomain of our fibered functor has a similar description as the stackification of a fiber product of prestacks.

Now, in terms of the above data, the map $G\backslash\!\backslash\mathfrak{X} \to H\backslash\!\backslash\mathfrak{X}$ sends

$$x \mapsto x, \quad (g, \xi) \mapsto (h(g), \xi),$$

while $G\backslash\!\backslash\mathfrak{X} \to G\backslash\!\backslash\mathrm{pt}$ sends

$$x \mapsto \mathrm{pt}, \quad (g, \xi) \mapsto g.$$

Thus, it is clear that, at the level of prestacks, our fibered functor is fully faithful and essentially surjective, and the result follows. □

Appendix B Low-dimensional euclidean supergeometry

In the category SM of supermanifolds, $\mathbb{R}^{1|1}$ has a (noncommutative) group structure given by

$$\mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}, \quad ((t, \theta), (t', \theta')) \mapsto (t + t' + \theta\theta', \theta + \theta').$$

The Lie algebra of left-invariant vector fields is free on one odd generator $D = \partial_\theta - \theta\partial_t$, and actions of $\mathbb{R}^{1|1}$ correspond (bijectively, modulo noncompactness issues) to odd vector fields. Similarly, $\mathbb{R}^{0|1}$ has a Lie algebra spanned by an odd element ∂_θ squaring to 0, and its actions correspond bijectively to homological vector fields, ie those odd Q such that $[Q, Q] = 0$.

The definition of euclidean structures on supermanifolds follows the philosophy of Felix Klein's Erlangen program. One starts by fixing a model space and a subgroup of diffeomorphisms, called the isometry group; a euclidean structure is then a maximal atlas whose transition maps are isometries. This idea is explained in detail in Stolz and Teichner [24, Sections 2.5 and 4.2]. In (real) dimensions $0|1$ and $1|1$, the model spaces are $\mathbb{R}^{0|1}$ and $\mathbb{R}^{1|1}$, respectively, with isometry groups

$$\mathrm{Isom}(\mathbb{R}^{0|1}) = \mathbb{R}^{0|1} \rtimes \mathbb{Z}/2, \quad \mathrm{Isom}(\mathbb{R}^{1|1}) = \mathbb{R}^{1|1} \rtimes \mathbb{Z}/2.$$

In both cases, $\mathbb{Z}/2$ acts by negating the odd coordinate and $\mathbb{R}^{d|1}$ acts by *left* multiplication (this choice influences our sign conventions, and dictates whether to work with left of right group actions at various places).

The differential form $d\theta \wedge d\theta$ on $\mathbb{R}^{0|1}$ is invariant under isometries, and therefore determines a canonical fiberwise 2–form ζ on any family $\Sigma \to S$ of euclidean $0|1$–manifolds. Conversely, any closed, nondegenerate, even fiberwise 2–form on Σ is locally of this form and determines a euclidean structure.

In dimension $1|1$, euclidean structures also admit ad hoc definitions in terms of sections of certain sheaves. In the remainder of this section, we discuss some of those alternative definitions, and study the stack of $1|1$–dimensional closed connected euclidean supermanifolds, which we will also call euclidean supercircles. This appendix is a survey of material I learned from Stephan Stolz, some of which does not seem to have appeared in the literature.

B.1 Euclidean structures in dimension $1|1$

bundle In Dumitrescu [9, Section 2.3], a conformal structure on a $1|1$–manifold X is defined to be a distribution $\mathcal{D}$ (ie a subsheaf of the tangent sheaf T_X) of rank $0|1$ fitting in a short exact sequence

$$0 \to \mathcal{D} \to T_X \to \mathcal{D}^{\otimes 2} \to 0 \tag{13}$$

(see also [12, Lecture 3]). A euclidean structure is then defined to be a choice, up to sign, of an odd vector field D generating $\mathcal{D}$. The fundamental example is the vector field $D = \partial_\theta - \theta\partial_t$ on $\mathbb{R}^{1|1}$. Note that it squares to $-\partial_t$, so in fact D and D^2 generate $T_{\mathbb{R}^{1|1}}$. More generally, conformal and euclidean structures on a family $X \to S$ of $1|1$–manifolds are appropriate splittings or sections of the vertical tangent bundle $T_{X/S}$.

