

Distortion of surfaces in graph manifolds

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Let $S \looparrowright N$ be an immersed horizontal surface in a 3–dimensional graph manifold. We show that the fundamental group of the surface S is quadratically distorted whenever the surface is virtually embedded (ie separable) and is exponentially distorted when the surface is not virtually embedded.

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1 Introduction

In the study of 3–manifolds, much attention has focused on the distinction between surfaces that lift to an embedding in a finite cover and those that do not. A π_1 –injective immersion $S \looparrowright N$ of a surface S into a 3–manifold N is a *virtual embedding* if (after applying a homotopy) the immersion lifts to an embedding of S into a finite cover of N . Due to work of Scott [18] and Przytycki and Wise [16], virtual embedding is equivalent to separability of the surface subgroup $\pi_1(S)$ in $\pi_1(N)$.

A major part of the solution of the virtual Haken conjecture is Wise and Agol’s theorem that every immersed surface in a finite-volume complete hyperbolic 3–manifold is virtually embedded [23; 1]. In contrast, Rubinstein and Wang [17] showed that many 3–dimensional graph manifolds contain immersed surfaces $g: S \looparrowright N$ that are not virtually embedded. The examples of Rubinstein and Wang are *horizontal* in the sense that in each Seifert component M of N , the intersection $g(S) \cap M$ is transverse to the Seifert fibration. Furthermore, they introduced a combinatorial invariant of horizontal surfaces called dilation, which they use to completely characterize which horizontal surfaces are virtually embedded.

If $H \leq G$ are groups with finite generating sets S and \mathcal{T} , the *distortion* of H in G is given by

$$\Delta_H^G(n) = \max\{|h|_{\mathcal{T}} \mid h \in H \text{ and } |h|_S \leq n\}.$$

Distortion does not depend on the choice of finite generating sets S and \mathcal{T} (up to a natural equivalence relation). The purpose of this article is to address the following question of Dani Wise:

Question 1.1 Given a 3-dimensional graph manifold N , which surfaces in N have nontrivial distortion?

Question 1.1 arises naturally in the study of cubulations of 3-manifold groups. The typical strategy for constructing an action of the fundamental group on a CAT(0) cube complex is to find a suitable collection of immersed surfaces and then to consider the CAT(0) cube complex dual to that collection of surfaces (see for instance Wise [22]).

Whenever a group G acts properly and cocompactly on a CAT(0) cube complex X , the stabilizer of each hyperplane must be an undistorted subgroup of G , since hyperplanes are convex. Hagen and Przytycki [7] show that chargeless graph manifolds do act cocompactly on CAT(0) cube complexes. It is clear from their construction that many graph manifolds contain undistorted surface subgroups.

However, the situation for horizontal surfaces turns out to be quite different. Our main result states that horizontal surfaces in graph manifolds always have a nontrivial distortion, and this distortion is directly related to virtual embedding in the following sense:

Theorem 1.2 *Let $S \looparrowright N$ be a horizontal surface properly immersed in a graph manifold N . The distortion of $\pi_1(S)$ in $\pi_1(N)$ is quadratic if S is virtually embedded, and exponential if S is not virtually embedded.*

The main tool used in the proof of this theorem is a simple geometric interpretation of Rubinstein and Wang's dilation in terms of slopes of lines in the JSJ planes of the universal cover.

We note that, throughout this paper, the term “graph manifold” specifically excludes two trivial cases: Seifert manifolds and Sol manifolds. By [Proposition 6.7](#), horizontal surfaces in Seifert manifolds are always undistorted. Distortion of surfaces in Sol manifolds is not addressed in this article.

Although [Theorem 1.2](#) deals only with horizontal surfaces in graph manifolds, understanding these is the key to a general understanding of all π_1 -injective surfaces in nongeometric 3-manifolds. The nongeometric 3-manifolds include both graph manifolds and also “mixed” type 3-manifolds, those whose JSJ decomposition contains at least one hyperbolic component and at least one JSJ torus.

Recently Yi Liu [11] has used Rubinstein and Wang's work on virtual embedding of horizontal surfaces in graph manifolds as the foundation for a study of virtual embedding of arbitrary surfaces in nongeometric 3-manifolds. Similarly, in a forthcoming article, the second author [13] uses [Theorem 1.2](#) as the foundation for a study of distortion of arbitrary surface subgroups in fundamental groups of nongeometric 3-manifolds.

As mentioned above, Hagen and Przytycki [7] have shown that chargeless graph manifolds act cocompactly on CAT(0) cube complexes. The cubulation they construct is dual to a family of properly immersed surfaces, none of which is entirely horizontal. More precisely, a key property of these surfaces is that they never contain two adjacent horizontal pieces.

The following corollary shows that in order to obtain a proper, cocompact cubulation (using any possible subgroups, not necessarily just surface subgroups) all surface subgroups must be of the type used by Hagen and Przytycki. The corollary follows from combining [Theorem 1.2](#) with [13].

Corollary 1.3 *Let G be the fundamental group of a graph manifold. Let $\{H_1, \dots, H_k\}$ be a collection of codimension-1 subgroups of G . Let X be the corresponding dual CAT(0) cube complex. If at least one H_i is the fundamental group of a surface containing two adjacent horizontal pieces, then the action of G on X is not proper and cocompact.*

After seeing an early version of this paper, Hung Cong Tran discovered an alternative proof of the quadratic distortion of certain horizontal surfaces whose fundamental groups can also be regarded as Bestvina–Brady kernels. This observation is part of a broader study by Tran [20] of the distortion of Bestvina–Brady kernels in right-angled Artin groups.

1.1 Connections to previous work

In [24], Woodhouse exploited strong parallels between graph manifolds and tubular spaces to study actions of tubular groups on CAT(0) cube complexes. In the tubular setting, immersed hyperplanes play the role of immersed surfaces in graph manifolds. Woodhouse extended the Rubinstein–Wang theory of dilation and found a connection between dilation and distortion. In particular, he proves that if an immersed hyperplane in a tubular group has nontrivial dilation then its distortion is at least quadratic. Inspired by Woodhouse's work, we use analogous techniques in the cleaner geometric setting of graph manifolds to obtain [Theorem 1.2](#).

We remark that the main proof of [Theorem 1.2](#) in [Sections 5](#) and [6](#) includes both virtually embedded and non-virtually embedded surfaces in a unified treatment that prominently uses a simple geometric interpretation of dilation in terms of “slopes of lines” in the Euclidean geometry of JSJ planes. This interpretation was inspired by Woodhouse’s earlier work on tubular groups. It seems likely that the main proofs here could be translated back to the tubular setting, where they may lead to new advances in that setting as well.

However there is an alternative (shorter) proof of the quadratic distortion of virtually embedded horizontal surfaces using the following result, which combines work of Gersten and Kapovich and Leeb. This alternative approach, while more direct, does not give any information about the structure of surfaces that are not virtually embedded.

Theorem 1.4 (Gersten; Kapovich and Leeb) *Let N be a graph manifold that fibers over the circle with fiber surface S . Then $\pi_1(S)$ is quadratically distorted in $\pi_1(N)$.*

Kapovich and Leeb implicitly use the quadratic upper bound (without stating it explicitly) in [\[10\]](#), where they attribute it to Gersten [\[5\]](#). The quadratic lower bound also follows easily from results of Gersten [\[5\]](#) and Kapovich and Leeb [\[10\]](#), but was not specifically mentioned by them. The discussion of [Theorem 1.4](#) in [\[10\]](#) and its precise derivation from [\[5\]](#) is brief and was not the main purpose of either article. For the benefit of the reader we have included a more detailed exposition of Kapovich and Leeb’s elegant proof of [Theorem 1.4](#) in [Section 7](#), which relies on Thurston’s geometric classification of 3-manifolds that fiber over the circle.

The virtually embedded case of [Theorem 1.2](#) can be derived as a corollary of [Theorem 1.4](#) as follows. If a horizontal surface $g: S \looparrowright N$ is virtually embedded then there exist finite covers $\hat{S} \rightarrow S$ and $\hat{N} \rightarrow N$ such that \hat{N} is an \hat{S} -bundle over S^1 (see for instance Wang and Yu [\[21\]](#)). By [Theorem 1.4](#), the distortion of $\pi_1(\hat{S})$ in $\pi_1(\hat{N})$ is quadratic. Distortion is unchanged when passing to subgroups of finite index, so the distortion of $\pi_1(S)$ in $\pi_1(N)$ is also quadratic.

Overview

In [Section 2](#) we review some concepts in geometric group theory. In [Section 3](#) we give several lemmas about curves on hyperbolic surfaces that will be used in [Section 5](#). [Section 4](#) is a review of background about graph manifolds and horizontal surfaces. A convenient metric on a graph manifold that we will use in this paper will be discussed.

In [Section 5](#) we prove the distortion of a horizontal surface in a graph manifold is at least quadratic. We also show that if the horizontal surface is not virtually embedded then the distortion is at least exponential. The strategy in this proof was inspired by the work of Woodhouse (see [Section 6](#) in [\[24\]](#)). In [Section 6](#) we prove that distortion of a horizontal surface in a graph manifold is at most exponential. Furthermore, if the horizontal surface is virtually embedded then the distortion is at most quadratic.

[Section 7](#) contains a detailed exposition of the proof of [Theorem 1.4](#).

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2 Preliminaries

In this section, we review some concepts in geometric group theory: quasi-isometry, distortion of a subgroup and the notions of domination and equivalence.

Definition 2.1 Let (X, d) be a metric space. A path $\gamma: [a, b] \rightarrow X$ is a *geodesic* if $d(\gamma(s), \gamma(t)) = |s - t|$ for all $s, t \in [a, b]$. A simple loop $f: S^1 \rightarrow X$ is a *geodesic loop* if f is an isometric embedding with respect to some length metric on S^1 .

Definition 2.2 (quasi-isometry) Let (X_1, d_1) and (X_2, d_2) be metric spaces. A (not necessarily continuous) map $f: X_1 \rightarrow X_2$ is an (L, C) -*quasi-isometric embedding* if there exist constants $L \geq 1$ and $C \geq 0$ such that for all $x, y \in X_1$ we have

$$\frac{1}{L}d_1(x, y) - C \leq d_2(f(x), f(y)) \leq Ld_1(x, y) + C.$$

If, in addition, there exists a constant $D \geq 0$ such that every point of X_2 lies in the D -neighborhood of the image of f , then f is an (L, C) -quasi-isometry. When such a map exists, X_1 and X_2 are quasi-isometric.

Let (X, d) be a metric space, and γ a path in X . We denote the length of γ by $|\gamma|$.

Definition 2.3 (quasigeodesic) A path γ in a metric space (X, d) is an (L, C) -quasigeodesic with respect to constants $L \geq 1$ and $C \geq 0$ if $|\gamma_{[x, y]}| \leq Ld(x, y) + C$ for all $x, y \in \gamma$. A quasigeodesic is a path that is (L, C) -quasigeodesic for some L and C .

Definition 2.4 A geodesic space (X, d) is δ -hyperbolic if every geodesic triangle with vertices in X is δ -thin in the sense that each side lies in the δ -neighborhood of the union of the other two sides.

Definition 2.5 (deviation) Let (X, d) be a geodesic space and $c \geq 0$ be a fixed number. Consider a pair of geodesic segments $[x, y]$ and $[z, t]$ such that $d(y, z) \leq c$. The c -deviation of $[x, y]$ and $[z, t]$, denoted by $\text{dev}_c([x, y], [z, t])$, is the quantity

$$\sup\{\max\{d(u, y), d(v, z)\} \mid u \in [x, y], v \in [z, t], d(u, v) \leq c\}.$$

The following theorem gives a criterion for determining that a piecewise geodesic in a δ -hyperbolic space is a quasigeodesic. The proof, which is very similar to the proof of Lemma 19 of [6, Chapter 5], is left as an exercise to the reader.

