

# The verbal width of acylindrically hyperbolic groups

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We show that the verbal width is infinite for acylindrically hyperbolic groups, which include hyperbolic groups, mapping class groups and  $\text{Out}(F_n)$ .

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## 1 Verbal subgroups

The *Brooks construction* is a method for constructing essential quasimorphisms on free groups. These Brooks quasimorphisms can be extended to general acylindrically hyperbolic groups and in this note we use these quasimorphisms to study *verbal subgroups*. A verbal subgroup generalizes the notion of a commutator subgroup. We begin with a precise definition. Suppose that  $G$  is a group and  $\mathbb{F}_k$  is the free group generated by  $x_1, \dots, x_k$ . For any choice of  $w \in \mathbb{F}_k$ , substitution defines a map, also denoted by  $w$ ,

$$w: G^k \rightarrow G.$$

In what follows we always assume that  $w$  is nontrivial.

The image of this function is the *verbal subset*, and is denoted by  $w[G]$ . The *verbal subgroup*  $w(G) < G$  is the subgroup generated by  $w[G]$ . For example, if  $k = 2$  and  $w = x_1x_2x_1^{-1}x_2^{-1}$  then  $w[G]$  is the set of commutators in  $G$  and  $w(G) = [G, G]$  is the commutator subgroup of  $G$ .

Let  $e_i$  be the sum of the exponents of  $x_i$  in  $w$ . For example, if  $w = x_2x_1x_2^{-2}$ , then  $e_1 = 1$ ,  $e_2 = 1 - 2 = -1$  and  $e_3 = \dots = e_k = 0$ . Let  $d(w) \geq 1$  be the gcd of the  $e_i$ . If they are all 0, define  $d(w) = 0$ . Note that the condition that  $d(w) = 0$  implies that  $w(G) \subset [G, G]$ , for when  $d(w) = 0$ ,  $w(G)$  will be in the kernel of any homomorphism from  $G$  to an abelian group.

For each  $g \in w(G)$ , define its *verbal length* by

$$\text{vl}_w(g) = \min\{n \mid g = g_1 \cdots g_n, g_i \in w[G]^{\pm 1}\}.$$

The *width* of  $w(G)$  is the supremum of  $vl_w(g)$  over all  $g \in w(G)$ . Note that if  $d(w) > 1$ , then  $g^d \in w[G]$ . (Replace each  $x_i$  in  $w$  with  $g^{a_i e_i}$ , where  $\sum a_i e_i = d$ .) In particular, if  $d = 1$  then  $w[G] = w(G) = G$  and the width is 1. The reader can consult Segal [12] for more information on the subject. If  $w$  is the commutator  $[x_1, x_2]$ , the verbal length is called the *commutator length*.

A group  $G$  is *acylindrically hyperbolic* if it has a nonelementary acylindrical action on a  $\delta$ -hyperbolic space; see Osin [10]. Recall that an isometric action of  $G$  on a metric space  $X$  is *acylindrical* if for all  $D > 0$  there exist  $L, N > 0$  such that if  $d(x, y) > L$  then the set

$$\{g \in G \mid d(x, gx) < D \text{ and } d(y, gy) < D\}$$

has  $< N$  elements. The first nontrivial example is due to Bowditch [4], who showed that the action of the mapping class group on the curve complex is acylindrical. There are now many examples, with the key point being that many seemingly weaker geometric criteria imply that the group is acylindrically hyperbolic. See Osin [10], Bestvina, Bromberg and Fujiwara [2] and Bestvina, Bromberg, Fujiwara and Sisto [3].

Here is our main result.

**Theorem 1.1** *Suppose that  $G$  is acylindrically hyperbolic and that  $d(w) \neq 1$ . Then the width of  $w(G)$  is infinite.*

This result generalizes the work of Rhemtulla [11] and Myasnikov and Nikolaev [9], who proved the theorem for free groups and hyperbolic groups, respectively.

Similarly to the stable commutator length, one can define the *stable verbal length* of  $g \in w(G)$ ,  $svl_w(g)$ , as follows:

$$svl_w(g) = \liminf_{n \rightarrow \infty} \frac{vl_w(g^n)}{n}.$$

If  $svl_w(g) > 0$  for some  $g$  then  $w(G)$  has infinite verbal width. However, if  $d(w) \geq 1$ , this method cannot be used, due to the following lemma:

**Lemma 1.2** (Calegari and Zhuang [6]) *If  $d = d(w) \geq 1$  then  $vl_w(g^n)$  is bounded and  $svl_w(g) = 0$  for all  $g \in w(G)$ .*

We may suppress  $w$  and write  $vl$  and  $svl$  instead of  $vl_w$  and  $svl_w$ .