We want to show that this is equivalent to the original definition. Denote by $\mathfrak{E}$ and $\mathfrak{E}'$ the stacks of families of $1|1$–dimensional euclidean manifolds according to the chart definition and the vector field definition, respectively. It is clear that we have a map $\mathfrak{E} \to \mathfrak{E}'$, since the transition maps of a euclidean chart preserve the canonical vector field D on $\mathbb{R}^{1|1}$ up to sign. Now, given an object in $\mathfrak{E}'$, the atlas from Proposition 22 below is indeed euclidean, by Propositions 20 and 21. This gives an inverse map $\mathfrak{E} \to \mathfrak{E}'$.

Proposition 20 *The subgroup of diffeomorphisms of* $\mathbb{R}^{1|1}$ *preserving the form* $\omega = dt - \theta\, d\theta$ *is precisely* $\mathrm{Isom}(\mathbb{R}^{1|1}) = \mathbb{R}^{1|1} \rtimes \mathbb{Z}/2$*, acting in the standard way on the left.*

A correct reading of this assertion requires that we think in families; thus, the claim is that the subsheaf of $\mathrm{Diff}(\mathbb{R}^{1|1}) \subset \underline{\mathrm{SM}}(\mathbb{R}^{1|1}, \mathbb{R}^{1|1})$ preserving ω is representable by the Lie group $\mathbb{R}^{1|1} \rtimes \mathbb{Z}/2$. Moreover, it will be clear from the proof that the proposition is true locally in $\mathbb{R}^{1|1}$, that is, if $U \subset \mathbb{R}^{1|1}$ is a connected domain, then the sheaf of embeddings $U \to \mathbb{R}^{1|1}$ preserving ω is $\mathbb{R}^{1|1} \rtimes \mathbb{Z}/2$.

Proof An S–family of diffeomorphisms of $\mathbb{R}^{1|1}$ is given by a diffeomorphism

$$\Phi\colon S \times \mathbb{R}^{1|1} \to S \times \mathbb{R}^{1|1}$$

commuting with the projections onto S. We can express this diffeomorphism in terms of a map $\phi\colon S \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$ by the formula

$$(s, x) \mapsto (s, \phi(s, x) \cdot x),$$

where s and x should be interpreted as T–points of S and $\mathbb{R}^{1|1}$ for a generic supermanifold T, and $\cdot$ indicates the usual group operation on $\mathbb{R}^{1|1}$. Writing $\phi = (r, \eta) \in (\mathbb{R} \times \mathbb{R}^{0|1})_{S \times \mathbb{R}^{1|1}}$ and $x = (t, \theta) \in (\mathbb{R} \times \mathbb{R}^{0|1})_T$ in terms of their components, the above formula becomes

$$(s, t, \theta) \mapsto (s, t + r(s, t, \theta) + \eta(s, t, \theta)\theta, \eta(s, t, \theta) + \theta).$$

Hence, the equation $\omega = \Phi^* \omega$ reads

$$dt - \theta\, d\theta = dt + dr - \theta\, d\theta - (2\theta + \eta)\, d\eta.$$

To analyze the restrictions imposed by this equation, let us write

$$r = r_0 + r_1\theta, \quad \eta = \eta_1 + \eta_0\theta, \quad \text{where } r_i, \eta_i \in C^\infty(S \times \mathbb{R})^i.$$

Then $dr - (2\theta + \eta)\, d\eta = 0$ gives us

$$(14) \quad 0 = (dr_0 - \eta_1\, d\eta_1) + (dr_1 + (2 + \eta_0)\, d\eta_1 - \eta_1\, d\eta_0)\theta + (r_1 - \eta_1\eta_0)\, d\theta \\ - (2 + \eta_0)\eta_0\theta\, d\theta.$$

Each individual term above vanishes. From the $\theta\, d\theta$ term, we get that $\eta_0 = 0$ or -2, since either η_0 or $(2 + \eta_0)$ has nonzero reduced part and hence is invertible, and the $d\theta$

term tells us that $r_1 = \eta_0\eta_1$. Plugging that into the θ term, we get $(2+2\eta_0)\,d\eta_1 = 0$, so $d\eta_1 = 0$ since the factor in front of it is a nonzero constant. Finally, (14) implies that $dr_0 = 0$.