Theorem 2.6 Let (X, d) be a δ -hyperbolic space. For any $\kappa \geq 0$ and $D \geq 0$, there exist constants $L = L(\delta, \kappa, D)$ and $C = C(\delta, \kappa, D)$ such that the following holds: Suppose a piecewise geodesic

$$c = [x_1, x_2] \cup [x_2, x_3] \cup \cdots \cup [x_{2m-1}, x_{2m}]$$

satisfies

- (1) $d(x_{2i}, x_{2i+1}) \leq \kappa$,
- (2) $d(x_{2i-1}, x_{2i}) \geq 11\kappa + 25\delta + 2D$, and
- (3) $\text{dev}_Q([x_{2i-1}, x_{2i}], [x_{2i+1}, x_{2i+2}]) \leq D$, where $Q = 4\delta + \kappa$.

Then c is an (L, C) -quasigeodesic. □

Definition 2.7 Let f and g be functions from positive reals to positive reals. The function f is dominated by g , denoted by $f \preceq g$, if there are positive constants A, B ,

C , D and E such that

$$f(x) \leq Ag(Bx + C) + Dx + E \quad \text{for all } x.$$

The functions f and g are *equivalent*, denoted by $f \sim g$, if $f \leq g$ and $g \leq f$.

The relation \sim is an equivalence relation. Polynomial functions with degree at least one are equivalent if and only if they have the same degree. Furthermore, all exponential functions are equivalent.

Definition 2.8 (subgroup distortion) Let $H \leq G$ be a pair of groups with finite generating sets \mathcal{T} and \mathcal{S} , respectively. The *distortion* of (H, \mathcal{T}) in (G, \mathcal{S}) is the function

$$\Delta_H^G(n) = \max\{|h|_{\mathcal{T}} \mid h \in H \text{ and } |h|_{\mathcal{S}} \leq n\}.$$

Up to equivalence, the function Δ_H^G does not depend on the choice of finite generating sets \mathcal{S} and \mathcal{A} .

Let $H \leq G$ be a pair of finitely generated groups. We say that the distortion Δ_H^G is *at least quadratic (exponential)* if it dominates a quadratic polynomial (exponential function). The distortion Δ_H^G is *at most quadratic (exponential)* if it is dominated by a quadratic polynomial (exponential function).

The following proposition is routine, and we leave the proof as an exercise for the reader.

Proposition 2.9 Let G , H and K be finitely generated groups with $K \leq H \leq G$.

- (1) If H is a finite-index subgroup of G then $\Delta_K^H \sim \Delta_K^G$.
- (2) If K is a finite-index subgroup of H then $\Delta_K^G \sim \Delta_H^G$. □

It is well known that a group acting properly, cocompactly and isometrically on a geodesic space is quasi-isometric to the space. The following corollary of this fact allows us to compute distortion using the geometries of spaces in place of word metrics.

Corollary 2.10 Let X and Y be compact geodesic spaces, and let $g: (Y, y_0) \rightarrow (X, x_0)$ be π_1 -injective. We lift the metrics on X and Y to geodesic metrics on the universal covers \tilde{X} and \tilde{Y} , respectively. Let $G = \pi_1(X, x_0)$ and $H = g_*(\pi_1(Y, y_0))$. Then the distortion Δ_H^G is equivalent to the function

$$f(n) = \max\{d_{\tilde{Y}}(\tilde{y}_0, h(\tilde{y}_0)) \mid h \in H \text{ and } d_{\tilde{X}}(\tilde{x}_0, h(\tilde{x}_0)) \leq n\}.$$

3 Curves in hyperbolic surfaces

In this section, we will give some results about curves on surfaces that will play an essential role in the proof of [Theorem 5.1](#).

Definition 3.1 (multicurves) A closed curve in a surface S is *essential* if it is not freely homotopic to a point or a boundary component. A *multicurve* in S is a finite collection of disjoint, essential simple closed curves such that no two are freely homotopic. If S has a metric, a multicurve is *geodesic* if each member of the family is a geodesic loop.

Lemma 3.2 Let S be a compact surface with negative Euler characteristic. Let \mathcal{C} be a multicurve in S . Let \mathcal{L} be the family of lines that are lifts of loops of \mathcal{C} or boundary loops of S . Equip S with any length metric d_S . Let d be the induced metric on the universal cover \tilde{S} . For any $r > 0$ there exists $D = D(r) < \infty$ such that for any two disjoint lines ℓ_1 and ℓ_2 of \mathcal{L} we have

$$\text{diam}(\mathcal{N}_r(\ell_1) \cap \mathcal{N}_r(\ell_2)) \leq D.$$

Proof Since $H = \pi_1(S)$ acts cocompactly on the universal cover \tilde{S} , there exists a closed ball $\bar{B}(x_0, R)$ whose H -translates cover \tilde{S} . Note that \mathcal{L} is locally finite in the sense that only finitely many lines of \mathcal{L} intersect the closed ball $B = \bar{B}(x_0, R + 2r)$. Since distinct lines of \mathcal{L} are not parallel, there exists a finite upper bound $D = D(r)$ on the diameter of the intersection $\mathcal{N}_r(\ell) \cap \mathcal{N}_r(\ell')$ for all lines $\ell \neq \ell' \in \mathcal{L}$ that intersect B .

Consider $\ell_1 \neq \ell_2 \in \mathcal{L}$. If $d(\ell_1, \ell_2) \geq 2r$ then their r -neighborhoods have empty intersection, and the result is vacuously true. Thus it suffices to assume that $d(\ell_1, \ell_2) < 2r$. By cocompactness, there exists $h \in H$ such that $h(\ell_1)$ intersects $\bar{B}(x_0, R)$. But then $h(\ell_1)$ and $h(\ell_2)$ both intersect B , so that

$$\text{diam}(\mathcal{N}_r(\ell_1) \cap \mathcal{N}_r(\ell_2)) \leq D,$$

as desired. □

Lemma 3.3 Let S be a compact hyperbolic surface with totally geodesic (possibly empty) boundary, and let \mathcal{C} be a nonempty geodesic multicurve. Then there exists a geodesic loop γ in S such that γ and \mathcal{C} have nonempty intersection.

Proof Choose any loop $c \in \mathcal{C}$. Cutting S along c produces a surface each of whose components has negative Euler characteristic since each of its components is a union of blocks of S joined along circles. For any such simple closed curve c in a compact surface S , we claim that there exists another closed curve γ whose geometric intersection number with c is nonzero. We leave the proof of this claim as an easy exercise for the reader (using, for example, the techniques in Sections 1.2.4 and 1.3 of [4]). After applying a homotopy, we can assume that γ is a geodesic loop. \square

Lemma 3.4 *Let S be a compact hyperbolic surface with totally geodesic (possibly empty) boundary, and let \mathcal{C} be a nonempty geodesic multicurve. Let \mathcal{L} be the family of lines that are lifts of loops of \mathcal{C} or boundary loops of S . For any $\kappa > 0$, there exist numbers μ , L and C such that the following holds:*

Consider a piecewise geodesic $c = \alpha_1\beta_1 \cdots \alpha_n\beta_n$ in the universal cover \tilde{S} . Suppose each segment α_j is contained in a line of \mathcal{L} , and each segment β_j meets the lines of \mathcal{L} only at its endpoints. If $|\beta_j| \leq \kappa$ and $|\alpha_j| \geq \mu$ for all j , then c is an (L, C) -quasigeodesic.

Proof In order to show that the piecewise geodesic c in the δ -hyperbolic space \tilde{S} is uniformly quasigeodesic, we will show that there exists a constant D such that whenever μ is sufficiently large, c satisfies the hypotheses of Theorem 2.6. Thus c is (L, C) -quasigeodesic for constants L and C not depending on c .

Let $Q = 4\delta + \kappa$. Let $D = D(Q)$ be the constant given by applying Lemma 3.2 to the hyperbolic surface S and the multicurve \mathcal{C} . Let $L = L(\delta, \kappa, D)$ and $C = C(\delta, \kappa, D)$ be the constants given by Theorem 2.6. We define $\mu = 11\kappa + 25\delta + 2D$. It follows that c satisfies conditions (1) and (2) of Theorem 2.6.

In order to verify Theorem 2.6(3), we need to show that $\text{dev}_Q(\alpha_j, \alpha_{j+1}) \leq D$. Let $u \in \alpha_j$ and $v \in \alpha_{j+1}$ such that $d(u, v) \leq Q$. Let u' and v' be the initial and terminal points of β_j . We need to show that $d(u, u') \leq D$ and $d(v, v') \leq D$. It is easy to see the elements u , v , u' and v' belong to $\mathcal{N}_Q(\alpha_j) \cap \mathcal{N}_Q(\alpha_{j+1})$ because $|\beta_j| \leq \kappa \leq Q$. We have that α_{j+1} and α_j do not belong to the same line of \mathcal{L} because any two distinct geodesics in \tilde{S} intersect at most one point. Hence,

$$\text{diam}(\mathcal{N}_Q(\alpha_j) \cap \mathcal{N}_Q(\alpha_{j+1})) \leq D$$

by Lemma 3.2. It follows that $d(u, u') \leq D$ and $d(v, v') \leq D$. It follows immediately from the definition of the deviation that $\text{dev}_Q(\alpha_j, \alpha_{j+1}) \leq D$. \square

4 Graph manifolds and horizontal surfaces

In this section, we review background about graph manifolds and horizontal surfaces. In addition, we discuss a convenient metric for a graph manifold that will be used in the next sections. We refer the reader to [17; 3; 10] for more details.

Definition 4.1 A *graph manifold* is a compact, irreducible, connected, orientable 3-manifold N that can be decomposed along embedded incompressible tori \mathcal{T} into finitely many Seifert manifolds. We specifically exclude Sol and Seifert manifolds from the class of graph manifolds. Up to isotopy, each graph manifold has a unique minimal collection of tori \mathcal{T} as above [8; 9]. This minimal collection is the *JSJ decomposition* of N , and each torus of \mathcal{T} is a *JSJ torus*.

Throughout this paper, a graph consists of a set \mathcal{V} of vertices and a set of \mathcal{E} of edges, each edge being associated to an unordered pair of vertices by a function ends given by $\text{ends}(e) = \{v, v'\}$, where $v, v' \in \mathcal{V}$. In this case we call v and v' the *endpoints* of the edge e and we also say v and v' are adjacent.

Definition 4.2 A *simple graph manifold* N is a graph manifold with the following properties:

- (1) Each Seifert component is a trivial circle bundle over an orientable surface of genus at least 2.
- (2) The intersection numbers of fibers of adjacent Seifert components have absolute value 1.

Theorem 4.3 [10] Any graph manifold N has a finite cover \hat{N} that is a simple graph manifold.

Definition 4.4 Let M be a Seifert manifold with boundary. A *horizontal surface* in M is an immersion $g: B \looparrowright M$ where B is a compact surface with boundary such that the image $g(B)$ is transverse to the Seifert fibration. We also require that g is *properly immersed*, ie $g(B) \cap \partial M = g(\partial B)$.

A *horizontal surface* in a graph manifold N is a properly immersed surface $g: S \looparrowright N$ such that for each Seifert component M , the intersection $g(S) \cap M$ is a horizontal surface in M . A horizontal surface g in a graph manifold N is always π_1 -injective and lifts to an embedding of S in the cover of N corresponding to the subgroup $g_*(\pi_1(S))$ by [17]. Consequently, g also lifts to an embedding $\tilde{S} \rightarrow \tilde{N}$ of universal covers.

