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On the homotopy types of Sp(n) gauge groups

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Let $\mathcal{G}_{k,n}$ be the gauge group of the principal $\mathrm{Sp}(n)$–bundle over S^4 corresponding to $k \in \mathbb{Z} \cong \pi_3(\mathrm{Sp}(n))$. We refine the result of Sutherland on the homotopy types of $\mathcal{G}_{k,n}$ and relate it to the order of a certain Samelson product in $\mathrm{Sp}(n)$. Then we classify the p–local homotopy types of $\mathcal{G}_{k,n}$ for $(p-1)^2+1 \geq 2n$.

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1 Introduction

Let G be a topological group and $P \to X$ be a principal G–bundle over a base space X. The gauge group of P, denoted by $\mathcal{G}(P)$, is the topological group of automorphisms of P, where an automorphism of P is a G–equivariant self-map of P covering the identity map of X. For fixed G and X, one has a collection of gauge groups $\mathcal{G}(P)$ as P ranges over all principal G–bundles over X, and we will be concerned with the classification of homotopy types in it.

Let G be a compact connected simple Lie group. Then there is a one-to-one correspondence between (isomorphism classes of) principal G–bundles over S^4 and $\pi_3(G) \cong \mathbb{Z}$. We denote by $\mathcal{G}_k(G)$ the gauge group of the bundle corresponding to $k \in \mathbb{Z} \cong \pi_3(G)$. Consider the classification of the homotopy type in the collection of gauge groups $\{\mathcal{G}_k(G)\}_{k\in\mathbb{Z}}$. The first classification was done by the second author [18] for $G = \mathrm{SU}(2)$, and since then, considerable effort has been made for the classification when G is of low rank; see Cutler [4], Hamanaka and Kono [8], Hamanaka, Kaji and Kono [7], Hasui, Kishimoto, Kono and Sato [9], Kamiyama, Kishimoto, Kono and Tsukuda [11], Kishimoto, Theriault and Tsutaya [16], Kono [18] and Theriault [23; 25; 26]. Properties of gauge groups related to the classification of the homotopy types have also been intensively studied; see Crabb and Sutherland [3] and Kishimoto and Kono [12], Kishimoto, Kono and Theriault [13], Kishimoto and Tsutaya [17], Kishimoto, Kono and Tsutaya [14; 15] and Theriault [24].

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In this paper, we study the classification of the homotopy types of $\mathcal{G}_k(\mathrm{Sp}(n))$. Let $\mathcal{G}_{k,n} = \mathcal{G}_k(\mathrm{Sp}(n))$. We will first consider Sutherland's homotopy invariant for $\mathcal{G}_{k,n}$ [21]: if $\mathcal{G}_{k,n}$ and $\mathcal{G}_{l,n}$ are homotopy equivalent, then $(k, n(2n+1)) = (l, n(2n+1))$ for n even and $(k, 4n(2n+1)) = (l, 4n(2n+1))$ for n odd. It seems that this invariant has indeterrminacy by a factor of 4 according to the parity of n, and we will refine Sutherland's result by removing this indeterminacy.

Theorem 1.1 *If $\mathcal{G}_{k,n}$ and $\mathcal{G}_{l,n}$ are homotopy equivalent, then $(k, 4n(2n+1)) = (l, 4n(2n+1))$.*

As for an explicit classification of $\mathcal{G}_{k,n}$, there are only two results for $n = 1, 2$: $\mathcal{G}_{k,1}$ and $\mathcal{G}_{l,1}$ are homotopy equivalent if and only if $(k, 12) = (l, 12)$ [18], and $\mathcal{G}_{k,2}$ and $\mathcal{G}_{l,2}$ are p–locally homotopy equivalent for any prime p if and only if $(k, 40) = (l, 40)$ [23]. The key fact that was used to prove these classifications is that $\mathcal{G}_k(G)$ is homotopy equivalent to the homotopy fiber of the map $G \to \Omega_0^3 G$ which is the adjoint of the Samelson product $S^3 \wedge G \to G$ of $k \in \mathbb{Z} \cong \pi_3(G)$ and the identity map of G. Actually, the integers 12 and 40 in the above classification are the orders of this Samelson product for $G = \mathrm{Sp}(1), \mathrm{Sp}(2)$, respectively. We will next show that the integer $4n(2n+1)$ in Theorem 1.1 is equal to the order of a certain Samelson product in $\mathrm{Sp}(n)$.

