E_2 structures and derived Koszul duality in string topology

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We construct an equivalence of E_2 algebras between two models for the Thom spectrum of the free loop space that are related by derived Koszul duality. To do this, we describe the functoriality and invariance properties of topological Hochschild cohomology.

Introduction

Chas and Sullivan started the subject of string topology with their observation that the homology of the free loop space LM of a closed oriented manifold M admits a Gerstenhaber structure that can be defined geometrically in terms of natural operations on loops and intersection of chains on the manifold. Contemporaneously, the solution to Deligne's Hochschild cohomology conjecture (by Kontsevich–Soibelman [14], McClure–Smith [23], Tamarkin [30], Voronov [31], Berger–Fresse [3], and perhaps others) established a Gerstenhaber algebra structure on the Hochschild cohomology of a ring or differential graded algebra or even an A_∞ ring spectrum. Cohen–Jones [8], in the course of giving a homotopical interpretation of the string topology product, connected these two ideas by relating a certain Thom spectrum of LM with the topological Hochschild cohomology THC(DM) of the Spanier–Whitehead dual DM. The homology of THC(DM) is canonically isomorphic to the Hochschild cohomology of the cochain algebra $C^*(M)$; Cohen–Jones [8] in particular produces a shifted isomorphism from the homology of the free loop space to the homology of THC(DM) that takes the string topology product to the cup product in Hochschild cohomology. Subsequent work of Malm [20] and Félix–Menichi–Thomas [10] give an isomorphism of Gerstenhaber algebras.

The Félix–Menichi–Thomas work derives from work of Keller [13] (see also [12]) building on unpublished work of Buchweitz. Keller [12] shows that (under mild hypotheses) the Hochschild cochains of Koszul dual dg algebras are equivalent as E_2

algebras (or, more specifically, B_∞ algebras; compare Gerstenhaber–Voronov [11] and Young [32]). When M is simply connected, the derived Koszul dual of C^*M is the cobar construction $\overline{\Omega}C_*M$, which is the Adams–Hilton model for the chains on the based loop space $C_*(\Omega M)$; Félix–Menichi–Thomas [10] constructs an isomorphism of Gerstenhaber algebras

$$HH^*(C^*M) \cong HH^*(\overline{\Omega}C_*M).$$

Since $HH^*(\overline{\Omega}C_*M)$ is isomorphic to the homology of $\mathrm{THC}(\Sigma^\infty_+\Omega M)$, in spectral models, we should look for an equivalence of E_2 ring spectra between $\mathrm{THC}(DM)$ and $\mathrm{THC}(\Sigma^\infty_+\Omega M)$. Our main result is the following theorem, proved in Section 5.

Theorem A *Let X be a simply connected finite cell complex; then $\mathrm{THC}(DX)$ and $\mathrm{THC}(\Sigma^\infty_+\Omega X)$ are weakly equivalent as E_2 ring spectra.*

Beyond the technical role of $HH^*(\overline{\Omega}C_*M)$ in the comparison of Gerstenhaber algebra structures, the spectral analogue $\mathrm{THC}(\Sigma^\infty_+\Omega M)$ in the previous theorem also provides a connection between string topology and topological field theory, as explained in Blumberg–Cohen–Teleman [4]. Furthermore, [4] sketches a relationship between $\mathrm{THC}(\Sigma^\infty_+\Omega M)$ and the wrapped Fukaya category of T^*M, motivated by the work of Abbondandolo–Schwarz [1] and Abouzaid [2]. (See also Seidel [28] for discussion of the significance of Hochschild cohomology of Fukaya categories.) Indeed, the previous theorem (and the machinery we develop to prove it) fills in results stated in [4] but deferred to a future paper.

In the discussion above and in the statement of Theorem A, we are using THC to denote a derived version of the topological Hochschild cohomology spectrum. What this means is slightly complicated by the fact that the standard cosimplicial construction is not functorial. In the setting of differential graded categories, Keller [13; 12] made sense of this for Hochschild cochains and proved limited functoriality and invariance results for Hochschild cohomology. Part of the purpose of this paper is to provide a spectral version of this theory.

For the E_2 structure, we use the McClure–Smith theory of [23; 24], which establishes the action of a specific E_2 operad $\mathcal{D}_2$ on the totalization (Tot) of the topological Hochschild construction of a strictly associative ring spectrum in any modern category of spectra (such as symmetric spectra, orthogonal spectra, or EKMM S–modules). We denote this point-set topological Hochschild cohomology construction as CC in the following theorem, proved in Section 6.

Theorem B *Let $\mathscr{S}$ denote either symmetric spectra, orthogonal spectra, or EKMM S–modules; let $\mathscr{S}[\mathcal{A}ss]$ be the category of associative ring spectra (in $\mathscr{S}$); and let $\mathrm{Ho}\,\mathscr{S}[\mathcal{A}ss]$ denote its homotopy category. Let $\mathrm{Ho}\,\mathscr{S}[\mathcal{A}ss]^{\simeq}$ denote the subcategory of $\mathrm{Ho}\,\mathscr{S}[\mathcal{A}ss]$ where the maps are isomorphisms. There is a contravariant functor THC from $\mathrm{Ho}\,\mathscr{S}[\mathcal{A}ss]^{\simeq}$ to the homotopy category of E_2 ring spectra together with canonical isomorphisms $\mathrm{THC}(R) \to \mathrm{CC}(R)$ for those R whose underlying objects of $\mathscr{S}$ are fibrant and cofibrant relative to the unit.*

The correct generality for THC is the setting of small spectral categories, which generalize associative ring spectra. In Section 3, we explain that the McClure–Smith theory extends to construct an E_2 structure on the Tot of the topological Hochschild–Mitchell construction $\mathrm{CC}(\mathscr{C})$ for a small spectral category $\mathscr{C}$. The natural weak equivalences for spectral categories are the *Dwyer–Kan equivalences*, or DK–equivalences. A spectral functor $\phi\colon \mathscr{D} \to \mathscr{C}$ is a *DK–embedding* if it induces a weak equivalence $\mathscr{D}(a,b) \to \mathscr{C}(\phi(a),\phi(b))$ for all objects a,b of $\mathscr{D}$; a DK–equivalence is a DK–embedding that induces an equivalence of *homotopy categories* $\pi_0\mathscr{D} \to \pi_0\mathscr{C}$.

The following theorem, proved in Section 6, is the natural generalization of Theorem B to this setting; the theorem roughly says that THC is functorial in DK–embeddings. In it, we use the condition for small spectral categories analogous to the condition we used for associative ring spectra in Theorem B: We say that a small spectral category $\mathscr{C}$ is pointwise relatively cofibrant if the mapping spectra $\mathscr{C}(c,c)$ are cofibrant relative to the unit for all objects c in $\mathscr{C}$ and the mapping spectra $\mathscr{C}(c,d)$ are cofibrant for all pairs of objects $c \neq d$ in $\mathscr{C}$. Similarly, we say a small spectral category is pointwise fibrant if each mapping spectrum $\mathscr{C}(c,d)$ is fibrant.

Theorem C *Let $\mathscr{SC}\mathrm{at}$ denote the category of small spectral categories, and let $\mathrm{Ho}(\mathscr{SC}\mathrm{at})$ denote the category obtained by formally inverting the DK–equivalences. Let $\mathrm{Ho}(\mathscr{SC}\mathrm{at})^{\mathrm{DK}}$ be the subcategory of $\mathrm{Ho}(\mathscr{SC}\mathrm{at})$ generated by the DK–embeddings. There is a contravariant functor THC from $\mathrm{Ho}(\mathscr{SC}\mathrm{at})^{\mathrm{DK}}$ to the homotopy category of E_2 ring spectra together with canonical isomorphisms $\mathrm{THC}(\mathscr{C}) \to \mathrm{CC}(\mathscr{C})$ for those $\mathscr{C}$ which are pointwise relatively cofibrant and pointwise fibrant.*

For any small spectral category $\mathscr{C}$, we can construct a functorial "thick closure" $\mathrm{Perf}(\mathscr{C})$ [7, Section 5] (after fixing a cardinal bound); roughly speaking, this is the full subcategory of spectral presheaves on $\mathscr{C}$ generated by the Yoneda image of $\mathscr{C}$ under finite homotopy colimits and retracts. A spectral functor $\phi\colon \mathscr{D} \to \mathscr{C}$ is a Morita

equivalence when the induced functor $\mathrm{Perf}(\mathscr{C}) \to \mathrm{Perf}(\mathscr{D})$ is a DK–equivalence. One reason for interest in the Morita equivalences is that the Bousfield localization of the category of small spectral categories at the Morita equivalences is a model for the ∞–category of small stable idempotent-complete ∞–categories; see Blumberg–Gepner–Tabuada [5, Remark 4.24]. The following theorem, proved in Section 5, shows that THC descends to a functor on a subcategory of this localization.

Theorem D *If $\phi\colon \mathscr{D} \to \mathscr{C}$ is a Morita equivalence, then $\mathrm{THC}(\phi)\colon \mathrm{THC}(\mathscr{C}) \to \mathrm{THC}(\mathscr{D})$ is an isomorphism in the homotopy category of E_2 ring spectra.*

Theorems B and D describe invariance properties of THC analogous to the well-established invariance properties of THH. However, THC in fact has more general invariance properties. For example, if $\mathscr{D}$ is a small spectral subcategory of the category of cofibrant-fibrant right $\mathscr{C}$–modules that factors the Yoneda embedding, then $\mathrm{THC}(\mathscr{C}) \to \mathrm{THC}(\mathscr{D})$ is an isomorphism in the homotopy category of E_2 ring spectra (see Example 5.3). The most general expression of this invariance we know can be expressed in terms of the double centralizer condition, which is also a generalization of derived Koszul duality.

Let $\mathscr{C}$ and $\mathscr{D}$ be small spectral categories and let $\mathscr{M}$ be a $(\mathscr{C}, \mathscr{D})$–bimodule (a commuting left $\mathscr{C}$–module and right $\mathscr{D}$–module structure; see Definition 1.8 or 2.5). Then there are canonical maps in the categories of homotopical $(\mathscr{C}, \mathscr{C})$–bimodules and homotopical $(\mathscr{D}, \mathscr{D})$–bimodules, respectively,

$$\mathscr{C} \to \mathbb{R}\,\mathrm{Hom}_{\mathscr{D}^{\mathrm{op}}}(\mathscr{M}, \mathscr{M}) \quad \text{and} \quad \mathscr{D} \to \mathbb{R}\,\mathrm{Hom}_{\mathscr{C}}(\mathscr{M}, \mathscr{M}).$$

A standard definition is that $\mathscr{M}$ satisfies the *double centralizer condition* when both these maps are weak equivalences. Working backward from this terminology, we say that $\mathscr{M}$ satisfies the *single centralizer condition for $\mathscr{C}$* when the first map (out of $\mathscr{C}$) is a weak equivalence and the *single centralizer condition for $\mathscr{D}$* when the second map (out of $\mathscr{D}$) is a weak equivalence. The following is the spectral version of the main theorem of Keller [12]; we prove it in Section 5.

Theorem E *Let $\mathscr{C}$ and $\mathscr{D}$ be small spectral categories and $\mathscr{M}$ a $(\mathscr{C}, \mathscr{D})$–bimodule that satisfies the single centralizer condition for $\mathscr{D}$; then there is a canonical map in the homotopy category of E_2 ring spectra $\mathrm{THC}(\mathscr{C}) \to \mathrm{THC}(\mathscr{D})$. If $\mathscr{M}$ satisfies the double centralizer condition, then $\mathrm{THC}(\mathscr{C}) \to \mathrm{THC}(\mathscr{D})$ is an isomorphism in the homotopy category of E_2 ring spectra.*

We deduce Theorem A in Section 5 as an immediate corollary of the previous theorem. Dwyer–Greenlees–Iyengar [9, Section 4.22] relates the double centralizer condition for the sphere spectrum $\mathbb{S}$ as a $(\Sigma_+^\infty \Omega X, DX)$–bimodule to the Eilenberg–Moore spectral sequence; in Section 3 of [6], we describe nice models of DX and $\Sigma_+^\infty \Omega X$ and explicitly prove the double centralizer condition when X is a simply connected finite cell complex for a bimodule whose underlying spectrum is equivalent to $\mathbb{S}$.

Theorems C and D have an ∞–categorical extension. Specifically, we prove the following theorem in Section 6.

Theorem F *Let $\mathscr{C}\mathrm{at}^{\mathrm{ex}}$ denote the ∞–category of small stable idempotent-complete ∞–categories and exact functors. Then* THC *extends to a functor from the subcategory of $\mathscr{C}\mathrm{at}^{\mathrm{ex}}$ where the morphisms are fully faithful inclusions to the ∞–category of E_2 ring spectra.*

Conventions

In this paper, $\mathscr{S}$ denotes either the category of symmetric spectra (of topological spaces), or the category of orthogonal spectra, or the category of EKMM S–modules. For brevity we call $\mathscr{S}$ the *category of spectra* and objects of $\mathscr{S}$ *spectra*. We regard the *stable category* as the homotopy category obtained from $\mathscr{S}$ be formally inverting the weak equivalences. (The words "spectrum" and "spectra" when used in this paper should not be construed as referring to any other notion or category.)

Acknowledgements

1 Bi-indexed spectra and the tensor-Hom adjunctions

The purpose of this section is to establish technical foundations for proving tensor-Hom adjunctions for modules over small spectral categories. To do this, we work here with the theory of "bi-indexed" spectra, which are like spectrally enriched directed graphs but where the source and target vertices can be in different sets, or like spectra parametrized over a discrete space of the form $A \times B$.

Definition 1.1 For sets A and B, an (A, B)–spectrum $\mathscr{X}$ consists of a choice of spectrum $\mathscr{X}(a, b)$ for each object (a, b) of $A \times B$; we call (a, b) a *bi-index*. A morphism of (A, B)–spectra $\mathscr{X} \to \mathscr{Y}$ consists of a map of spectra $\mathscr{X}(a, b) \to Y(a, b)$ for all bi-indexes $(a, b) \in A \times B$. A bi-indexed spectrum $\mathscr{X}$ is an (A, B)–spectrum for some A and B; we define the *source* of $\mathscr{X}$ (denoted by $S(\mathscr{X})$) to be B and the *target* of $\mathscr{X}$ to be A (denoted by $T(\mathscr{X})$). If $S(X) = S(Y)$ and $T(X) = T(Y)$, then the set of maps of bi-indexed spectra from $\mathscr{X}$ to $\mathscr{Y}$ is the set of maps of $(T(\mathscr{X}), S(\mathscr{X}))$–spectra from $\mathscr{X}$ to $\mathscr{Y}$; otherwise, it is empty.

For a bi-indexed spectrum $\mathscr{X}$, let $\mathscr{X}^{\mathrm{op}}$ denote the bi-indexed spectrum with

- $S(\mathscr{X}^{\mathrm{op}}) = T(\mathscr{X})$,
- $T(\mathscr{X}^{\mathrm{op}}) = S(\mathscr{X})$, and
- $\mathscr{X}^{\mathrm{op}}(s, t) = \mathscr{X}(t, s)$ for all $(t, s) \in T(\mathscr{X}) \times S(\mathscr{X})$.

We have written and typically write generic bi-indexed spectra with the target variable first and the source variable second; we refer to this as the TS–indexing convention. For the bi-indexed spectra associated to small spectral categories (see Definition 2.1), it is more usual to use the ST–indexing convention, writing the source variable first and the target variable second, and we follow this convention for spectral categories and their bimodules. When it is unclear from the context which indexing is used, we add a superscript st or ts, so

$$\mathscr{X}^{st}(a, b) = \mathscr{X}^{ts}(b, a).$$

We emphasize the distinction between $(-)^{st}$ and $(-)^{\mathrm{op}}$: $(-)^{st}$ just reverses the notation of source and target, while $(-)^{\mathrm{op}}$ reverses the notion of source and target.

As defined above, the category of bi-indexed spectra only admits maps between objects whose source sets agree and target sets agree, so it is sometimes useful to alter these sets.

Definition 1.2 Given functions $f\colon A' \to A$, $g\colon B' \to B$ and an (A, B)–spectrum $\mathscr{X}$, define the *restriction of $\mathscr{X}$ along* (f, g) to be the (A', B')–spectrum $R_{f,g}\mathscr{X}$ where

$$(R_{f,g}\mathscr{X})(a', b') = \mathscr{X}(f(a'), g(b')) \quad \text{for all } (a', b') \in A' \times B'.$$

We define the *target restriction of $\mathscr{X}$ along* f and the *source restriction of $\mathscr{X}$ along* g to be the (A', B)–spectrum $T_f \mathscr{X}$ and (A, B')–spectrum $S_g \mathscr{X}$ where

$$(T_f \mathscr{X})(a', b) = \mathscr{X}(f(a'), b) \quad \text{and} \quad (S_g \mathscr{X})(a, b') = \mathscr{X}(a, g(b'))$$

for all $(a', b) \in A' \times B$ and $(a, b') \in A \times B'$.