Now, recall that those formulas should be interpreted as equalities of S–families of differential forms on $\mathbb{R}^{1|1}$, ie sections of $\Omega^*(S \times \mathbb{R}^{1|1})$ modulo $\Omega^{\geq 1}(S)$. So in fact we have $r_0, \eta_1 \in C^\infty(S)$, and there is a locally constant function $a = 1 + \eta_0 \in (\mathbb{Z}/2)_S = \{\pm 1\}_S$. Therefore, the diffeomorphism Φ determines and is determined by $(r_0, \eta_1, a) \in (\mathbb{R}^{1|1} \rtimes \mathbb{Z}/2)_S$ via the correspondence

$$(r_0, \eta_1, a) \mapsto \phi_{r_0,\eta_1,a} = (r_0 + (a-1)\eta_1\theta, \eta_1 + (a-1)\theta) \in \mathbb{R}^{1|1}_{S\times\mathbb{R}^{1|1}}.$$

It is simple to check that any choice of (r_0, η_1, a) as above determines a diffeomorphism preserving ω, and that the choices $(r_0, \eta_1, 1)$ and $(0, 0, -1)$ act as translation by (r_0, η_1) and negation, respectively, of the odd variable. Therefore, to finish the proof, we just need to verify that given a second diffeomorphism Φ' prescribed, in a similar way, by (r_0', η_1', a'), the composition

$$(s,t,\theta) \overset{\Phi}{\longmapsto} \phi_{r_0,\eta_1,a}(s,\theta)\cdot(s,t,\theta) \overset{\Phi'}{\longmapsto} \phi_{r_0',\eta_1',a'}(s,\theta')\cdot\phi_{r_0,\eta_1,a}(s,\theta)\cdot(s,t,\theta),$$

where $\theta' = \eta + (a-1)\theta$ is the θ–component of the middle term, agrees with the action of the product $(r_0', \eta_1', a')\cdot(r_0, \eta_1, a)$; more explicitly,

$$\phi_{(r_0'+r_0+a'\eta_1'\eta_1,\eta_1'+a'\eta_1,a'a)}(s,\theta) = \phi_{r_0',\eta_1',a}(s, \eta + (a-1)\theta)\cdot\phi_{r_0,\eta_1,a}(s,\theta).$$

This is a tedious but straightforward calculation. □

Proposition 21 *A diffeomorphism of* $\mathbb{R}^{1|1}$ *preserves* $\omega = dt - \theta\, d\theta$ *if and only if it preserves* $D = \partial_\theta - \theta\partial_t$ *up to sign.*

Proof If an S–family of diffeomorphisms $\Phi\colon S\times\mathbb{R}^{1|1} \to S\times\mathbb{R}^{1|1}$ preserves ω, then it is determined by $\phi \in (\mathbb{R}^{1|1} \rtimes \mathbb{Z}/2)_S$ and it is easy to check that it sends D to either D or $-D$. Conversely, if $\Phi_* D = \pm D$, then $\Phi_* D^2 = (\pm D)^2$, so that

$$\langle D, \Phi^*\omega\rangle = \langle \Phi_* D, \omega\rangle = 0, \quad \langle D^2, \Phi^*\omega\rangle = \langle D^2, \omega\rangle.$$

Since D and D^2 generate $T_{\mathbb{R}^{1|1}}$ as a $C^\infty_{\mathbb{R}^{1|1}}$–module, it follows that $\Phi^*\omega = \omega$. □

Proposition 22 *Let* $X \to S$ *be an* S*–family of* $1|1$*–manifolds and* D *a vertical vector field generating a distribution as in* (13)*. Then* X *admits an atlas such that* D *can be written locally as* $\partial_\theta - \theta\partial_t$.

Proof We apply the Frobenius theorem [8, Lemma 3.5.2] to the vector field D^2. This gives us local charts $(t, \theta)\colon U \subset X \to S \times \mathbb{R}^{1|1}$ where D^2 gets identified with $-\partial_t$. With respect to one of those charts, we can write

$$D = f\partial_\theta + g\partial_t, \quad f = f_0 + f_1\theta, \quad g = g_1 + g_0\theta,$$

where $f_i, g_i \in C^\infty(S \times \mathbb{R})^i$, so that

$$D^2 = f(\partial_\theta f)\partial_\theta + f(\partial_\theta g)\partial_t + g(\partial_t f)\partial_\theta + g(\partial_t g)\partial_t$$