We set notation to state the result. Let $\epsilon\colon S^3 \to \mathrm{Sp}(n)$ be the bottom cell inclusion, so that it generates $\pi_3(\mathrm{Sp}(n)) \cong \mathbb{Z}$. Let Q_n be the quasiprojective space of rank n defined by James [10]. Then one has the inclusion $\iota_n\colon Q_n \to \mathrm{Sp}(n)$ such that the induced map in homology

$$\Lambda(\widetilde{H}_*(Q_n)) \to H_*(\mathrm{Sp}(n)) \tag{1}$$

is an isomorphism. We denote by $\langle \alpha, \beta \rangle$ the Samelson product of maps α and β.

Theorem 1.2 *The order of the Samelson product $\langle \epsilon, \iota_n \rangle$ in $\mathrm{Sp}(n)$ is $4n(2n+1)$.*

It is obvious that the order of the Samelson products $\langle \epsilon, 1_{\mathrm{Sp}(n)} \rangle$ is no less than the order of $\langle \epsilon, \iota_n \rangle$. Although we do not know these orders are equal, it is proved in [15] that if we localize at a large prime p, these orders are equal. Let $|g|$ denote the order of an element g of a group. For an integer $a = p^r q$ with $(p, q) = 1$, let $\nu_p(a) = p^r$.

Corollary 1.3 *If $(p-1)^2 + 1 \geq 2n$, then $\nu_p(|\langle \epsilon, 1_{\mathrm{Sp}(n)} \rangle|) = \nu_p(4n(2n+1))$.*

Remark The assumption in [15, Theorem 1.4], which is needed to prove Corollary 1.3, is $(p-1)(p-2)+1 \geq 2n$. But this assumption is actually too much and one can reduce it to $(p-1)^2+1 \geq 2n$ as in Corollary 1.3. This refinement will be explained in Section 2.

In [15] the classification of the p–local homotopy types of $\mathcal{G}_{k,n}$ for a large prime p is done in terms of the order of $\langle \epsilon, \iota_n \rangle$, by which one gets:

Corollary 1.4 *For $(p-1)^2+1 \geq 2n$, $\mathcal{G}_{k,n}$ and $\mathcal{G}_{l,n}$ are p–locally homotopy equivalent if and only if $\nu_p\big((k, 4n(2n+1))\big) = \nu_p\big((l, 4n(2n+1))\big)$.*

Remark Theriault [24] classified the p–local homotopy types of $\mathcal{G}_k(\mathrm{SU}(n))$ for $(p-1)^2+1 \geq n$ by using Toda's map $\Sigma^2\mathbb{C}P^{n-1} \to \mathbb{C}P^n$ for Bott periodicity. It may be possible to prove Corollaries 1.3 and 1.4 by modifying his method although Theorems 1.1 and 1.2 cannot. On the other hand, one can reprove Theriault's result by our method.

As in Friedlander [5], there is a p–local homotopy equivalence $B\mathrm{Spin}(2n+1) \simeq_{(p)} B\mathrm{Sp}(n)$ for any odd prime p, and we will see that this induces a p–local homotopy equivalence $\mathcal{G}_k(\mathrm{Spin}(2n+1)) \simeq_{(p)} \mathcal{G}_{k,n}$ for any odd prime p. On the other hand, it is shown in [12] that a p–local homotopy equivalence $\mathrm{Spin}(2n+2) \simeq_{(p)} \mathrm{Spin}(2n+1) \times S^{2n+1}$ for any odd prime p in Borel and Serre [2] induces a p–local homotopy equivalence between $\mathcal{G}_k(\mathrm{Spin}(2n+2))$ and the product of $\mathcal{G}_k(\mathrm{Spin}(2n+1))$ and a certain space for any odd prime p. Combining these results with Corollary 1.4, we get:

Corollary 1.5 *For $(p-1)^2+1 \geq 2n \geq 6$ and $\epsilon = 1, 2$, $\mathcal{G}_k(\mathrm{Spin}(2n+\epsilon))$ and $\mathcal{G}_l(\mathrm{Spin}(2n+\epsilon))$ are p–locally homotopy equivalent if and only if*

$$\nu_p\big((k, 4n(2n+1))\big) = \nu_p\big((l, 4n(2n+1))\big).$$

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2 Odd primary homotopy types of gauge groups

Let $\mathrm{map}(X, Y; f)$ be the path component of the mapping space $\mathrm{map}(X, Y)$ containing a map $f\colon X \to Y$. Let G be a compact connected simple Lie group. In [6; 1] it is shown that there is a homotopy equivalence

$$B\mathcal{G}_k(G) \simeq \mathrm{map}(S^4, BG; k\bar{\epsilon}), \tag{2}$$

where $\bar{\epsilon}$ corresponds to $1 \in \mathbb{Z} \cong \pi_4(BG)$. So, evaluating at the basepoint of S^4, one gets a homotopy fibration sequence

$$\mathcal{G}_k(G) \to G \xrightarrow{\partial_k} \Omega_0^3 G \to B\mathcal{G}_k(G) \to BG. \tag{3}$$

In particular, $\mathcal{G}_k(G)$ is homotopy equivalent to the homotopy fiber of ∂_k. Lang [19] identified ∂_k with a certain Samelson product in G. Let $\epsilon\colon S^3 \to G$ be the adjoint of $\bar{\epsilon}$.

