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Constructing the virtual fundamental class of a Kuranishi atlas

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Consider a space X, such as a compact space of J–holomorphic stable maps, that is the zero set of a Kuranishi atlas. This note explains how to define the virtual fundamental class of X by representing X via the zero set of a map $\mathscr{S}_M\colon M \to E$, where E is a finite-dimensional vector space and the domain M is an oriented, weighted branched topological manifold. Moreover, $\mathscr{S}_M$ is equivariant under the action of the global isotropy group Γ on M and E. This tuple $(M, E, \Gamma, \mathscr{S}_M)$ together with a homeomorphism from $\mathscr{S}_M^{-1}(0)/\Gamma$ to X forms a single finite-dimensional model (or chart) for X. The construction assumes only that the atlas satisfies a topological version of the index condition that can be obtained from a standard, rather than a smooth, gluing theorem. However, if X is presented as the zero set of an sc–Fredholm operator on a strong polyfold bundle, we outline a much more direct construction of the branched manifold M that uses an sc–smooth partition of unity.

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1 Introduction

1.1 Statement of main results

Let X be a compact space that is locally the zero set of a Fredholm operator $\mathcal{F}$ of index d, such as a moduli space of J–holomorphic stable curves. The question of how to define its fundamental class is central to symplectic geometry, since so much information about the properties of this geometry depends on the ability to "count" the number of elements in X. There are many possible approaches to this problem, eg Fukaya and Ono [3], Hofer [5], Hofer, Wysocki and Zehnder [6] and Tehrani and Fukaya [16]. In this note we develop the work of McDuff and Wehrheim [13; 14; 12] and Pardon [15] that uses atlases, in an attempt to clarify the passage from atlas to virtual fundamental class.

A d–dimensional atlas consists of a family of charts $\boldsymbol{K}_I$ indexed by subsets $I \subset \{1, \dots, N\} =: A$, together with coordinate changes $\widehat{\Phi}_{IJ}$ for $I \subset J$, where the chart $\boldsymbol{K}_I$ is a tuple

$$\boldsymbol{K}_I = (U_I, E_I, \Gamma_I, s_I, \psi_I),$$

consisting of a manifold U_I of dimension $d + \dim E_I$, a real vector space E_I, actions of the group Γ_I on U_I and on E_I, a Γ_I–equivariant map $s_I \colon U_I \to E_I$, and finally the footprint map $\psi_I \colon s_I^{-1}(0) \to X$ that induces a homeomorphism from $(s_I^{-1}(0))/\Gamma_I$ onto an open subset F_I of X. The charts $\boldsymbol{K}_i$ that are indexed by sets $\{i\}$ of length one are called basic charts, and we assume that their footprints $(F_i)_{1 \le i \le N}$ cover X, while the other charts $\boldsymbol{K}_I$ with $|I| > 1$ form transition data. In applications, the corresponding vector spaces E_i cover the cokernel of the Fredholm operator $\mathcal{F}$ at the points in the footprint $F_i \subset X$, and are called obstruction spaces because they obstruct the existence of solutions when $\mathcal{F}$ is deformed. The essence of the problem lies in trying to assemble these local finite-dimensional models for X into one structure that retains enough information to determine its fundamental class, which (when $d = 0$) one can think of as the number of solutions of a "generic" perturbation of $\mathcal{F}$.

The paper [12] explains one way to use a d–dimensional oriented atlas to define a Čech homology class $[X]^{\mathrm{vir}}_{\mathcal{K}} \in \check{H}_d(X; \mathbb{Q})$. Roughly speaking, the idea is this. Using the coordinate changes to identify different domains, one constructs a metrizable, Hausdorff space $|\mathcal{K}| = \bigcup_I U_I/{\sim}$ that supports a (generalized) orbibundle $|\boldsymbol{E}_{\mathcal{K}}| \to |\mathcal{K}|$ with a canonical section $|\mathfrak{s}| \colon |\mathcal{K}| \to |\boldsymbol{E}_{\mathcal{K}}|$ together with a natural identification

$$\iota_X \colon X \xrightarrow{\cong} |\mathfrak{s}|^{-1}(0).$$

With some difficulty, one then defines a multivalued perturbation section $|\nu|\colon |\mathcal{V}| \to |\boldsymbol{E}_{\mathcal{K}}|$, on a subset $|\mathcal{V}| \subset |\mathcal{K}|$, such that $|\mathfrak{s}+\nu|$ is transverse to 0. Finally, one shows that the perturbed zero set $|\mathfrak{s}+\nu|^{-1}(0)$ represents a unique element in $\check{H}_d(X;\mathbb{Q})$.

Because it uses the notion of transversality, the above construction requires that the atlas have some smoothness properties.[1] In particular, the transition maps between charts must satisfy the so-called tangent bundle (or index) condition. On the other hand, Pardon [15] introduces a new way to extract topological information from an atlas that satisfies a topological version of this condition that he calls the submersion axiom. Instead of gluing the chart domains together to form a topological space $|\mathcal{K}|$, Pardon works with K–homotopy sheaves of (co)chain complexes defined on homotopy colimits of spaces that are obtained from the chart domains. This gives a flexible way of assembling local homological information into a global object. Though this approach may be useful in many contexts, it is hard for a nonexpert in sheaf theory to understand where the technical difficulties are, and what actually has to be checked to ensure that the method works in any particular case. This becomes an issue if one wants to extend the method to cases (such as Hamiltonian Floer theory, or symplectic field theory) in which one must deal with a family of related moduli spaces and so should work on the chain level. The current paper was prompted by the desire to develop a different approach, that would replace Pardon's sophisticated sheaf theory by more elementary arguments that yet do not require smoothness.

This note only considers the simplest case, appropriate to Gromov–Witten theory, in which the aim is to construct a homology class $[X]^{\mathrm{vir}}_{\mathcal{K}} \in \check{H}_d(X;\mathbb{Q})$. Working with Pardon's submersion axiom, we define a consistent thickening of the domains of the atlas charts to make them all have the same dimension. In the case with trivial isotropy, one thereby constructs an oriented topological manifold M of dimension $D := d + \dim E_A$, together with a map $\mathscr{S}_M\colon M \to E_A$ whose zero set can be identified with X. If the isotropy is nontrivial, M is a branched manifold with a weighting function Λ and a global action of the total isotropy group Γ_A, and there is a homeomorphism $\mathscr{S}_M^{-1}(0)/\Gamma_A \xrightarrow{\cong} X$.[2] (A typical example of such a manifold (M,Λ) is the union of two circles, each of weight $\frac{1}{2}$, identified along a closed subarc A, so that the points

[1] See Castellano [1; 2] for a weak form of these requirements.

[2] Another way to say this is that $M := |\widehat{\boldsymbol{M}}|_{\mathcal{H}}$ is the Hausdorff realization of a topological groupoid $\widehat{\boldsymbol{M}}$ that is étale but not proper; see Sections 1.2 and 1.3 for relevant definitions. However, just as in the case of the construction of the zero set in [12], it is most natural to construct a topological category $\boldsymbol{M}$ in which not all morphisms are invertible, ie it is a monoid, rather than a groupoid.

$x \in A$ have weight $\Lambda(x) = 1$, while the others all have weight $\Lambda(x) = \frac{1}{2}$. See also Section 1.4.)

Here is the first main result. (See Theorem 1.3.4 for a more precise statement.)

Theorem A *Let $\mathcal{K}$ be a d–dimensional Kuranishi atlas on a compact space X that satisfies the submersion condition* (1.2.3) *and has total obstruction space $E_A := \prod_{i\in A} E_i$ and total isotropy group $\Gamma_A := \prod_{i\in A}\Gamma_i$. Let $D = d + \dim E_A$. Then there is an associated weighted branched D–dimensional manifold (M,Λ) with an action of Γ_A, and a Γ_A–equivariant map $\mathscr{S}_M\colon M \to E_A$ with a compact zero set $\mathscr{S}_M^{-1}(0)$. Moreover, there is a map $\psi\colon \mathscr{S}_M^{-1}(0) \to X$ that induces a homeomorphism $\mathscr{S}_M^{-1}(0)/\Gamma_A \xrightarrow{\cong} X$.*

It is immediate from the construction that the bordism class of a neighborhood of $\mathscr{S}_M^{-1}(0)$ in M depends only on the concordance class of $\mathcal{K}$.[3] Further, if $\mathcal{K}$ and hence (M,Λ) is oriented, we show in Lemma 2.3.4 that (M,Λ) carries a fundamental class μ_M in rational Čech homology $\check{H}_*$. Hence, we have the following:

Theorem B *If $\mathcal{K}$ is an oriented atlas on X as above, there is a unique element $[X]^{\mathrm{vir}}_{\mathcal{K}} \in \check{H}_d(X;\mathbb{Q})$ that is defined as follows. For $b \in \check{H}^d(X;\mathbb{Q})$ and $D = d + \dim E_A$, we have*

$$\langle [X]^{\mathrm{vir}}_{\mathcal{K}}, b\rangle := (\mathscr{S}_M)_*(\widehat{b}) \in \check{H}_{\dim E_A}(E_A, E_A \setminus \{0\};\mathbb{Q}) \cong \mathbb{Q}, \tag{1.1.1}$$

where $\widehat{b}$ is the image of b under the composite

$$\check{H}^d(X;\mathbb{Q}) \xrightarrow{\psi^*} \check{H}^d(\mathscr{S}_M^{-1}(0);\mathbb{Q}) \xrightarrow{\mathcal{D}} \check{H}_{\dim E_A}(M, M \setminus \mathscr{S}_M^{-1}(0);\mathbb{Q}),$$

and $\mathcal{D}$ is given by cap product with the fundamental class μ_M. Moreover, $[X]^{\mathrm{vir}}_{\mathcal{K}}$ depends only on the oriented concordance class of $\mathcal{K}$, and in the smooth case agrees with the class defined in [12].

A key element of the proof of Theorem A is Pardon's notion of *deformation to the normal cone*, which allows one to assemble different chart domains into a family of topological manifolds Y_J, albeit ones of the wrong dimension; see Proposition 2.1.1. The second key point is the existence of compatible collars for these manifolds Y_J. Remark 1.3.6 outlines the proof in more detail.

As we explain in Remark 2.2.5, if we start with a smooth atlas then the proofs of the above results can be somewhat simplified. In particular, by McDuff [9] we can

[3] Two atlases $\mathcal{K}^0$ and $\mathcal{K}^1$ on X are said to be concordant if there is an atlas $\mathcal{K}^{01}$ on $[0,1]\times X$ whose restriction to $\{\alpha\}\times X$ is $\mathcal{K}^\alpha$ for $\alpha = 0, 1$; see [13, Definition 4.1.6]. Note also that as here, when there is no danger of confusion, we often abbreviate "Kuranishi atlas" to "atlas".

construct M to be a simplicial complex, so that there is no need to use so much rational Čech homology when proving Theorem B. Further, if one works with polyfolds, then the proof can be radically simplified. Indeed, it is not difficult to define a smooth Kuranishi atlas on any space X that appears as the (compact) zero set of a polyfold bundle; see Hofer [5], Hofer, Wysocki and Zehnder [6], Yang [17] and McDuff and Wehrheim (work in progress). Because the polyfolds of Gromov–Witten theory support sc–smooth partitions of unity, if the isotropy is trivial, one can even define such an atlas with just one chart. In other words, one obtains a finite-dimensional model

$$(U, \mathbb{R}^N, s, \psi), \quad \psi\colon s^{-1}(0) \xrightarrow{\cong} X,$$

for the whole of X, in which U is a smooth manifold of dimension $d+N$ and $s\colon U \to \mathbb{R}^N$ is a smooth map. As we show in Remark 1.3.8, this construction can be adapted in the presence of isotropy. However, the domain of the single chart is no longer a manifold, but a branched manifold with action of the total isotropy group Γ_A.

Another simple example is the calculation of the Euler class of an oriented vector bundle $\pi\colon \mathcal{E} \to X$ of rank $2k$ over a compact manifold X. If $\mathcal{E}' \to X$ is an oriented complement to $\mathcal{E}$ of rank 2ℓ such that there is a vector bundle isomorphism $\phi\colon \mathcal{E} \oplus \mathcal{E}' \cong \mathbb{R}^N \times X$, where $N = 2k + 2\ell$, let

$$M = \mathcal{E}', \qquad \mathscr{S}\colon M \to \mathbb{R}^N, \quad (e', x) \mapsto \mathrm{pr}_{\mathbb{R}^N}(\phi(e', x)). \tag{1.1.2}$$

Then $\mathscr{S}^{-1}(0) \cong X$, and it is easy to check that the class $[X]^{\mathrm{vir}}_{\mathcal{K}}$ defined by (1.1.1) is Poincaré dual to the Euler class of $\mathcal{E} \to X$; see Lemma 1.4.1. This is an instance of the construction in Pardon [15, Defition 5.3.1] for the bundle $\pi\colon \mathcal{E} \to X$ with section $\mathfrak{s} \equiv 0$ in which the thickening $\lambda\colon \mathbb{R}^N \times X \to \mathcal{E}'$ is given by the projection.

Finally note that the methods of this paper should extend, eg to a more general notion of atlas, or to spaces more general than topological manifolds; see Remark 1.3.7.

1.2 Basic definitions and facts about atlases

A *weak Kuranishi atlas* $\mathcal{K}$ of dimension d on a compact metrizable space X consists of the following data:[4]

[4]These are essentially the same definitions as in [12], except that the smoothness requirements mentioned in Remarks 1.2.1(ii) below have been replaced by an equivariant version of Pardon's submersion axiom. The notion of topological atlas introduced in [13] is somewhat different; in particular the domains there need not be manifolds. For more details on all topics mentioned in this section, see the original papers [13; 14; 12] or McDuff [10].

- **Footprint cover** A finite open cover of X by nonempty sets $(F_i)_{i\in A}$.
- A *poset* $\mathcal{I}_{\mathcal{K}} = \{I \subset A \mid F_I := \bigcap_{i\in I} F_i \neq \varnothing\}$ that indexes the charts.
- **Charts** For all $I \in \mathcal{I}_{\mathcal{K}}$, F_I is the footprint of a chart $\boldsymbol{K}_I := (U_I, \Gamma_I, E_I, s_I, \psi_I)$, where
  - U_I is a finite-dimensional topological manifold of dimension $d + \dim E_I$;
  - $E_I := \prod_{i\in I} E_i$ is a product of even-dimensional[5] vector spaces such that $\dim U_I - \dim E_I = d$;
  - $\Gamma_I = \prod_{i\in I} \Gamma_i$ is a product of finite groups that acts on U_I, and acts by a product of linear actions on E_I;
  - $s_I \colon U_I \to E_I$ is a Γ_I–equivariant map;
  - the footprint map $\psi_I \colon s_I^{-1}(0) \to X$ induces a homeomorphism

$$s_I^{-1}(0)/\Gamma_I \xrightarrow{\cong} F_I. \tag{1.2.1}$$

- **Coordinate changes** If $I \subset J$ there is a coordinate change $\widehat{\Phi}_{IJ} \colon \boldsymbol{K}_I \to \boldsymbol{K}_J$ given by the following data, where we identify E_I as a subspace of E_J in the obvious way:
  - a relatively open, Γ_J–invariant subset $\widetilde{U}_{IJ}$ of $s_J^{-1}(E_I) \subset U_J$ containing $s_J^{-1}(0)$ and with a free action of $\Gamma_{J\setminus I}$;
  - a covering map $\rho_{IJ} \colon \widetilde{U}_{IJ} \to U_I$ that quotients out by the (free) action of $\Gamma_{J\setminus I}$ and is equivariant with respect to the projection $\Gamma_J \to \Gamma_I$, and, further,
$$s_I \circ \rho_{IJ} = s_J, \qquad \psi_J = \psi_I \circ \rho_{IJ} \quad \text{on } s_J^{-1}(0) \subset \widetilde{U}_{IJ};$$
  - if $I \subset J \subset K$, then
$$\rho_{IK} = \rho_{IJ} \circ \rho_{JK} \tag{1.2.2}$$
whenever both sides are defined;
  - in an *atlas* (rather than a weak atlas) we require in addition that the domain $\rho_{JK}^{-1}(\widetilde{U}_{IJ}) \cap \widetilde{U}_{JK}$ of $\rho_{JK} \circ \rho_{IJ}$ is a subset of the domain $\widetilde{U}_{IK}$ of ρ_{IK};
  - in a *tame atlas* we require that both sides of (1.2.2) have the same domain and that $\widetilde{U}_{IJ} = s_J^{-1}(E_I)$.

[5] For simplicity, we assume that E_i is even-dimensional, so that the orientation of a product of the E_i does not depend on their order. In the Gromov–Witten situation we may always choose the E_i to have a natural complex structure since the target of the linearized Cauchy–Riemann operator is a complex vector space of $(0,1)$–forms.

- **Equivariant submersion condition** For each $I \subset J$, each point $x \in \widetilde{U}_{IJ} \subset U_J$ has a product neighborhood that is compatible with the section $s_{J \smallsetminus I}$; more precisely, for each such x with stabilizer subgroup $\Gamma_x \subset \Gamma_I$, there is a Γ_x–equivariant local homeomorphism of the form

$$\phi^E_x \colon (E_{J \smallsetminus I, \delta} \times W_x, \{0\} \times W_x) \to (U_J, \widetilde{U}_{IJ}), \tag{1.2.3}$$

where $E_{J \smallsetminus I, \delta}$ is a δ–neighborhood of 0 in $E_{J \smallsetminus I}$ and W_x is a Γ_x–invariant neighborhood of x in $\widetilde{U}_{IJ}$, such that

$$s_{J \smallsetminus I} \circ \phi^E_x(e, y) = e, \quad e \in E_{J \smallsetminus I, \delta}.$$

Remarks 1.2.1 (i) Although the submersion axiom in [15] does not assume equivariance, this is needed in our set-up in order that M support an action of Γ_A. Notice that because $\Gamma_{J \smallsetminus I}$ acts freely on $\widetilde{U}_{IJ}$, the stabilizer Γ_x of $x \in \widetilde{U}_{IJ}$ lies in the subgroup Γ_I of $\Gamma_J \cong \Gamma_I \times \Gamma_{J \smallsetminus I}$. The standard proof of the submersion axiom for Gromov–Witten moduli spaces adapts easily to yield Γ_x–equivariance because it is an application of the gluing theorem at the stable map x. The process of gluing depends on various choices, for example of Riemannian metrics and of the complement to the image of the linearized Cauchy–Riemann operator at x, and these can always be chosen invariant under the finite stabilizer subgroup of x. This equivariance is built into the smooth index condition, since the latter is expressed in terms of the equivariant section maps $s_{J \smallsetminus I}$.

(ii) **The smooth case** In this case the manifolds U_I are assumed to be smooth, all structural maps (the group action on U_I, the section s_I and coordinate changes ρ_{IJ}) are smooth, and the submersion axiom is replaced by the requirement that $\widetilde{U}_{IJ}$ be a submanifold of U_J such that

(1.2.4) the derivative of $s_{J \smallsetminus I} \colon U_J \to E_{J \smallsetminus I}$ induces an isomorphism from the normal bundle of $\widetilde{U}_{IJ}$ in U_J to $E_{J \smallsetminus I} \times \widetilde{U}_{IJ}$.

In this case we claim that each of the maps τ_{IJ} in Proposition 1.3.3 can be chosen to be a local diffeomorphism onto its image, so that M is a smooth manifold if the isotropy is trivial, and otherwise is a smooth branched manifold. The construction of such an M is sketched in Remark 2.2.5.

(iii) **Orientations** We will consider an atlas to be oriented if each domain U_I (and each obstruction space E_I) has a Γ_I–invariant orientation that is respected by the coordinate changes. In fact, in the current situation, since we have assumed that the E_i

are all even-dimensional (eg that they are all complex vector spaces), then if they are also invariantly oriented, the E_I inherit natural orientations, and the local product structure given by the submersion condition permits the transfer of an orientation between charts. In the smooth case, a slightly more general notion of orientation is discussed extensively in [14; 12]. $\diamond$

We now briefly recall some other terminology that will be useful later. An atlas $\mathcal{K}' = (\boldsymbol{K}'_I, \widehat{\Phi}'_{IJ})$ is a *shrinking* of $\mathcal{K} = (\boldsymbol{K}_I, \widehat{\Phi}_{IJ})$ if

- it has the same index set $\mathcal{I}_{\mathcal{K}}$, obstruction spaces E_I and groups Γ_I,
- each chart domain U'_I is a precompact subset of U_I, written $U'_I \sqsubset U_I$,
- the coordinate changes are given by restriction.

For short, in this situation we write

$$\mathcal{U}' \sqsubset \mathcal{U}, \qquad \text{where} \quad \mathcal{U}' := \bigsqcup_{I \in \mathcal{I}_{\mathcal{K}}} U'_I, \quad \mathcal{U} := \bigsqcup_{I \in \mathcal{I}_{\mathcal{K}}} U_I. \tag{1.2.5}$$

It is shown in [13, Section 3.3; 12, Section 2.5] that every weak atlas has a tame shrinking $\mathcal{K}' \sqsubset \mathcal{K}$ that is unique up to a natural equivalence relation called concordance. A tame atlas $\mathcal{K}$ is called *preshrunk* if there is a double shrinking $\mathcal{K} \sqsubset \mathcal{K}' \sqsubset \mathcal{K}''$ such that both $\mathcal{K}$ and $\mathcal{K}'$ are tame.

Each atlas[6] $\mathcal{K}$ determines a topological category $\boldsymbol{B}_{\mathcal{K}}$ with

$$\operatorname{Obj}_{\boldsymbol{B}_{\mathcal{K}}} = \bigsqcup_{I \in \mathcal{I}_{\mathcal{K}}} U_I, \quad \operatorname{Mor}_{\boldsymbol{B}_{\mathcal{K}}} = \bigsqcup_{I \subset J} \widetilde{U}_{IJ} \times \Gamma_I, \tag{1.2.6}$$

$$s \times t \colon \operatorname{Mor}_{\boldsymbol{B}_{\mathcal{K}}} \to \operatorname{Obj}_{\boldsymbol{B}_{\mathcal{K}}} \times \operatorname{Obj}_{\boldsymbol{B}_{\mathcal{K}}}, \quad (I, J, y, \gamma) \mapsto \bigl(I, \gamma^{-1}(\rho_{IJ}(x)), (J, y)\bigr).$$

We denote by $|\mathcal{K}| := |\boldsymbol{B}_{\mathcal{K}}|$ its (geometric or naive) realization. Thus,

$$|\boldsymbol{B}_{\mathcal{K}}| := \bigsqcup_I U_I / {\sim},$$

where $\sim$ is the equivalence relation on $\operatorname{Obj}_{\boldsymbol{B}_{\mathcal{K}}}$ that is generated by the morphisms, ie $(I, x) \sim (J, y)$ if and only if there is a chain of morphisms

$$(I, x) = (I_0, x_1) \to (I_1, x_1) \leftarrow (I_2, x_2) \to \cdots \leftarrow (I_k, x_k) = (J, y).$$

[6] The extra assumption in the definition of atlas stated just after (1.2.2) implies that the set $\operatorname{Mor}_{\boldsymbol{B}_{\mathcal{K}}}$ defined below is closed under composition.

Though for a general atlas the quotient topology is non-Hausdorff, it is shown in [13, Theorem 3.1.9] (see also [12, Section 2.5]) that if $\mathcal{K}$ is preshrunk and tame, the quotient topology is Hausdorff and the natural maps

$$\pi_{\mathcal{K}}\colon U_I \to |\mathcal{K}| \tag{1.2.7}$$

induce homeomorphisms from U_I / Γ_I onto their images. Further, the quotient topology on $|\mathcal{K}'|$ restricts to a metrizable topology on $|\mathcal{K}|$ that agrees with the quotient topology on each set $\pi_{\mathcal{K}}(U_I)$. We will say that $\mathcal{K}$ is *good* if its realization $|\mathcal{K}|$ has these properties.[7]

From now on we assume that $\mathcal{K}$ is good in this sense, eg preshrunk and tame.

There is a similar category $\boldsymbol{E}_{\mathcal{K}}$ formed by the obstruction bundles with

$$\mathrm{Obj}_{\boldsymbol{E}_{\mathcal{K}}} = \bigsqcup_{I\in\mathcal{I}_{\mathcal{K}}} U_I \times E_I, \quad \mathrm{Mor}_{\boldsymbol{B}_{\mathcal{K}}} = \bigsqcup_{I\subset J} \widetilde{U}_{IJ} \times E_I \times \Gamma_I,$$

$$s\times t\colon \mathrm{Mor}_{\boldsymbol{E}_{\mathcal{K}}} \to \mathrm{Obj}_{\boldsymbol{E}_{\mathcal{K}}} \times \mathrm{Obj}_{\boldsymbol{E}_{\mathcal{K}}}, \quad (I, J, y, e, \gamma) \mapsto \big((I, \gamma^{-1}(\rho_{IJ}(y), e)), (J, y, e)\big).$$

The projections $\mathrm{pr}_I\colon U_I \times E_I \to U_I$, sections s_I and footprint maps ψ_I fit together to give functors

$$\mathrm{pr}\colon \boldsymbol{E}_{\mathcal{K}} \to \boldsymbol{B}_{\mathcal{K}}, \quad \mathfrak{s}\colon \boldsymbol{B}_{\mathcal{K}} \to \boldsymbol{E}_{\mathcal{K}}, \quad \psi\colon \mathfrak{s}^{-1}(0) \to \boldsymbol{X},$$

where $\boldsymbol{X}$ is the category with objects X and only identity morphisms, and one can show that ψ induces a homeomorphism $|\psi|\colon |\mathfrak{s}|^{-1}(0) \to X$.

Reductions and zero sets The situation when all the obstruction spaces E_I vanish is considered in McDuff [11]. In this case, the category $\boldsymbol{B}_{\mathcal{K}}$ is

- *étale*, ie the object and morphism spaces are manifolds, and the source and target maps are local homeomorphisms, and
- *proper*, ie the equivalence relation $\sim$ on the object space generated by the morphisms is closed.[8]

[7] The proof given in [13] that preshrunk and tame atlases are good is abstract, ie the argument only uses properties of the objects and maps in the category $\boldsymbol{B}_{\mathcal{K}}$. However, because the atlas domains are often constructed as subsets of an ambient Hausdorff metrizable space $\mathcal{S}$ (such as a space of stable maps), one can sometimes use the existence of $\mathcal{S}$ to bypass some of the arguments in [13].

[8] If $\mathrm{Obj}_{\boldsymbol{B}}$ is a separable, locally compact metric space (as is the case for the categories considered in this paper), then this properness condition implies that the realization $|\boldsymbol{B}|$ is Hausdorff; for a proof see [13, Lemma 3.2.4]. If in addition $\boldsymbol{B}$ is a groupoid, then this condition is equivalent to the more standard requirement that the map $s\times t\colon \mathrm{Mor}_{\boldsymbol{B}} \to \mathrm{Obj}_{\boldsymbol{B}} \times \mathrm{Obj}_{\boldsymbol{B}}$ is proper.

Moreover, by [11, Proposition 2.3] it has a natural completion to an EP (étale, proper) groupoid $\widehat{\boldsymbol{B}}_{\mathcal{K}}$ (ie a category in which all morphisms are invertible) that also has realization $|\mathcal{K}|$. Thus, $\widehat{\boldsymbol{B}}_{\mathcal{K}}$ provides an orbifold structure on $|\mathcal{K}|$.

If the obstruction spaces do not vanish, then the manifolds U_I have varying dimensions. However, if $\nu_I\colon U_I \to E_I$ is a perturbation section such that $s_I + \nu_I\colon U_I \to E_I$ is transverse to 0, then the perturbed zero set $Z_I := (s_I + \nu_I)^{-1}(0)$ has fixed dimension d. Hence, as is shown in [14, Lemma 7.2.7], if the isotropy groups vanish and if we can choose the ν_I compatibly, ie they form a functor

$$\nu\colon \boldsymbol{B}_{\mathcal{K}} \to \boldsymbol{E}_{\mathcal{K}},$$

then these zero sets fit together to form a manifold. However, in general the domains U_I overlap too much for there to be such a functor.[9]

We deal with this by passing to a *reduction* $\mathcal{V}$, ie a family of Γ_I–invariant, precompact open subsets $V_I \sqsubset U_I$ with the following properties:

(1.2.8)
- the footprints $\big(G_I := \psi_I(V_I \cap s_I^{-1}(0))\big)_{I \in \mathcal{I}_{\mathcal{K}}}$ cover X,
- $\pi_{\mathcal{K}}(\overline{V}_I) \cap \pi_{\mathcal{K}}(\overline{V}_J) \neq \varnothing$ only if $I \subset J$ or $J \subset I$,

where $\pi_{\mathcal{K}}\colon U_I \to |\mathcal{K}|$ is the projection in (1.2.7). In the construction given in Section 7.3 of [14] for the trivial isotropy case, we define the perturbation section as a functor

$$\nu\colon \boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}} \to \boldsymbol{E}_{\mathcal{K}}|_{\mathcal{V}}$$

on the full subcategory $\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}$ of $\boldsymbol{B}_{\mathcal{K}}$ with objects $\bigsqcup_I V_I$.

If the isotropy groups are nontrivial then it is (in general) no longer possible to choose a transverse equivariant section ν, even on a reduction $\mathcal{V}$. However, because the morphisms in $\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}$ are described so explicitly, we show in [12, Proposition 3.3.3] that we may construct the perturbation section as a (single-valued) functor

$$\nu\colon \boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma} \to \boldsymbol{E}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma},$$

where $\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}$ is the (nonfull, nonproper) subcategory of $\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}$ obtained by discarding the morphisms coming explicitly from the group actions. Thus,

$$\operatorname{Mor}_{\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\setminus\Gamma}} = \bigsqcup_{I \subset J} \widetilde{V}_{IJ}, \tag{1.2.9}$$

[9] See the beginning of [13, Section 7.1]. The relation between $\mathcal{U}$ and its reduction $\mathcal{V}$ is similar to that between the cover of a simplicial space by the stars of its vertices and the cover by the stars of its first barycentric subdivision. In particular, though the footprints $(F_i)_{1 \le i \le N}$ of the basic charts cover X, the corresponding sets $(G_i := F_i \cap |V_i|)_{1 \le i \le N}$ are disjoint and do not form a cover; see Figure 1.

where

$$s \times t\colon (I, J, y) \mapsto \bigl((I, \rho_{IJ}(y)), (J, y)\bigr) \quad \text{and} \quad \widetilde{V}_{IJ} = V_J \cap \rho_{IJ}^{-1}(V_I) \subset \widetilde{U}_{IJ}.$$

We show in [12, Theorem 3.2.8] that if $(s+\nu) \pitchfork 0$, the full subcategory of $\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\smallsetminus\Gamma}$ with objects

$$\bigsqcup_I (Z_I := (s_I + \nu_I)^{-1}(0))$$

with weights

$$\mathrm{wt}(Z_I) = 1/|\Gamma_I|$$

can be completed to a weighted étale groupoid whose realization is therefore a weighted branched manifold as defined in Section 1.3. We will see below that in the current context the branched manifold structure of M appears in a similar way.

1.3 The weighted branched manifold (M, Λ)

We will construct M from the realization of an étale category $\boldsymbol{M}$ whose objects are thickened versions of the domains V_I of a reduction $\mathcal{V}$ of the atlas $\mathcal{K}$, and whose morphisms have exactly the same structure as those in the category $\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\smallsetminus\Gamma}$ defined in (1.2.9). In particular, in general $\boldsymbol{M}$ is not proper, so that its realization $|\boldsymbol{M}|$ is not Hausdorff but rather branches along its locus of non-Hausdorff points (think of two copies of a circle attached along a subarc.)

We begin with some relevant definitions from [9]. If $\boldsymbol{G}$ is a wnb groupoid as described below, its realization $|\boldsymbol{G}|$ with the quotient topology is in general not Hausdorff. Hence, we consider its maximal Hausdorff quotient $|\boldsymbol{G}|_{\mathcal{H}}$, which has the following universal property: any continuous map from $|\boldsymbol{G}|$ to a Hausdorff space factors through $|\boldsymbol{G}|_{\mathcal{H}}$. In the following we write $|\boldsymbol{G}|$ for the realization $\mathrm{Obj}_{\boldsymbol{G}}/\sim$ of an étale groupoid $\boldsymbol{G}$, and $|\boldsymbol{G}|_{\mathcal{H}}$ for its maximal Hausdorff quotient.[10] We denote the natural maps by

$$\pi_{\boldsymbol{G}}\colon \mathrm{Obj}_{\boldsymbol{G}} \to |\boldsymbol{G}|, \quad \pi^{\mathcal{H}}_{|\boldsymbol{G}|}\colon |\boldsymbol{G}| \to |\boldsymbol{G}|_{\mathcal{H}}, \quad \pi^{\mathcal{H}}_{\boldsymbol{G}} := \pi^{\mathcal{H}}_{|\boldsymbol{G}|} \circ \pi_{\boldsymbol{G}}\colon \mathrm{Obj}_{\boldsymbol{G}} \to |\boldsymbol{G}|_{\mathcal{H}}.$$

Definition 1.3.1 [9, Definition 3.2] A *weighted nonsingular branched groupoid* (or *wnb groupoid*) of dimension d is a pair $(\boldsymbol{G}, \Lambda_{\boldsymbol{G}})$ consisting of a nonsingular,[11] étale

[10] The appendix to [12] gives succinct proofs of the results we use; in particular, the existence of $|\boldsymbol{G}|_{\mathcal{H}}$ is established in [12, Lemma A.2]. Lemma 2.3.2 gives an explicit description of $|\boldsymbol{G}|_{\mathcal{H}}$ in the case we need here.

[11] That is, there is at most one morphism between any two objects. Further, we restrict here to rational weights, but clearly this condition could be generalized.

groupoid $\boldsymbol{G}$ of dimension d, together with a rational weighting function $\Lambda_{\boldsymbol{G}}\colon |\boldsymbol{G}|_{\mathcal{H}} \to \mathbb{Q}^+ := \mathbb{Q} \cap (0,\infty)$ that satisfies the following compatibility conditions. For each $p \in |\boldsymbol{G}|_{\mathcal{H}}$ there is an open neighborhood $N \subset |\boldsymbol{G}|_{\mathcal{H}}$ of p, a collection $U_1,\dots,U_\ell$ of disjoint open subsets of $(\pi_{\boldsymbol{G}}^{\mathcal{H}})^{-1}(N) \subset \mathrm{Obj}_{\boldsymbol{G}}$ (called *local branches*) and a set of positive rational weights $m_1,\dots,m_\ell$ such that the following holds:

- **Cover** $(\pi_{|\boldsymbol{G}|}^{\mathcal{H}})^{-1}(N) = |U_1| \cup \dots \cup |U_\ell| \subset |\boldsymbol{G}|$.
- **Local regularity** For each $i = 1,\dots,\ell$ the projection $\pi_{\boldsymbol{G}}^{\mathcal{H}}|_{U_i}\colon U_i \to |\boldsymbol{G}|_{\mathcal{H}}$ is a homeomorphism onto a relatively closed subset of N.
- **Weighting** For all $q \in N$, the number $\Lambda_{\boldsymbol{G}}(q)$ is the sum of the weights of the local branches whose image contains q:

$$\Lambda_{\boldsymbol{G}}(q) = \sum_{i : q \in |U_i|_{\mathcal{H}}} m_i.$$

Now we can formulate the notion of a weighted branched manifold.[12]

Definition 1.3.2 A *weighted branched manifold* of dimension d is a pair (Z, Λ_Z) consisting of a topological space Z together with a function $\Lambda_Z\colon Z \to \mathbb{Q}^+$ and an equivalence class[13] of tuples $(\boldsymbol{G}, \Lambda_{\boldsymbol{G}}, f)$, where $(\boldsymbol{G}, \Lambda_{\boldsymbol{G}})$ is a d–dimensional wnb groupoid and $f\colon |\boldsymbol{G}|_{\mathcal{H}} \to Z$ is a homeomorphism that induces the function $\Lambda_Z := \Lambda_{\boldsymbol{G}} \circ f^{-1}$.

We define the weighted branched manifold (M, Λ) of Theorem A as the realization of a category $\boldsymbol{M}$ constructed as follows. First choose a Γ_i–invariant norm $\|\cdot\|$ on each E_i, and for any $J \subset A$ give the vector space $E_J := \prod_{i \in J} E_i$ the sup norm

$$\|e_J\| = \sup_{i \in J} \|e_i\|.$$

Further, let

$$E_{J,\varepsilon} := \{e_J \in E_J \mid \|e_J\| < \varepsilon\}, \tag{1.3.1}$$

[12] In distinction to [9; 12], we will not assume from the outset that a weighted branched manifold is oriented, since there is no need for this hypothesis until it comes to considering the fundamental class. Analogous definitions for cobordisms may be found in [12, Appendix].

[13] The precise notion of equivalence is given in [9, Definition 3.12]. In particular, it ensures that the induced function $\Lambda_Z := \Lambda_{\boldsymbol{G}} \circ f^{-1}$ and the dimension of $\mathrm{Obj}_{\boldsymbol{G}}$ is the same for equivalent structures $(\boldsymbol{G}, \Lambda_{\boldsymbol{G}}, f)$.

and

$$\underline{\varepsilon} := (\varepsilon_I)_{I \in \mathcal{I}_{\mathcal{K}}}, \tag{1.3.2}$$

where

$$I \subsetneq J \quad \Longrightarrow \quad 0 < \kappa \varepsilon_I < \varepsilon_J \quad \text{for } \kappa := \max\{|J| \mid J \in \mathcal{I}_{\mathcal{K}}\}.$$

Given a reduction $\mathcal{V}$ of an atlas $\mathcal{K}$ as in (1.2.8), for each $I \subset J$ we let

$$V_{IJ} = V_I \cap \pi_{\mathcal{K}}^{-1}(\pi_{\mathcal{K}}(V_J)) \subset V_I, \quad \widetilde{V}_{IJ} = V_J \cap \pi_{\mathcal{K}}^{-1}(\pi_{\mathcal{K}}(V_I)) \subset V_J, \tag{1.3.3}$$

where $\pi_{\mathcal{K}}\colon V_I \to |\mathcal{K}|$ is the obvious projection. Thus, $\rho_{IJ}(\widetilde{V}_{IJ}) = V_{IJ}$. Observe also that the group Γ_A acts on $E_{A \smallsetminus J, \varepsilon_J} \times V_J$ by

$$\gamma \cdot (e, x) = (\gamma|_{A \smallsetminus J}(e), \gamma|_J(x)), \quad \gamma \in \Gamma_A, \tag{1.3.4}$$

where $\gamma|_J$ denotes the projection of $\gamma \in \Gamma_A := \prod_{i \in A} \Gamma_i$ to $\Gamma_J := \prod_{i \in J} \Gamma_i$.

The following result is the key step in the proof of Theorem A. A more precise version is stated and proved in Proposition 2.2.2 below.

