A note on the (∞, n)–category of cobordisms

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In this extended note we give a precise definition of fully extended topological field theories à la Lurie. Using complete n–fold Segal spaces as a model, we construct an (∞, n)–category of n–dimensional bordisms, possibly with tangential structure. We endow it with a symmetric monoidal structure and show that we can recover the usual category of bordisms.

Introduction

Topological field theories (TFTs) arose as toy models for physical quantum field theories and have proven to be of mathematical interest, notably because they are a fruitful tool for studying topology. An n–dimensional TFT is a symmetric monoidal functor from the category of bordisms, which has closed $(n-1)$–dimensional manifolds as objects and n–dimensional bordisms as morphisms, to any other symmetric monoidal category, which classically is taken to be the category of vector spaces or chain complexes.

A classification of 1– and 2–dimensional TFTs follows from classification theorems for 1– and 2–dimensional compact manifolds with boundary; see Abrams [1]. In order to obtain a classification result for larger values of n one needs a suitable replacement of the classification of compact n–manifolds with boundary used in the low-dimensional cases. Moreover, as explained by Baez and Dolan [2], this approach requires passing to "extended" topological field theories. Here "extended" means that we need to be able to evaluate the n–TFT not only at n– and $(n-1)$–dimensional manifolds, but also at $(n-2)$–, ..., 1– and 0–dimensional manifolds. Thus, an extended n–TFT is a symmetric monoidal functor out of a higher category of bordisms. In light of the hope of computability of the invariants determined by an n–TFT, eg by a triangulation, it is natural to include this data. Furthermore, Baez and Dolan conjectured that, analogously to the 1–dimensional case, extended n–TFTs are fully determined by their value at a point, calling this the cobordism hypothesis. A definition of a suitable bicategory of n–bordisms and a proof of a classification theorem of extended TFTs for dimension 2 was given by Schommer-Pries [49].

In his expository manuscript [40], Lurie suggested passing to (∞, n)–categories for a proof of the cobordism hypothesis in arbitrary dimension n. He gave a detailed sketch of such a proof using a suitable higher category of bordisms, which, informally speaking, has zero-dimensional manifolds as objects, bordisms between objects as 1–morphisms, bordisms between bordisms as 2–morphisms, etc, and for $k > n$ there are only invertible k–morphisms given by diffeomorphisms and their isotopies. However, finding an explicit model for such a higher category poses one of the difficulties in rigorously defining these n–dimensional TFTs, which are called "fully extended".

In [40], Lurie gave a short sketch of a definition of this (∞, n)–category using complete n–fold Segal spaces as a model. Instead of using manifolds with corners and gluing them, his approach was to conversely use embedded closed (not necessarily compact) manifolds, following along the lines of Bökstedt and Madsen [12], Galatius [19] and

Galatius, Tillmann, Madsen and Weiss [21], and to specify points where they are cut into bordisms of which the embedded manifold is a composition. Whitney's embedding theorem ensures that every n–dimensional manifold M can be embedded into some large-enough vector space and suitable versions for manifolds with boundary can be adapted to obtain an embedding theorem for bordisms; see Section 8. Moreover, the rough idea behind the definition of the n–fold Segal space is that it includes the data, for $k_1, \ldots, k_n$, of the classifying space for diffeomorphisms of — in the ith direction k_i–fold — composable n–bordisms. Lurie's idea was to use the fact that the space of embeddings of M into $\mathbb{R}^\infty$ is contractible to justify the construction.

Modifying this approach, the main goal of this note is to provide a detailed construction of such an (∞, n)–category of bordisms, suitable for explicitly constructing an example of a fully extended n–TFT, which will be the content of a subsequent paper [13]. As we explain in Section 6.3, Lurie's sketch does not lead to an n–fold Segal space, as the essential constancy condition is violated. In our Definition 5.1, we propose a stronger condition on elements in the levels of the Segal space. We show that this indeed yields an n–fold Segal space PBord$_n$. Its completion Bord$_n$ defines an (∞, n)–category of n–bordisms and thus is a corrigendum to Lurie's n–fold simplicial space of bordisms from [40].

Furthermore, we endow it with a symmetric monoidal structure and also consider bordism categories with additional structure, eg orientations and framings, which allows us, in Section 10, to rigorously define fully extended topological field theories.

Our main motivation to have a precise definition of the (∞, n)–category of bordisms was the following: in the subsequent paper [13] we explicitly construct an example of a fully extended topological field theory. Given an E_n–algebra A we show that factorization homology with coefficients in A leads to a fully extended n–dimensional topological field theory with target category a suitable (∞, n)–Morita category with E_n–algebras as objects, bimodules as 1–morphisms, bimodules between bimodules as 2–morphisms, etc.

Organization of the paper In Part I, consisting of the first three sections, we recall the necessary tools from higher category theory needed to define fully extended TFTs.

Section 1 reviews the model for $(\infty, 1)$–categories given by complete Segal spaces and recalls some useful information about other models. In Section 2 we explain the model for (∞, n)–categories given by complete n–fold Segal spaces and introduce a model which is a hybrid between complete n–fold Segal spaces and Segal n–categories.

We propose two equivalent definitions of symmetric monoidal structures on complete n–fold Segal spaces in Section 3, one as a Γ–object in complete n–fold Segal spaces following Toën and Vezzosi [56] and one as a tower of suitable $(n+k)$–fold Segal spaces with one object, 1–morphism, ..., $(k-1)$–morphism for $k \geq 0$ following the delooping hypothesis.

Part II is devoted to the construction of Bord_n.

Our construction of the (∞, n)–category Bord_n of higher bordisms is based on a simpler complete Segal space Int of closed intervals, which we introduce in Section 4. The closed intervals correspond to places where we are allowed to cut the manifold into the bordisms it comprises. The fact that we prescribe closed intervals instead of just a point corresponds to fixing collars of the bordisms.

Section 5 is the central part of this article and consists of the construction of the complete n–fold Segal space Bord_n of bordisms. We discuss variants of Bord_n, including (∞, d)–categories of bordisms and tangles for arbitrary d, and compare our definition to Lurie's sketch in Section 6.

In Section 7 we endow Bord_n with a symmetric monoidal structure, both as a Γ–object and as a tower, and compare the two definitions.

In Section 8 we elaborate on the interpretation of the objects in Bord_n as n–bordisms. Furthermore we show that the homotopy category of the $(\infty, 1)$–category of bordisms is what one should expect, namely the homotopy category of the $(\infty, 1)$–category of n–bordisms $\mathrm{Bord}_n^{(\infty,1)}$ gives back the classical bordism category $n\mathrm{Cob}$.

Finally, in Section 9 we consider bordism categories with additional structure such as orientations, denoted by $\mathrm{Bord}_n^{\mathrm{or}}$, and framings, denoted by $\mathrm{Bord}_n^{\mathrm{fr}}$, which allows us to define fully extended n–dimensional topological field theories in Section 10.

Conventions

(1) Let *Space* denote the category of simplicial sets with its usual model structure. By *space* we mean a fibrant object in *Space*, ie a Kan complex.

(2) We denote the simplex category by Δ. Objects are finite ordered sets denoted by $[m] = (0 < \cdots < m)$ and morphisms are monotone maps. As an ordered set, we can view $[m]$ as a category. Functors between the associated categories arise exactly from monotone maps. Thus, we can take the nerve of $[m]$, which we will denote by Δ^m.

(3) The geometric realization $|\Delta^l|$ is the standard geometric simplex

$$\Big\{(x_0, \dots, x_l) \in \mathbb{R}^{l+1} : \sum_i x_i = 1,\ x_i \geq 0\Big\}.$$

We denote the extended simplex $\big\{(x_0, \dots, x_l) \in \mathbb{R}^{l+1} : \sum_i x_i = 1\big\}$ by $|\Delta^l|_e$.

Acknowledgements

Part I Symmetric monoidal (∞, n)–categories

A higher category, or n–category for $n \geq 0$, has not only objects and $(1-)$morphisms, but also k–morphisms between $(k-1)$–morphisms for $1 \leq k \leq n$. Strict higher categories can be rigorously defined, however most higher categories which occur in nature are not strict. Thus, we need to weaken some axioms and coherences between the weakenings become rather involved to formulate explicitly. Things turn out to become somewhat easier when using a geometric definition, in particular when furthermore allowing k–morphisms for all $k \geq 1$, which for $k \geq n$ are invertible. Such a higher category is called an (∞, n)–category. There are several models for such (∞, n)–categories, eg Segal n–categories (see [27]), Θ_n–spaces (see [46]) and complete n–fold Segal spaces (see [3]), which all are equivalent in an appropriate sense (see [53; 6]). For our purposes, the latter model turns out to be well suited and in this part we recall some basic facts about complete n–fold Segal spaces as higher categories. This is not at all exhaustive, and more details can be found in eg [10]. We also refer to [9], especially for their role in the proof of the cobordism hypothesis in [40].

Symmetric monoidal structures on (∞, n)–categories per se have not been very much studied in the literature for $n > 1$ (even though they are particular instances of commutative monoids in ∞–categories, which are extensively studied in [41]). We provide a

brief review of these in Section 3 and describe them in two different, but equivalent, ways: as Γ–objects on the one hand and using the delooping hypothesis on the other hand. A comparison follows from [23].

1 Models for $(\infty, 1)$–categories

We start with $n = 1$. An $(\infty, 1)$–category should be a category up to coherent homotopy which is encoded in the invertible higher morphisms. In this section, we will mention and give references for several models for $(\infty, 1)$–categories we will use in the later sections. A good overview on different models for $(\infty, 1)$–categories and their comparison can be found in [8]. It should be mentioned that, by [53], up to equivalence there is essentially only one theory of $(\infty, 1)$–categories; explicit equivalences between the models mentioned here have been proved eg in [17; 7; 5; 28]. One additional model which should be mentioned is that of Joyal's quasicategories. It has been intensively studied, most prominently in [38].

1.1 The homotopy hypothesis and $(\infty, 0)$–categories

The basic hypothesis upon which ∞–category theory is based goes back to Grothendieck [24] and is the following:

Hypothesis 1.1 (homotopy hypothesis) Spaces are models for ∞–groupoids, also referred to as $(\infty, 0)$–categories.

Given a space X, its points, ie 0–simplices, are thought of as objects of the $(\infty, 0)$–category, paths between points as 1–morphisms, homotopies between paths as 2–morphisms, homotopies between homotopies as 3–morphisms, and so forth. With this interpretation, it is clear that all n–morphisms are invertible up to homotopies, which are higher morphisms.

We take this hypothesis as the basic definition, and model "spaces" with simplicial sets rather than with topological spaces.

Definition 1.2 An $(\infty, 0)$–*category*, or ∞–*groupoid*, is a space. According to our conventions, it is a fibrant simplicial set, ie a Kan complex.

1.2 Topologically and simplicially enriched categories

Two particularly simple, but quite rigid models are topologically or simplicially enriched categories.

Definition 1.3 A *topological category* is a category enriched in topological spaces. A *simplicial category* is a category enriched in simplicial sets.

Topological and simplicial categories are discussed and used in [38; 55]. However, for our applications they turn out to be too rigid. We would also like to allow some flexibility for objects, not only morphisms, thus also requiring spaces of objects.

1.3 Segal spaces

Complete Segal spaces, first introduced by Rezk [45] as a model for $(\infty, 1)$–categories, turn out to be very well suited for geometric applications. We recall the definition in this section.

Definition 1.4 A (1–*fold*) *Segal space* is a simplicial space $X = X_\bullet$ which satisfies the *Segal condition*: for any $n, m \geq 0$, the commuting square

$$\begin{array}{ccc} X_{m+n} & \longrightarrow & X_m \\ \downarrow & & \downarrow \\ X_n & \longrightarrow & X_0 \end{array}$$

induced by the maps $[m] \to [m+n]$, $(0 < \cdots < m) \mapsto (0 < \cdots < m)$, and $[n] \to [m+n]$, $(0 < \cdots < n) \mapsto (m < \cdots < m+n)$, is a homotopy pullback square. In other words, the induced map

$$X_{m+n} \to X_m \times^h_{X_0} X_n$$

is a weak equivalence.

Defining a *map of Segal spaces* to be a map of the underlying simplicial spaces gives a category of Segal spaces $SeSp = SeSp_1$.

Remark 1.5 For any $m \geq 1$, consider the maps $g_\beta \colon [1] \to [m]$, $(0 < 1) \mapsto (\beta - 1 < \beta)$, for $1 \leq \beta \leq m$. Requiring the Segal condition is equivalent to requiring the condition that the maps

$$X_m \to X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1$$

induced by $g_1, \ldots, g_m$ are weak equivalences.

Remark 1.6 Following [40], we omit the Reedy fibrancy condition which often appears in the literature. In particular, this condition would guarantee that for $m, n \geq 0$ the canonical map

$$X_m \times_{X_0} X_n \to X_m \times^h_{X_0} X_n$$

is a weak equivalence. Our definition corresponds to the choice of the projective model structure instead of the injective (Reedy) model structure, which is slightly different (though Quillen equivalent) compared to [45]. We will explain this in more detail in Section 1.4.3.

Definition 1.7 We will refer to the spaces X_n as the *levels* of the Segal space X.

Example 1.8 Let $\mathcal{C}$ be a small topological category. Recall that its nerve is the simplicial set

$$N(\mathcal{C})_n = \mathrm{Hom}([n], \mathcal{C}) = \bigsqcup_{x_0,\ldots,x_n \in \mathrm{Ob}\,\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(x_0, x_1) \times \cdots \times \mathrm{Hom}_{\mathcal{C}}(x_{n-1}, x_n),$$

with face maps given by composition of morphisms, and degeneracies by insertions of identities. The nerve $N(\mathcal{C})$ is a Segal space. Moreover, a simplicial set, viewed as a simplicial space with discrete levels, satisfies the Segal condition if and only if it is the nerve of an (ordinary) category.

1.3.1 Segal spaces as $(\infty, 1)$–categories The above example motivates the following interpretation of Segal spaces as models for $(\infty, 1)$–categories. If $X_\bullet$ is a Segal space then we view the set of 0–simplices of the space X_0 as the set of objects. For $x, y \in X_0$ we view

$$\mathrm{Hom}_X(x, y) = \{x\} \times^h_{X_0} X_1 \times^h_{X_0} \{y\}$$

as the $(\infty, 0)$–category, ie the space, of arrows from x to y. More generally, we view X_n as the $(\infty, 0)$–category — ie the space — of n–tuples of composable arrows together with a composition. Note that given an n–tuple of composable arrows, the Segal condition implies that the corresponding fiber of the Segal map $X_n \to X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1$ is a contractible space. The map $X_n \to X_1$ determined by the functor $[1] \to [n]$, $0 < 1 \mapsto 0 < n$, can be thought of as "composition", and thus we can think of the n–tuple as having a contractible space of possible compositions. Moreover, one can interpret paths in the space X_1 of 1–morphisms as 2–morphisms, which are invertible up to homotopies, which in turn are 3–morphisms, and so forth.

1.3.2 The homotopy category of a Segal space To a higher category one can intuitively associate an ordinary category, its *homotopy category*, which has the same objects and whose morphisms are 2–isomorphism classes of 1–morphisms. For Segal spaces, one can realize this idea as follows.

Definition 1.9 The *homotopy category* $h_1(X)$ of a Segal space $X = X_\bullet$ is the (ordinary) category whose objects are the 0–simplices of the space X_0 and whose morphisms between objects $x, y \in X_0$ are

$$\mathrm{Hom}_{h_1(X)}(x, y) = \pi_0(\mathrm{Hom}_X(x, y)) = \pi_0(\{x\} \times^h_{X_0} X_1 \times^h_{X_0} \{y\}).$$

For $x, y, z \in X_0$, the following diagram induces the composition of morphisms, as weak equivalences induce bijections on π_0:

$$(\{x\} \times^h_{X_0} X_1 \times^h_{X_0} \{y\}) \times (\{y\} \times^h_{X_0} X_1 \times^h_{X_0} \{z\})$$
$$\to \{x\} \times^h_{X_0} X_1 \times^h_{X_0} X_1 \times^h_{X_0} \{z\} \xleftarrow{\simeq} \{x\} \times^h_{X_0} X_2 \times^h_{X_0} \{z\} \to \{x\} \times^h_{X_0} X_1 \times^h_{X_0} \{z\}.$$

Example 1.10 Given a small (ordinary) category $\mathcal{C}$, the homotopy category of its nerve, viewed as a simplicial space with discrete levels, is equivalent to $\mathcal{C}$:

$$h_1(N(\mathcal{C})) \simeq \mathcal{C}.$$

The above example motivates the following definition of equivalences of Segal spaces.

Definition 1.11 A map $f \colon X \to Y$ of Segal spaces is a *Dwyer–Kan equivalence* if

(1) the induced map $h_1(f) \colon h_1(X) \to h_1(Y)$ on homotopy categories is essentially surjective, and

(2) for each pair of objects $x, y \in X_0$ the induced map

$$\mathrm{Hom}_X(x, y) \to \mathrm{Hom}_Y(f(x), f(y))$$

is a weak equivalence.

1.4 Complete Segal spaces

We would like the equivalences of Segal spaces to be the Dwyer–Kan equivalences. However, instead of considering all Segal spaces and their the Dwyer–Kan equivalences, it turns out that we can consider a full subcategory of Segal spaces which satisfy an extra condition, called *completeness*, for which Dwyer–Kan equivalences have an equivalent, simpler description. To make sense of this, we need to first introduce the model categories involved.

1.4.1 The model structures of Segal spaces We now describe various model structures on the category *sSpace* of simplicial spaces. Ultimately, the goal is to have a

model category whose fibrant objects deserve to be called "$(\infty, 1)$–categories" and whose equivalences are analogs of equivalences of categories. We will first introduce model categories whose fibrant objects are Segal spaces. Then, in the next step, we will fix the weak equivalences. We refer to [45; 28] for more details.

Let us first consider the injective and projective model structures on the category of simplicial spaces, denoted by $sSpace_c$ and $sSpace_f$, respectively. Note that the fibrant objects in $sSpace_f$ are the levelwise fibrant ones, while the fibrant objects of $sSpace_c$ turn out to be the *Reedy fibrant* simplicial spaces.[1] Conversely, every object in $sSpace_c$ is cofibrant; see for example [26, Corollary 15.8.8.]. These model categories are Quillen equivalent (via the identity functor).

In the first step we perform left Bousfield localizations of the previous model structures $sSpace_c$ and $sSpace_f$ with respect to the morphisms

$$\Delta^1 \amalg_{\Delta^0} \cdots \amalg_{\Delta^0} \Delta^1 \to \Delta^n.$$

This provides two model categories, denoted by $sSpace_c^{\mathrm{Se}}$ and $sSpace_f^{\mathrm{Se}}$, which still are Quillen equivalent. For the injective model structure, it is immediate that fibrant objects in $sSpace_c^{\mathrm{Se}}$ satisfy $X_n \xrightarrow{\simeq} X_1 \times_{X_0} \cdots \times_{X_0} X_1$ and thus are Reedy fibrant Segal spaces. For the projective model structure, it follows from [28] that the fibrant objects in $sSpace_f^{\mathrm{Se}}$ satisfy $X_n \xrightarrow{\simeq} X_1 \times_{X_0}^h \cdots \times_{X_0}^h X_1$ and thus are Segal spaces.[2]

1.4.2 Complete Segal spaces   Although the model categories $sSpace_c^{\mathrm{Se}}$ and $sSpace_f^{\mathrm{Se}}$ have the (Reedy fibrant) Segal spaces as their fibrant objects, there are not enough weak equivalences: every weak equivalence between Segal spaces is indeed a Dwyer–Kan equivalence, but there are more Dwyer–Kan equivalences.

This problem can be circumvented by further localizing the model structures. For this new model structure, the weak equivalences between Segal spaces turn out to be exactly the Dwyer–Kan equivalences. We will see that these further localized model structures have fewer fibrant objects, which are the *complete* (Reedy fibrant) Segal spaces. We will focus on the case of the projective model structure, since the other case can be found spelled out in great detail in many references, for example the original [45], but

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[1]See for example [26, Theorem 15.8.7] for a proof that the injective and Reedy model structures coincide.

[2]This terminology is not consistent throughout the literature: often "Segal space" includes the Reedy fibrancy condition. Our examples will not be Reedy fibrant, which is the reason for our choice of terminology.

to our knowledge the former has so far only appeared in [28]. Moreover, although we will phrase it for the projective model structure, the first part works the same in the injective case. The difference appears when computing the involved mapping spaces explicitly; see the remark below.

Intuitively, the condition we would like to impose is that the underlying ∞–groupoid of invertible morphisms of the Segal space $X_\bullet$ is already encoded by the space X_0. To translate this, we first need to understand what the space of (homotopy) invertible morphisms of $X_\bullet$ is.

Let f be an element in X_1 with source and target x and y, ie its images under the two face maps $X_1 \rightrightarrows X_0$ are x and y. It is called *invertible* if its image under

$$\{x\}\times_{X_0}X_1\times_{X_0}\{y\}\to\{x\}\times^h_{X_0}X_1\times^h_{X_0}\{y\}\to\pi_0(\{x\}\times^h_{X_0}X_1\times^h_{X_0}\{y\})=\mathrm{Hom}_{h_1(X)}(x,y)$$

is an invertible morphism in $h_1(X)$, ie it has a left and right inverse.

To define the space of invertible morphisms, consider the walking isomorphism $I[1]$, which is the category with two objects and one invertible morphism between them,

$$\bullet \overset{\cong}{\longleftrightarrow} \bullet$$

Mapping the walking isomorphism into an arbitrary category $\mathcal{C}$ we get the isomorphisms of $\mathcal{C}$, and therefore the information about its underlying groupoid. Mimicking this procedure for a Segal space $X_\bullet$, we consider the derived mapping space

$$\mathrm{Map}_{sSpace^{Se}_f}(N(I[1]), X).$$

Moreover, an analog of [45, Lemma 5.8] shows that if an element in X_1 is invertible, any element in the same connected component will also be invertible. Thus we define the *space of invertible morphisms in $X_\bullet$* to be the homotopy pullback[3]

$$
\begin{array}{ccc}
X_1^{\mathrm{inv}} & \longrightarrow & X_1 \\
\downarrow {\scriptstyle h_\lrcorner} & & \downarrow \\
\pi_0\mathrm{Map}_{sSpace^{Se}_f}(N(I[1]), X) & \longrightarrow & \pi_0 X_1 = \pi_0\mathrm{Map}_{sSpace^{Se}_f}(\Delta^1, X)
\end{array}
$$

Here, the bottom arrow arises from the obvious functor $[1] \to I[1]$.

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[3]To compare with the definition in [28], note that the pullback is a homotopy pullback since the map $X_1 \to \pi_0(X_1)$ is a fibration.