Definition 4.5 A horizontal surface $g: S \looparrowright N$ in a graph manifold N is *virtually embedded* if g lifts to an embedding of S in some finite cover of N . By a theorem of Scott [18], a horizontal surface is virtually embedded if $g_*(\pi_1(S))$ is a *separable* subgroup of $\pi_1(N)$, ie it is equal to an intersection of finite-index subgroups. Przytycki and Wise have shown that the converse holds as well [15].

The following result about separability allows one to pass to finite covers in the study of horizontal surfaces, as explained in Corollary 4.7.

Proposition 4.6 [18, Lemma 1.1] *Let G_0 be a finite-index subgroup of G . A subgroup $H \leq G$ is separable in G if and only if $H \cap G_0$ is separable in G_0 .*

Corollary 4.7 *Let $q: (\hat{N}, \hat{x}_0) \rightarrow (N, x_0)$ be a finite covering of graph manifolds. Let $g: (S, s_0) \looparrowright (N, x_0)$ be a horizontal surface. Let $p: (\hat{S}, \hat{s}_0) \rightarrow (S, s_0)$ be the finite cover corresponding to the subgroup $g_*^{-1}q_*\pi_1(\hat{N}, \hat{x}_0)$. Then g lifts to a horizontal surface $\hat{g}: (\hat{S}, \hat{s}_0) \rightarrow (\hat{N}, \hat{x}_0)$. Furthermore, g is a virtual embedding if and only if \hat{g} is a virtual embedding.*

Definition 4.8 Suppose $g: S \looparrowright N$ is a horizontal surface in a graph manifold N with JSJ decomposition \mathcal{T} . Let \mathcal{T}_g denote the collection of components of $g^{-1}(\mathcal{T})$ in S . After applying a homotopy to g , we may assume that the image $g(c)$ of each curve $c \in \mathcal{T}_g$ is a multiple of a simple closed curve on the corresponding JSJ torus. The connected components of the splitting $S|g^{-1}(\mathcal{T})$ are the *blocks* B of S .

Remark 4.9 If $g: S \looparrowright N$ is a horizontal surface then \mathcal{T}_g is always nonempty. Indeed, $g(S)$ has nonempty intersection with each JSJ torus of N because a properly immersed horizontal surface in a connected graph manifold must intersect every fiber of every Seifert component (see Lemma 6.3 for details).

In [17], Rubinstein and Wang introduced the dilation of a horizontal surface, and proved that dilation is the obstruction to a surface being virtually embedded (see Theorem 4.11).

Definition 4.10 (dilation) Let $g: (S, s_0) \looparrowright (N, x_0)$ be a horizontal surface in a simple graph manifold N . Choose an orientation for the graph manifold N , an orientation for the fiber of each Seifert component, and an orientation for each curve $c \in \mathcal{T}_g$.

The *dilation* of a horizontal surface S in N is a homomorphism $w: \pi_1(S, s_0) \rightarrow \mathbb{Q}_+^*$ defined as follows. Choose $[\gamma] \in \pi_1(S, s_0)$ such that γ is transverse to \mathcal{T}_g . In the trivial case that γ is disjoint from the curves of the collection \mathcal{T}_g , we set $w(\gamma) = 1$. Let us assume now that this intersection is nonempty. Then \mathcal{T}_g subdivides γ into a

concatenation $\gamma_1 \cdots \gamma_m$ with the following properties: Each path γ_i starts on a circle $c_i \in \mathcal{T}_g$ and ends on the circle c_{i+1} . The image $g(\gamma_i)$ of this path in N lies in a Seifert component M_i . The image of the circle $g(c_i)$ in N lies in a JSJ torus T_i obtained by gluing a boundary torus \tilde{T}_i of M_{i-1} to a boundary torus \tilde{T}_i of M_i . Let \tilde{f}_i and \vec{f}_i be fibers of M_{i-1} and M_i in the torus T_i . By [Definition 4.2](#), the 1-cycles $[\tilde{f}_i]$ and $[\vec{f}_i]$ generate the integral homology group $H_1(T_i) \cong \mathbb{Z}^2$, so there exist integers a_i and b_i such that

$$[g(c_i)] = a_i[\tilde{f}_i] + b_i[\vec{f}_i] \quad \text{in } H_1(T_i).$$

Since the immersion is horizontal, these coefficients a_i and b_i must be nonzero. The dilation $w(\gamma)$ is the rational number $\prod_{i=1}^m |b_i/a_i|$. Note that $w(\gamma)$ depends only on the homotopy class of γ , since crossings of γ with a curve $c \in \mathcal{T}_g$ in opposite directions contribute terms to the dilation that cancel each other. For the rest of this paper we write w_γ instead of $w(\gamma)$.

The following result is a special case of [\[17, Theorem 2.3\]](#):

Theorem 4.11 *A horizontal surface $g: S \looparrowright N$ in a simple graph manifold is virtually embedded if and only if the dilation w is the trivial homomorphism.*

Remark 4.12 If $g: S \looparrowright N$ is a horizontal surface in a simple graph manifold M , then each block B is a connected surface with nonempty boundary and negative Euler characteristic. Indeed, the immersion $g: S \looparrowright N$ maps B to the corresponding Seifert component M with base surface F . The composition of $g|_B$ with the projection of M to F yields a finite covering map from B to F . Since $\chi(F) < 0$, it follows that $\chi(B) < 0$ as well.

We note that the collection \mathcal{T}_g is always a nonempty multicurve. Indeed, \mathcal{T}_g is nonempty by [Remark 4.9](#). Since the blocks of S have negative Euler characteristic, it follows that \mathcal{T}_g is a multicurve.

Remark 4.13 We now are going to describe a convenient metric on a simple graph manifold N introduced by Kapovich and Leeb [\[10\]](#). For each Seifert component $M_i = F_i \times S^1$ of N , we choose a hyperbolic metric on the base surface F_i so that all boundary components are totally geodesic of unit length, and then equip each Seifert component $M_i = F_i \times S^1$ with the product metric d_i such that the fibers have length one. The metrics d_i on M_i induce the product metrics on \tilde{M}_i , which by abuse of notation are also denoted by d_i .

Let M_i and M_j be adjacent Seifert components in the simple graph manifold N , and let $T \subset M_i \cap M_j$ be a JSJ torus. Each metric space (\tilde{T}, d_i) and (\tilde{T}, d_j) is a Euclidean plane. After applying a homotopy to the gluing map, we may assume that at each JSJ torus T , the gluing map ϕ from the boundary torus $\tilde{T} \subset M_i$ to the boundary torus $\tilde{T} \subset M_j$ is affine in the sense that the identity map $(\tilde{T}, d_i) \rightarrow (\tilde{T}, d_j)$ is affine. We now have a product metric on each Seifert component $M_i = F_i \times S^1$. These metrics may not agree on the JSJ tori but the gluing maps are bilipschitz (since they are affine). The product metrics on the Seifert components induce a length metric on the graph manifold N , denoted by d (see Section 3.1 of [2] for details). Moreover, there exists a positive constant K such that on each Seifert component $M_i = F_i \times S^1$ we have

$$\frac{1}{K}d_i(x, y) \leq d(x, y) \leq Kd_i(x, y)$$

for all x and y in M_i . (See Lemma 1.8 of [14] for a detailed proof of the last claim.) The metric d on N induces a metric on \tilde{N} , which is also denoted by d (by abuse of notation). Then, for all x and y in \tilde{M}_i , we have

$$\frac{1}{K}d_i(x, y) \leq d(x, y) \leq Kd_i(x, y).$$

The following remark introduces certain invariants of a horizontal surface g that will be used in the proof of [Theorem 6.1](#).

Remark 4.14 (1) Since N is compact, there exists a positive lower bound ρ for the distance between any two distinct JSJ planes in \tilde{N} .

(2) We recall that for each curve c_i in \mathcal{T}_g , there exist nonzero integers a_i and b_i such that

$$[g(c_i)] = a_i[\tilde{f}_i] + b_i[\vec{f}_i] \quad \text{in } H_1(T_i).$$

The *governor* of a horizontal surface $g: S \looparrowright N$ in a graph manifold is the quantity $\epsilon = \epsilon(g) = \max\{|a_i/b_i|, |b_i/a_i| \mid c_i \in \mathcal{T}_g\}$. (We use the term “governor” here in the sense of a device used to limit the top speed of a vehicle or engine. In this context the governor limits the rate of growth of the products used in calculating the dilation of curves in the surface.)

Proposition 4.15 For each $[\gamma] \in \pi_1(S, s_0)$ as in [Definition 4.10](#), we define

$$\Lambda_\gamma = \max \left\{ \prod_{i=j}^k |b_i/a_i| \mid 1 \leq j \leq k \leq m \right\}.$$

If the horizontal surface g is a virtual embedding, then there exists a positive constant Λ such that $\Lambda_\gamma \leq \Lambda$ for all $[\gamma] \in \pi_1(S, s_0)$.

Proof Let Γ_g be the graph dual to \mathcal{T}_g , and let n be the number of vertices in Γ_g . Each oriented edge e of Γ_g is dual to a curve $c \in \mathcal{T}_g$ and determines a slope $|b/a|$ as described in Definition 4.10 and Remark 4.14. By Theorem 4.11 the dilation is trivial for any loop in S . Therefore, for each cycle $e_1 \cdots e_m$ in Γ_g the corresponding product of slopes $\prod_{i=1}^m |b_i/a_i|$ is trivial. It follows that for any edge path $e_1 \cdots e_m$ the value of the product $\prod_{i=1}^m |b_i/a_i|$ depends only on the endpoints of the path. Each slope $|b_i/a_i|$ is bounded above by the governor ϵ of g . Since any two vertices of Γ_g are joined by a path of length less than n , the result follows, using $\Lambda = \epsilon^n$. \square

Remark 4.16 Let $g: S \looparrowright N$ be a horizontal surface in a simple graph manifold. Equip N with the metric d described in Remark 4.13. By [12, Lemma 3.1], the surface g can be homotoped to another horizontal surface $g': S \looparrowright N$ such that the following holds: For each curve c in $g'^{-1}(\mathcal{T})$, let T be the JSJ torus in N such that $g'(c) \subset T$. Then $g'(c)$ is *straight* in T in the sense that lifts of $g'(c)$ to \tilde{N} are straight lines in the JSJ planes containing it.

Remark 4.17 Let $f: S^1 \looparrowright T = S^1 \times S^1$ be a horizontal immersion. Then every fiber $\{x\} \times S^1$ in the torus T has nonempty intersection with $f(S^1)$. Indeed, this follows from the fact the composition of f with the natural projection of T to the first factor S^1 is a finite covering map.

5 A lower bound for distortion

The main theorem in this section is the following:

Theorem 5.1 Let $g: (S, s_0) \looparrowright (N, x_0)$ be a horizontal surface in a graph manifold M . Let $G = \pi_1(N, x_0)$ and $H = g_*(\pi_1(S, s_0))$. Then the distortion Δ_H^G is at least quadratic. Furthermore, if the horizontal surface g is not virtually embedded, then the distortion Δ_H^G is at least exponential.

To see the proof of Theorem 5.1, we need several lemmas. For the rest of this section we fix a horizontal surface $g: S \looparrowright N$ in a simple graph manifold N . We equip N with the metric d described in Remark 4.13 and equip S with a hyperbolic metric d_S such that the boundary (if nonempty) is totally geodesic and the simple closed curves of \mathcal{T}_g are geodesics.