Proposition 1.3.3 *Let $\mathcal{K}$ be a good atlas on X of dimension d. Then there is a reduction $\mathcal{V}$ and choice of constants $\underline{\delta} > 0$ such that the following holds.*

(i) *There is an étale category $\boldsymbol{M}$ of dimension $D := d + \dim E_A$ with*

$$\operatorname{Obj}_{\boldsymbol{M}} = \bigsqcup_{I \in \mathcal{I}_{\mathcal{K}}} M_J := E_{A \smallsetminus J, \delta_J} \times V_J, \quad \operatorname{Mor}_{\boldsymbol{M}} = \bigsqcup_{I \subset J,\, I, J \in \mathcal{I}_{\mathcal{K}}} \widetilde{M}_{IJ}, \tag{1.3.5}$$
$$s \times t\colon \operatorname{Mor}_{\boldsymbol{M}} \to \operatorname{Obj}_{\boldsymbol{M}} \times \operatorname{Obj}_{\boldsymbol{M}}, \quad (I, J, y) \mapsto \big((I, \tau_{IJ}(y)), (J, y)\big),$$

where $\widetilde{M}_{IJ} \subset M_J$ is an open Γ_A–invariant subset containing $\{0\} \times \widetilde{V}_{IJ}$ whose closure $\operatorname{cl}(\widetilde{M}_{IJ})$ is disjoint from $\operatorname{cl}(\widetilde{M}_{HJ})$ unless $I \subset H$ or $H \subset I$, and the map

$$\tau_{IJ}\colon \widetilde{M}_{IJ} \to M_{IJ} := E_{A \smallsetminus I, \delta_I} \times V_{IJ} \subset M_I$$

is a Γ_A–equivariant covering map onto $M_{IJ} \subset M_I$ such that

- *τ_{IJ} restricts to ρ_{IJ} on $\{0\} \times \widetilde{V}_{IJ}$;*
- *if $H \subset I \subset J$ then $\tau_{HJ} = \tau_{HI} \circ \tau_{IJ}$ on $\widetilde{M}_{IJ} \cap \widetilde{M}_{HJ}$, and*

$$\operatorname{graph} \tau_{IJ} \subset M_I \times M_J \quad \text{is closed.} \tag{1.3.6}$$

(ii) $\boldsymbol{M}$ *supports an action of* Γ_A *by* (1.3.4) *on objects, and by*

$$(I, J, y) \mapsto \gamma \cdot (I, J, y) := (I, J, \gamma^{-1} \cdot y), \quad \gamma \in \Gamma_A,\ y \in \widetilde{M}_{IJ},$$

on morphisms.

(iii) *There is a* Γ_A*–equivariant functor* $\mathscr{S}\colon \boldsymbol{M} \to \boldsymbol{E}_A$*, where the category* $\boldsymbol{E}_A$ *has objects* E_A *and only identity morphisms, that is given on objects by maps* $\mathscr{S}_J\colon M_J \to E_A$ *such that*

$$\mathscr{S}_J(0, x) = s_J(x), \quad \mathscr{S}_J^{-1}(E_J) \subset \{0\} \times V_J, \tag{1.3.7}$$

so that

$$(\mathscr{S}_J)^{-1}(0) = \{(0, x) \in E_{A \smallsetminus J} \times V_J : s_J(x) = 0\}.$$

The following result explains the construction and properties of the weighted branched manifold (M, Λ) mentioned in Theorem A. Note that $\mathscr{S}$ denotes a functor $\boldsymbol{M} \to \boldsymbol{E}_A$, while $\mathscr{S}_M\colon M \to E_A$ is the corresponding function on M.

Theorem 1.3.4 (i) *The category* $\boldsymbol{M}$ *constructed in Proposition 1.3.3 has a unique completion to a wnb groupoid* $\widehat{\boldsymbol{M}}$ *with the same objects as* $\boldsymbol{M}$ *and the same realization* $|\widehat{\boldsymbol{M}}| = |\boldsymbol{M}|$.

(ii) *If we denote the composite* $\mathrm{Obj}_{\boldsymbol{M}} \to |\boldsymbol{M}| \to |\widehat{\boldsymbol{M}}|_{\mathcal{H}}$ *by* $y \mapsto |y| \mapsto \pi^{\mathcal{H}}_{\boldsymbol{M}}(|y|)$*, the function* $\Lambda\colon M := |\widehat{\boldsymbol{M}}|_{\mathcal{H}} \to \mathbb{Q}^+$ *defined by*

$$\Lambda(p) := \frac{1}{|\Gamma_I|} \cdot \#\{y \in M_I \mid \pi^{\mathcal{H}}_{\boldsymbol{M}}(|y|) = p\} \quad \text{for } p \in |M_I|_{\mathcal{H}}$$

is a weighting function that gives (M, Λ) *the structure of a weighted branched manifold.*

(iii) *The group action by* Γ_A *and functor* $\mathscr{S}$ *extend to* $\widehat{\boldsymbol{M}}$*, so that there is a* Γ_A*–equivariant map* $\mathscr{S}_M\colon M \to E_A$*. Moreover, the zero set* $\mathscr{S}_M^{-1}(0)$ *is a compact subset of* M*, and the footprint maps* ψ_I *induce a homeomorphism*

$$\underline{\psi}\colon \mathscr{S}_M^{-1}(0)/\Gamma_A \xrightarrow{\cong} X.$$

(iv) *If* $\mathcal{K}$ *is oriented, so are* $\boldsymbol{M}$ *and* $\widehat{\boldsymbol{M}}$.

The category $\boldsymbol{M}$ has the same structure as the category $\boldsymbol{Z}^{\nu}$ considered in Theorem 3.2.8 of [12], formed by the perturbed zero set of the atlas $\mathcal{K}$; and the proof of Theorem 1.3.4 (which is given in Section 2.3) is essentially the same as the corresponding result for $\boldsymbol{Z}^{\nu}$.

Condition (1.3.6), that τ_{IJ} has closed graph, is automatically satisfied in the case of $\boldsymbol{Z}^\nu$, and is an important ingredient of the analysis of the branching structure of M. For example, if the isotropy groups are trivial, then the maps τ_{IJ} are homeomorphisms onto their images, and Lemma 2.3.1 implies that the only morphisms in the groupoid completion $\widehat{\boldsymbol{M}}$ are those given by the τ_{IJ} and their inverses. Hence, condition (1.3.6) implies that the equivalence relation on $\mathrm{Obj}_{\boldsymbol{M}}$ has closed graph, so that the quotient space $|\widehat{\boldsymbol{M}}|$ is Hausdorff, and therefore a manifold. An example with nontrivial isotropy is described in Example 1.4.3(IV).

Proof of Theorem A This is an immediate consequence of Theorem 1.3.4. □

Remark 1.3.5 Instead of taking M to be a weighted branched manifold with action of Γ_A, one could add the morphisms in Γ_A to the completed category $\widehat{\boldsymbol{M}}$ to obtain an étale groupoid $\widehat{\boldsymbol{M}} \times \Gamma_A$. In general, this groupoid is not proper. However, it does inherit a weighting function and so the realization $|\widehat{\boldsymbol{M}} \times \Gamma_A|_{\mathcal{H}}$ is a weighted branched orbifold M/Γ_A; for an explicit example see Example 1.4.3(VI). Note also that the action of the group Γ_A on $\boldsymbol{M}$ only affects the fundamental class $\mu_{\boldsymbol{M}}$ (and hence $[X]^{\mathrm{vir}}_{\mathcal{K}}$) via the weighting function Λ, whose values depend on the groups Γ_I as well as on the category $\boldsymbol{M}$. ◇

Remark 1.3.6 (outline of the argument) We will explain the main points of the proof of Proposition 1.3.3 in Section 2. The first step is to use "deformation to the normal cone" (see [15]) to construct manifolds $(Y_{\mathcal{U},J,\underline{\varepsilon}})_{J\in\mathcal{I}_{\mathcal{K}}}$ of dimension $d+\dim E_A+|J|-1$ with a natural boundary that lies over the boundary of a simplex Δ_J of dimension $|J|-1$. We next consider the open submanifold $Y_{\mathcal{V},J,\underline{\varepsilon}} \subset Y_{\mathcal{U},J,\underline{\varepsilon}}$ corresponding to a reduction, and show that this has a partial boundary collar with "corner control"; see Proposition 2.1.4. Then we use the collar to construct the covering maps $\tau_{IJ}\colon \widetilde{M}_{IJ} \to M_I$. Since the general definition of these maps is quite complicated, we explain in Example 2.2.1 how this works for an atlas with just three basic charts. Proposition 2.2.2 gives the general construction.

Section 3.1 contains technical details about compatible shrinkings, and the proof that each $Y_{\mathcal{U},J,\underline{\varepsilon}}$ is a manifold. The argument here is based on the existence of the local product structures provided by the submersion axiom. As we show in Step 1 of the proof of Proposition 2.1.4 in Lemma 3.2.1, this axiom also allows one to construct local collars that are compatible with the covering maps ρ_{IJ} and with projection to the vector spaces $E_{J\setminus I}$. In Step 2 of this proof we explain a standard method (described in Hatcher [4]) for assembling these local collars into a global collar for each $Y_{\mathcal{V},J,\underline{\varepsilon}}$, and show in Step 3 how to arrange that these collars have the consistency properties listed in Proposition 2.1.4 that are needed in the definition of the maps τ_{IJ}. This last step works under the assumption that the domains of the local collars are compatible with the reduction $\mathcal{V}$ and choice $\underline{\varepsilon}$ of thickening constants in a rather subtle way, which is summarized in the notion of compatible reduction $(\mathcal{V},\underline{\varepsilon})$ in Definition 3.1.9. $\diamond$

Remark 1.3.7 (generalizations) The construction of $\boldsymbol{M}$ could be generalized in various ways. The argument relies in an essential way on the submersion property in order to construct the collars in Proposition 2.1.4, ie on the fact that along $\widetilde{U}_{IJ}$ the space U_J is locally the product of the vector space $E_{J\setminus I}$ with the domain U_I. However, it does not use the fact that the domains U_I themselves are topological manifolds; for example, since all we want in the end is information on homology, it would no doubt suffice if they were (locally compact, metrizable) homology manifolds of dimension $\dim E_I + d$. One could also consider atlases (or equivalently categories $\boldsymbol{B}_{\mathcal{K}}$) whose charts are indexed by a poset more general than that given by the subsets of A. However, one does need to be able to restrict attention to a subcategory such as $\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}$ in which there are morphisms between the elements of two components of the domain only if the indices of those components are comparable in the given poset. Some possible generalizations of this kind are discussed in the last section of [10]. $\diamond$

Remark 1.3.8 (the polyfold approach) If X is the zero set of a Fredholm section $\mathfrak{s}$ of a polyfold bundle $\mathcal{E}\to\mathcal{S}$ of index d, then one can use the fact that the realization $|\mathcal{S}|$ supports partitions of unity to give a very simple construction for a weighted branched manifold M and section $\mathscr{S}$ whose corresponding relative Euler class agrees with that of $\mathfrak{s}\colon \mathcal{S}\to\mathcal{E}$. (In the applications of interest to us $\mathcal{S}$ is a category[14] whose realization is a space of stable maps with the Gromov topology; see Hofer [5] and Hofer, Wysocki and Zehnder [7].) Here is a very brief outline of the construction; for full details see McDuff and Wehrheim (work in progress).

Given $x\in X$ with stabilizer subgroup Γ_x, choose a lift $q_x\in \mathrm{Obj}_{\mathcal{S}}$, and a Γ_x–invariant open neighborhood $\mathcal{O}\subset \mathrm{Obj}_{\mathcal{S}}$ of q_x such that the map $\mathcal{O}\to|\mathcal{O}|\subset|\mathcal{S}|$ factors through a homeomorphism $\mathcal{O}/\Gamma_x \xrightarrow{\cong} |\mathcal{O}|$. Because $\mathfrak{s}$ is Fredholm, there is a Γ_x–equivariant linear map $\lambda\colon E\to \mathrm{Sect}(\mathcal{E}|_{\mathcal{O}})$ from a Γ_x–invariant normed linear space E to a subspace

[14] One can think of $\mathcal{S}$ as an infinite-dimensional version of an EP groupoid, where the objects $\mathrm{Obj}_{\mathcal{S}}$ do not form a set but nevertheless the quotient $|\mathcal{S}| = \mathrm{Obj}_{\mathcal{S}}/{\sim}$ is a topological space, where $\sim$ is defined by setting $x\sim y \iff \mathrm{Mor}_{\mathcal{S}}(x,y)\neq\varnothing$.

of sc$^+$–smooth sections that covers the cokernel of the linearization of $\mathfrak{s}$ at x. It follows that there is $\varepsilon > 0$ such that the set

$$U := \{(e,q) \in E \times \mathcal{O} \mid \mathfrak{s}(q) = \lambda(e,q), \|e\| < \varepsilon\} \tag{1.3.8}$$

is a manifold of dimension $d + \dim E$. (The proof involves a nontrivial amount of analytic detail that will appear in McDuff and Wehrheim (work in progress).) Choose a finite covering of the compact set $X := |\mathfrak{s}^{-1}(0)|$ by the footprints $(\psi_i(s_i^{-1}(0)))_{i\in A}$ of such charts

$$\boldsymbol{K}_i := (U_i, E_i, \Gamma_i, s_i, \psi_i), \quad s_i(e,q) = e,$$

and let $(|\mathcal{O}_i|)_{i\in A}$ be the associated open cover of a neighborhood of X in the ambient space $|\mathcal{S}|$. Just as in [11], one can use the groupoid structure of $\mathcal{S}$ to show that the $\boldsymbol{K}_i$ form the basic charts for a tame Kuranishi atlas $\mathcal{K}_{\mathcal{O},\lambda}$ whose transition charts are given by tuples of composable morphisms. Instead of giving more detail about this construction, we will outline how to modify these definitions so that the domains of the charts all have the same dimension $d + \dim E_A$.

First choose a family of bump functions $(\sigma_i)_{i\in A}$ with $\operatorname{supp}\sigma_i \subset |\mathcal{O}_i|$ such that

$$X = |\mathfrak{s}^{-1}(0)| \subset \bigcup_i \{x \mid \sigma_i(x) > 0 \text{ for some } i\}.$$

Then choose an ordering of the elements $i \in A$ and a reduction $(|\mathcal{W}_I|)_{I\in\mathcal{I}_\mathcal{K}}$ of the covering $(|\mathcal{O}_i|)_{i\in A}$ with the following properties:

- $|\mathcal{W}_I| \subset |\mathcal{O}_I| := \bigcap_{i\in I} |\mathcal{O}_i|$ for each $I \in \mathcal{I}_\mathcal{K}$;
- $X \subset \bigcup_{I\in\mathcal{I}_\mathcal{K}} |\mathcal{W}_I|$;
- $|\mathcal{W}_I| \cap |\mathcal{W}_J| \neq \varnothing \implies I \subset J$ or $J \subset I$;
- if $i \notin J$ then $\sigma_i \equiv 0$ on $|\mathcal{W}_J|$.

Then, given $I = \{i_0, \dots, i_k\}$ with $i_0 < i_1 < \cdots < i_k$, the space $M_I^\mathcal{W}$ consists of all tuples

$$\Big\{(e_A, q_{i_0}, \Psi_{q_{i_0}q_{i_1}}, \dots, q_{i_k}) \;\Big|\; |q_{i_0}| \in |\mathcal{W}_I|,\ \Psi_{qq'} \in \operatorname{Mor}(q,q'), \|e_A\| < \varepsilon,$$
$$\mathfrak{s}(q_{i_0}) = \sum_j \sigma_{i_j}(|q_{i_0}|)\Psi^*(\lambda_{i_j}(e_{i_j})(q_{i_j})) \in \mathcal{E}_{q_0}\Big\},$$

where $(q_{i_0}, \Psi_{q_{i_0}q_{i_1}}, q_{1_i}, \Psi_{q_{i_1}q_{i_2}}, \dots, q_{i_k})$ is a composable k–tuple of morphisms from a point $q_0 \in \mathcal{O}_{i_0}$ to $q_k \in \mathcal{O}_{i_k}$. By [7, Theorem 7.4], we may choose the σ_j so that for each $i, j \in A$ the function

$$\mathcal{O}_i \to [0,1], \quad q \mapsto \sigma_j(|q|),$$

is sc–smooth. It follows that if $\varepsilon > 0$ is suitably small, then, for each I, $M_I^{\mathcal{W}}$ is a manifold of dimension $d + \dim E_A$ with action of Γ_A. Moreover, much as in [11, Proposition 2.3], for each $I \subset J$ one can define a Γ_A–equivariant covering map

$$\tau_{IJ}\colon M_J^{\mathcal{W}} \supset \widetilde{M}_{IJ}^{\mathcal{W}} \to M_{IJ}^{\mathcal{W}} \subset M_I^{\mathcal{W}}$$

by taking an appropriate combination of the structural maps in $\mathcal{S}$ (such as compositions and source/target maps), where $M_{IJ}^{\mathcal{W}}$ (resp. $\widetilde{M}_{IJ}^{\mathcal{W}}$) consists of all elements in $M_I^{\mathcal{W}}$ (resp. $M_J^{\mathcal{W}}$) with $|q_{i_0}| \in |\mathcal{W}_I| \cap |\mathcal{W}_J|$. This gives a category $\boldsymbol{M}$ whose structure is precisely as described in Proposition 1.3.3. The resulting virtual fundamental class (VFC) $[X]^{\mathrm{vir}}$ is independent of all choices, and can be shown to agree with that defined by the polyfold Fredholm section $\mathfrak{s}\colon \mathcal{S} \to \mathcal{E}$.

Note that the equation satisfied by the elements in $M_I^{\mathcal{W}}$ involves the bump functions σ_j, while the equation (1.3.8) defining the chart domains of the atlas $\mathcal{K}_{\mathcal{O},\lambda}$ does not. Hence, the weighted branched manifold $(M^{\mathcal{W}}, \Lambda)$ constructed above is not identical to the manifold obtained from the atlas $\mathcal{K}_{\mathcal{O},\lambda}$ by the collaring construction described below. Nevertheless, these two constructions are closely related and, by adapting the arguments in Section 2.3, one can show that they define the same VFC $[X]^{\mathrm{vir}}$. For more details, see McDuff and Wehrheim (work in progress). $\diamond$

1.4 Examples

We end this introduction by giving some examples. Though these are not needed for the proofs of the main results, readers unfamiliar with the description of orbifolds via atlases might find it useful to read at least some of this section before proceeding further.

We begin by discussing the definition of the relative Euler class of an oriented vector bundle $\pi\colon \mathcal{E} \to W$ over a manifold that is equipped with a section $\mathfrak{s}\colon W \to \mathcal{E}$ whose zero set $X := \mathfrak{s}^{-1}(0)$ is compact. In particular, we explain why the method outlined in equation (1.1.2) does compute the Euler class $e(\mathcal{E})$ of $\mathcal{E} \to W$ when W is compact and $\mathfrak{s} \equiv 0$. In Remarks 1.4.2, we describe how to extend the construction to orbibundles. Finally, we show in detail how our main construction works to calculate the Euler class of the tangent bundle of S^2, starting from the atlas defined in [12]. Our approach easily generalizes to the football orbifold $S^2_{p,q}$, which is S^2 with orbifold points of orders p and q at the two poles.

Let $\pi\colon \mathcal{E} \to W$ be an oriented, vector bundle over the manifold W, together with a section $\mathfrak{s}\colon W \to \mathcal{E}$ with compact zero set $X \subset W$. As always (see Remark 1.2.1(iii)), we suppose that $\mathcal{E}$ has even rank to avoid problems with orientation.[15] We build a (Kuranishi) atlas whose charts are defined using tuples

$$(\mathcal{O}, E, \tau, s),$$

where

- $\mathcal{O} \subset W$ is open,
- E is an even-dimensional, oriented vector space,
- $\lambda\colon E \times \mathcal{O} \to \mathcal{E}|_{\mathcal{O}}$ is a surjective orientation-preserving bundle homomorphism over $\mathrm{id}_{\mathcal{O}}$, and
- λ pushes $s\colon \mathcal{O} \to E$ forward to $\mathfrak{s}|_{\mathcal{O}}$, ie $\lambda(s(x), x) = \mathfrak{s}(x) \in \mathcal{E}|_x$ for all $x \in \mathcal{O}$.

Given such a tuple, the corresponding chart

$$\boldsymbol{K} := (U, E, s, \psi), \quad \text{with footprint } F,$$

is defined by setting

$$U = \{(e, x) \in E \times \mathcal{O} \mid \lambda(e, x) = \mathfrak{s}(x)\}, \quad s(e, x) = e, \quad \psi(0, x) \mapsto x \in X.$$

One obtains an atlas as defined in Section 1.2 by taking the basic charts to be a finite family $(\boldsymbol{K}_i)_{i=1,\dots A}$ of charts of this form whose footprints (F_i) cover the compact set $X = \mathfrak{s}^{-1}(0)$, and the transition charts $(\boldsymbol{K}_I)_{I \in \mathcal{I}_{\mathcal{K}}}$ to be the corresponding charts (U_I, E_I, s_I, ψ_I) with footprints $F_I := \bigcap_i F_i$ that are formed just as above but now with $E_I = \prod_{i \in I} E_i$ and $\lambda_I = \sum_{i \in I} \lambda_i$. In particular,

$$U_I = \bigl\{((e_i), x) \in E_I \times \mathcal{O}_I \mid \textstyle\sum \lambda_i(e_i, x) = \mathfrak{s}(x)\bigr\}, \quad \text{where } \mathcal{O}_I := \displaystyle\bigcap_{i \in I} \mathcal{O}_i.$$

This gives an atlas in which the coordinate changes $\boldsymbol{K}_I \to \boldsymbol{K}_J$ are given by the obvious identifications

$$\widetilde{U}_{IJ} := \{(e, x) \in U_J \mid e \in E_I,\ x \in \mathcal{O}_J\} \xrightarrow{\cong} U_{IJ} = \{(e, x) \in U_I \mid Ax \in \mathcal{O}_J\}.$$

To see that the submersion condition holds, choose for each I a right inverse $\sigma_I\colon \mathcal{E}|_{\mathcal{O}_I} \to E_I \times \mathcal{O}_I$ to λ_I, so that $\lambda_I \circ \sigma_I = \mathrm{id}$, and define

$$\mathcal{E}'_{J \smallsetminus I} = \bigl\{\bigl(e' - \sigma_I(\mathfrak{s}(x)), x\bigr) \mid e' \in E_{J \smallsetminus I},\ x \in \mathcal{O}_{IJ}\bigr\} \subset E_J \times \mathcal{O}_J.$$

---

[15] Of course, over $\mathbb{Q}$ the Euler class vanishes for bundles of odd rank anyway.

Then $\mathcal{E}'_{J\smallsetminus I}$ is an affine subbundle of $E_J \times \mathcal{O}_J \to \mathcal{O}_J$, and we may identify U_J with the pullback of $\mathcal{E}'_{J\smallsetminus I}$ to $\widetilde{U}_{IJ}$ by the projection $\widetilde{U}_{IJ} \to U_J$, $(e,x) \mapsto x$.

Since there is such an atlas for every collection of charts $\boldsymbol{K}$ whose footprints cover X, any two such atlases $\mathcal{K}^0$ and $\mathcal{K}^1$ are *directly commensurate*, ie there is an atlas $\mathcal{K}$ whose charts include those of $\mathcal{K}^0$ and $\mathcal{K}^1$. Therefore, $\mathcal{K}^0$ and $\mathcal{K}^1$ are cobordant by [14, Section 6.2]. Hence, they define cobordant manifold models $(M, E_A, \mathscr{S})$ by Theorem A and the same class $[X]^{\mathrm{vir}}_{\mathcal{K}}$ by Theorem B.

If the bundle $\mathcal{E} \to W$ is smooth, then we can define the VFC either as in the proof of Theorem B given in Section 2.3, or via an inverse limit of the homology classes of the zero sets of a family of perturbed sections $\mathfrak{s} + \nu_k$ of $\mathcal{E} \to W$. As explained in the proof of Theorem B, these two approaches give the same answer. If W is just a topological manifold, it is of course easiest to represent the Euler class by starting with an atlas with just one basic chart (and hence just one chart). In this case, our general method of building an atlas gives the tuple described in (1.1.2). We now show that if $\mathfrak{s} \equiv 0$, so that $X = \mathfrak{s}^{-1}(0) = W$ is a compact manifold, then $[X]^{\mathrm{vir}}_{\mathcal{K}}$ as defined in (1.1.1) is Poincaré dual to the usual Euler class $e(\mathcal{E}) \in H^{2k}(X;\mathbb{Z})$, where $2k = \operatorname{rank} \mathcal{E}$. In the following lemma, we use simplicial (co)homology instead of the Čech theory discussed in the appendix, since all spaces are manifolds, and take coefficients $\mathbb{Z}$ since the isotropy is trivial.

Lemma 1.4.1 *If $\mathcal{E} \to X$ is an oriented $2k$–dimensional vector bundle over an oriented $(2k+d)$–dimensional manifold X with $\mathfrak{s} \equiv 0$ and atlas $\mathcal{K}$ as above, then*

$$[X]^{\mathrm{vir}}_{\mathcal{K}} = \mu_X \cap e(\mathcal{E}) \in H_d(X),$$

where μ_X is the fundamental class of X and $e(\mathcal{E}) \in H^{2k}(X;\mathbb{Z})$ is the Euler class of $\mathcal{E}$.

Proof By Theorem B and the above remarks, it suffices to calculate $[X]^{\mathrm{vir}}_{\mathcal{K}}$ using an atlas with one chart as in (1.1.2). Thus, we may take

$$M = \mathcal{E}', \qquad \mathscr{S}\colon M \to \mathbb{R}^N, \quad (e', x) \mapsto \mathrm{pr}_{\mathbb{R}^N}(\iota(e', x)),$$

where $\mathcal{E}'$ has rank 2ℓ, $N = 2k + 2\ell$, $\iota\colon M \to \mathcal{E}' \oplus \mathcal{E}$ is the inclusion and $\mathrm{pr}_{\mathbb{R}^N}$ is the projection

$$\mathrm{pr}_{\mathbb{R}^N}\colon \mathcal{E} \oplus \mathcal{E}' \cong \mathcal{O}_X^N := \mathbb{R}^N \times X \to \mathbb{R}^N.$$

Denote the Thom classes of $\mathcal{E}$ and $\mathcal{E}'$ by $\tau_{\mathcal{E}}$ and $\tau_{\mathcal{E}'}$ and their pullbacks to $\mathcal{O}^N_X$ by

$$\widetilde{\tau}_{\mathcal{E}} \in H^{2k}(\mathcal{O}^N_X, \mathcal{O}^N_X \smallsetminus \mathcal{E}'), \quad \widetilde{\tau}_{\mathcal{E}'} \in H^{2\ell}(\mathcal{O}^N_X, \mathcal{O}^N_X \smallsetminus \mathcal{E}).$$

Then, if $\tau_{\mathbb{R}^N} \in H^N(\mathbb{R}^N, \mathbb{R}^N \smallsetminus \{0\})$ is the canonical generator, we have

$$\mathscr{S}^*(\tau_{\mathbb{R}^N}) = \iota^*(\tau_{\mathcal{O}^N_X}) = \iota^*(\widetilde{\tau}_{\mathcal{E}} \cup \widetilde{\tau}_{\mathcal{E}'}) \in H^N(M, M \smallsetminus X).$$

We may identify $\mu_M \cap \tau_{\mathcal{E}'}$ with the fundamental class $\mu_X \in H_{2k+d}(X)$, where $\mu_M \in H_{2k+d}(M, M \smallsetminus X)$ is the restriction of the fundamental class of M. Then, for any class $b \in H^d(X)$, we use the cap product in (A.7) with $Y = M$, $A = \varnothing$ and $Y \smallsetminus U = X$, and the relation between cap and cup products for even-dimensional classes, to obtain

$$\begin{aligned}\langle [X]^{\mathrm{vir}}_{\mathcal{K}}, b\rangle := \langle \mathscr{S}_*(\mu_M \cap b), \tau_{\mathbb{R}^N}\rangle = \langle \iota_*(\mu_M \cap b), \widetilde{\tau}_{\mathcal{E}} \cup \widetilde{\tau}_{\mathcal{E}'}\rangle = \langle \iota_*(\mu_M \cap b \cap \tau_{\mathcal{E}'}), \widetilde{\tau}_{\mathcal{E}}\rangle \\ = \langle \mu_X \cap b, \iota_X^*(\tau_{\mathcal{E}})\rangle = \langle \mu_X \cap e(\mathcal{E}), b\rangle,\end{aligned}$$

where we have written $\iota_X \colon X \to \mathcal{E}$ for the inclusion and used the fact that $\iota_X^*(\tau_{\mathcal{E}})$ is the Euler class $e(\mathcal{E}) \in H^{2k}(X)$ of $\mathcal{E}$. □

Remarks 1.4.2 (i) The above construction easily adapts to the case of an oriented orbifold bundle $\mathcal{E} \to W$ over an oriented orbifold W, where now we should think of the spaces $\mathcal{E}$ and W as the realizations of suitable EP categories $\boldsymbol{E}$ and $\boldsymbol{W}$. Thus, one can build an atlas whose basic charts are as above with the addition of a group action, while the transition charts are made using composable tuples of morphisms in $\boldsymbol{E}$. For details, see [10, Section 5.2]. One can then piece the corresponding fattened charts together by the method explained in Sections 2 and 3 below to obtain a tuple $(M, E_A, \mathscr{S})$ as in Theorem A. However, we can also build the category $\boldsymbol{M}$ directly from the set of basic charts $(U_i, E_i, \Gamma_i, s_i, \psi_i)$, using a partition of unity, and an associated reduction as explained in Remark 1.3.8.

(ii) In Gromov–Witten theory it sometimes happens that the space of J–holomorphic maps in class A does form a compact manifold (or orbifold) X such that the rank of the cokernel of the linearized Cauchy–Riemann operator D_x at $x \in X$ is independent of x. In this case, these cokernels fit together to form a bundle $\mathcal{E} \to X$ such that the map $\mathfrak{s}$ induced by the Cauchy–Riemann operator is zero. We explain in [10, Remark 5.2.4] why one can choose a Gromov–Witten type atlas (constructed as in [10, Section 4] or [15]) with precisely the structure considered above.

(iii) Pardon [15, Proposition 5.3.4] proves the analog of Lemma 1.4.1 in the smooth case using a transverse perturbation of $\mathfrak{s}$ as in Step 3 of the proof of Lemma 2.3.4. ◇

Example 1.4.3 (the tangent bundle of the 2–sphere and the football) We now illustrate the construction in the proof of Theorem A in the case of the bundle $\pi\colon \mathrm{T}S^2 \to S^2$ with section $\mathfrak{s} \equiv 0$, starting from the Kuranishi atlas with two basic charts that was constructed in [12, Example 3.4.2]. We organize the details into several steps.

(I) (atlas for the tangent bundle of the 2–sphere) To build a Kuranishi atlas whose associated "bundle" pr: $|\boldsymbol{E}_{\mathcal{K}}| \to |\mathcal{K}|$ models $\mathrm{T}S^2$, cover S^2 by two copies D_1 and D_2 of the unit disc in $\mathbb{C}$, whose intersection $D_1 \cap D_2 =: D_{12} =: A \cong [0,1] \times S^1$ is an annulus, and for $i = 1, 2$ define

$$\boldsymbol{K}_i := (U_i := D_i,\ E_i := \mathbb{C},\ s_i := 0,\ \psi_i := \mathrm{id}).$$

For $i = 1, 2$, choose unitary trivializations $T_i\colon D_i \times \mathbb{C} \to \mathrm{T}S^2|_{D_i}$, $(x, e) \mapsto T_{i,x}(e)$, and then define the transition chart

$$\boldsymbol{K}_{12} := (U_{12} \subset E_1 \times E_2 \times A,\ E_1 \times E_2,\ s_{12} = \mathrm{pr}_{E_1 \times E_2},\ \psi_{12} = \mathrm{pr}_A|_{0 \times 0 \times A})$$

by setting

$$U_{12} := \{(e_1, e_2, x) \mid x \in A, T_{1,x}(e_1) + T_{2,x}(e_2) = 0\}.$$

The coordinate changes $\widehat{\Phi}_{i,12}$ are given by $U_{i,12} = \{(0,0)\} \times A$ and $\rho_{i,12}(0,0,x) = x$. To justify this choice of Kuranishi atlas, note that one can construct a commutative diagram

$$\begin{tikzcd} |\boldsymbol{E}_{\mathcal{K}}| \arrow[r] \arrow[d] & \mathrm{T}S^2 \arrow[d] \\ |\boldsymbol{B}_{\mathcal{K}}| \arrow[r] \arrow[u, bend left, "|\mathfrak{s}|"] & S^2 \arrow[u, bend left, "s\equiv 0"] \end{tikzcd}$$

where the top horizontal map restricts on $U_{12} \times E_{12}$ to the map

$$((e_1, e_2, x), e_1', e_2') \mapsto T_{1,x}(e_1') + T_{2,x}(e_2') \in \mathrm{T}_x S^2 \subset \mathrm{T}S^2|_A.$$

Thus, it takes

$$\operatorname{graph} s_{12} = \{((e_1, e_2, x), e_1, e_2) \mid (e_1, e_2, x) \in U_{12}\} \subset U_{12} \times E_{12}$$

to the zero section of $\mathrm{T}S^2$.

This construction is generalized to other (orbi)bundles in [10]. $\diamond$

(II) (calculating the Euler class) In order to calculate the Euler class of $\mathrm{T}S^2$ it is convenient to identify the annulus A with $[0,1] \times S^1$, and then consider the corresponding trivialization $\mathrm{T}S^2|_A \equiv A \times \mathbb{R}_t \times \mathbb{R}_\theta$, where $t \in [0,1]$ and $\theta \in S^1 \equiv \mathbb{R}/2\pi\mathbb{Z}$

are coordinates. Then, for $i = 1, 2$ there is a section $\nu_i\colon U_i \to E_i$ with one transverse zero such that

$$T_{i,x}(\nu_i(x)) = (x, 1, 0) \in A \times \mathbb{R}_t \times \mathbb{R}_\theta \equiv \mathrm{T}S^2|_A, \quad x \in A.$$

(Take suitably modified versions of the sections $\nu_1(z) = z$ and $\nu_2(z) = -z$, where $D_i \subset \mathbb{C}$.) Therefore, the ν_i fit together to give a global section of $\mathrm{T}S^2$ with two transverse zeros, and it follows that the Poincaré dual of $e(\mathrm{T}S^2)$ is represented by $2[\mathrm{pt}] \in H_0(S^2)$.

To see how $e(\mathrm{T}S^2)$ is calculated via the atlas, we start by choosing a reduction $\mathcal{G}$ of the footprint covering. For example, we may take $G_{12} = (\varepsilon, 1-\varepsilon) \times S^1 \sqsubset A$ for some $\varepsilon \in \bigl(0, \frac{1}{4}\bigr)$ and choose $G_i \sqsubset D_i$ so that

$$\widetilde{V}_{1,12} = (0,0) \times \bigl(\varepsilon, \tfrac{1}{4}\bigr) \times S^1 \subset U_{12}, \quad \widetilde{V}_{2,12} = (0,0) \times \bigl(\tfrac{3}{4}, 1-\varepsilon\bigr) \times S^1 \subset U_{12}.$$

Choose a cutoff function $\beta\colon [0,1] \times S^1 \xrightarrow{\mathrm{pr}} [0,1] \to [0,1]$ that equals 1 in $\bigl[0, \frac{1}{4}\bigr] \times S^1$ and 0 in $\bigl[\frac{3}{4}, 1\bigr] \times S^1$. Then the map $\nu_{12}\colon \widetilde{V}_{12} \to E_1 \times E_2$ given by

$$\nu_{12}(e_1, e_2, x) = \bigl(\beta(x)\nu_1(x), (1-\beta(x))\nu_2(x)\bigr) \in E_1 \times E_2$$

restricts to ν_i on $V_{i,12} \subset (0,0) \times A$ for $i = 1, 2$. Thus, the tuple (ν_1, ν_2, ν_{12}) is an admissible perturbation section in the sense of [12]. Moreover, $s_{12} + \nu_{12}$ does not vanish at any point $(e_1, e_2, x_0) \in V_{12}$ because the three equations

$$\begin{gathered} T_{1,x_0}(e_1) + T_{2,x_0}(e_2) = 0, \\ T_{1,x_0}(e_1) + \beta(x_0)(1,0) = T_{2,x_0}(e_2) + (1-\beta(x_0))(1,0) = 0 \in \{x_0\} \times \mathbb{R}_t \times \mathbb{R}_\theta \end{gathered}$$

together imply that the vector $(1,0) \in \mathbb{R}_t \times \mathbb{R}_\theta$ is zero, a contradiction. Hence, as before, the perturbed zero set consists of two points, each with weight one. $\diamond$

(III) (construction of the corresponding manifold M and section $\mathscr{S}_M\colon M \to E_{12}$) When, as in the case at hand, the isotropy groups are trivial, the current paper constructs from the above reduction $\mathcal{V}$ of $\mathcal{K}$ a manifold M that is the union of three components

$$M = ((M_1 = E_{2,\varepsilon} \times V_1) \sqcup (M_2 = E_{1,\varepsilon} \times V_2) \sqcup (M_{12} = V_{12}))/{\sim},$$

where $\sim$ identifies $(e_j, x) \in M_{i,12}$ with $\alpha_{i,12}(e_j, x) \in \widetilde{M}_{i,12} \subset M_{12}$ where $\alpha_{i,12} := \tau_{i,12}^{-1}$. The submersion axiom (1.2.3) implies that the submanifold $\widetilde{V}_{i,12}$ has local product neighborhoods in V_{12}. In Section 2 we will describe how to assemble these into a more global structure that can be used to relate the different components M_I. However, in the current situation there is an obvious global product structure that directly gives the needed attaching maps as follows. First, with $i=1$ and $j=2$, we define

$$\phi^E\colon (E_{2,\varepsilon}\times\widetilde{V}_{1,12}, \{0\}\times\widetilde{V}_{1,12})\to(V_{12},\widetilde{V}_{1,12}),\quad (e_2,x)\mapsto\bigl(-T_{1,x}^{-1}(T_{2,x}(e_2)),e_2,x\bigr).$$

Then, the attaching map $\alpha_{1,12}=\tau_{1,12}^{-1}$ is given by

$$\begin{aligned}\alpha_{1,12}\colon E_{2,\varepsilon}\times V_{1,12}&\to V_{12},\\ (e_2,x)&\mapsto x'=\phi^E(\lambda\cdot e_2,x)=\bigl(-T_{1,x}^{-1}(T_{2,x}(\lambda e_2)),\lambda e_2,x\bigr),\end{aligned}$$

where

$$\lambda:=\sqrt{\|e_2\|}.$$

Further, we take $\mathscr{S}_{12}=s_{12}$, where

$$s_{12}\bigl(-T_{1,x}^{-1}(T_{2,x}(e_2)),e_2,x\bigr)=\bigl(-T_{1,x}^{-1}(T_{2,x}(\lambda e_2)),\lambda e_2\bigr),$$

and then define $\mathscr{S}_1$ by pullback over $V_{1,12}$, extended over M_1 by a cutoff function

$$\mathscr{S}_1(e_2,x)=\beta_{1,12}(x)\bigl(-T_{1,x}^{-1}(T_{2,x}(\lambda e_2)),\lambda e_2\bigr)+(1-\beta_{1,12}(x)(0,e_2),$$

where $\beta_{1,12}\colon V_1\to[0,1]$ equals 0 near $x=0$ and 1 on $V_{1,12}$. Note that $\tau_{1,12}$ does have closed graph in $M_1\times M_2$ since M_1 contains no points (e_2,x) with $x\in\bigl\{\frac14\bigr\}\times S^1\subset A$, while M_2 contains no points (e_1,e_2,x) with $x\in\{0\}\times S^1\subset A$. There are similar formulas for $\alpha_{2,12}$ and $\mathcal{S}_2$.