Since bi-indexed spectra are determined by their constituent spectra on each bi-index, we have

$$T_g S_f = R_{f,g} = S_f T_g,$$

and for $f' \colon A'' \to A'$ and $g' \colon B'' \to B'$,

$$S_{f'} S_f = S_{f \circ f'}, \quad R_{(f',g')} R_{f,g} = R_{f \circ f', g \circ g'}, \quad \text{and} \quad T_{g'} T_g = T_{g \circ g'}.$$

We could use the preceding definition to define a more sophisticated category of bi-indexed sets incorporating nonidentity maps on source and target sets, but the advantage of the current approach is that this category of bi-indexed spectra has a partial monoidal structure, constructed as follows.

Construction 1.3 For $\mathscr{X}$ an (A, B)–spectrum and $\mathscr{Y}$ a (B, C)–spectrum, define $\mathscr{X} \otimes \mathscr{Y}$ to be the (A, C)–spectrum

$$(\mathscr{X} \otimes \mathscr{Y})(a, c) = \bigvee_{b \in B} \mathscr{X}(a, b) \wedge \mathscr{Y}(b, c).$$

For $\mathscr{Z}$ a (C, D)–spectrum, the associativity isomorphism for the smash product and the universal property of coproduct induce an associativity isomorphism

$$\alpha_{\mathscr{X}, \mathscr{Y}, \mathscr{Z}} \colon (\mathscr{X} \otimes \mathscr{Y}) \otimes \mathscr{Z} \cong \mathscr{X} \otimes (\mathscr{Y} \otimes \mathscr{Z}).$$

For a set A, let $\mathbb{S}_A$ be the (A, A)–spectrum where

$$\mathbb{S}(a_1, a_2) = \begin{cases} * & \text{if } a_1 \neq a_2, \\ \mathbb{S} & \text{if } a_1 = a_2. \end{cases}$$

The left and right unit isomorphism for the smash product induce left and right unit isomorphisms

$$\eta^{\ell}_{\mathscr{X}} \colon \mathbb{S}_A \otimes \mathscr{X} \cong \mathscr{X} \quad \text{and} \quad \eta^{r}_{\mathscr{X}} \colon \mathscr{X} \otimes \mathbb{S}_B \cong \mathscr{X}.$$

When viewing (A, B)–spectra as spectra parametrized over the discrete set $A \times B$, the previous construction gives a point-set version of the May–Sigurdsson composition $\odot$ of [22, Construction 17.1.3]. (For the construction of the right adjoints $\lhd$ and $\rhd$, ibid, see Construction 1.5 below.)

The coherence of associativity and unit isomorphisms for spectra then imply the following proposition.

Proposition 1.4 *The category of bi-indexed spectra is a partial monoidal category under $\otimes$: the object $\mathscr{X} \otimes \mathscr{Y}$ is defined when $T(\mathscr{Y}) = S(\mathscr{X})$, and whenever defined, the following associativity*

$$(\mathscr{W} \otimes \mathscr{X}) \otimes (\mathscr{Y} \otimes \mathscr{Z})$$

$$\alpha_{\mathscr{W} \otimes \mathscr{X}, \mathscr{Y}, \mathscr{Z}} \qquad \alpha_{\mathscr{W}, \mathscr{X}, \mathscr{Y} \otimes \mathscr{Z}}$$

$$((\mathscr{W} \otimes \mathscr{X}) \otimes \mathscr{Y}) \otimes \mathscr{Z} \qquad\qquad \mathscr{W} \otimes (\mathscr{X} \otimes (\mathscr{Y} \otimes \mathscr{Z}))$$

$$\alpha_{\mathscr{W}, \mathscr{X}, \mathscr{Y}} \otimes \mathrm{id}_{\mathscr{Z}} \qquad\qquad \mathrm{id}_{\mathscr{W}} \otimes \alpha_{\mathscr{X}, \mathscr{Y}, \mathscr{Z}}$$

$$(\mathscr{W} \otimes (\mathscr{X} \otimes \mathscr{Y})) \otimes \mathscr{Z} \longrightarrow \mathscr{W} \otimes ((\mathscr{X} \otimes \mathscr{Y}) \otimes \mathscr{Z})$$

$$\alpha_{\mathscr{W}, \mathscr{X} \otimes \mathscr{Y}, \mathscr{Z}}$$

and unit

$$(\mathscr{X} \otimes \mathbb{S}_B) \otimes \mathscr{Y} \xrightarrow{\ \alpha_{\mathscr{X}, \mathbb{S}_B, \mathscr{Y}}\ } \mathscr{X} \otimes (\mathbb{S}_B \otimes \mathscr{Y})$$

$$\eta^r_{\mathscr{X}} \qquad \eta^\ell_{\mathscr{Y}}$$

$$\mathscr{X} \otimes \mathscr{Y}$$

diagrams commute.

The tensor product has two partially defined right adjoints and we also construct a third closely related functor.

Construction 1.5 If $\mathscr{X}$ is an (A, B)–spectrum, $\mathscr{Y}$ is an (A, C)–spectrum, and $\mathscr{Z}$ is a (D, B)–spectrum, we define the (B, C)–spectrum $\mathrm{Hom}^\ell(\mathscr{X}, \mathscr{Y})$ as

$$(\mathrm{Hom}^\ell(\mathscr{X}, \mathscr{Y}))(b, c) = \prod_{a \in A} F(\mathscr{X}(a, b), \mathscr{Y}(a, c))$$

and the (D, A)–spectrum $\mathrm{Hom}^r(\mathscr{X}, \mathscr{Z})$ as

$$(\mathrm{Hom}^r(\mathscr{X}, \mathscr{Z}))(d, a) = \prod_{b \in B} F(\mathscr{X}(a, b), \mathscr{Z}(d, b))$$

(where F denotes the function spectrum construction, adjoint to the smash product). For $\mathscr{X}'$ an (A, B)–spectrum, we define the spectrum $\mathrm{Hom}^b(\mathscr{X}, \mathscr{X}')$ as

$$\mathrm{Hom}^b(\mathscr{X}, \mathscr{X}') = \prod_{(a,b) \in A \times B} F(\mathscr{X}(a, b), \mathscr{X}'(a, b)).$$

We note that Hom^b provides a *partial spectral enrichment* of bi-indexed spectra: when $\mathrm{Hom}^b(\mathscr{X}, \mathscr{X}')$ is defined, maps of spectra from $\mathbb{S}$ into $\mathrm{Hom}^b(\mathscr{X}, \mathscr{X}')$ are canonically in one-to-one correspondence with maps of bi-indexed spectra from $\mathscr{X}$ to $\mathscr{X}'$, and

when $\mathrm{Hom}^b(\mathscr{X}, \mathscr{X}')$ is not defined, the set of maps of bi-indexed spectra from $\mathscr{X}$ to $\mathscr{X}'$ is empty. An easy check of definitions shows the following adjunction property.

Proposition 1.6 *Let* $\mathscr{X}$ *be an* (A, B)*–spectrum,* $\mathscr{Y}$ *a* (B, C)*–spectrum, and* $\mathscr{Z}$ *an* (A, C)*–spectrum. Then there are canonical isomorphisms of spectra*

$$\mathrm{Hom}^b(\mathscr{X}, \mathrm{Hom}^r(\mathscr{Y}, \mathscr{Z})) \cong \mathrm{Hom}^b(\mathscr{X} \otimes \mathscr{Y}, \mathscr{Z}) \cong \mathrm{Hom}^b(\mathscr{Y}, \mathrm{Hom}^\ell(\mathscr{X}, \mathscr{Z})).$$

If $S(\mathscr{X}) = T(\mathscr{X}) = O$ for some set O, then $\mathscr{X}$ is precisely a small spectral O–graph (with the reverse convention on the order of variables, ie with the TS–indexing convention). The tensor product above restricts to a monoidal product on O–graphs and it is well known that the category of small spectral O–categories is isomorphic to the category of monoids for this monoidal product (see [26, Section 6.2]; compare [19, Section II.7]). We say more about this in Section 2. Partly to avoid confusion with the indexing conventions, we will call the monoids under this convention bi-indexed ring spectra.

Definition 1.7 A *bi-indexed ring spectrum* is a monoid for $\otimes$ in bi-indexed spectra. For a bi-indexed ring spectrum $\mathscr{X}$, the *object set* $O(\mathscr{X})$ is $S(\mathscr{X}) = T(\mathscr{X})$.

Note that with this definition, the natural morphisms for bi-indexed ring spectra only allow maps between small spectral categories with the same object sets. Instead of defining the analogue of spectral functors directly, it is more convenient to work with bimodules.

Definition 1.8 Let $\mathscr{X}$ and $\mathscr{Y}$ be bi-indexed ring spectra. An $(\mathscr{X}, \mathscr{Y})$–bimodule consists of a bi-indexed spectrum $\mathscr{M}$ together with a left $\mathscr{X}$–object structure (for $\otimes$) and a commuting right $\mathscr{Y}$–object structure. We write $\mathscr{M}\mathrm{od}_{\mathscr{X}, \mathscr{Y}}$ for the category of $(\mathscr{X}, \mathscr{Y})$–bimodules.

Commuting here means that for the left object structure $\xi \colon \mathscr{X} \otimes \mathscr{M} \to \mathscr{M}$ and the right object structure $\upsilon \colon \mathscr{M} \otimes \mathscr{Y} \to \mathscr{M}$, the diagram

$$
\begin{array}{ccc}
(\mathscr{X} \otimes \mathscr{M}) \otimes \mathscr{Y} & \xrightarrow{\ \alpha_{\mathscr{X}, \mathscr{M}, \mathscr{Y}}\ } & \mathscr{X} \otimes (\mathscr{M} \otimes \mathscr{Y}) \\
{\scriptstyle \xi \otimes \mathrm{id}_{\mathscr{Y}}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_{\mathscr{X}} \otimes \upsilon} \\
\mathscr{M} \otimes \mathscr{Y} & & \mathscr{X} \otimes \mathscr{M} \\
& {\scriptstyle \upsilon} \searrow \quad \swarrow {\scriptstyle \xi} & \\
& \mathscr{M} &
\end{array}
$$

commutes. The left and right object structures require (and are defined by the requirement that) the associativity

$$(\mathscr{X} \otimes \mathscr{X}) \otimes \mathscr{M} \xrightarrow{\alpha_{\mathscr{X},\mathscr{X},\mathscr{M}}} \mathscr{X} \otimes (\mathscr{X} \otimes \mathscr{M}) \xrightarrow{\mathrm{id}_{\mathscr{X}} \otimes \xi} \mathscr{X} \otimes \mathscr{M}$$

with $\mu_{\mathscr{X}} \otimes \mathrm{id}_{\mathscr{M}}$ and ξ down to $\mathscr{X} \otimes \mathscr{M} \xrightarrow{\xi} \mathscr{M}$

$$(\mathscr{M} \otimes \mathscr{Y}) \otimes \mathscr{Y} \xrightarrow{\alpha_{\mathscr{M},\mathscr{Y},\mathscr{Y}}} \mathscr{M} \otimes (\mathscr{Y} \otimes \mathscr{Y}) \xrightarrow{\mathrm{id}_{\mathscr{M}} \mu_{\mathscr{Y}}} \mathscr{M} \otimes \mathscr{Y}$$

with $\upsilon \otimes \mathrm{id}_{\mathscr{Y}}$ and υ down to $\mathscr{M} \otimes \mathscr{Y} \xrightarrow{\upsilon} \mathscr{M}$

and unit

$$\mathbb{S}_{O(\mathscr{X})} \otimes \mathscr{M} \xrightarrow{\eta_{\mathscr{X}} \otimes \mathrm{id}_{\mathscr{M}}} \mathscr{X} \otimes \mathscr{M} \xrightarrow{\xi} \mathscr{M}, \quad \eta^{\ell}$$

$$\mathscr{M} \otimes \mathbb{S}_{O(\mathscr{Y})} \xrightarrow{\mathrm{id}_{\mathscr{M}} \otimes \eta_{\mathscr{Y}}} \mathscr{M} \otimes \mathscr{Y} \xrightarrow{\upsilon} \mathscr{M}, \quad \eta^{r}$$

diagrams commute, where $\mu_{\mathscr{X}}$ and $\mu_{\mathscr{Y}}$ denote the multiplications and $\eta_{\mathscr{X}}$ and $\eta_{\mathscr{Y}}$ denote the units for the monoid structures on $\mathscr{X}$ and $\mathscr{Y}$.

Given a function $\phi \colon O(\mathscr{Y}) \to O(\mathscr{X})$, we get an $(O(\mathscr{X}), O(\mathscr{Y}))$–spectrum $S_{\phi}\mathscr{X} = \mathscr{X}(-, \phi(-))$, which has a canonical left $\mathscr{X}$–object structure, given by the monoid structure of $\mathscr{X}$. We explain in Section 2 why the following definition captures the correct notion of spectral functor.

Definition 1.9 Let $\mathscr{X}$ and $\mathscr{Y}$ be bi-indexed ring spectra. A *spectral functor* $\phi \colon \mathscr{Y} \to \mathscr{X}$ consists of a function $\phi \colon O(\mathscr{Y}) \to O(\mathscr{X})$ and an $(\mathscr{X}, \mathscr{Y})$–bimodule structure on the left $\mathscr{X}$–object $S_{\phi}\mathscr{X} = \mathscr{X}(-, \phi(-))$.

The Hochschild–Mitchell construction requires a version of Hom^{b} "over" a pair of small spectral categories and the adjunction of Proposition 1.6 suggests the utility of analogues of $\otimes$, Hom^{ℓ}, and Hom^{r} over small spectral categories.

Construction 1.10 Let $\mathscr{X}$, $\mathscr{Y}$, and $\mathscr{Z}$ be bi-indexed ring spectra. If $\mathscr{M}$ is an $(\mathscr{X}, \mathscr{Y})$–bimodule and $\mathscr{N}$ is a $(\mathscr{Y}, \mathscr{Z})$–bimodule, then we define the $(\mathscr{X}, \mathscr{Z})$–bimodule $\mathscr{M} \otimes_{\mathscr{Y}} \mathscr{N}$ to be the usual coequalizer

$$\mathscr{M} \otimes \mathscr{Y} \otimes \mathscr{N} \rightrightarrows \mathscr{M} \otimes \mathscr{N} \to \mathscr{M} \otimes_{\mathscr{Y}} \mathscr{N}$$

of the left and right $\mathscr{Y}$–actions. If $\mathscr{M}$ is an $(\mathscr{X}, \mathscr{Y})$–bimodule and $\mathscr{P}$ is an $(\mathscr{X}, \mathscr{Z})$–bimodule, we define the $(\mathscr{Y}, \mathscr{Z})$–bimodule $\mathrm{Hom}^{\ell}_{\mathscr{X}}(\mathscr{M}, \mathscr{P})$ to be the usual equalizer

$$\mathrm{Hom}^{\ell}_{\mathscr{X}}(\mathscr{M}, \mathscr{P}) \to \mathrm{Hom}^{\ell}(\mathscr{M}, \mathscr{P}) \rightrightarrows \mathrm{Hom}^{\ell}(\mathscr{X} \otimes \mathscr{M}, \mathscr{P}),$$

where one map is induced by the left $\mathscr{X}$–action on $\mathscr{M}$ and the other map is the composite of the left $\mathscr{X}$–action on $\mathscr{P}$ and the map

$$\mathrm{Hom}^{\ell}(\mathscr{M}, \mathscr{P}) \to \mathrm{Hom}^{\ell}(\mathscr{X} \otimes \mathscr{M}, \mathscr{X} \otimes \mathscr{P})$$

adjoint to the map

$$\mathscr{X} \otimes \mathscr{M} \otimes \mathrm{Hom}^{\ell}(\mathscr{M}, \mathscr{P}) \to \mathscr{X} \otimes \mathscr{P}$$

that applies the counit of the $\mathscr{M} \otimes (-)$, $\mathrm{Hom}(\mathscr{M}, -)$ adjunction. If $\mathscr{M}$ is an $(\mathscr{X}, \mathscr{Y})$–bimodule and $\mathscr{Q}$ is a $(\mathscr{Z}, \mathscr{Y})$–bimodule, then we define the $(\mathscr{Z}, \mathscr{X})$–bimodule $\mathrm{Hom}^{r}_{\mathscr{Y}}(\mathscr{M}, \mathscr{Q})$ to be the usual equalizer

$$\mathrm{Hom}^{r}_{\mathscr{Y}}(\mathscr{M}, \mathscr{Q}) \to \mathrm{Hom}^{r}(\mathscr{M}, \mathscr{Q}) \rightrightarrows \mathrm{Hom}^{r}(\mathscr{M} \otimes \mathscr{Y}, \mathscr{Q})$$

using the analogous pair of maps for Hom^{r}. If $\mathscr{M}$ is an $(\mathscr{X}, \mathscr{Y})$–bimodule and $\mathscr{M}'$ is an $(\mathscr{X}, \mathscr{Y})$–bimodule, we define the spectrum $\mathrm{Hom}^{b}_{\mathscr{X}, \mathscr{Y}}(\mathscr{M}, \mathscr{M}')$ to be the usual equalizer

$$\mathrm{Hom}^{b}_{\mathscr{X}, \mathscr{Y}}(\mathscr{M}, \mathscr{M}') \to \mathrm{Hom}^{b}(\mathscr{M}, \mathscr{M}') \rightrightarrows \mathrm{Hom}^{b}(\mathscr{X} \otimes \mathscr{M} \otimes \mathscr{Y}, \mathscr{M}')$$

with the analogous pair of maps for Hom^{b}.