**Proof** As observed above,  $g^d$  is in  $w[G]$  for all  $g \in G$ , so  $vl(g^{nd}) = 1$ . Since  $vl(gh) \leq vl(g) + vl(h)$ , this implies that  $vl(g^n)$  is bounded and  $svl(g) = 0$ .  $\square$

On the other hand, when  $d(w) = 0$  we have the following, which implies Theorem 1.1 in this case:

**Theorem 1.3** *If  $G$  is acylindrically hyperbolic and  $d(w) = 0$ , then  $\text{svl}_w(g) > 0$  for some element  $g \in w(G)$ .*

If  $G$  is a free group then this is Corollary 2.16 of [6].

### 1.1 An outline of the proof of Theorem 1.1 for free groups

To illustrate the main idea, we sketch the proof of Theorem 1.1 in the case that  $G = F$  is the free group with basis  $\{a, b\}$  and  $w = x_1x_2x_1x_2^{-1}$ , so that  $d = d(w) = 2$ . Consider  $g_i = ab^{2i}ab^{2i+1} \in G$  for  $i = 1, 2, \dots$ . This sequence has the property that distinct occurrences of any  $g_i$  in any reduced word have trivial overlap. Denote by  $H_i: G \rightarrow \mathbb{Z}$  the Brooks (counting) quasimorphism with respect to  $g_i$  (see Section 2 for the definition). For any  $y, z \in G$  we have  $|H_i(yz) - H_i(y) - H_i(z)| \leq 3$  by the usual tripod argument, since at most three copies of  $g_i$  along a tripod can have the tripod point in the interior. The key observation now is that in fact  $H_i(yz) - H_i(y) - H_i(z) = 0$  for all but at most three values of  $i$  by the nonoverlapping property of the  $g_i$  (the exceptional values of  $i$  depend on  $y$  and  $z$ ).

Now suppose that  $g \in w(G)$  has  $\text{vl}_w(g) = 1$ , so  $g = x_1x_2x_1x_2^{-1}$  for some  $x_1, x_2 \in G$ . Then, for all but  $3 \times 3 = 9$  values of  $i$  (since the length of the word  $g$  is 4 and there are at most three exceptional  $i$  for each multiplication),

$$H_i(g) = H_i(x_1) + H_i(x_2) + H_i(x_1) + H_i(x_2^{-1}) = 2H_i(x_1)$$

is even. Thus, to detect  $g$  with  $\text{vl}_w(g) > 1$  it suffices to ensure that  $H_i(g)$  is odd for 10 values of  $i$ . Similarly, to detect that  $\text{vl}_w(g)$  is large it suffices to ensure that  $H_i(g)$  is odd for sufficiently many  $i$ . An element such as

$$(ab)^2(ab^2)^2(ab^3)^2(ab^4)^2 \dots (ab^{N-1})^2(ab^N)^2$$

will do.

For a general acylindrically hyperbolic group  $G$  we perform the above construction on a suitable Schottky subgroup  $F \subset G$  and then use the method of Hull and Osin [8] to extend the quasimorphism from  $F$  to  $G$  (see Proposition 4.2). We will review their construction in Section 4 and show that the key observation above continues to hold for the extended quasimorphisms.

## 2 Extending Brooks quasimorphisms to acylindrically hyperbolic groups

We first recall the definition of a quasimorphism. Let  $G$  be a group. Then

$$H: G \rightarrow \mathbb{R}$$

is a *quasimorphism* if

$$\sup_{\alpha, \beta \in G} |H(\alpha\beta) - H(\alpha) - H(\beta)| = \Delta(H) < \infty.$$

The constant  $\Delta = \Delta(H)$  is the *defect* of  $H$ . Note that if  $\Delta = 0$  then  $H$  is a homomorphism. A quasimorphism is *antisymmetric* if  $H(\alpha^{-1}) = -H(\alpha)$ .

One way to construct a quasimorphism that is not a homomorphism is to start with a homomorphism and then add on a bounded function. Of course, this is not an interesting example. The *Brooks construction* is a way of building antisymmetric quasimorphisms, called *Brooks (counting) quasimorphisms*, that are not a bounded distance from a homomorphism.