This construction gives a 4–manifold M together with a map $\mathscr{S}_M\colon M\to E_{12}$ whose zero set is homeomorphic to S^2. In fact, we can identify M with a neighborhood of the zero section in $\mathrm{T}S^2$ that has width $\varepsilon>0$ over the discs $(V_i\setminus V_{i,12})_{i=1,2}$ and contains the whole of $\mathrm{T}S^2|_{G_{12}}$. This holds because V_{12} can be identified with $\mathrm{T}S^2|_{G_{12}}$. ◇

(IV) (the normal bundle of $\mathscr{S}_M^{-1}(0)\cong S^2$ in M is isomorphic to $\mathrm{T}S^2$) To see this, note that there is an embedding

$$M_1\cup_{\alpha_{1,12}}\widetilde{M}_{1,12}\to\mathbb{C}\times D_1$$

given on $M_1=E_{2,\varepsilon}\times V_1$ by the obvious inclusion (where we identify $E_2\equiv\mathbb{C}$) and on $\widetilde{M}_{1,12}$ by

$$\bigl(-T_{1,x}^{-1}(T_{2,x}(e_2)),e_2,x\bigr)\mapsto(\lambda^{-1}e_2,x)\in E_2\times A\subset\mathbb{C}\times D_1,\quad \lambda=\sqrt{\|e_2\|}.$$

Identifying A with $(\varepsilon,1-\varepsilon)\times S^1$ as above, we may extend this embedding over a neighborhood $\mathcal{N}_1\subset M_{12}$ of the set $\{(0,0)\}\times\bigl(\varepsilon,\frac12\bigr]\times S^1$ so that it equals

$$\bigl(-T_{1,x}^{-1}(T_{2,x}(e_2)),e_2,x\bigr)\mapsto(e_2,x)\quad\text{for all } x\in\bigl(\tfrac12-\delta,\tfrac12\bigr]\times S^1.$$

The similar embedding

$$(E_{1,\varepsilon}\times V_2)\cup_{\alpha_{1,12}}\mathcal{N}_2\to\mathbb{C}\times D_2$$

is given near the circle $\{\frac{1}{2}\}\times S^1$ by the map $\bigl(e_1,-T_{2,x}^{-1}(T_{1,x}(e_1)),x\bigr)\mapsto(e_1,x)$. Therefore, this bundle over S^2 is determined by the clutching map $x\mapsto -T_{2,x}^{-1}(T_{1,x})$, which is homotopic to the map $x\mapsto T_{2,x}^{-1}(T_{1,x})$ that determines $\mathrm{T}S^2$. $\diamond$

(V) (the case of the football orbifold $S^2_{p,q}$) This orbifold is topologically S^2, but has orbifold points of orders p and q at the two poles. Thus, the bundle $\pi\colon \mathrm{T}S^2_{p,q}\to S^2_{p,q}$ is again modeled by a Kuranishi atlas[16] with two basic charts $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ as above, with $\Gamma_1=\mathbb{Z}/p\mathbb{Z}$ acting by rotations on D_1 and E_1 and with $\Gamma_2=\mathbb{Z}/q\mathbb{Z}$ acting by rotations on D_2 and E_2. Since $s_i\equiv 0$ for $i=1,2$, the footprint maps

$$\psi_i\colon \mathfrak{s}_i^{-1}(0)=U_i\to S^2_{p,q},\quad x\mapsto |x|,$$

simply quotient out by the action of the group Γ_i. We choose the trivializations $T_{i,x}$ of $\mathrm{T}D_i$ to be equivariant under the rotation action of the isotropy groups, and will suppose for simplicity that $(p,q)=1$, so that the domain U_{12} of the transition chart is connected.[17] Then, in terms of the coordinates $(t,\theta)\in A$ introduced in (II) we have

$$U_{12}=\{(e_1,e_2,x)\in E_1\times E_2\times A\mid |T_{1,\rho_{1,12}(x)}(e_1)|+|T_{2,\rho_{2,12}(x)}(e_2)|=0\},$$
$$\rho_{1,12}(t,\theta)=(t,q\theta),\quad \rho_{2,12}(t,\theta)=(t,p\theta)\in A=[0,1]\times\mathbb{R}/\mathbb{Z},$$

where we denote the image of $(e,x)\in E_1\times D_1$ in $T_{|x|}S^2_{p,q}$ by $|T_{1,x}(e)|$, and the equation takes place in the tangent bundle of the orbifold. Because the maps $\rho_{i,ij}$ are equivariant by hypothesis, this equation is preserved by the action of Γ_{12} on U_{12} by

$$\Bigl(\frac{r}{p},\frac{s}{q}\Bigr)\cdot(e_1,e_2,(t,\theta))=\Bigl(\frac{r}{p}\cdot e_1,\frac{s}{q}\cdot e_2,\Bigl(t,\theta+\frac{kr}{p}+\frac{\ell s}{q}\Bigr)\Bigr),\quad kq+\ell p=1.$$

[16]The reader should beware that the words "orbifold atlas" or "good atlas" are usually used in orbifold theory with slightly different meaning, which is why [11] uses the words "strict atlas" to denote a Kuranishi atlas with trivial obstruction spaces. As explained in [11], a strict atlas $\mathcal{K}$ for an orbifold Z defines an EP groupoid $\boldsymbol{G}_{\mathcal{K}}$ whose realization is Z, and hence defines an orbifold structure on Z. Further, by [11, Proposition 3.3], $\boldsymbol{G}_{\mathcal{K}}$ is Morita equivalent to the category constructed from any standard orbifold atlas for Z. Finally one can obtain a standard orbifold atlas for Z from $\mathcal{K}$ by taking a collection of restrictions of the basic charts in $\mathcal{K}$ whose footprints cover Z, with transition maps induced by the morphisms in $\boldsymbol{G}_{\mathcal{K}}$.

[17]Since all points in U_{12} have trivial stabilizer, we need Γ_{12} to act freely on U_{12} in such a way that the projection $\rho_{j,12}$ quotients out by the action of Γ_i, which is possible for connected U_{12} only if $(p,q)=1$.

We may calculate the Euler class by using essentially the same perturbation section as before, since this may be chosen to be equivariant. But now the two zeros of the section count with weights, $\frac{1}{p}$ for the zero in V_1 and $\frac{1}{q}$ for the zero in V_2.

The corresponding category $\boldsymbol{M}$ has three components that are given by the same formulas as before. Again, the attaching maps $\tau_{i,12}\colon \widetilde{M}_{i,12} \to M_{i,12} \subset M_i$ are nontrivial covering maps. However, in distinction to the case of an atlas, the $\tau_{i,12}$ do *not* quotient by the induced action of Γ_j on $\widetilde{M}_{i,ij}$ since they are constructed to be Γ_{12} equivariant, and Γ_{12} acts (often effectively) on M_i, via

$$(\gamma_1, \gamma_2) \cdot (e_j, x_i) = (\gamma_j \cdot e_j, \gamma_i \cdot x_i).$$

However, as explained at the end of the proof of Proposition 2.2.2 (see for example (2.2.20)), they do quotient out by *some* action of Γ_j on $\widetilde{M}_{12}$ that extends its free action on $\widetilde{V}_{i,12} \subset \widetilde{M}_{i,12}$. For example, the map $\tau_{1,12}$ quotients out by the free action of Γ_q on $\widetilde{M}_{1,12} \subset E_1 \times E_2 \times \left(\varepsilon, \frac{1}{4}\right) \times S^1$ given by

$$\gamma \cdot (e_1, e_2, x) \mapsto (e_1, e_2, \gamma \cdot x).$$

Therefore, in the quotient space $M = |\boldsymbol{M}|$ there are q branches of M_{12} that come together over the 3–dimensional branching locus

$$\mathrm{Br}_1 := \left\{ |(e_1, e_2, x)| \in |M_{12}| \subset |\boldsymbol{M}|_{\mathcal{H}} \mid x \in \tfrac{1}{4} \times S^1 \right\}.$$

This is consistent with the requirements of Definition 1.3.1 since the component M_{12} has weight $\frac{1}{pq}$ while M_1 has weight $\frac{1}{p}$.

The construction of $\mathscr{S}_{\boldsymbol{M}}\colon M \to E_{12}$ is as before. Moreover, one can identify a neighborhood of its zero set $S^2_{p,q}$ with a neighborhood of the zero section of the tangent orbibundle to $S^2_{p,q}$. Hence, the Poincaré dual of $e(\mathrm{T}S^2_{p,q})$ is represented by

$$\left(\frac{1}{p} + \frac{1}{q}\right)[\mathrm{pt}] \in H_0(S^2_{p,q}). \qquad \diamond$$

(VI) (the quotient space $|\boldsymbol{M}|/\Gamma$ for $\mathrm{T}S^2_{p,q}$) The only morphisms in the category $\boldsymbol{M}$ come from the covering maps $\tau_{j,12}$. Since these are Γ_{12}–equivariant, we can add the action $\Gamma_{12} \times \mathrm{Obj}_{\boldsymbol{M}} \to \mathrm{Obj}_{\boldsymbol{M}}$ to the morphisms in $\boldsymbol{M}$. The resulting quotient space $|\boldsymbol{M}|/\Gamma_{12}$ has the following structure:

- It is covered by three branches M_1, M_2 and M_{12} with weights $1/p^2q$, $1/pq^2$ and $1/p^2q^2$.
- The two poles $[(0,0)] \in M_i/\Gamma_{12}$ have stabilizer subgroup Γ_{12}.

- The other points with nontrivial stabilizers lie on the two closed discs

$$\{0\}\times(\overline{V}_i \smallsetminus \{0\})/\Gamma_{12} \subset |M_i|/\Gamma_{12}, \quad i = 1,2,$$

  with isotropy subgroups Γ_j for $j \neq i$.

- For $i = 1,2$ there is branching of order $|\Gamma_j|$ over the 3–dimensional branching locus Br_i. For example, if $\Gamma_1 = \{\mathrm{id}\}$ and $\Gamma_2 = \mathbb{Z}/2\mathbb{Z}$, then $|M_1|/\Gamma_2$ is an orbifold with a 2–dimensional family of points with nontrivial stabilizer (corresponding to the points $\{0\}\times D_1 \subset E_2\times D_1$), while Γ_2 acts freely on M_{12} and the Γ_2–equivariant map $\rho_{1,12}\colon M_{1,12} \to M_1$ quotients out by a different free action of Γ_2 that lifts the rotation action on A via the projection $\widetilde{M}_{1,12} \subset E_{12}\times A \to A$. Thus, there is branching of order 2 along the boundary Br_1, which lies over the circle $t = \{\frac{1}{4}\}$.

We do not consider this space further, since it plays no role in the definition of the fundamental class. $\diamond$

2 The main arguments

In this section, we first explain how to construct an auxiliary family of collared manifolds and then explain in Section 2.2 how to use this family to prove Proposition 2.2.2 and hence Proposition 1.3.3. Finally, we prove Theorems A and B in Section 2.3.

The key notion is that of the manifold $Y_{\mathcal{U},J,\underline{\varepsilon}}$, which lies over the $(|J|-1)$–dimensional simplex Δ_J. Its open submanifold $Y_{\mathcal{V},J,\underline{\varepsilon}}$, corresponding to a choice of reduction $\mathcal{V} \subset \mathcal{U}$, has a partially defined boundary collar that is compatible both with shrinking of chart domains and with projection to Δ_J. We will define the attaching maps $\widetilde{M}_{IJ} \to M_{IJ}$ of the different components of $\mathrm{Obj}_{\boldsymbol{M}}$ by thinking of M_J as a subset of $Y_{\mathcal{V},J,\underline{\varepsilon}}$.

Although strictly speaking the construction of the category $\boldsymbol{M}$ only uses the manifolds $Y_{\mathcal{V},J,\underline{\varepsilon}}$, we also consider the manifolds $Y_{\mathcal{U},J,\underline{\varepsilon}}$ to clarify the exposition. The latter has elements that are relatively easy to understand (see (2.1.3)) and it has an easily described boundary, while, as we see from Proposition 2.1.4, the collar is supported on only a rather complicated part of the boundary of $Y_{\mathcal{V},J,\underline{\varepsilon}}$. Further, considering both $Y_{\mathcal{U},J,\underline{\varepsilon}}$ and $Y_{\mathcal{V},J,\underline{\varepsilon}}$ will allow us in Section 3 to introduce the many technical conditions satisfied by the pair $(\mathcal{V},\underline{\varepsilon})$ in stages, first some conditions on $(\mathcal{U},\underline{\varepsilon})$ needed for $Y_{\mathcal{U},J,\underline{\varepsilon}}$

to have good properties (Definition 3.1.1), and then more conditions needed to construct a suitable collar on $Y_{\mathcal{V},J,\underline{\varepsilon}}$ (Definition 3.1.9).

The first main results of this section are Proposition 2.1.1, which describes the structure of $Y_{\mathcal{U},J,\underline{\varepsilon}}$, and Proposition 2.1.4, which describes the properties of the boundary collars put on the manifolds $Y_{\mathcal{V},J,\underline{\varepsilon}}$. Proposition 2.2.2 then explains how to use these boundary collars to construct the attaching maps τ_{IJ} whose existence is claimed in Proposition 1.3.3. Since the general construction is quite complicated, we describe it first by example (see Example 2.2.1). Since the proofs of Theorems A and B in Section 2.3 depend only on the statement of Proposition 1.3.3, this subsection can be read independently of Sections 2.1 and 2.2.

2.1 The collared manifold Y

Suppose given a tame atlas $\mathcal{K}$ with set of chart domains $\mathcal{U} := (U_I)_{I\in\mathcal{I}_{\mathcal{K}}}$. The next definition uses a choice of constants $\underline{\varepsilon} = (\varepsilon_I)$ as in (1.3.2), and the following notation:

- $\Delta_J := \bigl\{t = (t_i)_{i\in J} \mid t_i \ge 0,\ |t| := \sum_{i\in J} t_i = 1\bigr\}$ is the $(|J|-1)$–simplex;
- for $\varnothing \ne I \subsetneq J$, we denote by $\iota_{IJ}\colon \Delta_I \to \Delta_J$ the natural inclusion with image

  $$\partial_{J\smallsetminus I}\Delta_J := \{t \in \Delta_J \mid t_j = 0,\ j \in J\smallsetminus I\} \subset \Delta_J$$

  (we often omit ι_{IJ} if there is no danger of confusion);
- $t\cdot e := \sum_{i\in J} t_i e_i$, where $t \in \Delta_J, e \in E_A$;
- $\kappa := \max\{|J| : J \in \mathcal{I}_{\mathcal{K}}\}$;
- $I(x) := \{j : s_j(x) \ne 0\} \subset J$ for $x \in U_J$; and
- $\underline{\varepsilon} := (\varepsilon_I)_{I\in\mathcal{I}_{\mathcal{K}}}$ is a set of positive constants such that $\kappa\varepsilon_I \le \varepsilon_J$ whenever $I \subsetneq J$.

Given $J \in \mathcal{I}_{\mathcal{K}}$, consider the set[18]

$$Y_J := Y_{\mathcal{U},J,\underline{\varepsilon}} = \bigl\{(e,x;t) \in E_A \times U_J \times \Delta_J \mid s_J(x) = t\cdot e,\ \|e\| < \kappa\varepsilon_{I(x)},\ \|s_i(x)\| < \varepsilon_{I(x)} \text{ for all } i \in J\bigr\}. \tag{2.1.1}$$

Here are some properties of this definition:

- Γ_A acts on $Y_{\mathcal{U},J,\underline{\varepsilon}}$ by

  $$\gamma\cdot(e,x;t) = (\gamma\cdot e, \gamma\cdot x; t).$$

[18] To begin with, readers should ignore the rather fussy conditions involving the constants $\underline{\varepsilon}$; in this connection see (2.1.6) and Corollary 2.1.2 below. Notice that we do need some such constants since the size of ε_J determines how thick the pieces M_J will be, and to construct $\boldsymbol{M}$ we need to embed (a covering of) M_{IJ} into M_J for all $I \subset J$.

- The condition $s_J(x) = t \cdot e$ implies that

$$I(x) := \{j : s_j(x) \neq 0\} \subset I(t) := \{i : t_i > 0\}. \tag{2.1.2}$$

  In particular, if $(e, x; t) \in Y_{\mathcal{U},J,\underline{\varepsilon}}$ we must have

$$x \in s_J^{-1}(E_{I(x)}) = \widetilde{U}_{I(x)J} \subset \widetilde{U}_{I(t)J}, \tag{2.1.3}$$

  where the equality holds because $\mathcal{K}$ is tame (see (1.2.2)). Further, the components of e in $E_{I(t)}$ are determined by the pair (x, t), while those in $E_{A \smallsetminus I(t)}$ can vary freely.

- There are three Γ_A–equivariant projections of $Y_{\mathcal{U},J,\underline{\varepsilon}}$ onto the factors of its domain:
  - $\mathrm{pr}_E\colon Y_{\mathcal{U},J,\underline{\varepsilon}} \to E_A$, $(e, x; t) \mapsto e$. For $I \subset A$, we denote by e_I the elements of E_I, and denote by pr_{E_I} the projection to E_I.
  - The projection $\mathrm{pr}_U\colon (e, x; t) \mapsto x \in U_J$ has contractible fibers that vary with $x \in U_J$.
  - The fibers of $\mathrm{pr}_\Delta\colon Y_{\mathcal{U},J,\underline{\varepsilon}} \to \Delta_J$, $(e, x; t) \mapsto t$, also depend on the image $t \in \Delta_J$. In particular, if for some $I \subsetneq J$ we have $t \in \mathrm{int}\, \Delta_I := \Delta_I \smallsetminus \partial \Delta_I \subset \partial \Delta_J$, then, for any $(e, x; t) \in \mathrm{pr}_\Delta^{-1}(t)$, we must have $x \in \widetilde{U}_{IJ}$ while the restriction $\mathrm{pr}_{E_{A \smallsetminus I}}(e)$ can vary freely.

- For each element of the form $(e, x; \iota_{IJ}(t)) \in Y_{\mathcal{U},J,\underline{\varepsilon}}$ there is a corresponding element $(e, \rho_{IJ}(x); t) \in Y_{\mathcal{U},I,\underline{\varepsilon}}$, where $\rho_{IJ}\colon \widetilde{U}_{IJ} \to U_{IJ}$ is part of the atlas coordinate change. Thus, if we define

$$\partial_{J \smallsetminus I} Y_J := \mathrm{pr}_\Delta^{-1}(\partial_{J \smallsetminus I} Y_J) := \{(e, x; t) \in Y_J \mid t_j = 0,\ j \notin I\}, \tag{2.1.4}$$

  there is a Γ_A–equivariant covering map

$$\partial_{J \smallsetminus I} Y_J \to Y_I \cap (E_A \times U_{IJ} \times \Delta_I) \subset Y_I. \tag{2.1.5}$$

  If the isotropy is trivial, we can therefore identify $\partial_{J \smallsetminus I} Y_J$ with an open subset of Y_I.

- The relevance of the conditions involving the constants $\underline{\varepsilon}$ are explained by the following remark. For each $x \in U_J$ such that $\|s_i(x)\| < \varepsilon_{I(x)}$ for all $i \in J$, and every H satisfying $I(x) \subset H \subset J$, there is a corresponding element

$$(e, x; \iota_{HJ}(b_H)) \in Y_{\mathcal{U},J,\underline{\varepsilon}}, \tag{2.1.6}$$

where b_H is the barycenter of Δ_H. Indeed, if we take $e := (|H| s_i(x))_{i\in A}$, then $e_j = 0$ for $j \notin I(x)$, by definition of $I(x)$, while for $i \in I(x)$ we have $\|e_i\| = |H|\,\|s_i(x)\| < \kappa\varepsilon_{I(x)}$, as required by (2.1.1).

The following result is proved in Corollary 3.1.4.

Proposition 2.1.1 *Let $\mathcal{U}^\Omega$ be a family of chart domains for an atlas on X. Without loss of generality, we may pass to a shrinking $\mathcal{U} \sqsubset \mathcal{U}^\Omega$ and choose constants $\underline{\varepsilon} > 0$ so that the following holds for all J :*

(i) $s_J(\overline{U}_J) \subset E_{J,\varepsilon_J}$.

(ii) *The space $Y_J := Y_{\mathcal{U},J,\underline{\varepsilon}}$ defined in* (2.1.1) *is a manifold of dimension $D + |J| - 1$, where $D := \dim E_A + d$.*

(iii) *Y_J has boundary given by*

$$\begin{aligned}\partial Y_J := Y_J \cap \mathrm{pr}_\Delta^{-1}(\partial\Delta_J) &= \bigcup_{I \subsetneq J} \partial_{J\smallsetminus I} Y_J\\ &= \bigcup_{I \subsetneq J} \{(e,x;t) \in Y_J : x \in \widetilde{U}_{IJ},\ t \in \partial_{J\smallsetminus I}\Delta_J\}.\end{aligned}$$

Corollary 2.1.2 *If Proposition 2.1.1 holds, then for all $I \subsetneq J$ there is an embedding $\iota_{EU}\colon E_{A\smallsetminus I,\varepsilon_I} \times \widetilde{U}_{IJ} \to Y_{\mathcal{U},J,\underline{\varepsilon}}$ given by*

$$\iota_{EU}\colon (e_{A\smallsetminus I}, x) \mapsto (e_{A\smallsetminus I} + b_I^{-1}\cdot s_I(x), x; b_I).$$

Proof Since $s_J(\overline{U}_J) \subset E_{J,\varepsilon_J}$ by (i), this holds by (2.1.6). □

Proposition 2.1.1 shows that the boundary of Y_J lies over that of Δ_J. It is well known that the boundary of every topological manifold can be collared. The next step is to show that we can construct this collar to have a special form, with control over the components in $E_{A\smallsetminus I}$ near the "corner" $\mathrm{pr}_\Delta^{-1}(\partial_{J\smallsetminus I}\Delta_J)$. However, to establish this we need to pass to a *reduction* $\mathcal{V} = (V_I)_{I\in\mathcal{I}_\mathcal{K}}$ of the atlas (see (1.2.8)), since this severely restricts the overlaps $\pi_\mathcal{K}(V_I) \cap \pi_\mathcal{K}(V_J)$ in $|\mathcal{K}|$ of the different chart domains. We define

$$Y_{\mathcal{V},J,\underline{\varepsilon}} := Y_{\mathcal{U},J,\underline{\varepsilon}} \cap (E_A \times V_J \times \Delta_J). \tag{2.1.7}$$

Since $Y_{\mathcal{V},J,\underline{\varepsilon}}$ is an open subset of $Y_{\mathcal{U},J,\underline{\varepsilon}}$, it is a manifold of dimension

$$d + \dim E_A + |J| - 1$$

with boundary

$$\partial Y_{\mathcal{V},J,\underline{\varepsilon}} = Y_{\mathcal{V},J,\underline{\varepsilon}} \cap \partial Y_{\mathcal{U},J,\underline{\varepsilon}} \subset \bigcup_{I \subsetneq J} E_A \times (V_J \cap \widetilde{U}_{IJ}) \times \partial_{J \smallsetminus I} \Delta_J .$$

We denote by

$$\iota_{EV}\colon E_{A\smallsetminus I,\varepsilon_I} \times \partial \widetilde{V}_{IJ} \to Y_{\mathcal{V},J,\underline{\varepsilon}}, \quad (e_{A\smallsetminus I}, x) \mapsto (e_{A\smallsetminus I} + b_I^{-1} \cdot s_I(x), x; b_I). \tag{2.1.8}$$

the restriction of the map ι_{EU} in Corollary 2.1.2, and will consider the projections

$$\begin{aligned} \mathrm{pr}_V\colon Y_{\mathcal{V},J,\underline{\varepsilon}} &\to V_J, \quad && (e,x;t) \mapsto x, \\ \mathrm{pr}_{|V|}\colon Y_{\mathcal{V},J,\underline{\varepsilon}} &\to |V_J|, \quad && (e,x;t) \mapsto |x| := \pi_{\mathcal{K}}(x), \end{aligned}$$

where $\pi_{\mathcal{K}}$ is as in (1.2.7).

There is a corresponding category with objects $\bigsqcup_{J\in\mathcal{I}_{\mathcal{K}}} Y_{\mathcal{V},J,\underline{\varepsilon}}$ and morphisms given by the covering maps

$$\begin{aligned} (\rho^Y_{IJ})_*\colon Y_{\mathcal{V},J,\underline{\varepsilon}} \cap (E_A \times \widetilde{V}_{IJ} \times \iota_{IJ}(\Delta_I)) &\to Y_{\mathcal{V},I,\underline{\varepsilon}}, \\ (e,x;\iota_{IJ}(t)) &\mapsto (e, \rho_{IJ}(x); t). \end{aligned} \tag{2.1.9}$$

This category has realization

$$\underline{Y}_{\mathcal{V}} := \bigcup_{J\in\mathcal{I}_{\mathcal{K}}} Y_{\mathcal{V},J,\underline{\varepsilon}}/{\sim},$$

where $(e,x;t)_I \sim (e',x';t')_J$ for $|I| \le |J|$ if $I \subset J$, $e' = e$, $t' = \iota_{IJ}(t)$ and $\rho_{IJ}(x') = x$. Notice that the projections to Δ_J induce a map

$$\mathrm{pr}_\Delta\colon \underline{Y}_{\mathcal{V}} \to \Delta_{\mathcal{K}} = \bigcup_{J\in\mathcal{I}_{\mathcal{K}}} \Delta_J/{\sim},$$

where the simplicial complex $\Delta_{\mathcal{K}}$ (with boundary identifications induced by the face inclusions ι_{IJ}) is the topological realization of the poset $\mathcal{I}_{\mathcal{K}}$.[19] There is also a projection

$$\mathrm{pr}_{|\mathcal{V}|}\colon \underline{Y}_{\mathcal{V}} \to |\mathcal{V}| \sqsubset |\mathcal{K}|, \quad [e,x;t] \mapsto |x|.$$

Remarks 2.1.3 (i) The projection $\mathrm{pr}_{|\mathcal{V}|} \times \mathrm{pr}_\Delta$ induces a map

$$\underline{Y}_{\mathcal{V}} \to \|\mathcal{V}\|' \subset |\mathcal{V}| \times \Delta_{\mathcal{K}},$$

[19] The topological realization of a topological category has one k–simplex for each length-k composable string of morphisms, with the "obvious" boundary identifications. Thus, $\Delta_{\mathcal{K}}$ has one k–simplex for each $I \in \mathcal{I}_{\mathcal{K}}$ with $|I| = k+1$. Observe that as the associated footprint covering $(F_I)_{i\in\mathcal{I}_{\mathcal{K}}}$ of the zero set X is refined, the space $\Delta_{\mathcal{K}}$ gives better and better approximations to the topology of X; indeed the Čech cohomology of $\Delta_{\mathcal{K}}$ converges to that of X.

whose image $\|\mathcal{V}\|'$ is closely related to, but not the same as, the topological realization $\|\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\succ\Gamma}\|$ of the category $\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\succ\Gamma}$ in (1.2.9). For example, if $x \in V_J$ is such that its image $|x| := \pi_{\mathcal{K}}(x)$ in $|\mathcal{K}|$ lies outside all the other sets $\mathrm{pr}_{\mathcal{K}}(V_I)$ for $I \neq J$, then it gives rise to a single point in $\|\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\succ\Gamma}\|$ (since the only morphism involving x is the identity morphism) while it corresponds to a whole simplex $x \times \Delta_J$ in $\|\mathcal{V}\|'$.[20] The partial boundary $\partial' Y_{\mathcal{V},J,\underline{\varepsilon}} \subset \partial Y_{\mathcal{V},J,\underline{\varepsilon}}$ that we consider below could be understood in terms of an embedding of $\|\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\succ\Gamma}\|$ into $\|\mathcal{V}\|'$. However, we will take a more naive, geometric point of view.

(ii) We saw in Remark 1.3.8 that in the polyfold setting one can use an sc–smooth partition of unity to construct a finite-dimensional branched manifold M with section $\mathscr{S}\colon M \to E_A$ that is a global chart for X. One can think of the extra coordinates $t \in \Delta_J$ (with $\sum t_i = 1$) as a kind of "external" partition of unity that gives a more indirect way to patch the different coordinate charts together. $\diamond$

The boundary collar We now consider lifts to $Y_{\mathcal{V},J,\underline{\varepsilon}}$ of the collar on $\partial\Delta_J$

$$c_J^{\Delta}\colon \partial\Delta_J \times [0,w) \to \Delta_J, \quad (t,r) \mapsto (1 - r|J|)t + r|J|b_J, \tag{2.1.10}$$

where $b_J = (1/|J|,\dots,1/|J|)$ is the barycenter of Δ_J and $w < 1/(4|J|)$; see Figure 5. Note that any $t \in \Delta_J$ with at least one component $t_i < w$ is in the image of this collar. In order to get maximal control over the collar we will not define it on all of $\partial Y_{\mathcal{V},J,\underline{\varepsilon}}$ since much of $\partial Y_{\mathcal{V},J,\underline{\varepsilon}}$ is irrelevant to the task at hand. Indeed, we are only interested in boundary points $(e,x;t)$ with $x \in \widetilde{V}_{IJ}$ for $I \subsetneq J$ while, by Proposition 2.1.1, a general boundary point has

$$x \in V_J \cap s_J^{-1}(E_I) = V_J \cap \widetilde{U}_{IJ},$$

a set that is usually strictly larger than the overlap $\widetilde{V}_{IJ}$ (which is defined in (1.3.3)). Although the submersion axiom (1.2.3) implies that each $\widetilde{V}_{IJ}$ is a submanifold in V_J of codimension $\dim(E_{J\smallsetminus I})$, we will make the following definition of the "boundary" of V_J:

$$\partial V_J := \bigcup_{H \subsetneq J} \widetilde{V}_{HJ}, \tag{2.1.11}$$

which lies over the "boundary" $\partial|V_J| = \bigcup_{H\subsetneq J} |V_{HJ}|$ of $|V_J|$.

[20] If the isotropy is trivial, there is an embedding $\|\boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\succ\Gamma}\| \to \|\mathcal{V}\|'$, whose image can be described using versions of the sets $\overline{\mathrm{st}}_J^{\Delta}(|x|)$ in (2.1.13) below.

We will define the collar

$$c^Y_J \colon \partial' Y_{\mathcal{V},J,\underline{\varepsilon}} \times [0, w_J) \to Y_{\mathcal{V},J,\underline{\varepsilon}}$$

over a subset $\partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$ of points $(e, x; t) \in \partial Y_{\mathcal{V},J,\underline{\varepsilon}}$ such that $x \in \partial V_J$ and t is restricted to lie in the set $\overline{\mathrm{st}}^{\Delta}_J(|x|)$ defined as follows. Recall that for each $x \in V_J$ the sets H such that $|x| := \pi_{\mathcal{K}}(x) \in \pi_{\mathcal{K}}(V_H)$ (where $\pi_{\mathcal{K}} \colon V_J \to |\mathcal{K}|$ is the projection (1.2.7)) form a chain

$$I := I_{\min}(|x|) = I_0(|x|) \subsetneq I_1(|x|) \subsetneq \cdots \subsetneq I_m(|x|) = I_{\max}(|x|) =: K. \tag{2.1.12}$$

If $J = I_n(|x|)$ with $n \le m$, we will write

$$\overline{\mathrm{st}}^{\Delta}_J(|x|) := \mathrm{conv}(b_{I_0}, b_{I_1}, \dots, b_{I_{n-1}}) \subset \partial_{J \smallsetminus I_{n-1}(|x|)} \Delta_J \tag{2.1.13}$$

for the convex hull of the barycenters of the simplices corresponding to the elements of this chain; see Figure 1. Note that $\overline{\mathrm{st}}^{\Delta}_J(|x|)$ lies in the boundary of Δ_J.

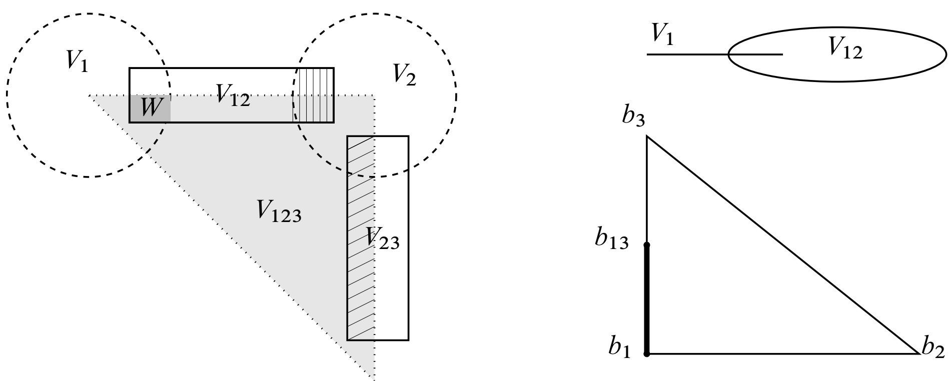

Figure 1: The figure on the left is schematic, showing the sets $|V_I|$ rather than their (disjoint) lifts V_I; the sets $V_{2,12} \subset V_2$ and $V_{23,123} \subset V_{23}$ are hatched, while for x in the shaded set W, we have $I_{\min}(|x|) = \{1\}$, $I_1(|x|) = \{1,2\}$, $I_{\max}(|x|) = \{1,2,3\}$. The top-right illustrates the change in dimension from V_1 to V_{12}, while the bottom-right shows $\overline{\mathrm{st}}^{\Delta}_J(|x|)$ for $x \in \widetilde{V}_{1,123} \cap \widetilde{V}_{13,123}$.

The domain $\partial' Y_{\mathcal{V},J,\underline{\varepsilon}} \subset \partial Y_{\mathcal{V},J,\underline{\varepsilon}}$ of the collar map c^Y_J contains all the points in the image of the injections ι_{EV} in (2.1.8), as well as the lifts to $Y_{\mathcal{V},J,\underline{\varepsilon}}$ of all points in $\mathrm{im}(c^Y_H)$ where $I \subsetneq H \subsetneq J$. To obtain points with more general t–coordinate we consider the following *rescaling operation*: Suppose given $t \in \Delta_J$ and a tuple $\mu_J = (\mu_j)_{j \in A}$ such that $\mu_j = 1$ if $j \notin J$, $\mu_j > 0$ for all j, and $\mu_J \cdot t \in \Delta_J$. Then, for any element $(e, x; t) \in Y_{\mathcal{V},J,\underline{\varepsilon}}$, there is a commutative diagram

$$\begin{tikzcd}
(e, x; t) \arrow[r, mapsto, "\mu_J \cdot"] \arrow[d, mapsto, "\mathrm{pr}_{E_{A\smallsetminus J}} \times \mathrm{pr}_V"'] & ((\mu_J)^{-1} \cdot e, x; \mu_J \cdot t) \arrow[d, mapsto, "\mathrm{pr}_{E_{A\smallsetminus J}} \times \mathrm{pr}_V"] \\
(e_{A\smallsetminus J}, x) \arrow[r, mapsto, "="] & (e_{A\smallsetminus J}, x)
\end{tikzcd} \tag{2.1.14}$$

where we assume $\|(\mu_J)^{-1} \cdot e\| < \kappa \varepsilon_{I(x)}$, so that the top arrow has target in $Y_{\mathcal{V},J,\underline{\varepsilon}}$.

The following result concerns a reduction $\mathcal{V}$ plus choice of constants $\underline{\varepsilon}$ that are *compatible* in the sense of Definition 3.1.9. In particular this means that property (i) in Proposition 2.1.1 holds, and that $(\mathcal{V}, \underline{\varepsilon})$ is compatible with a fixed choice of local product structures as in (1.2.3). The proof is given in Lemma 3.2.1 below.

Proposition 2.1.4 *Let $(\mathcal{V}, \underline{\varepsilon})$ be a compatible reduction of an atlas $\mathcal{K}$. Then, for each $J \in \mathcal{I}_{\mathcal{K}}$, there is an open subset $\partial' Y_{\mathcal{V},J,\underline{\varepsilon}} \subset \partial Y_{\mathcal{V},J,\underline{\varepsilon}}$, a constant $w_J > 0$ and a Γ_A–equivariant embedding*

$$c_J^Y : \partial' Y_{\mathcal{V},J,\underline{\varepsilon}} \times [0, w_J) \to Y_{\mathcal{V},J,\underline{\varepsilon}}, \quad ((e, x; t), r) \mapsto (e', x'; c_J^{\Delta}(t, r)), \tag{2.1.15}$$

with the following properties:[21]

- $\partial' Y_{\mathcal{V},J,\underline{\varepsilon}} \subset \{(e, x; t) :$ *for some* $I \subsetneq J$ *and* $x^0 \in \widetilde{V}_{IJ}$, $x \approx x^0$ *and* $t \in \overline{\mathrm{st}}_J^{\Delta}(|x^0|)\}$.
- c_J^Y *is* compatible with the projections *to* $E_{A\smallsetminus \bullet}$ *as follows: we have*

$$\iota_{EV}(E_{A\smallsetminus I,\varepsilon_I} \times \widetilde{V}_{IJ}) \subset \partial' Y_{\mathcal{V},J,\underline{\varepsilon}} \quad \text{for all } I \subsetneq J. \tag{2.1.16}$$

Further,

$$\begin{aligned} &c_J^Y((e, x; t), 0) = (e, x; t) \quad \text{for all } (e, x; t) \in \partial' Y_{\mathcal{V},J,\underline{\varepsilon}}, \\ &\mathrm{pr}_{E_{J\smallsetminus I}}(e) = 0 \implies c_J^Y((e, x; t), r) = (e, x; c_J^{\Delta}(t, r)), \end{aligned} \tag{2.1.17}$$

and

$$\mathrm{pr}_{E_{A\smallsetminus I}} \circ c_J^Y(\iota_{EV}(e, x), r) = \mathrm{pr}_{E_{A\smallsetminus I}}(e) \quad \text{for all } (e, x) \in E_{A\smallsetminus I,\varepsilon_I} \times \widetilde{V}_{IJ}. \tag{2.1.18}$$

- *The sets* $\partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$ *are* compatible with covering maps *as follows: if* $I \subsetneq H \subsetneq J$, *then the relevant part of the image of* c_H^Y *lifts to the domain* $\partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$ *of* c_J^Y. *More precisely, if* $(e, x; t) \in \partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$ *has* $x \in \rho_{HJ}^{-1}(\widetilde{V}_{IH}) \cap \widetilde{V}_{HJ}$[22] *and* $t \in \partial_{H\smallsetminus I} \Delta_H$,

[21] The precise definition of $\partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$ may be found in (3.2.19) and (3.2.20). By slight abuse of language we will call $\partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$ the *domain* of c_J^Y.