Finally, identity morphisms in $X_\bullet$ should be invertible. Indeed, the degeneracy map $s_0\colon [1] \to [0]$ factors as $[1] \to I[1] \to [0]$ and induces a map

$$X_0 \to X_1^{\mathrm{inv}}.$$

Definition 1.12 A Segal space $X_\bullet$ is *complete* if the map $X_0 \to X_1^{\mathrm{inv}}$ is a weak equivalence. We denote the full subcategory of $\mathcal{SeSp}$ whose objects are complete Segal spaces by $\mathcal{CSSp} = \mathcal{CSSp}_1$.

Example 1.13 Let $\mathcal{C}$ be a category. Then $N(\mathcal{C})$ is a complete Segal space if and only if there are no nonidentity isomorphisms in $\mathcal{C}$, ie the underlying groupoid of $\mathcal{C}$ is a set (viewed as a category with only identity morphisms).

In order to compute X_1^{inv} explicitly, we have to be able to describe the (derived) mapping space $\mathrm{Map}_{s\mathcal{S}pace_f^{\mathrm{Se}}}(N(I[1]), X)$.

Lemma 1.14 *We have a homotopy pullback square*

$$
\begin{array}{ccc}
\mathrm{Map}_{s\mathcal{S}pace_f^{\mathrm{Se}}}(N(I[1]), X) & \longrightarrow & X_3 \\
{\scriptstyle \{0,2\}\amalg\{1,3\}}\Big\downarrow & {\scriptstyle h\lrcorner} & \Big\downarrow{\scriptstyle \{0,2\}\amalg\{1,3\}} \\
X_0 \times X_0 & \longrightarrow & X_1 \times X_1
\end{array}
$$

Proof Note that since $X_\bullet$ was assumed to be a Segal space, it is fibrant, but $N(I[1])$ might not be cofibrant.[4] So, to compute the desired mapping space, we cofibrantly replace $N(I[1])$ and then compute the mapping space in the underlying category,

$$\mathrm{Map}_{s\mathcal{S}pace_f^{\mathrm{Se}}}(N(I[1]), X) \simeq \mathrm{Map}_{s\mathcal{S}pace}\big(\mathrm{cof}(N(I[1])), X\big).$$

To compute the cofibrant replacement, the crucial observation (originally by [45], reformulated by [6]) is that the nerve of $I[1]$ can be obtained by the pushout of simplicial sets

$$K = \Delta^3 \amalg_{\Delta^{\{0,2\}}\amalg\Delta^{\{1,3\}}} (\Delta^0 \amalg \Delta^0).$$

This can be seen as contracting the edges $\{0, 2\}$ and $\{1, 3\}$ in the 3–simplex

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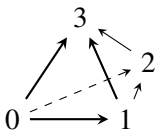
[4]For the injective model structure, it is cofibrant and therefore X_1^{inv} is just the subspace of X_1 of invertible morphisms.

We use an argument similar to that in [29, Remark 3.4], which observes the following: K is given by a strict pushout along a diagram of cofibrant objects of which one arrow is an inclusion. By [38, A.2.4.4], this is a homotopy pushout in the injective model structure and therefore homotopy equivalent to the homotopy pushout in the projective model structure. So a cofibrant replacement of K is given by taking the homotopy pushout of the same diagram,

$$\mathrm{cof}(K) = \Delta^3 \amalg^h_{\Delta^{\{0,2\}} \amalg \Delta^{\{1,3\}}} (\Delta^0 \amalg \Delta^0).$$

Finally, we obtain the space as the wanted homotopy pullback.[5] $\qquad\square$

1.4.3 Complete Segal spaces as fibrant objects There is a further model structure on the category of simplicial spaces which implements completeness. It is obtained by a further left Bousfield localization, with respect to the morphism

$$\Delta^0 \to N(I[1]).$$

This provides two Quillen equivalent model categories, denoted by $sSpace_c^{\mathrm{CSe}}$ and $sSpace_f^{\mathrm{CSe}}$. Fibrant objects in $sSpace_c^{\mathrm{CSe}}$ and $sSpace_f^{\mathrm{CSe}}$ are Reedy fibrant complete Segal spaces and complete Segal spaces, respectively.

Summarizing, we have the diagram

$$
\begin{array}{ccc}
sSpace_c & \longleftrightarrow & sSpace_f \\
\uparrow & & \uparrow \\
sSpace_c^{\mathrm{Se}} & \longleftrightarrow & sSpace_f^{\mathrm{Se}} \\
\uparrow & & \uparrow \\
sSpace_c^{\mathrm{CSe}} & \longleftrightarrow & sSpace_f^{\mathrm{CSe}}
\end{array}
$$

where the horizontal arrows are Quillen equivalences induced by the identity and the vertical arrows are localizations.

The following theorem shows that in the localized model structure Dwyer–Kan equivalences of Segal spaces indeed are weak equivalence, and therefore we have fixed the concern mentioned above. We refer to [28, Theorem 5.15] for a proof, which makes substantial use of the analogous result for Reedy fibrant Segal spaces in $sSpace_c^{\mathrm{CSe}}$ from [45, Theorem 7.7].

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[5]This can be compared to Rezk's definition using the zigzag category $0 \to 2 \leftarrow 1 \to 3$ and requiring the morphisms $0 \to 2$ and $1 \to 3$ to be identities.

Theorem 1.15 *Let X and Y be Segal spaces. A morphism $f \colon X \to Y$ is a weak equivalence in $sSpace_f^{\text{CSe}}$ if and only if it is a Dwyer–Kan equivalence.*

As a consequence the obvious inclusions induce the equivalences of categories

$$\mathcal{CSSp}[\textit{lwe}^{-1}] \to \mathcal{SeSp}[\mathcal{DK}^{-1}] \to \text{Ho}(sSpace_f^{\text{CSe}}),$$

where $\mathcal{DK}$ and $\textit{lwe}$ stand for the subcategories of Dwyer–Kan and levelwise weak equivalences, respectively.

This justifies the following definition.

Definition 1.16 An $(\infty, 1)$–*category* is a complete Segal space.

Remark 1.17 We denote the category of Reedy fibrant complete Segal spaces by $\mathcal{CSSp}_c$, that is to say the fibrant objects in $sSpace_c^{\text{CSe}}$. Remember that $sSpace_c^{\text{CSe}}$ and $sSpace_f^{\text{CSe}}$ are Quillen equivalent, so that the embedding $\mathcal{CSSp}_c \subset \mathcal{CSSp}$ induces an equivalence $\mathcal{CSSp}_c[\textit{lwe}^{-1}] \to \mathcal{CSSp}[\textit{lwe}^{-1}]$, of which an inverse is given by the Reedy fibrant replacement functor $(-)^R$. Sometimes it turns out to be more useful to work in the model category $sSpace_c^{\text{CSe}}$ as every object is cofibrant. Note that the Reedy fibrant replacement functor does not change the homotopy type of the levels.

Definition 1.18 The fibrant replacement functor in the model category $sSpace_f^{\text{CSe}}$ sending a Segal space to its fibrant replacement is called *completion*. In [45], Rezk gave a rather explicit construction of the completion of Segal spaces. He showed that there is a completion functor which to every Segal space X associates a complete Segal space $\widehat{X}$ together with a map $i_X \colon X \to \widehat{X}$, which is a Dwyer–Kan equivalence.

Remark 1.19 The completeness condition says that all invertible morphisms essentially are just identities up to the choice of a path. In this sense, one might like to think of complete Segal spaces as a homotopical version of skeletal[6] or *reduced* (see also [31]) category, and, since any category is equivalent to a reduced one, assuming this extra condition is harmless. However, the information on the invertible morphisms is merely encoded in a different way, namely in the spatial structure. Also, in the homotopical situation, this intuition might be misleading: indeed, instead of thinking of a complete Segal space as having few invertible morphisms, it is better to think of a complete Segal space as having a "maximal" space of objects. This is illustrated by [45, Corollary 6.6].

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[6]A category is called *skeletal* if each isomorphism class contains just one element; see for example [47].

A good example to keep in mind is a special case of [45, Remark 14.1]: given a group G, we can view as a category with one object, and consider its nerve. Its completion is the constant simplicial space BG.

Remark 1.20   The categories $sSpace_f$, $sSpace_c$, $sSpace_f^{Se}$, $sSpace_c^{Se}$, $sSpace_f^{CSe}$ and $sSpace_c^{CSe}$ are all Cartesian closed simplicial model categories. In particular, for any simplicial space X and any complete Segal space Y, the simplicial space Y^X is a complete Segal space.

1.4.4   The classification diagram: the Rezk or relative nerve   Many examples of (complete) Segal spaces arise by a construction, due to Rezk [45], which produces a (complete) Segal space from a simplicial model category. More generally, several authors [4; 37; 36] proved that this construction also gives a complete Segal space for far-reaching generalizations of model categories, namely for relative categories with certain weak conditions. For instance, categories of fibrant objects in the sense of Brown satisfy the conditions to obtain a Segal space; if they additionally are saturated, they lead to complete Segal spaces.

Definition 1.21   A *relative category* is a pair $(\mathcal{C}, \mathcal{W})$ consisting of a category $\mathcal{C}$ and a subcategory $\mathcal{W} \subseteq \mathcal{C}$ containing all objects of $\mathcal{C}$. The morphisms in $\mathcal{W}$ are called *weak equivalences*. A *relative functor* between two relative categories is a functor which preserves weak equivalences. Together they form a category $\mathcal{RelCat}$.

Definition 1.22   Let $(\mathcal{C}, \mathcal{W})$ be a relative category. Consider the simplicial object in categories $\mathcal{C}_\bullet$ given by $\mathcal{C}_n := \mathrm{Fun}([n], C)$. It has a subobject $\mathcal{C}_\bullet^\mathcal{W}$, where $\mathcal{C}_n^\mathcal{W} \subset \mathcal{C}_n$ is the subcategory which has the same objects and whose morphisms consist only of those composed of those in $\mathcal{W}$. Taking its nerve we obtain a simplicial space $N(\mathcal{C}, \mathcal{W})_\bullet$ with

$$N(\mathcal{C}, \mathcal{W})_n = N(\mathcal{C}_n^\mathcal{W}).$$

It is proved in [37] that this simplicial space satisfies the Segal condition if $(\mathcal{C}, \mathcal{W})$ admits a suitable homotopical three-arrow calculus. Moreover, it is complete if it additionally is saturated, ie a morphism is a weak equivalence if and only if it is an isomorphism in the homotopy category. However, it is not levelwise[7] fibrant unless we started with an ∞–groupoid. Its levelwise fibrant replacement is called the *Rezk or relative nerve* or the *classification diagram*, which, by abuse of notation, we again denote by $N(\mathcal{C}, \mathcal{W})$.

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[7]Strictly speaking we should call this a (complete) Segal simplicial set, since we defined a space to be fibrant.

Example 1.23 Let C be a small category. Then it is straightforward to see that $N(\mathcal{C}, \text{Iso}\,\mathcal{C})$ is a complete Segal space. Note that the natural morphism $N(\mathcal{C}) \to N(\mathcal{C}, \text{Iso}\,\mathcal{C})$ is a Dwyer–Kan equivalence. This exhibits $N(\mathcal{C}, \text{Iso}\,\mathcal{C})$ as a completion of $N(\mathcal{C})$.

Now we can apply this construction to the model category of complete Segal spaces from the previous section:

Definition 1.24 The $(\infty, 1)$–*category of* $(\infty, 1)$–*categories* is $N(\mathcal{CSSp}, \textit{lwe})$.

Remark 1.25 The inclusions of the relative categories of cofibrant-and-fibrant objects and of fibrant objects in a simplicial model category lead to equivalences of the classification diagrams. Thus, the inclusions of relative categories $(\mathcal{CSSp}, \textit{lwe}) \subset (\mathcal{SeSp}, \mathcal{DK}) \subset (sSpace_f^{\text{CSe}}, \mathcal{W})$ induce levelwise equivalences of complete Segal spaces, ie equivalences of $(\infty, 1)$–categories

$$N(\mathcal{CSSp}, \textit{lwe}) \to N(\mathcal{SeSp}, \mathcal{DK}) \to N(sSpace_f^{\text{CSe}}, \mathcal{W}).$$

Remark 1.26 Morphisms in the homotopy category of $(\infty, 1)$–categories are more subtle: they are zigzags $X \to X_1 \xleftarrow{\simeq} X_2 \to \cdots \to Y$, where the wrong-way arrows are weak equivalences and therefore more flexible.

We finally observe that a Quillen equivalence

$$L: \mathcal{M} \leftrightarrows \mathcal{N} : R$$

induces a weak equivalence of complete Segal spaces

$$X := N(\mathcal{M}, \textit{we}_\mathcal{M}) \simeq N(\mathcal{N}, \textit{we}_\mathcal{N}) =: Y$$

between the associated classification diagrams. Indeed, the left derived functor $\mathbb{L}L = L \circ Q$ (Q being a cofibrant replacement functor) preserves weak equivalence and thus induces a morphism $L: X \to Y$, which can be proven to be a Dwyer–Kan equivalence (this essentially follows from [16]).

1.5 Some other models for $(\infty, 1)$–categories

We very briefly recall some other models for $(\infty, 1)$–categories in this section which were the motivation for some definitions later on.

1.5.1 Segal categories Let us mention another model for $(\infty, 1)$–categories given by certain Segal spaces, which avoids having to require completeness by instead requiring

a discrete set of objects. This will be the motivation for the definitions of "hybrid n–fold Segal spaces" in Section 2.3.

Definition 1.27 A Segal (1–)category is a Segal space $X = X_\bullet$ such that X_0 is discrete, ie constant as a simplicial set. We denote by *SeCat* the full subcategory of *SeSp* consisting of Segal categories.

Segal categories also are the fibrant objects in a certain model category that is Quillen equivalent to $sSpace_f^{\mathrm{CSe}}$; see the above-mentioned [8] or [39] for more details and references. In particular, the embedding *SeCat* $\subset$ *SeSp* induces an equivalence of complete Segal spaces

$$N(SeCat, \mathcal{DK}) \to N(SeSp, \mathcal{DK}).$$

1.5.2 Relative categories Following [5], a rather weak notion of $(\infty, 1)$–category is given by relative categories from Definition 1.21. One should think of the weak equivalences $\mathcal{W}$ as being "formally inverted". We have already implicitly used this to define the $(\infty, 1)$–category of $(\infty, 1)$–categories.

Example 1.28 Let $\mathcal{C} = \mathrm{Ch}_R$ be the category of chain complexes over a ring R and let $\mathcal{W} \subseteq \mathcal{C}$ be the subcategory of chain complexes and quasi-isomorphisms.

RelCat admits a model structure exhibiting it as a model for $(\infty, 1)$–categories: in [4] the model structure of $sSpace_c^{\mathrm{CSe}}$ is transferred along a slight modification of the relative nerve,

$$N_\xi \colon RelCat \leftrightarrows sSpace_c^{\mathrm{CSe}} \colon K_\xi,$$

thus making the above adjunction into a Quillen equivalence.

1.5.3 Categories internal to simplicial sets Instead of enriching categories in a category of spaces as in Section 1.2, for certain applications it turns out to be useful to also have a space of objects (thus allowing more flexibility than in topological categories), but keeping strict composition (and thus having more rigidity than in Segal spaces). This philosophy is implemented when considering categories internal to spaces. We will use this model to construct examples of Segal spaces.

Definition 1.29 Let $\mathcal{S}$ be a category with finite limits. A *category internal to $\mathcal{S}$* or, for short, an *internal category in $\mathcal{S}$* consists of objects $\mathcal{C}_0$ and $\mathcal{C}_1$ together with source and target morphisms $s, t \colon \mathcal{C}_1 \rightrightarrows \mathcal{C}_0$, a degeneracy morphism $d \colon \mathcal{C}_0 \to \mathcal{C}_1$ satisfying

$s \circ d = t \circ d = \mathrm{id}_{\mathcal{C}_0}$ and a composition morphism $\circ \colon \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \to \mathcal{C}_1$ satisfying associativity and such that for any $x \in \mathcal{C}_0$, the maps $- \circ d(x)$ and $d(x) \circ -$ are the identity. Let $\mathcal{ICat}(\mathcal{S})$ denote the category of categories internal to $\mathcal{S}$, where morphisms from $(\mathcal{C}_0, \mathcal{C}_1)$ to $(\mathcal{D}_0, \mathcal{D}_1)$ are pairs of morphisms $\mathcal{C}_i \to \mathcal{D}_i$ for $i = 0, 1$ which are compatible with the additional structure in the obvious way.

For short, we call an *internal category* a category internal to $\mathcal{S} = sSet$. In this case, we use the notation $\mathcal{ICat} = \mathcal{ICat}(sSet)$.

Note that there is an equivalence of categories $\mathcal{ICat} \to Cat^{\Delta^{\mathrm{op}}}$. Composition with the levelwise nerve and swapping the simplicial directions gives a functor

$$N \colon \mathcal{ICat} \to Cat^{\Delta^{\mathrm{op}}} \xrightarrow{N(-)} Space^{\Delta^{\mathrm{op}}} \xrightarrow{\mathrm{swap}} sSpace \,.$$

In [28], similarly to $\mathcal{RelCat}$, the model structures of $sSpace_f^{\mathrm{Se}}$ and $sSpace_f^{\mathrm{CSe}}$ are transferred along N to endow $\mathcal{ICat}$ with a model structure, the latter exhibiting it as a model for $(\infty, 1)$–categories. Examples of fibrant objects in the former model category are given by the following strongly Segal internal categories [28, Proposition 5.13]; their nerves are Segal spaces.

Definition 1.30 A *strongly Segal internal category* is a category $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1)$ internal to $\mathcal{S} = Space \subset sSet$ such that the source and target maps $s, t \colon \mathcal{C}_1 \to \mathcal{C}_0$ are fibrations of simplicial sets. We denote by $\mathcal{ICat}^{\mathrm{Se}}$ the category of strongly Segal internal categories.

Remark 1.31 The condition that s and t be fibrations ensures that the pullback along them are homotopy pullbacks, and therefore $N\mathcal{C}$ is a Segal space. This condition is sufficient, but not necessary, for an internal category to be a fibrant object for the transferred model structure. On the other hand, the condition that $\mathcal{C}_0$ and $\mathcal{C}_1$ be Kan complexes is necessary.

Since the model structure was transferred, there is a Quillen equivalence given by the nerve,

$$N \colon \mathcal{ICat} \to sSpace_f^{\mathrm{Se}} \,.$$

Moreover, categorical equivalences of strongly Segal internal categories are (by definition) precisely the morphisms that are sent to Dwyer–Kan equivalences by the nerve. Thus, the induced morphism

$$N(\mathcal{ICat}_{\mathrm{Se}}, cat.eq.) \to N(\mathcal{SeSp}, \mathcal{DK})$$

is an equivalence of complete Segal spaces.

2 Models for (∞, n)–categories

As a model for (∞, n)–categories, we will use complete n–fold Segal spaces, which were first introduced by Barwick in his thesis and appeared prominently in Lurie [40]. Details can be found eg in [39; 6; 10]. As mentioned above, (∞, n)–categories are homotopical versions of weak n–categories. Recall that n–categories are inductively built by taking categories (weakly) enriched in $(n-1)$–categories. For $n = 2$ these are known as 2–categories (strict) or bicategories (weak). Alternatively, one could choose to consider categories *internal* to $(n-1)$–categories, ie they have a whole $(n-1)$–category of objects. For $n = 2$ these were first introduced under the name of double categories by Ehresmann [18] and have been thoroughly studied in category theory. Therefore, we will call the higher versions thereof n–*uple categories*.[8] Even though we present our main example as an n–fold Segal space in the next part, it actually arises from such an "n–uple" version, as we will see later on.

Moreover, it even comes from a more rigid model, namely from internal n–uple categories, which are n–uple categories internal to simplicial sets. This model is the easiest to define, which is why we start with it.

2.1 Internal n–uple categories

Iterating the approach in [28], one obtains a model for (∞, n)–uple categories given by n–uple categories internal to simplicial sets, ie categories internal to the category of $(n-1)$–uple categories internal to simplicial sets. Unraveling the definition for $n = 2$, there is a space of objects, a space of "horizontal" 1–morphisms, a space of "vertical" 1–morphisms and a space of 2–morphisms, together with unit maps and composition maps. For larger n, there is a space of objects and suitable spaces of higher morphisms "in all directions", again together with unit maps and composition maps. Equivalently, an n–uple category internal to simplicial sets is a simplicial object in (strict) n–fold categories. This model has been discussed in [15].

Our bordism category defined in the next part secretly is such an internal n–uple category, however, details on this model were not available at the time of writing this article, so we will present it in a different way here.

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[8]This is nonstandard: usually they are called n–fold categories. However, by an unfortunate choice of terminology, complete n–fold Segal spaces will correspond to n–categories. In order to hopefully reduce confusion we will instead be consistent in using "uple" for internal versions and reserve "fold" for the enriched, globular version.

Remark 2.1 Composition is well defined on the nose, as opposed to the models we will consider in the next sections.

2.2 *n*–uple and *n*–fold Segal spaces

Recall that an *n*–uple[9] simplicial space is a functor $X\colon (\Delta^{\mathrm{op}})^{\times n} \to \mathit{Space}$. An *n*–uple Segal space is an *n*–uple simplicial space with an extra condition ensuring it is the ∞–analog of an *n*–uple category.

Definition 2.2 An *n*–*uple Segal space* is an *n*–uple simplicial space $X = X_{\bullet,\ldots,\bullet}$ such that for every $1 \leq i \leq n$ and every $k_1,\ldots,k_{i-1},k_{i+1},\ldots,k_n \geq 0$,

$$X_{k_1,\ldots,k_{i-1},\bullet,k_{i+1},\ldots,k_n}$$

is a Segal space.

Defining a *map of n–uple Segal spaces* to be a map of the underlying *n*–uple simplicial spaces gives a category of *n*–uple Segal spaces, SeSp^n.

Imposing an extra globularity condition leads to a model for ∞–analogs of *n*–categories:

Definition 2.3 An *n*–uple simplicial space $X_{\bullet,\ldots,\bullet}$ is *essentially constant* if the map from the constant *n*–uple simplicial space $X_{0,\ldots,0}$ given by the degeneracy maps

$$X_{0,\ldots,0} \to X$$

is a weak equivalence of *n*–uple simplicial spaces.

Definition 2.4 An *n*–*fold Segal space* is an *n*–uple Segal space $X = X_{\bullet,\ldots,\bullet}$ such that for every $1 \leq i \leq n$ and every $k_1,\ldots,k_{i-1} \geq 0$, the $(n-i)$–uple simplicial space

$$X_{k_1,\ldots,k_{i-1},0,\bullet,\ldots,\bullet}$$

is essentially constant.[10]

Defining a *map of n–fold Segal spaces* to be a map of the underlying *n*–uple simplicial spaces gives a category of *n*–fold Segal spaces, SeSp_n.

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[9]Again, usually this is called an *n*–fold simplicial space, but we use this terminology to emphasize the difference.

[10]To be consistent with our choice of "uple" versus "fold", we could call an *n*–uple simplicial space which satisfies this extra condition an *n*–fold simplicial space.