Lemma 2.1 *The adjoint $S^3 \wedge G \to G$ of ∂_k is homotopic to the Samelson product $\langle k\epsilon, 1_G\rangle$.*

By linearity of Samelson products, we have $\langle k\epsilon, 1_G\rangle = k\langle \epsilon, 1_G\rangle$. We denote the k^{th} power map of $\Omega_0^3 G$ by the same symbol k. Then we get:

Corollary 2.2 $$\partial_k \simeq k \circ \partial_1.$$

Thus one sees that the order of the Samelson product $\langle \epsilon, 1_G\rangle$ is connected to the classification of the homotopy types of $\mathcal{G}_k(G)$. It is shown in [15] that, localized at a large prime, the calculation of the Samelson product $\langle \epsilon, 1_G\rangle$ reduces drastically and the homotopy types of $\mathcal{G}_k(G)$ are classified in terms of the order of $\langle \epsilon, 1_G\rangle$. We recall these results. Given a prime p, a space A is called a homology generating space of an H–space X if the following conditions hold:

(1) $H_*(X; \mathbb{Z}/p) = \Lambda(x_1, \dots, x_m)$.

(2) There is a map $\iota\colon A \to X_{(p)}$ which induces the inclusion of a generating set in mod p homology.

An H–space X is called retractible if it has a homology generating space A and the map $\Sigma\iota\colon \Sigma A \to \Sigma X_{(p)}$ has a left homotopy inverse. It is proved in [22] that if (G, p) is in Table 1, then $G_{(p)}$ is retractible, where we omit the cases $G = \mathrm{Spin}(2n)$ and $(G, p) = (\mathrm{G}_2, 3)$.

If G has a homology generating space A at a prime p, then the p–primary component of the order of $\langle \epsilon, \iota\rangle$ is obviously no less than that of $\langle \epsilon, 1_G\rangle$. In [15], if G is retractile in addition, then these two coincide. The assumption in [15] for this result is stronger than retractibility but one can easily follow its proof to see that only retractibility is used. So we record this result here with a weaker assumption.

Proposition 2.3 *If (G, p) is in Table 1, then $\nu_p(|\langle \epsilon, 1_G\rangle|) = \nu_p(|\langle \epsilon, \iota\rangle|)$.*

$\mathrm{SU}(n)$	$(p-1)^2+1 \geq n$
$\mathrm{Sp}(n), \mathrm{Spin}(2n+1)$	$(p-1)^2+1 \geq 2n$
$\mathrm{G}_2, \mathrm{F}_4, \mathrm{E}_6$	$p \geq 5$
$\mathrm{E}_7, \mathrm{E}_8$	$p \geq 7$

Table 1: Retractible Lie groups

Using this proposition, the following is proved in [15], where the assumption on the prime p can be weakened as well.

Theorem 2.4 *Suppose that (G,p) is in Table 1. Then $\mathcal{G}_k(G)$ and $\mathcal{G}_l(G)$ are p–locally homotopy equivalent if and only if $\nu_p((k,|\langle\epsilon,\iota\rangle|)) = \nu_p((l,|\langle\epsilon,\iota\rangle|))$.*

Proof of Corollary 1.3 Since (1) is an isomorphism, Q_n is a homology generating space of $\mathrm{Sp}(n)$ at any prime, and as in [22], $\mathrm{Sp}(n)$ is retractible with respect to Q_n at the prime p. Then Corollary 1.3 follows from Theorem 1.2 and Proposition 2.3. □

Proof of Corollary 1.4 This follows from Corollary 1.3 and Theorem 2.4. □

Proof of Corollary 1.5 We first consider the p–local homotopy type of the gauge group $\mathcal{G}_k(\mathrm{Spin}(2n+1))$ for any odd prime p. By [5], $B\mathrm{Sp}(n) \simeq_{(p)} B\mathrm{Spin}(2n+1)$. Then it follows from (2) that $\mathcal{G}_k(\mathrm{Spin}(2n+1)) \simeq_{(p)} \mathcal{G}_{k,n}$. Thus the result follows from Corollary 1.4.