[22] By (1.3.3), when $I \subsetneq H \subsetneq J$ any two of the sets $\widetilde{V}_{IJ}$, $\widetilde{V}_{HJ}$ and $\rho_{HJ}^{-1}(\widetilde{V}_{IH})$ determine the third.

then $(e, \rho_{HJ}(x); t)$ *is in the domain* $\partial' Y_{\mathcal{V},H,\underline{\varepsilon}}$ *of* c^Y_H *and for all* $r \in [0, w_H)$ *there is* $(e', x'; t') \in \partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$ *with* $x' \in \widetilde{V}_{HJ}$ *such that*

$$c^Y_H((e, x; t), r) = (e', \rho_{HJ}(x'); t') \in Y_{\mathcal{V},H,\underline{\varepsilon}}. \tag{2.1.19}$$

Further, the restriction of c^Y_H *to* $Y_{\mathcal{V},H,\underline{\varepsilon}} \cap \mathrm{pr}_V^{-1}(\widetilde{V}_{IH} \cap V_{HJ})$ *has a well defined lift (also called* c^Y_H*) to* $Y_{\mathcal{V},J,\underline{\varepsilon}}$ *such that, for all* $x \in \widetilde{V}_{IJ} \cap \widetilde{V}_{HJ}$*,*

$$(\mathrm{pr}^Y_{HJ})_*(c^Y_H(e, x; t), r) = \bigl(c^Y_H(e, \rho_{HJ}(x), t), r\bigr) \in Y_{\mathcal{V},H,\underline{\varepsilon}}, \quad r \in [0, w_H), \tag{2.1.20}$$

where $(\mathrm{pr}^Y_{HJ})_*$ *is as in* (2.1.9).

- *Each* $\partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$ *is* ***invariant under rescaling*** *as follows: if* $(e, x; t) \in \partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$*, where* $t \in \overline{\mathrm{st}}^{\Delta}_H(|x|)$*, then, for all* μ_H *as in* (2.1.14) *such that* $\mu_H \cdot t \in \overline{\mathrm{st}}^{\Delta}_H(|x|)$*, we have*

$$\mu_H \cdot (e, x; t) := (\mu_H^{-1} \cdot e, x; \mu_H \cdot t) \in \partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$$

  and

$$\begin{aligned} &\mathrm{pr}_{E_{A \smallsetminus H} \times V} \circ c^Y_J((e, x; t), r) \\ &\qquad = \mathrm{pr}_{E_{A \smallsetminus H} \times V} \circ c^Y_J((\mu_H^{-1} \cdot e, x; \mu_H \cdot t), r) \in E_{A \smallsetminus H} \times V_J. \end{aligned} \tag{2.1.21}$$

- *The collar maps* c^Y_J are compatible with shrinkings *as follows: if* $(\mathcal{V}', \underline{\varepsilon}') \sqsubset (\mathcal{V}, \underline{\varepsilon})$ *is another compatible reduction, then there are constants* $0 < w'_J < w_J$ *such that the restrictions of the maps* c^Y_J *to* $\partial' Y_{\mathcal{V}',J,\underline{\varepsilon}'} := \partial Y_{\mathcal{V}',J,\underline{\varepsilon}'} \cap \partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$ *have all the above properties with respect to the constants* w'_J*.*
- *If* $\mathcal{K}$ *is oriented then the collar map* c^Y_J *is compatible with the natural induced orientation on its domain and range.*

By Lemma 3.1.11, any reduction $\mathcal{V}''$ has a shrinking $\mathcal{V} \sqsubset \mathcal{V}''$ that is compatible with respect to some choice of constants $\underline{\varepsilon}$ and hence supports a collar $(c^Y_J)_{J \in \mathcal{I}_{\mathcal{K}}}$ as in Proposition 2.1.4. Further, we show in Corollary 3.2.3 that $(\mathcal{V}^\infty, \underline{\varepsilon}^\infty)$ has a further nested shrinking that is collar compatible in the following sense.

Definition 2.1.5 Let $(\mathcal{V}^\infty, \underline{\varepsilon}^\infty)$ be a compatible reduction, with collars $(c^{Y,\infty}_J)_{J \in \mathcal{I}_{\mathcal{K}}}$. We say that a shrinking $(\mathcal{V}, \underline{\varepsilon}) \sqsubset (\mathcal{V}^\infty, \underline{\varepsilon}^\infty)$ is *collar compatible* if it is compatible as in Definition 3.1.9 and if for all $J \in \mathcal{I}_{\mathcal{K}}$ the collar map $c^{Y,\infty}_J$ restricts to a collar $(c^Y_J)_J$ on $(\mathcal{V}, \underline{\varepsilon})$ whose widths w_J satisfy $\sqrt{\varepsilon_I} < w_J$ for all $I \subsetneq J$.

2.2 Construction of the category $\boldsymbol{M}$ and functor $\mathscr{S}\colon \boldsymbol{M} \to \boldsymbol{E_A}$

In (1.3.5), the component M_J of $\mathrm{Obj}_{\boldsymbol{M}}$ was defined as

$$M_J = E_{A\smallsetminus J,\varepsilon_J} \times V_J, \tag{2.2.1}$$

which is a manifold of dimension $d + \dim E_A$. We take $M_{IJ} := E_{A\smallsetminus J,\varepsilon_J} \times V_{IJ}$, and define the map $\tau_{IJ}\colon \widetilde{M}_{IJ} \to M_{IJ}$ that attaches M_J to M_I to have domain a suitable open subset $\widetilde{M}_{IJ} \subset M_J$ and to extend the atlas structural map

$$\rho_{IJ}\colon \{0\} \times \widetilde{V}_{IJ} \to \{0\} \times V_{IJ} \subset M_{IJ} \subset M_I.$$

We require that τ_{IJ} be a Γ_A–equivariant covering map, induced by a free action of $\Gamma_{J\smallsetminus I}$. Further, to obtain a category, these maps must be compatible with composition, ie for $I \subset H \subset J$ we need

$$\tau_{HJ} \circ \tau_{IH} = \tau_{IJ} \quad \text{on } \widetilde{M}_{IJ} \cap \widetilde{M}_{HJ} \cap \tau_{HJ}^{-1}(\widetilde{M}_{IH}) = \widetilde{M}_{IJ} \cap \widetilde{M}_{HJ}. \tag{2.2.2}$$

(Note that by (1.3.3) any two of the sets $\widetilde{M}_{IJ}$, $\widetilde{M}_{HJ}$ and $\tau_{HJ}^{-1}(\widetilde{M}_{IH})$ determine the third.) For maximal elements J of $\mathcal{I}_{\mathcal{K}}$, we then define $\mathscr{S}_J\colon M_J \to E_A$ as the projection

$$\mathscr{S}_J\colon M_J \to E_A, \quad (e_{A\smallsetminus J}, x) \mapsto (e_{A\smallsetminus J}, s_J(x)).$$

The above should be considered as the default formula for $\mathscr{S}_J$, which holds at points $(e_{A\smallsetminus J}, x) \in M_J$ where x is far from any overlap V_{JK} with $J \subsetneq K$. However, in general it must be modified in ways explained in Example 2.2.1 below.

Before giving the general formulas for μ_J, τ_{IJ} and $\mathscr{S}_J$, we discuss an example. Part (i) shows the role of the collar in constructing τ_{IJ}, and also how to achieve the closed graph condition in (1.3.6), while part (ii) explains the relevance of the collar's compatibility with projections and rescaling to the proof of the composition rule (2.2.2). The usefulness of considering multiple collar compatible shrinkings $(\mathcal{V}^n, \underline{\varepsilon}^n)$ will also become apparent. We will use cutoff functions $(\chi_{IJ}\colon V_I \to [0,1])_{I \subsetneq J}$ of the following form: if $\mathcal{V} \sqsubset \mathcal{V}'$, we have

$$\operatorname{supp}(\chi_{IJ}) \subset \bigcup_{I \subsetneq H \subset J} V'_{IH} \quad \text{and} \quad \bigcup_{I \subsetneq H \subset J} \overline{V}_{IH} \subset \operatorname{int}(\chi_{IJ}^{-1}(1)). \tag{2.2.3}$$

Example 2.2.1 (attaching the M_J) We begin by considering the case when the isotropy groups are trivial, so that $\tau_{IJ}\colon \widetilde{M}_{IJ} \to M_{IJ}$ is a homeomorphism. It is then easiest to define its inverse

$$\alpha_{IJ} := \tau_{IJ}^{-1}\colon M_{IJ} \to \widetilde{M}_{IJ},$$

since $M_{IJ} \subset M_I$ is defined to be the product $E_{A\smallsetminus I,\delta_I} \times V_{IJ}$ (where V_{IJ} is as defined in (1.3.3)) while $\widetilde{M}_{IJ}$ will simply be defined as the image $\alpha_{IJ}(M_{IJ})$. As in [14], we use the notation $\phi_{IJ} := \rho_{IJ}^{-1}\colon V_{IJ} \to \widetilde{V}_{IJ}$ for the inverse of the atlas structural map ρ_{IJ}.

(i) Consider the case when there are two basic charts with labels 1 and 2. Then $\boldsymbol{M}$ has three components:[23]

$$M_1 = E_{2,\delta_2} \times V_1, \quad M_2 = E_{1,\delta_1} \times V_2, \quad M_{12} := V_{12},$$

where we assume $(\mathcal{V}, \underline{\delta})$ is collar compatible as in Definition 2.1.5. In particular, this means that for $i = 1, 2$ we have $\delta_i < w_{12}^2$, where w_{12} is the width of the collar c_{12}^Y. We first define the attaching maps $\alpha_{1,12}$ and $\alpha_{2,12}$, then define the sections $\mathscr{S}_I$ and finally prune the sets M_{12} so as to satisfy the closed graph condition.

We define $\alpha_{1,12}$ as a composite $M_{1,12} := E_{2,\delta_2} \times V_{1,12} \to Y_{\mathcal{V},12,\underline{\delta}} \to M_{12}$:

$$\begin{aligned}
\text{(2.2.4)}\quad \alpha_{1,12}((e_2, x)) &= \mathrm{pr}_V\big(c_{12}^Y(\iota_{EV}(e_2, x), r)\big) \qquad\qquad (\text{with } r := \sqrt{\|e_2\|})\\
&= \mathrm{pr}_V\big(c_{12}^Y\big((s_1(x), e_2, \phi_{IJ}(x); b_1), r\big)\big)\\
&= \mathrm{pr}_V\big((e_1', e_2, x'; (1-r, r))\big)\\
&= x' \in V_{12} = M_{12},
\end{aligned}$$

where ι_{EV} is the map in (2.1.8), $b_1 = (1, 0)$ is the barycenter of Δ_1 considered as a point in Δ_2, we have used formula (2.1.10) for c_{12}^{Δ}, and we have used the fact from (2.1.18) that e_2 is unchanged by c_{12}^Y. We note the following:

- Because $(\mathcal{V}, \delta)$ is collar compatible, Definition 2.1.5 implies that the collar width satisfies $w_{12} > \sqrt{\|\delta_2\|} > r$. Hence, the element $c_{12}^Y\big((s_1(x), e_2, \phi_{IJ}(x); b_1), r\big)$ is well defined for all $(e_2, x) \in M_{1,12}$.
- Because the collar variable $r := \sqrt{\|e_2\|}$ vanishes for the points $(0, x) \in M_{1,12}$, the map $\alpha_{1,12}$ extends the inclusion $\phi_{IJ}\colon V_{IJ} \to \widetilde{V}_{IJ}$ by (2.1.17), as is required by Proposition 1.3.3(i). Further, for small enough δ_i the closures of the images of $\alpha_{1,12}$ and the similarly defined map $\alpha_{2,12}$ are disjoint.
- Because the points $(e, x; t) \in Y_{\mathcal{V},J,\underline{\varepsilon}}$ satisfy $s_J(x) = t \cdot e_J$ and we chose $r = \sqrt{\|e_2\|}$, we have

$$r\,\|e_2\| = (\|e_2\|)^{3/2} = \|s_2(x')\|,$$

so that $r = \|s_2(x')\|^{1/3}$ is determined by x'.

[23] Here we simplify notation by writing $M_{12} := M_{\{1,2\}}$, $M_{1,12} := M_{\{1\}\{1,2\}}$ and so on. For an example of this construction, see Section 1.4.

- To see that $\alpha_{1,12}$ is injective, notice that because c^Y_J is injective it suffices to check that the other elements, e'_1, e_2 and r, that appear in the tuple

$$(e'_1, e_2, x'; (1-r, r)) \in Y_{\mathcal{V},12,\underline{\varepsilon}}$$

are determined by $x' \in V_2$. But we saw above that $r = \|s_2(x')\|^{1/3}$, so that the equations $s_1(x) = (1-t)e'_1$ and $s_2(x) = te_2$ determine e'_1 and e_2.

We now define $\mathscr{S}_{12} := s_{12}\colon M_2 = V_2 \to E_{12}$, and define $\mathscr{S}_i$ on $\alpha^{-1}_{i,12}(\widetilde{M}_{i,12})$ by pullback; thus, on this set,

$$\mathscr{S}_i(e_j, x) = \big(\|e_j\|^{1/2} e_j, s_i(\alpha_{i,12}(e_j, x))\big), \quad i \neq j,$$

has the form claimed in (1.3.7). We then extend $\mathscr{S}_i$ to the rest of M_i by patching it to the default map $(e_j, x) \mapsto (e_j s_i(x)) \in E_j \times E_i = E_{12}$ via the cutoff χ_i in (2.2.3):

$$\text{(2.2.5)} \quad \mathscr{S}_i(e_j, x) = \chi_{i,12}(x)\big(\|e_j\|^{1/2} e_j, s_i(\alpha_{i,12}(e_j, x))\big) + (1 - \chi_{i,12}(x))(e_j, s_i(x)) \in E_{12}.$$

For this to be well defined, we need $\alpha_{i,12}$ to extend to a neighborhood of $M_{i,12}$ in M_i. But we can always assume that $\mathcal{V}$ is a shrinking of some other reduction $\mathcal{V}'$. Then, because the collar extends over $\mathcal{V}'$, we may extend $\alpha_{i,12}$ over the corresponding set $M'_{i,12}$ by using the above formula (2.2.4). It is then clear that $\mathscr{S}_i^{-1}(0) = \{0\} \times s_i^{-1}(0)$.

It remains to arrange that $\alpha_{i,12}$ has closed graph. Note that its restriction to $\{0\} \times V_{i,12}$ does have closed graph because $\mathcal{V}$ is a reduction of a good atlas $\mathcal{K}$, which among other things implies that the realization $|\mathcal{V}| \subset |\mathcal{K}|$ is Hausdorff; see the discussion around (1.2.7)–(1.2.8). Denote by

$$\text{(2.2.6)} \qquad \mathrm{Fr}(M_{i,12}) := \mathrm{cl}(M_{i,12}) \smallsetminus M_{i,12}$$

the frontier of $M_{i,12}$ in M_i, where, as usual, cl denotes the closure. As above, we may assume that $\alpha_{i,12}$ extends to a homeomorphism $\alpha_{i,12}\colon \mathrm{cl}(M_{i,12}) \to V'_{12}$, which evidently has a closed graph. Hence, it suffices to arrange that $V_{12} \cap \alpha_{i,12}(\mathrm{Fr}(M_{i,12})) = \varnothing$. But

$$V_{12} \cap \mathrm{cl}\big(\alpha_{i,12}(\mathrm{Fr}(M_{i,12}))\big) \subset \mathrm{cl}(\widetilde{V}_{i,12}) \smallsetminus \widetilde{V}_{i,12}$$

is a closed subset of V_{12} that is disjoint both from $\mathrm{cl}(\widetilde{V}_{j,12})$ (by the separation property of the sets $\widetilde{V}_{1,12}$ and $\widetilde{V}_{2,12}$) and from the zero set $s^{-1}_{12}(0)$ (because $|\mathcal{V}|$ is Hausdorff). Hence, as in Figure 2, if this set is nonempty, we can simply remove it from V_{12}, ie

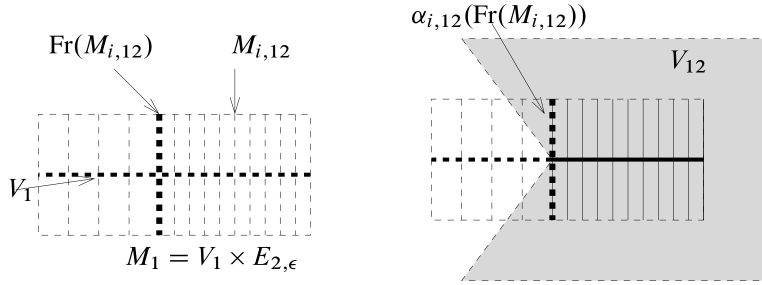

Figure 2: Removing points from V_{12} so that $\alpha_{i,12}$ has closed graph. Since V_{12} is open, the point where the two heavy lines cross is not in V_{12}. The set $\widetilde{M}_{i,12} \cong \widetilde{V}_{i,12} \times E_{i,\varepsilon_i} \subset V_{12}$ is hatched.

we replace V_{12} by

$$V_{12} \setminus \bigcup_{i=1,2} \mathrm{cl}\big(\alpha_{i,12}(\mathrm{Fr}(M_{i,12}))\big). \tag{2.2.7}$$

(ii) Now suppose that the atlas $\mathcal{K}$ has three basic charts with labels 1, 2 and 3, so that the sets V_I in the reduction $\mathcal{V}$ intersect as in Figure 1. We assume that the isotropy is trivial and all $E_i \neq 0$, and again explain how to choose the constants δ_i, and define the attaching maps α_{IJ} and sections $\mathscr{S}_I$ that involve the vertex 1, namely those with labels 1, 12, 13 and 123. It is now convenient to assume that we have four nested collar compatible shrinkings $(\mathcal{V}^1, \underline{\varepsilon}^1) \sqsubset (\mathcal{V}^2, \underline{\varepsilon}^2) \sqsubset (\mathcal{V}^3, \underline{\varepsilon}^3) \sqsubset (\mathcal{V}^4, \underline{\varepsilon}^4)$ of $\mathcal{V}'$. Correspondingly, for $I \subset \{1,2,3\}$ and $k \le \ell \le 4$ we define

$$M_I^k := E_{I,\varepsilon_i^k} \times V_I^k, \quad M_{IH}^{k,\ell} = E_{I,\varepsilon_i^k} \times V_{IH}^{k,\ell} \subset M_I^k,$$

where

$$V_{IH}^{k,\ell} = V_I^k \cap V_{IH}^\ell = V_I^k \cap \pi_{\mathcal{K}}^{-1}(\pi_{\mathcal{K}}(V_H^\ell)).$$

We aim to define a category with basic domains of the form $M_I^{|I|}$ and compatible morphisms $\alpha_{IH}\colon M_{IH}^{|I|,|H|} \to M_H^{|H|}$. However, to make these continuous and to define the corresponding maps $\mathscr{S}_I$ we have to define transition functions on larger sets such as $M_{IH}^{|I|+1,|H|}$. As in (i), we will first define suitable maps α_{IH} and sections $\mathscr{S}_I$, and then will prune domains to achieve the closed graph condition.

If $|I| = 1$ and $|H| = 2$, we define $\alpha_{IH}\colon M_{IH}^{1,2} \to M_H^2$ as in (i) above. These methods also easily adapt to define the maps α_{IH} for $|I| = 2$ and $\mathscr{S}_I\colon M_I \to E_A$ for $|I| \ge 2$. Indeed, if $J := \{1,2,3\}$ then

$$\alpha_{1i,J}\colon M_{1i,J}^{2,3} \to M_J^3$$

can be defined much as in (2.2.4). The only new point is that because Δ_{1i} is a 1–simplex, we have to decide how to lift $V_{1i,J}$ to $\partial Y_{\mathcal{V},J,\underline{\varepsilon}}$ in order to use the collar. For now, we use the default choice given by the embedding ι_{EV} in (2.1.8), ie we embed it over the barycenter b_{1i} of Δ_{1i}, which we identify with the corresponding point $\iota_{1i,J}(b_{1i})$ in Δ_J. Thus, for $i \neq j$, $i, j \in \{2,3\}$, we define

$$\text{(2.2.8)} \qquad \alpha_{1i,J}\colon M^{2,3}_{1i,J} \to M^3_J, \quad (e_j, x) \mapsto x',$$

as follows: with $(e_j, x) \in E_{3,\varepsilon_i^2} \times V^{2,3}_{1i}$ and $r = \sqrt{\|e_j\|}$,

$$\begin{aligned}(e_j, x) &\mapsto c^Y_J\big((\iota_{EV}(e,x)), r\big)\\ &\qquad\qquad =: (e'_{1i}, e_j, x'; c^\Delta_J(b_{1i}, r)) \in Y_{\mathcal{V}^3 J,\underline{\varepsilon}^3}\\ &\mapsto x' \in M^3_J.\end{aligned}$$

Since r depends on e_3 and hence on $s_3(x')$ as above, it follows as before that $\alpha_{1i,J}$ is injective. Notice also that if $x \in V^{2,\ell}_{1i,J}$ the point $\phi_{1i,J}(x)$ would lie in $\widetilde{V}^\ell_{1i,J}$ as would its image x' under the collar map since the collar maps preserve the shrinkings by Proposition 2.1.4. Taking $\ell = 4$ here, we may therefore define $\mathscr{S}_{12}$ by pullback from $\mathscr{S}_J$ on $M^{2,3}_{12,J}$, tapering it off to the product $s_{12} \times \mathrm{pr}_{E_j}$ outside the larger set $E_{j,\varepsilon_{1i}} \times V^{2,4}_{1i,J}$ by using the cutoff functions $\beta_{1i,J}$ as in (2.2.5).

The main new task is to define

$$\alpha_{1,J}\colon M^{1,3}_{1,J} \to M^3_J \qquad \text{so that} \quad \alpha_{1,J} := \alpha_{1i,J} \circ \alpha_{1,1i} \quad \text{in } M^{1,3}_{1,J} \cap M^{1,2}_{1,1i}.$$

If $x \in V^{1,3}_{1,J} \smallsetminus \bigcup_{i=2,3} V^{1,3}_{1,1i}$ (ie x is "far" from $V^{1,2}_{1,1i}$) then we may define

$$\text{(2.2.9)} \quad \alpha_{1,J}(e_{23}, x) = \mathrm{pr}_{E_3 \times V}\big(c^Y_J\big((s_1(x), e_{23}, \phi_{1,J}(x), b_1), r\big)\big), \quad r = \sqrt{\|e_{23}\|},$$

as in (2.2.4). Hence, the lift of $\alpha_{1,J}(e_{23}, x)$ to $Y_{\mathcal{V}^3,J,\underline{\varepsilon}^3}$ lies over the ray

$$c^\Delta_J(b_1 \times [0, w_0]) \subset \Delta_J.$$

On the other hand, the composite $\alpha_{1i,J} \circ \alpha_{1,1i}$ first uses the collar c^Y_{1i} for b_1 in Δ_{1i} and then the collar c^Y_J of b_{1i} in Δ_J, and hence its natural lift to $Y_{\mathcal{V}^3,J,\underline{\varepsilon}^3}$ is rather different. We interpolate between these two maps as follows, where we take $i = 2$ for clarity, and use cutoff functions $\beta_{1,12}$ as in (2.2.3), with support in $V^{1,3}_{1,12}$ and that equal 1 near the closed set $\overline{V}^{1,2}_{1,12} \sqsubset V^{1,3}_{1,12}$. Thus, with $x \in V^{1,3}_{1,12} \cap V^{1,3}_{1,J}$ and

$e_{23} = (e_2, e_3) \in E_2 3$, we write

$$r := \beta_{1,12}(x)\sqrt{\|e_2\|} \quad \text{and} \quad \bigl(r' := \max\bigl((1-\beta_{1,12}(x))\sqrt{\|e_2\|}, \sqrt{\|e_3\|}\bigr)\bigr)$$

and define

$$\begin{aligned}(2.2.10)\qquad (e_{23}, x) &\mapsto c^Y_{12}\bigl((s_1(x), e_{23}, \phi_{1,12}(x); b_1), r\bigr)\\ &\qquad\qquad =: (e_1', e_{23}, x'; 1-r, r) \in Y_{\mathcal{V}^3,12,\underline{\varepsilon}^3}\\ &\mapsto c^Y_J\bigl((e_1', e_{23}, \phi_{12,J}(x'); 1-r, r), r'\bigr)\\ &\qquad\qquad =: (e_1'', e_{23}, x''; t'') \in Y_{\mathcal{V}^3,J,\underline{\varepsilon}^3}\\ &\mapsto x'' =: \alpha_{1,J}((e_{23}, x)) \in M^3_J = V^3_J.\end{aligned}$$

Note the following:

- Here (as in (2.1.20)), we consider c^Y_{12} to be the lift to $\partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$ of the collar for $\partial' Y_{\mathcal{V},12,\underline{\varepsilon}}$, and the composite $c^Y_J \circ c^Y_{12}$ is defined by (2.1.19).
- The above map $(e_{23}, x) \mapsto x''$ is continuous, and equals that given in (2.2.9) when $\beta_{1,12}(x) = 0$ because $\|e_{23}\| = \max\{\|e_i\| : i = 2, 3\}$ by definition.
- If $x \in V^{1,3}_{1,J} \cap V^{1,2}_{1,12} \subset V^{1,3}_{1,J} \cap (\beta^{-1}_{1,12}(1))$, then

  $$\alpha_{1,12}(e_{23}, x) = \alpha_{12,J} \circ \alpha_{1,12}(e_{23}, x).$$

  Indeed, the invariance of the collar under rescaling in (2.1.21) shows that applying the second collar map at $(1-r, r)$ with $r' = \sqrt{\|e_3\|}$ and then projecting to M^3_J gives the same result as rescaling, then applying the second collar at b_{12} with the same r', and then projecting to M^3_J. Note that by (2.1.18) this last claim holds even if $e_3 = 0$, so that the second collar map has $r' = 0$ when $\beta_{1,12}(x) = 1$.
- It remains to check that this map $(e_{23}, x) \mapsto x''$ is injective. Since the first two maps in (2.2.9) are injective, it suffices to check that the projection

  $$(e_1'', e_{23}, x''; t'') \mapsto x''$$

  is injective. But both collar maps preserve e_2 and e_3 by the extended corner control in (2.1.18). Hence, for $i = 2, 3$ we know $\|e_i\|$ and therefore t_i'' from $s_i(x'') = t_i'' e_i$. Since $\sum_i t_i'' = 1$, we therefore know t'' and hence also $e'' = (e_1'', e_{23})$.

As before, we define $\mathscr{S}_1$ by pullback via $\alpha_{1,*}$ over $E_{23,\varepsilon_1} \times \bigcup_{\{1\}\subsetneq J} V_{1,J}$, extending to the rest of M_1 via a cutoff function $\beta_{1,J}$. However, to do this we need the pullback of $\mathscr{S}_1$ to be compatibly defined on a set that is larger than that on which we ultimately want $\mathscr{S}_1$ to equal the pullback. But we can arrange that the identity $\alpha_{1,J} = \alpha_{12,J} \circ \alpha_{1,12}$

actually holds on a neighborhood of the closure of $V^{1,2}_{1,12} \cap V^{1,3}_{1,J}$, since in (2.2.10) $\beta_{1,12} = 1$ on a neighborhood of $\overline{V}^{1,3}_{1,J}$, and we can always extend the domain of $\alpha_{1,12}$ to $V^{1,3}_{1,2}$. Therefore, we can imitate the formula in (2.2.5).

It remains to prune the domains M_J so as to achieve the closed graph condition for all maps α_{IJ}. We will do this by downwards recursion on I. Thus, first taking $|I| = 2$, we remove points from M_{123} so that the maps $\alpha_{I,123}$ have closed graph, and then with $I = \{i\}$ remove points from all M_J with $|J| \geq 2$ so that the maps $\alpha_{i,J}$ have closed graph. At each stage we use the analog of formula (2.2.7), removing from V_J all points in $\mathrm{cl}_{V_J}(\alpha_{IJ}(\mathrm{Fr}(M_{IJ})))$ where $\mathrm{Fr}(M_{IJ})$ is the frontier of $M_{IJ} = M^{|I|,|J|}_{IJ}$ in $M_I := M^{|I|,|I|}_I$. Since $\mathrm{Fr}(M_{IJ}) \subset M^{|I|,|J|+1}_{IJ}$, the points removed lie in the image of the extension of the collar over $Y_{J,\mathcal{V}^{|J|+1},\underline{\varepsilon}}$ but not in the image of the collar over $Y_{J,\mathcal{V}^{|J|},\underline{\varepsilon}}$. Hence, because the α_{IJ} are defined in terms of the collar map, these points do not lie in $\mathrm{im}\alpha_{HJ}$ for any $H \subsetneq I \subsetneq J$. Thus, the different steps do not interfere with each other.

(iii) If the isotropy is nontrivial, then we can still adopt the above approach, but now must interpret α_{IJ} as a local Γ_x–invariant inverse to τ_{IJ} and then define $\widetilde{M}_{IJ}$ to be the Γ_A–orbit of its image. Further, we must make equivariant constructions, but this is possible since the collar is equivariant, so that all the above formulas are appropriately equivariant. In particular, the sets that must be removed in order to achieve the closed graph condition for the local inverse α_{IJ} are Γ_x–invariant, so that we can arrange that τ_{IJ} has closed graph by removing its Γ_A orbit. $\diamond$

The next result is essentially a restatement of Proposition 1.3.3, though it gives a little more information on the nature of the map τ_{IJ}. Since the proof is rather complex, we describe the strategy here. As in Example 2.2.1, we define the maps α_{IJ} by downwards recursion on the cardinality $|I|$ of the index set I, shrinking domains at each step. In order to extend the interpolation formula for $\alpha_{IJ} = \tau^{-1}_{IJ}$ given in (2.2.10) to a chain of inclusions $I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_k$ of length $k > 1$, we apply an iterated sequence of collar maps over a family of paths $\mathcal{P}(e,x)$ in the simplex Δ_J as described in Step 2 below. We then define the attaching maps τ_{IJ} and $\mathscr{S}_I$, and check that they have the needed properties.

Proposition 2.2.2 *Suppose given a good atlas $\mathcal{K}$ on X. Then there is a reduction $\mathcal{V}$ and set of constants $\underline{\delta} = (\delta_I)_{I \in \mathcal{I}_\mathcal{K}} > 0$ such that the following properties hold with $M_I := E_{A \smallsetminus I,\delta_I} \times V_I$ and $M_{IJ} := E_{A \smallsetminus I,\delta_I} \times V_{IJ}$:*

(i) *For each $I \subset J$ there are open sets $\widetilde{M}_{IJ} \subset M_J$ and Γ_A–equivariant maps*

$$\tau_{IJ}\colon \widetilde{M}_{IJ} \to M_{IJ}$$

that restrict to ρ_{IJ} on $\{0\} \times \widetilde{V}_{IJ}$ and are such that:

- *$\widetilde{M}_{IJ}$ is a product $E_{A\smallsetminus J,\delta_I} \times \widetilde{M}^0_{IJ}$, where $\widetilde{V}_{IJ} \subset \widetilde{M}^0_{IJ} \subset V_J$, and $\mathrm{cl}(\widetilde{M}^0_{IJ})$ and $\mathrm{cl}(\widetilde{M}^0_{HJ})$ are disjoint unless I and H are nested.*
- *$\tau_{IJ} = \mathrm{id}_E \times \tau^0_{IJ}$ where $\tau^0_{IJ}\colon \widetilde{M}^0_{IJ} \to E_{J\smallsetminus I,\delta_I} \times V_{IJ}$ has the following properties:*
  - *$\tau^0_{IJ}(x) = (0, \rho_{IJ}(x))$ for $x \in \widetilde{V}_{IJ}$,*
  - *τ^0_{IJ} has closed graph, and*
  - *τ^0_{IJ} quotients out by a free action of $\Gamma_{J\smallsetminus I}$ that extends to a free action on a neighborhood of $\mathrm{cl}(\widetilde{M}^0_{IJ})$ in V_J.*

(ii) *For $I \subsetneq J \subsetneq K$ we have*

$$\tau_{JK}(\widetilde{M}_{IK} \cap \widetilde{M}_{JK}) = \widetilde{M}_{IJ} \cap M_{JK} \quad \text{and} \quad \tau_{IK} = \tau_{IJ} \circ \tau_{JK}. \tag{2.2.11}$$

(iii) *For each J there is $\mathscr{S}_J\colon M_J \to E_A$ such that, for all $J \subset K$, we have*

$$\mathscr{S}_J \circ \tau_{JK} = \mathscr{S}_K|_{\widetilde{M}_{JK}}, \quad \mathscr{S}_J^{-1}(E_J) \subset \{0\} \times V_J, \quad \mathscr{S}_J(0,x) = (0, s_J(x)). \tag{2.2.12}$$

(iv) *If the initial atlas $\mathcal{K}$ is oriented, then so is the category $\boldsymbol{M}$ defined by the above data as in* (1.3.5).

Corollary 2.2.3 *Proposition 1.3.3 holds.*

Proof If the category $\boldsymbol{M}$ is defined as in (1.3.5) using the above data, M_I, $\widetilde{M}_{IJ}$ and τ_{IJ}, then all the properties of Proposition 1.3.3 hold. □

Proof of Proposition 2.2.2 We proceed in five steps:

Step 1 (the set-up and basic strategy of proof) Fix a shrinking $\mathcal{G}^0 = (G^0_I)_{I\in\mathcal{I}_\mathcal{K}}$ of the footprint cover. By Corollary 3.2.3 we may choose a family of nested collar compatible shrinkings as above,

$$\psi^{-1}(\mathcal{G}^0) \sqsubset (\mathcal{V}^1, \underline{\varepsilon}^1) \sqsubset \cdots \sqsubset (\mathcal{V}^{\kappa+1}, \underline{\varepsilon}^{\kappa+1}) \sqsubset (\mathcal{V}^\infty, \underline{\varepsilon}^\infty) \sqsubset \mathcal{U}^\infty,$$

with collar widths that increase with m. The projection $\pi_{\mathcal{K}}\colon U_I^\infty \to |\mathcal{K}|$ quotients out by Γ_I and its restrictions to the $\mathcal{V}^m$ have the property that

$$\pi_{\mathcal{K}}(\overline{V_I^k}) \cap \pi_{\mathcal{K}}(\overline{V_J^\ell}) \neq \varnothing \iff I \subset J \text{ or } J \subset I.$$

For $m \le \ell$ we let $V_{IJ}^{m,\ell} := V_I^m \cap \pi_{\mathcal{K}}^{-1}(\pi_{\mathcal{K}}(V_J^\ell))$, and for $m \le |I|$ and $m \le \ell \le |J|$ define

$$M_I^m := E_{A\smallsetminus I,\varepsilon_I^m} \times V_I^m, \quad M_{IJ}^{m,\ell} = E_{A\smallsetminus I,\varepsilon_I^m} \times V_{IJ}^{m,\ell}. \tag{2.2.13}$$

For each $I \subsetneq J$ and $m \le |I|$ we will define $\mathscr{S}_J\colon M_J^{|J|} \to E_A$, a subset $\widetilde{M}_{IJ}^{m,\ell} \subset M_J^\ell$ and a Γ_A–equivariant covering map

$$\tau_{IJ}^{m,\ell}\colon \widetilde{M}_{IJ}^{m,\ell} \to M_{IJ}^{m,\ell}$$

with the following properties:

(a) $\tau_{IJ}^{m,\ell}$ has product form and closed graph as in (i), and quotients out by a free action of $\Gamma_{J\smallsetminus I}$ on $(\widetilde{M}_{IJ}^{m,\ell})^0$.

(b) For all $m \le m' \le |I|$ and $\ell \le \ell' \le |J|$, $\widetilde{M}_{IJ}^{m,\ell} \subset \widetilde{M}_{IJ}^{m',\ell'}$ and $\tau_{IJ}^{m',\ell'}|_{\widetilde{M}_{IJ}^{m,\ell}} = \tau_{IJ}^{m,\ell}$.

(c) If $I \subsetneq H \subsetneq J$ then $\tau_{IJ}^{|I|,|J|} = \tau_{HJ}^{|H|,|J|} \circ \tau_{IH}^{|I|,|H|}$ on their common domain; moreover, this domain maps onto

$$E_{A\smallsetminus I,\varepsilon_I} \times (V_{IJ}^{|I|,|J|} \cap \rho_{IJ}(V_{HJ}^{|H|,|J|})) \subset M_I^{|I|}.$$

(d) If $I \subsetneq J$ then $\mathscr{S}_I \circ \tau_{IJ} = \mathscr{S}_J$ on $\widetilde{M}_{IJ}^{|I|,|J|}$.

(e) $\mathscr{S}_J^{-1}(0) = \{0\} \times s_J^{-1}(0) \subset M_J^{|J|}$.

In the end we will take

$$M_I := M_I^{|I|}, \quad M_{IJ} := M_{IJ}^{|I|,|J|}$$

with the corresponding sets $\widetilde{M}_{IJ}^{|I|,|J|}$, and the restrictions of the maps τ_{IJ} and $\mathscr{S}_I$. In particular, $\delta_I = \varepsilon_I^{|I|}$.

For simplicity, we first assume that the isotropy groups are trivial. As in Example 2.2.1 (see in particular (2.2.10)) for $I \subsetneq J$ we will define a family of injective maps

$$\alpha_{IJ}\colon M_{IJ}^{|I|+1,|J|+1} \cap \{(e,x) \mid \|e_{J\smallsetminus I}\| < \varepsilon_I^{|I|}\} \to M_J^{|J|+1}, \quad \lambda \ge 1$$

(where $e_{J\smallsetminus I} := \mathrm{pr}_{E_{J\smallsetminus I}}(e)$), with well-defined restrictions

$$\alpha_{IJ} := \alpha_{IJ}|_{M_{IJ}^{m,k}} \colon M_{IJ}^{m,k} \to M_J^k, \quad m \le |I|+1,\ k \le |J|+1,\ m \le k, \tag{2.2.14}$$

such that

$$\alpha_{IJ} = \alpha_{HJ} \circ \alpha_{IH} \text{ on } M_{IJ}^{|I|,|J|} \cap \alpha_{IH}^{-1}(M_{HJ}^{|H|,|J|}) \quad \text{for all } I \subsetneq H \subsetneq J. \tag{2.2.15}$$

Then we define

$$\widetilde{M}_{IJ}^{m,\ell} = \alpha_{IJ}(M_{IJ}^{m,\ell}), \quad \tau_{IJ} = \alpha_{IJ}^{-1}.$$

With this, conditions (b)–(c) will hold and $\tau_{IJ} = \alpha_{IJ}^{-1}$ has the required product form. We will arrange the rest of (a) later.