Remark 2.5 Alternatively, one can formulate the conditions iteratively. First, an $n-$ *uple Segal space* is a simplicial object $Y_\bullet$ in $(n-1)$–uple Segal spaces which satisfies the Segal condition. Then, an n–fold Segal space is a simplicial object $Y_\bullet$ in $(n-1)$–fold Segal spaces which satisfies the Segal condition and such that Y_0 is essentially constant (as an $(n-1)$–fold Segal space). To get back the above definition, the ordering of the indices is crucial: $X_{k_1,\ldots,k_n} = (Y_{k_1})_{k_2,\ldots,k_n}$.

2.2.1 Interpretation as higher categories An n–fold Segal space can be thought of as a higher category in the following way.

The first condition means that this is an n–uple category, ie there are n different "directions" in which we can "compose". An element of $X_{k_1,\ldots,k_n}$ should be thought of as a composition consisting of k_i composable morphisms in the ith direction.

The second condition imposes that we indeed have a higher n–category, ie an $n-$ morphism has as source and target two $(n-1)$–morphisms which themselves have the "same" (in the sense that they are homotopic) source and target.

For $n = 2$ one can think of this second condition as "fattening" the objects in a bicategory. A 2–morphism in a bicategory can be depicted as
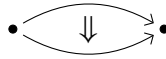

The top and bottom arrows are the source and target, which are 1–morphisms between the same objects.

In a 2–fold Segal space $X_{\bullet,\bullet}$, an element in $X_{1,1}$ can be depicted as
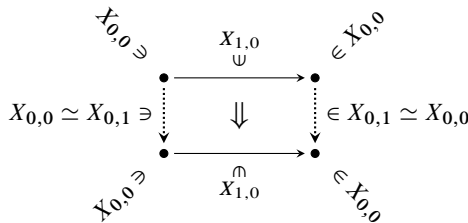
The images under the source and target maps in the first direction $X_{1,1} \rightrightarrows X_{1,0}$ are 1–morphisms which are depicted by the horizontal arrows. The images under the source and target maps in the second direction $X_{1,1} \rightrightarrows X_{0,1}$ are 1–morphisms, depicted by the dashed vertical arrows, which are essentially just identity maps, up to homotopy, since $X_{0,1} \simeq X_{0,0}$. Thus, here the source and target 1–morphisms (the horizontal ones) themselves do not have the same source and target anymore, but up to homotopy they do.

The same idea works with higher morphisms; in particular, one can imagine the corresponding diagrams for $n = 3$. A 3–morphism in a tricategory can be depicted as
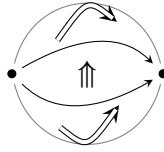
whereas a 3–morphism, ie an element in $X_{1,1,1}$ in a 3–fold Segal space X can be depicted as
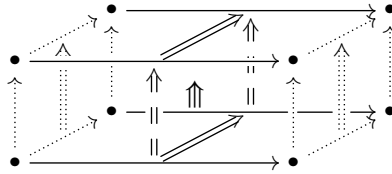
Here the dotted arrows are those in $X_{0,1,1} \simeq X_{0,0,1} \simeq X_{0,0,0}$ and the dashed ones are those in $X_{1,0,1} \simeq X_{1,0,0}$.

Thus, we should think of the set of 0–simplices of the space $X_{0,\dots,0}$ as the objects of our category, and elements of $X_{1,\dots,1,0,\dots,0}$ as i–morphisms, where $0 < i \leq n$ is the number of 1's. Pictorially, they are the ith "horizontal" arrows. Moreover, the other "vertical" arrows are essentially just identities of lower morphisms. Similarly to before, paths in $X_{1,\dots,1}$ should be thought of as $(n+1)$–morphisms, which therefore are invertible up to a homotopy, which itself is an $(n+2)$–morphism, and so forth.

2.2.2 The homotopy bicategory of a 2–fold Segal space
To any higher category one can intuitively associate a bicategory having the same objects and 1–morphisms, and with 2–morphisms being 3–isomorphism classes of the original 2–morphisms.

Definition 2.6 The *homotopy bicategory* $\mathrm{h}_2(X)$ of a 2–fold Segal space $X = X_{\bullet,\bullet}$ is defined as follows: objects are the points of the space $X_{0,0}$ and

$$\mathrm{Hom}_{\mathrm{h}_2(X)}(x, y) = \mathrm{h}_1(\mathrm{Hom}_X(x, y)) = \mathrm{h}_1(\{x\} \times^h_{X_{0,\bullet}} X_{1,\bullet} \times^h_{X_{0,\bullet}} \{y\})$$

as Hom categories. Horizontal composition is defined by

$$(\{x\} \times^h_{X_{0,\bullet}} X_{1,\bullet} \times^h_{X_{0,\bullet}} \{y\}) \times (\{y\} \times^h_{X_{0,\bullet}} X_{1,\bullet} \times^h_{X_{0,\bullet}} \{z\})$$
$$\to \{x\} \times^h_{X_{0,\bullet}} X_{1,\bullet} \times^h_{X_{0,\bullet}} X_{1,\bullet} \times^h_{X_{0,\bullet}} \{z\}$$
$$\xleftarrow{\simeq} \{x\} \times^h_{X_{0,\bullet}} X_{2,\bullet} \times^h_{X_{0,\bullet}} \{z\} \to \{x\} \times^h_{X_{0,\bullet}} X_{1,\bullet} \times^h_{X_{0,\bullet}} \{z\}.$$

The second arrow happens to go in the wrong way but it is a weak equivalence; therefore, after taking h_1, it turns out to be an equivalence of categories, and thus to have an inverse (assuming the axiom of choice).

A proof that this definition indeed gives a bicategory will be the subject of a subsequent article.

2.3 Complete and hybrid n–fold Segal spaces

As with (1–fold) Segal spaces, we need to impose an extra condition to ensure that invertible k–morphisms are paths in the space of $(k-1)$–morphisms. Again, there are several ways to include its information.

Definition 2.7 Let X be an n–fold Segal space and $1 \leq i, j \leq n$. It is said to satisfy
(CSS^i) if for every $k_1, \ldots, k_{i-1} \geq 0$,

$$X_{k_1,\ldots,k_{i-1},\bullet,0,\ldots,0}$$

is a complete Segal space;
(SC^j) if for every $k_1, \ldots, k_{j-1} \geq 0$,

$$X_{k_1,\ldots,k_{j-1},0,\bullet,\ldots,\bullet}$$

is discrete, ie a discrete space viewed as a constant $(n-j+1)$–fold Segal space.

Definition 2.8 An n–fold Segal space is

(1) *complete* if, for every $1 \leq i \leq n$, X satisfies (CSS^i).
(2) a *Segal n–category* if, for every $1 \leq j \leq n$, X satisfies (SC^j).
(3) *m–hybrid* for $m \geq 0$ if condition (CSS^i) is satisfied for $i > m$ and condition (SC^j) is satisfied for $j \leq m$.

Denote the full subcategory of $\mathcal{S}e\mathcal{S}p_n$ of complete n–fold Segal spaces by $\mathcal{C}\mathcal{S}\mathcal{S}p_n$.

Remark 2.9 An n–hybrid n–fold Segal space is a Segal n–category, while an n–fold Segal space is 0–hybrid if and only if it is complete.

For our purposes, the model of complete n–fold Segal spaces is well suited, which leads us to the following definition.

Definition 2.10 An (∞, n)–category is a complete n–fold Segal space.

2.3.1 The underlying model categories Similarly to Section 1.4.3 there are model categories running in the background. We can consider either the injective or projective model structure on the category of n–uple simplicial spaces $sSpace^n$, which we denote by $sSpace^n_c$ and $sSpace^n_f$, respectively. Bousfield localizations at the analogs of the Segal maps give model structures whose fibrant objects are (Reedy fibrant) n–uple Segal spaces; further localizing at maps governing essential constancy, the fibrant objects become (Reedy fibrant) n–fold Segal spaces, and a third localization at a map imposing completeness gives model structures $sSpace^{\mathrm{CSe}}_{n,c}$ and $sSpace^{\mathrm{CSe}}_{n,f}$, respectively, whose fibrant objects are (Reedy fibrant) complete n–fold Segal spaces; see [39; 6; 29, Appendix]. Note that, again, the identity map induces a Quillen equivalence between $sSpace^n_c$ and $sSpace^n_f$ which descends to the localizations.

Alternatively — and by [29, Appendix, Proposition A.9] equivalently — the construction of complete Segal objects for absolute distributors from [39] provides an iterative definition of these model categories by considering simplicial objects in a suitable model category (which is taken to be the appropriate localization of $sSpace_{n-1,c}$ or $sSpace_{n-1,f}$, respectively) and localizing at the maps governing the Segal condition, essential constancy and/or completeness in the new simplicial direction.

Lurie [39] also provides a model category whose fibrant objects are Segal category objects in some suitable underlying model category, thus allowing an iteration of the construction of Segal categories as well. Applying this construction m times to the above one for complete $(n-m)$–fold Segal spaces provides a model category whose fibrant objects are m–hybrid n–fold Segal spaces.

One can show (see eg [3; 39; 10; 11; 57]) that equivalences between (possibly noncomplete) n–fold Segal spaces for this model structure are exactly the *Dwyer–Kan equivalences*, which are defined inductively. For this we need the following inductive definition of the homotopy category of an n–fold Segal space:

Definition 2.11 The *homotopy category* $h_1(X)$ of an n–fold Segal space $X_{\bullet,\ldots,\bullet}$ is the following category: Its objects are the 0–simplices, ie the points of $X_{0,\ldots,0}$. For x and y two objects, we let

$$\mathrm{Hom}_X(x,y)_{\bullet,\ldots,\bullet} := \{x\} \times^h_{X_{0,\bullet,\ldots,\bullet}} X_{1,\bullet,\ldots,\bullet} \times^h_{X_{0,\bullet,\ldots,\bullet}} \{y\}$$

be the $(n-1)$–fold Segal space of morphisms[11] from x to y. Now let morphisms in $h_1(X)$ from x to y be the set of isomorphism classes of objects in $h_1(\mathrm{Hom}_X(x,y)_{\bullet,\ldots,\bullet})$,

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[11]We will revisit this notion in Section 2.4.4.

which is already defined by induction. Composition is defined using the Segal condition in the first index.

Definition 2.12 A morphism $f\colon X \to Y$ of n–fold Segal spaces is a *Dwyer–Kan equivalence* if

(1) the induced functor $h_1(f)\colon h_1(X) \to h_1(Y)$ is essentially surjective;

(2) for each pair of objects $x, y \in X_{0,\dots,0}$, the induced morphism $\mathrm{Hom}_X(x, y) \to \mathrm{Hom}_Y(f(x), f(y))$ is a Dwyer–Kan equivalence of $(n-1)$–fold Segal spaces.

Again we obtain equivalences of complete Segal spaces

$$N(\mathcal{CSSp}_n, \textit{we}) \to N(\mathcal{SeSp}_n, \mathcal{DK}) \to N(s\mathcal{Space}_n, \mathcal{W}_f^{\mathrm{CSe}}),$$

where $\mathcal{W}_f^{\mathrm{CSe}}$ is the subcategory of weak equivalences in the localization $s\mathcal{Space}_{n,f}^{\mathrm{CSe}}$.

Remark 2.13 $\mathcal{CSSp}_n$ is the subcategory of fibrant objects for a left Bousfield localization of $s\mathcal{Space}_{n,f}$ and weak equivalences of complete n–fold Segal spaces are levelwise weak equivalences. Denoting the category of fibrant objects in $s\mathcal{Space}_{n,c}^{\mathrm{CSe}}$, the Reedy fibrant complete n–fold Segal spaces, by $\mathcal{CSSp}_{n,c}$, the Quillen equivalence between $s\mathcal{Space}_{n,c}$ and $s\mathcal{Space}_{n,f}$ induces an equivalence $N(\mathcal{CSSp}_{n,c}, \textit{we}) \to N(\mathcal{CSSp}_n, \textit{we})$, whose inverse is given by Reedy fibrant replacement $(-)^R$.

Recall from [Remark 2.5](#) that we can think of an n–fold Segal space in an iterative way: we can view an n–fold Segal space as a Segal object in $(n-1)$–fold Segal spaces, which we in turn can think of a Segal object in Segal objects in $(n-2)$–fold Segal spaces, etc. Then condition (CSS^i) above means that the ith iteration is a *complete* Segal space object. For more on this point of view, see [39; 25].

Definition 2.14 Given an n–fold Segal space $X_{\bullet,\dots,\bullet}$, one can apply the completion functor iteratively to obtain a complete n–fold Segal space $\hat{X}_{\bullet,\dots,\bullet}$, its *(n–fold) completion*. This yields a map $X \to \hat{X}$, the *completion map*, which is universal among all maps (in the homotopy category) to complete n–fold Segal spaces. It is a left adjoint to the embedding of $\mathcal{CSSp}_n[\textit{we}^{-1}]$ into $\mathcal{SeSp}_n[\textit{we}^{-1}]$.

If an n–fold Segal space $X_{\bullet,\dots,\bullet}$ satisfies (SC^j) for $j \leq m$, we can apply the completion functor just to the last $n-m$ indices to obtain an m–hybrid n–fold Segal space $\hat{X}_{\bullet,\dots,\bullet}^m$, its *m–hybrid completion*.

2.4 Constructions of n–fold Segal spaces

We describe several intuitive constructions of (∞, n)–categories in terms of (complete) n–fold Segal spaces.

2.4.1 Truncation Given an (∞, n)–category, for $k \leq n$ its (∞, k)–*truncation, or k–truncation,* is the (∞, k)–category obtained by discarding the noninvertible m–morphisms for $k < m \leq n$.

In terms of n–fold Segal spaces, there is a functor $\tau_k \colon \mathcal{S}e\mathcal{S}p_n \to \mathcal{S}e\mathcal{S}p_k$ sending $X = X_{\bullet,\ldots,\bullet}$ to its k–truncation, the k–fold Segal space

$$\tau_k X = X_{\underbrace{\bullet,\ldots,\bullet}_{k \text{ times}},\underbrace{0,\ldots,0}_{n-k \text{ times}}}.$$

For economy, we will frequently use the notation $\mathbf{1}_k = (1, \ldots, 1)$ for k 1's, and similar. If X is m–hybrid then so is $\tau_k X$ by the definition of the conditions (CSS^i) and (SC^j). In particular, if X is complete, then $\tau_k X$ is as well, and, thus, the truncation of an (∞, n)–category is an (∞, k)–category.

Warning Truncation does not behave well with respect to completion, ie the truncation of the completion is not the completion of the truncation. However, we get a map in one direction

$$
\begin{array}{ccc}
\tau_k(X) & \longrightarrow & \tau_k(\hat{X}) \\
\downarrow & \nearrow & \\
\widehat{\tau_k(X)} & &
\end{array}
$$

In general, this map is not an equivalence. So in general one should always complete an n–fold Segal space before truncating it. For example, for $n = 1$ and a noncomplete Segal space X, the truncation $\tau_1(X) = X_0$ is just the zeroth space, but the truncation of the completion will be equivalent to the underlying ∞–groupoid X_1^{inv}. The map in this case is given by the degeneracy map. In the example $X = N(G)$ from Remark 1.19, the former is $N(G)_0 = \{*\}$ and the latter is BG, which are not equivalent in general.

Remark 2.15 As explained above, the k–truncation of an (∞, n)–category X should be the maximal (∞, k)–category contained in X. However, the image of the degeneracy

$$X_{\mathbf{1}_k,0,\ldots,0} \hookrightarrow X_{\mathbf{1}_m,0,\ldots,0}$$

consists exactly of the invertible m–morphisms for $k < m \leq n$ if and only if X satisfies (CSS^i) for $k < i \leq n$. For example, if $X = X_\bullet$ is a (1–fold) Segal space then X_0 is the underlying ∞–groupoid of invertible morphisms if and only if X is complete.

2.4.2 Extension Any (∞, n)–category can be viewed as an $(\infty, n+1)$–category with only identities as $(n+1)$–morphisms.

In terms of n–fold Segal spaces, any n–fold Segal space can be viewed as a constant simplicial object in n–fold Segal spaces, ie an $(n+1)$–fold Segal space which is constant in the first index. Explicitly, if $X_{\bullet,...,\bullet}$ is an n–fold Segal space, then $\varepsilon(X)_{\bullet,...,\bullet}$ is the constant simplicial object in the category of Segal spaces given by X, ie it is the $(n+1)$–fold Segal space such that for every $k \geq 0$,

$$\varepsilon(X)_{\bullet,...,\bullet,k} = X_{\bullet,...,\bullet}$$

and the face and degeneracy maps in the last index are identity maps.

Lemma 2.16 *If X is complete, then $\varepsilon(X)$ is complete.*

Proof Since X is complete, it satisfies (CSS^i) for $i > 1$. For $i = 0$, we have to show that $\varepsilon(X)_{\bullet,0,...,0}$ is complete. This is satisfied because

$$(\varepsilon(X)_{1,0,...,0})^{\mathrm{inv}} = \varepsilon(X)_{1,0,...,0} = X_{0,...,0} = \varepsilon(X)_{0,0,...,0},$$

since morphisms between elements x and y in the homotopy category of $\varepsilon(X)_{\bullet,k_2,...,k_n}$ are just connected components of the space of paths in $X_{k_2,...,k_n}$, and thus are always invertible. $\square$

We call ε the extension functor, which is left adjoint to τ_n. Moreover, the unit $\mathrm{id} \to \tau_1 \circ \varepsilon$ of the adjunction is the identity.

2.4.3 Inverting Given an (∞, n)–category, for $k \leq n$ we obtain an (∞, k)–category by inverting the noninvertible m–morphisms for $k < m \leq n$.

We saw that the extension functor ε had a right adjoint τ_n. It also has a left adjoint η, which formally inverts all $(n+1)$–morphisms. For an n–fold Segal space X, this is given by realizing the last index,

$$(\eta X)_{k_1,...,k_n} = |X_{k_1,...,k_n,\bullet}|.$$

Here geometric realization amounts to taking the diagonal of the bisimplicial set $X_{k_1,\dots,k_n,\bullet}$. Since the diagram

of right adjoints commutes, the diagram of left adjoints commutes as well. Therefore, completion and inverting commute.

2.4.4 The higher category of morphisms and loopings Given two objects x and y in an (∞, n)–category, morphisms from x to y should form an $(\infty, n-1)$–category.

This can be realized for n–fold Segal spaces, which is one of the main advantages of this model for (∞, n)–categories.

Definition 2.17 Let $X = X_{\bullet,\dots,\bullet}$ be an n–fold Segal space. As we have seen above one should think of objects as vertices of the space $X_{0,\dots,0}$. Let $x, y \in X_{0,\dots,0}$. The $(n-1)$–*fold Segal space of morphisms from x to y* is

$$\mathrm{Hom}_X(x, y)_{\bullet,\dots,\bullet} = \{x\} \times^h_{X_{0,\bullet,\dots,\bullet}} X_{1,\bullet,\dots,\bullet} \times^h_{X_{0,\bullet,\dots,\bullet}} \{y\}.$$

Remark 2.18 If X is m–hybrid, then $\mathrm{Hom}_X(x, y)$ is $(m-1)$–hybrid.

Example 2.19 (compatibility with extension) Let X be an $(\infty, 0)$–category, ie a space, viewed as an $(\infty, 1)$–category, ie a constant (complete) Segal space $\varepsilon(X)_\bullet$, where $\varepsilon(X)_k = X$. For any two objects $x, y \in \varepsilon(X)_0 = X$ the $(\infty, 0)$–category, ie the space, of morphisms from x to y is

$$\mathrm{Hom}_{\varepsilon(X)}(x, y) = \{x\} \times^h_{\varepsilon(X)_0} \varepsilon(X)_1 \times^h_{\varepsilon(X)_0} \{y\} = \{x\} \times^h_X \{y\} = \mathrm{Path}_X(x, y),$$

the path space in X, which coincides with what one expects by the interpretation of paths, homotopies, homotopies between homotopies, etc being higher invertible morphisms.

Definition 2.20 Let X be an n–fold Segal space, and $x \in X_0$ an object in X. Then the *looping of X at x* is the $(n-1)$–fold Segal space

$$\Omega_x(X)_{\bullet,\dots,\bullet} = \mathrm{Hom}_X(x, x)_{\bullet,\dots,\bullet} = \{x\} \times^h_{X_{0,\bullet,\dots,\bullet}} X_{1,\bullet,\dots,\bullet} \times^h_{X_{0,\bullet,\dots,\bullet}} \{x\}.$$

In the following, it will often be clear at which element we are looping, eg if there essentially is only one element, or at a unit for a monoidal structure, which we define in the next section. Then we omit the x from the notation and just write

$$\Omega X = \Omega(X) = \Omega_x(X).$$

We can iterate this procedure as follows.

Definition 2.21 Let $\Omega_x^0(X) = X$. For $1 \leq k \leq n$, let the k*–fold iterated looping* be the $(n-k)$–fold Segal space

$$\Omega_x^k(X) = \Omega_x(\Omega_x^{k-1}(X)),$$

where we view x as a trivial k–morphism via the degeneracy maps, ie an element in $\Omega_x^{k-1}(X)_{0\ldots,0} \to X_{\mathbf{1}_k,0,\ldots,0}$.

Looping k times commutes with taking the k–hybrid completion up to weak equivalence, since completion is taken index by index:

Let X be a k–hybrid n–fold Segal space. Then for the k–hybrid completion $\hat{X}$, which is the completion in the last $(n-k)$ variables, we have that $\Omega^k(\hat{X}) \xrightarrow{\simeq} \hat{X}_{\mathbf{1}_k,\bullet,\ldots,\bullet}$ is complete, so by the universal property of completion, the horizontal map in the following diagram exists:

$$\widehat{\Omega^k(X)} \dashrightarrow \Omega^k(\hat{X})$$
$$\Omega^k(X) \nearrow \uparrow$$

Lemma 2.22 *Let* X *be a* k*–hybrid* n*–fold Segal space. Then the induced map*

$$\widehat{\Omega^k(X)} \xrightarrow{\simeq} \Omega^k(\hat{X})$$

is a levelwise weak equivalence.