An easy check shows that the spectrum of maps $\mathrm{Hom}^{b}_{\mathscr{X}, \mathscr{Y}}$ provides a spectral enrichment of the category of $(\mathscr{X}, \mathscr{Y})$–bimodules.

Proposition 1.6 now generalizes to the following proposition. The proof is again purely formal.

Proposition 1.11 *Let $\mathscr{X}$, $\mathscr{Y}$, and $\mathscr{Z}$ be bi-indexed ring spectra. Let $\mathscr{M}$ be an $(\mathscr{X}, \mathscr{Y})$–bimodule, let $\mathscr{N}$ be a $(\mathscr{Y}, \mathscr{Z})$–bimodule, and let $\mathscr{P}$ be an $(\mathscr{X}, \mathscr{Z})$–bimodule. Then there are canonical isomorphisms of spectra*

$$\mathrm{Hom}^{b}_{\mathscr{X}, \mathscr{Y}}(\mathscr{M}, \mathrm{Hom}^{r}_{\mathscr{Z}}(\mathscr{N}, \mathscr{P})) \cong \mathrm{Hom}^{b}_{\mathscr{X}, \mathscr{Z}}(\mathscr{M} \otimes_{\mathscr{Y}} \mathscr{N}, \mathscr{P})$$
$$\cong \mathrm{Hom}^{b}_{\mathscr{Y}, \mathscr{Z}}(\mathscr{N}, \mathrm{Hom}^{\ell}_{\mathscr{X}}(\mathscr{M}, \mathscr{P})).$$

We note that $\otimes$ and $\otimes_{\mathscr{Y}}$ commute with target restriction (Definition 1.2) on the first variable and source restriction on the second variable; Hom^{ℓ} and $\mathrm{Hom}^{\ell}_{\mathscr{X}}$ convert

source restriction on the first variable to target restriction and preserve source restriction on the second variable. Likewise, Hom^r and $\mathrm{Hom}^r_{\mathscr{Y}}$ convert target restriction on the first variable to source restriction and preserve target restriction on the second variable.

The balanced tensor product produces the composition of spectral functors for the definition of spectral functors (Definition 1.9) above. Given a spectral functor ϕ from $\mathscr{Y}$ to $\mathscr{X}$ and a spectral functor θ from $\mathscr{Z}$ to $\mathscr{Y}$, using the $(\mathscr{X}, \mathscr{Y})$–bimodule structure on $S_\phi \mathscr{X}$ inherent in ϕ, we can make sense of the tensor product over $\mathscr{Y}$ on the right and construct a map of left $\mathscr{X}$–objects

$$S_\phi \mathscr{X} \otimes_{\mathscr{Y}} S_\theta \mathscr{Y} \to S_{\phi \circ \theta} \mathscr{X}.$$

This map is an isomorphism because $\otimes_{\mathscr{Y}}$ commutes with source restriction in the second variable; intrinsically, for every fixed $x \in O(\mathscr{X})$ and $z \in O(\mathscr{Z})$, the diagram

$$\coprod_{y_0, y_1 \in O(\mathscr{Y})} \mathscr{X}(x, \phi(y_0)) \wedge \mathscr{Y}(y_0, y_1) \wedge \mathscr{Y}(y_1, \theta(z)) \rightrightarrows \coprod_{y \in O(\mathscr{Y})} \mathscr{X}(x, \phi(y)) \wedge \mathscr{Y}(y, \theta(z))$$
$$\to \mathscr{X}(x, \phi(\theta(z)))$$

is a split coequalizer. Using the isomorphism to give $S_{\phi \circ \theta} \mathscr{X}$ a right $\mathscr{Z}$–action makes it an $(\mathscr{X}, \mathscr{Z})$–bimodule. We define the composite of the spectral functors $\phi \circ \theta$ to consist of the object function $\phi \circ \theta$ and this bimodule structure on $S_{\phi \circ \theta} \mathscr{X}$.

2 Small spectral categories and the tensor-Hom adjunctions

This section translates the work from the previous section to the framework of small spectral categories. When working in this framework, we use the ST–indexing convention as this is standard in this context. We begin by reviewing the definitions.

Definition 2.1 A *small spectral category* is a small category enriched over spectra. It consists of

(i)   a set of objects $O(\mathscr{C})$,

(ii) a spectrum $\mathscr{C}(a, b)$ for each pair of objects $a, b \in O(\mathscr{C})$,

(iii) a unit map $\mathbb{S} \to \mathscr{C}(a, a)$ for each object $a \in O(\mathscr{C})$, and

(iv) a composition map $\mathscr{C}(b, c) \wedge \mathscr{C}(a, b) \to \mathscr{C}(a, c)$ for each triple of objects $a, b, c \in O(\mathscr{C})$,

satisfying the usual associativity and unit properties. A *strict morphism* $\mathscr{C} \to \mathscr{C}'$ of small spectral categories with the same object set consists of a map of spectra

$\mathscr{C}(a,b) \to \mathscr{C}'(a,b)$ for every pair of objects $a, b \in O(\mathscr{C}) = O(\mathscr{C}')$ that commutes with the unit and composition maps; there are no strict morphisms between small spectral categories with different object sets.

We have inverse functors between the category of bi-indexed ring spectra and small spectral categories (with strict morphisms) defined as follows. For a bi-indexed ring spectrum $\mathscr{X}$, let $C_{\mathscr{X}}$ be the small spectral category defined by setting

(i) the object set $O(C_{\mathscr{X}}) = O(\mathscr{X})$,

(ii) the mapping spectra $C_{\mathscr{X}}(a,b) = \mathscr{X}^{st}(a,b) = \mathscr{X}^{ts}(b,a)$ for all $a, b \in O(C_{\mathscr{X}})$,

(iii) the unit $\mathbb{S} \to C_{\mathscr{X}}(a,a)$ to be the map induced by the monoid structure unit $\mathbb{S}_{O(\mathscr{X})} \to \mathscr{X}$ for all $a \in O(C_{\mathscr{X}})$, and

(iv) the composition

$$C_{\mathscr{X}}(b,c) \wedge C_{\mathscr{X}}(a,b) \to C_{\mathscr{X}}(a,c)$$

to be the map $\mathscr{X}(c,b) \wedge \mathscr{X}(b,a) \to \mathscr{X}(c,a)$ that appears as a wedge summand in the monoid structure multiplication $\mathscr{X} \otimes \mathscr{X} \to \mathscr{X}$.

Similarly, for a small spectral category $\mathscr{C}$, we define a bi-indexed ring spectrum $B_{\mathscr{C}}$ with the same object set by taking $B_{\mathscr{C}}(a,b) = \mathscr{C}(b,a)$ and the obvious unit and multiplication. These assignments are evidently functorial.

Proposition 2.2 *The functors C and B above are inverse isomorphisms of categories between the category of bi-indexed ring spectra and the category of small spectral categories (with strict morphisms).*

The more usual category of small spectral categories has morphisms given by *spectral functors*, which are simply the spectrally enriched functors. The following theorem relates this notion to Definition 1.9. We prove it at the end of the section after reviewing more of the theory of small spectral categories and their modules.

Theorem 2.3 *There is a canonical bijection between the set of spectral functors of small spectral categories $\mathscr{D} \to \mathscr{C}$ and the set of spectral functors of the corresponding bi-indexed ring spectra. This bijection is compatible with composition.*

Left and right modules are basic notions for small spectral categories that do not precisely correspond to left and right objects for bi-indexed ring spectra.

Definition 2.4 Let $\mathscr{C}$ be a small spectral category. The (spectrally enriched) category $\mathscr{M}\mathrm{od}_{\mathscr{C}}$ of *left $\mathscr{C}$–modules* is the (spectrally enriched) category of spectrally enriched functors from $\mathscr{C}$ to spectra; the (spectrally enriched) category $\mathscr{M}\mathrm{od}_{\mathscr{C}^{\mathrm{op}}}$ of *right $\mathscr{C}$–modules* is the (spectrally enriched) category of spectrally enriched contravariant functors from $\mathscr{C}$ to spectra.

For any one-point set $\{a\}$, the category of left $\mathscr{C}$–modules is isomorphic to the full subcategory category of left $B_{\mathscr{C}}$–objects with source set $\{a\}$ and is isomorphic as a spectrally enriched category to the category of $(B_{\mathscr{C}}, \mathbb{S}_{\{a\}})$–bimodules. Likewise, the category of right $\mathscr{C}$–modules is isomorphic to the full subcategory category of right $B_{\mathscr{C}}$–objects with target set $\{a\}$ and is isomorphic as a spectrally enriched category to the category of $(\mathbb{S}_{\{a\}}, B_{\mathscr{C}})$–bimodules. The category of left $B_{\mathscr{C}}$–objects is essentially the category of (singly) indexed left $B_{\mathscr{C}}$–modules: a left $B_{\mathscr{C}}$–object $\mathscr{M}$ consists of a left $\mathscr{C}$–module $\mathscr{M}^{st}(a, -)$ for each a in $S(\mathscr{M})$.

Bimodules for small spectral categories do correspond precisely with bimodules for bi-indexed ring spectra. In the context of bimodules of small spectral categories, just as in the context of bi-indexed spectra, we take the convention that the category on the left has the left action and the category on the right has the right action. However, as always in the context of small spectral category concepts, we follow the ST–indexing convention implicit in the definition below that the righthand variable is the covariant one while the lefthand variable is the contravariant one.

Definition 2.5 Let $\mathscr{C}$ and $\mathscr{D}$ be small spectral categories. Let $\mathscr{D}^{\mathrm{op}} \wedge \mathscr{C}$ be the small spectral category with objects $O(\mathscr{D}^{\mathrm{op}} \wedge \mathscr{C}) = O(\mathscr{D}) \times O(\mathscr{C})$, mapping spectra

$$(\mathscr{D}^{\mathrm{op}} \wedge \mathscr{C})((d, c), (d', c')) = \mathscr{D}(d', d) \wedge \mathscr{C}(c, c'),$$

unit induced by the units of $\mathscr{C}$ and $\mathscr{D}$ (and the canonical isomorphism $\mathbb{S} \wedge \mathbb{S} \cong \mathbb{S}$), and composition induced by the composition on $\mathscr{D}$ (performed backwards) and the composition on $\mathscr{C}$:

$$
\begin{aligned}
(\mathscr{D}^{\mathrm{op}} \wedge \mathscr{C})&((d', c'), (d'', c'')) \wedge (\mathscr{D}^{\mathrm{op}} \wedge \mathscr{C})((d, c), (d', c')) \\
&= (\mathscr{D}(d'', d') \wedge \mathscr{C}(c', c'')) \wedge (\mathscr{D}(d', d) \wedge \mathscr{C}(c, c')) \\
&\cong (\mathscr{D}(d', d) \wedge \mathscr{D}(d'', d')) \wedge (\mathscr{C}(c', c'') \wedge \mathscr{C}(c, c')) \\
&\to \mathscr{D}(d'', d) \wedge \mathscr{C}(c, c'') = (\mathscr{D}^{\mathrm{op}} \wedge \mathscr{C})((d, c), (d'', c'')).
\end{aligned}
$$

The (spectrally enriched) category $\mathscr{M}\mathrm{od}_{\mathscr{C}, \mathscr{D}}$ of *$(\mathscr{C}, \mathscr{D})$–bimodules* is the (spectrally enriched) category of spectrally enriched functors from $\mathscr{D}^{\mathrm{op}} \wedge \mathscr{C}$ to spectra.

Given a $(\mathscr{C}, \mathscr{D})$–bimodule $\mathscr{F}$, we write $B_{\mathscr{F}}$ for the $(O(\mathscr{C}), O(\mathscr{D}))$–spectrum

$$B_{\mathscr{F}}(c, d) = B_{\mathscr{F}}^{st}(d, c) = \mathscr{F}(d, c)$$

for $(c, d) \in O(\mathscr{C}) \times O(\mathscr{D})$. This has a canonical $(B_{\mathscr{C}}, B_{\mathscr{D}})$–bimodule structure with action maps induced by

$$B_{\mathscr{C}}(c, c') \wedge B_{\mathscr{F}}(c', d) = \mathscr{C}(c', c) \wedge \mathscr{F}(d, c') \to \mathscr{F}(d, c) = B_{\mathscr{F}}(c, d)$$

and

$$B_{\mathscr{F}}(c, d) \wedge B_{\mathscr{D}}(d, d') = \mathscr{F}(d, c) \wedge \mathscr{D}(d', d) \to \mathscr{F}(d', c) = B_{\mathscr{F}}(c, d').$$

This is evidently functorial, and indeed extends canonically to a spectrally enriched functor from the category of $(\mathscr{C}, \mathscr{D})$–bimodules to the category of $(B_{\mathscr{C}}, B_{\mathscr{D}})$–bimodules (in bi-indexed spectra).

Proposition 2.6 *The spectrally enriched functor B from $(\mathscr{C}, \mathscr{D})$–bimodules to $(B_{\mathscr{C}}, B_{\mathscr{D}})$–bimodules (in bi-indexed spectra) is an isomorphism of spectrally enriched categories.*

We write the inverse isomorphism as C; evidently, $C_{\mathscr{M}}(c, d) = \mathscr{M}(d, c)$ for all $(c, d) \in O(\mathscr{C}) \times O(\mathscr{D})$.

In light of the previous proposition, for $(\mathscr{C}, \mathscr{D})$–bimodules $\mathscr{F}$ and $\mathscr{G}$, we write $\mathrm{Hom}^b_{\mathscr{C}, \mathscr{D}}(\mathscr{F}, \mathscr{G})$ for the spectrum of bimodule maps from $\mathscr{F}$ to $\mathscr{G}$ and we more generally define $\otimes$, $\otimes_{\mathscr{D}}$, Hom^ℓ, $\mathrm{Hom}^\ell_{\mathscr{C}}$, Hom^r, and $\mathrm{Hom}^r_{\mathscr{D}}$ in terms of the inverse isomorphisms B and C (for spectral categories/bi-indexed ring spectra and bimodules). In explicit terms, we have:

Proposition 2.7 *Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be small spectral categories.*

(i) *For an $(\mathscr{A}, \mathscr{B})$–bimodule $\mathscr{F}$ and a $(\mathscr{B}, \mathscr{C})$–bimodule $\mathscr{G}$, the $(\mathscr{A}, \mathscr{C})$–bimodule $\mathscr{F} \otimes \mathscr{G} = C_{B_{\mathscr{F}} \otimes B_{\mathscr{G}}}$ satisfies*

$$(\mathscr{F} \otimes \mathscr{G})(c, a) = \bigvee_{b \in O(\mathscr{B})} \mathscr{F}(b, a) \wedge \mathscr{G}(c, b),$$

and $\mathscr{F} \otimes_{\mathscr{B}} \mathscr{G} = C_{B_{\mathscr{F}} \otimes_{B_{\mathscr{B}}} B_{\mathscr{G}}}$ is the coequalizer

$$\mathscr{F} \otimes \mathscr{B} \otimes \mathscr{G} \rightrightarrows \mathscr{F} \otimes \mathscr{G} \to \mathscr{F} \otimes_{\mathscr{B}} \mathscr{G}.$$

(ii) *For an $(\mathscr{A}, \mathscr{B})$–bimodule $\mathscr{F}$ and an $(\mathscr{A}, \mathscr{C})$–bimodule $\mathscr{G}$, the $(\mathscr{B}, \mathscr{C})$–bimodule $\mathrm{Hom}^\ell(\mathscr{F}, \mathscr{G}) = C_{\mathrm{Hom}^\ell(B_{\mathscr{F}}, B_{\mathscr{G}})}$ satisfies*

$$(\mathrm{Hom}^\ell(\mathscr{F}, \mathscr{G}))(c, b) = \prod_{a \in O(\mathscr{A})} F(\mathscr{F}(b, a), \mathscr{G}(c, a)),$$

and $\mathrm{Hom}^\ell_{\mathscr{A}}(\mathscr{F},\mathscr{G}) = C_{\mathrm{Hom}^\ell_{B_{\mathscr{A}}}(B_{\mathscr{F}},B_{\mathscr{G}})}$ *is the equalizer*

$$\mathrm{Hom}^\ell_{\mathscr{A}}(\mathscr{F},\mathscr{G}) \to \mathrm{Hom}^\ell(\mathscr{F},\mathscr{G}) \rightrightarrows \mathrm{Hom}^\ell(\mathscr{A} \otimes \mathscr{F},\mathscr{G}).$$

(iii) *For an* $(\mathscr{A},\mathscr{B})$*–bimodule* $\mathscr{F}$ *and a* $(\mathscr{C},\mathscr{B})$*–bimodule* $\mathscr{G}$*, the* $(\mathscr{C},\mathscr{A})$*–bimodule* $\mathrm{Hom}^r(\mathscr{F},\mathscr{G}) = C_{\mathrm{Hom}^r(B_{\mathscr{F}},B_{\mathscr{G}})}$ *satisfies*

$$(\mathrm{Hom}^r(\mathscr{F},\mathscr{G}))(a,c) = \prod_{b\in O(\mathscr{B})} F(\mathscr{F}(b,a),\mathscr{G}(b,c)),$$

and $\mathrm{Hom}^r_{\mathscr{B}}(\mathscr{F},\mathscr{G}) = C_{\mathrm{Hom}^r_{B_{\mathscr{B}}}(B_{\mathscr{F}},B_{\mathscr{G}})}$ *is the equalizer*

$$\mathrm{Hom}^r_{\mathscr{B}}(\mathscr{F},\mathscr{G}) \to \mathrm{Hom}^r(\mathscr{F},\mathscr{G}) \rightrightarrows \mathrm{Hom}^r(\mathscr{F} \otimes \mathscr{B},\mathscr{G}).$$

As an immediate consequence of Proposition 1.11, we obtain the corresponding adjunction in the context of small spectral categories.