(the remaining terms one could expect in this expansion involve ∂_θ^2, g^2 or $[\partial_\theta, \partial_t]$, so they vanish). Inspecting the coefficients of ∂_t, $\theta\partial_t$, ∂_θ and $\theta\partial_\theta$, respectively, we get

$$\begin{aligned} f_0 g_0 + g_1 g_1' &= -1, & f_1 g_0 + g_1 g_0' - g_0 g_1' &= 0, \\ -f_0 f_1 + g_1 f_0' &= 0, & g_1 f_1' + g_0 f_0' &= 0. \end{aligned}$$

The first equation implies that f_0 and g_0 are invertible, and the fourth equation implies that $g_1 g_0 f_0' = 0$. Multiplying the third equation by g_0 gives us $g_0 f_0 f_1 = 0$, so $f_1 = 0$. Using again the fourth equation, we conclude that $f_0' = 0$. Therefore, by the first equation, g_0' is a multiple of g_1 and the second equation reduces to $g_0 g_1' = 0 = g_1'$. Finally, we learn from the first equation that f_0 and $-g_0$ are inverses. To summarize, we have

$$D = f_0\partial_\theta - f_0^{-1}\theta\partial_t, \quad \text{where } f_0 \in C^\infty(S)^{\text{even}}.$$

Performing the change of coordinates $t \mapsto t$, $\theta \mapsto f_0^{-1}\theta$, we can assume $f_0 = 1$, which finishes the proof. □

B.2 Euclidean supercircles

We are interested in the stack $\mathfrak{K}$ of closed connected $1|1$–dimensional euclidean manifolds. Given a parameter supermanifold S and a map $l\colon S \to \mathbb{R}^{1|1}_{>0}$, we can form the S–family of supercircles of length l, $K_l = (S \times \mathbb{R}^{1|1})/\mathbb{Z}$, where the generator of the $\mathbb{Z}$–action is described, in terms of T–points of $S \times \mathbb{R}^{1|1}$, by $(s, u) \mapsto (s, l(s) \cdot u)$. Moreover, given any map $r\colon S \to \mathbb{R}^{1|1}$, the diffeomorphism of $S \times \mathbb{R}^{1|1}$, $(s, u) \mapsto (s, r(s) \cdot u)$, descends to an isometry $K_{r^{-1}lr} \to K_l$, and the flip fl: $\mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$ (the diffeomorphism negating the odd coordinate) descends to an isometry $K_{\mathrm{fl}(l)} \to K_l$, since fl is a group automorphism of $\mathbb{R}^{1|1}$.

We can assemble this collection of examples into a Lie groupoid as follows. Note that the right $\mathbb{R}^{1|1}$–action on itself by conjugation extends to an action of the semidirect product $\mathbb{Z}/2 \ltimes \mathbb{R}^{1|1}$, where $\mathbb{Z}/2$ acts via fl. It is then clear that we have a map of stacks $\mathbb{R}^{1|1}_{>0}/\!\!/(\mathbb{Z}/2 \ltimes \mathbb{R}^{1|1}) \to \mathfrak{K}$. To the S–point of the domain corresponding to a map $l\colon S \to \mathbb{R}^{1|1}_{>0}$, it assigns K_l, and to the morphism corresponding to the S–point (a, r) of $\mathbb{Z}/2 \ltimes \mathbb{R}^{1|1}$, it assigns the isometry $K_{r^{-1}\mathrm{fl}^a(l)r} \to K_l$. This only fails to be an equivalence of stacks because the S–family of morphisms $(0, l)\colon l \to l$ in the domain stack maps to the identity map of K_l.

Proposition 23 *The fibered functor $\mathbb{R}^{1|1}_{>0}/\!\!/(\mathbb{Z}/2 \ltimes \mathbb{R}^{1|1}) \to \mathfrak{K}$ is full and essentially surjective.*

Proof Any isometry $K_{l'} \to K_l$ lifts to an isometry of their covers $S \times \mathbb{R}^{1|1} \to S \times \mathbb{R}^{1|1}$. It follows from Proposition 20 that the isometry group of $\mathbb{R}^{1|1}$ is (no bigger than) $\mathbb{R}^{1|1} \rtimes \mathbb{Z}/2$, and this proves fullness.