Proof In the diagram

$$\widehat{\Omega^k(X)} \dashrightarrow \Omega^k(\hat{X})$$
$$\Omega^k(X) \nearrow \uparrow$$

we know that the vertical map is a DK-equivalence, since completions are DK-equivalences. Moreover, since X is hybrid, we have that $\Omega^k(X) \xrightarrow{\simeq} X_{\mathbf{1}_k,\bullet,\ldots,\bullet}$ and

$\Omega^k(\hat{X}) \xrightarrow{\simeq} \hat{X}_{\mathbf{1}_k,\bullet,\dots,\bullet}$, and by definition of (hybrid) completion, $X_{\mathbf{1}_k,\bullet,\dots,\bullet} \to \hat{X}_{\mathbf{1}_k,\bullet,\dots,\bullet}$ is just a completion, so it is a DK-equivalence. Thus, in the diagram

$$
\begin{array}{ccc}
\widehat{X_{\mathbf{1}_k,\bullet,\dots,\bullet}} & \dashrightarrow & \hat{X}_{\mathbf{1}_k,\bullet,\dots,\bullet} \\
\uparrow & \nearrow & \\
X_{\mathbf{1}_k,\bullet,\dots,\bullet} & &
\end{array}
$$

by the two-out-of-three property, the horizontal morphism is as well. But since both $\widehat{X_{\mathbf{1}_k,\bullet,\dots,\bullet}}$ and $\hat{X}_{\mathbf{1}_k,\bullet,\dots,\bullet}$ are complete, it is a levelwise equivalence. □

2.4.5 Extracting n–fold from n–uple Segal spaces

We can extract the maximal n–fold Segal space from an n–uple one by the following procedure. Let us recall and introduce some notation for various model structures on the category of n–uple simplicial spaces:

- $sSpace_{n,f}^{(C)\mathrm{Se}}$, where fibrant objects are (complete) n–fold Segal spaces.
- $sSpace_{n,c}^{(C)\mathrm{Se}}$, where fibrant objects are Reedy fibrant (complete) n–fold Segal spaces.
- $sSpace_{\mathrm{Se}}^{n,f}$, where fibrant objects are n–uple Segal spaces.
- $sSpace_{\mathrm{Se}}^{n,c}$, where fibrant objects are Reedy fibrant n–uple Segal spaces.

From now, let $* \in \{c, f\}$. There are (two) Quillen adjunctions

$$
sSpace_{n,*}^{\mathrm{Se}} \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightleftarrows}} sSpace_{\mathrm{Se}}^{n,*}.
$$

Let us denote (in a rather unusual way)

$$
\boldsymbol{L} := \mathbb{R}\mathrm{id} \colon N(sSpace_{n,*}^{\mathrm{Se}}, \mathrm{w.e.}) \to N(sSpace_{\mathrm{Se}}^{n,*}, \mathrm{w.e.}).
$$

Observe that on fibrant objects, $\boldsymbol{L}$ is nothing but the inclusion of (possibly Reedy fibrant) n–fold Segal spaces into (possibly Reedy fibrant) n–uple Segal spaces. After [25, Proposition 4.12], we know it has a right adjoint $\boldsymbol{R}$. For the given (possibly Reedy fibrant) n–uple Segal space X, we wish to compute $\boldsymbol{R}(X)$. By adjunction, we know that

$$
\boldsymbol{R}(X)_{\bullet,\dots,\bullet} \simeq \mathrm{Map}_{sSpace_{n,*}^{\mathrm{Se}}}^h (\Delta^{\vec{\bullet}}, \boldsymbol{R}(X)) \simeq \mathrm{Map}_{sSpace_{\mathrm{Se}}^{n,*}}^h (\boldsymbol{L}(\Delta^{\vec{\bullet}}), X),
$$

where $\Delta^{\vec{k}}$ for $\vec{k} = (k_1, \dots, k_n)$ is the n–fold simplicial set represented by

$$
[k_1] \times \cdots \times [k_n] \in \Delta^{\times n},
$$

and Map^h denotes the derived mapping space.

We will now find an explicit way to compute $R(X)$ by finding cofibrant replacements of $L(\Delta^{\vec{k}})$. We start by recalling certain strict n–categories of the desired shapes, which are all objects in Joyal's category Θ_n [30; 46].

For $\vec{k} = (k_1, \ldots, k_n)$, let $\Theta^{\vec{\bullet}}$ be the *walking $\vec{k}$–tuple of n–morphisms*, which is the strict n–category from [29, Definition 5.1]. We do not want to recall the full definition here, but rather the intuition:

- For $\vec{k} = (1, 0, \ldots, 0)$, the category $\Theta^{\vec{k}} = \bullet \to \bullet$ is the walking 1–morphism.
- For $\vec{k} = (2, 0, \ldots, 0)$, the category $\Theta^{\vec{k}} = \bullet \to \bullet \to \bullet$ is the walking composable pair of 1–morphisms.
- For $\vec{k} = (2, 1, \ldots, 0)$, the strict 2–category

$$\Theta^{\vec{k}} = \;\bullet\!\overset{\Uparrow}{\rightrightarrows}\!\bullet\!\overset{\Uparrow}{\rightrightarrows}\!\bullet$$

  is the walking horizontally composable pair of 2–morphisms.
- For $\vec{k} = (3, 2, \ldots, 0)$, we have the strict 2–category

$$\Theta^{\vec{k}} = \;\bullet\!\overset{\Uparrow}{\underset{\Uparrow}{\rightrightarrows}}\!\bullet\!\overset{\Uparrow}{\underset{\Uparrow}{\rightrightarrows}}\!\bullet\!\overset{\Uparrow}{\underset{\Uparrow}{\rightrightarrows}}\!\bullet$$

- More generally, for $\vec{k} = (k_1, \ldots, k_n)$, the strict n–category $\Theta^{\vec{\bullet}}$ has $k_1 \cdots k_n$ n–morphisms, which are composable following the pattern of a grid of dimension $k_1 \times \cdots \times k_n$.

The elementary building blocks for these categories are $\Theta^{(n)}$, where $(n) = (1_n, 0, \ldots, 0)$. All others are built by gluing these in a grid of dimension $k_1 \times \cdots \times k_n$. In [6], Barwick and Schommer-Pries use the following definition, which can been easily seen to be equivalent to the one in [29] by induction:

Definition 2.23 Let C^1 be the walking 1–morphism, ie the category with two objects and one nonidentity morphism from one object to the other, $C^1 = \{\bullet \to \bullet\}$. The strict n–category $\Theta^{(n)}$ is defined inductively by the pushout square

$$
\begin{array}{ccc}
\{0, 1\} \times \Theta^{(n-1)} & \longrightarrow & C^1 \times \Theta^{(n-1)} \\
\downarrow & & \downarrow \\
\{0, 1\} \times \{*\} & \longrightarrow & \Theta^{(n)}.
\end{array}
$$

This immediately implies the existence of a surjective "collapse" map $c_n \colon C^n \to \Theta^{(n)}$, where $C^n = (C^1)^{\times n}$ is the walking n–morphism as a strict n–uple category.

The n–fold nerve of $\Theta^{\vec{k}}$ is

- levelwise fibrant (because $\Theta^{\vec{k}}$ is discrete);
- a Segal space (because $\Theta^{\vec{k}}$ is a strict n–category);
- complete (because $\Theta^{\vec{k}}$ is reduced).

Let us thus abuse notation and still write $\Theta^{\vec{k}}$ for this (complete) n–fold Segal space. Now we can write the formula for the cofibrant replacement, and therefore the recipe for finding the underlying n–fold Segal space.

Theorem 2.24 *Given an n–uple Segal space X, its maximal underlying n–fold Segal space has levels, for $\vec{k} = (k_1, \ldots, k_n) \in (\Delta^{\mathrm{op}})^n$,*

$$\boldsymbol{R}(X)_{\vec{k}} = \mathrm{Map}^h_{sSpace^{n,*}_{\mathrm{Se}}}(\Theta^{\vec{k}}, X).$$

Since $\Theta^{\bullet}$ is an n–fold cosimplicial object in strict n–categories (see [29]), this defines a (complete) n–fold Segal space.

To prove this theorem, we need to understand what the cofibrant replacement $\boldsymbol{L}(\vec{\Delta}^{\bullet})$ is. The first step is a tool to compute the right-hand expression in the theorem, namely an explicit cofibrant replacement of $\Theta^{\vec{k}}$.

Proposition 2.25 *For $n = 1$, the category $\Theta^{(1)}$, or rather its nerve, is cofibrant in the projective model structure $sSpace^{1,f}_{\mathrm{CSe}}$. For $n > 1$, a cofibrant replacement of $\Theta^{(n)}$ in the projective model structure of n–uple Segal spaces $sSpace^{n,f}_{\mathrm{Se}}$ is given inductively by replacing the pushouts in the definition by homotopy pushouts and $\Theta^{(n-1)}$ by its (inductively already defined) cofibrant replacement.*

Proof Similarly to Section 1.4, we use an argument similar to that in [29, Remark 3.4], which observes the following: $\Theta^{(2)}$ is given by a strict pushout along a diagram of cofibrant objects of which one arrow is an inclusion. By [38, A.2.4.4], this is a homotopy pushout in the injective model structure and therefore homotopy equivalent to the homotopy pushout in the projective model structure. So a cofibrant replacement of $\Theta^{(2)}$ is given by taking the homotopy pushout of the same diagram,

$$
\begin{array}{ccc}
\{0,1\} \times C^1 & \hookrightarrow & C^2 \\
\downarrow & & \downarrow \\
\{0,1\} \times \{*\} & \longrightarrow & \mathrm{cof}(\Theta^{(2)})
\end{array}
$$

$h_{\lrcorner}$

Now we proceed by induction. Assume we have shown the statement for $k < n$ and we have a cofibrant replacement $\text{cof}(\Theta^{(k)})$ given as in the proposition. Then, since the map $\{0, 1\} \hookrightarrow C^1$ is a cofibration in the projective model structure, the map

$$\{0, 1\} \times \text{cof}(\Theta^{(n-1)}) \hookrightarrow C^1 \times \text{cof}(\Theta^{(n-1)})$$

is a cofibration. Moreover, $\{0, 1\} \times \text{cof}(\Theta^{(n-1)})$, $C^1 \times \text{cof}(\Theta^{(n-1)})$, and $\{0, 1\} \times \{*\}$ are all cofibrant, so we can use the above-mentioned [38, A.2.4.4] again to see that the strict pushout, which is weakly equivalent to $\Theta^{(n)}$, is a homotopy pushout, and moreover cofibrant. Summarizing, it is a cofibrant replacement of $\Theta^{(n)}$. □

Remark 2.26 Similarly, we can obtain cofibrant replacements for $\Theta^{\vec{k}}$ as defined in [29] by replacing the pushouts in the definition by homotopy pushouts.

The remaining ingredient in the proof of the theorem is the following lemma.

Lemma 2.27 *The natural map* $\Delta^{\vec{k}} \to \Theta^{\vec{k}}$ *is a weak equivalence in* $sSpace_{n,*}^{\text{Se}}$.

Proof We need to show that for any fibrant object Y in $sSpace_{n,*}^{\text{Se}}$ the induced map $\text{Map}_{sSpace_{n,*}^{\text{Se}}}^h(\Theta^{\vec{k}}, Y) \to \text{Map}_{sSpace_{n,*}^{\text{Se}}}^h(\Delta^{\vec{k}}, Y)$ is a weak equivalence of simplicial sets.

We show the claim for $\vec{k} = (k)$ proceeding by induction using the explicit cofibrant replacement from the previous proposition. For $k = 1$, this is true, since $\Theta^{(1)} = \Delta^{(1)} = \Delta^1$. Assume we have proven the statement for $l < k$. Then

$$\text{Map}^h(\Theta^{(k)}, Y) \xrightarrow{\simeq} \text{Map}^h(C^1 \times \Theta^{(k-1)}, Y) \times_{\text{Map}^h(\{0,1\} \times \Theta^{(k-1)}, Y)}^h \text{Map}^h(\{0, 1\}, Y)$$

$$\simeq \text{Map}^h(C^1 \times \Theta^{(k-1)}, Y) \times_{Y_{0,\bullet,\ldots,\bullet}^{\times 2}}^h Y_{0,\ldots,0}^{\times 2}$$

$$\simeq \text{Map}^h(C^1 \times \Theta^{(k-1)}, Y)$$

$$\simeq \text{Map}^h(\Theta^{(k-1)}, \text{Hom}(C^1, Y)).$$

Here the first equivalence uses that the cofibrant replacement of $\Theta^{(k)}$ is the homotopy pushout as described in the previous proposition, the next equivalence computes the mapping spaces on the right and below the times symbol, the third equivalence uses essential constancy of $Y_{0,\bullet,\ldots,\bullet}$ and the last one uses that n–fold Segal spaces are Cartesian closed.

By the induction hypothesis, the natural map $\Delta^{(k-1)} \to \Theta^{(k-1)}$ induces an equivalence

$$\text{Map}^h(\Theta^{(k-1)}, \text{Hom}(C^1, Y)) \xrightarrow{\simeq} \text{Map}^h(\Delta^{(k-1)}, \text{Hom}(C^1, Y)) \simeq \text{Map}^h(\Delta^{(k)}, Y)$$
$$\simeq Y_{(k)}.$$

A similar argument works for general $\vec{k}$. □

Remark 2.28 The above lemma is equivalent to the observation that the model structure $sSpace^{\mathrm{Se}}_{n,*}$ can be obtained as the left Bousfield localization of $sSpace^{n,*}_{\mathrm{Se}}$ along $\Delta^{\vec{k}} \to \Theta^{\vec{k}}$.

Proof of Theorem 2.24 The following equivalences are compatible with the cosimplicial structure of $\Delta^{\vec{\bullet}}$ and $\Theta^{\vec{\bullet}}$:

$$R(X)_{\vec{k}} \cong \mathrm{Map}^h(\Delta^{\vec{k}}, R(X)) \simeq \mathrm{Map}^h(L(\Delta^{\vec{k}}), X)$$

$$\xleftarrow[\simeq]{\text{Lemma 2.27}} \mathrm{Map}^h(L(\Theta^{\vec{k}}), X) \simeq \mathrm{Map}^h(\Theta^{\vec{k}}, X). \qquad \square$$

3 Symmetric monoidal structures

3.1 Definition via Γ–objects

Following [52; 56], we define a symmetric monoidal n–fold Segal space in analogy to Segal's Γ–spaces from [50]. This is a special case of commutative monoid in an arbitrary $(\infty, 1)$–category as defined in [41].

Definition 3.1 Segal's category Γ is the category whose objects are the finite sets

$$\langle m \rangle = \{0, \ldots, m\}$$

for $m \geq 0$ which are pointed at 0. Morphisms are pointed functions, ie for $k, m \geq 0$, functions

$$f \colon \langle m \rangle \to \langle k \rangle, \quad f(0) = 0.$$

For every $m \geq 0$, there are m canonical morphisms

$$\gamma_\beta \colon \langle m \rangle \to \langle 1 \rangle, \quad j \mapsto \delta_{\beta j},$$

for $1 \leq \beta \leq m$, called the *Segal morphisms*.

Remark 3.2 Γ is a skeleton of the category of finite pointed sets Fin_*. In his original paper [50], Segal defined Γ to be the opposite category of Fin_*. However, in the literature, Γ has often appeared with the above convention.

Recall from Section 2.3.1 that the $(\infty, 1)$–category of (∞, n)–categories is presented by a model category in which the fibrant objects are complete n–fold Segal spaces.

More precisely, the $(\infty, 1)$–category of (∞, n)–categories is defined to be the complete Segal space

$$N(\mathcal{CSSp}_n, \mathit{we}) \simeq N(\mathcal{SeSp}_n, \mathcal{DK}) \simeq N(s\mathcal{Space}_n, \mathcal{W}_f^{\mathrm{CSe}}).$$

We would now like to define a symmetric monoidal (∞, n)–category to be an $(\infty, 1)$–functor from Γ, viewed as an $(\infty, 1)$–category, eg as $N(\Gamma, \mathrm{Iso}\,\Gamma)$, to the $(\infty, 1)$–category of (∞, n)–categories satisfying certain properties.

Using the strictification theorem of Toën and Vezzosi from [54] the $(\infty, 1)$–category of such functors can be computed using the model category $(s\mathcal{Space}_{n,f}^{\mathrm{CSe}})^{\Gamma}$ of Γ–diagrams in $s\mathcal{Space}_{n,f}^{\mathrm{CSe}}$ endowed with the projective model structure,

$$N((s\mathcal{Space}_{n,f}^{\mathrm{CSe}})^{\Gamma}, \mathcal{W}) \xrightarrow{\simeq} \mathrm{Map}^h_{s\mathcal{Space}_{n,f}^{\mathrm{CSe}}}(N(\Gamma, \mathrm{Iso}\,\Gamma), N(s\mathcal{Space}_n, \mathcal{W}_f^{\mathrm{CSe}})).$$

Fibrant objects in the former are strict functors from Γ to $\mathcal{CSSp}_n$. Thus, the following definition suffices.

Definition 3.3 A *symmetric monoidal complete n–fold Segal space* is a (strict) functor from Γ to the (strict) category of complete n–fold Segal spaces $\mathcal{CSSp}_n$,

$$A \colon \Gamma \to \mathcal{CSSp}_n,$$

such that for every $m \geq 0$, the induced map

$$A\left(\prod_{1 \leq \beta \leq m} \gamma_\beta\right) \colon A\langle m \rangle \to (A\langle 1 \rangle)^m$$

is an equivalence of complete n–fold Segal spaces.

The complete n–fold Segal space $X = A\langle 1 \rangle$ is called the complete n–fold Segal space *underlying* A, and by abuse of language we will sometimes call a complete n–fold Segal space X symmetric monoidal if there is a symmetric monoidal complete n–fold Segal space A such that $A\langle 1 \rangle = X$.

Remark 3.4 In particular, for $m = 0$, this implies that $A\langle 0 \rangle$ is levelwise equivalent to a point, viewed as a constant n–fold Segal space, which we will denote by $\mathbb{1}$.

Remark 3.5 We can define symmetric monoidal n–fold Segal spaces in a similar way, by replacing $\mathcal{CSSp}_n$ by $\mathcal{SeSp}_n$.

Definition 3.6 The $(\infty, 1)$–category, ie complete Segal space, of functors from Γ to $\mathcal{CSS}p_n$, which as mentioned above can be computed using the model category of Γ–diagrams in $s\mathcal{S}pace_{n,f}^{\mathrm{CSe}}$, has a full sub-$(\infty, 1)$–category of symmetric monoidal complete n–fold Segal spaces. Similarly to Section 2.3.1, this $(\infty, 1)$–category can be realized as the localization of the projective model structure on $(s\mathcal{S}pace_{n,f}^{\mathrm{CSe}})^\Gamma$ with respect to the Segal morphisms; see [29, Example A.11]. A 1–morphism in this $(\infty, 1)$–category is called a *symmetric monoidal functor of (∞, n)–categories*.

The completion map $X \to \hat{X}$ is a weak equivalence. Moreover, since Dwyer–Kan equivalences are closed under products, completion commutes with finite products of Segal spaces (up to weak equivalence). Therefore we obtain the following lemma.

Lemma 3.7 *If $A\colon \Gamma \to \mathcal{SeSp}_n$ is a symmetric monoidal n–fold Segal space, then*

$$\hat{A}\colon \Gamma \to \mathcal{CSS}p_n, \quad \langle m \rangle \mapsto \widehat{A\langle m \rangle},$$

is a symmetric monoidal complete n–fold Segal space.

Example 3.8 Let $A\colon \Gamma \to \mathcal{SeSp}$ be a symmetric monoidal Segal space. Consider the product of maps $\gamma_1 \times \gamma_2$ and the map $\gamma\colon \langle 2 \rangle \to \langle 1 \rangle$, $1, 2 \mapsto 1$. They induce a span

$$A\langle 1 \rangle \times A\langle 1 \rangle \xleftarrow[\simeq]{A(\gamma_1) \times A(\gamma_2)} A\langle 2 \rangle \xrightarrow{A(\gamma)} A\langle 1 \rangle.$$

Passing to the homotopy category, we obtain a map

$$h_1(A\langle 1 \rangle) \times h_1(A\langle 1 \rangle) \to h_1(A\langle 1 \rangle).$$

Toën and Vezzosi [56] showed that this is a symmetric monoidal structure on the category $h_1(A\langle 1 \rangle)$. Roughly speaking, this uses functoriality of A. Associativity uses the Segal space $A\langle 3 \rangle$; $A\langle 0 \rangle$ corresponds to the unit; and the map $c\colon \langle 2 \rangle \to \langle 2 \rangle$, $1 \mapsto 2$, $2 \mapsto 1$, induces the braiding and commutativity.

Example 3.9 Truncations and extensions of symmetric monoidal (∞, n)-categories are again symmetric monoidal. Let A be a symmetric monoidal n–fold (complete) Segal space. Since τ_k and ε are functorial and preserve weak equivalences and products (since they are right adjoints), the assignments

$$\tau_k(A)\langle m \rangle = \tau_k(A\langle m \rangle), \quad \varepsilon(A)\langle m \rangle = \varepsilon(A\langle m \rangle)$$

can be extended to functors $\tau_k(A)$ and $\varepsilon(A)$, and the images of $A\left(\prod_{1 \leq \beta \leq m} \gamma_\beta\right)$ are again weak equivalences. Thus, they endow the k–truncation and extension with a symmetric monoidal structure.

Example 3.10 Given a symmetric monoidal (possibly complete) n–fold Segal space $A\colon \Gamma \to \mathcal{SeSp}_n$, recall that $A\langle 0 \rangle$ is weakly equivalent to the point $\mathbb{1}$, viewed as a constant n–fold Segal space. For every $m \geq 0$ there is a unique map $\langle 0 \rangle \to \langle m \rangle$, which induces a map $\mathbb{1} \simeq A\langle 0 \rangle \to A\langle m \rangle$ which picks out a distinguished object $\mathbb{1}_{\langle m \rangle} \in A\langle m \rangle$. The looping of A with respect this object is also symmetric monoidal, with

$$\Omega(A)\langle m \rangle = \Omega_{\mathbb{1}_{\langle m \rangle}}(A\langle m \rangle),$$

which extends to a symmetric monoidal structure similarly to in the previous example. Note that since the space of choices for the unit $\mathbb{1}_{\langle m \rangle}$ is contractible, different choices lead to equivalent loopings.

Example 3.11 Important examples come from the classification diagram construction. Let $\mathcal{C}$ be a small symmetric monoidal category and let $\mathcal{W} = \mathrm{Iso}\,\mathcal{C}$. As we saw in Definition 1.22, this gives a complete Segal space $N(\mathcal{C}, \mathcal{W})$. The symmetric monoidal structure of $\mathcal{C}$ endows $N(\mathcal{C}, \mathcal{W})$ with the structure of a symmetric monoidal complete Segal space:

First note that $\mathcal{W}^{\times m} = \mathrm{Iso}(\mathcal{C}^{\times m})$ for every m. On objects, let $A\colon \Gamma \to \mathcal{CSSp}$ be given by $A\langle m \rangle = N(\mathcal{C}^{\times m}, \mathcal{W}^{\times m})_\bullet$. We explain the image of the map $\langle 2 \rangle \to \langle 1 \rangle$, $1, 2 \mapsto 1$, which should be a map $A\langle 2 \rangle \to A\langle 1 \rangle$. The image of an arbitrary map $\langle m \rangle \to \langle l \rangle$ can be defined similarly.