Lemma 5.2 *Let γ be any geodesic loop in S such that γ and \mathcal{T}_g have nonempty intersection and such that $w_\gamma \geq 1$. There exists a positive number $A = A(\gamma)$ such that for all $\mu > 0$ the following holds: Let $\{c_1, \dots, c_m\}$ be the sequence of curves of \mathcal{T}_g crossed by γ . The image of the circle $g(c_i)$ in M lies in a JSJ torus T_i . For each $i = 1, 2, \dots, m$, let a_i and b_i be the integers such that*

$$[g(c_i)] = a_i[\vec{f}_i] + b_i[\vec{f}_i] \quad \text{in } H_1(T_i; \mathbb{Z}).$$

Extend the sequence a_1, \dots, a_m to a periodic sequence $\{a_j\}_{j=1}^\infty$ with $a_{j+m} = a_j$ for all $j > 0$, and similarly extend b_1, \dots, b_m to an m -periodic sequence $\{b_j\}_{j=1}^\infty$. Then there exists a (nonperiodic) sequence of integers $\{t(j)\}_{j=1}^\infty$, depending on our choice of the constant μ and the loop γ , with the following properties:

- (1) $|t(j)| \geq \mu$ for all j .
- (2) $|t(j)a_j + t(j-1)b_{j-1}| \leq A$ for all $j > 1$.
- (3) The partial sum $f(n) = \sum_{j=1}^n |t(j)|$ satisfies $f(n) \geq n^2 + w_\gamma^n$.

Proof By the definition of the dilation function, we have $w_\gamma = \prod_{i=1}^m |b_i/a_i|$.

Let $A = \max\{1 + |a_i| \mid i = 1, 2, \dots, m\}$. Set $\xi = \min\{|1/a_j|\}$, and choose $\lambda \in (0, 1]$ so that $\lambda \leq |b_{j-1}/a_j|$ for all $j > 1$. Starting from an initial value $t(1) \geq \mu/\lambda^{m-1}$, we recursively construct an infinite sequence $\{t(j)\}$ satisfying (2). Suppose that $t(j-1)$ has been defined for some $j > 1$, and we wish to define $t(j)$. As $A \geq 1 + |a_j|$, we have

$$1 \leq \frac{A-1}{|a_j|} = \frac{A + |t(j-1)b_{j-1}|}{|a_j|} - \frac{1 + |t(j-1)b_{j-1}|}{|a_j|}.$$

It follows that there is an integer $t(j)$ such that

$$(\clubsuit) \quad \frac{1 + |t(j-1)b_{j-1}|}{|a_j|} \leq |t(j)| \leq \frac{A + |t(j-1)b_{j-1}|}{|a_j|},$$

which is equivalent to

$$1 \leq |t(j)a_j| - |t(j-1)b_{j-1}| \leq A.$$

Furthermore, we are free to choose the sign of $t(j)$ so that $t(j)a_j$ and $t(j-1)b_{j-1}$ have opposite signs, which immediately gives (2). By induction, the sequence $\{t(j)\}$ satisfies both (\clubsuit) and (2) for all $j > 1$.

We next show that any sequence $\{t(j)\}$ satisfying (\clubsuit) also satisfies (3) . Indeed, the first inequality of (\clubsuit) implies

$$(\diamond) \quad |t(j)| \geq \frac{1}{|a_j|} + \left| \frac{b_{j-1}}{a_j} \right| |t(j-1)| \geq \left| \frac{b_{j-1}}{a_j} \right| |t(j-1)| \quad \text{for all } j > 1.$$

For any $j > m$, we apply (\diamond) iteratively m times (and use that the sequences $\{a_j\}$ and $\{b_j\}$ are m -periodic) to get

$$\begin{aligned} (\heartsuit) \quad |t(j)| &\geq \frac{1}{|a_j|} + \left| \frac{b_{j-1}}{a_j} \right| |t(j-1)| \\ &\geq \frac{1}{|a_j|} + \left| \frac{b_{j-1}b_{j-2}}{a_j a_{j-1}} \right| |t(j-2)| \\ &\quad \vdots \\ &\geq \frac{1}{|a_j|} + \left| \frac{b_1 \cdots b_m}{a_1 \cdots a_m} \right| |t(j-m)| \\ &\geq \xi + w_\gamma |t(j-m)|. \end{aligned}$$

Further applying (\heartsuit) iteratively k times (and using that $w_\gamma \geq 1$) gives

$$\begin{aligned} (\spadesuit) \quad |t(j)| &\geq \xi + w_\gamma |t(j-m)| \\ &\geq \xi + w_\gamma (\xi + w_\gamma |t(j-2m)|) \\ &\geq 2\xi + w_\gamma^2 |t(j-2m)| \\ &\quad \vdots \\ &\geq \xi k + w_\gamma^k |t(j-km)| \quad \text{for all } j > km. \end{aligned}$$

The inequality (\spadesuit) can be rewritten in the form

$$|t(km+1)| \geq \xi k + |t(1)| w_\gamma^k \quad \text{for all } k > 0.$$

Finally, for each positive n we observe that

$$\sum_{j=1}^{nm} |t(j)| \geq \sum_{k=1}^{n-1} |t(km+1)| \geq \xi \sum_{k=1}^{n-1} k + |t(1)| \sum_{k=1}^{n-1} w_\gamma^k,$$

which implies (3) , as desired.

In order to establish (1) , recall that γ satisfies $w_\gamma \geq 1$. Therefore, (\heartsuit) implies that $|t(j)| \geq |t(j-m)|$ for any $j > m$. In particular, it follows that the terms of the sequence $\{t(j)\}$ have absolute value bounded below by the absolute values of the first m terms, $t(1), \dots, t(m)$.

Using our choice of $t(1) \geq \mu/\lambda^{m-1}$ and the fact $\lambda \leq |b_{j-1}/a_j|$, the inequality (\diamond) shows that for all $i = 1, \dots, m$ we have

$$|t(i)| \geq \lambda^{i-1} |t(1)| \geq \frac{\mu}{\lambda^{m-i}}.$$

As $\lambda \leq 1$, we conclude that $|t(i)| \geq \mu$, completing the proof of (1). \square

Definition 5.3 (spirals) Let γ be a closed curve in S satisfying the conclusion of Lemma 3.3. Let $\{c_1, \dots, c_m\}$ be the sequence of curves of \mathcal{C} crossed by γ . Let $\{\gamma_1, \dots, \gamma_m\}$ be the sequence of subpaths of γ introduced in Definition 4.10. Extend the finite sequences of curves c_1, \dots, c_m and $\gamma_1, \dots, \gamma_m$ to m -periodic infinite sequences $\{c_j\}_{j=1}^\infty$ and $\{\gamma_j\}_{j=1}^\infty$. Let $\{t(j)\}$ be a sequence of integers. We write $\alpha_j = c_j^{t(j)}$.

Choose the basepoint $s_0 \in S$ to be a point of intersection between γ and one of the curves of the family \mathcal{C} . For each $n \in \mathbb{N}$, we define a *spiral* loop σ_n in S based at s_0 as a concatenation,

$$\sigma_n = \alpha_1 \gamma_1 \cdots \alpha_{nm} \gamma_{nm},$$

and a *double spiral* loop ρ_n of σ_n in S based at s_0 as $\rho_n = \sigma_n \alpha'_{nm+1} \sigma_n^{-1}$, where $\alpha'_{nm+1} = c_{nm+1}^{t(1)}$.

Lemma 5.4 Let $\{t(j)\}_{j=1}^\infty$ be the sequence of integers given by Lemma 5.2. Let ρ_n be the double spiral loop of σ_n corresponding to the curve γ and the sequence $\{t(j)\}$. Let $\tilde{\rho}_n$ be the lift of ρ_n in \tilde{S} . Then the distance in \tilde{N} between the endpoints of $\tilde{g}(\tilde{\rho}_n)$ is bounded above by a linear function of n .

Proof First we describe informally the idea of the proof, which is illustrated in Figure 1. The lift $\tilde{\sigma}_n$ of σ_n is the spiral-shaped curve running around the outside of the left-hand diagram. The path $\tilde{\sigma}_n$ alternates between long segments $\tilde{\alpha}_j$ belonging to JSJ planes and short segments $\tilde{\gamma}_j$ belonging to Seifert components. Each long segment $\tilde{\alpha}_j$ is one side of a large triangle in the JSJ plane whose other two sides are fibers of the adjacent blocks, which meet in a corner y_j opposite to $\tilde{\alpha}_j$. Connecting each pair of adjacent corners y_j produces a thin trapezoid that interpolates between two adjacent JSJ triangles. We will see that the sequence of exponents $\{t(j)\}$ in the construction of $\tilde{\sigma}_n$ was chosen carefully to ensure that the distances between adjacent corners y_j are all short, ie bounded above. Thus the path running around the inside of the spiral has at most a linear length — except for its last segment, which is a long side of a large JSJ triangle. The double spiral gives rise to a diagram similar to the spiral diagram,

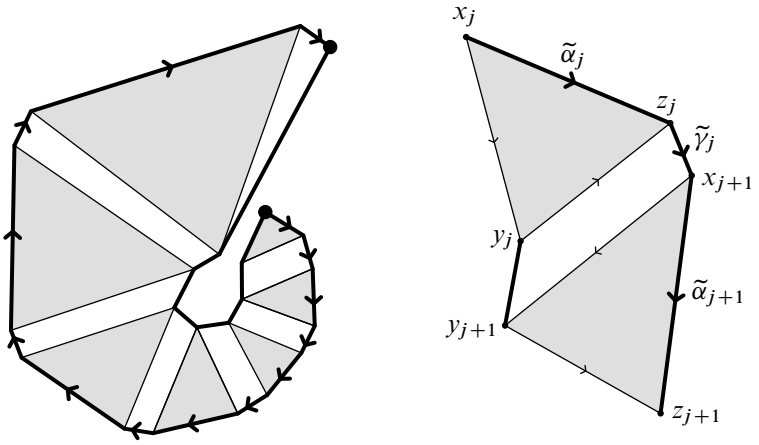


Figure 1: On the left, the piecewise geodesic $\tilde{\sigma}_n$ is the spiral-shaped path running around the outside of the diagram. On the right is a magnified portion of the left-hand picture showing more detail.

except that it has been doubled along this long triangular side, so that the long side no longer appears on the boundary of the diagram but rather appears in its interior (see Figure 2).

To be more precise, let M_j be the Seifert component of N containing γ_j , with its given product metric as a hyperbolic surface crossed with a circle of length one. Let (\tilde{M}_j, d_j) be the Seifert component of \tilde{N} that contains $\tilde{\gamma}_j$, with the product metric d_j induced by lifting the given metric on M_j . Let \tilde{T}_j be the JSJ plane containing $\tilde{\alpha}_j$. Recall

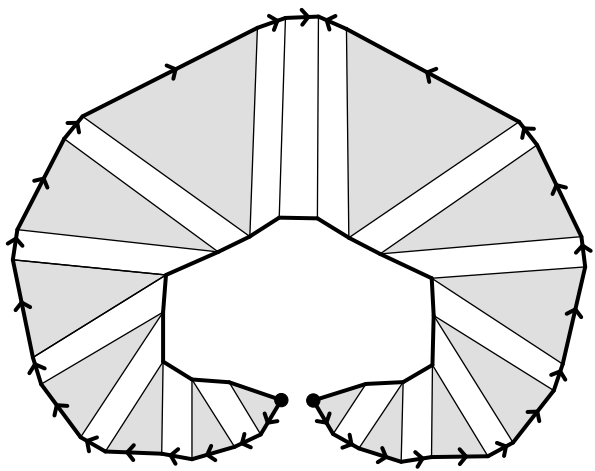


Figure 2: The outer path represents the piecewise geodesic $\tilde{\rho}_n$.

that \tilde{T}_j is a topological plane that is a subspace of both \tilde{M}_{j-1} and \tilde{M}_j . However, the metrics d_{j-1} and d_j typically do not agree on the common subspace \tilde{T}_j . Each metric space $\tilde{E}_j = (\tilde{T}_j, d_{j-1})$ and $\tilde{E}_j = (\tilde{T}_j, d_j)$ is a Euclidean plane and the identity map $\tilde{E}_j \rightarrow \tilde{E}_j$ is affine.