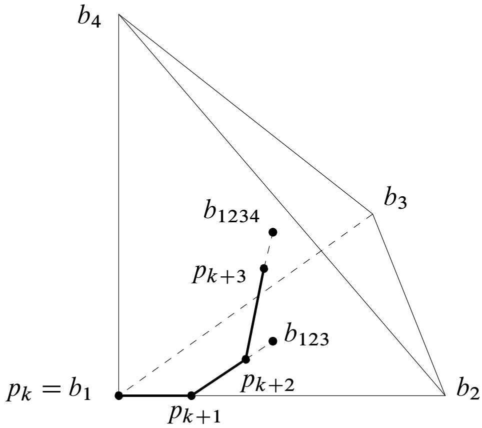

Figure 3: The path $\mathcal{P}(e,x)$ with $I_k = \{1\}, \dots, I_{k+3} = \{1,2,3,4\}$

Step 2 (definition of α_{IJ} via the paths $\mathcal{P}(e,x)$) To define $\alpha_{IJ}(e,x)$ we consider the chain of length $m = m(|x|)$ formed by the sets H such that $|x| \in |V_H|^{|H|+1}$,

$$I_{\min}(|x|) = I_0(|x|) \subsetneq I_1(|x|) \subsetneq \cdots \subsetneq I_m(|x|) = I_{\max}(|x|), \tag{2.2.16}$$

modifying the definition of $\overline{\mathrm{st}}_J^{\Delta}(|x|)$ from (2.1.13) accordingly. Extending the procedure in (2.2.10), if $I = I_k(|x|)$ we define $\alpha_{IJ}(e,x)$ by applying collar maps in $Y_{\mathcal{V}^{\kappa+1}, I_m, \underline{\varepsilon}^{\kappa+1}}$ a total of $m-k$ times with initial points $p_{n-1} \in \mathrm{pr}_{\Delta}^{-1}(\Delta_{I_n})$ and collar lengths r_n for $n = k+1, \dots, m = m(|x|)$. In fact, it is useful to think of applying the iterated collar map that lies over the path $\mathcal{P}(e,x)$ in $\overline{\mathrm{st}}_J^{\Delta}(|x|)$ with the vertices

$$p_k = b_{I_k}, \quad p_n = (1-r_n)p_{I_{n-1}} + r_n b_{I_n} = c_{I_n}^{\Delta}(p_{n-1}, r_n), \quad k < n \le m(|x|)$$

(see Figure 3), where the r_n are as described below. Note that by the collar compatibility with covering maps in (2.1.20) it makes no difference whether at the n^{th} step we apply the collar map over the segment $[p_{n-1}, p_n]$ in $Y_{\mathcal{V},I_n,\underline{\varepsilon}}$ (where $\mathcal{V} := \mathcal{V}^\kappa$) and then lift to the next level $Y_{\mathcal{V},I_{n+1},\underline{\varepsilon}}$, or whether we first lift all the way to $Y_{\mathcal{V},I_m,\underline{\varepsilon}}$ (where $I_m = I_{\max}$), and then apply the collar maps. We take the second approach, first lifting the initial point $(e_{A\smallsetminus I_k}, x)$ to

$$(e_{A\smallsetminus I_k} + b_{I_k}^{-1} \cdot s_{I_k}(x), \phi_{I_k I_m}(x); b_{I_k}) \in \partial_{I_m\smallsetminus I_k} Y_{\mathcal{V},I_m,\underline{\varepsilon}^{\kappa+1}} \cap \partial'_{I_m\smallsetminus I_0} Y_{\mathcal{V},I_m,\underline{\varepsilon}^{\kappa+1}}$$

and then applying successive collar maps that remain in the boundary $\partial' Y_{\mathcal{V},I_m,\underline{\varepsilon}^{\kappa+1}}$ until the very last step. Note that by the collar compatibility with shrinkings we can work in $\mathcal{V} := \mathcal{V}^\kappa$ rather than in the different $\mathcal{V}^i$.

To complete this definition of $\alpha_{IJ}(e_{A\smallsetminus I}, x)$ it remains to define the lengths $r_n = r_n(x)$ for $k+1 \le n \le m$. To achieve consistency with coordinate changes, for each $I \in \mathcal{I}_{\mathcal{K}}$, we choose a cutoff function $\chi_I \colon |\mathcal{K}| \to [0,1]$ such that

$$\operatorname{supp}(\chi_I) \subset \pi_{\mathcal{K}}(V_I^{|I|+1}), \quad \chi_I^{-1}(1) \subset \pi_{\mathcal{K}}(V_I^{|I|}), \tag{2.2.17}$$

and for each J denote its pullback to the set $V_J^{|J|+1}$ by the same letter. Then, writing $a_n := \sqrt{\|e_{I_n\smallsetminus I_{n-1}}\|}$ and $\chi_i := \chi_{I_i}$, we define

$$\begin{aligned}
r_{m+1}(x) &:= \chi_{m+1}(x) a_{m+1}, \\
r_{m+2}(x) &:= \chi_{m+2}(x) \max\bigl((1-\chi_{m+1}(x))a_{m+1}, a_{m+2}\bigr), \\
&\;\vdots \\
r_n(x) &:= \chi_n(x)\bigl(\max_{m<j\le n} \lambda_j a_j\bigr), \\
\lambda_j &:= \prod_{i=j}^{n-1} (1-\chi_i(x)), \quad j<n, \\
\lambda_n &:= 1.
\end{aligned}$$

To check that $\alpha_{IJ}(e_{A\smallsetminus I}, x)$ is well defined we note the following:

- The path $\mathcal{P}(e,x)$ depends both on the position of $|x|$ with respect to the sets $|V_H|^{|H|+1}$ in the chain (2.2.16), and on the relative sizes a_k of the relevant components of e; see equations (2.2.10).
- In order for the collar maps to be defined over $\mathcal{P}(e,x)$, we must have $r_n(x) < w_{I_n}$ for all n. But

  $$r_n \le \max_{m<j\le n} \sqrt{\|e_{I_n\smallsetminus I_{n-1}}\|} < \sqrt{\|e_{A\smallsetminus I}\|} < \sqrt{\varepsilon_{I_m}} < w_{I_n}$$

  for all $m > n$ because $(\mathcal{V}, \underline{\varepsilon})$ is collar compatible; see Definition 2.1.5.

- Further, at each stage we need the image of the iterated collar map to lie in the domain of the next collar map It follows from (2.1.16) that the initial point
$$(e_{A\smallsetminus I_k} + b_{I_k}^{-1}\cdot s_{I_k}(x), \phi_{I_k I_m}(x); b_{I_k})$$
of $\mathcal{P}(e,x)$ does lie in $\partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$. One then uses the fact that these domains $\partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$ are compatible with covering maps, as explained in (2.1.19)–(2.1.20).
- To see that the path $\mathcal{P}(e,x)$ varies continuously with x, it suffices to check continuity for a sequence of points $x^\nu \to x^\infty$ for which just one of the functions χ — say χ_s — changes from a positive value to zero. But in this case (assuming that e is fixed) the functions $r_i(x)$ are continuous for $i < s$, while for $i \geq s$ we have
$$\begin{gathered} a_i^\nu = a_i^\infty, \quad i \neq s, s+1, \qquad a_s^\infty = \max(a_s^\nu, a_{s+1}^\nu), \\ \lim_\nu r_i(x^\nu) = r_i(x^\infty), \quad i < s, \qquad \lim_\nu r_s(x^\nu) = 0, \\ \lim_\nu r_i(x^\nu) = r_{i-1}(x^\infty), \quad i > s. \end{gathered}$$
- If $x \in V_{IH}^{|I|+1,|H|}$ for $H = I_s$ with $m < s < n$, then $\chi_s(x) = 1$. In this case, we can divide $\mathcal{P}(e,x)$ into two independent segments at the point p_s, because the lengths $r_n(x)$ for $n > s$ no longer depend on a_i for $i \leq s$ since $\lambda_i = 0$ for $i \leq s$. Further, the second part of $\mathcal{P}(e,x)$ projects to the path $\mathcal{P}(\phi_{IH}(x))$ under the natural projection
$$(\mathrm{conv}(b_{I_0},\dots,b_{I_{\max}(|x|)})) \smallsetminus (\mathrm{conv}(b_{I_0},\dots,b_{I_s})) \to \mathrm{conv}(b_{I_s},\dots,b_{I_{\max}}).$$

Step 3 (definition of the maps α_{IJ} and sections $\mathscr{S}_I$ in the case of trivial isotropy) With these formulas in hand, we now define the maps α_{IJ} and sections $\mathscr{S}_I$ by downwards recursion on $|I|$. For $|J| = \kappa := \max\{|J| : J \in \mathcal{I}_{\mathcal{K}}\}$, we define
$$\mathscr{S}_J := \mathscr{S}'_J,$$
where
$$\mathscr{S}'_J\colon M_J \to E_A, \quad (e_{A-J}, x) \mapsto (e_{A\smallsetminus J}, s_J(x)).$$
If $|I| = \kappa - 1$, for $x \in V_{IJ}^{|I|+1,|J|}$ the path $\mathcal{P}(|x|)$ has one segment of length $\chi_I a_k := \chi_I\sqrt{\|e_{J\smallsetminus I}\|}$, and we define $\alpha_{IJ}\colon M_{IJ}^{|I|,|J|} \to M_J^{|J|}$ by applying the collar map as in (2.2.8). For these values of x we have $\chi_I(x) = 1$. However, the fact that we have defined α_{IJ} over the larger set $V_{IJ}^{|I|+1,|J|}$ means that the function

$$\mathscr{S}_I := \prod_{J : I \subsetneq J} (1 - \chi_J)\mathscr{S}'_I + \sum_{J : I \subsetneq J} \chi_J \alpha_{IJ}^*(\mathscr{S}_J)\colon V_I^{|I|} \to E_I \tag{2.2.18}$$

is well defined and is compatible under pullback from V_J.

Let us now suppose that maps $\alpha_{IJ}\colon V_{IJ}^{|I|+1,|J|+1} \to V_J^{|J|+1}$, and functions $\mathscr{S}_I\colon V_I^{|I|} \to E_A$ have been defined for all $I \subsetneq J$ with $|I| > k$ so as to satisfy conditions (2.2.14)–(2.2.15), and consider I with $|I| = k$. Because there are no transition functions $\alpha_{II'}$ between these sets V_I, we can work separately with each such I. Then define $\alpha_{IJ}(x)$ for $x \in V_{IJ}^{|I|+1,|J|+1}$ by applying the collar maps c_{HJ}^Y as described in Step 2 over the part, called $\mathcal{P}_{IJ}(x)$ below, of the path $\mathcal{P}(e,x)$ from $p_k = b_I$ (where $I = I_k(|x|)$) to p_q, where $J = I_q(|x|)$.

We check the properties of α_{IJ} as follows.

- The map α_{IJ} depends continuously on x because we saw above that the path $\mathcal{P}(e,x)$ depends continuously on x, and because by (2.1.18) the collar map along a path segment of length 0 is the identity.
- Both $\widetilde{M}_{IJ}$ and α_{IJ} have the product form required by (a) because the collar map c_J^Y does not change the components of $e_{A\smallsetminus I}$ that lie in $E_{A\smallsetminus J}$; see (2.1.18).
- We repeatedly use the fact that the collar is compatible with all the shrinkings to show that (b) holds.
- To prove the composition formula (c), we use the fact proved above that when $x \in M_{IH}^{|I|+1,|H|}$, the path $\mathcal{P}_{IJ}(x)$ divides into two independent segments, the first of which is simply $\mathcal{P}_{IH}(x)$, while the second projects onto $\mathcal{P}_{HJ}(\phi_{IJ}(x))$. Now use the invariance of the collar map under rescaling (2.1.21).
- To see that α_{IJ} is injective, notice first that the path $\mathcal{P}_{IJ}(x)$ is determined by x. Hence, the collar maps applied to the lift $(e', \phi_{IJ}(x); b_I)$ of $(e,x) \in M_{II}$ to $Y = Y_{\mathcal{V},J,\underline{\varepsilon}}$ give a point in Y that lies over a point $t_x \in \Delta_J$, which is determined by $\mathcal{P}(e,x)$ because the collar c_J^Y lifts c_J^Δ by (2.1.15). But the collar maps are injective, as is the projection $Y_{\mathcal{V},J,\underline{\varepsilon}} \cap \mathrm{pr}_\Delta^{-1}(t_x)$ to M_J.

Finally, we define $\mathscr{S}_I$ as in (2.2.18). This clearly has the properties required in (iii).

Step 4 (completion of the proof in the case of trivial isotropy) The first claims in (i), namely that τ_{IJ}^0 extends ρ_{IJ} and that $\widetilde{M}_{IJ}$ is a product of the form $E_{A\smallsetminus J,\delta_I} \times \widetilde{M}_{IJ}^0$, are clear. To establish the separation claim, namely that $\mathrm{cl}(\widetilde{M}_{IJ}^0) \cap \mathrm{cl}(\widetilde{M}_{HJ}^0) = \varnothing$ unless I and H are nested, notice that the intersections of $\mathrm{cl}(\widetilde{M}_{IJ}^0)$ and $\mathrm{cl}(\widetilde{M}_{HJ}^0)$ with $\{0\} \times V_J$ certainly have this property by definition of a reduction. Hence, starting with maximal $|I|$ as usual, we may, if necessary, shrink the constants δ_I so that this property holds. Further, (iv) holds, because if $\boldsymbol{K}$ is oriented, then so are all the manifolds $Y_{\mathcal{U},J,\underline{\varepsilon}}$ and M_I. Since the structural maps in $\mathcal{K}$ preserve orientation by definition, and the collar maps c_J^Y preserve orientation by Proposition 2.1.4, so do the maps τ_{IJ} constructed above.

It remains to arrange that the maps α_{IJ} have closed graph. We do this by the method described in Example 2.2.1. Given $I \subsetneq J$, recall that $M_{IJ} := M_{IJ}^{|I|,|J|} \subset M_{IJ}^{|I|,|J|+1}$ and define $\mathrm{Fr}(M_{IJ}) := \mathrm{cl}(M_{IJ}) \setminus M_{IJ}$, where we take the closure in $M_{IJ}^{|I|,|J|+1}$. Then the maps α_{IJ} extend to give a compatible family of embeddings defined over $\mathrm{cl}(M_{IJ})$. The images $\alpha_{IJ}(\mathrm{cl}(M_{IJ}))$ and $\alpha_{HJ}(\mathrm{cl}(M_{HJ}))$ are disjoint unless I and H are nested. Moreover, if H and I are nested, the intersection $\alpha_{IJ}(M_{IJ}) \cap \alpha_{HJ}(\mathrm{Fr}(M_{HJ}))$ is empty because $\alpha_{IJ}(M_{IJ}) \subset M_J = M_J^{|J|}$ while $\alpha_{HJ}(\mathrm{Fr}(M_{HJ})) \subset \mathrm{Fr}(M_J) \subset M_J^{|J|+1} \setminus M_J^{|J|}$. Hence, if we define

$$M_J' := M_J \setminus \bigcup_{I \subsetneq J} \mathrm{cl}(\alpha_{IJ}(\mathrm{Fr}(M_{IJ}))), \tag{2.2.19}$$

the maps α_{IJ} for $I \subsetneq J$ have image in M_J' and closed graph. Moreover, if we have already arranged that all the maps $\alpha_{JK}\colon M_{JK} \to M_K$ for $J \subsetneq K$ have closed graph and satisfy the compatibility conditions (2.2.15) for all $I \subset J \subset K$, then if we replace the domain M_{JK} by $M_J' \cap M_{JK}$, the maps $\alpha_{JK}\colon M_{JK}' \to M_K$ will still have these properties. Hence, we may arrange that all the maps α_{IJ} have closed graph by applying these two steps for each J, starting with J such that $|J|$ is maximal and then working down.

This completes the proof if the isotropy groups are trivial.

Step 5 (the case of nontrivial isotropy) To construct the maps τ_{IJ} in general, we argue as above, taking $\phi_{IJ}(x)$ to be the local inverse to the covering map ρ_{IJ} at $x \in \widetilde{V}_{IJ}$, and then defining τ_{IJ} to be the Γ_A–equivariant extension of α_{IJ}^{-1} to a neighborhood of the orbit of $(0,x)$ in $\widetilde{M}_{IJ}$. To see that this definition is consistent and independent of the choice of $x \in \rho_{IJ}^{-1}(\rho_{IJ}(x_0)$, note that the collar map is equivariant and, once the shrinkings $(\mathcal{V}^k, \underline{\varepsilon}^k)$ are chosen, the only other choice in the above construction is that of the cutoff functions χ_I in (2.2.17), whose pullbacks to the sets V_I are also equivariant. Hence, the local inverse $\phi_{IJ}(x_0)$ is invariant under the stabilizer group Γ_x, and so the extension is well defined.

Since all the previous arguments apply without essential change, it remains to check that τ_{IJ}^0 quotients out by a free action of $\Gamma_{J\setminus I}$ on $\widetilde{M}_{IJ}^0$ that extends to a neighborhood of $\mathrm{cl}(\widetilde{M}_{IJ}^0)$. To establish this, we must define an appropriate action of $\Gamma_{J\setminus I}$ on $\widetilde{M}_{IJ}^0$. If $\Gamma_{J\setminus I}$ acts trivially on $E_{J\setminus I}$, then this action is simply the restriction of the given action of $\Gamma_{J\smallsetminus I}$ on V_J. However, in general this is not the case, and the new action

$$\Gamma_{J\smallsetminus I} \times \widetilde{M}^0_{IJ} \to \widetilde{M}^0_{IJ}, \quad (\gamma, x) \mapsto \gamma * x,$$

is described as follows. Notice first that because the collar c^Y_J is Γ_A–equivariant and injective, each point $x_0 \in \widetilde{V}_{IJ} \subset \widetilde{M}^0_{IJ}$ with $\tau^0_{IJ}(x_0) = (0, x_0') \in E_{J\smallsetminus I,\delta_I} \times V_{IJ}$ has a neighborhood $\mathcal{N}(x_0)$ on which τ^0_{IJ} is injective and has image $\mathcal{N}'$ of product form, namely $\mathcal{N}' = E_{J\smallsetminus I,\delta_I} \times \mathcal{O}' \subset E_{J\smallsetminus I,\delta_I} \times V_{IJ}$. Further, $\Gamma_{J\smallsetminus I}$ acts on $\mathcal{N}'$ via its action on $E_{J\smallsetminus I,\delta_I}$, since it fixes the points of $V_{IJ} \subset V_I$. If $\tau^{-1}_{IJ,x_0}\colon \mathcal{N}' \to \mathcal{N}(x_0)$ is the local inverse to τ_{IJ} at x_0, we now define

$$\gamma * x := \gamma \cdot_J \tau^{-1}_{IJ,x_0}(\gamma^{-1} \cdot_I \tau_{IJ}(x)), \quad x \in \mathcal{N}(x_0), \tag{2.2.20}$$

where for clarity we have written $x \mapsto \gamma \cdot_I x$ (resp. $x \mapsto \gamma \cdot_J x$) for the standard action of $\gamma \in \Gamma_{J\smallsetminus I}$ on $E_{J\smallsetminus I,\delta_I} \times V_{IJ} \subset E_{J\smallsetminus I,\delta_I} \times V_I$ (resp. on $\widetilde{M}^0_{IJ} \subset V_J$). Then

$$\begin{aligned}\tau_{IJ}(\gamma * x) &= \tau_{IJ}\big(\gamma \cdot_J \tau^{-1}_{IJ,x_0}(\gamma^{-1} \cdot_I \tau_{IJ}(x))\big)\\ &= \gamma \cdot_I \big(\tau_{IJ} \circ \tau^{-1}_{IJ,x_0}(\gamma^{-1} \cdot_I \tau_{IJ}(x))\big) = \tau_{IJ}(x),\end{aligned}$$

where the second equality uses the equivariance of τ_{IJ} with respect to the actions $\cdot_J$ and $\cdot_I$. Now extend this action over the whole orbit by setting $\delta * (\gamma * x) := (\delta\gamma) * x$. This new action $x \mapsto \gamma * x$ is free, since $\Gamma_{J\smallsetminus I}$ acts freely on $\widetilde{V}_{IJ}$. Further, this action extends to a free action on a neighborhood of the closure of $\widetilde{M}^0_{IJ}$ since it is determined by τ_{IJ}, and hence by the collar, both of which can be extended. □

Lemma 2.2.4 *The action $x \mapsto \gamma * x$ of $\Gamma_{J\smallsetminus I}$ on $\widetilde{M}_{IJ}$ has the following properties:*

- (i) *If $H, I \subset J$, then the action of $\Gamma_{J\smallsetminus I}$ on $\widetilde{M}_{IJ}$ preserves the subset $\widetilde{M}_{IJ} \cap \widetilde{M}_{HJ}$.*
- (ii) *If $H \subset I \subset J$ then the restriction to $\widetilde{M}_{IJ} \cap \widetilde{M}_{HJ}$ of the action of $\Gamma_{J\smallsetminus I}$ on $\widetilde{M}_{IJ}$ agrees with that obtained by considering $\Gamma_{J\smallsetminus I}$ as a subgroup of $\Gamma_{J\smallsetminus H}$ and restricting the corresponding action from $\widetilde{M}_{HJ}$ to $\widetilde{M}_{IJ} \cap \widetilde{M}_{HJ}$.*
- (iii) *If $H \subset I \subset J$ and $y \in \widetilde{M}_{IJ} \cap \widetilde{M}_{HJ}$, then*

$$\tau_{IJ}(\gamma_{J\smallsetminus H} * y) = (\gamma_{J\smallsetminus H}|_{I\smallsetminus H}) * \tau_{IJ}(y),$$

where $\gamma_{J\smallsetminus H}|_{I\smallsetminus H}$ is the image of $\gamma_{J\smallsetminus H}$ under the projection $\Gamma_{J\smallsetminus H} \to \Gamma_{I\smallsetminus H}$.

- (iv) *Properties* (i) *and* (ii) *continue to hold for the extension of the action to the closure* $\mathrm{cl}(\widetilde{M}_{IJ})$ *of $\widetilde{M}_{IJ}$ in M_J.*

Proof (i) follows from (2.2.20) because the action $x \mapsto \gamma \cdot_J x$ preserves the sets $\widetilde{M}_{HJ}$ for all $H \subset J$. (ii) also follows immediately from (2.2.20) and the fact that $\tau_{HJ} = \tau_{HI} \circ \tau_{IJ}$ on $\widetilde{M}_{IJ} \cap \widetilde{M}_{HJ}$. (iii) holds because the maps τ_{IJ} are equivariant with respect to the projection $\Gamma_J \to \Gamma_I$ and take $\widetilde{M}_{IJ} \cap \widetilde{M}_{HJ}$ to $M_{IJ} \cap \widetilde{M}_{HI}$ by (2.2.11). Finally, (iv) holds because the extended action is defined by extending the domain of the maps in (2.2.20). □

Remark 2.2.5 (the smooth case) Note first that if we apply the above construction to a smooth atlas (ie one that satisfies the smooth submersions condition in (1.2.4)), then the charts used in (3.1.4) to give $Y_{\mathcal{V},J,\underline{\varepsilon}}$ the structure of a topological manifold do not have differentiable inverses. A related problem may also be seen in Example 2.2.1: the attaching map $\alpha_{1,12}$ in (2.2.4) is given by the collar, which by (3.2.3) has the form $(e_2, x) \mapsto x' := \phi(\|e_2\|^{1/2} e_2, x)$, where ϕ is the local product structure along $\widetilde{V}_{IJ}$ in (1.2.3). Thus, even if ϕ were a diffeomorphism, $\alpha_{1,12}$ would not have a smooth inverse along the submanifold $e_2 = 0$. Thus, just as in standard blow-up constructions, in order to obtain a smooth category $\boldsymbol{M}$ from a smooth atlas one needs to choose a smoothing of $Y_{\mathcal{V},J,\underline{\varepsilon}}$ along its boundary.

Alternatively, one could use a different construction that avoids introducing the manifold Y. Instead, one can construct the all-important collar structure used to define the maps τ_{IJ} by using the exponential map with respect to a suitable family of metrics on the sets V_J. Indeed, recall that by the smooth tangent bundle condition (1.2.4) the derivative $\mathrm{d}s_{J \setminus I}$ induces an isomorphism from the normal bundle $T^{\perp}(\widetilde{V}_{IJ})$ of $\widetilde{V}_{IJ}$ in V_J to the product $E_{J \setminus I, \varepsilon_I} \times \widetilde{V}_{IJ}$. To explain the idea, let us suppose for simplicity that the cover $\mathcal{V}$ is refined so that the group $\Gamma_{J \setminus I}$ acts freely on the components on $\widetilde{V}_{IJ}$, so that the restriction of τ_{IJ} to each component is a diffeomorphism onto V_{IJ}. Then we can think of V_{IJ} as a subset of $\widetilde{V}_{IJ}$ and the task is to define a consistent family of injections $\alpha_{IJ}\colon E_{J \setminus I, \varepsilon_I} \times V_{IJ} \to V_J$. To this end, choose a family of Γ_I–invariant Riemannian metrics g_I on V_I and constants ε_I that are compatible in the following sense:

- For each $I \subsetneq J$, $\widetilde{V}_{IJ}$ is a totally geodesic submanifold of (V_J, g_J) and
$$(\rho_{IJ})_*(g_J|_{\widetilde{V}_{IJ}}) = g_I|_{V_{IJ}}.$$
- $0 < \varepsilon_I < \varepsilon_J$ if $I \subsetneq J$.
- For each $I \subsetneq J$, the g_J–exponential map along directions perpendicular to $\widetilde{V}_{IJ}$ defines an embedding $\alpha_{IJ}\colon E_{J \setminus I, \varepsilon_I} \times V_{IJ} \to V_J$.

- The corners are locally flat, ie if $x \in \widetilde{V}_{IJ} \cap \widetilde{V}_{HJ}$ for $I \subsetneq H \subsetneq J$ then

$$\alpha_{IJ}(e_{J \smallsetminus H} + e_{H \smallsetminus I}, x) = \alpha_{HJ}(e_{J \smallsetminus H}, \alpha_{IH}(e_{H \smallsetminus I}, x)).$$

The last condition means that the composition rule holds directly, without having to introduce analogs of the paths $\mathcal{P}(e,x)$. Of course, the choice of the g_I and ε_I requires some attention to detail as in the proof of Lemma 3.1.11 below; see also the construction of the perturbation section in [14, Section 7.3]. Thus, one begins with a family of shrinkings $\mathcal{V}^\kappa \sqsubset \cdots \sqsubset \mathcal{V}^1 \sqsubset \mathcal{V}^0$ of an initial reduction $\mathcal{V}^0$, where $\kappa := \max\{|J| \mid J \in \mathcal{I}_\mathcal{K}\}$, and then chooses metrics g_J on $V_J^{|J|}$, starting with J of length $|J| = 1$, that satisfy the above conditions for the submanifolds $\widetilde{V}_{IJ}^{|J|}$ of $V_J^{|J|}$ for some constant $\varepsilon_I' > 0$. Finally, once g_J is defined on V_J^κ for all J, one chooses suitable constants ε_J, now starting with maximal $|J|$ and working down. Further details are left to the reader, as is the proof that the resulting branched manifold is cobordant to the topological one constructed in detail above. For this last step one would need to adapt the proof of uniqueness in Step 2 of the proof of Theorem B in Section 2.3.

Besides obtaining a smooth rather than topological branched manifold, there are no real advantages to this construction unless one wants to work with the virtual fundamental class on the chain level using de Rham cochains. Another point is that by [9] we can construct the branched manifold M to be a simplicial complex, so we could simplify the proof of Lemma 2.3.4 by using (locally finite) singular homology instead of Čech homology. However, because we know nothing about X except that it is compact and Hausdorff, the VFC has still to be considered as an element in Čech homology. For further discussions of the smooth case, see Step 3 of the proof of Theorem B. ◇

2.3 Proof of Theorems A and B

To prove Theorem A, we must show that the category $\boldsymbol{M}$ constructed in Section 2.2 has a unique completion to a weighted branched groupoid, and then analyze the structure of this groupoid and the associated weighted branched manifold (M, Λ). The arguments needed here are very similar, but not identical, to those in [11, Proposition 2.3] (which considers the case of the category $\boldsymbol{B}_\mathcal{K}$ defined by an atlas with trivial obstruction spaces) and in [12, Section 3.3] (which analyzes the zero set of a transverse perturbation section). Theorem B has two parts. It first states that if $\mathcal{K}$ is oriented the weighted branched manifold (M, Λ) carries a natural fundamental class $[X]^{\mathrm{vir}}_\mathcal{K}$, a result that was proven in [9] in the case when $\boldsymbol{M}$ is smooth and compact, with or without boundary. Although smoothness is assumed throughout [9], the only place where this condition is essential is in the construction of the fundamental class in the proof of [9, Proposition 3.25]. In this case, we may replace $\boldsymbol{M}$ by an equivalent wnb groupoid that is tame in the sense that its branching loci are piecewise smooth and hence triangulable, which allows us to work with singular homology in the proof of Lemma 2.3.4. In the present case, we must use rational Čech cohomology, and the appropriate dual homology theory for noncompact manifolds as described in the appendix. The second and more substantial part of the proof of Theorem B explains why $[X]^{\mathrm{vir}}_{\mathcal{K}}$ is independent of all choices made in its construction, and why, in the smooth case, the new definition is consistent with the previous definition via perturbation sections.

We begin with a lemma about groupoid completions of étale categories; for definitions, see Sections 1.2 and 1.3. As usual we denote by $\mathcal{I}$ a collection of subsets of a finite set A, and say that $I, H \in \mathcal{I}$ are nested if $I \subset H$ or $H \subset I$. We state the condition *identity* below for completeness; it follows immediately from the fact that every category has identity morphisms.

Lemma 2.3.1 *Let $\mathcal{I}$ be a collection of subsets of a finite set A and $\boldsymbol{M}$ be an étale category with*

$$\mathrm{Obj}_{\boldsymbol{M}} = \bigsqcup_{I \in \mathcal{I}} M_J, \quad \mathrm{Mor}_{\boldsymbol{M}} = \bigsqcup_{I \subset J,\, I,J \in \mathcal{I}} \widetilde{M}_{IJ},$$
$$s \times t\colon \mathrm{Mor}_{\boldsymbol{M}} \to \mathrm{Obj}_{\boldsymbol{M}} \times \mathrm{Obj}_{\boldsymbol{M}}, \quad (I, J, y) \mapsto \big((I, \tau_{IJ}(y)), (J, y)\big),$$

where $\widetilde{M}_{IJ} \subset M_J$ is an open subset and the maps $\tau_{IJ}\colon \widetilde{M}_{IJ} \to M_{IJ}$ satisfy the following conditions:

- **Identity** *For all $I \in \mathcal{I}$, $\tau_{II} = \mathrm{id}$ on $\widetilde{M}_{II} = M_{II} = M_I$.*
- **Composition** *For all $H \subset I \subset J$, $\tau_{IJ}^{-1}(\widetilde{M}_{HI} \cap M_{IJ}) = \widetilde{M}_{HJ} \cap \widetilde{M}_{IJ}$ and $\tau_{HJ} = \tau_{HI} \circ \tau_{IJ}$ on $\widetilde{M}_{HJ} \cap \widetilde{M}_{IJ}$; hence, if $z \in \widetilde{M}_{IJ} \cap \widetilde{M}_{HJ}$, where $H \subset I \subset J$, we have $(H, I, y) \circ (I, J, z) = (H, J, z)$.*
- **Separation** $\mathrm{cl}(\widetilde{M}_{IJ}) \cap \mathrm{cl}(\widetilde{M}_{HJ}) = \varnothing$ *unless I and H are nested.*
- **Group actions** *For each $i \in A$ there is a finite group Γ_i such that, for all $I \subset J$, τ_{IJ} quotients out by the restriction to $\widetilde{M}_{IJ}$ of a free action of $\Gamma_{J \smallsetminus I}$ on* $\mathrm{cl}(\widetilde{M}_{IJ}) \subset M_J$, *where $\Gamma_{J \smallsetminus I} := \prod_{i \in J \smallsetminus I} \Gamma_i$. Moreover, these actions $x \mapsto \gamma * x$ satisfy the compatibility conditions listed in Lemma 2.2.4.*

Then, there is a unique nonsingular groupoid $\widehat{\boldsymbol{M}}$ with the same object space and realization as $\boldsymbol{M}$. Its morphism spaces for $I \subset J$ are

$$\begin{aligned}(2.3.1)\qquad \operatorname{Mor}_{\widehat{\boldsymbol{M}}}(M_I, M_J) &:= \bigcup_{\varnothing\neq F\subset I} (\widetilde{M}_{IJ} \cap \widetilde{M}_{FJ}) \times \Gamma_{I\smallsetminus F},\\ &= \{(y,\gamma) \in M_J \times \Gamma_J \mid y \in \widetilde{M}_{IJ},\ \gamma \in \Gamma_{I\smallsetminus H_y}\},\end{aligned}$$

where $H_y := \min\{H : y \in \widetilde{M}_{HJ}\}$, with

$$\begin{aligned}(2.3.2)\qquad s\times t(I,J,y,\gamma_{I\smallsetminus H_y}) &= \big((I, \gamma^{-1}_{I\smallsetminus H_y} * \tau_{IJ}(y)), (J,y)\big),\\ (I,I,y,\gamma_{I\smallsetminus H_y})^{-1} &= (I,I,\gamma^{-1}_{I\smallsetminus H_y} * y, \gamma^{-1}_{I\smallsetminus H_y}).\end{aligned}$$

In particular, $\widehat{\boldsymbol{M}}$ is étale, and there is an injective functor $\boldsymbol{M} \to \widehat{\boldsymbol{M}}$.

Proof Observe first that because a nonsingular category has at most one morphism between any two objects, $\widehat{\boldsymbol{M}}$ must have precisely one morphism between any two objects (I,x) and (J,y) that are equivalent under the equivalence relation $\sim_{\boldsymbol{M}}$ on $\operatorname{Obj}_{\boldsymbol{M}}$ generated by $\operatorname{Mor}_{\boldsymbol{M}}$. Hence, there is precisely one nonsingular groupoid with $\operatorname{Obj}_{\widehat{\boldsymbol{M}}} = \operatorname{Obj}_{\boldsymbol{M}}$ and $|\widehat{\boldsymbol{M}}| = |\boldsymbol{M}|$. Since there is an injection

$$\operatorname{Mor}_{\boldsymbol{M}}(M_I, M_J) = \widetilde{M}_{IJ} \hookrightarrow \bigcup_{\varnothing\neq F\subset I} (\widetilde{M}_{IJ} \cap \widetilde{M}_{FJ}) \times \Gamma_{I\smallsetminus F}$$

when $I \subset J$, and the structural maps described in (2.3.2) are étale, it remains to check that the formula (2.3.1) does describe $\operatorname{Mor}_{\widehat{\boldsymbol{M}}}(M_I, M_J)$.

The separation property implies that for each $y \in M_J$ the set of F such that $|y| \in |\widetilde{M}_{FJ}| \subset |\boldsymbol{M}|$ is nested. Let H_y be the minimal such element. By Lemma 2.2.4, for all $H_y \subset I \subset J$ the group $\Gamma_{I\smallsetminus H_y}$ acts freely on $\widetilde{M}_{H_y I}$. Hence, each element in $\bigcup_{\varnothing\neq F\subset I} (\widetilde{M}_{IJ} \cap \widetilde{M}_{FJ}) \times \Gamma_{I\smallsetminus F}$ has a unique description of the form (y,γ) with $y \in \widetilde{M}_{IJ}$ and $\gamma \in \Gamma_{I\smallsetminus H_y}$. Further, given such (y,γ) it follows from Lemma 2.2.4(iii) that $\tau_{H_y J}(y) = \tau_{H_y I}(\gamma^{-1} * \tau_{IJ}(y))$ for any I with $y \in \widetilde{M}_{IJ}$, so that

$$(J,y) \sim_{\boldsymbol{M}} (H_y, \tau_{H_y J}(y)) = \big(H_y, \tau_{H_y J}(\gamma^{-1} * \tau_{IJ}(y))\big) \sim_{\boldsymbol{M}} (I, \tau_{IJ}(y)).$$

Hence, for each I with $y \in \widetilde{M}_{IJ}$ and each $\gamma \in \Gamma_{I\smallsetminus H_y}$ there must be a morphism m in $\widehat{\boldsymbol{M}}$ from $(I, \gamma^{-1} * \tau_{IJ}(y))$ to (J,y). To see that these are the only morphisms in $\widehat{\boldsymbol{M}}$ it remains to observe that each equivalence class $[(J,y)]$ contains a unique element of the form (H_y, z) where $z \in M_{H_y}$; further if $H_y \subset I \subset J$ then $[(J,y)] \cap M_I = \{x \in \rho^{-1}_{H_y I}(z)\}$ consists of the $\Gamma_{I\smallsetminus H_y}$–orbit of $\tau_{IJ}(y)$. Since each morphism is uniquely specified by its source and target, there is no need to write out the composition rule in $\widehat{\boldsymbol{M}}$ explicitly. □

Lemma 2.3.2 *Suppose in the situation of Lemma 2.3.1 that for each $I \subset J$ the map τ_{IJ} has closed graph. Then:*

(i) *The maximal Hausdorff quotient $|\boldsymbol{M}|_{\mathcal{H}}$ is the realization of a nonsingular groupoid $\widehat{\boldsymbol{M}}_{\mathcal{H}}$ with objects $\mathrm{Obj}_{\boldsymbol{M}}$ and morphisms from M_I to M_J with $I \subset J$ given by*

$$\mathrm{Mor}_{\widehat{\boldsymbol{M}}_{\mathcal{H}}}(M_I, M_J) := \bigcup_{\varnothing \neq F \subset I} (\widetilde{M}_{IJ} \cap \mathrm{cl}(\widetilde{M}_{FJ})) \times \Gamma_{I \smallsetminus F}.$$

(ii) *For each I, the map $\pi_I^{\mathcal{H}}\colon M_I \to |\widehat{\boldsymbol{M}}_{\mathcal{H}}|$ is a local homeomorphism with open image, and in particular is a proper map onto its image.*

(iii) *The space $M := |\boldsymbol{M}|_{\mathcal{H}} = |\widehat{\boldsymbol{M}}_{\mathcal{H}}|$ can be given the structure of a weighted nonsingular branched manifold with weighting function $\Lambda_{\boldsymbol{M}}\colon M \to \mathbb{Q}^+ = \mathbb{Q} \cap (0, \infty)$ given for $p \in |M_I|_{\mathcal{H}}$ by*

$$\Lambda_{\boldsymbol{M}}(p) := \frac{1}{|\Gamma_I|} \#\{y \in M_I \mid \pi_{\boldsymbol{M}}^{\mathcal{H}}(y) = p\} = \frac{|\Gamma_{I \smallsetminus F_y}|}{|\Gamma_I|},$$

where $F_y := \min\{F : y \in \mathrm{cl}(\widetilde{M}_{FI})\} = \min\{F : \pi_{\boldsymbol{M}}^{\mathcal{H}}(y) \in \mathrm{cl}(\pi_{\boldsymbol{M}^{\mathcal{H}}}(M_F)\}$. Moreover, the wnb manifold M is oriented if $\boldsymbol{M}$ is.