An l–simplex in $A\langle 2 \rangle_0 = N(\mathcal{C} \times \mathcal{C}, \mathcal{W} \times \mathcal{W})_0$ is a pair

$$C_0 \xrightarrow{w_1} \cdots \xrightarrow{w_l} C_l, \quad D_0 \xrightarrow{w_1'} \cdots \xrightarrow{w_l'} D_l,$$

and is sent to

$$C_0 \otimes D_0 \xrightarrow{w_1 \otimes w_1'} \cdots \xrightarrow{w_l \otimes w_l'} C_l \otimes D_l.$$

Observe that $w_i \otimes w_i'$ is again in $\mathcal{W}$. More generally, an l–simplex in

$$A\langle 2 \rangle_k = N(\mathcal{C} \times \mathcal{C}, \mathcal{W} \times \mathcal{W})_k$$

is a pair of diagrams

$$
\begin{array}{ccccccc}
C_{0,0} & \xrightarrow{f_{10}} & C_{1,0} & \xrightarrow{f_{20}} & \cdots & \xrightarrow{f_{k0}} & C_{k,0} \\
\downarrow{\scriptstyle w_{01}} & & \downarrow{\scriptstyle w_{11}} & & & & \downarrow{\scriptstyle w_{k1}} \\
C_{0,1} & \xrightarrow{f_{11}} & C_{1,1} & \xrightarrow{f_{21}} & \cdots & \xrightarrow{f_{k1}} & C_{k,1} \\
\downarrow{\scriptstyle w_{02}} & & \downarrow{\scriptstyle w_{21}} & & & & \downarrow{\scriptstyle w_{k2}} \\
\vdots & & \vdots & & & & \vdots \\
\downarrow{\scriptstyle w_{0l}} & & \downarrow{\scriptstyle w_{1l}} & & & & \downarrow{\scriptstyle w_{kl}} \\
C_{0,l} & \xrightarrow{f_{1l}} & C_{1,l} & \xrightarrow{f_{2l}} & \cdots & \xrightarrow{f_{k,l}} & C_{k,l}
\end{array}
\qquad
\begin{array}{ccccccc}
D_{0,0} & \xrightarrow{g_{10}} & D_{1,0} & \xrightarrow{g_{20}} & \cdots & \xrightarrow{g_{k0}} & D_{k,0} \\
\downarrow{\scriptstyle v_{01}} & & \downarrow{\scriptstyle v_{11}} & & & & \downarrow{\scriptstyle v_{k1}} \\
D_{0,1} & \xrightarrow{g_{11}} & D_{1,1} & \xrightarrow{g_{21}} & \cdots & \xrightarrow{g_{k1}} & D_{k,1} \\
\downarrow{\scriptstyle v_{02}} & & \downarrow{\scriptstyle v_{21}} & & & & \downarrow{\scriptstyle v_{k2}} \\
\vdots & & \vdots & & & & \vdots \\
\downarrow{\scriptstyle v_{0l}} & & \downarrow{\scriptstyle v_{1l}} & & & & \downarrow{\scriptstyle v_{kl}} \\
D_{0,l} & \xrightarrow{g_{1l}} & D_{1,l} & \xrightarrow{g_{2l}} & \cdots & \xrightarrow{g_{k,l}} & D_{k,l}
\end{array}
$$

which is sent to the diagram

$$
\begin{array}{ccccc}
C_{0,0} \otimes D_{0,0} & \xrightarrow{f_{10} \otimes g_{10}} & C_{1,0} \otimes D_{1,0} & \xrightarrow{f_{20} \otimes g_{20}} & \cdots \\
\downarrow{\scriptstyle w_{01} \otimes v_{01}} & & \downarrow{\scriptstyle w_{11} \otimes v_{11}} & & \\
C_{0,1} \otimes D_{0,1} & \xrightarrow{f_{11} \otimes g_{11}} & C_{1,1} \otimes D_{1,1} & \xrightarrow{f_{21} \otimes g_{21}} & \cdots \\
\downarrow & & \downarrow & & \\
\vdots & & \vdots & &
\end{array}
$$

of componentwise tensor products.

Finally, we need to check that $A\big(\prod_{1 \le \beta \le m} \gamma_\beta\big)$ is a weak equivalence. This follows from the fact that

$$
(A\langle m \rangle)_k = N(\mathcal{C}^{\times m}, \mathcal{W}^{\times m})_k = (N(\mathcal{C}, \mathcal{W})_k)^{\times m} = (A\langle 1 \rangle_k)^m.
$$

Remark 3.12 More generally, if we start with a symmetric monoidal relative category $(\mathcal{C}, \mathcal{W})$ (a definition can eg be found in [14]) such that all $N(\mathcal{C}^{\times m}, \mathcal{W}^{\times m})$ are (complete) Segal spaces, then the above construction for $(\mathcal{C}, \mathcal{W})$ yields a symmetric monoidal (complete) Segal space $N(\mathcal{C}, \mathcal{W})$.

3.2   Definition via towers of $(n+i)$–fold Segal spaces

Recall that a monoidal category can be seen as a bicategory with just one object. Similarly, a k–monoidal n–category should be the same as a connected $(k+n)$–category with only one object, one 1–morphism, one 2–morphism, and so on up to one $(k-1)$–morphism.

We will base our definitions for (symmetric) monoidal (∞, n)–categories in this section on this guiding principle, which often goes by the name "delooping hypothesis".

Moreover, in our simplicial setting this principle turns out to be true almost by definition: we will use that associative monoids in a (higher) category $\mathcal{C}$ can be described as simplicial objects in $\mathcal{C}$ satisfying Segal conditions. This motivates the following definition of a $(k-)$monoidal complete n–fold Segal space.

3.2.1 Monoidal n–fold complete Segal spaces To implement the above idea, we first need to explain what "having (essentially) one object" means.

Definition 3.13 A *connected* or 0–*connected* n–fold Segal space X is a pointed object in n–fold Segal spaces, ie a morphism $* \to X$ from the constant n–fold Segal space consisting of a point to X such that the map

$$* \to X_{0,\bullet,\ldots,\bullet}$$

is a weak equivalence of $(n-1)$–fold Segal spaces. In particular, a connected n–fold Segal space has a contractible space of objects.

Definition 3.14 A *monoidal* complete n–fold Segal space is a 1–hybrid $(n+1)$–fold Segal space $X^{(1)}$ which is connected. Note that as $X^{(1)}$ is 1–hybrid, $X^{(1)}_{0,\bullet,\ldots,\bullet}$ is constant with values a discrete space. Thus, to be connected implies that $X^{(1)}_{0,\bullet,\ldots,\bullet}$ is equal to the point viewed as a constant n–fold Segal space; we again denote the unique object by $*$. We say that this endows the complete n–fold Segal space

$$X = \Omega(X^{(1)}) = \Omega_*(X^{(1)})$$

with a monoidal structure and that $X^{(1)}$ is a *delooping* of X.

Remark 3.15 Without the completeness condition, we could define a monoidal n–fold Segal space to be an $(n+1)$–fold Segal space $X^{(1)}$ which is connected. Then $\Omega_*(X^{(1)}) = \mathrm{Hom}_{X^{(1)}}(*, *)$ is independent of the choice of point $* \in X_{0,\ldots,0}$ and we can say that this endows the n–fold Segal space $X = \Omega(X^{(1)}) = \Omega_*(X^{(1)})$ with a monoidal structure. However, for a complete Segal space X, the space $X_{0,\ldots,0}$ will not be contractible (unless it is trivial). Thus, we need a model for $(\infty, n+k)$–categories which can have a point as the set of objects, 1–morphisms, et cetera. This motivates our use of hybrid Segal spaces.

Remark 3.16 Let X be an m–hybrid n–fold Segal space with $m > 0$ which is connected. Then $X_{0,\bullet,\ldots,\bullet} = *$, and therefore the looping just is

$$\Omega(X)_{\bullet,\ldots,\bullet} = \{*\} \times^h_{\{*\}} X_{1,\bullet,\ldots,\bullet} \times^h_{\{*\}} \{*\} \simeq X_{1,\bullet,\ldots,\bullet}.$$

A similar definition works for hybrid Segal spaces.

Definition 3.17 A *monoidal m–hybrid n–fold Segal space* is an $(m+1)$–hybrid $(n+1)$–fold Segal space $X^{(1)}$ which is connected. We say that this endows the m–hybrid n–fold Segal space

$$X = \Omega(X^{(1)})$$

with a monoidal structure and that $X^{(1)}$ is a *delooping* of X.

Remark 3.18 Definitions 3.14 and 3.17 are special cases of the following more general construction of monoids in a model category. Let $\mathcal{M}$ be a left proper cellular model category, and consider the projective model structure on the category $\mathcal{M}^{\Delta^{op}}$ of simplicial objects in $\mathcal{M}$. By the strictification theorem by Toën and Vezzosi [54], the $(\infty, 1)$–category of $(\infty, 1)$–functors between the $(\infty, 1)$–categories represented by Δ^{op} and $\mathcal{M}$ is equivalent to $N(\mathcal{M}^{\Delta^{op}}, \mathit{lwe})$. We say that an object $X_\bullet \in \mathcal{M}^{\Delta^{op}}$ is a *weak monoid* if the Segal maps

$$X_n \to X_1^n$$

are weak equivalences. One can show that the $(\infty, 1)$–category of monoids in the $(\infty, 1)$–category $N(\mathcal{M}, \mathit{we})$ — which is, as usual, obtained by a localization of the model structure on $\mathcal{M}^{\Delta^{op}}$ with respect to the maps governing the Segal morphisms — is equivalent to the relative nerve of the relative category of weak monoids in $\mathcal{M}$ and levelwise weak equivalences. Monoidal m–hybrid n–Segal spaces are exactly the weak monoids in m–hybrid n–Segal spaces.

Example 3.19 Let $\mathcal{C}$ be a small monoidal category and let $\mathcal{W} = \mathrm{Iso}\,\mathcal{C}$. As we saw in Definition 1.22, this gives a complete Segal space $N(\mathcal{C}, \mathcal{W})$. The monoidal structure of $\mathcal{C}$ endows $N(\mathcal{C}, \mathcal{W})$ with the structure of a monoidal complete Segal space:

Recall that $\mathcal{C}_\bullet$ was the simplicial object in categories given by $\mathcal{C}_n := \mathrm{Fun}([n], \mathcal{C})$. Let $\mathcal{C}_{m,n} = \mathcal{C}_n^{\otimes m}$ be the category which has objects of the form

$$C_{01} \otimes \cdots \otimes C_{0m} \xrightarrow{c_1} \cdots \xrightarrow{c_n} C_{n0} \otimes \cdots \otimes C_{nm}$$

for $c_i = c_{i1} \otimes \cdots \otimes c_{im}$ and morphisms of the form

$$
\begin{array}{ccc}
C_{01} \otimes \cdots \otimes C_{0m} \xrightarrow{c_1} \cdots \xrightarrow{c_n} C_{n0} \otimes \cdots \otimes C_{nm} \\
\Big\downarrow{f^0} \qquad\qquad\qquad\qquad \Big\downarrow{f^n} \\
D_{01} \otimes \cdots \otimes D_{0m} \xrightarrow{d_1} \cdots \xrightarrow{d_n} D_{n0} \otimes \cdots \otimes D_{nm}
\end{array}
$$

where $c_1, \ldots, c_n$, $d_1, \ldots, d_n$ and $f^0, \ldots, f^n$ are products of m morphisms in $\mathcal{C}$.

Consider its subcategory $\mathcal{C}^{\mathcal{W}}_{m,n} \subset \mathcal{C}_{m,n}$ which has the same objects, and vertical morphisms involving only the ones in $\mathcal{W} = \mathrm{Iso}\,\mathcal{C}$, ie $f^0, \ldots, f^n$ are products of morphisms in $\mathcal{W}$.

Now let

$$
\mathcal{C}^{(1)}_{m,n} = N(\mathcal{C}^{\mathcal{W}}_{m,n})
$$

be the (ordinary) nerve. By a direct verification one sees that the collection $\mathcal{C}^{(1)}_{\bullet,\bullet}$ is a 2–fold Segal space. Moreover,

(1) $\mathcal{C}^{(1)}_{0,n} = N(\mathcal{C}^{\otimes 0}_n) = *$, so $\mathcal{C}^{(1)}_{0,\bullet}$ is discrete and equal to the point viewed as a constant Segal space, and

(2) for every $m \geq 0$, we get that $\mathcal{C}^{(1)}_{m,\bullet} = N(\mathcal{C}^{\mathcal{W}}_{m,\bullet}) = N((\mathcal{C}^{\otimes m}_\bullet)^{\mathcal{W}})$ is a complete Segal space.

Summarizing, $\mathcal{C}^{(1)}$ is a 1–hybrid 2–fold Segal space which is connected and endows $\Omega(\mathcal{C}^{(1)})_\bullet \simeq \mathcal{C}^{(1)}_{1,\bullet} \simeq N(\mathcal{C}, \mathcal{W})_\bullet$ with the structure of a monoidal complete Segal space.

3.2.2 k–monoidal n–fold complete Segal spaces

To encode braided or symmetric monoidal structures, we can push this definition even further.

Definition 3.20 An n–fold Segal space X is called j–*connected* if

$$
X_{\mathbf{1}_j, 0, \bullet, \ldots, \bullet}
$$

is weakly equivalent to the point viewed as a constant n–fold Segal space.

Remark 3.21 Being j–connected implies being i–connected for every $0 \leq i < j$.

Definition 3.22 A k–*monoidal* m–hybrid n–fold Segal space is an $(m+k)$–hybrid $(n+k)$–fold Segal space $X^{(k)}$ which is $(k-1)$–connected.

Remark 3.23 Since $X^{(k)}$ is $(m+k)$–hybrid, $X^{(k)}_{\mathbf{1}_i,0,\bullet,\dots,\bullet}$ is discrete for every $0 \leq i < k$. Thus, being $(k-1)$–connected implies that $X^{(k)}_{\mathbf{1}_i,0,\bullet,\dots,\bullet}$ is equal to the point viewed as a constant $(n-i+1)$–fold Segal space for every $0 \leq i < k$.

By the following proposition, this definition satisfies the delooping hypothesis. In practice we can use it to define a k–monoidal n–fold complete Segal space step-by-step by defining a tower of monoidal i–hybrid $(n+i)$–fold Segal spaces for $0 \leq i < k$.

Proposition 3.24 *The data of a k–monoidal n–fold complete Segal space is the same as a tower of monoidal i–hybrid $(n+i)$–fold Segal spaces $X^{(i+1)}$ for $0 \leq i < k$ together with weak equivalences*

$$X^{(j)} \simeq \Omega(X^{(j+1)})$$

for every $0 \leq j < k-1$.

Definition 3.25 We say that these equivalent data endow the complete n–fold Segal space

$$X = X^{(0)} \simeq \Omega(X^{(1)})$$

with a k–monoidal structure. The $(n+i+1)$–fold Segal space $X^{(i+1)}$ is called an i–fold delooping of X.

Before proving the proposition, we need the following lemmas.

Lemma 3.26 *If X is a k–monoidal m–hybrid n–fold Segal space and $0 \leq l \leq k$, then X is an l–monoidal $(m+k-l)$–hybrid $(n+k-l)$–fold Segal space.*

Proof Since X is a k–monoidal m–hybrid n–fold Segal space, X is an $(m+k)$–hybrid $(n+k)$–fold Segal space such that

$$X_{\mathbf{1}_{k-1},0,\dots,0} = *.$$

This implies that $X_{\mathbf{1}_{l-1},0,\dots,0} = *$.   □

Lemma 3.27 *Let X be a k–monoidal m–hybrid n–fold Segal space. Then $\Omega(X) = \Omega_*(X)$ is a $(k-1)$–monoidal $(m-1)$–hybrid n–fold Segal space.*

Proof This follows from

$$\Omega(X)_{\bullet,\dots,\bullet} = \mathrm{Hom}_X(*,*)_{\bullet,\dots,\bullet} = \{*\} \times^h_{X_{0,\bullet,\dots,\bullet}} X_{1,\bullet,\dots,\bullet} \times^h_{X_{0,\bullet,\dots,\bullet}} \{*\} \simeq X_{1,\bullet,\dots,\bullet},$$

since $X_{0,\bullet,\dots,\bullet} = \{*\}$.   □

Proof of Proposition 3.24 Let Y be a k–monoidal n–fold complete Segal space. By Lemma 3.26 it is a monoidal $(k-1)$–hybrid $(n+k-1)$–fold Segal space and we define the top layer of our tower to be $X^{(k)} = Y$.

Now let $X^{(k-1)} = \Omega(X^{(k)})$. By Lemmas 3.27 and 3.26, this is a monoidal $(k-2)$–hybrid $(n+k-2)$–fold Segal space.

Inductively, define $X^{(i)} = \Omega(X^{(i+1)})$ for $1 \leq i \leq k-1$. Similarly to above, by Lemmas 3.27 and 3.26, this is a monoidal $(i-1)$–hybrid $(n+i-1)$–fold Segal space.

Conversely, assume we are given a tower $X^{(i)}$ as in the proposition. Since $Y = X^{(k)}$ is a monoidal $(k-1)$–hybrid $(n+k-1)$–fold Segal space,

$$(1) \qquad\qquad Y_{0,\bullet,\ldots,\bullet} = X^{(k)}_{0,\bullet,\ldots,\bullet} = *.$$

Since $X^{(k-1)}$ is a monoidal $(k-2)$–hybrid $(n+k-2)$–fold Segal space and by (1),

$$(2) \qquad Y_{1,0,\bullet,\ldots,\bullet} = X^{(k)}_{1,0,\bullet,\ldots,\bullet} = \{*\} \times^h_{X^{(k)}_{0,0,\bullet,\ldots,\bullet}} X^{(k)}_{1,0,\bullet,\ldots,\bullet} \times^h_{X^{(k)}_{0,0,\bullet,\ldots,\bullet}} \{*\}$$

$$= \Omega(X^{(k)})_{0,\bullet,\ldots,\bullet}$$

$$\simeq X^{(k-1)}_{0,\bullet,\ldots,\bullet} = *.$$

Since $X^{(k)}$ is k–hybrid, $Y_{1,0,\bullet,\ldots,\bullet} = X^{(k)}_{1,0,\bullet,\ldots,\bullet}$ is discrete and so $Y_{1,0,\bullet,\ldots,\bullet} = *$.

Inductively, for $0 \leq i < k$, since $X^{(k-i)}$ is a monoidal $(k-i-1)$–hybrid $(n+k-i-1)$–fold Segal space and by (1), (2), $\ldots$,

$$Y_{\mathbf{1}_i,0,\bullet,\ldots,\bullet} = X^{(k)}_{\mathbf{1}_i,0,\bullet,\ldots,\bullet}$$

$$\simeq \{*\} \times^h_{X^{(k)}_{0,\mathbf{1}_{i-1},0,\bullet,\ldots,\bullet}} X^{(k)}_{\mathbf{1}_i,0,\bullet,\ldots,\bullet} \times^h_{X^{(k)}_{0,\mathbf{1}_{i-1},0,\bullet,\ldots,\bullet}} \{*\}$$

$$= \Omega(X^{(k)})_{\mathbf{1}_{i-1},0,\bullet,\ldots,\bullet}$$

$$\simeq X^{(k-1)}_{\mathbf{1}_{i-1},0,\bullet,\ldots,\bullet} = \cdots \simeq X^{(k-i)}_{0,\bullet,\ldots,\bullet} = *.$$

Again, since $X^{(k)}$ is k–hybrid, we find that $Y_{\mathbf{1}_i,0,\bullet,\ldots,\bullet} = X^{(k)}_{\mathbf{1}_i,0,\bullet,\ldots,\bullet}$ is discrete and so $Y_{\mathbf{1}_i,0,\bullet,\ldots,\bullet} = *$. $\qquad\square$

Given a bicategory $\mathcal{C}$ and an object x in $\mathcal{C}$, the endomorphism, or loop, category $\mathrm{End}_{\mathcal{C}}(x) = \Omega_x(\mathcal{C})$ is monoidal. Its monoidal structure comes from composition of endomorphisms, which is encoded in the full sub-bicategory of $\mathcal{C}$ which has only the object x. In analogy with topology, one can call this delooping $B\Omega_x(\mathcal{C})$. We now prove that a similar statement holds for n–fold Segal spaces. With the definition of

"symmetric monoidal" appearing in the next section, it will become clear that this provides an analog of Example 3.10 in this setting.

Recall from Section 2.4 the constructions of the truncation of an n–fold Segal space to an $(n-1)$–fold Segal space and its left adjoint, extension. Truncation also has a right adjoint, which is taking the 0th coskeleton:

$$(\mathrm{cosk}_0(X))_{k_1,\bullet,\ldots,\bullet} = X^{(k_1+1)}_{\bullet,\ldots,\bullet};$$

face and degeneracy maps are given by partial projections and partial diagonals. Given an n–fold Segal space X and $1 \leq l \leq n$, we can first truncate l times and then take the coskeleton l times to obtain an n–fold Segal space, which we abbreviate by $\mathbf{cosk}_0^l(X)$.

Definition 3.28 Fix $1 \leq l \leq n$. Let $X_{\bullet,\ldots,\bullet}$ be an n–fold Segal space and $x \in X_{0,\ldots,0}$. The object x determines a map $x \colon * \to \mathbf{cosk}_0^l(X)$ using the degeneracy maps. We define a new n–fold Segal space $\mathrm{pre}\,\mathscr{L}_x^l(X)$ as the homotopy pullback

$$
\begin{array}{ccc}
\mathrm{pre}\,\mathscr{L}_x^l(X)_{\bullet,\ldots,\bullet} & \longrightarrow & X_{\bullet,\ldots,\bullet} \\
\downarrow & & \downarrow{\scriptstyle S} \\
* & \xrightarrow{\ \ x\ \ } & \mathbf{cosk}_0^l(X)
\end{array}
$$

For $\mathrm{pre}\,\mathscr{L}_x^l(X)$ we have that, for $1 \leq i \leq l$,

$$\mathrm{pre}\,\mathscr{L}_x^l(X)_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n} \simeq * \cong \{x\}.$$

To obtain an l–hybrid Segal space, we discretize these spaces, ie we define

$$
\mathscr{L}_x^l(X)_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n} = \begin{cases} * \cong \{x\} & \text{if } k_i = 0 \text{ for } 1 \leq i \leq l, \\ \mathrm{pre}\,\mathscr{L}_x^l(X)_{k_1,\ldots,k_n} & \text{otherwise,} \end{cases}
$$

with the obvious modified face and degeneracy maps.