We next consider the p–local homotopy type of $\mathcal{G}_k(\mathrm{Spin}(2n+2))$. Note that we are now assuming $p \geq 5$. Then it follows from [12] that there is a p–local homotopy equivalence

$$\mathcal{G}_k(\mathrm{Spin}(2n+2)) \simeq_{(p)} \mathcal{G}_k(\mathrm{Spin}(2n+1)) \times S^{2n+1} \times \Omega^4 S^{2n+1}.$$

So the above case of $\mathcal{G}_k(\mathrm{Spin}(2n+1))$ implies that

$$\mathcal{G}_k(\mathrm{Spin}(2n+2)) \simeq_{(p)} \mathcal{G}_l(\mathrm{Spin}(2n+2))$$

whenever $\nu_p\big((k,4n(2n+1))\big) = \nu_p(k,4n(2n+1))$. By [21], $\pi_{4n+1}(\mathcal{G}_{k,n})_{(p)} \cong \mathbb{Z}/\nu_p\big((k,4n(2n+1))\big)$. The order of $\pi_{4n+1}(S^{2n+1} \times \Omega^4 S^{2n+1})_{(p)}$ is finite, say M, implying that the order of

$$\pi_{4n+1}\big(\mathcal{G}_k(\mathrm{Spin}(2n+2))\big)_{(p)} \cong \pi_{4n+1}(\mathcal{G}_{k,n} \times S^{2n+1} \times \Omega^4 S^{2n+1})_{(p)}$$

is $M\nu_p\big((k,4n(2n+1))\big)$. Thus we get that $\nu_p\big((k,4n(2n+1))\big) = \nu_p\big((l,4n(2n+1))\big)$ whenever $\mathcal{G}_k(\mathrm{Spin}(2n+2)) \simeq_{(p)} \mathcal{G}_l(\mathrm{Spin}(2n+2))$, completing the proof. □

3 Unstable $K\mathrm{Sp}$–theory

If a space Z is low-dimensional, then the homotopy set $[Z,\mathrm{U}(n)]$ is isomorphic to $\widetilde{K}^{-1}(Z)$. So we call $[Z,\mathrm{U}(n)]$ unstable K–theory. In [8], for $\dim Z \le 2n$, a method for computing $[Z,\mathrm{U}(n)]$ is given by comparing it with $\widetilde{K}^{-1}(Z)$. We call the homotopy set $[Z,\mathrm{Sp}(n)]$ unstable $K\mathrm{Sp}$–theory as well, and Nagao [20] considered the analogous method for computing unstable $K\mathrm{Sp}$–theory. We will use Nagao's method to calculate Samelson products in $\mathrm{Sp}(n)$, so we recall it here.

The cohomology of $B\mathrm{Sp}(n)$ and $\mathrm{Sp}(n)$ are given by

$$H^*(B\mathrm{Sp}(n)) = \mathbb{Z}[q_1,\dots,q_n], \quad H^*(\mathrm{Sp}(n)) = \Lambda(x_3,\dots,,x_{4n-1}),$$

where q_i is the i^{th} symplectic Pontrjagin class and $x_{4i-1} = \sigma(q_i)$ for the cohomology suspension σ. Let $X_n = \mathrm{Sp}(\infty)/\mathrm{Sp}(n)$. By an easy inspection, one sees that $H^*(X_n) = \Lambda(\bar{x}_{4n+3},\bar{x}_{4n+7},\dots)$ for $\pi^*(\bar{x}_{4i-1}) = x_{4i-1}$, where $\pi\colon \mathrm{Sp}(\infty)\to X_n$ is the projection. Then we get that ΩX_n is $(4n{+}1)$–connected and $H^{4n+2}(\Omega X_n) = \mathbb{Z}\{a_{4n+2}\}$, where $\sigma(\bar{x}_{4n+3}) = a_{4n+2}$ and $R\{z_1,z_2,\dots\}$ means the free R–module with a basis $\{z_1,z_2,\dots\}$. In particular, the map $a_{4n+2}\colon \Omega X_n \to K(\mathbb{Z},4n+2)$ is a loop map and is a $(4n{+}3)$–equivalence. So if $\dim Z \le 4n+2$, the map $(a_{4n+2})_*\colon [Z,\Omega X_n] \to H^{4n+2}(Z)$ is an isomorphism of groups. Moreover, it is shown in [20] that the composite

$$\widetilde{K\mathrm{Sp}}^{-2}(Z) = [Z,\Omega\mathrm{Sp}(\infty)] \xrightarrow{(\Omega\pi)_*} [Z,\Omega X_n] \xrightarrow{(a_{4n+2})_*} H^{4n+2}(Z)$$