Proof Denote by $\approx_{\boldsymbol{M}}$ the equivalence relation on $\mathrm{Obj}_{\boldsymbol{M}}$ corresponding to the quotient map $\mathrm{Obj}_{\boldsymbol{M}} \to |\boldsymbol{M}|_{\mathcal{H}}$. Its graph is the closure in $\mathrm{Obj}_{\boldsymbol{M}} \times \mathrm{Obj}_{\boldsymbol{M}}$ of the graph of $\sim_{\boldsymbol{M}}$. First consider the component in $\mathrm{Mor}_{\widehat{\boldsymbol{M}}}$ consisting of morphisms from M_I to M_J for $I \subset J$ with $\gamma = \mathrm{id}$. This set can be identified with $\widetilde{M}_{IJ}$ and has closed graph by hypothesis. Next consider the set of morphisms in $\widehat{\boldsymbol{M}}$ from M_J to M_J,

$$\mathrm{Mor}_{\widehat{\boldsymbol{M}}}(M_J, M_J) := \bigcup_{\varnothing \neq F \subset J} (\widetilde{M}_{FJ}) \times \Gamma_{J \smallsetminus F},$$
$$s \times t(J, J, y, \gamma) = ((J, \gamma^{-1} * y), (J, y)).$$

These morphisms form a group with closure

$$\begin{aligned}\mathrm{Mor}_{\widehat{\boldsymbol{M}}_{\mathcal{H}}}(M_J, M_J) &:= \bigcup_{\varnothing \neq F \subset J} (M_J \cap \mathrm{cl}(\widetilde{M}_{FJ})) \times \Gamma_{J \smallsetminus F} \\ &= \{(y, \gamma) \mid y \in M_J,\ \gamma \in \Gamma_{J \smallsetminus F_y}\},\end{aligned}$$

where

$$s \times t(J, J, y, \gamma) = ((J, \gamma^{-1} * y), (J, y)).$$

Next observe that every morphism $(I, J, y, \gamma) \in \mathrm{Mor}_{\widehat{M}}(M_I, M_J)$, where $I \subset J$ may be written as the composite

$$(J, J, y, \gamma) \circ (I, J, \gamma^{-1} * y, \mathrm{id})$$

of a morphism of the second type followed by one of the first type. Therefore, because the action of $\Gamma_{I \smallsetminus F}$ on $\widetilde{M}_{FI} \cap \widetilde{M}_{IJ} \subset M_J$ (where $F \subset I \subset J$) extends to an action on $\mathrm{cl}(\widetilde{M}_{FJ}) \cap \widetilde{M}_{IJ}$ by Lemma 2.2.4(iii), the limit of a convergent sequence of morphisms also is such a composite. Claim (i) then follows easily.

Since $|\widehat{M}_{\mathcal{H}}|$ has the quotient topology, to establish (ii) it suffices to show that the inverse image $(\pi_J^{\mathcal{H}})^{-1}(\pi_I^{\mathcal{H}}(M_I))$ is open in M_J for all J. This set is empty unless $I \subset J$ or $J \subset I$. In the former case, $(\pi_J^{\mathcal{H}})^{-1}\pi_I^{\mathcal{H}}(M_I) = \tau_{IJ}^{-1}(M_{IJ}) = \widetilde{M}_{IJ}$, which is open. In the latter case, $(\pi_J^{\mathcal{H}})^{-1}\pi_I^{\mathcal{H}}(M_I) = \tau_{IJ}(\widetilde{M}_{IJ})$, which is also open. This proves (ii).

To prove (iii), note that by (ii) we may define the local branches at $p = \pi_I^{\mathcal{H}}(y) \in |M_I|_{\mathcal{H}}$ to be the image under $\Gamma_{I \smallsetminus F_p}$ of an open neighborhood $U \subset M_I$ of y that is disjoint from $\mathrm{cl}(\widetilde{M}_{FI})$ unless $F_y \subset F$ and is also disjoint from its images under $\Gamma_{I \smallsetminus F_y}$. Each such local branch is given weight $1/|\Gamma_I|$. It then follows easily from Lemma 2.2.4 that Λ_M is well defined and has the required properties. For more details, see [12, Lemma 3.2.10]. Finally, the statement about orientations is clear. □

Remark 2.3.3 As in [12, Lemma 3.2.10], it follows from part (ii) of Lemma 2.3.2 that the topology on $|\widehat{M}_{\mathcal{H}}|$ is second countable, locally compact and metrizable. ◇

With these preliminaries in hand, it is easy to show that in, the oriented case, $M = |\widehat{M}_{\mathcal{H}}|$ has a fundamental class.

Lemma 2.3.4 *Let M be oriented with corresponding oriented wnb groupoid $\widehat{M}_{\mathcal{H}}$ constructed as in Lemma 2.3.2, and let $M := |\widehat{M}_{\mathcal{H}}|$. Then there is a class $\mu_M \in \check{H}_N^{\infty}(M)$ with the following property: if $U := \pi_I^{\mathcal{H}}(M_I)$ for some $I \in \mathcal{I}_{\mathcal{K}}$, then*

$$\rho_{M,U}(\mu_M) = \frac{1}{|\Gamma_I|}(\pi_I^{\mathcal{H}})_*(\mu_I) \in \check{H}_N^{\infty}(U), \tag{2.3.3}$$

where $\mu_I \in \check{H}_N^{\infty}(M_I)$ is the fundamental class in (A.3) *and $\rho_{M,U}$ is the restriction map on homology in* (A.4).

Proof It follows from Lemma 2.3.2 that the statement of the lemma makes sense: the class μ_I exists by property (a$'$) in the appendix, the restriction exists by (b$'$) because U is open, and the pushforward exists by (c$'$) because the map $\pi_I^{\mathcal{H}}\colon M_I \to |M|$ is proper. We prove the lemma by showing that for $k = 1, 2, \dots$ there is a class μ_k on $W_k := \bigcup_{I:|I|\le k} \pi_I^{\mathcal{H}}(M_I)$ such that

$$\mu_k|_{\pi_I^{\mathcal{H}}(M_I)} = (\pi_I^{\mathcal{H}})_*\left(\frac{1}{|\Gamma_I|}\mu_I\right) \quad \text{for all } I,\ |I| \le k.$$

When $k = 1$, W_1 is a disjoint union of sets $\pi_I^{\mathcal{H}}(M_I)$, where $|I| = 1$, and we simply define μ_1 to be the given pushforward. Let us suppose that μ_k is constructed, and consider the definition of μ_{k+1}. Since the sets $(\pi_J^{\mathcal{H}}(M_J))_{|J|=k+1}$ are disjoint, it follows from (e$'$) that we can consider each of them separately. Further, by applying Mayer–Vietoris with $U = W_k$ and $V = \pi_I^{\mathcal{H}}(M_J)$ it suffices to show that the classes $\mu_k \in \check{H}_N^{\infty}(W_k)$ and $(\pi_J^{\mathcal{H}})_*((1/|\Gamma_J|)\mu_J) \in \check{H}_N^{\infty}(V)$ have the same restriction to $W_k \cap V = \bigcup_{I \subsetneq J} \pi_I^{\mathcal{H}}(M_{IJ})$. But because restriction commutes with pushforward by (d$'$), it suffices to prove the corresponding statement for the fundamental classes of the spaces M_J. Namely, we must check that

$$\frac{1}{|\Gamma_J|}(\tau_{IJ})_*(\mu_J|_{\widetilde{M}_{IJ}}) = \frac{1}{|\Gamma_I|}(\mu_I|_{M_{IJ}}).$$

But on manifolds the homology theory $\check{H}^{\infty}$ agrees with the usual locally compact singular homology. Hence, the above property holds because, by hypothesis, the maps $\tau_{IJ}\colon \widetilde{M}_{IJ} \to M_{IJ}$ are orientation-preserving covering maps of degree $|\Gamma_J|/|\Gamma_I|$. □

We are now in a position to prove the main theorems. We begin with the proof of Theorem 1.3.4, which as already noted in Section 1.3 immediately implies Theorem A.

Proof of Theorem 1.3.4 Given the oriented atlas $\mathcal{K}$ we construct the category $\boldsymbol{M}$ as in Proposition 2.2.2. We saw in Lemma 2.3.2(i) that this category has a unique Hausdorff groupoid completion $\widehat{\boldsymbol{M}}_{\mathcal{H}}$, which proves (i). Part (ii) follows immediately from Lemma 2.3.2(iii). Further, the action of the group Γ_A on $\boldsymbol{M}$ and E_A induces an action on $\widehat{\boldsymbol{M}}_{\mathcal{H}}$ and $\boldsymbol{E}_A$, and the functor $\mathscr{S}\colon \boldsymbol{M} \to \boldsymbol{E}_A$ extends to a Γ_A–equivariant functor $\widehat{\boldsymbol{M}}_{\mathcal{H}} \to \boldsymbol{E}_A$. Therefore, it remains to check that the induced map $\mathscr{S}_M\colon M \to E_A$ on the realizations has compact zero set $\mathscr{S}_M^{-1}(0)$ and that there is a map $\psi\colon \mathscr{S}_M^{-1}(0) \to X$ that induces a homeomorphism $\mathscr{S}_M^{-1}(0)/\Gamma_A \xrightarrow{\cong} X$.

Since $M_I \cap \mathscr{S}_I^{-1}(0) = \{0\} \times (V_I \cap s_I^{-1}(0))$ by (1.3.7), the full subcategory of $\boldsymbol{M}$ with objects $\bigsqcup_I M_I \cap \mathscr{S}_I^{-1}(0)$ includes into the full subcategory of $\boldsymbol{B}_{\mathcal{K}}$ with objects

$\bigsqcup_I V_I \cap s_I^{-1}(0)$. Hence, there is an induced map on the realizations

$$|\boldsymbol{M}| \cap \mathscr{S}^{-1}(0) \to |\mathcal{K}| \cap |\mathfrak{s}|^{-1}(0) \cong X.$$

This is continuous and surjective, but not injective because we have not yet quotiented out by the group actions. Nevertheless, because X is Hausdorff, the universal property satisfied by the Hausdorff quotient implies that it factors through a map $\psi\colon |\widehat{\boldsymbol{M}}_{\mathcal{H}}| \cap \mathscr{S}^{-1}(0) = \mathscr{S}_M^{-1}(0) \to X$. Further, because ψ_I induces a homeomorphism $V_I \cap s_I^{-1}(0)/\Gamma_I \to G_I \subset X$ (where G_I is the footprint of the reduced chart domain V_I) and $\Gamma_{A\smallsetminus I}$ acts trivially on $\mathscr{S}_I^{-1}(0) = V_I \cap s_I^{-1}(0)$, this map $\psi\colon \mathscr{S}_M^{-1}(0) \to X$ does factor through a bijective and continuous map $\mathscr{S}_M^{-1}(0)/\Gamma_A \to X$. To see that it is a homeomorphism, it suffices to check that $\mathscr{S}_M^{-1}(0)$ is compact.

To this end, notice first that because the topology on M is metrizable by Remark 2.3.3, we need only check that $\mathscr{S}_M^{-1}(0)$ is sequentially compact. Thus, consider a sequence of points $p_k \in \mathscr{S}_M^{-1}(0)$. Because M is the union of the finite number of sets $|M_I|_{\mathcal{H}}$, we may suppose that $p_k \in |M_I|_{\mathcal{H}}$ for all k. Choose a sequence $y_k \in M_I \cap \mathscr{S}_I^{-1}(0)$ such that $\pi_I^{\mathcal{H}}(y_k) = p_k$. Then $y_k = (0, z_k) \in E_{A\smallsetminus I,\varepsilon_I} \times (V_I \cap s_I^{-1}(0))$, and $\psi(y_k) = \psi_I(z_k) \in X$. By passing to a subsequence, we may suppose that the sequence $\psi_I(z_k)$ converges to $x_\infty \in X$. Since the footprints $G_J := \psi_J(s_J^{-1}(0))$ of the reduced charts form an open covering of X, we may further suppose that there is J such that $\psi_I(z_k) \in G_J$ for all k and that this sequence has limit $x_\infty = \psi_J(z_\infty) \in G_J$. Because $G_I \cap G_J \neq \varnothing$, the sets I and J are nested, and the original sequence $p_k \in |M_I|_{\mathcal{H}}$ must lie in the intersection $p_k \in |M_I|_{\mathcal{H}} \cap |M_J|_{\mathcal{H}}$. Therefore, the p_k also have lifts $y_k' = (0, z_k') \in E_{A\smallsetminus J,\varepsilon_J} \times (V_J \cap s_J^{-1}(0))$, and now it follows from the fact that the map $V_J \cap s_J^{-1}(0) \to G_J \subset X$ is finite-to-one that some subsequence of the z_k' must converge to a some point z_∞' in the finite set $(V_J \cap s_J^{-1}(0)) \cap \psi_J^{-1}(x_\infty)$. Hence, (p_k) has a subsequence that converges to $\pi_J^{\mathcal{H}}(0, z_\infty') \in |M_J|_{\mathcal{H}} \subset M$. This completes the proof of Theorem A. □

The proof of Theorem B is somewhat longer, and hence we restate it for the convenience of the reader. Here we assume that $\mathcal{K}$ and X are as in Theorem A.

Theorem B *If $\mathcal{K}$ is oriented, there is a unique element $[X]_{\mathcal{K}}^{\mathrm{vir}} \in \check{H}_d(X;\mathbb{Q})$ that is defined as follows. For $b \in \check{H}^d(X;\mathbb{Q})$ and $D = d + \dim E_A$, we have*

$$\langle [X]_{\mathcal{K}}^{\mathrm{vir}}, b\rangle := (\mathscr{S}_M)_*(\widehat{b}) \in \check{H}^c_{\dim E_A}(E_A, E_A \smallsetminus \{0\};\mathbb{Q}) \cong \mathbb{Q},$$

where $\widehat{b}$ *is the image of* b *under the composite*

$$\check{H}^d(X;\mathbb{Q}) \xrightarrow{\psi^*} \check{H}^d(\mathscr{S}_M^{-1}(0);\mathbb{Q}) \xrightarrow{\mathcal{D}} \check{H}^c_{\dim E_A}(M, M \smallsetminus \mathscr{S}_M^{-1}(0);\mathbb{Q}),$$

and $\mathcal{D}$ *is given by cap product with the fundamental class* $\mu_M \in H^{d+\dim E_A}(M)$ *constructed in Lemma 2.3.4. Moreover,* $[X]^{\mathrm{vir}}_{\mathcal{K}}$ *depends only on the oriented concordance class of* $\mathcal{K}$*, and in the smooth case agrees with the class defined in* [12].

Proof We proceed in three steps:

Step 1 (definition of $[X]^{\mathrm{vir}}_{\mathcal{K}}$) Since the fundamental class μ_M exists by Lemma 2.3.4, and an appropriate cap product exists by point (f′) in the appendix, in order to see that $\langle [X]^{\mathrm{vir}}_{\mathcal{K}}, b\rangle$ is well defined it remains to note that the map

$$(\mathscr{S}_M)_*\colon \check{H}^c_{\dim E_A}(M, M \smallsetminus \mathscr{S}_M^{-1}(0);\mathbb{Q}) \to \check{H}^c_{\dim E_A}(E_A, E_A \smallsetminus \{0\};\mathbb{Q}) \cong \mathbb{Q}$$

is well defined. Further, it takes values in $\mathbb{Q}$, because E_A is oriented by the definition in Remark 1.2.1(iii) and the theory $\check{H}^c_*$ coincides with singular homology theory on simplicial spaces.

Step 2 (proof of uniqueness) To prove the uniqueness of $[X]^{\mathrm{vir}}_{\mathcal{K}}$, one must state and prove the analog of Proposition 1.3.3 for cobordism atlases, and also prove that all choices made in the construction are unique modulo oriented cobordism. For the constructions that involve atlases, such results are proved in [13; 14; 12]; see [13, Proposition 4.2.3] for different choices of metrics and [13, Theorem 4.2.7] for different choices of tame shrinkings, [13, Theorem 5.1.6] for a discussion of reductions, [14, Section 8] for orientations (in particular [14, Theorem 8.1.12]) and [12, Appendix] for weighted branched cobordisms. The present construction also requires a choice of local product structures (as in (1.2.3)) and partition of unity (as in (3.2.17)) in order to define the collar of the manifolds $Y_{\mathcal{V},J,\underline{\varepsilon}}$. However, in distinction to the smooth case, it is not necessary to arrange that cobordism atlases have specified collars (ie local product structures) near the two boundary components because the VFC $[X]^{\mathrm{vir}}_{\mathcal{K}}$ is now defined via diagram (A.7), which involves restriction to the boundary rather than via a perturbation section that must be extended from the boundary to the interior.

Thus, we define a cobordism atlas $\mathcal{K}^{01}$ over $[0,1]\times X$ between two d–dimensional atlases $\mathcal{K}^0$ and $\mathcal{K}^1$ on X to be an atlas $\mathcal{K}^{01}$ over $[0,1]\times X$ of dimension $d+1$ such that:

(i) The charts whose footprints intersect $\partial([0,1]\times X) = \bigsqcup_\alpha \alpha\times X$ are manifolds with boundary.

(ii) For $\alpha = 0, 1$ there are functorial inclusions

$$\iota_\alpha\colon \mathcal{K}^\alpha \to \mathcal{K}^{01}, \quad \iota_\alpha^{\mathcal{I}}\colon \mathcal{I}_{\mathcal{K}^\alpha} \to \mathcal{I}_{\mathcal{K}^{01}}, \qquad \alpha = 0, 1,$$

that (for simplicity) we assume to have disjoint images, and for each $I \in \mathcal{I}_{\mathcal{K}^\alpha}$ take the chart domain U_I^α onto the boundary $\partial U_{I'}^{01}$ of the corresponding chart in $\mathcal{K}^{01}$, where $I' := \iota_\alpha^{\mathcal{I}}(I)$, preserving orientation for $\alpha = 1$ and reversing it for $\alpha = 0$.

(iii) We further require that the local product structures in (1.2.3) for the chart domains in $\mathcal{K}^\alpha$ extend to local product structures near the boundary points of the corresponding chart domains in $\mathcal{K}^{01}$.

We show in [14, Theorem 7.1.5] that any pair of reductions $\mathcal{V}^\alpha$ of $\mathcal{K}^\alpha$ may be extended to a reduction $\mathcal{V}^{01}$ of $\mathcal{K}^{01}$ such that there are natural inclusions $\iota_\alpha^V\colon |\mathcal{V}^\alpha| \to |\mathcal{V}^{01}|$ that are homeomorphisms to their image. Further, if $J \in \mathcal{I}_{\mathcal{K}^\alpha}$ for $\alpha = 0, 1$, then for suitable small $\underline{\varepsilon}^\alpha > 0$ there is a commutative diagram

$$\begin{tikzcd} E_{A^{01}\smallsetminus A^\alpha,\underline{\varepsilon}^\alpha} \times Y_{\mathcal{V}^\alpha,J,\underline{\varepsilon}} \arrow[r, "\iota_\alpha^Y"] \arrow[d, "\mathrm{pr}_V"'] & Y_{\mathcal{V}^{01},\iota^\alpha(J),\underline{\varepsilon}} \arrow[d, "\mathrm{pr}_V"'] \\ V_J^\alpha \arrow[r, "\iota_\alpha^V"] & V^{01}_{\iota^\alpha(J)} \end{tikzcd}$$

Notice here that we take the product of $Y_{\mathcal{V}^\alpha,J,\underline{\varepsilon}}$ with the extra obstruction spaces $E_{A^{01}\smallsetminus A^\alpha,\underline{\varepsilon}^\alpha}$ in order to increase its dimension to that of $Y_{\mathcal{V}^{01},\iota^\alpha(J),\underline{\varepsilon}}$. Because the maps (1.2.3) in the submersion axiom for $\mathcal{V}^{01}$ extend those for $\mathcal{V}^\alpha$, we can choose the covering and partition of unity in Step 2 of the proof of Lemma 3.2.1 for $\mathcal{V}^{01}$ to extend those already chosen for $\mathcal{V}^\alpha$. Therefore, we can construct the collars on $Y_{\mathcal{V}^{01},\iota^\alpha(J),\underline{\varepsilon}}$ to extend already constructed collars on the sets $Y_{\mathcal{V}^\alpha,J,\underline{\varepsilon}}$. Hence, after possibly shrinking $\underline{\varepsilon} > 0$, we can arrange that for small $\underline{\varepsilon}^\alpha > 0$ there are embeddings

$$\iota_\alpha^M\colon E_{A^{01}\smallsetminus A^\alpha,\underline{\varepsilon}^\alpha} \times M^\alpha \to M^{01} \quad \text{such that } \bigsqcup_\alpha \mathrm{im}(\iota_\alpha^M) = \partial M^{01},$$

and also that the map $\mathscr{S}_M^{01}\colon M^{01} \to E_A$ satisfies

$$\mathscr{S}_M^{01} \circ \iota_\alpha^M = \mathscr{S}_M^\alpha \circ \mathrm{pr}_M^\alpha\colon E_{A^{01}\smallsetminus A^\alpha,\underline{\varepsilon}^\alpha} \times M^\alpha \to E_A, \tag{2.3.4}$$

where $\mathrm{pr}_M^\alpha\colon E_{A^{01}\smallsetminus A^\alpha,\underline{\varepsilon}^\alpha} \times M^\alpha \to M^\alpha$ is the projection.

Because M^{01} is constructed from an atlas for the product $[0,1]\times X$, the natural projection $(\mathscr{S}_M^{01})^{-1}(0)\to[0,1]\times X$ factors through a homeomorphism

$$(\mathscr{S}_M^{01})^{-1}(0)/\Gamma^{01}\xrightarrow{\cong}[0,1]\times X.$$

Notice here that for $\alpha=0,1$, the group Γ_{01} decomposes as a product, which we will write as $\Gamma'_{01\smallsetminus\alpha}\times\Gamma_\alpha$, where $\Gamma'_{01\smallsetminus\alpha}$ acts trivially on $(\mathscr{S}_M^{01})^{-1}(0)\cap(\mathrm{im}\iota_\alpha^M)$. Therefore, there are natural identifications

$$((\mathscr{S}_M^{01})^{-1}(0)\cap(\mathrm{im}\iota_M^\alpha))/\Gamma^{01}\cong((\mathscr{S}_M^\alpha)^{-1}(0))/\Gamma^\alpha\cong\{\alpha\}\times X\subset[0,1]\times X.$$

Thus, M^{01} is an oriented branched manifold of dimension $N^{01}+1$, where $N^{01}=d+\dim E_{A^{01}}$, with boundary that decomposes as a union

$$\partial M^{01}=\bigsqcup_{\alpha=0,1}EM^\alpha,\quad\text{where } EM^\alpha:=\iota_M^\alpha(E_{A^{01}\smallsetminus A_\alpha,\underline{\varepsilon}^\alpha}\times M^\alpha).\tag{2.3.5}$$

For $\alpha=0,1$, the branched manifold EM^α carries a fundamental class

$$\mu_{EM^\alpha}:=\mu_{E_{A^{01}\smallsetminus A_\alpha}}\times\mu_{M^\alpha}.$$

Because the isotropy group of the boundary chart labeled I_α in $\mathcal{K}_\alpha$ equals that of the corresponding cobordism chart in $\mathcal{K}^{01}$, the equation (2.3.3) is consistent with the boundary map in the long exact sequence (A.5) for the pair $(M^{01},\partial M^{01})$. Hence, the proof of Lemma 2.3.4 adapts to show that the interior of M^{01} also carries a fundamental class

$$\mu_{M^{01}}\in\check{H}^\infty_{N^{01}+1}(M^{01}\smallsetminus\partial M^{01})\tag{2.3.6}$$

such that

$$\partial(\mu_{M^{01}})=(\mu_{EM^1},-\mu_{EM^0})\in\check{H}^\infty_{N^{01}}(EM^0)\oplus\check{H}^\infty_{N^{01}}(EM^1)\cong\check{H}_{N^{01}}(\partial M^{01}),$$

where ∂ is the boundary map in the long exact sequence in (A.5).

We now apply the cap product in (A.7) with

$$Y=M^{01},\quad U=(\mathscr{S}_M^{01})^{-1}(E_{A^{01}}\smallsetminus\{0\})\subset M^{01},\quad A=\bigsqcup_{\alpha=0,1}EM^\alpha.$$

Then $Y\smallsetminus U=(\mathscr{S}_M^{01})^{-1}(0)$ is compact with a natural projection to $[0,1]\times X$ and hence to X. Since these maps are proper, any class $b\in\check{H}^d(X)$ pulls back to a class $b_Y\in\check{H}^d(Y\smallsetminus U)$ such that $\iota^*(b_Y)=b_A$ where $\iota\colon A\to Y$ is the inclusion, and

$b_A = (b_0, b_1)$, where b_α can be identified with the pullback of b to $(\mathscr{S}_M^\alpha)^{-1}(0) \subset M_\alpha$. Hence, the cap product

$$(\partial \mu_{M^{01}}) \cap b_A \in \check{H}^c_{N^{01}}(A, U \cap A)$$

is in the image of the map ∂' in (A.7) and hence vanishes when pushed forward to $\check{H}^c_{N^{01}}(Y, U)$. But there is a commutative diagram

$$\begin{array}{ccccc}
(\partial \mu_{M^{01}}) \cap b_A & \quad & \check{H}^c_{N^{01}}(A, U \cap A) & \xrightarrow{\mathscr{S}_M} & \check{H}^c_{N^{01}}(E_{A^{01}}, E_{A^{01}} \smallsetminus \{0\}) \\
\iota_* \Big\downarrow & & \Big\downarrow \iota_* & & \Big\downarrow = \\
0 & & \check{H}^c_{N^{01}}(Y, U) & \xrightarrow{\mathscr{S}_M} & \check{H}^c_{N^{01}}(E_{A^{01}}, E_{A^{01}} \smallsetminus \{0\})
\end{array}$$

Hence, $(\mathscr{S}_M)_*((\partial \mu_{M^{01}}) \cap b_A) = 0$. Since $(\partial \mu_{M^{01}}) \cap b_A$ measures the difference between the two classes $\mu_{EM^\alpha} \cap b_\alpha$, these classes have the same image in

$$\check{H}^c_{N^{01}}(E_{A^{01}}, E_{A^{01}} \smallsetminus \{0\}),$$

as claimed.

Step 3 (agreement with previous definition in the smooth case) It remains to show that in the smooth case the class $[X]^{\text{vir}}_{\mathcal{K}}$ constructed here agrees with that constructed in [12, Section 3]. The idea there was to construct a small smooth perturbation functor[24]

$$\nu = (\nu_I)\colon \boldsymbol{B}_{\mathcal{K}}|^{\smallsetminus\Gamma}_{\mathcal{V}} \to \boldsymbol{E}_{\mathcal{K}}|^{\smallsetminus\Gamma}_{\mathcal{V}}$$

such that $s_I + \nu_I$ is transverse to zero for all I, and then assemble the resulting zero sets $Z^\nu_I := (s_1 + \nu_I)^{-1}(0) \subset V_I$ into a weighted branched manifold $Z^\nu := |\hat{Z}^\nu_{\mathcal{H}}|$. Note that Z^ν is oriented and has a weight-preserving natural inclusion into M, ie each branch of Z^ν is a submanifold of a branch of M with the same weight. Now choose a sequence ν_k of perturbation sections with $\|\nu_k\| \to 0$. There is a corresponding nested sequence of neighborhoods $B_{\varepsilon_k}(\iota_{\mathcal{K}}(X))$ of the zero set $X \cong \iota_{\mathcal{K}}(X) \subset |\mathcal{V}|$ with intersection equal to $\iota_{\mathcal{K}}(X)$. Then the zero sets Z^{ν_k} map to $B_{\varepsilon_k}(\iota_{\mathcal{K}}(X)) \subset |\mathcal{V}|$, and we showed in [12, Theorem 3.3.5] that for all $\ell > k$ the two branched manifolds Z^{ν_ℓ} and Z^{ν_k} are cobordant in $B_{\varepsilon_k}(\iota_{\mathcal{K}}(X))$ and hence represent the same homology class in $B_{\varepsilon_k}(\iota_{\mathcal{K}}(X))$. It follows from the tautness property of rational Čech homology (see property (h$'$) in the appendix) that the inverse limit of this sequence of classes in $B_{\varepsilon_k}(\iota_{\mathcal{K}}(X))$ determines a unique element of $\check{H}_d(\iota_{\mathcal{K}}(X); \mathbb{Q}) \cong \check{H}_d(X; \mathbb{Q})$, which we called $[X]^{\text{vir}}_{\mathcal{K}}$ and showed to be independent of all choices.

[24] For notation see (1.2.9).

We now interpret this construction in the current setting. As above, fix a compact neighborhood[25] $\overline{\mathcal{N}}_0$ of $\mathscr{S}_M^{-1}(0)$, so that

$$\delta_0 := \inf\{\|\mathscr{S}_M(x)\| : x \in \mathrm{Fr}(\mathcal{N}_0) := \overline{\mathcal{N}}_0 \smallsetminus \mathcal{N}_0\} > 0,$$

and choose a nested sequence $\overline{\mathcal{N}}_k$ of compact neighborhoods of $\mathscr{S}_M^{-1}(0)$ such that

$$\bigcap_k \overline{\mathcal{N}}_k = \mathscr{S}_M^{-1}(0), \quad \mathscr{S}_M(\overline{\mathcal{N}}_k) \subset E_{A,\delta_k}, \quad \text{where } \delta_{k+1} < \delta_k < \delta_0.$$

Choose a corresponding sequence of transverse perturbation sections $\nu_k = (\nu_{k,I})$ such that the perturbed zero set $(s_I + \nu_{k,I})^{-1}(0)$ is contained in $V_I \cap \pi_{\mathcal{K}}^{-1}(\mathcal{N}_k)$ for all I, and for each k, consider the map

$$\hat{\nu}_k \colon M \to E_A, \quad \hat{\nu}_k(\pi_I(e_{A\smallsetminus I}, x)) = \nu_k(x) \in E_I \subset E_A.$$

This is well defined because $\nu_k \colon \boldsymbol{B}_{\mathcal{K}}|_{\mathcal{V}}^{\smallsetminus\Gamma} \to \boldsymbol{E}_{\mathcal{K}}|_{\mathcal{V}}^{\smallsetminus\Gamma}$ is a functor. Then

$$\mathrm{pr}_{E_{A\smallsetminus I}}\big((\mathscr{S}_M + \hat{\nu}_k)(\pi_I(e_{A\smallsetminus I}, x))\big) = \mathrm{pr}_{E_{A\smallsetminus I}}\big(\mathscr{S}_M(\pi_I(e_{A\smallsetminus I}, x))\big) \neq 0 \quad \text{if } e_{A\smallsetminus I} \neq 0,$$

while

$$\mathrm{pr}_{E_I}\big((\mathscr{S}_M + \hat{\nu}_k)(\pi_I(e_{A\smallsetminus I}, x))\big) = (s_I + \nu_{k,I})(x).$$

Therefore, we may identify the weighted branched manifold Z^{ν_k} with the perturbed zero set

$$(\mathscr{S}_M + \hat{\nu}_k)^{-1}(0) \subset \mathcal{N}_k \subset \mathscr{S}_M^{-1}(E_{A,\delta_k}).$$

Given $b \in \check{H}^d(X;\mathbb{Q})$, choose a sequence $b_k \in \check{H}^d(\overline{\mathcal{N}}_k;\mathbb{Q})$ such that $\varinjlim b_k = \psi^*(b)$, where $\psi \colon \mathscr{S}_M^{-1}(0) \to X$ is the footprint map, and let $\iota_k \colon Z^{\nu^k} \to \mathcal{N}_k$ be the inclusion. We must show that

$$\lim_k \langle \mu_{Z^{\nu_k}}, \iota_k^*(b_k)\rangle = (\mathscr{S}_M)_*(\mu_M \cap \psi^*(b)) \in \mathbb{Q}.$$

Consider the diagram below, in which the top and bottom square commute while the middle homotopy commutes:

[25] One important difference between $|\mathcal{V}|$ and M is that the zero set $|\mathfrak{s}|^{-1}(0)$ does *not* have compact neighborhoods in $|\mathcal{V}|$ by [14, Example 6.1.11], while it does in the branched manifold M.

$$\begin{tikzcd}
(M, M \setminus \mathscr{S}_M^{-1}(0)) \arrow[r, "\mathscr{S}_M"] & (E_A, E_A \setminus \{0\}) \arrow[d, equal] \\
(M, M \setminus \overline{\mathcal{N}}_k) \arrow[u, "\iota"] \arrow[r, "\mathscr{S}_M"] \arrow[d, equal] & (E_A, E_A \setminus \{0\}) \arrow[d, equal] \\
(M, M \setminus \overline{\mathcal{N}}_k) \arrow[r, "\mathscr{S}_M + \nu_k"] \arrow[d, "\iota"] & (E_A, E_A \setminus \{0\}) \arrow[d, equal] \\
(M, M \setminus Z^{\nu_k}) \arrow[r, "\mathscr{S}_M + \nu_k"] & (E_A, E_A \setminus \{0\})
\end{tikzcd}$$

Because Z^{ν_k} is a weighted branched smooth submanifold of M with orientation and weights compatible with that of E_A and M, its fundamental class $\mu_{Z^{\nu_k}}$ satisfies

$$\mu_{Z^{\nu_k}} = \mu_M \cap ((\mathscr{S}_M + \nu_k)^*(\mathfrak{o}_E)) \in H_d(Z^{\nu_k}, \mathbb{Q}),$$

where $\mathfrak{o}_E \in H^{\dim E_A}(E_A, E_A \setminus \{0\})$ is the natural generator.[26] This immediately implies that

$$\langle \mu_{Z^{\nu_k}}, \iota_k^*(b_k)\rangle = \langle (\mathscr{S}_M + \nu_k)_*(\mu_M \cap \iota_k^*(b_k)), \mathfrak{o}_E\rangle \in \mathbb{Q}.$$

Now note that the commutativity of the above diagram implies that

$$\varprojlim (\mathscr{S}_M + \nu_k)_*(\mu_M \cap \iota_k^*(b_k)) = (\mathscr{S}_M)_*(\mu_M \cap \psi^*(b)) \in H_{\dim E_A}(M, M \setminus \mathscr{S}_M^{-1}(0)).$$

The result follows. □

With a little more work, we can prove that our construction extends to atlases for compact pairs (W, X) as in [15, Lemma 5.2.4]. The following lemma defines

$$[W]^{\mathrm{vir}}_{\mathcal{K}} \in \check{H}^{\infty}_{d+1}(W \setminus X) = \mathrm{Hom}(\check{H}^{d+1}(W \setminus X); \mathbb{Q}).$$

Note that $\check{H}^{\infty}_{d+1}(W \setminus X) = \check{H}^{c}_{d+1}(W, X)$ by property (g′) in the appendix.

Lemma 2.3.5 *Given an oriented $(d+1)$–dimensional Kuranishi atlas $\mathcal{K}^W$ with boundary on a compact pair $(W, X := \partial W)$, there is an associated virtual fundamental class $[W]^{\mathrm{vir}}_{\mathcal{K}} \in \check{H}^{\infty}_{d+1}(W \setminus X)$ such that*

$$\partial([W]^{\mathrm{vir}}_{\mathcal{K}}) = [X]^{\mathrm{vir}}_{\mathcal{K}} \in \check{H}^{\infty}_{d}(X) = \check{H}^{c}_{d}(X), \tag{2.3.7}$$

[26] We can use singular homology since we can assume that Z^{ν_k} and M are simplicial complexes by [9].

where ∂ is the differential in the long exact sequence (A.5). *In particular, the image of $[X]^{\mathrm{vir}}_{\mathcal{K}}$ in $\check{H}^{\infty}_{d}(W) = \check{H}^{c}_{d}(W)$ is zero.*

Proof We define the notion of an oriented $(d+1)$–dimensional Kuranishi atlas $(\mathcal{K}, \partial\mathcal{K})$ for the pair $(W, \partial W)$ by replacing $[0,1]\times X$ by W in the above definition of a cobordism atlas. Thus, we take $\mathcal{K}^{01} =: \mathcal{K}^{W}$ to be an atlas for W, $\mathcal{K}^{1} =: \mathcal{K}^{X}$ an atlas for X and $\mathcal{K}^{0}$ to be empty, and assume the obvious analogs of (i)–(iii) above. Then, given a branched manifold (M^X, Λ^X) constructed from $\mathcal{K}^X$, we may construct a branched manifold (M^W, Λ^W) with boundary

$$\partial(M^W) = E_{A^W \smallsetminus A^X} \times M^X,$$

and extend $\mathrm{id}\times\mathscr{S}_X$ from $\partial(M^W)$ to a map $\mathscr{S}_W\colon M^W \to E_{A^W}$ that satisfies the analogs of equations (2.3.4) and (2.3.5) above. Further, using the fundamental class $\mu^W_M \in H^{\infty}_{N^W}(M^W \smallsetminus \partial M^W)$ defined as in (2.3.6), we define an element

$$[W]^{\mathrm{vir}}_{\mathcal{K}} \in \check{H}^{\infty}_{d+1}(W \smallsetminus X)$$

by setting

$$\langle [W]^{\mathrm{vir}}_{\mathcal{K}}, b\rangle := (\mathscr{S}_{M^W})_*(\widehat{b}) \in \check{H}_{\dim E_{A^W}}(E_{A^W}, E_{A^W} \smallsetminus \{0\}; \mathbb{Q}) \cong \mathbb{Q},$$

where $\widehat{b}$ is defined as follows. Let

$$Y = M^W, \quad A = \partial M^W, \quad U = \mathscr{S}_W^{-1}(E_{A^W} \smallsetminus \{0\}).$$

Then the pullback $\psi^* b \in \check{H}^{d+1}(Y \smallsetminus (U \cup A); \mathbb{Q})$ of $b \in \check{H}^{d+1}(W \smallsetminus X; \mathbb{Q})$ determines

$$\widehat{b} := \mu^W_M \cap \psi^* b \in \check{H}^{c}_{\dim E_{A^W}}(Y \smallsetminus A, U \smallsetminus A; \mathbb{Q}),$$

where $\cap\colon \check{H}^{\infty}_{p+q}(Y \smallsetminus A) \otimes \check{H}^{p}(Y \smallsetminus U) \to \check{H}^{c}_{q}(Y, U \cup A)$ is as in (A.6) with $A = \varnothing$.

To prove (2.3.7), note that in the diagram (with the same Y, U and A)

$$\begin{tikzcd}
\check{H}^{\infty}_{p+q+1}(Y \smallsetminus A) \otimes \check{H}^{p+1}(Y \smallsetminus (A \cup U)) \arrow[r, "\cap"] \arrow[d, "\partial"] & \check{H}^{c}_{q}(Y \smallsetminus A, U \smallsetminus A) \arrow[r, "\iota_*"] & \check{H}^{c}_{q}(Y, U) \\
\check{H}^{\infty}_{p+q}(A) \otimes \check{H}^{p}(A \smallsetminus U) \arrow[u, "\delta"] \arrow[r, "\cap"] & \check{H}^{c}_{q}(A, A \cap U) \arrow[ur, "j_*"] &
\end{tikzcd}$$

we have

$$j_*((\partial\mu^W_M) \cap b') = \iota_*(\mu^W_M \cap (\delta b')) \in \check{H}^{c}_{q}(Y, U)$$

for all $\mu \in \check{H}^{\infty}_{p+q+1}(Y \smallsetminus A)$ and $b' \in \check{H}^{p}(A \smallsetminus U)$, where δ is as in (A.1).[27] Since ψ^* commutes with δ, this implies

$$\langle \partial([W]^{\mathrm{vir}}_{\mathcal{K}}), b\rangle = \langle [W]^{\mathrm{vir}}_{\mathcal{K}}, \delta b\rangle \quad \text{for all } b \in \check{H}^{d}(X).$$

The result follows. □

3 Further details and constructions

In Section 3.1 we first define the notion of a compatible shrinking $(\mathcal{U}, \underline{\varepsilon})$ and prove Proposition 2.1.1. We then introduce the more intricate notion of a compatible reduction $(\mathcal{V}, \underline{\varepsilon})$, which involves not only the compatibility of $\mathcal{V}$ with a set of constants $\underline{\varepsilon}$ but also its compatibility with a suitable cover of the set of overlaps in $|\mathcal{V}|$, properties that are essential for the proof in Section 3.2 that $Y_{\mathcal{V},J,\underline{\varepsilon}}$ has a collar that satisfies the conditions listed in Proposition 2.1.4.