The plane \tilde{T}_j universally covers a JSJ torus T_j obtained by identifying boundary tori \tilde{T}_j and \tilde{T}_j of Seifert components M_{j-1} and M_j . The initial and terminal points x_j and z_j of $\tilde{\alpha}_j$ are contained in Euclidean geodesics $\tilde{\ell}_j \subset \tilde{E}_j$ and $\tilde{\ell}_j \subset \tilde{E}_j$ that project to fibers $\tilde{f}_j \subset \tilde{T}_j$ and $\tilde{f}_j \subset \tilde{T}_j$, respectively. The lines $\tilde{\ell}_j$ and $\tilde{\ell}_j$ intersect in a unique point y_j . Similarly, we consider the subpath $\tilde{\alpha}_j^{-1}$ in the double spiral $\tilde{\rho}_n$. Let \bar{y}_j be the intersection point of the two fibers that contain its endpoints.

Our goal is to find a linear upper bound for the distance in (N, d) between the endpoints of $\tilde{\rho}_n$. By the triangle inequality it suffices to produce an upper bound for the distance between successive points of the linear sequence $y_1, \dots, y_{nm}, \bar{y}_{nm}, \dots, \bar{y}_1$. Recall that the inclusions $(\tilde{M}_j, d_j) \rightarrow (\tilde{N}, d)$ are K -bilipschitz for some universal constant K , as explained in [Remark 4.13](#). Thus it is enough to bound the distance between these points with respect to the given product metrics on each Seifert component.

Let η be the maximum of the lengths of $\tilde{\gamma}_j$ with respect to the metric d_j . Let A be the constant given by [Lemma 5.2](#). We claim that

$$(\star) \quad d_j(y_j, y_{j+1}) \leq \eta + A.$$

We prove this claim by examining the quadrilateral with vertices y_j, z_j, x_{j+1} and y_{j+1} , illustrated [Figure 1](#), right. This quadrilateral is a trapezoid in the sense that the opposite sides $[y_j, z_j]$ and $[y_{j+1}, x_{j+1}]$ lie in fibers of \tilde{M}_j that are parallel lines in the product metric d_j .

To find the lengths of these parallel sides, we must first examine the JSJ triangle $\Delta(x_j, y_j, z_j)$, also shown in [Figure 1](#), right. We consider this triangle to be a pair of homotopic paths $\tilde{\alpha}_j$ and τ_j from x_j to y_j in the plane \tilde{T}_j , where $\tau_j = [x_j, y_j] \cup [y_j, z_j]$ is a segment of $\tilde{\ell}_j$ concatenated with a segment of $\tilde{\ell}_j$. In particular, $\tilde{\alpha}_j$ and τ_j project to homotopic loops in T_j of the form $g(\alpha_j)$ and $(\tilde{f}_j)^{r_j}(\tilde{f}_j)^{s_j}$ for some $r_j, s_j \in \mathbb{Z}$. Recall that the fiber \tilde{f}_j has length one in the product metric on M_{j-1} . Similarly, the fiber \tilde{f}_j has length one in M_j . It follows that $d_{j-1}(x_j, y_j) = |r_j|$ and $d_j(y_j, z_j) = |s_j|$. Since $g(\alpha_j) = g(c_j)^{t(j)}$, the homology relation (written additively)

$$t(j)[g(c_j)] = t(j)a_j[\tilde{f}_j] + t(j)b_j[\tilde{f}_j] \quad \text{in } H_1(T_j; \mathbb{Z}) \approx \pi_1(T_j)$$

implies that $r_j = t(j)a_j$ and $s_j = t(j)b_j$.

Let β_j be the lift of γ_j based at y_j . The fiber which contains y_{j+1} and x_{j+1} will intersect β_j at exactly one point. We denote this point by u_j . It follows that $d_j(u_j, y_j) \leq \eta$ and $d_j(u_j, y_{j+1}) = |t(j+1)a_{j+1} + t(j)b_j|$. Using the triangle inequality for $\Delta(y_j, u_j, y_{j+1})$, we have

$$d_j(y_j, y_{j+1}) \leq d_j(y_j, u_j) + d_j(u_j, y_{j+1}) \leq \eta + |t(j+1)a_{j+1} + t(j)b_j| \leq \eta + A$$

by [Lemma 5.2\(2\)](#), completing the proof of (\star) .

For a similar reason, the distance in the corresponding Seifert component between \bar{y}_j and \bar{y}_{j-1} is at most $\eta + A$. Thus it suffices to find an upper bound for $d_{nm}(y_{nm}, \bar{y}_{nm})$. Let R be the length of \tilde{c}_1 in the metric d_{nm} . Let \bar{z}_{nm} be the initial point of $\tilde{\alpha}_{nm}^{-1}$. Since $\alpha'_{nm} = c_{nm+1}^{t(1)}$, the d_{nm} -distance between the endpoints of $\tilde{\alpha}'_{nm+1}$ is at most $R|t(1)|$. By the triangle inequality, $d_{nm}(z_{nm}, \bar{z}_{nm}) \leq 2\eta + R|t(1)|$.

We note that $[z_{nm}, y_{nm}]$ and $[\bar{z}_{nm}, \bar{y}_{nm}]$ lie in parallel fibers of \tilde{M}_{nm} . Furthermore, they are oriented in the same direction with respect to the fiber and have the same length. Therefore, they form opposite sides of a Euclidean parallelogram in \tilde{M}_{nm} . In particular, the distances $d_{nm}(y_{nm}, \bar{y}_{nm})$ and $d_{nm}(z_{nm}, \bar{z}_{nm})$ are equal, so that $d_{nm}(y_{nm}, \bar{y}_{nm}) \leq 2\eta + R|t(1)|$. \square

Proof of Theorem 5.1 Any graph manifold has a simple finite cover by [Theorem 4.3](#). By [Corollary 4.7](#) and [Proposition 2.9](#), it suffices to prove the theorem for all horizontal surfaces in this cover. Thus we assume, without loss of generality, that N is a simple graph manifold.

Let γ be any geodesic loop in S such that γ and \mathcal{T}_g have nonempty intersection. The existence of such a loop is guaranteed by [Lemma 3.3](#). Replacing γ with γ^{-1} if necessary, we may assume that $w_\gamma \geq 1$. Let κ be the maximum of lengths of γ_i with respect to the metric d_S .

Let μ , L and C be the constants given by [Lemma 3.4](#). Let $\{t(j)\}$ be the sequence of integers given by [Lemma 5.2](#). For each n , let σ_n be the spiral loop corresponding to the curve γ and the sequence $\{t(j)\}$. Let ρ_n be the double spiral of σ_n . Let \mathcal{L} be the family of lines that are lifts of loops of \mathcal{T}_g or boundary loops of S . Since $\tilde{\rho}_n$ satisfies the hypotheses of [Lemma 3.4](#), it is an (L, C) -quasigeodesic in \tilde{S} .

Let h_n be the homotopy class of loop ρ_n at the basepoint s_0 . We claim that

$$d_{\tilde{S}}(\tilde{s}_0, h_n(\tilde{s}_0)) \geq n^2 + w_\gamma^n.$$

Indeed, let r be the minimum of the lengths of c_i with respect to the metric d_S . By the construction of the spiral loop σ_n we have $|\tilde{\sigma}_n| \geq r \sum_{j=1}^{nm} |t(j)|$. By Lemma 5.2(3), we have $|\tilde{\sigma}_n| \geq n^2 + w_\gamma^n$. It is obvious that $|\tilde{\rho}_n| \geq n^2 + w_\gamma^n$ because $|\tilde{\rho}_n| \geq 2|\tilde{\sigma}_n|$. Since $\tilde{\rho}_n$ is an (L, C) -quasigeodesic, it follows that $d_{\tilde{S}}(\tilde{s}_0, h_n(\tilde{s}_0)) \geq n^2 + w_\gamma^n$.

Furthermore, we have $d(\tilde{x}_0, h_n(\tilde{x}_0)) \leq n$ by Lemma 5.4. Therefore, $n^2 + w_\gamma^n \leq \Delta_H^G$ by Corollary 2.10. It follows that Δ_H^G is at least quadratic. If the horizontal surface is not virtually embedded, then we may choose the geodesic loop γ in S so that $w_\gamma > 1$, by Theorem 4.11. In this case w_γ is an exponential function, and $w_\gamma^n \leq \Delta_H^G$. \square

6 Upper bound of distortion

In this section, we will find the upper bound of the distortion of horizontal surface. The main theorem in this section is the following:

Theorem 6.1 *Let $g: (S, s_0) \looparrowright (N, x_0)$ be a horizontal surface in a graph manifold N . Let $G = \pi_1(N, x_0)$ and $H = g_*(\pi_1(S, s_0))$. Then the distortion Δ_H^G is at most exponential. Furthermore, if the horizontal surface g is virtually embedded then Δ_H^G is at most quadratic.*

Definition 6.2 Lift the JSJ decomposition of the simple graph manifold N to the universal cover \tilde{N} , and let T_N be the tree dual to this decomposition of \tilde{N} . Lift the collection \mathcal{T}_g to the universal cover \tilde{S} . The tree dual to this decomposition of \tilde{S} will be denoted by T_S . The map \tilde{g} induces a map $\zeta: T_S \rightarrow T_N$.

The following lemma plays an important role in the proof of Theorem 6.1:

Lemma 6.3 *Let F be a connected compact surface with nonempty boundary and $\chi(F) < 0$. Let $M = F \times S^1$. Let $g: (B, b) \looparrowright (M, x)$ be a horizontal surface. Then each fiber in \tilde{M} intersects with $\tilde{g}(\tilde{B})$ exactly at one point.*

Proof According to Lemma 2.1 in [17], there exists a finite covering map $p: B \times S^1 \rightarrow M$ and an embedding $i: B \rightarrow B \times S^1$ given by $i(x) = (x, 1)$ such that $g = p \circ i$. Let $\tilde{i}: \tilde{B} \rightarrow \tilde{B} \times \mathbb{R}$ be the lift of i such that $\tilde{i}(\tilde{b}) = (\tilde{b}, 0)$. Let $\phi: \tilde{B} \rightarrow B$ and $\Psi: \tilde{F} \rightarrow F$ be the universal covering maps. Let $\pi: \mathbb{R} \rightarrow S^1$ be the usual covering space. Since $\tilde{B} \times \mathbb{R}$ and $\tilde{F} \times \mathbb{R}$ both universal cover M , there exists a homeomorphism $\omega: (\tilde{B} \times \mathbb{R}, (\tilde{b}, 0)) \rightarrow (\tilde{F} \times \mathbb{R}, \tilde{g}(\tilde{b}))$ such that $(\Psi \times \pi) \circ \omega = p \circ (\phi \times \pi)$. By the

unique lifting property, we have $\omega \circ \tilde{\iota} = \tilde{g}$. It follows that $\tilde{g}(\tilde{B}) = \omega(\tilde{B} \times \{0\})$. Since ω maps each fiber in $\tilde{B} \times \mathbb{R}$ to a fiber in $\tilde{F} \times \mathbb{R}$, it follows that each fiber in $\tilde{F} \times \mathbb{R}$ intersects $\omega(\tilde{B} \times \{0\})$ exactly at one point. \square

Proposition 6.4 *The map ζ is bijective.*

Proof A simplicial map between trees is bijective if it is locally bijective. Thus it suffices to show that the map ζ is locally injective and locally surjective (see [19] for details).