Let  $F = \langle a, b \rangle$  be the free group on two generators and let  $w$  be a reduced nonempty word in  $F$ . For  $x \in F$ , let  $N_w(x)$  be the number of copies of  $w$  without overlapping in  $x$  when  $x$  is written as a reduced word and let  $H_w(x) = N_w(x) - N_{w^{-1}}(x)$ . Note that  $H_w(x^{-1}) = -H_w(x)$ . Brooks proved the following:

**Theorem 2.1** *The function  $H_w$  is an antisymmetric quasimorphism and if  $w$  is not a power of  $a$  or  $b$  then it is not a bounded distance from a homomorphism. If  $w$  is cyclically reduced, then  $H_w(w^n) = n$ .*

Note that if  $N$  is a finite group then the Brooks quasimorphisms can be extended to  $F \times N$  by choosing them to be constant on the second factor.

In [7], Dahmani, Guirardel and Osin show that an acylindrically hyperbolic group  $G$  contains a copy of a *hyperbolically embedded*  $F \times N$ , where  $N$  is the maximal finite normal subgroup of  $G$ . In [8], Hull and Osin show that any antisymmetric quasimorphism on a hyperbolically embedded subgroup extends to a quasimorphism of the entire group. Combing these two results we have the following theorem:

**Theorem 2.2** *Let  $G$  be acylindrically hyperbolic. Then there exists a free group  $F = \langle a, b \rangle < G$  such that for every Brooks quasimorphism  $H_w$  with  $w$  nonempty and reduced there is a quasimorphism  $H: G \rightarrow \mathbb{R}$  such that  $H|_F = H_w$ .*

**Remark** There is a weaker version of this theorem (which would be good enough for our applications here) that follows from [2]. The approach in [2] is more direct as it does not go through the theory of hyperbolically embedded subgroups. We also note that both approaches use the projection complex from [1] in an essential way.

As a demonstration of our methods we first give a proof of Theorem 1.3:

**Proof of Theorem 1.3** Given a quasimorphism  $H$  with the defect  $\Delta$  and an element  $g = g_1 \dots g_n$ , by repeatedly applying the quasimorphism bound we have

$$\left| H(g) - \sum_{i=1}^n H(g_i) \right| \leq (n - 1)\Delta.$$

If  $g = w(g_1, \dots, g_k)$  and  $H$  is antisymmetric, this becomes

$$\left| H(g) - \sum_{i=1}^n e_i H(g_i) \right| \leq (|w| - 1)\Delta,$$

where  $e_i$  is the sum of the exponents of  $x_i$  in  $w$ , and when  $d(w) = 0$  (so all the  $e_i$  are 0) this becomes  $|H(g)| \leq (|w| - 1)\Delta$  for  $g \in w[G]$ . More generally, for  $g \in w(G)$  we have  $|H(g)| \leq (\text{vl}(g)|w| - 1)\Delta$  and therefore, if  $H(g^n) \geq n$  for all  $n > 0$ , we have  $\text{svl}(g) \geq (|w|\Delta)^{-1} > 0$ .

We will use the Brooks construction (and the Hull–Osin extension) to find a  $g \in w(G)$  with  $H_g(g^n) \geq n$  for all  $n > 0$ . To do this we need to find a cyclically reduced word in  $w(F) \subset w(G) \cap F$ . Pick a nontrivial element  $h \in w(F)$ . If it is cyclically reduced, let  $g = h$  and we are done. If not, then  $h = a \dots a^{-1}$  for a basis element  $a$  (or its inverse). Let  $h'$  be obtained from  $h$  by swapping  $a$ 's and  $b$ 's (with  $b$  another basis element). Then  $hh'$  is cyclically reduced and still in  $w(F)$ , so  $g = hh'$  is the desired element. □

If  $w \in [G, G]$  then, for any  $g \in w(G)$ ,  $\text{cl}(g) \leq \text{cl}(w)\text{vl}(g)$ , so  $\text{scl}(g) \leq \text{cl}(w)\text{svl}(g)$ . In particular, if  $\text{scl}(g) > 0$  then  $\text{svl}(g) > 0$  and Theorem 1.3 would follow if we knew that every verbal subgroup of an acylindrically hyperbolic group had an element  $g$  with  $\text{scl}(g) > 0$ . However, proving this does not seem any easier than the more general proof above.