3.1 Shrinkings and the manifold *Y*

We assume given an ambient preshrunk tame[28] atlas $\mathcal{K}^{\Omega}$ with chart domains $\mathcal{U}^{\Omega}$, together with a tame shrinking $\mathcal{U}^{\infty} \sqsubset \mathcal{U}^{\Omega}$, and then choose a further shrinking $\mathcal{F}^0$ of the footprints $\mathcal{F}^{\infty}$ of $\mathcal{U}^{\infty}$. For short we write $\psi^{-1}(\mathcal{F}^0) \sqsubset \mathcal{U}^{\infty} \sqsubset \mathcal{U}^{\Omega}$. By the submersion axiom and the precompactness of $\widetilde{U}^{\infty}_{IK}$ in $\widetilde{U}^{\Omega}_{IK}$ for each $I \subsetneq K$, we may choose a finite set of points $z_\alpha \in \widetilde{U}^{\Omega}_{IK}$, constants $\varepsilon_\alpha > 0$ and Γ_{z_α}–equivariant local homeomorphisms

$$\phi^{E}_{IK,z_\alpha}\colon E_{K\smallsetminus I,\varepsilon_\alpha} \times \widetilde{W}_{IK,z_\alpha} \to U^{\Omega}_{K}, \quad 1 \le \alpha \le A_{IK}, \tag{3.1.1}$$

where $\widetilde{W}_{IK,z_\alpha} \subset \widetilde{U}^{\Omega}_{IK}$ is open, such that

$$\begin{gathered} s_{K\smallsetminus I} \circ \phi^{E}_{IK,z_\alpha}(e, y) = e, \\ \widetilde{U}^{\infty}_{IK} \subset \bigcup_{1\le\alpha\le A_{IK}} \widetilde{W}_{IK,z_\alpha} \subset \widetilde{U}^{\Omega}_{IK} \quad \text{for all } I \subsetneq K. \end{gathered} \tag{3.1.2}$$

We may and will assume that each ϕ^{E}_{IK,z_α} is Γ_K–equivariant. (To do this, first shrink the $\widetilde{W}_{IK,z_\alpha}$ so that they have disjoint images under the group $\Gamma_K/\Gamma_{z_\alpha}$, and then replace them by their orbit under $\Gamma_K/\Gamma_{z_\alpha}$.)

[27] This extension to property (B5) on [8, page 336] holds by combining properties (B4) and (B6).

[28] For terminology see Section 1.2.

Our first task is to make a preliminary choice of shrinking so that the space $Y_{\mathcal{U},J,\underline{\varepsilon}}$ is a manifold with boundary. In the following definition, condition (b) ensures that the charts are compatible with a fixed shrinking of the zero sets, while conditions (a) and (c) have already been used in the proof of Corollary 2.1.2. Some version of condition (d) is an essential ingredient in the proof that $Y_{\mathcal{U},J,\underline{\varepsilon}}$ is a manifold with boundary; see (3.1.4) below. As we saw in (2.1.4) and (2.1.5), the elements $(e,x;t) \in \partial_{J \smallsetminus I} Y_{\mathcal{U},J,\underline{\varepsilon}}$ have $x \in \widetilde{U}_{IJ}$ and $\|e\| < \kappa \varepsilon_{I(x)}$, where $I(x) \subset I \subsetneq J$. Therefore, in order for the domain of the local chart in (3.1.4) to include all the boundary points of $Y_{\mathcal{U},J,\underline{\varepsilon}}$, we do need the map $\phi^E_{IK,\varepsilon}$ to be defined using a constant ε that is $> \kappa \varepsilon_I$, and we have chosen to use $(\kappa+1)\varepsilon_I$ for convenience and precision. Note also that we do not insist that the image of the map ϕ^E_{IK} in (3.1.3) below is contained in $\mathcal{U}$ or even in $\mathcal{U}^\infty$. Such a requirement comes later; see (3.1.9).

Definition 3.1.1 Given $\psi^{-1}(\mathcal{F}_0) \sqsubset \mathcal{U}^\infty \sqsubset \mathcal{U}^\Omega$ as above, we will say that a shrinking $\mathcal{U}$ and set of positive constants $\underline{\varepsilon} := (\varepsilon_K)_{K \in \mathcal{I}_\mathcal{K}}$ are $(\mathcal{G}_0, \mathcal{U}^\infty)$*–compatible* if the following holds:

(a) $0 < \kappa \varepsilon_I < \varepsilon_K$ if $I \subsetneq K$ (see (1.3.2)).

(b) $\psi^{-1}(\mathcal{F}^0) \sqsubset \mathcal{U} \sqsubset \mathcal{U}^\infty$.

(c) $s_I(\overline{U}_I) \subset E_{I,\varepsilon_I}$ for all I.

(d) For all $I \subsetneq K$, each $z \in \widetilde{U}_{IK} \subset U_K$ has a neighborhood $\widetilde{\mathcal{O}}_{IK} \subset \widetilde{U}_{IK}$ such that one of the homeomorphisms ϕ^E_{IK,z_α} in (3.1.1) restricts to give a map

$$\phi^E_{IK} \colon E_{K \smallsetminus I,(\kappa+1)\varepsilon_I} \times \widetilde{\mathcal{O}}_{IK} \to U^\Omega_K \tag{3.1.3}$$

that is a homeomorphism to its image, where $\kappa := \max\{|K| : K \in \mathcal{I}_\mathcal{K}\}$.

For simplicity we call the pair $(\mathcal{U}, \underline{\varepsilon})$ a *compatible shrinking*.

Lemma 3.1.2 *Suppose given $\psi^{-1}(\mathcal{G}_0) \sqsubset \psi^{-1}(\mathcal{F}_0) \sqsubset \mathcal{U}^\infty \sqsubset \mathcal{U}^\Omega$ as above. Then there is an $(\mathcal{F}_0, \mathcal{U}^\infty)$–compatible shrinking $(\mathcal{U}, \underline{\varepsilon})$.*

Proof First choose any tame shrinking $\mathcal{U}'$ such that $\psi^{-1}(\mathcal{F}^0) \sqsubset \mathcal{U}' \sqsubset \mathcal{U}^\infty$, which is possible by [13, Proposition 3.3.5]. Then each set U'_I is covered by a finite number of the sets $\widetilde{W}_{IK,z_\alpha}$ in (3.1.1) and we choose any set of constants $\underline{\varepsilon}$ satisfying (a) and also so that $\varepsilon_I < \varepsilon_\alpha/(\kappa+1)$ for all relevant α. Then, if we define $U_I := U'_I \cap s_I^{-1}(E_{I,\varepsilon_I})$, property (d) holds. Further, $\mathcal{U} := (U_I)$ is a tame shrinking of $\mathcal{U}^\infty$ because the coordinate changes commute with the section maps s_I and preserve the norms $\|\cdot\|$ on E_A. (More precisely,

$$\widehat{\phi}_{IK}\circ s_I\circ \rho_{IK}=s_K\colon \widetilde{U}_{IK}\to E_K,$$

where the canonical inclusion $\widehat{\phi}_{IK}\colon E_I\to E_K$ preserves $\|\cdot\|$, ie $\|\widehat{\phi}_{IK}(e)\|=\|e\|$.) Hence, $\mathcal{U}$ satisfies (c) and (b), as required. □

From now on, we fix $(\mathcal{F}^0,\mathcal{U}^\infty)$, and hence cease to refer to them explicitly.

Lemma 3.1.3 *If $(\mathcal{U},\underline{\varepsilon})$ is compatible, then for each J, $Y_{\mathcal{U},J,\underline{\varepsilon}}$ is a manifold of dimension $D:=d+\dim E_A+|J|-1$, with boundary equal to*

$$Y_{\mathcal{U},J,\underline{\varepsilon}}\cap \mathrm{pr}_\Delta^{-1}(\partial\Delta)=\bigcup_{I\subsetneq J}\partial_{J\smallsetminus I}Y_{\mathcal{U},J,\underline{\varepsilon}}=\bigcup_{I\subsetneq J}\{(e,x;t):x\in\widetilde{U}_{IJ},\ t\in\partial_{J\smallsetminus I}\Delta_J\}.$$

Proof We show that each point $(e,x;t)\in Y_{\mathcal{U},J,\underline{\varepsilon}}$ has a neighborhood homeomorphic to an open subset of $(\mathbb{R}_{\geq 0})^k\times\mathbb{R}^{D-k}$, where $k=\#\{j\in J\mid t_j=0\}$. Thus, the projection $\mathrm{pr}_\Delta\colon Y_{\mathcal{U},J,\underline{\varepsilon}}\to\Delta_J$ is compatible with the boundary structure of Δ_J.

First consider a point $(e,x;t)\in Y_{\mathcal{U},J,\underline{\varepsilon}}$ with $t_i\neq 0$ for all $i\in J$. Then the coordinates e_j for $j\in J$ are determined by (x,t) via the requirement $s_J(x)=t\cdot e|_J$ while the components of $e|_{A\smallsetminus J}:=(e_i)_{i\in A\smallsetminus J}$ can vary freely. Hence, the tuple $(e,x;t)$ is uniquely determined by the point $(e|_{A\smallsetminus J},x;t)\in E_{A\smallsetminus J}\times U_J\times\operatorname{int}\Delta_J$, and so has a manifold neighborhood of dimension D.

It remains to define boundary charts at the points $(e^0,x^0,t^0)\in Y_{\mathcal{U},J,\varepsilon}$ with

$$t^0\in\partial\Delta_J=\bigcup_{I\subsetneq J}\partial_{J\smallsetminus I}\Delta_J=:\bigcup_{I\subsetneq J}\operatorname{int}\Delta_I.$$

Suppose first that

$$I(x^0):=\{i:s_i(x^0)\neq 0\}=\{i:t_i^0>0\}=:I(t^0)=:I,$$

so that $x^0\in\widetilde{U}_{IJ}$. By (3.1.2), there is a neighborhood $\widetilde{\mathcal{O}}$ of x^0 in $\widetilde{U}_{IJ}$ that is contained in one of the sets $\widetilde{W}_{IJ,z_\alpha}$ in (3.1.1), and below we denote by ϕ the associated map ϕ^E_{IJ,z_α}. There is a corresponding neighborhood of (e^0,x^0,t^0) in

$$\partial_{J\smallsetminus I}Y_{\mathcal{U},J,\underline{\varepsilon}}\cap\{(e,x;t):e|_{J\smallsetminus I}=0\}$$

given by

$$\widetilde{\mathcal{O}}'_{I,J,\underline{\varepsilon}}:=\{(e_{A\smallsetminus J}+t_I^{-1}\cdot s_I(x),x;t_I)\mid x\in\widetilde{\mathcal{O}},\ t_I\approx t_I^0,\ \|e_{A\smallsetminus J}\|<\kappa\varepsilon_I\}\subset\partial_{J\smallsetminus I}Y_{\mathcal{U},J,\underline{\varepsilon}}.$$

Now consider the map

$$\psi\colon E_{J\smallsetminus I,(\kappa+1)\varepsilon_I} \times [0,\delta)^{|J\smallsetminus I|} \times \widetilde{\mathcal{O}}'_{I,J,\underline{\varepsilon}} \to Y_{\mathcal{U}^\Omega,J,\underline{\varepsilon}}, \tag{3.1.4}$$

$$\big(e_{J\smallsetminus I}, r_{J\smallsetminus I}, (e_{A\smallsetminus J} + t_I^{-1}\cdot s_I(x), x; t_I)\big) \\ \mapsto (e_{A\smallsetminus J} + e_{J\smallsetminus I} + (\lambda t_I)^{-1}\cdot s_I(x'), x'; \lambda t_I + r_{J\smallsetminus I}),$$

where

$$x' := \phi(r_{J\smallsetminus I}\cdot e_{JI}, x) \quad \text{for } \phi := \phi^E_{IJ,z_\alpha} \quad \text{and} \quad \lambda := 1 - |r_{J\smallsetminus I}| = 1 - \sum_{j\in J\smallsetminus I} r_j.$$

To see that ψ does have image in $Y_{\mathcal{U}^\Omega,J,\underline{\varepsilon}}$ for sufficiently small $\delta > 0$ and $\widetilde{\mathcal{O}}$, we check the conditions in (2.1.1) as follows:

- By (3.1.2),

  $$r_{J\smallsetminus I}\cdot e_{J\smallsetminus I} = s_{J\smallsetminus I}\circ\phi(r_{J\smallsetminus I}\cdot e_{JI}, x) = s_{J\smallsetminus I}(x'),$$

  so that the image $(e, x; t)$ of ψ does satisfy the equation $s_J(x) = t\cdot e$ if $x' \approx x^0$ and $\delta > 0$ is sufficiently small.

- Next, we check that $s_J(x') \in E_{A,\varepsilon_{I(x')}}$. To this end, note first that because we started by assuming $I(x^0) = I$, the definition of $Y_{\mathcal{U},J,\underline{\varepsilon}}$ implies $\|s_I(x^0)\| < \varepsilon_I$. Second, because $s_I(x') \approx s_I(x^0)$, we have $s_I(x') < \varepsilon_I$ for sufficiently small δ and $\widetilde{\mathcal{O}}$. But if $r_{J\smallsetminus I} \neq 0$ we have $I(x') \supsetneq I(x^0)$, so that $\varepsilon_I < \frac{1}{\kappa}\varepsilon_{I(x')}$ by (1.3.2). Therefore, because $\lambda \approx 1$ and we use the sup norm on the product E_A, we have

  $$s_J(x') = e_{J\smallsetminus I} + (\lambda t_I)^{-1}\cdot s_{J\smallsetminus I}\circ\phi(r_{J\smallsetminus I}\cdot e_{J\smallsetminus I}, x) \in E_{A,\varepsilon_{I(x')}}$$

  for sufficiently small $\delta > 0$.

- Since elements in the domain of ψ have $\|e_{A\smallsetminus J}\| < \kappa\varepsilon_I < \varepsilon_{I(x')}$, elements in its image also satisfy this condition.

It is now easy to check that ψ is a local homeomorphism that equals the identity map when $r_{J\smallsetminus I} = 0$ since $\phi(0, x) = x$. Hence, its restriction to a suitable open subset of its domain provides a local boundary chart for $Y_{\mathcal{U},J,\underline{\varepsilon}}$ at (e^0, x^0, t^0).

It remains to consider the case when $I = I(x^0) \subsetneq H = I(t^0)$. In this case, write $t^0 = t^0_I + t^0_{H\smallsetminus I}$. Then the above formula for ψ must be modified as follows: Denote the elements of $I(t^0)$ by $t'_H = t'_I + t'_{H\smallsetminus I}$. Then, for $r_{J\smallsetminus I} \approx 0$, we define

$$\psi\big(e_{J\smallsetminus I}, r_{J\smallsetminus I}, (e_{A\smallsetminus J} + (t'_H)^{-1}\cdot s_I(x), x; t'_H)\big) \\ = (e_{A\smallsetminus J} + e_{J\smallsetminus I} + (t''_I)^{-1}\cdot s_I(x''), x''; t''_J), \tag{3.1.5}$$

where

- x varies in a neighborhood $\widetilde{\mathcal{O}} \subset \widetilde{U}_{HJ}$ of x^0;
- $\lambda < 1$ is chosen so that $t''_J := ((t_i)'')_{i\in J}$ has $|t''_J| := \sum_{i\in J} t''_i = 1$, where

$$t''_i = \lambda t'_i \quad \text{if } i \in I, \qquad t''_h = \lambda t'_h + r_h if i \in H \smallsetminus I,$$
$$t''_j = r_j \quad \text{if } j \in J \smallsetminus H;$$

- $x'' = \phi(t''_{J\smallsetminus I} \cdot e_{J\smallsetminus I}, x) \in V_J$.

Then one can check as above that $\operatorname{im}\psi$ is a neighborhood of (e^0, x^0, t^0) in $Y_{\mathcal{U},J,\underline{\varepsilon}}$. This completes the proof. □

Corollary 3.1.4 *Proposition 2.1.1 holds.*

Proof Combine Lemmas 3.1.2 and 3.1.3. □

Remark 3.1.5 In (3.1.5) the coordinates $r_{H\smallsetminus I} \in \mathbb{R}^{H\smallsetminus I}$ parametrize directions tangent to $\partial_{J\smallsetminus H} Y_{\mathcal{U},J,\underline{\varepsilon}}$, while the coordinates $r_{J\smallsetminus H} \in \mathbb{R}^{J\smallsetminus H}$ parametrize the directions normal to the codimension-$|J\smallsetminus H|$ face $\partial_{J\smallsetminus H} Y_{\mathcal{U},J,\underline{\varepsilon}}$. ◇

We now define and construct *compatible reductions* $(\mathcal{V}, \underline{\varepsilon})$. In order to prove Proposition 2.1.4, it turns out that we need more control over the sets $\mathcal{O}_{IK}$ in Definition 3.1.1(d). Indeed, because of the consistency requirements on the collar, it is not sufficient to choose the $\mathcal{O}_{IK}$ separately for each pair $I \subsetneq K$; rather they must be chosen consistently for all pairs, as we now describe. Further, because the collar has fixed width and image in $Y_{\mathcal{V},J,\underline{\varepsilon}}$, the product maps in (d) must have image in V_K rather than in V_K^Ω. Then, as we will see in the first step of the proof of Lemma 3.2.1 below, they can be used to provide local collars along the boundary of $Y_{\mathcal{V},J,\underline{\varepsilon}}$.

Note first that because the local product structures

$$\phi^E_{IK,z_\alpha}\colon E_{K\smallsetminus I,\varepsilon_\alpha} \times \widetilde{W}_{IK,z_\alpha} \to U_K^\Omega, \quad 1 \le \alpha \le A_{IK}, \tag{3.1.6}$$

in (3.1.1) are equivariant and satisfy $s_{K\smallsetminus I} \circ \phi^E_{IK,z_\alpha}(e,y) = e$, they descend via ρ_{HK} whenever $I \subsetneq H \subsetneq K$. More precisely, for such H,

$$\phi^E_{IK,z_\alpha}\colon E_{H\smallsetminus I,\varepsilon_\alpha} \times (\widetilde{U}_{HK} \cap \widetilde{W}_{IK,z_\alpha}) \to \widetilde{U}^\Omega_{HK} = s_K^{-1}(E_H)$$

is the lift of a well-defined map

$$\phi^E_{IH,\rho_{HK}(z_\alpha)}\colon E_{H\smallsetminus I,\varepsilon_\alpha} \times \rho_{HK}(\widetilde{U}_{HK} \cap \widetilde{W}_{IK,z_\alpha}) \to U^\Omega_H. \tag{3.1.7}$$

Before defining the notion of compatible reduction, we describe certain covers of the set $\overline{\mathcal{OL}}(|\mathcal{V}|)$ of "overlaps" in $|\mathcal{V}|$, which is the image in $|\mathcal{V}|$ of the relevant part of the boundary of $\bigcup_J Y_{\mathcal{V},J,\underline{\varepsilon}}$. See Figure 4.

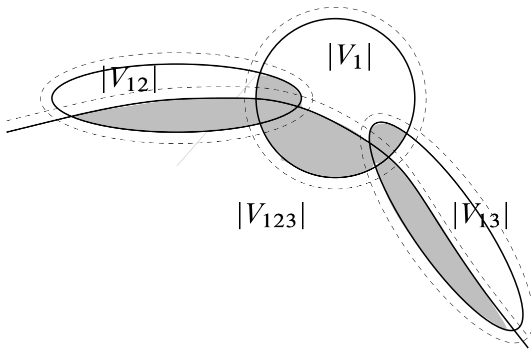

Figure 4: Here $\overline{\mathcal{OL}}(|\mathcal{V}|)$ is shaded, and the sets V_I^Ω are given by dotted lines; note that sets $|V_I^\Omega|$ and $|V_J^\Omega|$ are disjoint unless I and J are nested.

Definition 3.1.6 Given a subset $|W| \subset |\mathcal{V}^\Omega|$ we say that $W \subset V_I^\Omega$ is a *lift* of $|W|$ if

$$\pi_{\mathcal{K}}(W) = |W|, \quad W = V_I^\Omega \cap \pi_{\mathcal{K}}^{-1}(|W|),$$

ie W is a "full" inverse image of $|W|$ in $\mathcal{V}_I^\Omega$.

Lemma 3.1.7 *If $(\mathcal{U}, \underline{\varepsilon})$ is compatible, and $\mathcal{V} \sqsubset \mathcal{V}^\Omega \sqsubset \mathcal{U}$ is any nested reduction, denote by*

$$\overline{\mathcal{OL}}(|\mathcal{V}|) := \bigcup_{I \subsetneq K} |\overline{V}_{IK}| \subset |\mathcal{V}^\Omega|, \tag{3.1.8}$$

the closure of the set of overlaps in $|\mathcal{V}|$. Then we may cover $\overline{\mathcal{OL}}(|\mathcal{V}|)$ by a finite number of sets $(|W_\alpha|)_{1\le\alpha\le N}$, where for each α there is $\widetilde{W}_{IK,z_\alpha}$ as in (3.1.6) *such that*

$$W_\alpha := \widetilde{V}_{IK}^\Omega \cap \pi_{\mathcal{K}}^{-1}(|W_\alpha|) \subset \widetilde{W}_{IK,z_\alpha}$$

is a lift of $|W_\alpha|$. Moreover, we require that I is minimal and K is maximal in the sense that

(i) *W_α is an open subset of $\widetilde{V}_{IK}^\Omega$,*

(ii) $|V_H| \cap |W_\alpha| \ne \varnothing \Longrightarrow I \subset H \subset K$.

In this situation, we say that $\mathcal{V}$ is ***adapted to the cover*** $(|W_\alpha|)_{1\le\alpha\le N}$.

Proof Choose compatible shrinkings $\mathcal{V} \sqsubset \mathcal{V}^1 \sqsubset \cdots \sqsubset \mathcal{V}^\kappa \sqsubset \mathcal{V}^\Omega \sqsubset \mathcal{U}$. Work by downwards induction on $|I| = \ell \le \kappa - 1$, so that at the ℓ^{th} stage we have a covering $(|W_\alpha^\ell|)_{\alpha\in B_\ell}$ of

$$\bigcup_{I \subsetneq K, \ell \le |I|} |V_{IK}|$$

with lifts W_α^ℓ satisfying (i) and such that (ii) holds if $|H| \ge \ell$. When $\ell = \kappa - 1$, the existence of the finite covering holds by the precompactness of $|\mathcal{V}|$ in $|\mathcal{U}|$ while (ii) is easy to arrange because the sets $|V_H|$ with $|H| = \ell$ are disjoint. Now let us suppose that this holds for $\ell + 1$ with the sets $(|W_\alpha^{\ell+1}|)_{\alpha\in B_{\ell+1}}$ and consider the statement for ℓ.

The covering $(|W_\alpha^\ell|)$ will consist of sets of two kinds:

- If $|W_\alpha^{\ell+1}|$ lifts to $W_\alpha^{\ell+1} \subset V_I^\Omega$, where $|I| \ge \ell + 1$ is as in (i), then we take the set $|W'_\alpha|$ where
$$W'_\alpha := W_\alpha^{\ell+1} \smallsetminus \bigcup_{|H|=\ell} \mathrm{cl}(\widetilde{V}_{HI}^\ell).$$
This is open in V_I^Ω since we have removed a closed set, and satisfies (ii) for ℓ. These sets cover
$$\Bigl(\bigcup_{I \subsetneq K,\, \ell+1 \le |I|} |V_{IK}^\ell|\Bigr) \smallsetminus \Bigl(\bigcup_{H \subsetneq K,\, |H| = \ell} \mathrm{cl}(|V_{HK}^\ell|)\Bigr).$$
- Next add a finite cover of the compact set $\bigcup_{H \subsetneq K, |H|=\ell} \mathrm{cl}(|V_{HK}^\ell|)$ by sets $|W_\alpha|$ whose lifts lie in V_{HK}^Ω, where $|H| = \ell$. These obviously satisfy (ii).

This completes the proof. □

Remarks 3.1.8 (i) If $\mathcal{V}$ is adapted to the cover $(|W_\alpha|)_{1\le\alpha\le N}$, and $\mathcal{V}' \sqsubset \mathcal{V}$ is any shrinking, then $\mathcal{V}'$ is also adapted to the cover $(|W_\alpha|)_{1\le\alpha\le N}$.

(ii) If $I \subsetneq H$ then in general $\widetilde{V}_{IH}$ is not closed in V_H. Therefore, in order to cover $\overline{\mathcal{OL}}(|\mathcal{V}|)$ by sets $|W_\alpha|$ that satisfy condition (ii) in Lemma 3.1.7 one cannot insist that each set $|W_\alpha|$ lift to an open subset of some V_I, but rather as in Lemma 3.1.7(i) that it have a lift to an open subset of some $V_I^\Omega \sqsupset V_I$. ◇

Definition 3.1.9 Suppose that $\psi^{-1}\mathcal{G}^0 \sqsubset \mathcal{V}^\Omega \sqsubset \mathcal{U}$, where $\mathcal{G}^0 \sqsubset \mathcal{F}$ is a reduction of the footprint cover (ie $G_I^0 \sqsubset F_I$ for all I and $\bigcup_I G_I^0 = X$), and choose a shrinking $\mathcal{V}^\infty \sqsubset \mathcal{V}^\omega$ that is adapted to the cover $(|W_\alpha|)_{1\le\alpha\le N}$ where $|W_\alpha| \subset |\mathcal{V}^\Omega|$. With these choices fixed, we then say that the pair $(\mathcal{V}, \underline{\varepsilon})$ is *precompatible* if the following conditions hold:

(a′) $0 < \kappa\varepsilon_I < \varepsilon_J$ for all $I \subsetneq J$.

(b′) $\psi^{-1}(\mathcal{G}^0) \sqsubset \mathcal{V} \sqsubset \mathcal{V}^\infty$.

(c′) $s_I(\overline{V}_I) \subset E_{I,\varepsilon_I}$ for all I.

(d′) For all α with $W_\alpha \subset \widetilde{V}^\Omega_{IK}$ and $I \subsetneq H \subset K$,

$$\phi^E_{IH,\alpha}\big(E_{H \smallsetminus I,(\kappa+1)\varepsilon_I} \times (\widetilde{V}_{IH} \cap \rho_{HK}(W_\alpha \cap \widetilde{V}_{HK}))\big) \subset V_H. \tag{3.1.9}$$

Further, we say that $(\mathcal{V}, \underline{\varepsilon})$ is *compatible* if it is precompatible and if $(\mathcal{V}, \underline{\varepsilon}) \sqsubset (\mathcal{V}', \underline{\varepsilon}')$, where $(\mathcal{V}', \underline{\varepsilon}')$ is also precompatible and $\underline{\varepsilon} \le \underline{\varepsilon}'$, ie $\varepsilon_J \le \varepsilon'_J$ for all $J \in \mathcal{I}_\mathcal{K}$.

Remark 3.1.10 If $(\mathcal{V}, \underline{\varepsilon})$ is compatible, so that it is a shrinking of the precompatible $(\mathcal{V}', \underline{\varepsilon}')$, then we may assume that each set $|W_\alpha|$ of the associated covering of $|\mathcal{V}|$ lifts to some subset $\widetilde{V}'_{IK}$. In other words, we can equivalently define $(\mathcal{V}, \underline{\varepsilon})$ to be compatible if $(\mathcal{V}, \underline{\varepsilon}) \sqsubset \mathcal{V}^\infty$ is precompatible as above for some reduction $\mathcal{V}^\infty$ that is provided with constants $\underline{\varepsilon}^\infty \ge \underline{\varepsilon}$ such that (a′) and (c′) hold. ◇

The next lemma shows that the hypothesis in Proposition 2.1.4 can be satisfied.

Lemma 3.1.11 *Suppose given $\psi^{-1}(\mathcal{G}_0) \sqsubset \psi^{-1}(\mathcal{F}_0) \sqsubset \mathcal{V}^\infty \sqsubset \mathcal{V}^\Omega$ such that $\mathcal{V}^\infty$ is adapted to the covering $(|W_\alpha|)_{1 \le \alpha \le N}$, where $|W_\alpha| \subset |\mathcal{V}^\Omega|$. Then:*

(i) *There is a precompatible shrinking $(\mathcal{V}, \underline{\varepsilon}) \sqsubset \mathcal{V}^\infty$.*

(ii) *Any precompatible $(\mathcal{V}', \underline{\varepsilon}')$ has a compatible shrinking $(\mathcal{V}, \underline{\varepsilon}) \sqsubset (\mathcal{V}', \underline{\varepsilon}')$.*

Proof The proof of (i) is somewhat similar to that of Lemmas 3.1.2 and 3.1.7, except that now we have to make sure that (d′) holds, ie that we can choose $\mathcal{V}$ so that the image of $\phi^E_{IH,\alpha}$ lies in V_H for all $I \subsetneq H \subset K$ rather than just in the fixed ambient space U^Ω_J as in (3.1.3). Claim (ii) then follows by the same argument, with $\mathcal{V}^\infty$ replaced by $\mathcal{V}'$.

To prove (i), we first choose any reduction $\mathcal{V}^\kappa$ of $\mathcal{U}$, where $(\mathcal{U}, \underline{\varepsilon}^\kappa)$ is compatible, so that $(\mathcal{V}, \underline{\varepsilon})$ satisfies (a′), (b′) and (c′). We then work by downwards induction on $\ell := |J|$, so that after the ℓ^{th} stage we have chosen a reduction $(\mathcal{V}^\ell, \underline{\varepsilon}^\ell)$ with

$$\psi^{-1}(\mathcal{G}^0) \sqsubset \mathcal{V}^\ell \sqsubset \mathcal{V}^\kappa, \quad \underline{\varepsilon}^\ell \le \underline{\varepsilon}^\kappa$$

that satisfies (a′), (b′) and (c′) for all I and K, and satisfies (d′) for all I with $|I| \ge \ell$. Since (d′) is vacuous when $\ell = \kappa$, it suffices to suppose that we have found suitable

$(\mathcal{V}^{\ell+1}, \underline{\varepsilon}^{\ell+1})$ for some $1 < \ell+1 \le \kappa$, and consider the construction of $(\mathcal{V}^{\ell}, \underline{\varepsilon}^{\ell})$. Our method gives $\underline{\varepsilon}^{\ell}$, where $\varepsilon_J^{\ell} = \varepsilon_J^{\ell+1}$ if $|J| > \ell$ and $\varepsilon_I^{\ell} \le \varepsilon_I^{\ell+1}$ if $|I| \le \ell$. Further, for $|J| > \ell$ we construct V_J^{ℓ} by removing some points in $\widetilde{V}_{IJ}^{\ell+1}$ from $V_J^{\ell+1}$ for $|I| = \ell$. Note that removing these points does not affect the validity of (d′) for pairs $I \subsetneq K$ with $|I| \ge \ell+1$.

Choose an intermediate reduction $\mathcal{V}'$ such that $\mathcal{V}^0 \sqsubset \mathcal{V}' \sqsubset \mathcal{V}^{\ell+1}$. Because the subsets $\pi_{\mathcal{K}}(V_I^{\infty}) \subset |\mathcal{K}|$ with $|I| = \ell$ are disjoint, we may work separately with each such I. Given $x \in V_I$ with $I \subsetneq K = I_{\max}(|x|)$, the set $\widetilde{V}'_{IK} = V'_K \cap \pi_{\mathcal{K}}^{-1}(\pi_{\mathcal{K}}(V'_I))$ is precompact in $\widetilde{V}_{IK}^{\ell+1} = V_K^{\ell+1} \cap \pi_{\mathcal{K}}^{-1}(\pi_{\mathcal{K}}(V_I^{\ell+1}))$ and hence there is $0 < \varepsilon_I^{\ell} \le \varepsilon_I^{\ell+1}$ such that for each α with $W_\alpha \subset V_I^{\Omega}$ and each $I \subsetneq H \subset K$ we have

$$\phi_{IJ}^{E}\big(E_{H \smallsetminus I, (\kappa+1)\varepsilon_I} \times (\widetilde{V}'_{IH} \cap \rho_{IH}^{-1}(W_\alpha))\big) \subset V_H^{\ell+1}. \tag{3.1.10}$$

For J with $|J| > \ell$ we now define

$$V_J^{\ell} := V_J^{\ell+1} \smallsetminus \bigcup_{I \subset J, |I| = \ell} \big(s_J^{-1}(E_I) \cap (V_J^{\ell+1} \smallsetminus V'_J)\big).$$

Then V_J^{ℓ} is an open subset of $V_J^{\ell+1}$, since we have removed a closed subset. Now choose ε_J^{ℓ} for $|J| < \ell$ so as to satisfy (a′) and then define

$$V_J^{\ell} := \big\{x \in V_J^{\ell+1} \mid s_H(x) < \tfrac{1}{2}\varepsilon_J^{\ell}\big\}, \quad |J| \le \ell.$$

Then (c′) holds, and (b′) still holds for J with $|J| > \ell$ because it holds for $\mathcal{V}'$, and it holds when $|J| \le \ell$ because we did not change the zero sets $s_J^{-1}(0)$. Moreover, (d′) holds because when $|J| > \ell$ the only points in $V_J^{\ell+1}$ that were removed to form V_J^{ℓ} lie in $s_J^{-1}(E_I)$ for $I = \ell$. But this does not affect the validity of (3.1.10) (and hence (3.1.9)) because

$$\phi_{IJ}^{E}((E_{J \smallsetminus I, (\kappa+1)\varepsilon_I} \smallsetminus \{0\}) \times \{z\}) \cap s_J^{-1}(E_I) = \varnothing$$

by the first equation in (3.1.2). This completes the proof. □

3.2 Construction of the boundary collar

It remains to establish the existence of a collar with the properties stated in Proposition 2.1.4. Recall from (2.1.10) that Δ_J has a collar of the form[29]

$$c_J^{\Delta} \colon \partial\Delta_J \times [0, \delta] \to \Delta_J, \quad (t^{\partial}, r) \mapsto (1 - r|J|)t^{\partial} + r|J|b_J, \tag{3.2.1}$$

[29]Here for the sake of clarity we write t^{∂} for the coordinate of a general point in $\partial\Delta_J$, while t could be any point in Δ_J.

where b_J is the barycenter of Δ_J and $0 < \delta < \frac{1}{4}$; see Figure 5. It is convenient to write

$$\mathcal{N}^{\Delta}_{\delta}(\partial_{J \smallsetminus I}\Delta) := \{t \in \Delta_J \mid t_j < \delta \text{ for all } j \in J \smallsetminus I\}.$$

Notice that

$$c^{\Delta}_J\big((\partial\Delta \cap \mathcal{N}^{\Delta}_{\delta}(\partial_{J \smallsetminus I}\Delta)) \times [0,\delta)\big) \subset \mathcal{N}_{2\delta}(\partial_{J \smallsetminus I}\Delta); \tag{3.2.2}$$

ie the width-δ collar of the corner $\partial\Delta \cap \mathcal{N}^{\Delta}_{\delta}(\partial_{J \smallsetminus I}\Delta)$ lies in $\mathcal{N}^{\Delta}_{2\delta}(\partial_{J \smallsetminus I}\Delta)$. We now show that for each J this collar lifts to a (partial) collar for $\partial Y_{\mathcal{V},J,\underline{\varepsilon}}$ with the properties stated in Proposition 2.1.4.

Lemma 3.2.1 *Suppose that $(\mathcal{V},\underline{\varepsilon})$ is a compatible reduction. Then, for each $J \in \mathcal{I}_{\mathcal{K}}$, there is a constant $w_J > 0$, subset $\partial' Y_{\mathcal{V},J,\underline{\varepsilon}} \subset \partial Y_{\mathcal{V},J,\underline{\varepsilon}}$ and map c^Y_J as in* (2.1.15) *with the properties detailed in Proposition 2.1.4.*

Proof The proof has three steps.

Step I (construction of local collars) As in Remark 3.1.10 we will assume that $(\mathcal{V},\underline{\varepsilon}) \sqsubset (\mathcal{V}^{\infty},\underline{\varepsilon}^{\infty})$ is precompatible, where each set $|W_\alpha|$ lifts to some $\widetilde{V}^{\infty}_{IK}$. In this step, we fix α, $I = I_\alpha$, and $K = K_\alpha$, and define a local collar of width w_α over a subset $\mathcal{O}^{\infty}_{K,\alpha}$ of $\partial Y_{\mathcal{V}^{\infty},K,\underline{\varepsilon}^{\infty}}$. This subset is determined by the set $W_\alpha \subset \widetilde{V}^{\infty}_{IK}$, and is the inverse image of an open subset $|\mathcal{O}^{\infty}_{K,\alpha}|$ of the set of overlaps $\overline{\mathcal{OL}}(|\mathcal{V}^{\infty}|)$ in (3.1.8).

To this end, consider the coordinate chart for $Y_{\mathcal{V}^{\infty},K,\underline{\varepsilon}^{\infty}}$ given much as in (3.1.4) by

$$\begin{aligned} \psi\colon E_{A \smallsetminus I,(\kappa+1)\varepsilon_I} \times W_\alpha \times [0,\delta_\alpha]^{|K \smallsetminus I|} &\to Y_{\mathcal{V}^{\infty},K,\underline{\varepsilon}^{\infty}}, \\ (e_{A \smallsetminus I}, x, r_{K \smallsetminus I}) &\mapsto (e_{A \smallsetminus I} + (\lambda b_I)^{-1} \cdot s_I(x'), x'; \lambda b_I + r_{K \smallsetminus I}), \end{aligned} \tag{3.2.3}$$

where

$$x' = \phi^E_{IK,z_\alpha}(r_{K \smallsetminus I} \cdot e_{K \smallsetminus I}, x), \quad \lambda := 1 - |r_{K \smallsetminus I}| =: 1 - \sum_{j \in K \smallsetminus I} r_j.$$

For each $x \in W_\alpha := W_\alpha \cap \widetilde{V}^{\infty}_{IK}$, restrict to those $r^{\partial}_{K \smallsetminus I}$ such that

$$\lambda^{\partial} b_I + r^{\partial}_{K \smallsetminus I} \in \overline{\mathrm{st}}^{\Delta}_K(|x|) \subset \partial\Delta_K,$$

where the superscript ∂ indicates that the corresponding point lies in the boundary. The above map provides coordinates

$$\mathcal{C}^{\delta}\colon (e_{A \smallsetminus I}, x, r^{\partial}_{K \smallsetminus I}) \mapsto \psi(e_{A \smallsetminus I}, x, r^{\partial}_{K \smallsetminus I}) = (e_{A \smallsetminus I} + e''_I, x''; t^{\partial}) \tag{3.2.4}$$

for an open subset

$$\mathcal{O}^{\infty}_{K,\alpha} \subset \{(e, x; t^{\partial}) : t^{\partial} \in \overline{\mathrm{st}}^{\Delta}_{K}(|x|),\ t^{\partial} \approx 0\} \tag{3.2.5}$$

of the boundary $\partial Y_{\mathcal{V}^{\infty},K,\underline{\varepsilon}}$. We will assume, as we may, that $\mathcal{O}^{\infty}_{K,\alpha} = \mathrm{pr}_{\mathcal{V}}^{-1}(|\mathcal{O}^{\infty}_{K,\alpha}|)$, where $|\mathcal{O}^{\infty}_{K,\alpha}|$ is open in $\overline{\partial \mathcal{L}}(|\mathcal{V}^{\infty}|) \subset |\mathcal{V}|$.