Suppose by way of contradiction that ζ is not locally injective. Then there exist three distinct blocks \tilde{B}_1 , \tilde{B}_2 and \tilde{B}_3 in \tilde{S} such that $\tilde{B}_1 \cap \tilde{B}_2 \neq \emptyset$ and $\tilde{B}_2 \cap \tilde{B}_3 \neq \emptyset$ and such that the images $\tilde{g}(\tilde{B}_1)$ and $\tilde{g}(\tilde{B}_3)$ lie in the same block \tilde{M}_1 of \tilde{M} . Let \tilde{M}_2 denote the block containing the image $\tilde{g}(\tilde{B}_2)$. We write $\ell_1 = \tilde{B}_1 \cap \tilde{B}_2$ and $\ell_3 = \tilde{B}_2 \cap \tilde{B}_3$. We have that $\tilde{g}(\ell_1)$ and $\tilde{g}(\ell_3)$ are subsets of the JSJ plane $\tilde{T} = \tilde{M}_1 \cap \tilde{M}_2$. Since the lines ℓ_1 and ℓ_3 are disjoint and the map \tilde{g} is an embedding, it follows that $\tilde{g}(\ell_1)$ and $\tilde{g}(\ell_3)$ are disjoint lines in the plane \tilde{T} . By Remark 4.17, any fiber in the plane \tilde{T} intersects $\tilde{g}(\ell_1)$ and $\tilde{g}(\ell_3)$ at distinct points. This contradicts Lemma 6.3 because $\tilde{g}(\ell_1)$ and $\tilde{g}(\ell_3)$ are subsets of $\tilde{g}(\tilde{B}_2)$. Therefore, ζ is locally injective.

By Lemma 6.3, the block $\tilde{g}(\tilde{B})$ must intersect every fiber of the Seifert component \tilde{M} containing it. In particular, $\tilde{g}(\tilde{B})$ intersects every JSJ plane adjacent to \tilde{M} . Therefore, the map ζ is locally surjective. \square

The following corollary is a combination of Lemma 6.3 and Proposition 6.4:

Corollary 6.5 *Each fiber of \tilde{N} intersects $\tilde{g}(\tilde{S})$ in one point.*

Remark 6.6 Let G and H be finitely generated groups with generating sets \mathcal{A} and \mathcal{B} , respectively. Let $\phi: G \rightarrow H$ be a homomorphism. Then there exists a positive number L such that $|\phi(g)|_{\mathcal{B}} \leq L|g|_{\mathcal{A}}$ for all g in G . Indeed, suppose that $\mathcal{A} = \{g_1, g_2, \dots, g_n\}$; we define $L = \max\{|\phi(g_i)|_{\mathcal{B}} \mid i = 1, 2, \dots, n\}$. Since ϕ is a homomorphism, it is not hard to see that $|\phi(g)|_{\mathcal{B}} \leq L|g|_{\mathcal{A}}$ for all $g \in G$.

The following proposition shows that the distortion of a horizontal surface in a trivial Seifert manifold is linear:

Proposition 6.7 *Let F be a connected compact surface with nonempty boundary and $\chi(F) < 0$. Let $g: (B, b) \looparrowright (M, x)$ be a horizontal surface, where $M = F \times S^1$. Let $H = g_*(\pi_1(B, b))$ and $G = \pi_1(M, x)$. Then $H \hookrightarrow G$ is a quasi-isometric embedding.*

Proof We first choose generating sets for $\pi_1(B)$, $\pi_1(F)$ and $\pi_1(S^1)$. The generating sets of $\pi_1(F)$ and $\pi_1(S^1)$ induce a generating set on $\pi_1(M)$. Let $g_1: B \rightarrow F$ and $g_2: B \rightarrow S^1$ be the maps such that $g = (g_1, g_2)$. We have that $g_1: B \rightarrow F$ is a finite covering map because g is a horizontal surface in M . It follows that $g_{1*}(\pi_1(B))$ is a finite-index subgroup of $\pi_1(F)$. As a result, g_{1*} is an $(L', 0)$ -quasi-isometry for some constant L' . Since $g_* = (g_{1*}, g_{2*})$ we have $|g_*(h)| \geq |g_{1*}(h)| \geq |h|/L'$ for all $h \in \pi_1(B)$. Applying [Remark 6.6](#) to the homomorphism g_* , the constant L' can be enlarged so that we can show that g_* is an $(L', 0)$ -quasi-isometric embedding. \square

For the rest of this section, we fix $g: (S, s_0) \looparrowright (N, x_0)$ a horizontal surface in a simple graph manifold N , the metric d given by [Remark 4.13](#), and the hyperbolic metric d_S on S described in [Section 5](#). We also assume that, for each curve c in \mathcal{T}_g , its image $g(c)$ is straight in the sense of [Remark 4.16](#) in the JSJ torus that contains it. We define a metric $d_{\tilde{g}(\tilde{S})}$ on $\tilde{g}(\tilde{S})$ as the following: for any $u = \tilde{g}(x)$ and $v = \tilde{g}(y)$, we define $d_{\tilde{g}(\tilde{S})}(u, v) = d_{\tilde{S}}(x, y)$. The following corollary follows by combining [Proposition 6.7](#) with several earlier results, using the fact that S has only finitely many blocks and N has only finitely many Seifert components.

Corollary 6.8 *There exist numbers L and C such that the following holds: For each block \tilde{B} in \tilde{S} , let $\tilde{M} = \tilde{F} \times \mathbb{R}$ be the Seifert component of \tilde{N} such that $\tilde{g}(\tilde{B}) \subset \tilde{M}$. The map $\tilde{g}|_{\tilde{B}}: \tilde{B} \rightarrow \tilde{M} = \tilde{F} \times \mathbb{R}$ can be expressed as a pair of maps $\tilde{g}_1: \tilde{B} \rightarrow \tilde{F}$ and $\tilde{g}_2: \tilde{B} \rightarrow \mathbb{R}$. Then \tilde{g}_1 and $\tilde{g}|_{\tilde{B}}$ are (L, C) -quasi-isometric embeddings, and*

$$|\tilde{g}_2(u) - \tilde{g}_2(v)| \leq L d_M(\tilde{g}_1(u), \tilde{g}_1(v)) + C$$

for all $u, v \in \tilde{B}$.

Proof The map $g|_B: B \rightarrow F \times S^1$ can be written as (g_1, g_2) . Since the map $g_1: B \rightarrow F$ is a finite covering map, the lift \tilde{g}_1 is a quasi-isometry. It follows from [Proposition 6.7](#) that $\tilde{g}|_{\tilde{B}}$ is a quasi-isometric embedding. The fact that \tilde{g}_1 and $\tilde{g}|_{\tilde{B}}$ are quasi-isometrically embedded implies the final claim. \square

Now we describe informally the strategy of the proof of [Theorem 6.1](#) in the case N is a simple graph manifold. For each $n \in \mathbb{N}$, let $u \in \tilde{g}(\tilde{S})$ be such that $d(\tilde{x}_0, u) \leq n$. We would like to find an upper bound (either quadratic or exponential as appropriate) for $d_{\tilde{S}}(\tilde{x}_0, u)$ in terms of n . We first show the existence of a path ξ in \tilde{N} connecting \tilde{x}_0 to u and passing through k Seifert components $\tilde{M}_0, \dots, \tilde{M}_{k-1}$ such that ξ intersects the plane $\tilde{T}_i = \tilde{M}_{i-1} \cap \tilde{M}_i$ at exactly one point, denoted by y_i . We will show that k is bounded above by a linear function of n . Therefore, it suffices to find an upper

bound for $d_{\tilde{g}}(\tilde{x}_0, u)$ in terms of k . Let d_i be the given product metric on \tilde{M}_i . By [Corollary 6.5](#) the fiber of \tilde{M}_{i-1} passing through y_i intersects $\tilde{g}(\tilde{S})$ in a unique point, which is denoted by x_i . Similarly, the fiber of \tilde{M}_i passing through y_i intersects $\tilde{g}(\tilde{S})$ in one point, which is denoted by z_i .

On the one hand, proving that the distance in $\tilde{g}(\tilde{S})$ between the endpoints of ξ is dominated by the sum $\sum_{i=1}^{k-1} (d_{i-1}(y_i, x_i) + d_i(y_i, z_i))$ is easy. On the other hand, finding a quadratic or exponential upper bound for this sum as a function of k requires more work. Our strategy is to analyze the growth of the sequence of numbers

$$\begin{aligned} d_0(y_1, x_1), d_1(y_1, z_1), d_1(y_2, x_2), \dots, \\ d_{i-2}(y_{i-1}, x_{i-1}), d_{i-1}(y_{i-1}, z_{i-1}), d_{i-1}(y_i, x_i), \dots, \\ d_{k-2}(y_{k-1}, x_{k-1}), d_{k-1}(y_{k-1}, z_{k-1}). \end{aligned}$$

A relation between $d_{i-1}(y_{i-1}, z_{i-1})$ and $d_{i-1}(y_i, x_i)$ in the Seifert component \tilde{M}_{i-1} will be described in [Lemma 6.9](#). The ratio of $d_{i-1}(y_{i-1}, z_{i-1})$ to $d_{i-2}(y_{i-1}, x_{i-1})$ in the JSJ plane \tilde{T}_{i-1} will be described in [Lemma 6.10](#).

Lemma 6.9 (crossing a Seifert component) *There exists a positive constant L' such that the following holds: For each block \tilde{B} in \tilde{S} , let $\tilde{M} = \tilde{F} \times \mathbb{R}$ be the Seifert component such that $\tilde{g}(\tilde{B}) \subset \tilde{M}$. Let d_M be the given product metric on \tilde{M} and let \tilde{T} and \tilde{T}' be two disjoint JSJ planes in the Seifert component \tilde{M} . For any two points $y \in \tilde{T}$ and $y' \in \tilde{T}'$, let $\ell \subset \tilde{T}$ and $\ell' \subset \tilde{T}'$ be the lines that project to the fiber S^1 in M such that $y \in \ell$ and $y' \in \ell'$. Let x (resp. x') be the unique intersection point of ℓ (resp. ℓ') with $\tilde{g}(\tilde{B}) \cap \tilde{M}$ given by [Lemma 6.3](#). Then*

$$d_M(y', x') \leq d_M(y, x) + L' d_M(y, y').$$

Proof Let ρ and K be the constants given by [Remarks 4.13](#) and [4.14](#), respectively, and let $D = \rho/K$. Let L and C be the constants given by [Corollary 6.8](#). Let $L' = L + 2 + C/D$.