Proposition 2.8 *Let* $\mathscr{A}$*,* $\mathscr{B}$*, and* $\mathscr{C}$ *be small spectral categories. Let* $\mathscr{F}$ *be an* $(\mathscr{A},\mathscr{B})$*–bimodule,* $\mathscr{G}$ *be a* $(\mathscr{B},\mathscr{C})$*–bimodule, and* $\mathscr{H}$ *be an* $(\mathscr{A},\mathscr{C})$*–bimodule. Then there are canonical isomorphisms of spectra*

$$\mathrm{Hom}^b_{\mathscr{A},\mathscr{B}}(\mathscr{F},\mathrm{Hom}^r_{\mathscr{C}}(\mathscr{G},\mathscr{H})) \cong \mathrm{Hom}^b_{\mathscr{A},\mathscr{C}}(\mathscr{F} \otimes_{\mathscr{B}} \mathscr{G},\mathscr{H})$$
$$\cong \mathrm{Hom}^b_{\mathscr{B},\mathscr{C}}(\mathscr{G},\mathrm{Hom}^\ell_{\mathscr{A}}(\mathscr{F},\mathscr{H})).$$

Comparing the formulas in Proposition 2.7 with the intrinsic definition of the spectral enrichment of a category of spectral functors reveals the following relationship between $\mathrm{Hom}^\ell_{\mathscr{C}}$ and the spectral enrichment on the category of left $\mathscr{C}$–modules, which is essentially a special case of the observation on $\mathrm{Hom}^\ell_{\mathscr{X}}$ and source restriction in the previous section. An analogous result holds for $\mathrm{Hom}^r_{\mathscr{C}}$ and the spectral enrichment on the category of right $\mathscr{C}$–modules.

Proposition 2.9 *In the notation of Proposition 2.7(ii),*

$$(\mathrm{Hom}^\ell_{\mathscr{A}}(\mathscr{F},\mathscr{G}))^{st}(c,b) \cong \mathscr{M}\mathrm{od}_{\mathscr{A}}(\mathscr{F}(b,-),\mathscr{G}(c,-))$$

for all $b \in O(\mathscr{B})$ *and* $c \in O(\mathscr{C})$*.*

Finally, we return to Theorem 2.3.

Proof of Theorem 2.3 Let $\mathscr{X}$ and $\mathscr{Y}$ be bi-indexed ring spectra and let $\mathscr{C}$ and $\mathscr{D}$ denote the corresponding small spectral categories.

Given a spectral functor $\psi \colon \mathscr{D} \to \mathscr{C}$, let $\mathscr{F}_\psi$ denote the $(\mathscr{C}, \mathscr{D})$–bimodule with component spectra $\mathscr{F}_\psi^{st}(d, c) = \mathscr{C}(\psi(d), c)$ and the evident bimodule structure. Then the underlying left $\mathscr{X}$–object of the $(\mathscr{X}, \mathscr{Y})$–bimodule $B_{\mathscr{F}_\psi}$ is $S_\psi \mathscr{X}$. Let B_ψ be the spectral functor $\mathscr{X} \to \mathscr{Y}$ that uses the underlying object function of ψ as the function on object sets and $B_{\mathscr{F}_\psi}$ as specifying the bimodule structure on $S_\psi \mathscr{X}$.

Given a spectral functor $\phi \colon \mathscr{Y} \to \mathscr{X}$, we obtain a spectral functor C_ϕ from $\mathscr{D}$ to $\mathscr{C}$ using the same object function and the map on morphism spectra defined as follows. The map of left $\mathscr{X}$–objects $S_\phi \mathscr{X} \otimes \mathscr{Y} \to S_\phi \mathscr{X}$ is adjoint to a map of bi-indexed spectra $\mathscr{Y} \to \operatorname{Hom}_{\mathscr{X}}^{\ell}(S_\phi \mathscr{X}, S_\phi \mathscr{X})$. Because Hom^ℓ converts source restriction in the first variable and preserves source restriction in the second variable, we have a canonical isomorphism

$$\operatorname{Hom}_{\mathscr{X}}^{\ell}(S_\phi \mathscr{X}, S_\phi \mathscr{X}) \cong R_{\phi,\phi} \operatorname{Hom}_{\mathscr{X}}^{\ell}(\mathscr{X}, \mathscr{X}) \cong R_{\phi,\phi} \mathscr{X}.$$

The map of bi-indexed spectra $\mathscr{Y} \to R_{\phi,\phi} \mathscr{X}$ then specifies a map $\mathscr{D}(a, b) \to \mathscr{C}(\phi(a), \phi(b))$ for all $a, b \in O(\mathscr{D})$. In light of Proposition 2.9, this map is the composite

$$\mathscr{D}(a, b) \to F_{\mathscr{C}}(\mathscr{C}(\phi(b), -), \mathscr{C}(\phi(a), -)) \cong \mathscr{C}(\phi(a), \phi(b))$$

of the adjoint of $\mathscr{C}(\phi(b), -) \wedge \mathscr{D}(a, b) \to \mathscr{C}(\phi(a), -)$ and the enriched Yoneda lemma isomorphism. From here it follows easily that the constructed map on morphism spectra preserves units and composition.

It is clear that $B_{C_\phi} = \phi$ and $C_{B_\psi} = \psi$, and moreover that B preserves composition of spectral functors.   □

Using the enriched form of the Yoneda lemma, it is straightforward to check that natural transformations of spectral functors between small spectral categories correspond to maps of bimodules for spectral functors between bi-indexed ring spectra; we do not use this result.

3   Hochschild–Mitchell and McClure–Smith constructions

In this section, we review the point-set construction of topological Hochschild cohomology of a small spectral category in terms of the Hochschild–Mitchell complex CC. We then observe that this fits into the framework of the McClure–Smith approach to the Deligne conjecture; in particular, there is a canonical E_2 ring spectrum structure on CC.

Construction 3.1 (topological Hochschild–Mitchell construction) Let $\mathscr{C}$ be a small spectral category and $\mathscr{M}$ a $(\mathscr{C}, \mathscr{C})$–bimodule. Let $\mathrm{CC}^{\bullet}(\mathscr{C}; \mathscr{M})$ be the cosimplicial spectrum $\mathrm{Hom}^{b}_{\mathscr{C}, \mathscr{C}}(B_{\bullet}(\mathscr{C}; \mathscr{C}; \mathscr{C}), \mathscr{M})$, where $B_{\bullet}$ denotes the two-sided bar construction for the monoidal product $\otimes$. More concretely, $\mathrm{CC}^{\bullet}(\mathscr{C}; \mathscr{M})$ is the cosimplicial spectrum which in cosimplicial degree n is

$$\mathrm{Hom}^{b}_{\mathscr{C}, \mathscr{C}}(\mathscr{C} \otimes \underbrace{\mathscr{C} \otimes \cdots \otimes \mathscr{C}}_{n \text{ factors}} \otimes \mathscr{C}, \mathscr{M})$$

with coface map δ^{i} induced by the product $\mathscr{C} \otimes \mathscr{C} \to \mathscr{C}$ (see Proposition 2.2 and Definition 1.7) on the ith and $(i+1)^{\text{st}}$ factors (starting the count from zero outside the braces) and codegeneracy σ^{i} maps the unit map $C_{\mathbb{S}_{O(\mathscr{C})}} \to \mathscr{C}$ (see Proposition 1.4) inserting the $\mathscr{C}$ as the ith factor. We write $\mathrm{CC}^{\bullet}(\mathscr{C})$ for $\mathrm{CC}^{\bullet}(\mathscr{C}; \mathscr{C})$ in the case $\mathscr{M} = \mathscr{C}$. Let $\mathrm{CC}(\mathscr{C}; \mathscr{M})$ and $\mathrm{CC}(\mathscr{C})$ denote the spectra obtained by applying Tot.

Construction 3.1 is evidently covariantly functorial in maps of the bimodule $\mathscr{M}$ and contravariantly functorial in spectral functors of the small spectral category $\mathscr{C}$ (pulling back the bimodule structure along the spectral functor). Without hypotheses on $\mathscr{C}$ and $\mathscr{M}$, the topological Hochschild–Mitchell construction may not preserve weak equivalences. However, when $\mathscr{C}$ is pointwise relatively cofibrant (see Definition 4.5) and $\mathscr{M}$ is pointwise fibrant, CC preserves weak equivalences in each variable; see Proposition 7.4 and Theorem 7.5.

The free, forgetful adjunction arising from the interpretation of small spectral categories as monoids for $\otimes$ (Proposition 2.2) allows us to rewrite the cosimplicial object in Construction 3.1 more explicitly as

$$\mathrm{CC}^{0}(\mathscr{C}, \mathscr{M}) = \prod_{c} \mathscr{M}(c, c)$$

and

$$(3.2) \qquad \mathrm{CC}^{n}(\mathscr{C}; \mathscr{M}) \cong \mathrm{Hom}^{b}(\underbrace{\mathscr{C} \otimes \cdots \otimes \mathscr{C}}_{n \text{ factors}}, \mathscr{M})$$

$$\cong \prod_{c_0, \ldots, c_n} F(\mathscr{C}(c_1, c_0) \wedge \cdots \wedge \mathscr{C}(c_n, c_{n-1}), \mathscr{M}(c_n, c_0)).$$

In this form, the cofaces $\delta^{1}, \ldots, \delta^{n-1} \colon \mathrm{CC}^{n-1} \to \mathrm{CC}^{n}$ are induced by the composition in the category with δ^{0} and δ^{n} induced by the bimodule structure on $\mathscr{M}$; the codegeneracies $\sigma^{i} \colon \mathrm{CC}^{n} \to \mathrm{CC}^{n+1}$ are induced by $\mathbb{S} \to \mathscr{C}(c_i, c_i)$, inserting the identity map

in the ith position. This is the usual explicit description of the Hochschild–Mitchell construction for an enriched category.

Thinking in terms of the partial monoidal category of bi-indexed spectra (Section 1), the description of $\mathrm{CC}^{\bullet}(\mathscr{C})$ as $\mathrm{Hom}^b(\mathscr{C} \otimes \cdots \otimes \mathscr{C}, \mathscr{C})$ identifies $\mathrm{CC}^{\bullet}(\mathscr{C})$ as the (nonsymmetric) endomorphism operad $\mathcal{E}\mathrm{nd}^b(B_{\mathscr{C}})$ of the corresponding bi-indexed spectrum $B_{\mathscr{C}}$. Because $B_{\mathscr{C}}$ is a monoid for the monoidal product, there is a canonical map $\mathbb{S} \to \mathcal{E}\mathrm{nd}^b(B_{\mathscr{C}})(n)$ for all n induced by the iterated multiplication

$$B_{\mathscr{C}} \otimes \cdots \otimes B_{\mathscr{C}} \to B_{\mathscr{C}}.$$

These assemble to a map of nonsymmetric operads $\mathcal{A}\mathrm{ss} \to \mathcal{E}\mathrm{nd}^b(B_{\mathscr{C}})$, where $\mathcal{A}\mathrm{ss}$ denotes the nonsymmetric associative operad in spectra $\mathcal{A}\mathrm{ss}(n) = \mathbb{S}$. This is precisely a "operad with multiplication" in the terminology of McClure–Smith [24, Definition 10.1]. The point of identifying this structure is that it is the data required in the McClure–Smith theory to induce an E_2 ring structure on Tot. Specifically, as a consequence of [24, Theorem 9.1, Proposition 10.3], we can immediately deduce the following proposition.

Proposition 3.3 *The topological Hochschild–Mitchell construction $\mathrm{CC}(\mathscr{C}))$ has a canonical structure of a $\mathcal{D}_2$–algebra (E_2 ring spectrum), where $\mathcal{D}_2$ is the E_2 operad of McClure–Smith [24, Section 9].*

Proof In terms of the maps $e \colon \mathbb{S} \to \mathcal{E}\mathrm{nd}^b(B_{\mathscr{C}})(0)$ and $\mu \colon \mathbb{S} \to \mathcal{E}\mathrm{nd}^b(B_{\mathscr{C}})(2)$, under the isomorphism of $\mathcal{E}\mathrm{nd}^b(B_{\mathscr{C}})(n)$ with $\mathrm{CC}^n(\mathscr{C})$, the coface and codegeneracy maps for $\mathrm{CC}(\mathscr{C})$ above coincide with the ones described on page 1136 of [24] in the proof of Theorem 10.3: for $f \in \mathcal{E}\mathrm{nd}^b(B_{\mathscr{C}})$

$$\sigma^i f = f \circ_{i+1} e \qquad \text{and} \qquad \delta^i f = \begin{cases} \mu \circ_2 f & \text{if } i = 0, \\ f \circ_i \mu & \text{if } i = 1, \dots, n, \\ \mu \circ_1 f & \text{if } i = n+1. \end{cases} \qquad \square$$

There is no naturality statement in the preceding proposition because $\mathrm{CC}(\mathscr{C})$ is not functorial in $\mathscr{C}$ (on the point-set level) in any reasonable way. It does have a very limited functoriality for spectral functors that induce isomorphisms on mapping spectra; we refer to these as *strictly fully faithful* spectral functors. For a strictly fully faithful spectral functor $\phi \colon \mathscr{D} \to \mathscr{C}$, there is a restriction map $\phi^* \colon \mathcal{E}\mathrm{nd}^b(\mathscr{C}) \to \mathcal{E}\mathrm{nd}^b(\mathscr{D})$, which is a map of operads with multiplication, constructed as follows.

On arity n, the map takes the form

$$\mathcal{E}nd^b(\mathscr{C})(n) = \prod_{c_0,\ldots,c_n} F(\mathscr{C}(c_1,c_0) \wedge \cdots \wedge \mathscr{C}(c_n,c_{n-1}), \mathscr{C}(c_n,c_0))$$

$$\to \prod_{d_0,\ldots,d_n} F(\mathscr{D}(d_1,d_0) \wedge \cdots \wedge \mathscr{D}(d_n,d_{n-1}), \mathscr{D}(d_n,d_0)) = \mathcal{E}nd^b(\mathscr{D})(n),$$

where on the $d_0,\ldots,d_n$ factor of the target, we use projection onto the $c_i = \phi(d_i)$ factor of the source and the isomorphism $\mathscr{C}(\phi(d_i),\phi(d_j)) \cong \mathscr{D}(d_i,d_j)$ of the strictly fully faithful spectral functor ϕ.

It is straightforward to verify that this map is compatible with the operad structures and commutes with the inclusion of $\mathcal{A}ss$.

Proposition 3.4 *The topological Hochschild–Mitchell construction* CC *extends to a functor from the category of small spectral categories and strictly fully faithful spectral functors to $\mathcal{D}_2$–algebras (E_2 ring spectra).*

4 Homotopy theory of objects and bimodules over small spectral categories

Proposition 3.4 established a very limited naturality for the functor CC. To extend this functoriality to the more general derived statement of Theorems C and E, we need to introduce in the next section some conditions on bimodules that we call centralizer conditions. These are phrased in terms of the derived functors of $\mathrm{Hom}^{\ell}_{\mathscr{C}}$ and $\mathrm{Hom}^r_{\mathscr{D}}$; the purpose of this section is to set up the homotopical algebra and review some conditions that ensure that the point-set functors represent the derived functors.

Before beginning the discussion of model category structures, it is convenient to introduce terminology for extending conditions and properties on spectra and maps of spectra to small spectral categories, left and right objects, and bimodules.