Remark 3.29 Unraveling the coskeleton, for $k_1,\ldots,k_l > 0$ the $(n-l)$–fold Segal space $\mathscr{L}_x^l(X)_{k_1,\ldots,k_l,\bullet,\ldots,\bullet}$ is a homotopy fiber

$$
\begin{array}{ccc}
\mathscr{L}_x^l(X)_{k_1,\ldots,k_l,\bullet,\ldots,\bullet} & \longrightarrow & X_{k_1,\ldots,k_l,\bullet,\ldots,\bullet} \\
\downarrow & & \downarrow{\scriptstyle S} \\
* & \xrightarrow{\ \ x\ \ } & X^{\times(k_1+1)\cdots(k_n+1)}_{0,\ldots,0,\bullet,\ldots,\bullet}
\end{array}
$$

where $S \colon X_{k_1,\ldots,k_l,\bullet,\ldots,\bullet} \to X^{\times(k_1+1)\cdots(k_n+1)}_{0,\ldots,0,\bullet,\ldots,\bullet}$ is the product of all maps arising from the maps $f_i \colon [0] \to [k_i]$. The remaining face maps send everything to the point $*$, which

we identify with x, or, more precisely, its image under the appropriate composition of degeneracy maps. The remaining degeneracy maps $d_{\bullet,\dots,\bullet}\colon \mathscr{L}_x^l(X)_{k_1,\dots,k_l,\bullet,\dots,\bullet} \to \mathscr{L}_x^l(X)_{k_1,\dots,k_i-1,\dots,k_l,\bullet,\dots,\bullet}$ satisfy $d_{\bullet,\dots,\bullet}(*) = d_{\bullet,\dots,\bullet}(x)$, where again we identify x with its image under the appropriate composition of degeneracy maps. Since X is an n–uple simplicial space, $\mathscr{L}_x^l(X)$ is well defined as an n–uple simplicial space. The Segal condition is preserved, and, if X satisfied condition (CSS^i) for some $i > l$, then $\mathscr{L}_x^l(X)$ does too.

Lemma 3.30 *Let $1 \le l \le n$ and let $X_{\bullet,\dots,\bullet}$ be an n–fold Segal space which satisfies* (CSS^i) *for $i > l$. Then, for any $x \in X_{0,\dots,0}$, the n–fold Segal space $\mathscr{L}_x^l(X)$ is an l–monoidal complete $(n-l)$–fold Segal space which endows $\Omega_x^l(X)$ with an l–monoidal structure.*

Proof By construction, $\mathscr{L}_x^l(X)$ is $(l-1)$–connected. Since X satisfies (CSS^i) for $i > l$, it is an l–hybrid n–fold Segal space. Finally, $\Omega_x^l(X) = \Omega_x^l(\mathscr{L}_x^l(X))$ by the following lemma. □

Lemma 3.31 *For an n–fold Segal space X, an object $x \in X_{0,\dots,0}$ and $0 \le l \le n$, we have an equivalence*

$$\Omega_x^l(\mathscr{L}_x^l(X)) \simeq \Omega_x^l(X).$$

Proof This can be checked levelwise: exactly the parts of X which involve x remain in $\mathscr{L}_x^l(X)$, and when looping at x that's the part that is seen. □

This lemma gives a method for finding a k–monoidal structure as a tower.

Proposition 3.32 *Let $Y^{(0)}$ be an n–fold Segal space. Assume we are given, for $1 \le l \le k$, an $(n+l)$–fold Segal space $Y^{(l)}$ together with an object $y_l \in Y^{(l)}$ such that*

$$\Omega_{y_l}(Y^{(l)}) \simeq Y^{(l-1)}.$$

Then $Y^{(0)}$ has a k–monoidal structure. If all $Y^{(l)}$ satisfy (CSS^i) *for $i > l$, then $Y^{(0)}$ is a k–monoidal complete Segal space.*

Proof The monoidal $(n+l-1)$–fold Segal space $\mathscr{L}_{y_l}(Y^{(l)})$ endows $Y^{(l-1)}$ with a monoidal structure. Proposition 3.24 finishes the proof. □

3.2.3 Symmetric monoidal n–fold complete Segal spaces The stabilization hypothesis, first formulated in [2], states that an n–category which is monoidal of a

sufficiently high degree cannot be made "more monoidal" and, moreover, is symmetric monoidal. For Tamasani's weak n–categories, a proof was given by Simpson [51]; for general n–categories a proof follows from Lurie's proof of Dunn's additivity in [41]; see [23] for details.

For (∞, n)–categories, we cannot expect stabilization: for instance, k–monoidal $(\infty, 0)$–categories are (∞, k)–categories with one object, one morphism, etc up to one $(n-1)$–morphism, which, in turn, are E_k–algebras (in *Space*). Note that since there are E_k–algebras which are not E_{k+1}–algebras, there are k–monoidal $(\infty, 0)$–categories which are not $(k+1)$–monoidal. However, this motivates the following definition:

Definition 3.33 A *symmetric monoidal structure* on a complete n–fold Segal space X is a tower of monoidal i–hybrid $(n+i)$–fold Segal spaces $X^{(i+1)}$ for $i > 0$ such that if we set $X^{(0)} = X$, we have that, for every $i \geq 0$,

$$X^{(i)} \simeq \Omega(X^{(i+1)}).$$

3.3 Comparing the two definitions

In this section we show that every symmetric monoidal (∞, n)–category defined as in Section 3.1 gives one as defined in Section 3.2. The converse is also true, but we do not go into the details here. Essentially this is a consequence of Dunn's additivity (see [41]): starting with the definition via a tower of $(\Delta^k)^{\mathrm{op}}$–monoids, one can replace them by E_k–monoids, which in turn, when letting k go to ∞, lead to a commutative monoid. See also [23, Corollary 6.3.13].

We start with a symmetric monoidal (∞, n)–category defined as in Section 3.1, a symmetric monoidal complete n–fold Segal space $X \colon \Gamma \to \mathcal{CSSp}_n$. We will precompose it with the functor

$$f \colon \Delta^{\mathrm{op}} \to \Gamma, \quad [m] \mapsto \langle m \rangle,$$

which sends a map $(f \colon [n] \to [m])$ in Δ to $\widetilde{f} \colon \langle m \rangle \to \langle n \rangle$, where $\widetilde{f}(0) = 0$ and, for $j \neq 0$,

$$\widetilde{f}(j) = \begin{cases} \min\{i : f(i) = j\} & \text{if it exists,} \\ 0 & \text{otherwise.} \end{cases}$$

The composition

$$\widetilde{X}^{(1)} \colon \Delta^{\mathrm{op}} \xrightarrow{f} \Gamma \xrightarrow{X} \mathbf{CSSp}_n$$

is an $(n+1)$–fold simplicial space. Moreover, since f sends the maps g_β from Remark 1.5 to the Segal morphisms γ_β from Definition 3.1, $\widetilde{X}^{(1)}$ is an $(n+1)$–fold

Segal space. It satisfies (CSS^i) for $i > 1$. Moreover, $\widetilde{X}^{(1)}$ is connected. However, it does not satisfy (SC^j) for $j = 1$ since $X^{(1)}_{0,\bullet,\ldots,\bullet}$ may not be discrete. We can easily remedy this problem: choose an object $x \in X^{(1)}_{0,\ldots,0}$ and consider the $(n+1)$–fold Segal space $X^{(1)} = \mathscr{L}_x(\widetilde{X}^{(1)})$. Unraveling the definition, we have that

$$X^{(1)}_{k_1,\bullet,\ldots,\bullet} = \begin{cases} * & \text{if } k_1 = 0, \\ \widetilde{X}^{(1)}_{k_1,\bullet,\ldots,\bullet} & \text{if } k_1 \neq 0, \end{cases}$$

as complete n–fold Segal spaces. Note that choosing different x's leads to equivalent complete n–fold Segal spaces. Lemma 3.30 implies that $X^{(1)}$ is a monoidal complete n–fold Segal space.

The higher layers of the tower are obtained from the maps $\Gamma^k \to \Gamma$ coming from taking the smash product of finite pointed sets, ie taking their product and identifying anything containing a basepoint. Then, composing with f^k we obtain

$$\widetilde{X}^{(k)} \colon (\Delta^{\mathrm{op}})^k \xrightarrow{f^k} \Gamma^k \to \Gamma \to \mathcal{CSSp}_n.$$

Similarly, $\widetilde{X}^{(k)}$ is $(k-1)$–connected, but might not satisfy (SC^j) for $j \leq k$. Choosing any object $x \in X^{(k)}_{0,\ldots,0}$, then $X^{(k)} = \mathscr{L}^k_x(\widetilde{X}^{(k)})$ is the desired k–monoidal complete n–fold Segal space.

Part II   The (∞, n)–category of bordisms

To rigorously define fully extended topological field theories we need a suitable (∞, n)–category of bordisms, which, informally speaking, has zero-dimensional manifolds as objects, bordisms between objects as 1–morphisms, bordisms between bordisms as 2–morphisms, etc, and for $k > n$ there are only invertible k–morphisms. Finding an explicit model for such a higher category, ie defining a complete n–fold Segal space of bordisms, is the main goal of this part and this paper. We endow it with a symmetric monoidal structure and also consider bordism categories with additional structure, eg orientations and framings, which allows us, in Section 10, to rigorously define fully extended topological field theories.

4   The complete n–fold Segal space of closed intervals

In this section we define a complete Segal space Int$_\bullet$ of closed intervals in $\mathbb{R}$ which will form the basis of the n–fold Segal space of bordisms. It will be a tool to record

where (in the time direction) the bordisms can be cut. In particular, there will be a forgetful functor from bordisms to these closed intervals. We start by defining an internal category of closed intervals in $\mathbb{R}$, whose nerve will give a complete Segal space of certain tuples of closed intervals. However, for our model of the bordism category, to avoid having to deal with manifolds with corners, we will instead want to interpret the tuples of intervals as being closed in an open interval of finite length (instead of $\mathbb{R}$). This will be explained in Section 4.3. Finally, we could have chosen that open interval to always be $(0,1)$ and thus fix the "length" in the time direction of the bordism and its collars to be 1. This choice requires rescaling and will be explained in Section 4.5.

4.1 Int^c as an internal category

We first define a category internal to topological spaces $\mathfrak{Int}_c$ which gives rise to a strongly Segal internal category Int^c of closed intervals in $\mathbb{R}$.

The topological space of objects of $\mathfrak{Int}_c$ is

$$(3) \qquad\qquad \mathfrak{Int}_0^c = \{(a,b) : a < b\} \subset \mathbb{R}^2$$

with the standard topology from $\mathbb{R}^2$. We interpret an element $(a,b) \in \mathfrak{Int}_0^c$ as the closed interval $I = [a,b]$. This interpretation gives a bijection from the set of points of the topological space $\mathfrak{Int}_0^c$ to the set of closed bounded intervals

$$\mathfrak{Int}_0^c \longleftrightarrow \{\text{closed bounded intervals } I = [a,b] \text{ in } \mathbb{R} \text{ with nonempty interior}\}$$

which we use as an identification. In fact, $\mathfrak{Int}_0^c$ is a submanifold of $\mathbb{R}^2$ and to get the desired Kan complex Int_0^c, we take smooth singular simplices (see eg [35]), ie for $l \geq 0$, the l–simplices are pairs of smooth maps $a, b \colon |\Delta^l|_e \to \mathbb{R}$ such that $a(s) < b(s)$ for every $s \in |\Delta^l|_e$. Faces and degeneracies are the usual ones. We view such an l–simplex as a *closed interval bundle* and denote it by $[a,b] \to |\Delta^l|_e$ or $(I(s))_{s \in |\Delta^l|_e} = (a(s), b(s))_{s \in |\Delta^l|_e}$.

The topological space of morphisms of $\mathfrak{Int}_c$ is

$$(4) \quad \mathfrak{Int}_1^c = \{(a_0, a_1, b_0, b_1) : a_j < b_j \text{ for } j = 0, 1 \text{ and } a_0 \leq a_1, b_0 \leq b_1\} \subset \mathbb{R}^4,$$

again with the standard topology from $\mathbb{R}^4$. Now we interpret $(a_0, a_1, b_0, b_1) \in \mathfrak{Int}_1^c$ as a pair of ordered closed intervals $I_0 \leq I_1$, where $I_0 = [a_0, b_0]$ and $I_1 = [a_1, b_1]$.

Here "ordered" means that $a_0 \leq a_1$ and $b_0 \leq b_1$. This gives an identification of the points of the topological space with certain pairs of intervals:

$$\mathfrak{Int}_1^c \longleftrightarrow \{I_0 \leq I_1 : I_j = [a_j, b_j] \text{ with } a_j < b_j \text{ for } j = 0, 1 \text{ and } a_0 \leq a_1, b_0 \leq b_1\}.$$

As above, $\mathfrak{Int}_1^c$ has the structure of a submanifold of $\mathbb{R}^4$ and by taking smooth singular simplices we obtain a Kan complex Int_1^c whose l–simplices now are quadruples of smooth maps $a_0, a_1, b_0, b_1 \colon |\Delta^l|_e \to \mathbb{R}$ such that $a_j(s) < b_j(s)$ for $j = 0, 1$, $a_0(s) \leq a_1(s)$ and $b_0(s) \leq b_1(s)$ for every $s \in |\Delta^l|_e$. We view such an l–simplex as a *closed interval bundle with two closed subintervals* and denote it by $([a_0, b_0] \leq [a_1, b_1]) \to |\Delta^l|_e$ or $(I_0(s) \leq I_1(s))_{|\Delta^l|_e}$.

The face and degeneracy maps

$$\mathfrak{Int}_0^c \overset{s}{\underset{t}{\overset{\longleftarrow}{\underset{\longleftarrow}{- d \to}}}} \mathfrak{Int}_1^c$$

arise from forgetting and repeating an interval, respectively:

$$s \colon [a_0, b_0] \leq [a_1, b_1] \longmapsto [a_0, b_0],$$
$$t \colon [a_0, b_0] \leq [a_1, b_1] \longmapsto [a_1, b_1],$$

and

$$d \colon [a, b] \longmapsto [a, b] \leq [a, b].$$

Composition is given by remembering the outer intervals:

$$([a_0, b_0] \leq [a_1, b_1]) \circ ([a_1, b_1] \leq [a_2, b_2]) = ([a_0, b_0] \leq [a_2, b_2]).$$

Here s, t and d are smooth maps, so $\mathfrak{Int}^c$ is a category internal to manifolds. Thus, when taking smooth singular simplices to get Int^c, all the above assignments are well defined for l–simplices as well and commute with the faces and degeneracies. Moreover, s and t are fibrations since they are restrictions of projections.

Remark 4.1 Even though we like to think of the l–simplices in Int_0^c and Int_1^c as "closed interval bundles", we do not treat them as such: face and degeneracy maps are not defined to be pullbacks of the bundles, which would only be defined up to isomorphism; instead, they are defined explicitly at the level of spaces to ensure that simplicial functoriality holds.

Summarizing, we obtain:

Lemma 4.2 Int^c *is a strongly Segal internal category.*

Moreover, the spaces of objects and morphisms are contractible:

Lemma 4.3 $\mathrm{Int}_0^c \simeq \mathrm{Int}_1^c \simeq *.$

Proof The underlying topological space is contractible as a subspace of $\mathbb{R}^{2k}$, so the associated Kan complex given by taking smooth simplices is also contractible.   □

4.2 Int^c as a complete Segal space

We defined Int^c as a strongly Segal internal category in the previous section. Its nerve, constructed in Section 1.5.3, is a Segal space $\mathrm{Int}_\bullet^c = N(\mathrm{Int}^c)_\bullet$. Let us spell out this Segal space in more detail to become more familiar with it.

For an integer $k \geq 0$, let

$$(5) \quad \mathfrak{Int}_k^c = \{(\underline{a},\underline{b}) = (a_0,\ldots,a_k,b_0,\ldots,b_k) : a_j < b_j \text{ for } 0 \leq j \leq k \text{ and}$$
$$a_{j-1} \leq a_j \text{ and } b_{j-1} \leq b_j \text{ for } 1 \leq j \leq k\} \subset \mathbb{R}^{2k}$$

with the subspace topology. As above, one can extract Kan complexes Int_k^c by taking smooth simplices. Note that for $k = 0, 1$ this coincides with (3) and (4) above. As before, we interpret an element $(\underline{a}, \underline{b})$ as an ordered $(k+1)$–tuple of closed intervals $\underline{I} = I_0 \leq \cdots \leq I_k$ with left endpoints a_j and right endpoints b_j such that I_j has nonempty interior. By "ordered", ie $I_j \leq I_{j'}$, we mean that the endpoints are ordered, ie $a_j \leq a_{j'}$ and $b_j \leq b_{j'}$ for $j \leq j'$.

Spatial structure of the levels The spatial structure of a level Int_k^c comes from taking smooth singular simplices of the submanifold of $\mathbb{R}^{2k}$. Thus, an l–simplex consists of smooth maps

$$|\Delta^l|_e \to \mathbb{R}, \quad s \mapsto a_j(s), b_j(s),$$

for $j = 0,\ldots,k$ such that for every $s \in |\Delta^l|_e$, the following inequalities hold:

$$a_i(s) < b_i(s) \quad \text{for } i = 0,\ldots,k,$$
$$a_{i-1}(s) \leq a_i(s),$$
$$b_{i-1}(s) \leq b_i(s) \quad \text{for } i = 1,\ldots,k.$$

We denote an l–simplex by $(I_0 \leq \cdots \leq I_k) \to |\Delta^l|_e$ or $(I_0(s) \leq \cdots \leq I_k(s))_{s \in |\Delta^l|}$ and call it a *closed interval bundle with $(k + 1)$ subintervals*.

For a morphism $f \colon [m] \to [l]$ in the simplex category Δ, ie a (weakly) order-preserving map, let $|f| \colon |\Delta^m|_e \to |\Delta^l|_e$ be the induced map between standard simplices. Let

f^Δ be the map sending an l–simplex in Int_k^c to the m–simplex in Int_k^c given by precomposing with $|f|$:

$$f^\Delta \colon (I_0(s) \leq \cdots \leq I_k(s))_{s \in |\Delta^l|_e} \mapsto \big(I_0(|f|(s)) \leq \cdots \leq I_k(|f|(s))\big)_{s \in |\Delta^m|_e}.$$

Notation 4.4 We denote the *spatial face and degeneracy maps* of Int_k^c by d_j^Δ and s_j^Δ for $0 \leq j \leq l$.

The following lemma is a straightforward generalization of Lemma 4.3.

Lemma 4.5 *Each level* Int_k^c *is a contractible Kan complex.*

Simplicial structure: the simplicial space $\mathrm{Int}_\bullet^c$ By construction, since Int^c was strongly Segal, its nerve is a functor $\mathrm{Int}_\bullet^c \colon \Delta^{\mathrm{op}} \to \mathcal{Space}$. Let us recall that to a morphism $g \colon [m] \to [k]$ in Δ, it assigns

$$\mathrm{Int}_k \xrightarrow{g^*} \mathrm{Int}_m, \quad (I_0(s) \leq \cdots \leq I_k(s))_{s \in |\Delta^l|_e} \mapsto (I_{g(0)}(s) \leq \cdots \leq I_{g(m)}(s))_{s \in |\Delta^l|_e}.$$

One could alternatively see this directly by observing that the assignment is clearly functorial and f^Δ and g^* commute for all morphisms f and g in Δ.

Notation 4.6 We denote the *simplicial face* and *degeneracy maps* by d_j and s_j for $0 \leq j \leq k$.

Explicitly, they are given by the following formulas. The jth degeneracy map is given by doubling the jth interval, and the jth face map is given by deleting the jth interval:

$$\mathrm{Int}_k \xrightarrow{s_j} \mathrm{Int}_{k+1}, \quad (I_0 \leq \cdots \leq I_k) \mapsto (I_0 \leq \cdots \leq I_j \leq I_j \leq \cdots \leq I_k),$$

$$\mathrm{Int}_k \xrightarrow{d_j} \mathrm{Int}_{k-1}, \quad (I_0 \leq \cdots \leq I_k) \mapsto (I_0 \leq \cdots \leq \widehat{I}_j \leq \cdots \leq I_k).$$

The complete Segal space $\mathrm{Int}_\bullet^c$

Proposition 4.7 $\mathrm{Int}_\bullet^c$ *is a complete Segal space. Moreover, the inclusion* $* \hookrightarrow \mathrm{Int}_\bullet^c$ *given by degeneracies, where* $*$ *is seen as a constant complete Segal space, is an equivalence of complete Segal spaces.*

Proof We have seen in Lemma 4.5 that every Int_k^c is contractible. This ensures the Segal condition, namely that

$$\mathrm{Int}_k^c \xrightarrow{\simeq} \mathrm{Int}_1^c \times_{\mathrm{Int}_0^c}^h \cdots \times_{\mathrm{Int}_0^c}^h \mathrm{Int}_1^c,$$

completeness, and also that the given inclusion is a levelwise equivalence. $\qquad \square$

4.3 The internal category or complete Segal space Int of ordered closed intervals in an open one

We now change our interpretation of the spaces (5): we do not identify them with the spaces of ordered closed bounded intervals $I_0 \leq \cdots \leq I_k$ anymore, but as ordered intervals which are closed in (a_0, b_k), ie we interpret the elements as

$$\widetilde{I}_0 \leq \cdots \leq \widetilde{I}_k,$$

where $\widetilde{I}_j = I_j \cap (a_0, b_k)$ for $0 \leq j \leq k$. Thus, in the generic case when $a_j \neq a_0$ for $0 < j \leq k$ and $b_j \neq b_k$ for $0 \leq j < k$, then $\widetilde{I}_0 \leq \cdots \leq \widetilde{I}_k$ are the half-open or closed intervals

$$(a_0, b_0] \leq [a_1, b_1] \leq \cdots \leq [a_{k-1}, b_{k-1}] \leq [a_k, b_k).$$

If we view the elements in (5) in this way, we will denote the internal category (or analogously the Segal space) by Int.

Note that the identity gives an isomorphism of complete Segal spaces describing the change of interpretation,

$$\mathrm{Int}_k^c \to \mathrm{Int}_k, \quad (I_0 \leq \cdots \leq I_k) \mapsto (\widetilde{I}_0 \leq \cdots \leq \widetilde{I}_k),$$

where $\widetilde{I}_j = I_j \cap (a_0, b_k)$ for $j = 0, \dots, k$. Conversely, $I_j = \mathrm{cl}_{\mathbb{R}}(\widetilde{I}_j)$, the closure of $\widetilde{I}_j$ in $\mathbb{R}$.

Definition 4.8 Let

$$\mathrm{Int}^n_{\bullet,\dots,\bullet} = (\mathrm{Int}_\bullet)^{\times n}.$$

We denote an element in $\mathrm{Int}^n_{k_1,\dots,k_n}$ by

$$\bar{\underline{I}} = (\bar{a}, \bar{b}) = (I_0^i \leq \cdots \leq I_{k_i}^i)_{1 \leq i \leq n}.$$

Lemma 4.9 *The n–fold simplicial space $\mathrm{Int}^n_{\bullet,\dots,\bullet}$ is a complete n–fold Segal space. Moreover, the inclusion $* \hookrightarrow \mathrm{Int}^n_{\bullet,\dots,\bullet}$ given by degeneracies, where $*$ is seen as a constant complete Segal space, is an equivalence of complete n–fold Segal spaces.*

Proof The Segal condition and completeness follow from the Segal condition and completeness for $\mathrm{Int}_\bullet$. Since every Int_k is contractible by Lemma 4.5, $(\mathrm{Int}_\bullet)^{\times n}$ satisfies essential constancy, so Int^n is a complete n–fold Segal space. It also ensures that the given inclusion is a levelwise equivalence.   □

4.4 The boxing maps

We will need the following maps for convenience later:

Definition 4.10 Fix $k \geq 0$. The map of spaces

$$B \colon \mathrm{Int}_k \to \mathrm{Int}_0, \quad \underline{I} = (I_0 \leq \cdots \leq I_k) \to |\Delta^l|_e \mapsto B(\underline{I}) = B(\underline{a}, \underline{b}) = (a_0, b_k) \to |\Delta^l|_e,$$

is called the *boxing map*.