One can also ask if  $\text{scl}(g) = 0$  implies that  $\text{svl}_w(g) = 0$  for all  $w$ . Here the answer is negative. For example, take  $w = [[x, y], [z, u]]$  and

$$G = \langle a, b, c, d, t \mid t[[a, b], [c, d]]t^{-1} = [[a, b], [c, d]]^{-1} \rangle.$$

Then for  $g = [[a, b], [c, d]]$  we have that  $g$  is conjugate to  $g^{-1}$ , which forces  $\text{scl}(g) = 0$ . On the other hand, we claim that  $\text{svl}_w(g) > 0$ . Indeed, if  $\text{svl}_w(g) = 0$ , then  $g^n$  can be written as a product of a sublinear number of double commutators, which would imply that  $\text{scl}_H(g) = 0$ , where  $H = [G, G]$ . We now argue that  $\text{scl}_H(g) > 0$ . In fact we will show that there is an index 2 subgroup  $N < G$  with  $H < N$  and such that  $N$  surjects to the free group  $F_4 = \langle a, b, c, d \rangle$  with  $g$  mapping to  $[[a, b], [c, d]]$ . Since nontrivial elements in free groups have positive scl (see [5]), the claim follows. In fact, it shows  $\text{scl}_N(g) > 0$ . It immediately implies  $\text{scl}_H(g) > 0$ .

We let  $N$  be the kernel of the map  $G \rightarrow \mathbb{Z}/2$  that sends  $t$  to 1 and  $a, b, c$  and  $d$  to 0. The corresponding double cover  $Y$  of the presentation 2-complex  $X$  of  $G$  consists of the disjoint union of two roses  $R_i$  with petals labeled  $a_i, b_i, c_i$  and  $d_i$  for  $i = 1, 2$ , with edges  $t_i$  connecting the vertex of  $R_i$  to the vertex of  $R_{3-i}$ . The map to  $X$  is the obvious one, sending  $a_i$  to  $a$  etc. The relation 2-cell in  $X$  lifts to two 2-cells in  $Y$ , with attaching maps  $t_i[[a_{i+1}, b_{i+1}], [c_{i+1}, d_{i+1}]]t_i^{-1}[[a_i, b_i][c_i, d_i]]$  with indices taken mod 2. Now map the 1-skeleton of  $Y$  to the rose corresponding to  $\langle a, b, c, d \rangle$  via  $t_i \mapsto *$ ,  $a_1, c_2 \mapsto a$ ,  $b_1, d_2 \mapsto b$ ,  $c_1, a_2 \mapsto c$  and  $d_1, b_2 \mapsto d$ . We then extend this to the two 2-cells. This is possible since via the attaching maps the boundary of the 2-cells are mapped to  $[[c, d], [a, b]][[a, b], [c, d]]$  and  $[[a, b], [c, d]][[c, d], [a, b]]$ , which are trivial in  $\langle a, b, c, d \rangle$  since  $[x, y]^{-1} = [y, x]$ .

### 3 Some facts about the Hull–Osin extension

Unfortunately, rather than just the statement of Theorem 2.2 we need some elements of the proof in [8] (our Lemma 3.1). We review them now. In this section  $F$  can be any hyperbolically embedded subgroup in  $G$ . A pair is always ordered unless otherwise stated.

It is convenient to replace the quasimorphism with a function on  $G \times G$ , called a bicombing in [8]. If  $H$  is a quasimorphism we define  $r(x, y) = H(x^{-1}y)$ . Note that  $r(zx, zy) = r(x, y)$  and  $|r(x, y) + r(y, z) + r(z, x)|$  is bounded by the defect of  $H$ . On the other hand if we are given a map  $r(x, y)$  (satisfying the properties from the previous sentence) then the map  $x \mapsto r(1, x)$  is a quasimorphism so  $r$  determines  $H$  just as  $H$  determines  $r$ . In particular, to construct  $\tilde{H}$ , an extension of  $H$ , in [8] they first construct  $\tilde{r}: G \times G \rightarrow \mathbb{R}$ , which is an extension of  $r: F \times F \rightarrow \mathbb{R}$ . To construct  $\tilde{r}$ , for each  $x, y \in G$  and each coset  $aF$  is associated a finite collection of pairs  $E(x, y; aF) = \{(x_i, y_i)\}$ , where  $x_i, y_i \in F$ . For the convenience of the reader

we briefly review the construction of the sets and then state the key properties that we will need.