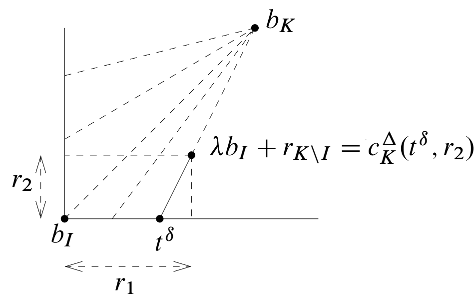

Figure 5: Here $K = I \cup \{1,2\}$ and t^{δ} lies on the boundary with $t_2 = 0$. Hence, $r_{K\smallsetminus I} = (r_1, r_2)$, where r_2 is the collar coordinate along the ray from t^{δ} to b_K, while $t^{\delta} = c^{\Delta}_K(b_I, r)$ for $r = (t^{\delta})_1$.

We now define a collar over $\mathcal{O}^{\infty}_{K,\alpha}$ of width $w_{\alpha} < \frac{1}{2}\delta_{\alpha}$ (see (3.2.2)) as follows. Given

$$(t^{\partial}, r) \in \overline{\mathrm{st}}^{\Delta}_{K}(|x|) \times [0,\delta), \quad \text{where } (t^{\partial}, r) \approx (b_I, 0),$$

choose $r^{\partial}_{K\smallsetminus I}$ and $r_{K\smallsetminus I}$ (both ≈ 0) so that

$$t^{\partial} := \lambda^{\partial} b_I + r^{\partial}_{K\smallsetminus I}, \quad c^{\Delta}_K(t^{\partial}, r) = \lambda b_I + r_{K\smallsetminus I}, \tag{3.2.6}$$

where $\lambda^{\partial} := 1 - |r^{\partial}_{K-1}|$ and $\lambda := 1 - |r_{K-1}|$; see Figure 5. Then, with $\mathcal{C}^{\delta}$ as in (3.2.4), define

$$\begin{gathered} c^{Y}_{K,\alpha} \colon \mathcal{O}^{\infty}_{K,\alpha} \times [0, w_{\alpha}) \to Y_{\mathcal{V}^{\infty},K,\underline{\varepsilon}}, \\ ((e_{A\smallsetminus I} + e''_I, x''; t^{\partial}), r) \xmapsto{(\mathcal{C}^{\delta})^{-1}\times \mathrm{id}} ((e_{A\smallsetminus I}, x, r^{\partial}_{K\smallsetminus I}), r) \mapsto \psi(e_{A\smallsetminus I}, x, r_{K\smallsetminus I}), \end{gathered} \tag{3.2.7}$$

where $r_{K\smallsetminus I} \in [0,\delta)^{K\smallsetminus I}$ is the function of $r^{\delta}_{K\smallsetminus I}$ and δ is as defined in (3.2.6). In particular, if $|K \smallsetminus I| = 1$ then $r_{K\smallsetminus I}$ has only one component, and so is the same as the collar variable r, while $t^{\delta} = b_I$. Therefore, the collar is simply given by ψ:

$$\begin{gathered} c^{Y}_{I\cup\{j\},\alpha} \colon \mathcal{O}^{\infty}_{I\cup\{j\},\alpha} \times [0, w_{\alpha}) \to Y_{\mathcal{V}^{\infty},I\cup\{j\},\underline{\varepsilon}}, \\ ((e_{A\smallsetminus I} + e_I, x; b_I), r) \mapsto \psi(e_{A\smallsetminus I}, x, r). \end{gathered} \tag{3.2.8}$$

The next task is to extend the domain of this collar to

$$\overline{\mathrm{st}}(\mathcal{O}^{\infty}_{K,\alpha}) := \{(\mu_H \cdot (e,x;t) \mid (e,x;t) \in \mathcal{O}_\alpha,\ \mu_H \cdot t \in \overline{\mathrm{st}}^{\Delta}_K(|x|)\} \tag{3.2.9}$$

by rescaling as follows. Consider a tuple μ_H (as in (2.1.21)), where $I \subset H \subsetneq K$, and point $t^\delta \in \overline{\mathrm{st}}^{\Delta}_K(|x|) \cap (\{b_I\} \times [0,\delta]^{|K \smallsetminus I|})$ such that

$$\mu_H \cdot t^\partial \in \overline{\mathrm{st}}^{\Delta}_K(|x|) \cap (\{b_I\} \times [0,\delta]^{|K \smallsetminus I|}),$$

and let $\mu'_H\cdot$ with $(\mu'_H)_i = 1$ for $i \notin H$ give the corresponding rescaling in the coordinates $\Delta_I \times [0,\delta_\alpha]^{|K \smallsetminus I|}$. Thus, if $c^{\Delta}(t^\partial, r) = (1 - |r_{K\smallsetminus I}|)b_I + r_{K \smallsetminus I}$ as in (3.2.6), we have

$$c^{\Delta}_K(\mu_H \cdot t^\partial, r) = \mu'_H \cdot (\lambda b_I + r_{K \smallsetminus I}). \tag{3.2.10}$$

Note that this rescaling in the boundary $\partial_{K \smallsetminus H}\Delta_K$ does not affect the collar variable r along this part of the boundary. Then the following diagram commutes, where we write $e'_I = (t_I)^{-1} \cdot s_I(x')$ and $y := (e_I, x; t_I) \in \partial Y$:

$$\begin{tikzcd}
(e_{A\smallsetminus I}, y, r_{K\smallsetminus I}) \arrow[r, mapsto, "\psi"] \arrow[d, mapsto, "\mu'_H\cdot"'] & (e_{A\smallsetminus I} + e'_I, x' = \phi(r_{K\smallsetminus I}\cdot e_{K\smallsetminus I}, x); \lambda b_I + r_{K\smallsetminus I}) \arrow[d, mapsto, "\mu'_H\cdot"'] \\
((\mu'_H)^{-1}\cdot e_{A\smallsetminus I}, y, \mu'_H\cdot r_{K\smallsetminus I}) \arrow[r, mapsto, "\psi"] & ((\mu'_H)^{-1}\cdot(e_{A\smallsetminus I} + e'_I), x'; \mu'_H\cdot(\lambda b_I + r_{K\smallsetminus I}))
\end{tikzcd}$$

because the rescaling on the left does not affect the image $x' = \phi(r_{K\smallsetminus I} \cdot e_{K \smallsetminus I}, x) \in V_K$ on the right. Therefore, because $c^Y_{K,\alpha}$ is a composite of ψ^{-1} (at $r = 0$) with ψ, and because rescaling does not affect the collar variable r, the following diagram commutes:

$$\begin{tikzcd}
((e_{A\smallsetminus I} + e''_I, x''; t^\partial), r) \arrow[r, mapsto, "c^Y_{K,\alpha}"] \arrow[d, mapsto, "\mu_H\cdot"'] & (e', x'; t') \arrow[d, mapsto, "\mu_H\cdot"'] \\
\big((\mu_H^{-1} \cdot (e_{A\smallsetminus I} + e''_I), x''; \mu_H \cdot t^\partial), r\big) \arrow[r, mapsto, "c^Y_{K,\alpha}"] & ((\mu_H)^{-1} \cdot e', x', \mu_H \cdot t')
\end{tikzcd} \tag{3.2.11}$$

In other words, if we apply the collar and then rescale (a little) by μ_H, we get the same result as rescaling by μ_H and then applying the collar. It follows that we can unambiguously extend the domain of the local collar to $\overline{\mathrm{st}}(\mathcal{O}^{\infty}_{K,\alpha})$ by defining

$$c^Y_{K,\alpha}((e_{A\smallsetminus I} + e''_I, x''; t), r) := \mu_H^{-1} \cdot c^Y_{K,\alpha}(\mu_H \cdot (e', x', t')), \tag{3.2.12}$$

where μ_H is chosen so that $\mu_H \cdot (e', x', t')$ lies in the domain of the map in (3.2.7). Note that $c^Y_{K,\alpha}$ is equivariant because the maps in (3.1.6) and (3.2.3) used to construct it are equivariant.

Although we assumed in the above construction that K was maximal, so that $W_\alpha \subset V^\infty_{IK}$ this condition was not used in any essential way in the above construction. Thus, for any J such that $I \subsetneq J \subset K$, by using the map in (3.1.7) instead of (3.1.6) we can define a collar $c^Y_{J,\alpha}$ over

$$c^Y_{J,\alpha} \colon \overline{\mathrm{st}}(\mathcal{O}^\infty_{J,\alpha}) \times [0, w_\alpha) \to Y_{\mathcal{V}^\infty, J, \underline{\varepsilon}^\infty}, \tag{3.2.13}$$

where

$$\overline{\mathrm{st}}(\mathcal{O}^\infty_{J,\alpha}) := \{(e, \rho_{JK}(x), t^\partial) \in \partial Y_{\mathcal{V}^\infty, J, \underline{\varepsilon}^\infty} \mid x \in \widetilde{V}_{JK} \cap W_\alpha, (e, x; t^\partial) \in \overline{\mathrm{st}}(\mathcal{O}^\infty_{K,\alpha})\},$$

and $\overline{\mathrm{st}}(\mathcal{O}^\infty_{K,\alpha})$ is as defined in (3.2.9).

Further, we can restrict these collars to the corresponding subsets $\overline{\mathrm{st}}(\mathcal{O}_{J,\alpha})$ of $\partial Y_{\mathcal{V},J,\underline{\varepsilon}}$ for all $I \subsetneq J \subset K$, obtaining a set of locally defined collars of width w_α. Note that this collar still has width w_α because we used the constant ε_I in (3.2.3) rather than ε^∞_I. Hence, although $\underline{\varepsilon} < \underline{\varepsilon}^\infty$ in general, when we restrict the domain of ϕ in (3.2.3) to the points in $\partial Y_{\mathcal{V},K,\underline{\varepsilon}}$ the image of ϕ lies in $Y_{\mathcal{V},K,\underline{\varepsilon}}$ by condition (d$'$) in Definition 3.1.9.

We claim that these collars satisfy all the conditions in Proposition 2.1.4. In particular, if $I \subsetneq H \subsetneq K$ the domain of $c^Y_{K,\alpha}$ contains the image of the collar $c^Y_{H,\alpha}$ by (3.2.5). They are compatible with projections and invariant under rescaling by construction.

The domains $\overline{\mathrm{st}}(\mathcal{O}_{J,\alpha})$ of these collars are not open in $\partial Y_{\mathcal{V},J,\underline{\varepsilon}}$ because of the restriction $t \in \overline{\mathrm{st}}^\Delta_J(|x|)$, and because the condition that $(e, x; t^\partial) \in \overline{\mathrm{st}}(\mathcal{O}_{K,\alpha})$ places certain extra (but unimportant) restrictions on $\|\mathrm{pr}_{E_{K \setminus I}} e\|$ when t^∂ has been rescaled far from b_I. However, modulo these provisos, for each such J they consist of the full inverse image in $\partial Y_{\mathcal{V},J,\underline{\varepsilon}}$ of the open subset $|\mathcal{O}_\alpha| := |\mathcal{O}^\infty_{K,\alpha}|$ of the "boundary" $\partial |V^\infty_K|$ of $|V^\infty_K|$,

$$|\mathcal{O}_\alpha| := |\mathcal{O}^\infty_{K,\alpha}| \subset \partial |V^\infty_K| := \bigcup_{H \subsetneq K} |V^\infty_{HK}| \subset \overline{\mathcal{OL}}(|\mathcal{V}^\infty|), \tag{3.2.14}$$

where $\mathcal{O}^\infty_{K,\alpha}$ is as defined in (3.2.5).

Step 2 (construction of a global collar from a covering by local collars) We now explain a method from [4, Proposition 3.42] that combines local collars

$$(c_\alpha \colon \mathcal{U}_\alpha \times [0, w_\alpha) \to Y)_{1 \le \alpha \le N}$$

defined over open subsets $\mathcal{U}_\alpha \subset \partial Y$ of the boundary of a manifold Y into a global collar over $\partial' Y$ of width w, where $\partial' Y$ is any precompact subset of $\bigcup_\alpha \mathcal{U}_\alpha$ and $w < \min_\alpha \frac{1}{2} w_\alpha$.

To this end, choose a partition of unity $(\lambda_\alpha)_\alpha$ subordinate to the covering of $\partial' Y$ by the sets $(U_\alpha)_\alpha$, and define

$$Y' := Y \cup_\theta (\partial' Y \times [-w, 0]),$$

where θ identifies $\partial' Y \times \{0\}$ with $\partial' Y$ in the obvious way. We claim that there is a homeomorphism

$$\Psi\colon (Y', \partial' Y \times [-w, 0]) \to \Bigl(Y, \bigcup_\alpha c_\alpha(\mathcal{U}_\alpha \times [0, 2w))\Bigr).$$

Granted this, we define the collar by

$$c^Y\colon \partial' Y \times [0, w) \to Y, \quad (y, r) \mapsto \Psi_J(y, r - w).$$

The homeomorphism Ψ is a composite

$$\Psi = \Psi_N \circ \cdots \circ \Psi_1$$

of homeomorphisms

$$\Psi_\ell\colon Y'\Bigl(-1 + \sum_{\alpha<\ell} \lambda_\alpha\Bigr) \to Y'\Bigl(-1 + \sum_{\alpha\le\ell} \lambda_\alpha\Bigr),$$

where for any function $\sigma\colon \partial' Y \to [0, 1]$ we define

$$Y'(-1 + \sigma) := Y \cup_\theta \{(y, r) \mid y \in \partial' Y,\ (-1 + \sigma(y))w \le r \le 0\}.$$

To define Ψ_ℓ, first extend the product structure of the external collar $\partial Y \times [-w, 0]$ via the local collar c_ℓ to obtain an extended collar neighborhood

$$\hat{c}_\ell\colon \mathcal{U}_\ell \times [-w, w_\ell) \to Y'.$$

Then define

$$\Psi_\ell(\hat{c}_\ell(y, r)) = \hat{c}_\ell(y, f_{y,\ell}(r)),$$

where

$$f_{y,\ell}\colon \Bigl[\Bigl(-1 + \sum_{\alpha<\ell} \lambda_\alpha(y)\Bigr)w, 2w\Bigr] \to \Bigl[\Bigl(-1 + \sum_{\alpha\le\ell} \lambda_\alpha(y)\Bigr)w, 2w\Bigr]$$

is a homeomorphism that translates by $\lambda_\ell(y)$ if $r \le \sum_{\alpha<\ell} \lambda_\alpha(y)w$. This completes the construction.

Remark 3.2.2 If each local collar c_α lifts a map $\mathrm{pr}_\Delta\colon (Y,\partial Y)\to([0,1),\{0\})$, then the global collar does as well; ie we have

$$\mathrm{pr}_\Delta\circ c(y,r)=r.$$

This holds because each $f_{y,\ell}$ is a translation by $\lambda_\ell(y)w$ on the relevant part of its domain, where $\sum_\ell \lambda_\ell(y)=1$. Further, if for some map $\mathrm{pr}_E\colon Y\to E$ we have $c_\alpha(y,r)=\mathrm{pr}_E(y)$, then the global collar also satisfies $c^Y(y,r)=\mathrm{pr}_E(y)$. $\diamond$

Step 3 (completion of the proof) Once the cover and partition of unity are chosen, the construction in Step 2 depends only on the ordering of the sets in the cover. Even though we saw in Step 1 that the local covers satisfy all the compatibility conditions required in Proposition 2.1.4, we will have to organize the construction rather carefully in order to achieve this for the global collars.

Recall from the discussion of (1.2.7) that because the atlas $\mathcal{K}$ is assumed tame and preshrunk and hence good, the subspace topology on $|\mathcal{V}^\infty|$ (considered as a subset of $|\mathcal{K}|$) is metrizable, and so we may fix a metric on $|\mathcal{V}^\infty|$. Since the sets $|V_I|$ and $|V_J|$ have disjoint closures unless $I\subset J$ or $J\subset I$, we may choose

(3.2.15) $\delta_0>0$ smaller than half the distance between any two such sets.

We next order the sets $|W_\alpha|_{1\le\alpha\le N}$ of the cover of $\overline{\mathcal{OL}}(\mathcal{V})$ so that as α increases the cardinality $|I_\alpha|$ of the minimal set I in Lemma 3.1.7(i) increases. Thus, we assume that there are numbers $0=n_0\le n_1\le n_2\le\cdots\le n_{\kappa-1}=N$ such that

$$N_{k-1}<\alpha\le N_k \implies |I_\alpha|=k.$$

By (3.2.14), the sets $(|\mathcal{O}_\alpha|)_{1\le\alpha\le N}$ cover a neighborhood of the compact subset $\overline{\mathcal{OL}}(|\mathcal{V}|)$ in $|\mathcal{V}^\infty|$. Further, by condition (ii) in Lemma 3.1.7 and our choice of N_k, if $\alpha>N_k$, the set $|\mathcal{O}_\alpha|$ does not meet any $|V_I|$ with $|I|\le k$. Hence, we may choose $\delta_0>\delta_1>0$ so that for each k, the sets $(|\mathcal{O}_\alpha|)_{1\le\alpha\le N_k}$ cover the closed δ_1–neighborhood

$$\overline{\mathcal{N}}_{\delta_1}(k):=\overline{\mathcal{N}}_{\delta_1}\Bigl(\bigcup_{|I|\le k,\,L\in\mathcal{I}_\mathcal{K}}|\overline{V}_{IL}|\Bigr)\subset\overline{\mathcal{OL}}(|\mathcal{V}|)$$

of the compact subset $\bigcup_{|I|\le k,\,L\in\mathcal{I}_\mathcal{K}}|\overline{V}_{IK}|$. By shrinking the sets $\mathcal{O}_\alpha$ to $\mathcal{O}'_\alpha$, we may then assume in addition that for some $0<\delta_2<\delta_1$ we have

(3.2.16) $$(\alpha>N_k)\implies|\mathcal{O}'_\alpha|\cap\overline{\mathcal{N}}_{\delta_2}(k)=\varnothing\quad\text{for all }k.$$

For each $k \le \kappa$, choose a partition of unity $(\lambda^k_\alpha)_{1\le\alpha\le N_k}$ for $\overline{\mathcal{N}}_{\delta_2}(k)$ with respect to the covering by $(|\mathcal{O}'_\alpha|)_{1\le\alpha\le N_k}$ such that

$$1 \le \alpha \le N_{k-1} \implies \lambda^k_\alpha = \lambda^{k-1}_\alpha. \tag{3.2.17}$$

Finally, choose $w' > 0$ such that

$$2w' < \min_\alpha w_\alpha. \tag{3.2.18}$$

Now define

$$\partial^k Y_{\mathcal{V},J,\underline{\varepsilon}} = \bigcup_{1\le\alpha\le N_k} \{(e,x;t) \mid (e,x;t) \in \overline{\mathrm{st}}(\mathcal{O}'_{J,\alpha})\}, \tag{3.2.19}$$

where $\overline{\mathrm{st}}(\mathcal{O}'_{J,\alpha})$ is defined as in (3.2.13) but with $\mathcal{O}^\infty_{K,\alpha}$ replaced by $\mathcal{O}^\infty_{K,\alpha} \cap \pi_{\mathcal{K}}^{-1}(|\mathcal{O}'_\alpha|)$. Then, for each $I \subsetneq J$ with $|I| = k$, we may use the local collars $c^Y_{J,\alpha}$ together with the partition of unity on $\partial^k Y_{\mathcal{V},J,\underline{\varepsilon}}$ obtained by pulling back (λ^k_α) to construct a collar

$$c^Y_{J,k}\colon \partial^k Y_{\mathcal{V},J,\underline{\varepsilon}} \times [0, w'_J) \to Y_{\mathcal{V},J,\underline{\varepsilon}}$$

as in Step 2. Condition (3.2.17) implies that $c^Y_{J,k}$ agrees with $c^Y_{J,k-1}$ on their common domain of definition. Hence, the collars fit together to give a well-defined collar

$$c^Y_J\colon \partial' Y_{\mathcal{V},J,\underline{\varepsilon}} \times [0, w'_J) \to Y_{\mathcal{V},J,\underline{\varepsilon}}, \quad \text{where } \partial' Y_{\mathcal{V},J,\underline{\varepsilon}} := \bigcup_{k<|J|} \partial^k Y_{\mathcal{V},J,\underline{\varepsilon}}. \tag{3.2.20}$$

Note that c^Y_J lifts c^Δ_J by Remark 3.2.2. Thus, it does have the form required by (2.1.15).

It remains to check that we can choose collar widths $w_J \le w'_J$ so that the resulting collars have all the required properties.

- The maps c^Y_J are equivariant, because the local collars are, and the partition of unity is pulled back from $|\mathcal{V}^\infty|$.

- To see that the c^Y_J are compatible with projection to $E_{A\smallsetminus\bullet}$, suppose that $I \subsetneq J$ has $|I| = k < |J|$. Then c^Y_J has the properties in (2.1.17) because all the local collars do. Further, the points $\iota_{EV}(e,x) = (b_I^{-1}\cdot e, x; b_I)$ mentioned in (2.1.18) lie in $\partial^k Y_{\mathcal{V},J,\underline{\varepsilon}}$. Therefore, $c^Y_J(\iota_{EV}(e,x), r)$ is made by combining the local collars $(c^Y_{J,\alpha})_{\alpha\le N_k}$. But we saw in Step 1 that all these local collars satisfy (2.1.18) for $E_{A\smallsetminus I}$. It follows that the combined collar formed in Step 2 must also satisfy (2.1.18) for $E_{A\smallsetminus I}$.

- Similarly, the fact that the relevant local collars that form c^Y_J are invariant under rescaling as in (2.1.21) implies that c^Y_J also satisfies (2.1.21).

- To prove that the pairs (c^Y_J, w_J) are compatible with covering maps we need to check two things:

(a) that their domains are large enough (ie that (2.1.19) holds for all $I \subsetneq H \subsetneq J$), and

(b) that when $H \subsetneq J$, the collar c^Y_H has a natural lift to $Y_{\mathcal{V},J,\underline{\varepsilon}}$.

Claim (b) again follows because the local collars used to form c^Y_H (as well as the partition of unity) can be lifted in this way. (This is just a consequence of equivariance.) Claim (a) has two parts. The first claims that if $(e,x;t) \in \partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$ has $x \in \widetilde{V}_{IH} \cap \widetilde{V}_{HJ}$, where $I \subsetneq H \subsetneq J$, then $(e, \rho_{HJ}(x); t)$ is in the domain $\partial' Y_{\mathcal{V},H,\underline{\varepsilon}}$ of c^Y_H. To see this, note that $\partial' Y_{\mathcal{V},J,\underline{\varepsilon}}$ is the union over k of the sets $\partial^k Y_{\mathcal{V},J,\underline{\varepsilon}}$ of (3.2.19). But we have

$$\begin{aligned}\partial^k Y_{\mathcal{V},J,\underline{\varepsilon}} \cap \{(e,x;t) \mid x \in \widetilde{V}_{IH} \cap \widetilde{V}_{HJ}\} &= \partial^{|H|} Y_{\mathcal{V},J,\underline{\varepsilon}} \cap \{(e,x;t) \mid x \in \widetilde{V}_{IH} \cap \widetilde{V}_{HJ}\} \\ &= \{(e,x;t) \mid (e,\rho_{HK}(x);t) \in \partial^{|H|} Y_{\mathcal{V},H,}\},\end{aligned}$$

where the first equality holds by (3.2.16), while the second holds because the sets $\overline{\mathrm{st}}(\mathcal{O}^\infty_{J,\alpha})$ are compatible with the covering maps ρ_{HJ} by (3.2.13).

The second part of (a) concerns the choice of suitable widths $w_H \le w'_H$ for all $H \in \mathcal{I}_{\mathcal{K}}$. Since the domains of the collars are by now fixed, we can choose each w_H independently: its choice depends only on the domains of the collars c^Y_J for $J \supsetneq H$. Notice that by the definition of the set $\mathcal{O}^\infty_{K,\alpha}$ in (3.2.5), it holds (with $w_H = \frac{1}{2}\delta_\alpha$ for example) for the original domains $\mathcal{O}^\infty_{K,\alpha}$ of the local collars. Moreover, because $\delta_2 < \delta_0$ (where δ_0 is the separation distance in (3.2.15)), this property is not affected by the shrinking from $|\mathcal{O}^\infty_{K,\alpha}|$ to $|\mathcal{O}'_\alpha|$ in (3.2.16). Hence, it is easy to see that one can choose suitable w_H for the global collars.

- We must check that this collar restricts to any compatible shrinking $(\mathcal{V}', \underline{\varepsilon}') \sqsubset (\mathcal{V}, \underline{\varepsilon})$. But this is immediate since the above construction depends only on the choice of coordinate charts in (3.2.3), which restrict to $(\mathcal{V}', \underline{\varepsilon}')$ by the definition of compatibility, and the choice of an appropriate partition of unity, which we can also restrict to $\mathcal{V}'$.

- Finally we must check that if $\mathcal{K}$ is oriented, the collar map c^Y_J preserves the natural induced orientation on its domain and range. But this is clear from its construction.

This completes the proof of Lemma 3.2.1. □

Corollary 3.2.3 *Any reduction $\mathcal{V}'$ has a collar compatible shrinking $(\mathcal{V}, \underline{\varepsilon})$.*

Proof By Definition 2.1.5, it suffices to construct a compatible (V_J, ε_J) such that

(e) for all pairs $I \subsetneq J$ we have $\varepsilon_I \le w^2_J$, where w_J is the collar width for V_J.

Without loss of generality, let us suppose that $(\mathcal{V}', \underline{\varepsilon}')$ is compatible, with collars c^Y_J of widths w'_J. As in the proof of Lemma 3.1.11 we work by downwards induction on $|J|$. Hence, at the k^{th} stage, we assume that we have compatible $(\mathcal{V}^{k+1}, \underline{\varepsilon}^{k+1})$ such that condition (e) holds for all $I \subsetneq J$ with $|I| \geq k+1$, and aim to construct compatible $(\mathcal{V}^k, \underline{\varepsilon}^k, w^k_J)$ so that (e) holds whenever $|I| \geq k$. As before we take $(V^k_J, \varepsilon^k_J, w^k_J) = (V^{k+1}_J, \varepsilon^{k+1}_J, w^{k+1}_J)$ if $|J| \geq k+1$. The key point is this: if we shrink the set $(V^{k+1}_I, \varepsilon^{k+1}_I)$, where $|I| \leq k$, by decreasing ε^{k+1}_I and hence V^{k+1}_I (because of condition (c) in Definition 3.1.1), then this does not decrease the collar width $c^Y_{J,k+1}$ of any V^{k+1}_J with $I \subsetneq J$, since this change only affects points that either lie in the boundary of $Y_{\mathcal{V}^{k+1},J,\underline{\varepsilon}^{k+1}}$ or are interior points with $I(x) = \{i \mid s_i(x) \neq 0\} \subset I$ that do not occur in $\operatorname{im}(c^Y_{J,k+1})$ because of its construction. Hence, it makes sense to choose $\varepsilon^k_I \leq \varepsilon^{k+1}_I$ for the elements $|I| = k$ so that condition (e) holds at level k, and then shrink V^{k+1}_I to a set V^k_I that satisfies (a)–(c). As usual, this can be done independently for each I at level k. To complete the inductive step, we then make appropriate choices for lower-level I as in Lemma 3.1.11 to obtain a compatible shrinking $(\mathcal{V}^k, \underline{\varepsilon}^k)$ that satisfies (e) at levels $\geq k$. This completes the proof. □

Appendix Rational Čech cohomology and homology

We briefly describe the properties of the (co)homology theories in [8] that are based on the properties of Alexander–Spanier cochains. We do not need the full generality of this theory because the space $M = |\boldsymbol{M}|_{\mathcal{H}}$ is locally compact and Hausdorff. Throughout we assume that Y is locally compact and Hausdorff, with $A \subset Y$ closed and $U \subset Y$ open, and take coefficients in $\mathbb{Q}$. Further, we denote these theories by $\check{H}$ to distinguish them from singular (co)cohomology.[30]

We need the following properties of the cohomology theory:

(a) [8, Theorem 3.21] If Y is a connected orientable n–dimensional manifold then $\check{H}^i(Y) = 0$ unless $i = n$, in which case $\check{H}^n(Y) = \mathbb{Q}$, ie $\check{H}^*$ is like rational singular cohomology with compact supports.

(b) [8, Section 1.2] If $f\colon A \to Y$ is proper, there is an induced map $f^*\colon \check{H}^i(Y) \to \check{H}^i(A)$.

[30]In [8, Chapter 10] the theory we call $\check{H}^*$ below is denoted by H^*_c to distinguish it from another theory that does not concern us here.

(c) [8, Section 1.3] If $U \subset Y$ is open, there is an induced map $f_*\colon \check{H}^i(U) \to \check{H}^i(Y)$. Further, if Y is as in (a) and U is an open n–disc, then f_* is an isomorphism.

(d) [8, Theorem 1.6] If $A \subset Y$ is closed then there is an exact sequence

$$\cdots \to \check{H}^i(Y \setminus A) \to \check{H}^i(Y) \to \check{H}^i(A) \xrightarrow{\delta} \check{H}^{i+1}(Y \setminus A) \to \cdots, \tag{A.1}$$

ie the group $\check{H}^i(A)$ plays the role of the relative group $H^i(Y, Y \setminus A)$.

The dual homology theory developed in [8, Chapter 4] is denoted by H_*^∞ in [8, Chapter 10] to emphasize that it is analogous to locally finite singular homology; we shall call it $\check{H}_*^\infty$. It follows from the universal coefficient theorem [8, Theorem 4.17] that

$$\check{H}_k^\infty(X) = \mathrm{Hom}(\check{H}^k(X); \mathbb{Q}). \tag{A.2}$$

Further, because Y is locally compact and Hausdorff, it follows from the uniqueness property for $\check{H}_c^*$ stated in [8, Section 6.7] that the dual theory $\check{H}_k^\infty$ is isomorphic to rational Borel–Moore homology.

As shown by the following, the functorial properties of $\check{H}_*^\infty$ are different from the usual singular theory.

(a′) If Y is a connected orientable n–manifold, then $\check{H}_i^\infty(Y) = 0$ unless $i = n$, in which case $\check{H}_n^\infty(Y) = \mathbb{Q}$; more generally, any orientable n–manifold has a fundamental class

$$\mu_Y \in \check{H}_n^\infty(Y). \tag{A.3}$$

(b′) [8, Section 4.6] If $U \subset Y$ is open, there is an induced restriction

$$\rho_{Y,U}\colon \check{H}_i^\infty(Y) \to \check{H}_i^\infty(U); \tag{A.4}$$

moreover, for $U_1 \subset U_2 \subset Y$ we have $\rho_{Y,U_1} = \rho_{U_2,U_1} \circ \rho_{Y,U_2}$.

(c′) [8, Section 4.6] If $f\colon A \to Y$ is continuous and proper, then there is an induced pushforward $f_*\colon \check{H}_i^\infty(A) \to \check{H}_i^\infty(Y)$; moreover, given a proper inclusion $\iota\colon A \to Y$, there is a functorial long exact sequence

$$\cdots \to \check{H}_i^\infty(A) \xrightarrow{\iota_*} \check{H}_i^\infty(Y) \xrightarrow{\rho_{Y,Y\setminus A}} \check{H}_i^\infty(Y \setminus A) \xrightarrow{\partial} \check{H}_{i-1}^\infty(A) \to \cdots. \tag{A.5}$$

(d′) [8, Section 4.3(3c)] If $f\colon A \to Y$ is proper and U is open in Y, then the following diagram commutes:

$$\begin{tikzcd} \check{H}_i^\infty(A) \arrow[r, "f_*"] \arrow[d, "\rho_{A,A\cap f^{-1}(U)}"] & \check{H}_i^\infty(Y) \arrow[d, "\rho_{Y,U}"] \\ \check{H}_i^\infty(A\cap f^{-1}(U)) \arrow[r, "f_*"] & \check{H}_i^\infty(U) \end{tikzcd}$$

(e′) [8, Section 4.9(6)] If $Y = U \cup V$, where U and V are open, then there is an exact Mayer–Vietoris sequence of the form

$$\cdots \to \check{H}_{i+1}^\infty(U\cap V) \to \check{H}_i^\infty(Y) \to \check{H}_i^\infty(U) \oplus \check{H}_i^\infty(V) \to \check{H}_i^\infty(U\cap V) \to \cdots.$$

In particular, if U is the disjoint union of a finite number of sets of U_i, then

$$\check{H}_*^\infty(U) \cong \oplus_i \check{H}_*^\infty(U_i).$$

(f′) [8, page 334] If $U \subset Y$ is open while $A \subset Y$ is closed, there is a cap product

$$\cap\colon \check{H}_{p+q}^\infty(Y \setminus A) \otimes \check{H}^p(Y \setminus U) \to \check{H}_q^c(Y, U \cup A). \tag{A.6}$$

This takes values in compactly supported Čech homology, a theory whose functorial properties are analogous to those of the usual singular homology. In particular, if the triple $(U\cup A; U, A)$ is *excisive* for $\check{H}^c$ (ie $\check{H}_q^c(A, U\cap A) \cong \check{H}_q^c(U\cup A, U)$), then there is a commutative diagram

$$\begin{tikzcd} \check{H}_{p+q+1}^\infty(Y\setminus A) \otimes \check{H}^p(Y\setminus U) \arrow[r, "\cap"] \arrow[d, "\partial\otimes(-1)^p\iota^*"'] & \check{H}_{q+1}^c(Y, U\cup A) \arrow[d, "\delta"'] \\ \check{H}_{p+q}^\infty(A) \otimes \check{H}^p(A\setminus U) \arrow[r, "\cap"] & \check{H}_q^c(A, U\cap A) \end{tikzcd} \tag{A.7}$$

Note that the above diagram exists when Y is locally compact, A is closed and $Y \setminus U$ is compact. To see this, choose a nested sequence $\mathcal{N}_k$ of precompact open neighborhoods of $Y\setminus U$ in Y with

$$Y\setminus U = \bigcap_k \mathcal{N}_k, \quad U = \bigcup_k (Y \setminus \mathcal{N}_k).$$

Since by definition

$$\check{H}_*^c(Y, U\cup A) = \varprojlim \check{H}_*^c(Y, (Y\setminus\mathcal{N}_k)\cup A), \quad \check{H}_*^c(A, U\cap A) = \varprojlim \check{H}_*^c(A, A\setminus\mathcal{N}_k),$$

and the triple of closed sets $(Y, Y \smallsetminus \mathcal{N}_k, A)$ is excisive by [8, Corollary 9.5], it follows that $(Y, Y \smallsetminus U, A)$ is excisive, as required.

(g′) [8, Exercise 5, page 272] If X is Hausdorff and $X \smallsetminus A$ is a precompact open subset of X, then $\check{H}^{\infty}_*(X \smallsetminus A) = \check{H}^{c}_*(X, A)$.

(h′) This homology is *taut*; ie if $X \subset Y$ is closed, where Y is locally compact and Hausdorff, and $N_{k+1} \subset N_k$ is a nested sequence of closed neighborhoods of X in Y, then (by [8, Theorem 6.4])

$$\check{H}^{\infty}_d(X) = \varprojlim(\check{H}^{\infty}_d(N_k)).$$

List of symbols

I Related to atlases

Theorem A	$\mathcal{K},\ X,\ E_A,\ \Gamma_A,\ \mathcal{S}_M\colon M \to E_A$
Theorem B	$[X]^{\mathrm{vir}}_{\mathcal{K}}$
Section 1.2	$\mathcal{K},\ A,\ \mathcal{I}_{\mathcal{K}}$
	for $I \in \mathcal{I}_{\mathcal{K}},\ \boldsymbol{K}_I = (U_I, E_I, \Gamma_I, s_I, \psi_I),\ F_I \subset X$
	for $I \subset J,\ \widetilde{U}_{IJ} \subset U_J,\ U_{IJ} \subset U_I,\ \rho_{IJ}\colon \widetilde{U}_{IJ} \to U_{IJ}$
(1.3.4)	$\gamma\|_J, \Gamma_A$
(1.2.3)	ϕ^E_x
(1.3.2)	$\underline{\varepsilon}$
(1.3.1)	$E_{I,\varepsilon}$ for $I \subset A$
(1.2.5)	$\mathcal{K}' \sqsubset \mathcal{K},\ \mathcal{U}' \sqsubset \mathcal{U} = (U_I)_{I \in \mathcal{I}_{\mathcal{K}}}$
Section 3, beginning	$\mathcal{F} = (F_I)_{I \in \mathcal{I}_{\mathcal{K}}}$
(1.2.6) ff	$\boldsymbol{B}_{\mathcal{K}}$ and $\boldsymbol{E}_{\mathcal{K}}$
(1.2.7)	$\|\mathcal{K}\| := \|\boldsymbol{B}_{\mathcal{K}}\|,\ \pi_{\mathcal{K}}\colon U_I \to \|\mathcal{K}\|$
(1.2.8) ff	reduction $\mathcal{V} = (V_I),\ G_I,\ \boldsymbol{B}_{\mathcal{K}}\|_{\mathcal{V}}$
(1.3.3)	$V_{IJ} \subset V_I,\ \widetilde{V}_{IJ} \subset V_J$ and $\pi_{\mathcal{K}}\colon V_I \to \|\mathcal{K}\|$

II Related to wnb manifolds

Definition 1.3.1	$(\boldsymbol{G}, \Lambda_{\boldsymbol{G}}),\ \|\boldsymbol{G}\|_{\mathcal{H}},\ \pi^{\mathcal{H}}_{\boldsymbol{G}}$
Proposition 1.3.3	$(\boldsymbol{M}, \Lambda_{\boldsymbol{M}}),\ (M, \Lambda),\ M_I,\ M_{IJ},\ \widetilde{M}_{IJ},\ \tau_{IJ},\ \mathscr{S}_J,\ \mathscr{S},\ \mathscr{S}_M$
Theorem 1.3.4	$M = \|\widehat{\boldsymbol{M}}\|_{\mathcal{H}},\ \|\widehat{\boldsymbol{M}}\|,\ \|\widehat{\boldsymbol{M}}_{\mathcal{H}}\|$

III Related to the manifolds Y

Section 3.1, beginning	$t \in \Delta_J$, $\partial_{J \smallsetminus I} \Delta_J$, ι_{IJ}, $t \cdot e$, κ
(2.1.2)	$I(x)$, $I(t)$
(2.1.1)	$(e, x; t) \in Y_J = Y_{\mathcal{U}, J, \underline{\varepsilon}}$
(2.1.7)	$Y_{\mathcal{V}, J, \underline{\varepsilon}}$
(2.1.3)	pr_E, pr_U, pr_Δ
(2.1.5)	$\partial_{J \smallsetminus I} Y_J$
(2.1.6)	$b_H \in \Delta_H$
Corollary 2.1.2	ι_{EU}
(2.1.8)	ι_{EV}

IV Related to the collar

(2.1.10)	c_J^Δ
(2.1.15)	$c_J^Y \colon \partial' Y_{\mathcal{V}, J, \underline{\varepsilon}} \times [0, w_j) \to Y_{\mathcal{V}, J, \underline{\varepsilon}}$
(2.1.13)	$\overline{\mathrm{st}}_J^\Delta(\lvert x \rvert)$
(2.1.11)	∂V_J
(2.2.6)	Fr, cl

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