Its n–fold product gives, for every $k_1, \ldots, k_n \geq 0$, a map $B \colon \mathrm{Int}_{k_1, \ldots, k_n}^n \to \mathrm{Int}_0^n$ which sends an l–simplex to the (family of) smallest open box(es) containing all intervals,

$$\underline{I} = (I_0^i \leq \cdots \leq I_{k_i}^i)_{1 \leq i \leq n} \to |\Delta^l|_e$$
$$\mapsto B(\underline{I}) = B(\overline{a}, \overline{b}) = (a_0^1, b_{k_1}^1) \times \cdots \times (a_0^n, b_{k_n}^n) \to |\Delta^l|_e.$$

We will usually view the total space of $B(\underline{I}) \to |\Delta^l|_e$ as sitting inside $\mathbb{R}^n \times |\Delta^l|_e$ as $\bigcup_{s \in |\Delta^l|_e} B(\underline{I}(s)) \times \{s\}$.

We will also require the following rescaling maps.

Definition 4.11 For an element $\underline{I} \in \mathrm{Int}_{k_1, \ldots, k_n}^n$, let $\rho(\underline{I}) \colon B(\underline{I}) \to (0, 1)^n$ be the restriction of the product of the affine maps $\mathbb{R} \to \mathbb{R}$ sending a_0^i to 0 and b_k^i to 1. We call it the *box rescaling map*.

4.5 A variant: closed intervals in $(0, 1)$

One might prefer to restrict to intervals which lie in $(0, 1)$, modifying the definition to

$$\mathrm{Int}_k^{(0,1)} = \{(\underline{a}, \underline{b}) = (a_0, \ldots, a_k, b_0, \ldots, b_k) : a_j < b_j \text{ for } 0 \leq j \leq k,$$
$$0 = a_0 \leq a_1 \leq \cdots \leq a_k \text{ and } b_0 \leq \cdots \leq b_{k-1} \leq b_k = 1\} \subset \mathrm{Int}_k$$

The simplicial structure now has to be modified to ensure that the outer endpoints always are 0 and 1. This is provided by composition with an affine rescaling map: Let $g \colon [m] \to [k]$ be a morphism in Δ. Then, let

$$\mathrm{Int}_k^{(0,1)} \xrightarrow{g^*} \mathrm{Int}_m^{(0,1)}, \quad (I_0 \leq \cdots \leq I_k) \to |\Delta^l|_e \mapsto \rho_g(I_{g(0)} \leq \cdots \leq I_{g(m)}) \to |\Delta^l|_e,$$

where the rescaling map $\rho_g = \rho(I_{g(0)} \leq \cdots \leq I_{g(m)})$ is the unique affine transformation $\mathbb{R} \to \mathbb{R}$ sending $a_{g(0)}$ to 0 and $b_{g(m)}$ to 1.

Lemma 4.12 $\mathrm{Int}_\bullet^{(0,1)}$ *is a complete Segal space.*

Proof The only thing which is not completely analogous to Int^c is checking that it is a simplicial space. Given two maps $[m] \xrightarrow{g} [k] \xrightarrow{\widetilde{g}} [p]$, and $I_0 \leq \cdots \leq I_p$, the rescaling map $\rho_{\widetilde{g} \circ g}$ and the composition of the rescaling maps $\rho_{\widetilde{g}} \circ \rho_g$ both send $a_{\widetilde{g} \circ g(0)}$ to 0 and $b_{\widetilde{g} \circ g(m)}$ to 1 and, since affine transformations $\mathbb{R} \to \mathbb{R}$ are uniquely determined by the image of two points, this implies that they coincide. Thus, this gives a functor $\Delta^{\text{op}} \to \mathcal{S}pace$. $\qquad\square$

Note that the degeneracy maps are the same ones, given by repeating an interval. However, the face maps need to modified: after deleting an end interval we have to rescale the remaining intervals linearly to $(0, 1)$. Explicitly, for $j = 0$, the rescaling map is the affine map ρ_0 sending $(a_1, 1)$ to $(0, 1)$, $\rho_0(x) = (x - a_1)/(1 - a_1)$ and, for $j = k$, it is the affine map $\rho_k \colon (0, b_{k-1}) \to (0, 1)$, $\rho_k(x) = x/b_{k-1}$. Then,

$$\text{Int}_k^{(0,1)} \xrightarrow{d_j} \text{Int}_{k-1}^{(0,1)},$$

$$I_0 \leq \cdots \leq I_k \longmapsto \begin{cases} I_0 \leq \cdots \leq \widehat{I_j} \leq \cdots \leq I_k & \text{if } j \neq 0, k, \\ (0, (b_1 - a_1)/(1 - a_1)] \leq \cdots \leq [(a_k - a_1)/(1 - a_1), 1) & \text{if } j = 0, \\ (0, (b_0)/(b_{k-1})] \leq \cdots \leq [(a_{k-1})/(b_{k-1}), 1) & \text{if } j = k. \end{cases}$$

Remark 4.13 An advantage of this "reduced" version is that the space of objects is just a point: for $k = 0$, the condition on the endpoints of the intervals becomes $a_0 = 0$ and $b_0 = 1$, so the only element is $(0, 1) \in \text{Int}_0$. In particular, Int_0 is discrete.

Remark 4.14 The boxing maps applied to $\text{Int}_k^{(0,1)}$ are trivial: for $\underline{I} = I_0 \leq \cdots \leq I_k$, we always have that $B(\underline{I}) = (0, 1)$. Moreover, $\text{Int}_k^{(0,1)}$ is the preimage of $(0, 1)$ under the boxing maps. Finally, note that the simplicial structure is defined exactly as the composition

$$\text{Int}_k^{(0,1)} \xrightarrow{\iota} \text{Int}_k \xrightarrow{g^*} \text{Int}_m \xrightarrow{\rho} \text{Int}_m^{(0,1)},$$

where $\rho \colon \underline{I} \mapsto (\rho(\underline{I}))(\underline{I})$ consists of applying the box rescaling maps. Moreover, since $\rho \circ \iota = \text{id}$, the diagram

$$
\begin{array}{ccc}
\text{Int}_k & \underset{\iota}{\overset{\rho}{\rightleftarrows}} & \text{Int}_k^{(0,1)} \\
\downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g^*} \\
\text{Int}_m & \xrightarrow{\rho} & \text{Int}_m^{(0,1)}
\end{array}
$$

commutes and shows that the simplicial structure is defined exactly in a way to ensure that we a natural transformation of simplicial spaces

$$\rho \colon \text{Int} \to \text{Int}^{(0,1)},$$

which is a weak equivalence of complete Segal spaces.

5   The (∞, n)–category of bordisms Bord_n

In this section we define an n–fold Segal space PBord_n in several steps. However, it will turn out not to be complete in general. By applying the completion functor we obtain a complete n–fold Segal space, the (∞, n)–category of bordisms Bord_n.

Let V be a finite-dimensional vector space. We first define the levels relative to V with elements being certain submanifolds of the (finite-dimensional) vector space $V \times \mathbb{R}^n \cong V \times B$, where B is an open box, ie a product of n bounded open intervals in $\mathbb{R}$. Then we vary V, ie we take the limit over all finite-dimensional vector spaces lying in some fixed infinite-dimensional vector space, eg $\mathbb{R}^\infty$. The idea behind this process is that by Whitney's embedding theorem, every manifold can be embedded in some large enough vector space, so in the limit, we include representatives of every n–dimensional manifold. We use $V \times B$ instead of $V \times \mathbb{R}^n$ as in this case the spatial structure is easier to write down explicitly.

5.1   The sets of 0–simplices of $(\mathrm{PBord}_n^V)_{k_1, \ldots, k_n}$

The intuition behind the following definition should be the following. An element (ie 0–simplex) in the space $(\mathrm{PBord}_n^V)_{1, \ldots, 1}$ should be an n–fold bordism, ie a manifold for which there are n "time" directions singled out and whose boundary is decomposed into an incoming and an outgoing part in each of these time directions. This is a picture of a simple example for $n = 2$:
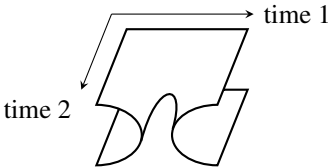

An element in the space $(\mathrm{PBord}_n^V)_{k_1, \ldots, k_n}$ should be an n–fold bordism, which is the composition of k_1 bordisms in the first "time" direction, k_2 bordisms in the second direction, and so on. This is a picture of an example for $n = 2$ and $k_1 = k_2 = 2$:
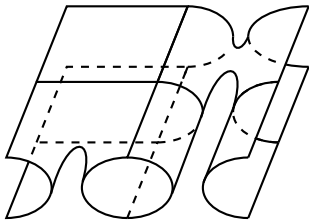
The pictures both depict the bordisms as embedded into $\mathbb{R}$ times the two time directions. We would like to point out that the time directions have a preferred ordering, as we will discuss in more detail later.

More generally, we will choose the bordisms to be equipped with an embedding into some finite-dimensional real vector space V times n time directions, which we single out to track where the bordism is allowed to be cut into the individual composed bordisms. Furthermore, to keep track of the "cuts", we need to remember the data of the grid in the time directions:
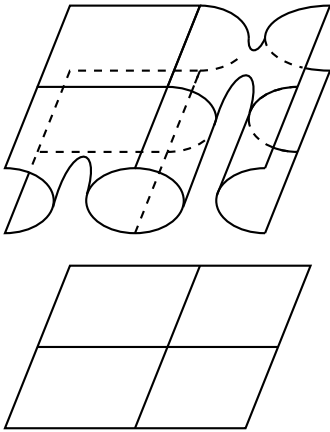
In practice, we will keep track of little intervals surrounding the grid instead of the grid itself. This should be thought of as remembering little collars around the cuts rather than the cuts themselves:
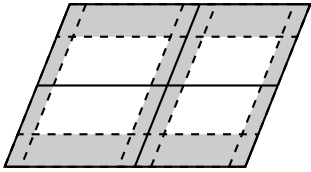
We will explain how to recover the cuts and how to interpret the following definition in the example and remark right after the definition. Moreover, we will relate it to more classical definitions of (higher) bordisms in Section 8.

For $S \subseteq \{1, \dots, n\}$ denote the projection from $\mathbb{R}^n$ onto the coordinates indexed by S by $\pi_S \colon \mathbb{R}^n \to \mathbb{R}^S$. We will now define the sets of 0–simplices of $(\mathrm{PBord}_n^V)_{k_1, \dots, k_n}$ and denote them by $(\mathbb{P}\mathrm{Bord}_n^V)_{k_1, \dots, k_n}$ to avoid adding an extra index. This notation will only appear in this and the next subsection.

Definition 5.1 Let V be a finite-dimensional $\mathbb{R}$–vector space, which we identify with some $\mathbb{R}^r$. For every n–tuple $k_1, \ldots, k_n \geq 0$, let $(\mathbb{P}\mathrm{Bord}_n^V)_{k_1, \ldots, k_n}$ be the collection of tuples $(M, \bar{I} = (I_0^i \leq \cdots \leq I_{k_i}^i)_{1 \leq i \leq n})$, satisfying the following conditions:

(1) For $1 \leq i \leq n$,
$$(I_0^i \leq \cdots \leq I_{k_i}^i) \in \mathrm{Int}_{k_i}.$$

(2) M is a closed and bounded n–dimensional submanifold of $V \times B(\bar{I})$ and the composition $\pi \colon M \hookrightarrow V \times B(\bar{I}) \twoheadrightarrow B(\bar{I})$ is a proper map.[12]

(3) For every $S \subseteq \{1, \ldots, n\}$, let $p_S \colon M \xrightarrow{\pi} B(\bar{I}) \xrightarrow{\pi_S} \mathbb{R}^S$ be the composition of π with the projection π_S onto the S–coordinates. Then, for every $1 \leq i \leq n$ and $0 \leq j_i \leq k_i$, at every $x \in p_{\{i\}}^{-1}(I_{j_i}^i)$, the map $p_{\{i, \ldots, n\}}$ is submersive.

Example 5.2 An example of an element in $(\mathbb{P}\mathrm{Bord}_1^\mathbb{R})$ is depicted below. It represents a composition of three 1–bordisms, the first one of which is "degenerate", ie a trivial 1–bordism between two points:
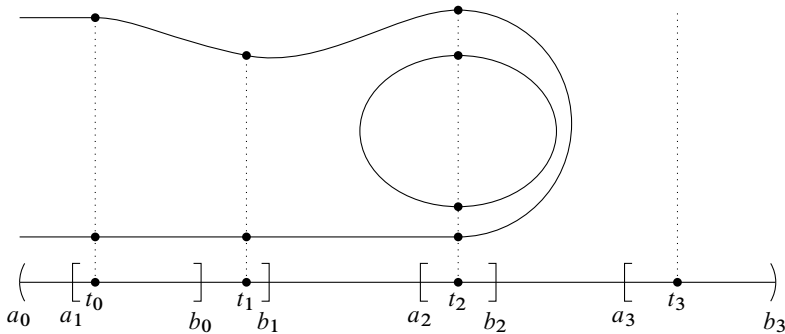
Remark 5.3 For $k_1, \ldots, k_n \geq 0$, one should think of an element in $(\mathbb{P}\mathrm{Bord}_n^V)_{k_1, \ldots, k_n}$ as a collection of $k_1 \cdots k_n$ composed bordisms, with k_i composed bordisms with collars in the ith direction. They can be understood as follows:

• Condition (3) in particular implies that for every $1 \leq i \leq n$, at every $x \in p_{\{i\}}^{-1}(I_j^i)$, the map $p_{\{i\}}$ is submersive. So if we choose $t_j^i \in I_j^i$, it is a regular value of $p_{\{i\}}$, and therefore $p_{\{i\}}^{-1}(t_j^i)$ is an $(n-1)$–dimensional manifold. The embedded manifold M should be thought of as a composition of n–bordisms where $p_{\{i\}}^{-1}(t_j^i)$ is one of the $(n-1)$–bordisms (or a composition thereof) in the composition:

---

[12]Recall the boxing map from Section 4.4.
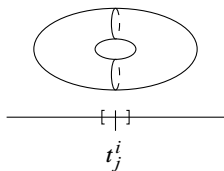
- For any $t_j^{n-1} \in I_j^{n-1}$ and $t_l^{n-1} \in I_l^{n-1}$, there is an inclusion of the preimages

$$p_{\{n-1,n\}}^{-1}((t_j^{n-1}, t_l^n)) \subset p_{\{n-1\}}^{-1}(I_j^{n-1}),$$

and by condition (3) the map $p_{\{n-1,n\}}$ is submersive there. Therefore, the preimage $p_{\{n-1,n\}}^{-1}((t_j^{n-1}, t_l^n))$ is an $(n-2)$–dimensional manifold, which should be thought of as one of the $(n-2)$–bordisms which are connected by the composition of n–bordisms M. Moreover, again since $p_{\{n-1,n\}}$ is submersive everywhere in $p_{\{n-1\}}^{-1}(I_j^{n-1})$, a variant of Ehresmann's fibration theorem shows that the preimage $p_{\{n-1\}}^{-1}(t_j^{n-1})$ is a trivial fibration and thus a trivial $(n-1)$–bordism between the $(n-2)$–bordisms it connects:
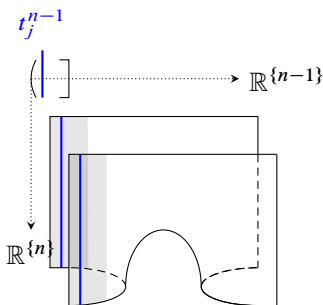
- Similarly, for $(t_{j_k}^k, \ldots, t_{j_n}^n) \in I_{j_k}^k \times \cdots \times I_{j_n}^n$, the preimage

$$p_{\{k,\ldots,n\}}^{-1}((t_{j_k}^k, \ldots, t_{j_n}^n))$$

is a $(k-1)$–dimensional manifold, which should be thought of as one of the $(k-1)$–bordisms which is connected by the composition of n–bordisms M.

- Moreover, the following proposition shows that different choices of "cutting points" $t_j^i \in I_j^i$ lead to diffeomorphic bordisms. In the case when $b_j^i < a_{j+1}^i$, one should thus think of the n–bordisms we compose as $\pi^{-1}\left(\prod_{i=1}^n [b_j^i, a_{j+1}^i]\right)$ and the preimages of the specified intervals as collars of the bordisms along which they are composed. Otherwise, one should think of that n–bordism in the composition as being "degenerate", ie of being a trivial n–bordism.

We will come back to this interpretation in Section 8 when we compute homotopy categories.

Proposition 5.4 *Let* $(M, \bar{I}) \in (\mathbb{P}\mathrm{Bord}_n^V)_{k_1,\ldots,k_n}$. *Fix* $1 \leq i \leq n$ *and* $0 \leq j \leq j' \leq k_i$. *Then, for any* $u_j^i, v_j^i \in I_j^i$ *and* $u_{j'}^i, v_{j'}^i \in I_{j'}^i$ *such that* $u_j^i < u_{j'}^i$ *and* $v_j^i < v_{j'}^i$, *there is a diffeomorphism*

$$p_{\{i\}}^{-1}([u_j^i, u_{j'}^i]) \to p_{\{i\}}^{-1}([v_j^i, v_{j'}^i]).$$

Proof Since the map $p_{\{i\}}$ is submersive in I_j^i and $I_{j'}^i$, we can apply the Morse lemma, which we recall in Theorem 8.11, to $p_{\{i\}}$ twice to obtain diffeomorphisms

$$p_{\{i\}}^{-1}([u_j^i, u_{j'}^i]) \to p_{\{i\}}^{-1}([v_j^i, u_{j'}^i]) \to p_{\{i\}}^{-1}([v_j^i, v_{j'}^i]). \qquad \square$$

Applying the proposition successively for $i = 1, \ldots, n$ yields:

Corollary 5.5 *Let* $(M, \bar{I}) \in (\mathbb{P}\mathrm{Bord}_n^V)_{k_1,\ldots,k_n}$ *and let* $B_1, B_2 \subseteq \mathbb{R}^n$ *be products of nonempty closed bounded intervals with endpoints lying in the same specified intervals, ie* $B_1 = \prod_i [u_j^i, u_{j'}^i]$ *and* $B_2 = \prod_i [v_j^i, v_{j'}^i]$, *where* $0 \leq j \leq j' \leq k_i$ *and* $u_j^i, v_j^i \in I_j^i$ *and* $u_{j'}^i, v_{j'}^i \in I_{j'}^i$ *such that* $u_j^i < u_{j'}^i$ *and* $v_j^i < v_{j'}^i$ *for every* $1 \leq i \leq n$. *Then there is a diffeomorphism*

$$\pi^{-1}(B_1) \to \pi^{-1}(B_2).$$

5.2 The spaces $(\mathrm{PBord}_n)_{k_1 \ldots, k_n}$

The level sets $(\mathbb{P}\mathrm{Bord}_n^V)_{k_1,\ldots,k_n}$ form the underlying sets of 0–simplices of spaces which we construct in this subsection. Ultimately, we want the space to encode the diffeomorphisms of n–fold bordisms which are the composition of k_i bordisms in the ith direction. More precisely, it should be the disjoint union of classifying spaces thereof. It will only become apparent that the space we define is the desired one in Section 8.2, in particular Proposition 8.17.

5.2.1 The topological space $(\mathfrak{P}\mathfrak{B}\mathfrak{o}\mathfrak{r}\mathfrak{d}_n^V)_{k_1,\ldots,k_n}$ We endow the set

$$(\mathbb{P}\mathrm{Bord}_n^V)_{k_1,\ldots,k_n}$$

with the following topology, coming from modifications of the Whitney C^∞–topology on $\mathrm{Emb}(M, V \times (0,1)^n)$.

In [19], spelled out in more detail in [20], a topology is constructed[13] on the set of closed (not necessarily compact) n–dimensional submanifolds $M \subseteq V \times (0,1)^n$, which

---

[13] Both [19] and [20] use the notation $\Psi(V \times (0,1)^n) = \mathrm{Sub}(V \times (0,1)^n)$.

we identify with the quotient

$$\mathrm{Sub}(V \times (0,1)^n) \xleftarrow{\;\cong\;} \bigsqcup_{[M]} \mathrm{Emb}(M, V \times (0,1)^n)/\mathrm{Diff}(M),$$

where the coproduct is taken over diffeomorphism classes of n–manifolds. It is given by defining the neighborhood basis at M to be

$$\{N \subset V \times (0,1)^n : N \cap K = j(M) \cap K, \; j \in W\},$$

where $K \subset V \times (0,1)^n$ is compact and $W \subseteq \mathrm{Emb}(M, V \times (0,1)^n)$ is a neighborhood of the inclusion $M \hookrightarrow V \times (0,1)^n$ in the Whitney C^∞–topology. Thus, we obtain a topology on

$$\mathrm{Sub}(V \times (0,1)^n) \times \mathfrak{Int}^n_{k_1,\dots,k_n},$$

where we view $\mathfrak{Int}^n_{k_1,\dots,k_n}$ as a (topological) subspace of $\mathbb{R}^{2k}$ as in Section 4.1.

For an element $\underline{I} \in \mathfrak{Int}^n_{k_1,\dots,k_n}$, recall from Definition 4.11 the box rescaling map $\rho(\underline{I}) \colon B(\underline{I}) \to (0,1)^n$. Then we identify an element $(M, \underline{I}) \in (\mathfrak{PBord}^V_n)_{k_1,\dots,k_n}$ whose underlying submanifold is the image of an embedding $\iota \colon M \hookrightarrow V \times B(\underline{I})$ with the element $([\rho(\underline{I}) \circ \iota], \rho(\underline{I}))$ in the above space. This identification gives an inclusion

$$(\mathfrak{PBord}^V_n)_{k_1,\dots,k_n} \subseteq \mathrm{Sub}(V \times (0,1)^n) \times \mathfrak{Int}^n_{k_1,\dots,k_n},$$

which we use to topologize the left-hand side.

5.2.2 The space $(\mathrm{PBord}^V_n)_{k_1,\dots,k_n}$ To model the levels of the bordism category as spaces, ie as Kan complexes, we can start with the above version as a topological space and take singular simplices of this topological space. However, *smooth maps* from a smooth manifold X to $\mathrm{Sub}(V \times (0,1)^n)$ as defined in [20, Definition 2.16, Lemma 2.17] are easier to handle. By Lemma 2.18 in the same paper, every continuous map from a smooth manifold — in particular from $|\Delta^l|_e$ — to $(\mathfrak{PBord}^V_n)_{k_1,\dots,k_n}$ can be perturbed to a smooth one, so the homotopy type when considering smooth singular simplices does not change.