Let a and b be the projection points of y and y' to \tilde{F} , respectively. We write $\tilde{g}|_{\tilde{B}} = (\tilde{g}_1, \tilde{g}_2)$, where $\tilde{g}_1: \tilde{B} \rightarrow \tilde{F}$ and $\tilde{g}_2: \tilde{B} \rightarrow \mathbb{R}$. Since $\tilde{g}|_{\tilde{B}}$ is an embedding map and $x, x' \in \tilde{g}(\tilde{B})$, there exist $a', b' \in \tilde{B}$ such that $\tilde{g}(a') = x$ and $\tilde{g}(b') = x'$. It follows that $\tilde{g}_1(a') = a$ and $\tilde{g}_1(b') = b$. By [Corollary 6.8](#) we have

$$|\tilde{g}_2(a') - \tilde{g}_2(b')| \leq L d_M(\tilde{g}_1(a'), \tilde{g}_1(b')) + C = L d_M(a, b) + C.$$

Since $\rho \leq d(a, b) \leq Kd_M(a, b)$, it follows that $D \leq d_M(a, b)$. Therefore,

$$|\tilde{g}_2(a') - \tilde{g}_2(b')| \leq \left(L + \frac{C}{D}\right)d_M(a, b) \leq \left(L + \frac{C}{D}\right)d_M(y, y').$$

With respect to the orientation of the factor \mathbb{R} of \tilde{M} , let $\Delta(y, a)$ and $\Delta(y', b)$ be the displacements of pairs of points (y, a) and (y', b) , respectively. We would like to show that $|\Delta(y, a) - \Delta(y', b)| \leq 2d_M(y, y')$. Indeed, let s and t be the real numbers such that $y = (a, s)$ and $y' = (b, t)$. We note that $\Delta(y, a) = -s$ if $s \geq 0$ and $\Delta(y, a) = s$ if $s \leq 0$, as well as $\Delta(y', b) = -t$ if $t \geq 0$ and $\Delta(y', b) = t$ if $t \leq 0$. Since $d_M(a, b) \leq d_M(y, y')$, it follows that $|\Delta(y, a) - \Delta(y', b)| \leq 2d_M(y, y')$. Moreover, we have that $d_M(y, x) = |\tilde{g}_2(a') + \Delta(y, a)|$ and $d_M(y', x') = |\tilde{g}_2(b') + \Delta(y', b)|$. Therefore, the previous inequalities imply

$$\begin{aligned} d_M(y', x') - d_M(y, x) &\leq |\tilde{g}_2(a') - \tilde{g}_2(b')| + |\Delta(y, a) - \Delta(y', b)| \\ &\leq \left(L + 2 + \frac{C}{D}\right)d_M(y, y') = L'd_M(y, y'). \quad \square \end{aligned}$$

Lemma 6.10 (crossing a JSJ plane) *Let \tilde{M} and \tilde{M}' be the two adjacent Seifert components. Let d_M and $d_{M'}$ be the given product metrics on \tilde{M} and \tilde{M}' , respectively. Let $\tilde{T} = \tilde{M} \cap \tilde{M}'$ and let $\alpha \subset \tilde{g}(\tilde{S}) \cap \tilde{T}$ be the line such that α universally covers a curve $g(c)$ for some c in \mathcal{T}_g . We moreover assume that α is a straight line in (\tilde{T}, d_M) . For any two points x and z in the line α , let $\tilde{\ell} \subset (\tilde{T}, d_M)$ and $\tilde{\ell}' \subset (\tilde{T}, d_{M'})$ be the Euclidean geodesics such that $x \in \tilde{\ell}$ and $z \in \tilde{\ell}$ and they project to fibers $\tilde{f} \subset \tilde{T}$ and $\tilde{f}' \subset \tilde{T}$, respectively. Let y be the unique intersection point of $\tilde{\ell}$ and $\tilde{\ell}'$. Let a and b be the integers such that*

$$[g(c)] = a[\tilde{f}] + b[\tilde{f}'] \quad \text{in } H_1(T; \mathbb{Z}),$$

where T is the JSJ torus obtained from gluing \tilde{T} to \tilde{T}' . Then

$$d_M(y, x) = \left| \frac{a}{b} \right| d_{M'}(y, z).$$

Proof Let κ be the positive constant such that length of the fiber \tilde{f} with respect to the metric d_M equals to $1/\kappa$. Let \tilde{c} be a path lift of c such that $\tilde{g}(\tilde{c}) \subset \alpha$. Let x' and z' be the initial point and the terminal point of $\tilde{g}(\tilde{c})$, respectively. Let $\tilde{\ell}'$ and $\tilde{\ell}'$ be the Euclidean geodesics in (\tilde{T}, d_M) and $(\tilde{T}, d_{M'})$, respectively, such that $x' \in \tilde{\ell}'$ and $z' \in \tilde{\ell}'$ and both lines $\tilde{\ell}'$ and $\tilde{\ell}'$ project to fibers in \tilde{T} and \tilde{T}' , respectively. Let y' be the unique intersection point of $\tilde{\ell}'$ and $\tilde{\ell}'$. It was shown in the proof of [Lemma 5.4](#) that $d_M(y', x') = |a|$ and $d_{M'}(y', z') = |b|$. By the definition of κ , it follows that $d_{M'}(y', z') = \kappa d_M(y', z')$.

In the Euclidean plane (\tilde{T}, d_M) , consider the similar triangles

$$\Delta(x, y, z) \quad \text{and} \quad \Delta(x', y', z').$$

Since $d_M(y, x)/d_M(y, z) = d_M(y', x')/d_M(y', z') = \kappa|a/b|$, it follows that

$$d_M(y, x) = \left| \frac{a}{b} \right| \kappa d_M(y, z) = \left| \frac{a}{b} \right| d_{M'}(y, z). \quad \square$$

Proof of Theorem 6.1 We may assume that N is a simple graph manifold for the same reason as in the first paragraph of the proof of Theorem 5.1. Let K be the constant given by Remark 4.13. Let L and C be the constants given by Corollary 6.8. Moreover, the constant L can be enlarged so that $L \geq K$ and L is greater than the constant L' given by Lemma 6.9. Moreover, we assume that the basepoint s_0 belongs to a curve in the collection \mathcal{T}_g .

For any $h \in \pi_1(S, s_0)$ such that $d(\tilde{x}_0, h(\tilde{x}_0)) \leq n$, we will show that $d_{\tilde{S}}(\tilde{s}_0, h(\tilde{s}_0))$ is bounded above by either a quadratic or exponential function in terms of n . The theorem is confirmed by an application of Corollary 2.10. We consider the following cases:

Case 1 (the points \tilde{s}_0 and $h(\tilde{s}_0)$ belong to the same block \tilde{B}) In this degenerate case, we will show that $d_{\tilde{S}}(\tilde{s}_0, h(\tilde{s}_0)) \leq n$. Indeed, let \tilde{M} be the Seifert component such that $\tilde{g}(\tilde{B}) \subset \tilde{M}$. Let $d_{\tilde{M}}$ be the given product metric on \tilde{M} . Since $\tilde{g}|_{\tilde{B}}$ is an (L, C) -quasi-isometric embedding by Corollary 6.8, it follows that

$$d_{\tilde{g}(\tilde{B})}(\tilde{x}_0, h(\tilde{x}_0)) \leq L d_{\tilde{M}}(\tilde{x}_0, h(\tilde{x}_0)) + C \leq L^2 d(\tilde{x}_0, h(\tilde{x}_0)) + C \leq L^2 n + C.$$

Therefore,

$$d_{\tilde{S}}(\tilde{s}_0, h(\tilde{s}_0)) = d_{\tilde{g}(\tilde{S})}(\tilde{x}_0, h(\tilde{x}_0)) \leq d_{\tilde{g}(\tilde{B})}(\tilde{x}_0, h(\tilde{x}_0)) \leq L^2 n + C.$$

Case 2 (the points \tilde{s}_0 and $h(\tilde{s}_0)$ belong to distinct blocks of \tilde{S}) Let \mathcal{L} be the family of lines in \tilde{S} that are lifts of curves of \mathcal{T}_g . Since we assume that s_0 belongs to a curve in the collection \mathcal{T}_g , there are distinct lines α and α' in \mathcal{L} such that $\tilde{s}_0 \in \alpha$ and $h(\tilde{s}_0) \in \alpha'$. Let e and e' be the nonoriented edges in the tree T_S corresponding to the lines α and α' , respectively. Consider the nonbacktracking path joining e to e' in the tree T_S , with ordered vertices v_0, v_1, \dots, v_{k-1} , where v_1 is not a vertex on the edge e and v_{k-2} is not a vertex on the edge e' . Consider the corresponding vertices $\zeta(v_0), \dots, \zeta(v_{k-1})$ of the tree T_N (see Definition 6.2). We denote the Seifert components corresponding to the vertices $\zeta(v_i)$ by \tilde{M}_i with $i = 0, 1, \dots, k-1$. We note that the Seifert components \tilde{M}_i

are distinct because ζ is injective by [Proposition 6.4](#). Let d_i be the given product metric on the Seifert component \tilde{M}_i .

For convenience, relabel \tilde{x}_0 by y_0 and $h(\tilde{x}_0)$ by y_k . Let γ be a geodesic in (\tilde{N}, d) joining y_0 to y_k . Replace the path γ by a new path ξ , described as follows: For $i = 1, 2, \dots, k-1$, let $\tilde{T}_i = \tilde{M}_{i-1} \cap \tilde{M}_i$, and let $t_i = \sup\{t \in [0, 1] \mid \gamma(t) \in \tilde{T}_i\}$. Let $y_i = \gamma(t_i)$. Let ξ_i be the geodesic segment in (\tilde{M}_i, d_i) joining y_i to y_{i+1} . Let ξ be the concatenation $\xi_0 \xi_1 \cdots \xi_{k-1}$. Since $d_{i-1}(y_{i-1}, y_i) \leq Ld(y_{i-1}, y_i)$, it follows that $|\xi| \leq L|\gamma| \leq Ln$. Here $|\cdot|$ denotes the length of a path with respect to the metric d .

For each i , let $\tilde{\ell}_i$ and $\bar{\ell}_i$ be the Euclidean geodesics in (\tilde{T}_i, d_{i-1}) and (\tilde{T}_i, d_i) passing through y_i such that they project to fibers $\tilde{f}_i \subset \tilde{T}_i$ and $\bar{f}_i \subset \tilde{T}_i$, respectively. Let $\alpha_i = \tilde{g}(\tilde{S}) \cap \tilde{T}_i$. By [Corollary 6.5](#), the lines $\tilde{\ell}_i$ and $\bar{\ell}_i$ intersect α_i at points, denoted by x_i and z_i , respectively. Let ρ be the minimum distance between two distinct JSJ planes in \tilde{N} (see [Remark 4.14](#)).

Claim 1 *There exists a linear function J , not depending on the choices of n , h or ξ , such that*

$$d_{\tilde{g}(\tilde{S})}(y_0, y_k) \leq J\left(\sum_{i=1}^{k-1} d_i(y_i, z_i) + n\right).$$

Let $z_0 = y_0$ and $z_k = y_k$. We first show that, for each $i = 0, \dots, k-1$,

$$d_i(z_i, z_{i+1}) \leq L^2(d_i(z_i, y_i) + |\xi_i| + d_{i+1}(y_{i+1}, z_{i+1})).$$

Indeed,

$$\begin{aligned} d_i(z_i, z_{i+1}) &\leq d_i(z_i, y_i) + d_i(y_i, y_{i+1}) + d_i(y_{i+1}, z_{i+1}) \\ &\leq d_i(z_i, y_i) + L|\xi_i| + d_i(y_{i+1}, z_{i+1}) \\ &\leq d_i(z_i, y_i) + L|\xi_i| + Ld(y_{i+1}, z_{i+1}) \\ &\leq d_i(z_i, y_i) + L|\xi_i| + L^2d_{i+1}(y_{i+1}, z_{i+1}) \quad (\text{because } L \geq 1) \\ &\leq L^2(d_i(z_i, y_i) + |\xi_i| + d_{i+1}(y_{i+1}, z_{i+1})). \end{aligned}$$

Using [Corollary 6.8](#) we obtain

$$\begin{aligned} d_{\tilde{g}(\tilde{S})}(y_0, y_k) &\leq \sum_{i=0}^{k-1} d_{\tilde{g}(\tilde{S})}(z_i, z_{i+1}) \leq \sum_{i=0}^{k-1} Ld_i(z_i, z_{i+1}) + kC \\ &\leq 2L^3 \sum_{i=0}^k d_i(z_i, y_i) + L^3|\xi| + kC \end{aligned}$$

$$= 2L^3 \sum_{i=1}^{k-1} d_i(z_i, y_i) + L^3 |\xi| + kC.$$

Since $d(\tilde{x}_0, h(\tilde{x}_0)) \leq n$, it follows that $k\rho \leq n$, and therefore $k \leq n/\rho$. We note that $|\xi| \leq Ln$. Letting $A = \max\{2L^3, L^4 + C/\rho\}$ and $J(x) = Ax$, Claim 1 is confirmed.