Let  $\Gamma$  be a Cayley graph for  $G$  formed from a generating set that contains every element of  $F$ . An edge labeled by an element in  $F$  is called an  $F$ -edge. Given  $x, y \in G$ , let  $\gamma$  be a geodesic in  $\Gamma$  from  $x$  to  $y$ . Each  $F$ -coset has diameter 1 in  $\Gamma$  so  $\gamma$  will intersect a given coset  $aF$  in at most two points  $x'$  and  $y'$ . We say that  $\gamma$  *essentially penetrates*  $aF$  if any path in  $\Gamma$  from  $x'$  to  $y'$  that doesn't contain any  $F$ -edges has length  $\geq C$ , where  $C$  is a constant that only depends on  $G$  and  $F$ . We let  $S(x, y)$  be the set of cosets  $aF$ , where there is some geodesic from  $x$  to  $y$  that essentially penetrates  $aF$ . A central fact from [8] is that if there is one geodesic that essentially penetrates then every geodesic from  $x$  to  $y$  must intersect  $aF$ . For each coset  $aF \in S(x, y)$ , we let  $E(x, y; aF)$  be the set of pairs  $(x', y') \in F$  such that  $ax'$  and  $ay'$  are the entry and exit points for a geodesic from  $x$  to  $y$  in  $\Gamma$ . For each coset the particular representative  $a$  is not important except that the choice needs to be fixed for once and for all. If  $aF \notin S(x, y)$  then  $E(x, y; aF)$  is empty.

We now define

$$\tilde{r}(x, y) = \sum_{aF \in S(x, y)} \left( \frac{1}{|E(x, y; aF)|} \sum_{(x', y') \in E(x, y; aF)} r(x', y') \right).$$

For this to be well defined we need the sum to be finite. The inside sum is finite by Lemma 3.8 of [8] and the outside sum is finite since  $S(x, y)$  is finite by their Corollary 3.4. Note that while in [8] it is only stated that the size of  $E(x, y; aF)$  is finite, it is in fact uniformly bounded, which will be important in the proof of Lemma 4.1 later (see the footnote there). Note that “uniformly bounded” in this discussion means that it depends only on  $G$  and  $F$ .

The following lemma is a combination of Lemma 3.9 and (the proof of) Lemma 4.7 in [8]. We fix the word metric with respect to a finite generating set on  $F$  and denote the distance between  $x$  and  $y$  by  $|x - y|$ .

**Lemma 3.1** *Given  $x, y, z \in G$ , for all but at most two cosets  $aF$  exactly one of the following three possibilities holds:*

- (1)  $E(x, y; aF) = \emptyset$ .
- (2)  $E(x, y; aF) = E(x, z; aF) \neq \emptyset$  and  $E(y, z; aF) = \emptyset$ .
- (3)  $E(x, y; aF) = E(z, y; aF) \neq \emptyset$  and  $E(x, z; aF) = \emptyset$ .

If  $aF$  doesn't satisfy the above then one of the following holds:

- (A) All of  $E(x, y; aF)$ ,  $E(x, z; aF)$  and  $E(y, z; aF)$  are nonempty and, for any pairs  $(x', y') \in E(x, y; aF)$ ,  $(x'', z') \in E(x, z; aF)$  and  $(y'', z'') \in E(y, z; aF)$ ,  $|x' - x''|$ ,  $|y' - y''|$  and  $|z' - z''|$  are uniformly bounded.
- (B) Only  $E(x, z; aF)$  is empty and, for any pairs  $(x', y') \in E(x, y; aF)$  and  $(z', y'') \in E(z, y; aF)$ ,  $|x' - z'|$  and  $|y' - y''|$  are uniformly bounded or the same statement holds with  $y$  and  $z$  swapped.
- (C) Only  $E(x, y; aF)$  is nonempty and, for all pairs  $(x', y') \in E(x, y; aF)$ ,  $|x' - y'|$  is uniformly bounded.

**Proof** We explain how we deduce this statement from [8]. The first half is just a reformulation of their Lemma 3.9. The second half is in the proof of their Lemma 4.7. Our notation corresponds to theirs as follows:  $(G, F) \leftrightarrow (G, H_\lambda)$  and  $x, y, z \leftrightarrow f, g, h$ . We ignore the suffix  $\lambda$  since we have only one subgroup  $F$  while they have a collection  $\{H_\lambda\}$ . They use  $\hat{d}_\lambda$  to measure the distance between two points  $p$  and  $q$  on  $aF$  (for them,  $xH_\lambda$ ), but we use  $|p - q|$ . What is important is that for any constant  $C > 0$  there exists  $D > 0$  such that if  $\hat{d}_\lambda(p, q) \leq C$  then  $|p - q| \leq D$ . This is because  $\hat{d}_\lambda$  is a proper metric. That said, the cases (A), (B) and (C) corresponds to their cases (1), (2) and (3), respectively. For example, in case (1), they obtain that the distance of two concerned points is at most  $3C$  in (14), where  $C$  is a constant that depends only on  $G$  and  $H_\lambda$ . This gives a desired estimate for our case (A). The other two cases are similar.  $\square$