is given by $(-1)^{n+1}(2n+1)!\,\mathrm{ch}_{4n+2}(u^{-1}c'(\xi))$ for $\xi\in\widetilde{K\mathrm{Sp}}^{-2}(Z)$, where ch_k denotes the $2k$–dimensional part of the Chern character, u is a generator of $\widetilde{K}(S^2)\cong\mathbb{Z}$ and $c'\colon K\mathrm{Sp}\to K$ is the complexification. Now we apply $[Z,-]$ to the homotopy fibration sequence $\Omega\mathrm{Sp}(\infty)\to\Omega X_n\to\mathrm{Sp}(n)\to\mathrm{Sp}(\infty)$ and get an exact sequence of groups

$$\widetilde{K\mathrm{Sp}}^{-2}(Z) \to [Z,\Omega X_n] \to [Z,\mathrm{Sp}(n)] \to \widetilde{K\mathrm{Sp}}^{-1}(Z).$$

Then, by the above identification of $[Z,\Omega X_n]$, Nagao [20] obtained:

Theorem 3.1 *If Z is a CW–complex of dimension $\le 4n+2$, then there is an exact sequence of groups*

$$\widetilde{K\mathrm{Sp}}^{-2}(Z) \xrightarrow{\Phi} H^{4n+2}(Z) \to [Z,\mathrm{Sp}(n)] \to \widetilde{K\mathrm{Sp}}^{-1}(Z)$$

such that, for $\xi\in\widetilde{K\mathrm{Sp}}^{-2}(Z)$,

$$\Phi(\xi) = (-1)^{n+1}(2n+1)!\,\mathrm{ch}_{4n+2}(u^{-1}c'(\xi)).$$

This is also useful for computing the Samelson products in $\mathrm{Sp}(n)$, as follows. Let $\gamma\colon \mathrm{Sp}(n)\wedge\mathrm{Sp}(n)\to\mathrm{Sp}(n)$ be the reduced commutator map. Since $\mathrm{Sp}(\infty)$ is homotopy commutative, the composite $\mathrm{Sp}(n)\wedge\mathrm{Sp}(n)\xrightarrow{\gamma}\mathrm{Sp}(n)\to\mathrm{Sp}(\infty)$ is null-homotopic. Then, since there is a homotopy fibration $\Omega X_n\to\mathrm{Sp}(n)\to\mathrm{Sp}(\infty)$, γ lifts to a map $\tilde\gamma\colon \mathrm{Sp}(n)\wedge\mathrm{Sp}(n)\to\Omega X_n$. In [20], a specific lift is constructed as:

Proposition 3.2 *There is a lift* $\tilde\gamma\colon \mathrm{Sp}(n)\wedge\mathrm{Sp}(n)\to\Omega X_n$ *of* γ *satisfying*

$$\tilde\gamma^*(a_{4n+2})=\sum_{i+j=n+1} x_{4i-1}\otimes x_{4j-1}.$$

Thus, by Theorem 3.1, one gets:

Corollary 3.3 *Let* A *and* B *be CW–complexes such that* $\dim A+\dim B\le 4n+2$. *The order of the Samelson product of maps* $\alpha\colon A\to\mathrm{Sp}(n)$ *and* $\beta\colon B\to\mathrm{Sp}(n)$ *is equal to the order of*

$$\sum_{i+j=n+1}\alpha^*(x_{4i-1})\otimes\beta^*(x_{4j-1})$$

in the cokernel of the map $\Phi\colon \widetilde{K\mathrm{Sp}}^{-2}(A\wedge B)\to H^{4n+2}(A\wedge B)$ *of Theorem 3.1*

The following data of $\tilde K^*(Q_n)$ and $\widetilde{K\mathrm{Sp}}^*(Q_n)$ will be used to apply the above results to our case. Let $\iota_n\colon Q_n\to\mathrm{Sp}(n)$ be the inclusion and $\theta_1\colon \Sigma Q_2\to B\mathrm{Sp}(\infty)$ be the composite of the adjoint $\Sigma Q_2\to B\mathrm{Sp}(2)$ of ι_2 and the inclusion $B\mathrm{Sp}(2)\to B\mathrm{Sp}(\infty)$. Let θ_2 be the composite of the pinch map onto the top cell $\Sigma Q_2\to S^8$ and a generator of $\pi_8(B\mathrm{Sp}(\infty))\cong\mathbb{Z}$. Put $y_{4j-1}=\iota_n^*(x_{4j-1})$. Then $H^*(Q_n)=\mathbb{Z}\{y_3,\dots,y_{4n-1}\}$ and

$$\mathrm{ch}(c'(\theta_1))=\Sigma y_3-\tfrac16\Sigma y_7,\quad \mathrm{ch}(c'(\theta_2))=2\Sigma y_7. \tag{4}$$