In order to complete the proof, it suffices to find an appropriate upper bound for the sum appearing in the conclusion of Claim 1. In the general case, we need an exponential upper bound, but in the special case of trivial dilation we require a quadratic upper bound.

By [Lemma 6.9](#) we immediately see

$$(*) \qquad d_{i-1}(y_i, x_i) \leq d_{i-1}(y_{i-1}, z_{i-1}) + L^2 |\xi_{i-1}|.$$

Observe that α_i universally covers a closed curve $g(c_i) \in T_i$ for some $c_i \in \mathcal{T}_g$ and $[g(c_i)] = a_i[\tilde{f}_i] + b_i[\tilde{f}_j]$ in $H_1(T_i; \mathbb{Z})$ for some $a_i, b_i \in \mathbb{Z}$. By [Lemma 6.10](#), we conclude that

$$(\dagger) \qquad d_i(y_i, z_i) = \left| \frac{b_i}{a_i} \right| d_{i-1}(y_i, x_i).$$

Claim 2 *Suppose g is not virtually embedded. There exists a function F , not depending on the choices of n , h or ξ , such that*

$$\sum_{j=1}^{k-1} d_j(y_j, z_j) \leq F(n)$$

and $F(n) \sim 2^n$.

Since g is not virtually embedded, the governor $\epsilon = \epsilon(g)$, defined in [Remark 4.14](#), must be strictly greater than 1. We will show by induction on $j = 0, \dots, k-1$ that

$$d_j(y_j, z_j) \leq L^3 n \sum_{i=1}^j \epsilon^i.$$

The base case of $j = 0$ is trivial since $y_0 = z_0$, so both sides of the inequality equal zero. For the inductive step, we use [\(*\)](#), [\(\dagger\)](#) and the facts $|\xi_{j-1}| \leq Ln$ and $|b_j/a_j| \leq \epsilon$ to see that

$$\begin{aligned} d_j(y_j, z_j) &= \left| \frac{b_j}{a_j} \right| d_{j-1}(y_j, x_j) \\ &\leq \epsilon d_{j-1}(y_j, x_j) \end{aligned}$$

$$\begin{aligned}
 &\leq \epsilon(d_{j-1}(y_{j-1}, z_{j-1}) + L^2|\xi_{j-1}|) \\
 &\leq \epsilon d_{j-1}(y_{j-1}, z_{j-1}) + \epsilon L^3 n \\
 &\leq \epsilon(\epsilon + \epsilon^2 + \cdots + \epsilon^{j-1})L^3 n + \epsilon L^3 n \\
 &\leq (\epsilon + \epsilon^2 + \cdots + \epsilon^j)L^3 n.
 \end{aligned}$$

Summing this geometric series gives

$$d_j(y_j, z_j) \leq \frac{L^3 n}{\epsilon - 1} \epsilon^{j+1}.$$

Summing a second time over j , we obtain

$$\sum_{j=1}^{k-1} d_j(y_j, z_j) \leq \frac{L^3 n}{\epsilon - 1} (\epsilon^2 + \cdots + \epsilon^k) \leq \frac{\epsilon L^3}{(\epsilon - 1)^2} n \epsilon^k \leq \frac{\epsilon L^3}{(\epsilon - 1)^2} n \epsilon^{n/\rho},$$

which is equivalent to an exponential function of n , establishing Claim 2. (Recall that for any polynomial $p(n)$, we have $p(n)2^n \leq 2^n 2^n = 2^{2n}$ for sufficiently large n .)

Claim 3 Assume that g is virtually embedded. There exists a quadratic function Q , not depending on the choices of n , h or ξ , such that

$$\sum_{j=1}^{k-1} d_j(y_j, z_j) \leq Q(n).$$

For any $1 \leq i \leq j$, let

$$\Theta_{i,j} = \left| \frac{b_i}{a_i} \right| \cdot \left| \frac{b_{i+1}}{a_{i+1}} \right| \cdots \left| \frac{b_j}{a_j} \right|.$$

Let Λ be the constant given by [Proposition 4.15](#). In order to prove Claim 3, we mimic the argument of Claim 2—using $\Theta_{i,j}$ in place of the terms of the form ϵ^ℓ . This change gives tighter results than those obtained in Claim 2, since $\Theta_{i,j}$ is bounded above by the constant Λ . This upper bound applies only in the virtually embedded case, as explained in [Proposition 4.15](#). The key recursive property satisfied by $\Theta_{i,j}$ is

$$\Theta_{i,j-1} \left| \frac{b_j}{a_j} \right| = \Theta_{i,j}.$$

We will show by induction on $j = 0, \dots, k-1$ that

$$d_j(y_j, z_j) \leq L^2 \sum_{i=1}^j |\xi_{i-1}| \Theta_{i,j}.$$

As before, the base case $j = 0$ is trivial. For the inductive step, we use (\dagger) and $(*)$ to see that

$$\begin{aligned} d_j(y_j, z_j) &= d_{j-1}(y_j, x_j) \left| \frac{b_j}{a_j} \right| \\ &\leq (d_{j-1}(y_{j-1}, z_{j-1}) + L^2 |\xi_{j-1}|) \left| \frac{b_j}{a_j} \right| \\ &= d_{j-1}(y_{j-1}, z_{j-1}) \left| \frac{b_j}{a_j} \right| + L^2 |\xi_{j-1}| \left| \frac{b_j}{a_j} \right| \\ &\leq \left(L^2 \sum_{i=1}^{j-1} |\xi_{i-1}| |\Theta_{i,j-1}| \right) \left| \frac{b_j}{a_j} \right| + L^2 |\xi_{j-1}| \left| \frac{b_j}{a_j} \right| \\ &= L^2 \sum_{i=1}^{j-1} |\xi_{i-1}| |\Theta_{i,j}| + L^2 |\xi_{j-1}| |\Theta_{j,j}| \\ &= L^2 \sum_{i=1}^j |\xi_{i-1}| |\Theta_{i,j}|. \end{aligned}$$

Since $\Theta_{i,j}$ is bounded above by Λ , and $\sum_{i=1}^j |\xi_{i-1}| \leq |\xi| \leq Ln$, we have

$$d_j(y_j, z_j) \leq L^2 \sum_{i=1}^j |\xi_{i-1}| |\Theta_{i,j}| \leq \Lambda L^3 n.$$

Summing over j , we obtain

$$\sum_{j=1}^{k-1} d_j(y_j, z_j) \leq (k-1) \Lambda L^3 n \leq \left(\frac{n}{\rho} - 1 \right) \Lambda L^3 n,$$

which is a quadratic function of n , establishing Claim 3.

If g is not virtually embedded, we combine Claims 1 and 2 to get an exponential upper bound for $d_{\tilde{g}(\tilde{S})}(y_0, y_k)$. In the virtually embedded case, we combine Claims 1 and 3 to get a quadratic upper bound. The theorem now follows from [Corollary 2.10](#). \square

7 Fiber surfaces have quadratic distortion

In this section we show in detail how results of Gersten [\[5\]](#) and Kapovich and Leeb [\[10\]](#) can be combined with Thurston's geometric description of 3-manifolds that fiber over the circle to prove [Theorem 1.4](#). This section is an elaboration of ideas that are implicitly used by Kapovich and Leeb in [\[10\]](#) but not stated explicitly there. As explained in

the introduction, [Theorem 1.4](#) is the main step in an alternative proof of the virtually embedded case of [Theorem 1.2](#).

The following theorem relates distortion of normal subgroups to the notion of divergence of groups. Roughly speaking, divergence is a geometric invariant that measures the circumference of a ball of radius n as a function of n . (See [\[5\]](#) for a precise definition.)

Theorem 7.1 [\[5, Theorem 4.1\]](#) *If $G = H \rtimes_{\phi} \mathbb{Z}$, where G and H are finitely generated, then the divergence of G is dominated by the distortion Δ_H^G .*

Let H be generated by a finite set \mathcal{T} . An automorphism $\phi \in \text{Aut}(H)$ has *polynomial growth* of degree at most d if there exist constants α and β such that

$$|\phi^i(t)|_{\mathcal{T}} \leq \alpha n^d + \beta$$

for all $t \in \mathcal{T}$ and all i with $|i| \leq n$.

Gersten claims the following result in the case that H is a free group. However, his proof uses only that H is finitely generated, so we get the following result using the same proof:

Theorem 7.2 [\[5, Proposition 4.2\]](#) *If $G = H \rtimes_{\phi} \mathbb{Z}$, where G and H are finitely generated and $\phi \in \text{Aut}(H)$ has polynomial growth of degree at most d , then $\Delta_H^G \leq n^{d+1}$.*

Proof of Theorem 1.4 Let N be a graph manifold that fibers over S^1 with surface fiber S . Then N is the mapping torus of a homeomorphism $f \in \text{Aut}(S)$. In particular, if we let $G = \pi_1(N)$ and $H = \pi_1(S)$, then $G = H \rtimes_{\phi} \mathbb{Z}$, where $\phi \in \text{Aut}(H)$ is an automorphism induced by f . Passing to finite covers, we may assume without loss of generality that N and S are orientable and that the map f is orientation-preserving.

Kapovich and Leeb show that the divergence of the fundamental group of any graph manifold is at least quadratic [\[10, Section 3\]](#). Therefore, by [Theorem 7.1](#) the distortion of H in G is also at least quadratic.

By [Theorem 7.2](#), in order to establish a quadratic upper bound for Δ_H^G , it suffices to show that ϕ has linear growth. We will first apply the Nielsen–Thurston classification of surface homeomorphisms to the map f (see for example Corollary 13.2 of [\[4\]](#)). By the Nielsen–Thurston theorem, there exists a multicurve $\{c_1, \dots, c_k\}$ with the following properties: The curves c_i have disjoint closed annular neighborhoods S_1, \dots, S_k . Let S_{k+1}, \dots, S_{k+l} be the closures of the connected components of $S - \bigcup_{i=1}^k S_i$. Then

there is a map g isotopic to f and a positive number m such that g^m leaves each subsurface S_i invariant. Furthermore, g^m is a product of homeomorphisms $g_1 \cdots g_{k+l}$ such that each g_i is supported on S_i . For $i = 1, \dots, k$, the map g_i (supported on the annulus S_i) is a power of a Dehn twist about c_i . For $i = k+1, \dots, k+l$, each g_i is either the identity or a map that restricts to a pseudo-Anosov map of S_i .

Consider the mapping torus \hat{N} for the homeomorphism g^m of S , which finitely covers N . Apply Thurston's geometric classification of mapping tori to \hat{N} to conclude that the family of tori $c_i \times S^1$ in the mapping torus \hat{N} is equal to the family of JSJ tori of \hat{N} . It follows that for each $i = k+1, \dots, k+l$ the map g_i is equal to the identity on S . Indeed, if any g_i were pseudo-Anosov, then the corresponding JSJ component of \hat{N} would be atoroidal and hyperbolic, which is impossible in a graph manifold.

Therefore, g^m is a product of powers of Dehn twists about disjoint curves. In particular, the automorphism ϕ^m has linear growth. Clearly ϕ itself must also have linear growth. Therefore, the distortion of H in G is at most quadratic, as desired. \square

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