Definition Schema 4.1 For any property or condition on spectra or maps of spectra, we say the property holds *pointwise* on a bi-indexed spectrum, bi-indexed spectrum with extra structure, or a map of bi-indexed spectra with extra structure when it holds at every bi-index. We say that such a property holds *pointwise* for a small spectral category, bimodule, strict morphism of small spectral categories, or map of bimodules when it holds for the underlying bi-indexed spectrum or map of bi-indexed spectra.

For the model structures on the category of bi-indexed spectra and for bi-indexed ring spectra $\mathscr{X}$ and $\mathscr{Y}$, the categories of left $\mathscr{X}$–objects, right $\mathscr{Y}$–objects, and $(\mathscr{X}, \mathscr{Y})$–bimodules, we take the weak equivalences to be the pointwise weak equivalences and the fibrations to be the pointwise fibrations. To describe the cofibrations, let I denote the standard set of generating cofibrations for the model category of spectra. Then given sets A and B, elements $a \in A$ and $b \in B$, and element $i \colon C \to D$ in I, let $C_{A,B;a,b;i}$ and $D_{A,B;a,b;i}$ be the (A, B)–spectra that are C and D (respectively) on (a, b) and $*$ elsewhere, and let $f_{A,B;a,b;i} \colon C_{A,B;a,b;i} \to D_{A,B;a,b;i}$ be the map of bi-indexed spectra that does $i \colon C \to D$ on (a, b). Although the collection BI of such maps does not form a small set, for any given bi-indexed spectrum $\mathscr{Z}$, the collection of maps from the domains of the elements of BI does form a small set (or is isomorphic to one) since only those $C_{A,B;a,b;i}$ with $A = T(\mathscr{Z})$ and $B = S(\mathscr{Z})$ admit a map to $\mathscr{Z}$. This allows the small object argument to be applied with the collection BI. The cofibrations of bi-indexed spectra are exactly the pointwise cofibrations (maps that are cofibrations at each bi-index (a, b)) and these are exactly the maps that are retracts of relative cell complexes built by attaching cells from BI (in the sense of [21, Definition 5.4]). The cofibrations in the category of left $\mathscr{X}$–objects, right $\mathscr{Y}$–objects, and $(\mathscr{X}, \mathscr{Y})$–bimodules are the retracts of relative cell complexes (in the sense of [21, Definition 5.4]) built by attaching cells of the form $\mathscr{X} \otimes f_{O(\mathscr{X}),B;a,b;i}$, $f_{A,O(\mathscr{Y});a,b;i} \otimes \mathscr{Y}$, and $\mathscr{X} \otimes f_{O(\mathscr{X}),O(\mathscr{Y});a,b;i} \otimes \mathscr{Y}$, respectively. The usual arguments (eg [21, Section 5]) prove the following proposition.

Proposition 4.2 *The category of bi-indexed spectra and, for bi-indexed ring spectra $\mathscr{X}$ and $\mathscr{Y}$, the categories of left $\mathscr{X}$–objects, right $\mathscr{Y}$–objects, and $(\mathscr{X}, \mathscr{Y})$–bimodules are topologically enriched closed model categories with fibrations and weak equivalences the pointwise fibrations and pointwise weak equivalences, and cofibrations the retracts of relative cell complexes. The category of $(\mathscr{X}, \mathscr{Y})$–bimodules is a spectrally enriched closed model category.*

Proposition 4.3 *For small spectral categories $\mathscr{C}$ and $\mathscr{D}$, the category of $(\mathscr{C}, \mathscr{D})$–bimodules is a spectrally enriched closed model category with fibrations and weak equivalences the pointwise fibrations and pointwise weak equivalences and cofibrations the retracts of relative cell complexes.*

In the statement "topologically enriched" or "spectrally enriched" means that the categories satisfy the topological or spectral version of Quillen's Axiom SM7, which is called the "Enrichment Axiom" in [15, Section 3].

For identifying cofibrant resolutions in Section 7, we need to know when cofibrant bimodules are pointwise cofibrant; a sufficient condition is for the small spectral categories in question to be pointwise "semicofibrant": Recall from Lewis–Mandell [15, Definition 1.2, Proposition 6.4] that a spectrum X is *semicofibrant* when $X \wedge (-)$ preserves cofibrations and acyclic cofibrations, or equivalently, $F(X, -)$ preserves fibrations and acyclic fibrations. Cofibrant spectra are in particular semicofibrant; in the standard model structure on symmetric spectra and orthogonal spectra, the sphere spectrum is cofibrant and all semicofibrant objects are cofibrant. In the positive stable model category and in EKMM S–modules, the sphere spectrum is not cofibrant but is only semicofibrant. It follows formally that a weak equivalence of semicofibrant spectra $X \to X'$ induces a weak equivalence $X \wedge Y \to X' \wedge Y$ for any spectrum Y (to see this, smash with a cofibrant approximation of $\mathbb{S}$) and a weak equivalence $F(X', Z) \to F(X, Z)$ for any fibrant spectrum Z [15, Theorem 6.2]. The explicit description of cofibrations in the bimodule model structures implies the following proposition.

Proposition 4.4 *A cofibrant bi-indexed spectrum is pointwise cofibrant. If $\mathscr{X}$ and $\mathscr{Y}$ are pointwise semicofibrant bi-indexed ring spectra then the cofibrant objects in left $\mathscr{X}$–objects, right $\mathscr{Y}$–objects, and $(\mathscr{X}, \mathscr{Y})$–bimodules are pointwise cofibrant. In particular if $\mathscr{C}$ and $\mathscr{D}$ are pointwise semicofibrant small spectral categories then cofibrant $(\mathscr{C}, \mathscr{D})$–bimodules are pointwise cofibrant.*

In later work, we use the following slightly stronger hypothesis on the small spectral categories.

Definition 4.5 A small spectral category $\mathscr{C}$ is pointwise relatively cofibrant when for every object c in $O(\mathscr{C})$, the unit map $\mathbb{S} \to \mathscr{C}(c, c)$ is a cofibration of spectra, and for every pair of distinct objects c and d in $O(\mathscr{C})$, the mapping spectrum $\mathscr{C}(c, d)$ is cofibrant as a spectrum.

Pointwise relatively cofibrant small spectral categories are in particular pointwise semicofibrant [15, Proposition 1.3(c)]. The following proposition produces sufficient examples of such small spectral categories. We made the following observation in [7, Propositions 2.6 and 2.7] based on the earlier work of Schwede–Shipley [26, Proposition 6.3], extending functoriality in strict morphisms to functoriality in arbitrary spectral functors. Although stated there in the context of symmetric spectra of simplicial sets, the same arguments prove it for the other modern categories of spectra.

Proposition 4.6 *Let $\mathscr{C}$ be a small spectral category. There are functorial spectral categories $\mathscr{C}^{\mathrm{Cell}}$ and $\mathscr{C}^{\mathrm{Cell},\Omega}$ and natural DK–equivalences that are isomorphisms on object sets (or, equivalently, strict morphisms that are pointwise weak equivalences)*

$$\mathscr{C} \leftarrow \mathscr{C}^{\mathrm{Cell}} \rightarrow \mathscr{C}^{\mathrm{Cell},\Omega}$$

such that $\mathscr{C}^{\mathrm{Cell}}$ is pointwise relatively cofibrant and $\mathscr{C}^{\mathrm{Cell},\Omega}$ is pointwise relatively cofibrant and pointwise fibrant. Moreover, if $\mathscr{C}$ is pointwise fibrant, then so is $\mathscr{C}^{\mathrm{Cell}}$.

Next we move on to derived functors. We concentrate on the case of

$$\mathrm{Hom}^\ell_{\mathscr{X}}(-,-)\colon (\mathscr{O}\mathrm{bj}^\ell_{\mathscr{X}})^{\mathrm{op}} \times \mathscr{O}\mathrm{bj}^\ell_{\mathscr{X}} \to \mathscr{BS},$$

which takes a pair of left $\mathscr{X}$–objects to a bi-indexed spectrum, where $\mathscr{X}$ is an arbitrary bi-indexed ring spectrum. The discussion for Hom^r has an exact parallel for Hom^ℓ, switching left/right and source/target, with all corresponding results holding. In essence, Section 5 of [15] discusses this kind of derived functor, although the story here is complicated because bi-indexed spectra and categories of left objects are not enriched over spectra, but are only partially enriched: once we fix a source set A, the full subcategory of left $\mathscr{X}$–objects with source set A is isomorphic to the category of $(\mathscr{X}, \mathbb{S}_A)$–bimodules and then is enriched, while there are no maps between left $\mathscr{X}$–objects with different source sets. Therefore, constructing "partially enriched" derived functors of two variables for the entire category of left $\mathscr{X}$–objects is equivalent to constructing enriched derived functors of two variables on each pair of these categories of bimodules. Applying [15, Theorem 5.8], we have the following result.

Theorem 4.7 *Let $\mathscr{X}$ be a bi-indexed ring spectrum. For the functor*

$$\mathrm{Hom}^\ell_{\mathscr{X}}(-,-)\colon (\mathscr{O}\mathrm{bj}^\ell_{\mathscr{X}})^{\mathrm{op}} \times \mathscr{O}\mathrm{bj}^\ell_{\mathscr{X}} \to \mathscr{BS},$$

the partially enriched right derived functor $\mathbb{R}\,\mathrm{Hom}^\ell_{\mathscr{X}}(-,-)$ exists and is constructed by cofibrant replacement of the first variable and fibrant replacement of the second variable.

Proof As indicated above, we restrict the source set to A for the first variable (the contravariant variable) and the source set to B for the second variable (the covariant variable) to apply [15, Theorem 5.8] directly. In the statement of [15, Theorem 5.8], to reach this conclusion, we need to observe (1) that $\mathrm{Hom}^\ell_{\mathscr{X}}$ fits into an enriched parametrized adjunction, (2) that each adjunction of one variable is a Quillen adjunction when the parametrizing variable is cofibrant, and (3) that the left adjoint preserves

weak equivalences between cofibrant objects in the parametrizing variable when the adjunction variable is cofibrant (or, equivalently, the analogous condition for fibrant objects on the right adjoint). In this case, the enriched parametrized left adjoint is given by the functor $\otimes$ that takes a left $\mathscr{X}$–object with source set A and an (A, B)–spectrum to a left $\mathscr{X}$–object with source set B. (Here the (A, B)–spectrum is the adjunction variable and the left $\mathscr{X}$–object with source set A is the parametrizing variable.) From the explicit description of cofibrations, it is clear that $\otimes$ preserves cofibrations in each variable when the other is cofibrant. Since smash product of spectra preserves acyclic cofibrations of spectra and the smash product with a cofibrant spectrum preserves arbitrary weak equivalences, it is clear from the formula for $\otimes$ that it preserves acyclic cofibrations in each variable when the other variable is cofibrant. This then verifies the hypotheses of [15, Theorem 5.8]. $\qquad\square$

When $\mathscr{X} \to \mathscr{X}'$ is a map of bi-indexed ring spectra, we obtain a canonical forgetful or pullback functor from left $\mathscr{X}'$–objects to left $\mathscr{X}$–objects, which induces a natural transformation $\mathrm{Hom}^{\ell}_{\mathscr{X}'} \to \mathrm{Hom}^{\ell}_{\mathscr{X}}$ and a natural transformation of derived functors $\mathbb{R}\,\mathrm{Hom}^{\ell}_{\mathscr{X}'} \to \mathbb{R}\,\mathrm{Hom}^{\ell}_{\mathscr{X}}$. The argument for [15, Theorem 8.3] then implies the following result.

Proposition 4.8 *If $\mathscr{X} \to \mathscr{X}'$ is a weak equivalence of bi-indexed ring spectra, then the forgetful functor from left $\mathscr{X}'$–objects to left $\mathscr{X}$–objects is the right adjoint of a Quillen equivalence and the natural map $\mathbb{R}\,\mathrm{Hom}^{\ell}_{\mathscr{X}'} \to \mathbb{R}\,\mathrm{Hom}^{\ell}_{\mathscr{X}}$ is a natural isomorphism in the homotopy category of bi-indexed spectra.*

We prefer to phrase the centralizer conditions in the next section in terms of small spectral categories and bimodules over small spectral categories. One technical wrinkle that arises (and indeed is the main issue studied by [15] as a whole) is that when we plug bimodules into $\mathrm{Hom}^{\ell}_{\mathscr{X}}$ and consider functors of the form

$$(4.9) \qquad \mathrm{Hom}^{\ell}_{\mathscr{X}} \colon (\mathscr{M}\mathrm{od}_{\mathscr{X},\mathscr{Y}})^{\mathrm{op}} \times \mathscr{M}\mathrm{od}_{\mathscr{X},\mathscr{Z}} \to \mathscr{M}\mathrm{od}_{\mathscr{Y},\mathscr{Z}},$$

even when the enriched right derived functor exists, it may not agree with the derived functor $\mathbb{R}\,\mathrm{Hom}^{\ell}_{\mathscr{X}}$ of Theorem 4.7 without hypotheses on $\mathscr{Y}$. We return to this question below.

The technical issue just mentioned causes some awkwardness in trying to state a version of Theorem 4.7 for small spectral categories. We dealt with this in the introduction by phrasing the centralizer conditions in terms of homotopical bimodules, which are defined as follows. By neglect of structure, small spectral categories become

small categories enriched over the stable category; in the definition below $\mathscr{D}^{\mathrm{op}} \wedge^{\mathbb{L}} \mathscr{C}$ denotes the small category enriched over the stable category that is defined analogously to $\mathscr{D}^{\mathrm{op}} \wedge \mathscr{C}$ in Definition 2.5, but using the smash product in the stable category.

Definition 4.10 A homotopical $(\mathscr{C}, \mathscr{D})$–bimodule is an enriched functor from $\mathscr{D}^{\mathrm{op}} \wedge^{\mathbb{L}} \mathscr{C}$ to the stable category. More generally, for a category $\mathscr{C}$at (partially) enriched over spectra or over the stable category, homotopical left $\mathscr{C}$–modules in $\mathscr{C}$at, homotopical right $\mathscr{D}$–modules in $\mathscr{C}$at, and homotopical $(\mathscr{C}, \mathscr{D})$–bimodules in $\mathscr{C}$at are functors enriched over the stable category from $\mathscr{C}$, $\mathscr{D}^{\mathrm{op}}$, and $\mathscr{D}^{\mathrm{op}} \wedge^{\mathbb{L}} \mathscr{C}$ into $\mathscr{C}$at, respectively.

When $\mathscr{M}$ is a $(\mathscr{C}, \mathscr{D})$–bimodule, by neglect of structure it is a homotopical right $\mathscr{D}$–module in the homotopy category of left $\mathscr{C}$–objects and any cofibrant approximation in the category of left $\mathscr{C}$–objects inherits the canonical structure of a homotopical right $\mathscr{D}$–module. Similar observations apply to the fibrant approximation of a $(\mathscr{C}, \mathscr{E})$–bimodule, giving

$$(4.11) \qquad \mathbb{R} \operatorname{Hom}^{\ell}_{\mathscr{C}}(\mathscr{M}, \mathscr{N}) = \mathbb{R} \operatorname{Hom}^{\ell}_{B_{\mathscr{C}}}(B_{\mathscr{M}}, B_{\mathscr{N}})$$

the canonical structure of a homotopical $(\mathscr{D}, \mathscr{E})$–bimodule, for $\mathbb{R} \operatorname{Hom}^{\ell}_{B_{\mathscr{C}}}$ the right derived functor in Theorem 4.7.

We now return to the question of when the right derived functor of (4.9) exists and is compatible with the right derived functor in Theorem 4.7. Although written in the context of symmetric monoidal categories, Theorems 1.7(a) and 1.11(a) of [15] show that both of these hold when the underlying bi-indexed spectrum of $\mathscr{Y}$ is pointwise semicofibrant. In our context, the following gives the most convenient statement; as always, the analogous result for $\operatorname{Hom}^{r}_{\mathscr{D}}$ also holds.

Theorem 4.12 Let $\mathscr{C}$ and $\mathscr{D}$ be small spectral categories and assume that $\mathscr{D}$ is pointwise semicofibrant.

(i) The forgetful functor from $(\mathscr{C}, \mathscr{D})$–bimodules to left $\mathscr{C}$–objects preserves cofibrations (and all weak equivalences).

(ii) The enriched right derived functor of

$$\operatorname{Hom}^{\ell}_{\mathscr{C}} \colon (\mathscr{M}\mathrm{od}_{\mathscr{C}, \mathscr{D}})^{\mathrm{op}} \times \mathscr{M}\mathrm{od}_{\mathscr{C}, \mathscr{D}} \to \mathscr{M}\mathrm{od}_{\mathscr{D}, \mathscr{D}}$$

exists and is constructed by cofibrant replacement of the contravariant variable and fibrant replacement of the covariant variable.

(iii) Moreover, the underlying functor to homotopical $(\mathscr{D}, \mathscr{D})$–bimodules of the right derived functor of (ii) agrees with the derived functor of (4.11).