Let $\rho_1=q(u^2c'(\theta_1))\in\widetilde{K\mathrm{Sp}}(\Sigma^5Q_2)$, where $q\colon K\to K\mathrm{Sp}$ is the quaternionization. Let $\rho_2\in\widetilde{K\mathrm{Sp}}(\Sigma^5Q_2)$ be the composite of the pinch map to the top cell $\Sigma^5Q_2\to S^{12}$ and a generator of $\pi_{12}(B\mathrm{Sp}(\infty))\cong\mathbb{Z}$. Then we have

$$\mathrm{ch}(c'(\rho_1))=2\Sigma^5y_3+\tfrac13\Sigma^5y_7,\quad \mathrm{ch}(c'(\rho_2))=\Sigma^5y_7.$$

Lemma 3.4
$$\widetilde{K\mathrm{Sp}}(\Sigma^iQ_2)=\begin{cases}\mathbb{Z}\{\theta_1,\theta_2\} & \text{if } i=1,\\ \mathbb{Z}\{\rho_1,\rho_2\} & \text{if } i=5,\\ 0 & \text{if } i\equiv 0 \mod 4.\end{cases}$$

Proof A homotopy cofibration $S^4 \to \Sigma Q_2 \to S^8$ induces a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{K\mathrm{Sp}}(S^8) & \longrightarrow & \widetilde{K\mathrm{Sp}}(\Sigma Q_2) & \longrightarrow & \widetilde{K\mathrm{Sp}}(S^4) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle c'=2} & & \downarrow{\scriptstyle c'} & & \downarrow{\scriptstyle c'=1} & & \\ 0 & \longrightarrow & \widetilde{K}(S^8) & \longrightarrow & \widetilde{K}(\Sigma Q_2) & \longrightarrow & \widetilde{K}(S^4) & \longrightarrow & 0 \end{array}$$

Then we get the first equality by (4) and $\widetilde{K\mathrm{Sp}}(S^{4m}) \cong \mathbb{Z}$. The remaining equalities are seen by the same argument. □

The complexification $c'\colon B\mathrm{Sp}(\infty) \to B\mathrm{U}(\infty)$ restricts to a map $\Sigma Q_n \to \Sigma^2\mathbb{C}P^{2n-1}$, which we denote by the same symbol c'. Let $\eta \in \widetilde{K}(\mathbb{C}P^{2n-1})$ be the Hopf bundle minus the trivial line bundle, and put $\xi_i = (c')^*(u\eta^i) \in \widetilde{K}(\Sigma Q_n)$. Then we have

$$\mathrm{ch}(c'(\xi_i)) \equiv \Sigma x_{4i-1} \mod (\Sigma x_{4j-1} \mid j > i).$$

Thus, by the skeletal argument analogous to the proof of Lemma 3.4, one gets:

Lemma 3.5 $$\widetilde{K}(\Sigma Q_n) = \mathbb{Z}\{\xi_1, \dots, \xi_n\}.$$

Proposition 3.6 *$\widetilde{K\mathrm{Sp}}(\Sigma^5 Q_n) = \mathbb{Z}\{\zeta_1, \dots, \zeta_n\}$, where $\zeta_1 = q(u^2\xi_1)$ and*

$$\mathrm{ch}(c'(\zeta_i)) \equiv \epsilon_i \Sigma^5 y_{4i-1} \mod (\Sigma^5 y_{4j-1} \mid j > i)$$

for $i > 1$ with $\epsilon_i = 1$ for i even and $\epsilon_i = 2$ for i odd.

Proof The case $n = 2$ is proved in Lemma 3.4. Consider the commutative diagram with exact rows induced from the homotopy cofibration sequence $\Sigma^5 Q_{n-1} \to \Sigma^5 Q_n \to S^{4n+4}$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{K\mathrm{Sp}}(S^{4n+4}) & \longrightarrow & \widetilde{K\mathrm{Sp}}(\Sigma^5 Q_n) & \longrightarrow & \widetilde{K\mathrm{Sp}}(\Sigma^5 Q_{n-1}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle c'=\epsilon_n} & & \downarrow{\scriptstyle c'} & & \downarrow{\scriptstyle c'} & & \\ 0 & \longrightarrow & \widetilde{K}(S^{4n+4}) & \longrightarrow & \widetilde{K}(\Sigma^5 Q_n) & \longrightarrow & \widetilde{K}(\Sigma^5 Q_{n-1}) & \longrightarrow & 0 \end{array}$$