Now, let  $B(x, y)$  be the collection of cosets that don't satisfy Lemma 3.1(1)–(3) and let  $B(x, y, z)$  be the union of  $B(x, y)$ ,  $B(y, z)$  and  $B(z, x)$ . By Lemma 3.1,  $B(u, v)$  contains at most two cosets so  $B(x, y, z)$  contains at most six. Cosets in  $B(x, y, z)$  are of type (A), (B) or (C) depending on which of the conditions in Lemma 3.1 they satisfy.

We are interested in the sum  $\tilde{r}(x, y) + \tilde{r}(y, z) + \tilde{r}(z, x)$ . It will be convenient to define new sets  $E(x, y, z; aF)$  to be the product of the sets of pairs  $E(x, y; aF)$ ,  $E(y, z; aF)$  and  $E(z, x; aF)$ . Note that one or more of the sets may be empty, in which case the product would be empty. (In fact for at most one coset at least one of the sets will be empty.) To get around this, if  $E(u, v; aF)$  is empty, we make it nonempty by adding the “empty pair”  $(\emptyset, \emptyset)$  and we define  $r(\emptyset, \emptyset) = 0$ . With this modification



$E(x, y, z; aF)$  will always be a triple of pairs in  $F \cup \{\emptyset\}$ . Next we define

$$\rho(x, y, z; aF) = \frac{1}{|E(x, y, z; aF)|} \sum_{E(x, y, z; aF)} r(x_-, y_+) + r(y_-, z_+) + r(z_-, x_+),$$

where  $(x_-, y_+) \in E(x, y; aF)$ ,  $(y_-, z_+) \in E(y, z; aF)$  and  $(z_-, x_+) \in E(z, x; aF)$ , and observe that

$$\tilde{r}(x, y) + \tilde{r}(y, z) + \tilde{r}(z, x) = \sum_{aF} \rho(x, y, z; aF).$$

To show that  $\tilde{r}$  determines a quasimorphism, Hull and Osin show that for nearly all cosets the expression  $\rho(x, y, z; aF)$  is zero and for the finitely many when it is not it is uniformly bounded.

**Corollary 3.2** *If  $aF \notin B(x, y, z)$  then  $\rho(x, y, z; aF) = 0$ .*

**Proof** If  $aF \notin B(x, y, z)$  then either  $E(x, y, z; aF)$  is the triple of empty pairs and  $\rho(x, y, z; aF) = 0$  or all the terms in the sum cancel and again  $\rho(x, y, z; aF) = 0$ .  $\square$

## 4 Many independent quasimorphisms

For the remainder of the paper we can assume that  $d(w) > 1$ . If  $H: G \rightarrow \mathbb{Z}$  is a homomorphism, an easy calculation gives that for any  $g \in w(G)$  we have that  $H(g)$  is divisible by  $d(w)$ . We will construct a family of quasimorphisms where this is true for nearly all the quasimorphisms in the family, where the number of exceptions is bounded above by the  $\text{vl}(g)$ .

Let  $F \times N$  be hyperbolically embedded in  $G$ , where  $F$  is a free group of rank at least 2 and  $N$  is a finite group. For simplicity we suppose the rank of  $F$  is 2 in the following. We now fix a sequence of words that we will use to build Brooks’s quasimorphisms on  $F$ , then extend it to  $F \times N$ , trivially on  $N$ . Let  $g'_i = ab^{2i}$ ,  $g''_i = ab^{2i+1}$  and  $g_i = g'_i g''_i$  and let  $H_i = H_{g_i}$  be the Brooks quasimorphism and  $r_i$  the corresponding bicombings.

We fix the word metric with respect a finite generating set on  $F \times N$  and denote the distance between  $x$  and  $y$  by  $|x - y|$ .