Proof The explicit description of the cofibrations shows that when $\mathscr{D}$ is pointwise semi-cofibrant, a cofibration of $(\mathscr{C}, \mathscr{D})$–bimodules forgets to a cofibration of left $\mathscr{C}$–objects. From here it is straightforward to check the conditions of [15, Theorem 5.4] that ensure the existence of the enriched right derived functor, and the comparison with the derived functor of (4.11) is immediate. $\square$

5 Centralizer conditions, maps of CC, and the proof of the main theorem

In this section we begin the process of extending the functoriality of CC by constructing zigzags associated to bimodules that satisfy centralizer conditions that we review below. We do enough work that we can prove the main theorem of the introduction, Theorem A, which gives an equivalence of E_2 ring spectra for the two THC constructions commonly studied in string topology. We also prove Theorems D and E.

We begin with the centralizer conditions.

Definition 5.1 Let $\mathscr{C}$ and $\mathscr{D}$ be small spectral categories and let $\mathscr{F}$ be a $(\mathscr{C}, \mathscr{D})$–bimodule. The *centralizer map for $\mathscr{D}$* is the map in the category of homotopical $\mathscr{D}$–modules

$$\mathscr{D} \to \mathbb{R}\operatorname{Hom}^{\ell}_{\mathscr{C}}(\mathscr{F}, \mathscr{F})$$

adjoint to the map $\mathscr{F} \otimes \mathscr{D} \to \mathscr{F}$, where $\mathbb{R}\operatorname{Hom}^{\ell}_{\mathscr{C}}$ is as in (4.11) (that is, the right derived functor in Theorem 4.7). The *centralizer map for $\mathscr{C}$* is the analogous map

$$\mathscr{C} \to \mathbb{R}\operatorname{Hom}^{r}_{\mathscr{D}}(\mathscr{F}, \mathscr{F}).$$

We say that:

(i) $\mathscr{F}$ satisfies the *double centralizer condition* when both centralizer maps are weak equivalences.

(ii) $\mathscr{F}$ satisfies the *single centralizer condition* for $\mathscr{C}$ or $\mathscr{D}$ when the centralizer map for $\mathscr{C}$ or $\mathscr{D}$ (resp.) is a weak equivalence.

We have the following motivating examples.

Example 5.2 (DK–embeddings) If $\phi \colon \mathscr{D} \to \mathscr{C}$ is a spectral functor and $\mathscr{F}$ is the bimodule $\mathscr{F}_{\phi} = \mathscr{C}(\phi(-), -)$ then the enriched form of the Yoneda lemma shows that $\operatorname{Hom}^{\ell}_{\mathscr{C}}(\mathscr{F}, \mathscr{F})$ is canonically isomorphic to $\mathscr{C}(\phi(-), \phi(-))$ and the centralizer

map $\mathscr{D} \to \operatorname{Hom}^{\ell}_{\mathscr{C}}(\mathscr{F}, \mathscr{F})$ is the map $\phi \colon \mathscr{D}(-, -) \to \mathscr{C}(\phi(-), \phi(-))$; moreover, $\mathscr{C}(\phi(-), \phi(-))$ also represents the derived functor $\mathbb{R}\operatorname{Hom}^{\ell}_{\mathscr{C}}(\mathscr{F}, \mathscr{F})$. It follows that $\mathscr{F}$ satisfies the single centralizer condition for $\mathscr{D}$ if and only if ϕ is a DK–embedding. Moreover, if ϕ is a DK–equivalence then the enriched Yoneda lemma in the homotopy category shows that $\mathscr{C} \to \mathbb{R}\operatorname{Hom}^{r}_{\mathscr{D}}(\mathscr{F}, \mathscr{F})$ is a weak equivalence and $\mathscr{F}$ satisfies the double centralizer condition.

Example 5.3 ($\mathscr{D}$ and $\operatorname{Perf}(\mathscr{D})$) Let $\mathscr{D}$ be a pointwise fibrant small spectral category, and $\mathscr{C}$ be a small full spectral subcategory of the category of right $\mathscr{D}$–modules consisting of only cofibrant-fibrant objects. Assume the Yoneda embedding factors $\phi \colon \mathscr{D} \to \mathscr{C}$, and let $\mathscr{F} = \mathscr{F}_{\phi}$. For example, $\mathscr{C} = \operatorname{Perf}(\mathscr{D})$ (for any large enough cardinality) fits into this context. Then the bimodule $\mathscr{F}$ satisfies the double centralizer condition. Since ϕ is a DK–embedding, as per the previous example, $\mathscr{F}$ satisfies the single centralizer condition for $\mathscr{D}$. To see that the centralizer map for $\mathscr{C}$ is a weak equivalence, we consider the map

$$\mathscr{C}(x, y) \to \mathbb{R}\operatorname{Hom}^{r}_{\mathscr{D}}(\mathscr{F}(-, x), \mathscr{F}(-, y))$$

for fixed x and y. Recalling that x and y are $\mathscr{D}$–modules, the enriched Yoneda lemma gives isomorphisms $x(d) \cong \mathscr{F}(d, x)$ and $y(d) \cong \mathscr{F}(d, y)$ for all d in $\mathscr{D}$, and hence an isomorphism

$$\operatorname{Hom}^{r}_{\mathscr{D}}(\mathscr{F}(-, x), \mathscr{F}(-, y)) \cong \operatorname{Hom}^{r}_{\mathscr{D}}(x(-), y(-)) \cong \mathscr{M}\mathrm{od}_{\mathscr{D}^{\mathrm{op}}}(x, y) = \mathscr{C}(x, y)$$

(see Proposition 2.9). Since we have assumed that x and y are cofibrant-fibrant right $\mathscr{D}$–modules, the point-set functor represents the right derived functor, and we see that $\mathscr{F}$ also satisfies the single centralizer condition for $\mathscr{C}$.

Example 5.4 (Morita contexts) Let $\mathscr{M}$ be a cofibrant $(\mathscr{C}, \mathscr{D})$–bimodule (and we assume without loss of generality that $\mathscr{C}$ and $\mathscr{D}$ are pointwise semicofibrant). Then the left derived functor of $\mathscr{M} \otimes_{\mathscr{D}} (-)$ from the derived category of $\mathscr{D}$–modules to the derived category of $\mathscr{C}$–modules is an equivalence of homotopy categories if and only if $\mathscr{M} \otimes_{\mathscr{D}} (-)$ restricts to a DK–equivalence $\operatorname{Perf}(\mathscr{D}) \to \operatorname{Perf}(\mathscr{C})$ (for models of large enough cardinality). When this holds, the derived functor of the right adjoint $\operatorname{Hom}^{\ell}_{\mathscr{C}}(\mathscr{M}, -)$ induces the inverse equivalence and represents the right derived functor $\mathbb{R}\operatorname{Hom}^{\ell}_{\mathscr{C}}(\mathscr{M}, -)$ in Definition 5.1. In particular, the unit of the derived adjunction for $\mathscr{D}$ is the centralizer map $\mathscr{D} \to \mathbb{R}\operatorname{Hom}^{\ell}_{\mathscr{C}}(\mathscr{M}, \mathscr{M})$ and so $\mathscr{M}$ satisfies the single centralizer condition for $\mathscr{D}$. Although written in the context of associative ring

spectra, the proof of Theorem 4.1.2 of [27] implies that there exists a cofibrant $(\mathscr{D}, \mathscr{C})$–bimodule $\mathscr{N}$ such that $\mathscr{M} \otimes_{\mathscr{D}} \mathscr{N}$ is weakly equivalent to $\mathscr{C}$ as a $(\mathscr{C}, \mathscr{C})$–bimodule and $\mathscr{N} \otimes_{\mathscr{C}} \mathscr{N}$ is weakly equivalent to $\mathscr{D}$ as a $(\mathscr{D}, \mathscr{D})$–bimodule. It then follows that the left derived functor of $(-) \otimes_{\mathscr{C}} \mathscr{M}$ from the derived category of right $\mathscr{C}$–modules to the derived category of right $\mathscr{D}$–modules is an equivalence of categories, which implies that $\mathscr{M}$ satisfies the single centralizer condition for $\mathscr{C}$. Thus, $\mathscr{M}$ satisfies the double centralizer condition.

Example 5.5 (DX and ΩX) Let X be a simply connected finite cell complex, or equivalently (up to homotopy) the geometric realization of a reduced finite simplicial set. In [6, Section 3], we consider the Kan loop group model GX for ΩX and describe an explicit $(\Sigma^{\infty}_{+} GX, DX)$–bimodule SP (whose underlying spectrum is equivalent to $\mathbb{S}$) that we show satisfies the double centralizer condition. (This example is originally due to Dwyer–Greenlees–Iyengar [9, Section 4.22], at least after extension of scalars to a field.)

To construct the zigzag, we use the following construction of Keller [12, Section 4.5].

Construction 5.6 Let $\mathscr{C}$ and $\mathscr{D}$ be small spectral categories and $\mathscr{F}$ a $(\mathscr{C}, \mathscr{D})$–bimodule. Let $\mathscr{C}\mathrm{at}_{\mathscr{F}}$ be the small spectral category with objects $O(\mathscr{C}) \amalg O(\mathscr{D})$, mapping spectra

$$\mathscr{C}\mathrm{at}_{\mathscr{F}}(a, b) = \begin{cases} \mathscr{C}(a, b) & \text{for } a, b \in O(\mathscr{C}), \\ * & \text{for } a \in O(\mathscr{C}) \text{ and } b \in O(\mathscr{D}), \\ \mathscr{F}(a, b) & \text{for } a \in O(\mathscr{D}) \text{ and } b \in O(\mathscr{C}), \\ \mathscr{D}(a, b) & \text{for } a, b \in O(\mathscr{D}) \end{cases}$$

with units coming from the units of $\mathscr{C}$ and $\mathscr{D}$, and composition coming from the composition in $\mathscr{C}$ and $\mathscr{D}$ and the bimodule structure of $\mathscr{F}$.

The construction comes with canonical strictly fully faithful spectral functors $\mathscr{C} \to \mathscr{C}\mathrm{at}_{\mathscr{F}}$ and $\mathscr{D} \to \mathscr{C}\mathrm{at}_{\mathscr{F}}$, which by Proposition 3.4 induce maps of $\mathcal{D}_2$–algebras

$$\mathrm{CC}(\mathscr{D}) \leftarrow \mathrm{CC}(\mathscr{C}\mathrm{at}_{\mathscr{F}}) \to \mathrm{CC}(\mathscr{C}).$$

The following theorem ties in the double centralizer condition. We prove it in Section 7.

Theorem 5.7 *Assume $\mathscr{C}$ and $\mathscr{D}$ are pointwise relatively cofibrant and pointwise fibrant small spectral categories and let $\mathscr{F}$ be a pointwise semicofibrant-fibrant $(\mathscr{C}, \mathscr{D})$–bimodule. Then:*

(i)   If $\mathscr{F}$ satisfies the single centralizer condition for $\mathscr{D}$, then the map $\mathrm{CC}(\mathscr{C}\mathrm{at}_{\mathscr{F}}) \to$ $\mathrm{CC}(\mathscr{C})$ is a weak equivalence.

(ii) If $\mathscr{F}$ satisfies the single centralizer condition for $\mathscr{C}$, then the map $\mathrm{CC}(\mathscr{C}\mathrm{at}_{\mathscr{F}}) \to$ $\mathrm{CC}(\mathscr{D})$ is a weak equivalence.

If we take for granted that a functor THC exists as in Theorem B, then the previous theorem combined with the examples above gives just what we need to prove Theorems A, D, and E.

Proof of Theorem A   As per the statement of Theorem B, for any associative ring spectrum A, $\mathrm{THC}(A)$ may be constructed as $\mathrm{CC}(A')$ for an associative ring spectrum A' whose underlying spectrum is fibrant and for which the inclusion of the unit $\mathbb{S} \to A$ is a cofibration of spectra (eg applying cofibrant and fibrant replacement functors in the category of associative ring spectra). Indeed, in all previous literature discussing $\mathrm{THC}(DX)$ and $\mathrm{THC}(\Sigma^{\infty}_{+}\Omega X)$, this was always done tacitly. Using such a model DX' for DX and R for $\Sigma^{\infty}_{+}\Omega X$ (or GX as in Example 5.5), we have a cofibrant bimodule SP satisfying the double centralizer condition, as in the example. The required chain of weak equivalences of E_2 ring spectra is then given by the zigzag

$$\mathrm{CC}(DX') \leftarrow \mathrm{CC}(\mathscr{C}\mathrm{at}_{\mathrm{SP}}) \to \mathrm{CC}(R)$$

of weak equivalences of $\mathcal{D}_2$–algebras. $\qquad\square$

Proof of Theorems D and E   By Examples 5.2 and 5.3, Theorem D is a special case of Theorem E. The proof of Theorem E is identical to the special case given by Theorem A: apply both parts of Theorem 5.7 to appropriate pointwise relatively cofibrant-fibrant replacements as in Proposition 4.6. $\qquad\square$

6   The construction of THC (proof of Theorems B and C)

The purpose of this section is to construct topological Hochschild cohomology as a homotopical functor. We begin by constructing THC as a functor on the homotopy category level from a subcategory of the homotopy category of small spectral categories to the homotopy category of E_2 ring spectra. Using work of Lindsey [16], we then show that essentially the same argument actually constructs THC as a functor from a subcategory of the $(\infty, 1)$–category $\mathscr{C}\mathrm{at}^{\mathrm{ex}}$ of small stable idempotent-complete $(\infty, 1)$–categories to the $(\infty, 1)$–category of E_2 ring spectra. Throughout, we work

with quasicategories as a model for $(\infty, 1)$–categories and rely on the foundational setup of Joyal and Lurie [17; 18].

Definition 6.1 For a small spectral category $\mathscr{C}$, let

$$\mathrm{THC}(\mathscr{C}) = \mathrm{CC}(\mathscr{C}^{\mathrm{Cell},\Omega}),$$

where $\mathscr{C}^{\mathrm{Cell},\Omega}$ is the functorial pointwise relatively cofibrant-fibrant replacement of Proposition 4.6.

Given a DK–embedding $\phi \colon \mathscr{D} \to \mathscr{C}$, by functoriality we get a DK–embedding

$$\widetilde{\phi} \colon \mathscr{D}^{\mathrm{Cell},\Omega} \to \mathscr{C}^{\mathrm{Cell},\Omega}$$

and the bimodule $\mathscr{F}_{\widetilde{\phi}}$ representing this functor (see Definition 1.9 and Theorem 2.3) satisfies the single centralizer condition for $\mathscr{C}^{\mathrm{Cell},\Omega}$ (see Example 5.2). Writing $\mathscr{C}\mathrm{at}_{\widetilde{\phi}}$ as an abbreviation for $\mathscr{C}\mathrm{at}_{\mathscr{F}_{\widetilde{\phi}}}$, Theorem 5.7 then gives us a zigzag of maps of $\mathcal{D}_2$–algebras

(6.2) $\mathrm{CC}(\mathscr{C}^{\mathrm{Cell},\Omega}) \xleftarrow{\simeq} \mathrm{CC}(\mathscr{C}\mathrm{at}_{\widetilde{\phi}}) \to \mathrm{CC}(\mathscr{D}),$

which we interpret as a map in the homotopy category of E_2 ring spectra,

$$\mathrm{THC}(\mathscr{C}) \to \mathrm{THC}(\mathscr{D}).$$

This gives the next step in the construction of THC as a functor, the definition on maps.

Definition 6.3 For a DK–embedding $\phi \colon \mathscr{D} \to \mathscr{C}$, define $\mathrm{THC}(\phi)$ to be the map $\mathrm{THC}(\mathscr{C}) \to \mathrm{THC}(\mathscr{D})$ in the homotopy category of E_2–ring spectra arising from the zigzag of (6.2).

To check that this definition respects composition and unit maps, we use the following construction.

Definition 6.4 Let $\phi_1 \colon \mathscr{C}_0 \to \mathscr{C}_1, \ldots, \phi_n \colon \mathscr{C}_{n-1} \to \mathscr{C}_n$ be a composable sequence of spectral functors. Define $\mathscr{C}\mathrm{at}_{\phi_1,\ldots,\phi_n}$ to be the small spectral category with objects the disjoint union of the objects of $\mathscr{C}_i$ for all i and with mapping spectra

$$\mathscr{C}\mathrm{at}_{\phi_1,\ldots,\phi_n}(a, b) = \begin{cases} \mathscr{C}_j(\phi_{i,j}(a), b) & \text{if } i \leq j, \\ * & \text{if } i > j \end{cases}$$

for $a \in O(\mathscr{C}_i)$ and $b \in O(\mathscr{C}_j)$, where $\phi_{i,j} = \mathrm{id}$ if $i = j$ and $\phi_{i,j} = \phi_{j-1} \circ \cdots \circ \phi_i$ for $i < j$. Composition is induced by composition in $\mathscr{C}_0, \ldots, \mathscr{C}_n$ and the functors ϕ_i, and units come from the units in $\mathscr{C}_0, \ldots, \mathscr{C}_n$.