Induct on n. Then we get that $\widetilde{K\mathrm{Sp}}(\Sigma^5 Q_n)$ is a free abelian group and the upper exact sequence splits. Thus we obtain the desired $\zeta_1, \dots, \zeta_n$, completing the proof. □

4 Proofs of the main theorems

To prove Theorem 1.1, we need several lemmas. Let $\widetilde{\partial}_k\colon \mathrm{Sp}(n) \to \Omega^4 X_n$ be the adjoint of the map $\widetilde{\gamma} \circ (\epsilon \wedge 1_{\mathrm{Sp}(n)})\colon S^3 \wedge \mathrm{Sp}(n) \to \Omega X_n$, where $\widetilde{\gamma}$ is as in Proposition 3.2. Then $\widetilde{\partial}_k$ is a lift of ∂_k, so by (3) and Theorem 3.1, we get the following commutative diagram with exact columns and rows, where $\delta_k = (a_{4n+2} \circ \widetilde{\partial}_k)_*$:

$$
\begin{array}{ccccccc}
 & & \widetilde{K\mathrm{Sp}}^{-2}(\Sigma^{4n-5}Q_2) & & & & \\
 & & \downarrow{\scriptstyle\Phi} & & & & \\
\widetilde{K\mathrm{Sp}}^{-1}(\Sigma^{4n-8}Q_2) & \xrightarrow{\delta_k} & H^{4n+2}(\Sigma^{4n-5}Q_2) & & & & \\
\| & & \downarrow & & & & \\
\widetilde{K\mathrm{Sp}}^{-1}(\Sigma^{4n-8}Q_2) & \xrightarrow{(\partial_k)_*} & [\Sigma^{4n-5}Q_2, \mathrm{Sp}(n)] & \to & [\Sigma^{4n-8}Q_2, B\mathcal{G}_{k,n}] & \to & \widetilde{K\mathrm{Sp}}(\Sigma^{4n-8}Q_2) \\
 & & \downarrow & & & & \\
 & & \widetilde{K\mathrm{Sp}}^{-1}(\Sigma^{4n-5}Q_2) & & & &
\end{array}
\tag{5}
$$

Lemma 4.1 $$[\Sigma^{4n-8}Q_2, B\mathcal{G}_{k,n}] \cong \mathrm{Coker}(\partial_k)_*.$$

Proof By Lemma 3.4, one has $\widetilde{K\mathrm{Sp}}(\Sigma^{4n-8}Q_2) = 0$, so the lemma follows from (5). □

Lemma 4.2 $[\Sigma^{4n-5}Q_2, \mathrm{Sp}(n)] \cong \mathbb{Z}/\big(\frac{1}{3}(2n+1)!\big)$ *for n even.*

Proof Since $\widetilde{K\mathrm{Sp}}^{-1}(\Sigma^{4n-5}Q_2) = 0$ by Lemma 3.4, we get $[\Sigma^{4n-5}Q_2, \mathrm{Sp}(n)] \cong \mathrm{Coker}\,\Phi$ by (5). Since n is even, $\widetilde{K\mathrm{Sp}}^{-2}(\Sigma^{4n-5}Q_2) \cong \widetilde{K\mathrm{Sp}}(\Sigma^5 Q_2)$. Then it follows from Theorem 3.1 and Lemma 3.4 that $\mathrm{Im}\,\Phi = \mathbb{Z}\big\{\frac{1}{3}(2n+1)!\,\Sigma^{2n-5}y_7\big\}$. Thus, for $H^{4n+2}(\Sigma^{4n-5}Q_2) = \mathbb{Z}\{\Sigma^{4n-5}y_7\}$, the proof is done. □

Lemma 4.3 $\mathrm{Im}(\partial_k)_* \cong \mathbb{Z}/\big((2n+1)!/\big(3(k, 4n(2n+1))\big)\big)$ *for n even.*

Proof Since $\widetilde{K\mathrm{Sp}}^{-1}(\Sigma^{4n-5}Q_2) = 0$ by Lemma 3.4, we have $\mathrm{Im}(\partial_k)_* = \mathrm{Im}\,\delta_k/\mathrm{Im}\,\Phi$ by (5). We calculate $\mathrm{Im}\,\delta_k$, where $\mathrm{Im}\,\Phi$ has already been calculated in the proof of Lemma 4.2. Let $\widehat{\alpha}\colon \Sigma^{4n-8}Q_2 \to \mathrm{Sp}(\infty)$ be the adjoint of $\alpha \in \widetilde{K\mathrm{Sp}}(\Sigma^{4n-7}Q_2)$. By definition, we have $\delta_k(\alpha) = k\Sigma^3\widehat{\alpha}^*(x_{4n-1})$, so we calculate $\widehat{\alpha}^*(x_{4n-1})$. Let $\mathrm{ch}(c'(\alpha)) = a\Sigma^{4n-7}y_3 + b\Sigma^{4n-7}y_7$ for $a, b \in \mathbb{Q}$. By the Newton formula, $\mathrm{ch}_{4n} = -(1/(2n-1)!)c_{2n} + \text{decomposables}$, implying that $(-1)^n\alpha^*(q_n) = (c' \circ \alpha)^*(c_{2n}) = -b(2n-1)!\,\Sigma^{4n-7}y_7$. Then, by taking the adjoint, we get