**Lemma 4.1** *Given a triple of pairs  $(x_-, x_+)$ ,  $(y_-, y_+)$  and  $(z_-, z_+)$  in  $F \times N$  with  $|x_- - x_+|$ ,  $|y_- - y_+|$  and  $|z_- - z_+|$  bounded by  $L$ , there are at most*

- $L$  of the  $r_i$  such that  $r_i(x_-, x_+) \neq 0$ ;

- $2L$  of the  $r_i$  such that  $r_i(x_-, y_+) + r_i(y_-, x_+) \neq 0$ ;
- $3L + 3$  of the  $r_i$  such that  $r_i(x_-, y_+) + r_i(y_-, z_+) + r_i(z_-, x_+) \neq 0$ .

Therefore, there is a uniform bound on the number of  $\rho_i$  where  $\rho_i(x, y, z; aF) \neq 0$ .

**Proof** We only discuss the case that  $N$  is trivial. The general case is similar.

If  $r_i(x_-, x_+) \neq 0$ , there is a translate of the word  $g_i$  in the segment between  $x_-$  and  $x_+$  in the Cayley graph (with the standard generators). Since two  $g_i$  can't intersect in a segment, a (very bad) upper bound for the number of  $r_i$  with  $r_i(x_-, x_+) \neq 0$  is  $|x_- - x_+| \leq L$ .

The triple  $x_-, x_+$  and  $y_+$  form a tripod in the Cayley graph and let  $m$  be the central vertex. Then  $r_i(x_-, y_+) + r_i(y_+, x_+) = 0$  unless there is a translate of  $g_i$  in the segment from  $y_+$  to  $x_-$  that intersects the segment from  $m$  to  $x_-$  or a translate in the segment from  $y_+$  to  $x_+$  that intersects the segment from  $m$  to  $x_+$ . Again using the fact that two  $g_i$  can't overlap in a segment, an upper bound for the number of the  $r_i$  with  $r_i(x_-, y_+) + r_i(y_+, x_+) \neq 0$  is  $|x_- - m| + |x_+ - m| = |x_- - x_+| \leq L$ . Similarly, there are at most  $L$  of the  $r_i$  such that  $r_i(x_+, y_+) + r_i(y_+, x_+) \neq 0$  or, equivalently,  $r_i(y_+, x_+) = r_i(y_-, x_+)$  for all but  $L$  of the  $r_i$  and therefore  $r_i(x_-, y_+) + r_i(y_-, x_+) = 0$  for all but  $2L$  of the  $r_i$ .

Now we examine the tripod formed by  $x_-, y_-$  and  $z_-$ . As with Brooks's original argument, the sum

$$r_i(x_-, y_-) + r_i(y_-, z_-) + r_i(z_-, x_-)$$

is zero unless the a translate of the word  $g_i$  intersects the central vertex of the tripod. At most three such words can intersect the central vertex so the sum is nonzero for at most three of the  $r_i$ . As above, for at most  $L$  of the  $r_i$  we have  $r_i(x_-, y_-) = r_i(x_i, y_+)$ , etc. Therefore,

$$r_i(x_-, y_+) + r_i(y_-, z_+) + r_i(z_-, x_+) = 0$$

for all but at most  $3L + 3$  of the  $r_i$ .

Since  $F$  is hyperbolically embedded in  $G$ , if  $aF \notin B(x, y, z)$  then  $\rho_i(x, y, z; aF) = 0$  for all  $i$  by Corollary 3.2. For cosets  $aF \in B(x, y, z)$  of type (A) there will be at most  $(3L + 3)|E(x, y, z; aF)|$  of the  $\rho_i$  with  $\rho_i(x, y, z; aF) \neq 0$ , for cosets of type (B) at most  $2L|E(x, y, z; aF)|$  and for cosets of type (C) at most  $L|E(x, y, z; aF)|$ .

By Lemma 3.8 of [8],  $|E(u, v; aF)|$  is uniformly bounded<sup>1</sup> and therefore so is  $|E(x, y, z; aF)|$ . It follows that there is uniform bound on the number of  $\rho_i$  with  $\rho_i(x, y, z; aF) \neq 0$ .  $\square$

Since  $F \times N$  is hyperbolically embedded in  $G$ , let  $\tilde{H}_i: G \rightarrow \mathbb{R}$  be the Hull–Osin extension of the  $H_i$  and  $\tilde{r}_i$  the corresponding bicomings.