We note that for a single morphism, $\mathscr{C}\mathrm{at}_\phi$ is $\mathscr{C}\mathrm{at}_{\mathscr{F}_\phi}$ for the bimodule $\mathscr{F}_\phi$ associated to ϕ, which is consistent with the notation we used in (6.2). We deduce from Theorem 5.7 the following corollary.

Corollary 6.5 *With notation as in Definition 6.4, assume each $\mathscr{C}_i$ is pointwise relatively cofibrant-fibrant and that ϕ_1 is a DK–embedding. Then the inclusion of $\mathscr{C}_0$ in $\mathscr{C}\mathrm{at}_{\phi_1,\dots,\phi_n}$ induces a weak equivalence $\mathrm{CC}(\mathscr{C}\mathrm{at}_{\phi_1,\dots,\phi_n}) \to \mathrm{CC}(\mathscr{C}_0)$.*

Proof Let $\psi\colon \mathscr{C}_0 \to \mathscr{C}\mathrm{at}_{\phi_2,\dots,\phi_n}$ be the composite of ϕ_1 with the inclusion of $\mathscr{C}_1$ in $\mathscr{C}\mathrm{at}_{\phi_2,\dots,\phi_n}$. We then have a canonical isomorphism of small spectral categories from $\mathscr{C}\mathrm{at}_\psi$ to $\mathscr{C}\mathrm{at}_{\phi_1,\dots,\phi_n}$. Since ϕ_1 is a DK–embedding, so is ψ, and Theorem 5.7 implies that the induced map $\mathrm{CC}(\mathscr{C}\mathrm{at}_\psi) \to \mathrm{CC}(\mathscr{C}_0)$ is a weak equivalence. □

We can now prove Theorems B and C.

Proof of Theorems B and C The proofs of the two theorems are essentially the same; for the proof of Theorem B, simply restrict to the subcategory of small spectral categories consisting of the associative ring spectra. (Note that even in the case of Theorem B, the argument still requires use of CC of small spectral categories, namely, the small spectral categories $\mathscr{C}\mathrm{at}_{\mathscr{F}_\phi}$.)

We have defined THC on objects and morphisms in Definitions 6.1 and 6.3; we need to show that THC preserves composition and units. Given $\phi_1\colon \mathscr{C}_0 \to \mathscr{C}_1$ and $\phi_2\colon \mathscr{C}_1 \to \mathscr{C}_2$, let $\tilde{\phi}_1$ and $\tilde{\phi}_2$ denote the induced functors on $\mathscr{C}_i^{\mathrm{Cell},\Omega}$. We then have the strictly commuting diagram of strictly fully faithful morphisms
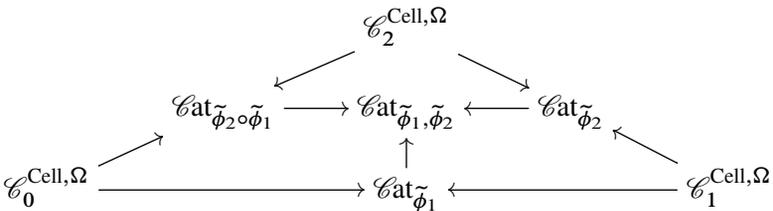
from which we get the commutative diagram of $\mathcal{D}_2$–algebras
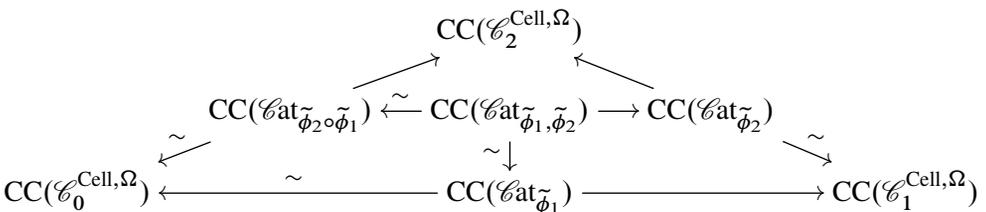
The arrows marked with "$\sim$" are weak equivalences by Theorem 5.7, Corollary 6.5, and the 2-out-of-3 property. Since $\text{THC}(\phi_1)$, $\text{THC}(\phi_2)$, and $\text{THC}(\phi_2 \circ \phi_1)$ are defined by the outer zigzags in the diagram above, we see that

$$\text{THC}(\phi_2 \circ \phi_1) = \text{THC}(\phi_1) \circ \text{THC}(\phi_2).$$

Although $\text{THC}(\text{id}_{\mathscr{C}})$ is not defined to be the identity map, part (ii) of Theorem 5.7 shows that $\text{THC}(\text{id}_{\mathscr{C}})$ is an isomorphism (in the homotopy category), which together with the fact just shown that $\text{THC}(\text{id}_{\mathscr{C}}) = \text{THC}(\text{id}_{\mathscr{C}}) \circ \text{THC}(\text{id}_{\mathscr{C}})$, proves that $\text{THC}(\text{id}_{\mathscr{C}})$ is the identity map for any small spectral category $\mathscr{C}$. □

Finally, we prove Theorem F by explaining how to refine THC into an functor of ∞–categories. For the source, for simplicity, we take the nerve of the category of small spectral categories and DK–embeddings, $\mathscr{SC}\text{at}^{\text{DK}}$; the functor will take DK–equivalences (and indeed Morita equivalences) to equivalences in the target, and so one can from there factor through an ∞–categorical Bousfield localization. For the target category, we will use the homotopy coherent nerve of a pointwise fibrant replacement of the Dwyer–Kan hammock localization of the category of $\mathcal{D}_2$–algebras, $N^{\text{hc}}L\mathscr{S}[\mathcal{D}_2]$. We do not get a point-set map of quasicategories, however, because although our construction above takes morphisms of small spectral categories to zigzags of $\mathcal{D}_2$–algebras, which are honest morphisms in the hammock localization, it does not preserve composition strictly. If we think in terms of zigzags in the original category of $\mathcal{D}_2$–algebras, the construction $\text{CC}(\mathscr{C}\text{at}_{\phi_1,\dots,\phi_n})$ gives n–simplex zigzags associated to a sequence of composable morphisms. Zachery Lindsey studied this kind of ∞–functoriality in his 2018 Indiana University thesis [16]; in the notation there, we construct a map

$$N(\mathscr{SC}\text{at}^{\text{DK}}) \to \text{Zig}(N^{\text{hc}}L\mathscr{S}[\mathcal{D}_2], N^{\text{hc}}L\mathscr{S}[\mathcal{D}_2]^{\simeq})$$

as follows:

(0) A 0–simplex of $N(\mathscr{SC}\text{at}^{\text{DK}})$ is a small spectral category $\mathscr{C}$, and it maps to $\text{CC}(\mathscr{C}^{\text{Cell},\Omega})$.

(1) A 1–simplex of $N(\mathscr{SC}\text{at}^{\text{DK}})$ is a DK–embedding $\phi\colon \mathscr{C}_0 \to \mathscr{C}_1$, and it maps to the zigzag (6.2).

(2) An n–simplex of $N(\mathscr{SC}\text{at}^{\text{DK}})$ consists, in general, of n composable DK–embeddings $\phi_i\colon \mathscr{C}_{i-1} \to \mathscr{C}_i$, and it maps to the n–simplex zigzag for $\mathscr{C}\text{at}_{\phi_1,\dots,\phi_n}$ generalizing the 2–simplex zigzag pictured in the proof of Theorem C.

Although both the source and target simplicial sets are quasicategories, this is not a map of quasicategories because it only preserves face maps and not degeneracy maps. The work of Steimle [29] (see Theorems 1.2 and 1.4) allows us to correct this to construct a functor from $N(\mathscr{SC}\mathrm{at}^{\mathrm{DK}})$ to $\mathrm{Zig}(N^{\mathrm{hc}}L\mathscr{S}[\mathcal{D}_2], N^{\mathrm{hc}}L\mathscr{S}[\mathcal{D}_2]^{\simeq})$.

Lindsey [16] shows that the inclusion of a quasicategory $\mathscr{Q}$ in the quasicategory $\mathrm{Zig}(\mathscr{Q}, \mathscr{Q}^{\simeq})$ is a categorical equivalence. This then proves the following theorem.

Theorem 6.6 *The preceding construction constructs a zigzag of maps of quasicategories from $N\mathscr{SC}\mathrm{at}^{\mathrm{DK}}$ to $N^{\mathrm{hc}}L\mathscr{S}[\mathcal{D}_2]$, providing a functor* THC *from the category of small spectral categories and DK–embeddings to the category of E_2 ring spectra that sends Morita equivalences to weak equivalences.*

Since THC sends Morita equivalences to weak equivalences, it factors through the Bousfield localization of $N\mathscr{SC}\mathrm{at}^{\mathrm{DK}}$ at the Morita equivalences; using the equivalence of [5, Theorem 4.23] between the localization of $N\mathscr{SC}\mathrm{at}$ at the Morita equivalences and $\mathscr{C}\mathrm{at}^{\mathrm{ex}}$ then proves Theorem F.

7 Proof of Theorem 5.7

This section is devoted to the proof of Theorem 5.7. The basic idea is to compare $\mathrm{CC}(\mathscr{C}\mathrm{at}_{\mathscr{F}})$ to a construction of the form $\mathrm{Hom}^b_{\mathscr{C},\mathscr{D}}(\mathscr{R}, \mathscr{F})$, where $\mathscr{R}$ is a certain simplicial object resolving $\mathscr{F}$. We start with the following simplicial construction.

Construction 7.1 Let $\mathscr{C}$ and $\mathscr{D}$ be small spectral categories and let $\mathscr{G}$ be a $(\mathscr{C}, \mathscr{D})$–bimodule. The simplicial $(\mathscr{C}, \mathscr{D})$–bimodule $\mathscr{R}_\bullet(\mathscr{C}; \mathscr{G}; \mathscr{D})$ is defined by

$$\mathscr{R}_n(\mathscr{C}; \mathscr{G}; \mathscr{D}) = \bigvee_{j=0,\ldots,n+1} \underbrace{\mathscr{C} \otimes \cdots \otimes \mathscr{C}}_{j \text{ factors}} \otimes \mathscr{G} \otimes \underbrace{\mathscr{D} \otimes \cdots \otimes \mathscr{D}}_{n+1-j \text{ factors}}$$

(a total of $n+2$ summands each with $n+2$ factors), where the face map d_i multiplies the ith and $(i+1)^{\mathrm{st}}$ factors using the multiplication of $\mathscr{C}$ or $\mathscr{D}$ or action on $\mathscr{G}$ and the degeneracy map s_i is induced by the map $\mathbb{S}_{O(\mathscr{C})} \to \mathscr{C}$ in the $(i+1)^{\mathrm{st}}$ factor on the jth summand for $i < j$ and induced by the map $\mathbb{S}_{O(\mathscr{D})} \to \mathscr{D}$ in the $(i+1)^{\mathrm{st}}$ factor on the jth summand for $i \geq j$. We write $\mathscr{R}_\bullet(\mathscr{G})$ when $\mathscr{C}$ and $\mathscr{D}$ are clear, and we write $\mathscr{R}(\mathscr{G})$ for the geometric realization.

For example, $\mathscr{R}_0(\mathscr{G}) = \mathscr{G} \otimes \mathscr{D} \vee \mathscr{C} \otimes \mathscr{G}$ and the degeneracy map s_0 is

$$\mathscr{G} \otimes \mathscr{D} \cong \mathscr{G} \otimes \mathbb{S}_{O(\mathscr{D})} \otimes \mathscr{D} \to \mathscr{G} \otimes \mathscr{D} \otimes \mathscr{D}$$

on the 0th summand and

$$\mathscr{C} \otimes \mathscr{G} \cong \mathscr{C} \otimes \mathbb{S}_{O(\mathscr{C})} \otimes \mathscr{G} \to \mathscr{C} \otimes \mathscr{C} \otimes \mathscr{G}$$

on the 1st summand.

We have an augmentation map of $(\mathscr{C}, \mathscr{D})$–bimodules $\epsilon: \mathscr{R}_\bullet(\mathscr{G}) \to \mathscr{G}$ induced by multiplying all the $\mathscr{C}$ and $\mathscr{D}$ factors through.

Proposition 7.2 *The augmentation $\epsilon: \mathscr{R}_\bullet(\mathscr{G}) \to \mathscr{G}$ is a homotopy equivalence of simplicial bi-indexed spectra.*

Proof In the category of bi-indexed spectra, the simplicial object $\mathscr{R}_\bullet(\mathscr{G})$ has an "extra degeneracy" in the sense of [25, Section 4.5]: Define $s_{-1}: \mathscr{R}_n(\mathscr{G}) \to \mathscr{R}_{n+1}\mathscr{G}$ to be the map

$$\mathscr{R}_n(\mathscr{G}) \cong \mathbb{S}_{O(\mathscr{C})} \otimes \mathscr{R}_n(\mathscr{G}) \to \mathscr{C} \otimes \mathscr{R}_n(\mathscr{G}) \subset \mathscr{R}_{n+1}(\mathscr{G}).$$

These maps satisfy

$$s_{-1}s_i = s_{i+1}s_{-1}, \quad s_{-1}d_i = d_{i+1}s_{-1}, \quad s_0s_{-1} = s_{-1}s_{-1}, \quad \text{and} \quad d_0s_{-1} = \text{id}.$$

The map $s: \mathscr{G} \to \mathscr{R}_0(\mathscr{G})$ given by

$$\mathscr{G} \cong \mathbb{S}_{O(\mathscr{C})} \otimes \mathscr{G} \to \mathscr{C} \otimes \mathscr{G} \subset \mathscr{R}_0(\mathscr{G})$$

splits the map ϵ and (with s_{-1}) exhibits ϵ as the split coequalizer of $d_0, d_1: \mathscr{R}_1(\mathscr{G}) \to \mathscr{R}_0(\mathscr{G})$. Meyer's theorem [25, Lemma 4.5.1] now gives the result. $\square$

The Reedy model structures on simplicial and cosimplicial spectra are convenient for identifying when maps of simplicial spectra realize to cofibrations and maps of cosimplicial spectra Tot to fibrations. The following proposition follows the usual outline of similar results, which are proved from the pushout-product property of the smash product of spectra and the construction of the latching object of a simplicial spectrum as a sequence of pushouts.

Proposition 7.3 *If $\mathscr{C}$ and $\mathscr{D}$ are pointwise relatively cofibrant small spectral categories and $\mathscr{G}$ is a cofibrant $(\mathscr{C}, \mathscr{D})$–bimodule then the geometric realization of $\mathscr{R}_\bullet(\mathscr{G})$ is a cofibrant $(\mathscr{C}, \mathscr{D})$–bimodule and for every fibrant $(\mathscr{C}, \mathscr{D})$–bimodule $\mathscr{F}$, the cosimplicial spectrum $\mathrm{Hom}^b_{\mathscr{C}, \mathscr{D}}(\mathscr{R}_\bullet(\mathscr{G}), \mathscr{F})$ is Reedy fibrant.*

The same kind of observation applied to the two-sided bar construction $B(\mathscr{C};\mathscr{C};\mathscr{C})$ in the construction of CC proves the following proposition.

Proposition 7.4 *If $\mathscr{C}$ is a pointwise relatively cofibrant small spectral category and $\mathscr{M}$ is a fibrant $(\mathscr{C},\mathscr{C})$–bimodule, then the cosimplicial spectrum $\mathrm{CC}^\bullet(\mathscr{C},\mathscr{M})$ is Reedy fibrant.*

The previous proposition together with the formula for $\mathrm{CC}(\mathscr{C};\mathscr{M})$ in (3.2) proves invariance under weak equivalences of fibrant $\mathscr{M}$ for $\mathscr{C}$ satisfying the hypothesis. Although this is all we need for the proof of Theorem 5.7, we state a more general invariance theorem for convenience of future reference.