$$\widehat{\alpha}^*(x_{4n-1}) = (-1)^{n+1}b(2n-1)!\,\Sigma^{4n-8}y_7.$$

Since n is even, we have $\widetilde{KSp}(\Sigma^{4n-7}Q_2)\cong\widetilde{KSp}(\Sigma Q_2)$. Thus, by Lemma 3.4, we obtain $\operatorname{Im}\delta_k=\mathbb{Z}\{\frac{1}{6}k(2n-1)!\,\Sigma^{4n-5}y_7\}$. Therefore, the proof is completed. □

Lemma 4.4 *If n is even and $n>2$, then* $[\Sigma^{4n-8}Q_2, B\mathcal{G}_{k,n}]\cong\mathbb{Z}/(k,4n(2n+1))$.

Proof Combine Lemmas 4.1, 4.2 and 4.3. □

Proof of Theorem 1.1 By the result of Sutherland [21] mentioned above, it is sufficient to prove the theorem for n even. When $n=2$, the result of Theriault [23] mentioned above implies the theorem. Assume that $n>2$ and $\mathcal{G}_{k,n}\simeq\mathcal{G}_{l,n}$. Then, since $[\Sigma^{4n-8}Q_2, B\mathcal{G}_{m,n}]\cong[\Sigma^{4n-9}Q_2,\mathcal{G}_{m,n}]$ for any m, we have $[\Sigma^{4n-8}Q_2, B\mathcal{G}_{k,n}]\cong[\Sigma^{4n-8}Q_2, B\mathcal{G}_{l,n}]$, so the theorem follows from Lemma 4.4. □

Proof of Theorem 1.2 As $\dim\Sigma^3Q_n=4n+2$, we apply Corollary 3.3 to the Samelson product $\langle\epsilon,\iota_n\rangle$ in $\mathrm{Sp}(n)$. Then, for $\sum_{i+j=n+1}\epsilon^*(x_{4i-1})\otimes\iota_n^*(x_{4j-1})=\Sigma^3y_{4n-1}$, it is sufficient to show that the image of $\Phi\colon\widetilde{KSp}^{-2}(\Sigma^3Q_n)\to H^{4n+2}(\Sigma^3Q_n)$ is generated by $4n(2n+1)\Sigma^3y_{4n-1}$. For $\zeta_1\in\widetilde{KSp}(\Sigma^5Q_n)$ of Proposition 3.6, we have

$$\mathrm{ch}_{4n+2}(u^{-1}c'(\zeta_1))=\mathrm{ch}_{4n+2}((1+t)(u\xi_1))=\frac{2}{(2n-1)!}\Sigma^3y_{4n-1},$$

so $4n(2n+1)\Sigma^3y_{4n-1}\in\operatorname{Im}\Phi$, where $t\colon K\to K$ is the complex conjugation. On the other hand, by Lemmas 3.5 and 3.6, $c'(\widetilde{KSp}(\Sigma^5Q_n))$ is included in

$$\mathbb{Z}\{c'(\zeta_1),u^2\xi_2,\dots,u^2\xi_n\}\subset\widetilde{K}(\Sigma^5Q_n).$$

By definition, we have

$$\mathrm{ch}_{4n+2}(u\wedge\xi_k)=\sum_{\substack{r_1+\cdots+r_k=2n-1\\ r_1\ge1,\dots,r_k\ge1}}\frac{(2n-1)!}{r_1!\cdots r_k!}\cdot\frac{1}{(2r_1-1)!\cdots(2r_k-1)!}\Sigma^3y_{4n-1}.$$

For $k\ge2$, the coefficients of $(2n+1)!\,\mathrm{ch}_{4n+2}(u\xi_k)$ are divisible by $4n(2n+1)$. Then $\operatorname{Im}\Phi$ is included in the submodule generated by $4n(2n+1)\Sigma^3y_{4n-1}$. Thus we obtain that $\operatorname{Im}\Phi$ is generated by $4n(2n+1)\Sigma^3y_{4n-1}$, as desired. Therefore, the proof is completed. □

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