It remains to show that our fibered functor is essentially surjective. Pick any $K \in \mathfrak{K}_S$. Restricting to a neighborhood in S if needed, fix a section $x\colon S \to K$ and a vector field D_K specifying the euclidean structure. Then D_K gives us an action $\mu\colon \mathbb{R}^{1|1} \times K \to K$; composing with x, we get a map of S–families

$$\mu_x\colon \mathbb{R}^{1|1} \times S \xrightarrow{\mathrm{id} \times x} \mathbb{R}^{1|1} \times K \xrightarrow{\mu} K.$$

Since the generators D and D^2 of the Lie algebra of $\mathbb{R}^{1|1}$ are μ–related to the linearly independent vector fields D_K and D_K^2, μ_x is a local diffeomorphism. Thus, we can find a function $l\colon S \to \mathbb{R}^{1|1}_{>0}$ which is minimal, pointwise in S, among those l such that $\mu(l, x) = x$. Therefore, μ_x factors through a diffeomorphism

$$K_l = (\mathbb{R}^{1|1} \times S)/\mathbb{Z}l \to K. \qquad \square$$

Remark 24 At least locally, an S–family in $\mathfrak{K}$ is determined, up to isomorphism, by a conjugacy class in $(\mathbb{R}^{1|1}_{>0})_S$. However, an actual "length" function $l\colon S \to \mathbb{R}^{1|1}_{>0}$ is extra information, determined for instance by a basepoint (ie a section of the submersion $K \to S$). In particular, the coarse moduli space of euclidean supercircles is not a representable supermanifold.

Each conjugation-invariant (generalized) submanifold of $\mathbb{R}^{1|1}_{>0}$ gives rise to a full substack of $\mathbb{R}^{1|1}_{>0}/\!\!/(\mathbb{Z}/2 \ltimes \mathbb{R}^{1|1})$, and therefore to a full substack of $\mathfrak{K}$. Here we are interested in the choice $\{1\} \subset \mathbb{R}^{1|1}_{>0}$, and we let $\mathfrak{K}_1 \subset \mathfrak{K}$ denote the substack of supercircles of length 1. Recall also the definition of $\mathfrak{B}^{\mathbb{T}} = \mathfrak{B}^{\mathbb{T}}(\mathrm{pt})$ from Section 3.2.

Theorem 25 *There in an equivalence of stacks* $\mathfrak{K}_1 \cong \mathfrak{B}^{\mathbb{T}}$*.*

Proof The fibered functor of Proposition 23 factors through an equivalence

$$\mathrm{pt}/\!\!/(\mathbb{Z}/2 \ltimes \mathbb{T}^{1|1}) \to \mathfrak{K}_1.$$

On the other hand, consider the S–point of $\mathfrak{B}^{\mathbb{T}}$ determined by the trivial family $\Sigma = S \times \mathbb{R}^{0|1}$ and the trivial bundle $P = \Sigma \times \mathbb{T}$ with the standard connection $\omega = dt - \theta\, d\theta$. It is easy to see, using Proposition 20, that the automorphism group of $(P, \Sigma) \in \mathfrak{B}^{\mathbb{T}}_S$ is precisely $\mathrm{Isom}(\mathbb{T}^{1|1})$. This determines a fully faithful fibered functor

$$\mathrm{pt}/\!\!/(\mathbb{Z}/2 \ltimes \mathbb{T}^{1|1}) \to \mathfrak{B}^{\mathbb{T}}.$$

It only remains to check that it is also essentially surjective. For contractible S and any object of $\mathfrak{B}^{\mathbb{T}}_S$, we can assume the underlying bundles $\Sigma \to S$ and $P \to \Sigma$ are trivial. So we just need to prove that the connection ω on P can be taken to be the standard one.

In general, a (fiberwise) connection on P can be written as $\omega = dt + (f_1 + f_0\theta)\, d\theta$ for functions $f_i \in C^\infty(S)$ of parity i. The curvature condition imposes that $f_0 = -1$. Under the gauge transformation of $P = S \times \mathbb{R}^{0|1} \times \mathbb{T}$ given by $(s, \theta, t) \mapsto (s, \theta, t - f_1(s)\theta)$, the connection ω pulls back to the standard $dt - \theta\, d\theta$. □

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Max Planck Institute for Mathematics
Bonn, Germany

`astoffel@mpim-bonn.mpg.de`

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