Theorem 7.5 *Let $\mathscr{C}$ be a pointwise relatively cofibrant small spectral category and let $\mathscr{M}$ be a fibrant $(\mathscr{C},\mathscr{C})$–bimodule.*

(i) *$\mathrm{CC}(\mathscr{C};\mathscr{M})$ represents the derived functor $\mathbb{R}\operatorname{Hom}^b_{\mathscr{C},\mathscr{C}}(\mathscr{C},\mathscr{M})$. In particular, $\mathrm{CC}(\mathscr{C};-)$ preserves weak equivalences between fibrant $(\mathscr{C},\mathscr{C})$–bimodules.*

(ii) *Assume that $\mathscr{C}'$ is a pointwise relatively cofibrant small spectral category. If $\phi\colon \mathscr{C}' \to \mathscr{C}$ is a DK–equivalence, then the induced map $\mathrm{CC}(\mathscr{C};\mathscr{M}) \to \mathrm{CC}(\mathscr{C}';\phi^*M)$ is a weak equivalence.*

Proof The hypothesis on $\mathscr{C}$ implies that the inclusion of the degree-zero part of the bar construction

$$\mathscr{C} \otimes \mathscr{C} \to B_\bullet(\mathscr{C};\mathscr{C};\mathscr{C})$$

is a Reedy cofibration of $(\mathscr{C},\mathscr{C})$–bimodules, and it follows that $B(\mathscr{C};\mathscr{C};\mathscr{C})$ is a semicofibrant $(\mathscr{C},\mathscr{C})$–bimodule. Part (i) is then [15, Corollary 6.3]. Part (ii) follows immediately from part (i). □

Proposition 7.4 does not apply directly to $\mathscr{C}at_{\mathscr{F}}$ under the hypotheses of Theorem 5.7 unless we further require $\mathscr{F}$ be pointwise cofibrant (which is not the case in the main example of interest, $\mathscr{F} = \mathscr{F}_\phi$ for a DK–embedding $\phi\colon \mathscr{D} \to \mathscr{C}$). Nevertheless, the same argument applies to prove the following proposition.

Proposition 7.6 *If $\mathscr{C}$, $\mathscr{D}$, and $\mathscr{F}$ satisfy the hypotheses of Theorem 5.7, then the cosimplicial spectrum $\mathrm{CC}^\bullet(\mathscr{C}at_{\mathscr{F}})$ is Reedy fibrant.*

Construction 7.7 Let $\mathscr{C}$ and $\mathscr{D}$ be small spectral categories and $\mathscr{F}$ a $(\mathscr{C}, \mathscr{D})$–bimodule. We construct a map of cosimplicial spectra

$$\gamma^{\bullet} \colon \mathrm{CC}^{\bullet}(\mathscr{C}\mathrm{at}_{\mathscr{F}}) \to \mathrm{Hom}^b_{\mathscr{C},\mathscr{D}}(\mathscr{R}_{\bullet}(\mathscr{F}), \mathscr{F})$$

as follows. In cosimplicial degree zero, we have

$$\mathrm{CC}^0(\mathscr{C}\mathrm{at}_{\mathscr{F}}) \cong \prod_{d \in O(\mathscr{D})} \mathscr{D}(d,d) \times \prod_{c \in O(\mathscr{C})} \mathscr{C}(c,c),$$

while

$$\mathrm{Hom}^b_{\mathscr{C},\mathscr{D}}(\mathscr{R}_0(\mathscr{F}), \mathscr{F})$$
$$= \mathrm{Hom}^b_{\mathscr{C},\mathscr{D}}(\mathscr{F} \otimes \mathscr{D} \vee \mathscr{C} \otimes \mathscr{F}, \mathscr{F})$$
$$\cong \mathrm{Hom}^b_{\mathbb{S}_{O(\mathscr{D})},\mathscr{D}}(\mathscr{D}, \mathrm{Hom}^{\ell}_{\mathscr{C}}(\mathscr{F}, \mathscr{F})) \times \mathrm{Hom}^b_{\mathscr{C},\mathbb{S}_{O(\mathscr{C})}}(\mathscr{C}, \mathrm{Hom}^r_{\mathscr{D}}(\mathscr{F}, \mathscr{F}))$$
$$\cong \prod_{d \in O(\mathscr{D})} (\mathrm{Hom}^{\ell}_{\mathscr{C}}(\mathscr{F}, \mathscr{F}))(d,d) \times \prod_{c \in O(\mathscr{C})} (\mathrm{Hom}^r_{\mathscr{D}}(\mathscr{F}, \mathscr{F}))(c,c),$$

and we define γ^0 to be the product of the centralizer maps. For $n > 0$, for any $j = 1, \ldots, n$, given $c_0, \ldots, c_{j-1} \in O(\mathscr{C})$ and $d_j, \ldots, d_n \in O(\mathscr{D})$, let

$$(\mathscr{C}, \mathscr{F}, \mathscr{D})_{n,j}(c_0, \ldots, c_{j-1}, d_j, \ldots, d_n)$$
$$= \mathscr{C}(c_1, c_0) \wedge \cdots \wedge \mathscr{C}(c_{j-1}, c_{j-2}) \wedge \mathscr{F}(d_j, c_{j-1}) \wedge \mathscr{D}(d_{j+1}, d_j) \wedge \cdots \wedge \mathscr{D}(d_n, d_{n-1}),$$

where we understand the righthand side of this formula as

$$\mathscr{F}(d_1, c_0) \wedge \mathscr{D}(d_2, d_1) \wedge \cdots \wedge \mathscr{D}(d_n, d_{n-1})$$

when $j = 1$ and

$$\mathscr{C}(c_1, c_0) \wedge \cdots \wedge \mathscr{C}(c_{n-1}, c_{n-2}) \wedge \mathscr{F}(d_n, c_{n-1})$$

when $j = n$. Then, in this notation,

$$\mathrm{CC}^n(\mathscr{C}\mathrm{at}_{\mathscr{F}})$$
$$\cong \prod_{d_0, \ldots, d_n \in O(\mathscr{D})} F(\mathscr{D}(d_1, d_0) \wedge \cdots \wedge \mathscr{D}(d_n, d_{n-1}), \mathscr{D}(d_n, d_0))$$
$$\times \prod_{j=1}^{n} \prod_{\substack{c_0, \ldots, c_{j-1} \in O(\mathscr{C}) \\ d_j, \ldots, d_n \in O(\mathscr{D})}} F((\mathscr{C}, \mathscr{F}, \mathscr{D})_{n,j}(c_0, \ldots, c_{j-1}, d_j, \ldots, d_n), \mathscr{F}(d_n, c_0))$$
$$\times \prod_{c_0, \ldots, c_n \in O(\mathscr{C})} F(\mathscr{C}(c_1, c_0) \wedge \cdots \wedge \mathscr{C}(c_n, c_{n-1}), \mathscr{C}(c_n, c_0)),$$

while

$$
\operatorname{Hom}^b_{\mathscr{C},\mathscr{D}}(\mathscr{R}_n(\mathscr{F}),\mathscr{F})
$$

$$
\cong \prod_{d_0,\ldots,d_n\in O(\mathscr{D})} F\big(\mathscr{D}(d_1,d_0)\wedge\cdots\wedge\mathscr{D}(d_n,d_{n-1}),(\operatorname{Hom}^\ell_{\mathscr{C}}(\mathscr{F},\mathscr{F}))(d_n,d_0)\big)
$$

$$
\times \prod_{j=1}^{n}\prod_{\substack{c_0,\ldots,c_{j-1}\in O(\mathscr{C})\\ d_j,\ldots,d_n\in O(\mathscr{D})}} F\big((\mathscr{C},\mathscr{F},\mathscr{D})_{n,j}(c_0,\ldots,c_{j-1},d_j,\ldots,d_n),\mathscr{F}(d_n,c_0)\big)
$$

$$
\times \prod_{c_0,\ldots,c_n\in O(\mathscr{C})} F\big(\mathscr{C}(c_1,c_0)\wedge\cdots\wedge\mathscr{C}(c_n,c_{n-1}),(\operatorname{Hom}^r_{\mathscr{D}}(\mathscr{F},\mathscr{F}))(c_n,c_0)\big),
$$

and we define γ^n to be the map induced by the centralizer maps $\mathscr{D}\to\operatorname{Hom}^\ell_{\mathscr{C}}(\mathscr{F},\mathscr{F})$ and $\mathscr{C}\to\operatorname{Hom}^r_{\mathscr{D}}(\mathscr{F},\mathscr{F})$ on the outer factors and the identity on the inner factors. The maps $\gamma^\bullet$ clearly commute with the codegeneracy maps and all but the zeroth and last coface maps. A tedious but straightforward check of the definitions verifies that the $\gamma^\bullet$ also commute with the zeroth and last coface maps. Let γ denote the map on Tot induced by $\gamma^\bullet$.

Proof of Theorem 5.7 Fix $\mathscr{C}$, $\mathscr{D}$, and $\mathscr{F}$ as in the statement. Let $\mathscr{F}'\to\mathscr{F}$ be a cofibrant replacement in the category of $(\mathscr{C},\mathscr{D})$–bimodules, and consider the composite map

$$
\gamma'\colon \mathrm{CC}(\mathscr{C}\mathrm{at}_{\mathscr{F}})\xrightarrow{\gamma}\operatorname{Hom}^b_{\mathscr{C},\mathscr{D}}(\mathscr{R}(\mathscr{F}),\mathscr{F})\to\operatorname{Hom}^b(\mathscr{R}(\mathscr{F}'),\mathscr{F}).
$$

We note that γ' can be described in terms of the Tot of a cosimplicial map with formula analogous to γ. The inclusions of the summands

$$
\mathscr{F}'\otimes\mathscr{D}\otimes\cdots\otimes\mathscr{D}\quad\text{and}\quad\mathscr{C}\otimes\cdots\otimes\mathscr{C}\otimes\mathscr{F}'
$$

in $\mathscr{R}_\bullet(\mathscr{F}')$ assemble to a map of simplicial $(\mathscr{C},\mathscr{D})$–bimodules

$$
B_\bullet(\mathscr{F}';\mathscr{D};\mathscr{D})\vee B_\bullet(\mathscr{C};\mathscr{C};\mathscr{F})\to\mathscr{R}_\bullet(\mathscr{C};\mathscr{F}';\mathscr{D}),
$$

where B denotes the two-sided bar construction for $\otimes$. The hypotheses on $\mathscr{C}$, $\mathscr{D}$, and $\mathscr{F}'$ are sufficient for this map to be a Reedy cofibration, and so it induces a fibration

$$
\operatorname{Hom}^b_{\mathscr{C},\mathscr{D}}(\mathscr{R}(\mathscr{F}'),\mathscr{F})\to\operatorname{Hom}^b_{\mathscr{C},\mathscr{D}}(B(\mathscr{F}';\mathscr{D};\mathscr{D})\vee B(\mathscr{C};\mathscr{C};\mathscr{F}'),\mathscr{F})
$$

since $\mathscr{F}$ is fibrant. We then have a canonical isomorphism

$$\mathrm{Hom}^b_{\mathscr{C},\mathscr{D}}(B(\mathscr{F}';\mathscr{D};\mathscr{D}) \vee B(\mathscr{C};\mathscr{C};\mathscr{F}'), \mathscr{F})$$
$$\cong \mathrm{Hom}^b_{\mathscr{C},\mathscr{D}}(B(\mathscr{F}';\mathscr{D};\mathscr{D}), \mathscr{F}) \times \mathrm{Hom}^b_{\mathscr{C},\mathscr{D}}(B(\mathscr{C};\mathscr{C};\mathscr{F}'), \mathscr{F})$$
$$\cong \mathrm{Hom}^b_{\mathscr{C},\mathscr{D}}(\mathscr{F}' \otimes_{\mathscr{D}} B(\mathscr{D};\mathscr{D};\mathscr{D}), \mathscr{F}) \times \mathrm{Hom}^b_{\mathscr{C},\mathscr{D}}(B(\mathscr{C};\mathscr{C};\mathscr{C}) \otimes_{\mathscr{C}} \mathscr{F}', \mathscr{F})$$
$$\cong \mathrm{Hom}^b_{\mathscr{D},\mathscr{D}}(B(\mathscr{D};\mathscr{D};\mathscr{D}), \mathrm{Hom}^\ell_{\mathscr{C}}(\mathscr{F}', \mathscr{F})) \times \mathrm{Hom}^b_{\mathscr{C},\mathscr{C}}(B(\mathscr{C};\mathscr{C};\mathscr{C}), \mathrm{Hom}^r_{\mathscr{D}}(\mathscr{F}', \mathscr{F}))$$
$$\cong \mathrm{CC}(\mathscr{D}; \mathrm{Hom}^\ell_{\mathscr{C}}(\mathscr{F}', \mathscr{F})) \times \mathrm{CC}(\mathscr{C}; \mathrm{Hom}^r_{\mathscr{D}}(\mathscr{F}', \mathscr{F})).$$

Opening up the construction of γ, we see that diagram

$$
\begin{array}{ccc}
\mathrm{CC}(\mathscr{C}\mathrm{at}_{\mathscr{F}}) & \longrightarrow & \mathrm{CC}(\mathscr{D}) \times \mathrm{CC}(\mathscr{C}) \\
\gamma' \downarrow & & \downarrow \\
\mathrm{Hom}^b_{\mathscr{C},\mathscr{D}}(\mathscr{R}(\mathscr{F}'), \mathscr{F}) & \longrightarrow & \mathrm{CC}(\mathscr{D}; \mathrm{Hom}^\ell_{\mathscr{C}}(\mathscr{F}', \mathscr{F})) \times \mathrm{CC}(\mathscr{C}; \mathrm{Hom}^r_{\mathscr{D}}(\mathscr{F}', \mathscr{F}))
\end{array}
$$

commutes, where the right vertical map is induced by the double centralizer maps on the bimodule variables of CC. We have observed that the bottom horizontal map is a fibration and in particular the Tot of a Reedy fibration of cosimplicial spectra; the top horizontal map is also a fibration and the Tot of a Reedy fibration of cosimplicial spectra. The map on horizontal fibers is the Tot of the cosimplicial map that in each degree is the weak equivalence

$$\prod_{j=1}^{n} \prod_{\substack{c_0,\ldots,c_{j-1} \in O(\mathscr{C}) \\ d_j,\ldots,d_n \in O(\mathscr{D})}} F\big((\mathscr{C}, \mathscr{F}, \mathscr{D})_{n,j}(c_0,\ldots,c_{j-1},d_j,\ldots,d_n), \mathscr{F}(d_n,c_0)\big)$$
$$\to \prod_{j=1}^{n} \prod_{\substack{c_0,\ldots,c_{j-1} \in O(\mathscr{C}) \\ d_j,\ldots,d_n \in O(\mathscr{D})}} F\big((\mathscr{C}, \mathscr{F}', \mathscr{D})_{n,j}(c_0,\ldots,c_{j-1},d_j,\ldots,d_n), \mathscr{F}(d_n,c_0)\big).$$

Tot takes this degreewise weak equivalence of Reedy fibrant objects to a weak equivalence of spectra. It follows that the square above is homotopy cartesian. Both maps

$$\mathrm{Hom}^b(\mathscr{R}(\mathscr{F}'), \mathscr{F}) \to \mathrm{Hom}^b_{\mathscr{C},\mathscr{D}}(B(\mathscr{F}';\mathscr{D};\mathscr{D}), \mathscr{F}) \cong \mathrm{CC}(\mathscr{D}; \mathrm{Hom}^\ell_{\mathscr{C}}(\mathscr{F}', \mathscr{F})),$$
$$\mathrm{Hom}^b(\mathscr{R}(\mathscr{F}'), \mathscr{F}) \to \mathrm{Hom}^b_{\mathscr{C},\mathscr{D}}(B(\mathscr{C};\mathscr{C};\mathscr{F}'), \mathscr{F}) \cong \mathrm{CC}(\mathscr{C}; \mathrm{Hom}^r_{\mathscr{D}}(\mathscr{F}', \mathscr{F}))$$

are weak equivalences since the maps $B(\mathscr{F}';\mathscr{D};\mathscr{D}) \to \mathscr{R}(\mathscr{F})$ and $B(\mathscr{C};\mathscr{C};\mathscr{F}') \to \mathscr{R}(\mathscr{F})$ are weak equivalences of cofibrant $(\mathscr{C}, \mathscr{D})$–bimodules. Since $\mathscr{F}'$ is cofibrant

as both a left $\mathscr{C}$–object and right $\mathscr{D}$–object (Theorem 4.12(i) and the right object version), $\mathrm{Hom}^{\ell}_{\mathscr{C}}(\mathscr{F}', \mathscr{F})$ and $\mathrm{Hom}^{r}_{\mathscr{D}}(\mathscr{F}', \mathscr{F})$ are pointwise fibrant. It follows that when $\mathscr{D} \to \mathrm{Hom}^{\ell}_{\mathscr{C}}(\mathscr{F}', \mathscr{F})$ is a weak equivalence, so is the map $\mathrm{CC}(\mathscr{C}\mathrm{at}_{\mathscr{F}}) \to \mathrm{CC}(\mathscr{C})$; likewise, when $\mathscr{C} \to \mathrm{Hom}^{r}_{\mathscr{D}}(\mathscr{F}', \mathscr{F})$ is a weak equivalence, so is the map $\mathrm{CC}(\mathscr{C}\mathrm{at}_{\mathscr{F}}) \to \mathrm{CC}(\mathscr{D})$. By Theorem 4.12(iii) (and its analogue for $\mathrm{Hom}^{r}_{\mathscr{D}}$), both $\mathrm{Hom}^{\ell}_{\mathscr{C}}(\mathscr{F}', \mathscr{F})$ and $\mathrm{Hom}^{r}_{\mathscr{D}}(\mathscr{F}', \mathscr{F})$ represent the derived functors in the statement of the centralizer conditions. The theorem now follows. □

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