**Proposition 4.2** *There exists an  $M > 0$  such that, for any  $x, y \in G$ ,*

$$\tilde{H}_i(xy) - \tilde{H}_i(x) - \tilde{H}_i(y) = 0$$

*holds except for at most  $M$  of the  $\tilde{H}_i$ . It follows that, for any  $a_1, \dots, a_k \in G$ ,*

$$\tilde{H}_i(a_1 \cdots a_k) - \tilde{H}_i(a_1) - \cdots - \tilde{H}_i(a_k) = 0$$

*holds except for at most  $M(k - 1)$  of the  $\tilde{H}_i$ .*

**Proof** First observe

$$\tilde{H}_i(xy) - \tilde{H}_i(x) - \tilde{H}_i(y) = \tilde{r}_i(id, xy) + \tilde{r}_i(x, id) + \tilde{r}_i(xy, x),$$

so we can instead show that

$$\tilde{r}_i(x, y) + \tilde{r}_i(y, z) + \tilde{r}_i(z, x) = 0$$

for all but  $M$  of the  $\tilde{r}_i$ . But this follows from Lemma 4.1, as, for all but at most six cosets,  $\rho_i(x, y, z; aF) = 0$  for all  $i$  and for each of these bad cosets there is a uniform bound on the number of  $\rho_i$  with  $\rho_i(x, y, z; aF) \neq 0$ .  $\square$

**Lemma 4.3** *For each  $K$  there exists  $g \in w(F \times N)$  such that  $H_i(g) = 1$  for all  $1 \leq i \leq K$ .*

**Proof** We will find an element  $g \in w(F)$ . Recall that we are assuming that  $d = d(w) > 1$  and therefore, for any  $f \in F$ , we have  $f^d \in w[F]$ . Let  $h'_i = (g'_i)^d$  and  $h''_i = (g''_i)^d$ . Then the product  $h_i = h'_i h''_i$  contains a single copy of  $g_i$ . Let  $g = h_1 h_2 \cdots h_K$ . Note that  $g$  is already reduced since there are only positive powers of  $a$  and  $b$  in the  $h'_i$  and  $h''_i$ . Furthermore by our construction of the  $g_i$  there will be exactly one copy of  $g_i$  in  $g$  and no copies of  $g_i^{-1}$ . Therefore,  $H_i(g) = 1$  for  $1 \leq i \leq K$ .  $\square$

<sup>1</sup>In [8, Lemma 3.8] it is only claimed that  $|E(x, y; aF)|$  is finite, however it is easy to see that their proof shows that the bound is uniform since the constant  $C$  in Lemmas 3.3 and 2.4 is uniform.

We now give a proof of Theorem 1.1 when  $d(w) > 1$ :

**Proof** We first show that for any  $g \in w(G)$  we have that  $\tilde{H}_i(g)$  is divisible by  $d(w) > 1$  for all but a bounded number of the  $\tilde{H}_i$ , where the bound only depends on  $\text{vl}(g)$ . To see this we first observe that if  $g = w(g_1, \dots, g_k) \in w[G]$  then, by Proposition 4.2,

$$\tilde{H}_i(w(g_1, \dots, g_k)) = \sum_{j=1}^k e_j \tilde{H}_i(g_j)$$

for all but  $M(|w| - 1)$  of the  $\tilde{H}_i$ . In particular, for any  $g \in w[G]$  there are at most  $M(|w| - 1)$  of the  $i$  such that  $\tilde{H}_i(g)$  isn't divisible by  $d(w)$ . Similarly, if  $g \in w(G)$  is a product of  $\text{vl}(g)$  elements  $g_j \in w[G]$  then

$$\tilde{H}_i(g) = \sum_{j=1}^{\text{vl}(g)} \tilde{H}_i(g_j)$$

for all but  $M(\text{vl}(g) - 1)$  of the  $\tilde{H}_i$ . If all of the  $\tilde{H}_i(g_j)$  are divisible by  $d(w)$  then so is  $\tilde{H}_i(g)$ , so we have that  $\tilde{H}_i(g)$  is divisible by  $d(w)$  for all but at most  $M(\text{vl}(g) - 1) + \text{vl}(g)(M(|w| - 1))$  of the  $\tilde{H}_i$ . In particular, a bound on  $\text{vl}(g)$  gives a bound on the number of  $\tilde{H}_i$  where  $\tilde{H}_i(g)$  is not divisible by  $d(w) > 1$ .

On the other hand, by Lemma 4.3, for any  $K > 0$  we can find a word  $h_K$  such that  $\tilde{H}_i(h_K) = H_i(h_K) = 1$  for  $1 \leq i \leq K$ . Therefore,  $\text{vl}(h_K) \rightarrow \infty$  as  $K \rightarrow \infty$ .  $\square$

From the above proof we see that  $\text{vl}(h_K) \geq K/(M|w|)$ . We know of no examples where this bound is sharp.

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