We could directly define the space $(\mathrm{PBord}^V_n)_{k_1,\dots,k_n}$ to be the smooth singular space of $(\mathfrak{PBord}^V_n)_{k_1,\dots,k_n}$. However, we will first give a very explicit description of it.

Definition 5.6 An l–simplex of $(\mathrm{PBord}_n^V)_{k_1,\ldots,k_n}$ consists of tuples

$$\big(M, \bar{I}(s) = (I_0^i(s) \leq \cdots \leq I_{k_i}^i(s))_{s\in|\Delta^l|_e}\big)$$

such that:

(1) $\bar{I} = (I_0^i \leq \cdots \leq I_{k_i}^i)_{1\leq i\leq n} \to |\Delta^l|_e$ is an l–simplex in $\mathrm{Int}_{k_1,\ldots,k_n}^n$.

(2) M is a closed and bounded $(n{+}l)$–dimensional submanifold of

$$V \times B(\bar{I}(s))_{s\in|\Delta^l|_e} \subset V \times \mathbb{R}^n \times |\Delta^l|_e$$

such that[14]

   (a) the composition $\pi\colon M \hookrightarrow V \times B(\bar{I}(s))_{s\in|\Delta^l|_e} \twoheadrightarrow B(\bar{I}(s))_{s\in|\Delta^l|_e}$ of the inclusion with the projection is proper,

   (b) its composition with the projection onto $|\Delta^l|_e$ is a submersion $M \to |\Delta^l|_e$ which is trivial outside $|\Delta^l| \subset |\Delta^l|_e$.

(3) For every $S \subseteq \{1,\ldots,n\}$, let

$$p_S\colon M \xrightarrow{\pi} B(\bar{I}(s))_{s\in|\Delta^l|_e} \subset \mathbb{R}^n \times |\Delta^l|_e \xrightarrow{\pi_S} \mathbb{R}^S \times |\Delta^l|_e$$

be the composition of π with the projection π_S onto the S–coordinates. Then, for every $1 \leq i \leq n$ and $0 \leq j_i \leq k_i$, at every $x \in p_{\{i\}}^{-1}\big(\bigcup_{s\in|\Delta^l|_e} I_{j_i}^i(s) \times \{s\}\big)$, the map $p_{\{i,\ldots,n\}}$ is submersive.

From the definition of smooth map in [20, Definition 2.16, Lemma 2.17] we immediately get:

Lemma 5.7 *An l–simplex of* $(\mathrm{PBord}_n^V)_{k_1,\ldots,k_n}$ *is exactly a smooth l–simplex of* $(\mathfrak{PBord}_n^V)_{k_1,\ldots,k_n}$.

Remark 5.8 For $l = 0$ we recover Definition 5.1. Moreover, for every $s \in |\Delta^l|_e$ the fiber M_s of $M \to |\Delta^l|_e$ determines an element in $(\mathrm{PBord}_n^V)_{k_1,\ldots,k_n}$,

$$(M_s) = \big(M_s \subset V \times B(\bar{I}(s)), \bar{I}(s)\big).$$

We will use the notation $\pi_s\colon M_s \to B(\bar{I}(s))$ for the composition of the embedding and the projection.

Remark 5.9 The conditions (2)(a)–(b) and (3) imply that $M \to |\Delta^l|_e$ is a smooth fiber bundle, and, since $|\Delta^l|_e$ is contractible, even a trivial fiber bundle. The proof is a more elaborate version of the argument after [21, Definition 2.6].

---

[14]Recall that we view the total space of $B(\bar{I}) \to |\Delta^l|_e$ as sitting inside $\mathbb{R}^n \times |\Delta^l|_e$ as the union $\bigcup_{s\in|\Delta^l|_e} B(\bar{I}(s)) \times \{s\}$.

We now use the simplicial maps of $\mathrm{Int}^n_{k_1,\ldots,k_n}$ to explain those of $(\mathrm{PBord}^V_n)_{k_1,\ldots,k_n}$.

Definition 5.10 Fix $k \geq 0$ and let $f\colon [m] \to [l]$ be a morphism in the simplex category Δ, ie a (weakly) order-preserving map. Then let $|f|\colon |\Delta^m|_e \to |\Delta^l|_e$ be the induced map between standard simplices.

Let f^Δ be the map sending an l–simplex in $(\mathrm{PBord}^V_n)_{k_1,\ldots,k_n}$ to the m–simplex which consists of:

(1) For $1 \leq i \leq n$, the m–simplex in Int_{k_i} obtained by applying f^Δ,

$$f^\Delta\big((I^i_0(s) \leq \cdots \leq I^i_{k_i}(s))_{s \in |\Delta^l|_e}\big) = \big(I_0(|f|(s)) \leq \cdots \leq I_k(|f|(s))\big)_{s \in |\Delta^m|_e}.$$

(2) The $(n+m)$–dimensional submanifold $f^\Delta M \subseteq V \times B(\bar{I}(s))_{s \in |\Delta^m|_e}$ obtained by the pullback of $M \to |\Delta^l|_e$ along $|f|$. Note that its fiber at $s \in |\Delta^m|_e$ is $(f^\Delta M)_s = M_{|f|(s)}$ and

$$f^\Delta M = \bigcup_{s \in |\Delta^m|_e} M_{|f|(s)} \times \{s\}.$$

The above assignment is indeed well defined since the underlying assignment for the underlying intervals is well defined and, since the map $|f|$ is a submersion, the pullback of $M \to |\Delta^l|_e$ along $|f|$ is also a submersion. Moreover, the assignment is functorial, since pullback commutes contravariantly with composition, and thus $(\mathrm{PBord}^V_n)_{k_1,\ldots,k_n}$ is a simplicial set.

Proposition 5.11 The simplicial set $(\mathrm{PBord}^V_n)_{k_1,\ldots,k_n}$ is the smooth singular space of $(\mathfrak{PBord}^V_n)_{k_1,\ldots,k_n}$. In particular, it is a space.

Proof By definition, the simplicial maps f^Δ are induced precisely by the maps $|f|\colon |\Delta^m|_e \to |\Delta^l|_e$. □

Notation 5.12 We denote the *spatial face and degeneracy maps* of $(\mathrm{PBord}^V_n)_{k_1,\ldots,k_n}$ by d^Δ_j and s^Δ_j for $0 \leq j \leq l$.

Example 5.13 We now construct an example of a path. It shows that cutting-off part of the collar of a bordism yields an element which is connected to the original one by a path.

Let $(M) = (M, \bar{I} = (I^i_0 \leq \cdots \leq I^i_{k_i})_{i=1,\ldots,n}) \in (\mathrm{PBord}^V_n)_{k_1,\ldots,k_n}$ and fix $1 \leq i \leq n$. We show that cutting off a short enough piece in the ith direction at an end of an

element of $(\mathrm{PBord}_n^V)_{k_1,\ldots,k_n}$ leads to an element which is connected by a path to the original one. Fix $1 \leq i \leq n$ and let $\varepsilon < b_0^i - a_0^i$.

Choose a smooth, increasing, bijective function $[0, 1] \to [0, \varepsilon]$, $s \mapsto \varepsilon(s)$, with vanishing derivative at the endpoints.

For $0 \leq j \leq k_i$ and $s \in [0, 1] \subset |\Delta^1|_e$, let

$$I_j^i(s) = (a_0^i + \varepsilon(s), b_{k_i}^i) \cap I_j^i,$$

and then $B(\bar{I}(s)) = (a_0^i + \varepsilon(s), b_{k_i}^i) \subset B(\bar{I})$. For $s \leq 0$ and $s \geq 1$ let the family be constant. Then let $M(\varepsilon)$ be the preimage of the subset $\bigcup_{s \in |\Delta^1|_e} B(\bar{I}(s)) \times \{s\} \subseteq B(\bar{I}) \times |\Delta^1|_e$ of $M \times |\Delta^1|_e \to B(\bar{I}) \times |\Delta^1|_e$, ie the submanifold

$$
\begin{array}{ccc}
M(\varepsilon) & \lhook\joinrel\longrightarrow & M \times |\Delta^1|_e \\
\downarrow & & \downarrow \\
\bigcup_{s \in |\Delta^1|_e} B(\bar{I}(s)) \times \{s\} & \longrightarrow & B(\bar{I}) \times |\Delta^1|_e
\end{array}
$$

Then $(M(\varepsilon), \bar{I}(s))$ is a 1–simplex in $(\mathrm{PBord}_n^V)_{k_1,\ldots,k_n}$ with fibers

$$M(\varepsilon)_s = p_{\{i\}}^{-1}\big((a_0^i + \varepsilon(s), b_k^i)\big).$$

Remark 5.14 In the above example we constructed a path from an element in $(\mathrm{PBord}_n^V)_{k_1,\ldots,k_n}$ to its *cutoff*, where we cut off the preimage of $p_i^{-1}((a_0^i, \varepsilon])$ for suitably small ε. The same argument holds for cutting off the preimage of $p_i^{-1}([b_{k_i}^i - \delta, b_{k_i}^i))$ for suitably small δ. Moreover, we can iterate the process and cut off ε_i and δ_i strips in all i directions. Choosing $\varepsilon_i = \frac{1}{2}(b_0^i - a_0^i)$ and $\delta_i = \frac{1}{2}(b_{k_i}^i - a_{k_i}^i)$ yields a path to its *cutoff* with underlying submanifold

$$\mathrm{cut}(M) = \pi^{-1}\bigg(\prod_{i=1}^n \big(\tfrac{1}{2}(a_0^i + b_0^i), \tfrac{1}{2}(a_{k_i}^i + b_{k_i}^i)\big)\bigg).$$

5.3 The n–fold simplicial space $(\mathrm{PBord}_n)_{\bullet,\cdots,\bullet}$

We make the collection of spaces $(\mathrm{PBord}_n^V)_{\bullet,\ldots,\bullet}$ into an n–fold simplicial space by lifting the simplicial structure of $\mathrm{Int}_{\bullet,\ldots,\bullet}^{\times n}$. We first need to extend the assignment

$$([k_1], \ldots, [k_n]) \mapsto (\mathrm{PBord}_n^V)_{k_1,\ldots,k_n}$$

to a functor from $(\Delta^{\mathrm{op}})^n$.

Definition 5.15 For every $1 \leq i \leq n$, let $g_i \colon [m_i] \to [k_i]$ be a morphism in Δ, and denote by $g = (g_i)_i$ their product in Δ^n. Then

$$(\mathrm{PBord}_n^V)_{k_1,\ldots,k_n} \xrightarrow{\ g^* \ } (\mathrm{PBord}_n^V)_{m_1,\ldots,m_n}$$

applies g_i^* to the ith tuple of intervals and perhaps cuts the manifold. Explicitly, on l–simplices, g^* sends an element

$$\left(M \subset V \times B(\underline{\bar{I}}(s))_{s \in |\Delta^l|_e}, \ \underline{\bar{I}}(s) = (I_0^i(s) \leq \cdots \leq I_{k_i}^i(s))_{i=1}^n\right)$$

to

$$\left(g^*M = \pi^{-1}\big(B(g^*\underline{\bar{I}}(s))_{s \in |\Delta^l|_e}\big) \subset V \times B(\underline{\bar{I}}(s))_{s \in |\Delta^l|_e},\right.$$
$$\left. g^*(\underline{\bar{I}})(s) = (I_{g(0)}^i(s) \leq \cdots \leq I_{g(m_i)}^i(s))_{i=1}^n\right),$$

where $\pi \colon M \subset V \times B(\underline{\bar{I}}(s))_{s \in |\Delta^l|_e} \twoheadrightarrow B(\underline{\bar{I}}(s))_{s \in |\Delta^l|_e}$. Note that $(g^*M)_s = g^*M_s$.

Note that as the manifold g^*M is the preimage of the new box, we just cut off the part of the manifold outside the new box. This is functorial, as it is functorial on the intervals, and, if $\widetilde{g}_i \colon [k_i] \to [\widetilde{k}_i]$ and $\widetilde{g} = (\widetilde{g}_i)_i$, the following diagram commutes by construction:

$$
\begin{array}{ccccc}
M & \supseteq & g^*M & \supseteq & \widetilde{g}^*g^*M \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
B(\underline{\bar{I}}(s))_{s\in|\Delta^l|_e} & \supseteq & B(g^*(\underline{\bar{I}}(s)))_{s\in|\Delta^l|_e} & \supseteq & B(\widetilde{g}^*g^*(\underline{\bar{I}}(s)))_{s\in|\Delta^l|_e}
\end{array}
$$

Notation 5.16 We denote the *(simplicial) face* and *degeneracy maps* by

$$d_j^i \colon (\mathrm{PBord}_n^V)_{k_1,\ldots,k_n} \to (\mathrm{PBord}_n^V)_{k_1,\ldots,k_i-1,\ldots,k_n},$$
$$s_j^i \colon (\mathrm{PBord}_n^V)_{k_1,\ldots,k_n} \to (\mathrm{PBord}_n^V)_{k_1,\ldots,k_i+1,\ldots,k_n}$$

for $0 \leq j \leq k_i$.

Notation 5.17 Recall from [Remark 5.3](#) that for $k_1, \ldots, k_n \geq 0$, one should think of a 0–simplex in $(\mathrm{PBord}_n^V)_{k_1,\ldots,k_n}$ as a collection of $k_1 \cdots k_n$ composed bordisms with k_i composed bordisms with collars in the ith direction. These composed collared bordisms are the images under the maps

$$D(j_1, \ldots, j_k) \colon (\mathrm{PBord}_n^V)_{k_1,\ldots,k_n} \to (\mathrm{PBord}_n^V)_{1,\ldots,1}$$

for $(1 \leq j_i \leq k_i)_{1 \leq i \leq n}$ arising as compositions of inert face maps, ie $D(j_1, \ldots, j_k)$ is the map determined by the maps

$$d(j_i): [1] \rightarrow [k_i], \quad (0 < 1) \mapsto (j_i - 1 < j_i),$$

in the category Δ. This should be thought of as sending an element to the $(j_1, \ldots, j_k)^{\text{th}}$ collared bordism in the composition. Moreover, we will later use the notation

$$D^i(j_i): (\text{PBord}_n^V)_{k_1, \ldots, k_n} \rightarrow (\text{PBord}_n^V)_{k_1, \ldots, 1, \ldots, k_n}$$

for the maps induced by just $d(j_i)$. By abuse of notation, we will denote the submanifold $d(j_i)^* M$ by $D^i(j_i)(M)$.

Proposition 5.18 *The spatial and simplicial structures of* $(\text{PBord}_n^V)_{\bullet, \ldots, \bullet}$ *are compatible, ie for* $f: [l] \rightarrow [p]$, $g_i: [m_i] \rightarrow [k_i]$ *for* $1 \leq i \leq n$*, the induced maps*

$$f^\Delta \quad \text{and} \quad g^*$$

commute. We thus obtain an n*–fold simplicial space* $(\text{PBord}_n^V)_{\bullet, \cdots, \bullet}$.

Proof Since Int^n is a simplicial space, it is enough to show that the maps commute on the manifold part, ie

$$g^* f^\Delta M = f^\Delta g^* M.$$

This follows from the commuting of the following diagram, in which all sides arise from taking preimages:

The preimages are taken over $B(g^* \bar{\underline{I}}(s))_{s \in |\Delta^m|_e} \subset B(\bar{\underline{I}}(s))_{s \in |\Delta^m|_e}$ and $|f|: |\Delta^m|_e \rightarrow |\Delta^l|_e$, respectively, which affect different components of

$$V \times \bigcup_{s \in |\Delta^m|_e} \left(B(\bar{\underline{I}}(s)) \times \{s\}\right) \subset V \times \mathbb{R}^n \times |\Delta^m|_e,$$

so they commute. $\qquad \square$

5.4 The complete n–fold Segal space Bord_n

We will now prove that PBord_n^V leads to an (∞, n)–category, ie a complete n–fold Segal space of bordisms.

Proposition 5.19 $(\mathrm{PBord}_n^V)_{\bullet,\ldots,\bullet}$ is an n–fold Segal space.

Proof We need to prove that the Segal condition is satisfied and globularity.

Step 1: Segal condition Fix fixed $k_1, \ldots, k_n \geq 0$. We need to show that for every $1 \leq i \leq n$, and $k_i = m + l$, the Segal map

$$\gamma_{m,l} \colon (\mathrm{PBord}_n^V)_{k_1,\ldots,k_i,\ldots,k_n}$$
$$\to (\mathrm{PBord}_n^V)_{k_1,\ldots,m,\ldots,k_n} \times^h_{(\mathrm{PBord}_n^V)_{k_1,\ldots,0,\ldots,k_n}} (\mathrm{PBord}_n^V)_{k_1,\ldots,l,\ldots,k_n}$$

is a weak equivalence. From now on we will often omit writing out the indices for $\alpha \neq i$ for clarity.

Since every level set $(\mathrm{PBord}_n^V)_{k_1,\ldots,k_n}$ is a Kan complex by Proposition 5.11, ie fibrant, the homotopy fiber product on the right-hand side can be chosen to be the space of triples consisting of two points and a path between their target and source, respectively.

Note that an element in this space is given by a triple consisting of

$$(M, \underline{\bar{I}}) = (\iota \colon M \subset V \times B(\underline{\bar{I}}), \underline{\bar{I}} = (I_0^i \leq \cdots \leq I_m^i, I_0^j \leq \cdots \leq I_{k_j}^j)_{1 \leq j \leq n, j \neq i}),$$

$$(N, \underline{\bar{J}}) = (\kappa \colon N \subset V \times B(\underline{\bar{J}}), \underline{\bar{J}} = (J_0^i \leq \cdots \leq J_l^i, J_0^j \leq \cdots \leq J_{k_j}^j)_{1 \leq j \leq n, j \neq i}),$$

together with a path h from the target

$$D^i(m)(M, \underline{\bar{I}}) = (D^i(m)(M), I_m^i, (I_0^j \leq \cdots \leq I_{k_j}^j)_{1 \leq j \leq n, j \neq i})$$

of $(M, \underline{\bar{I}})$ in the ith direction to the source

$$D^i(1)(N, \underline{\bar{J}}) = (D^i(1)(N), J_0^i, (J_0^j \leq \cdots \leq J_{k_j}^j)_{1 \leq j \leq n, j \neq i})$$

of $(N, \underline{\bar{J}})$ in the ith direction (using Notation 5.17).

The Segal map $\gamma_{m,l}$ factors as a composition

(6)
$$\begin{array}{ccc}
(\mathrm{PBord}_n^V)_{k_i} & \xrightarrow{\gamma_{m,l}} & (\mathrm{PBord}_n^V)_m \times^h_{(\mathrm{PBord}_n^V)_0} (\mathrm{PBord}_n^V)_l \\
\downarrow & & \uparrow \\
(\mathrm{PBord}_n^V)^{m,l} & \longrightarrow & P_{k_i}^{m,l}
\end{array}$$

as follows: Informally, the lower right-hand corner is the subspace of triples for which, for the directions besides the ith, the tuples of intervals agree and the path of intervals is constant. The lower left-hand corner is the subspace thereof, for which in addition, in the ith direction, $I_m^i = J_0^i$, and along the path this interval stays constant. We will define these spaces below. Our strategy to prove that $\gamma_{m,l}$ is a weak equivalence is to show that all three maps are weak equivalences. Here the left vertical map is the main step of the proof — this is where "composing" the bordisms happens, as we will see below. That the bottom and right vertical map are weak equivalences follows from a rescaling procedure. Let us first define the two spaces in question.

For the lower right-hand corner, for $1 \leq j \leq n$ and $j \neq i$, consider the jth forgetful map

$$\mathrm{PBord}_n^V \to \mathrm{Int}, \quad (M, \bar{I}) \mapsto \underline{I}^j.$$

The canonical maps from the pullback to the homotopy pullback

$$\mathrm{Int}_\bullet \cong \mathrm{Int}_\bullet \times_{\mathrm{Int}_\bullet} \mathrm{Int}_\bullet \to \mathrm{Int}_\bullet \times_{\mathrm{Int}_\bullet}^h \mathrm{Int}_\bullet$$

(which is a weak equivalence since a deformation retract is straightforward to write down and rescales the second tuple of intervals) for varying j induce a (strict) pullback square

$$
\begin{array}{ccc}
P_{\bullet,\ldots,\bullet}^{m,l} & \longrightarrow & (\mathrm{PBord}_n^V)_{\bullet,\ldots,\bullet,m,\bullet,\ldots,\bullet} \times_{(\mathrm{PBord}_n^V)_{\bullet,\ldots,\bullet,0,\bullet,\ldots,\bullet}}^h (\mathrm{PBord}_n^V)_{\bullet,\ldots,\bullet,l,\bullet,\ldots,\bullet} \\
\downarrow & & \downarrow \\
\mathrm{Int}_{\bullet,\ldots,\bullet}^{\times(n-1)} & \xrightarrow{\ \simeq\ } & \mathrm{Int}_{\bullet,\ldots,\bullet}^{\times(n-1)} \times_{\mathrm{Int}_{\bullet,\ldots,\bullet}^{\times(n-1)}}^h \mathrm{Int}_{\bullet,\ldots,\bullet}^{\times(n-1)}
\end{array}
$$

The strict pullback of this diagram consists of exactly those pairs whose j^{th} tuples of intervals agree for every $j \neq i$, and is constant along the path (but the embedded manifold can still vary).[15]

For the lower left-hand corner, consider the canonical map

$$\mathrm{Int}_m \times_{\mathrm{Int}_0} \mathrm{Int}_l \to \mathrm{Int}_m \times_{\mathrm{Int}_0}^h \mathrm{Int}_l$$

(which is a weak equivalence since both sides are contractible).

---

[15] Since the right vertical map is a weak equivalence, if the diagram were also a homotopy pullback diagram, we would immediately see that the left vertical map is a weak equivalence as well. However, neither map in the diagram is a fibration (nor even a "sharp map" à la Rezk [44]), so we need to find a different strategy.

Now form the (strict) pullback

$$
\begin{array}{ccc}
P^{m,l}_{\bullet,\dots,\bullet} & \longrightarrow & \mathrm{Int}_m \times^h_{\mathrm{Int}_0} \mathrm{Int}_l \\
\uparrow & & \simeq\uparrow \\
(\mathrm{PBord}^V_n)^{m,l}_{\bullet,\dots,\bullet} & \longrightarrow & \mathrm{Int}_m \times_{\mathrm{Int}_0} \mathrm{Int}_l
\end{array}
$$

It consists of exactly those pairs whose j^{th} tuples of intervals agree for every $j \neq i$ and is constant along the path (but the embedded manifold can still vary), and, in addition, in the ith direction, the last interval of the first element is the first interval of the second element.[16]

The left vertical map in (6) is a weak equivalence   We first fix once and for all a "smoothed diagonal" $D \subset [0,1]^2$: it is the graph of a map $\varsigma\colon [0,1] \to [0,1]$, which has vanishing derivative in $\left[0,\frac{1}{3}\right]$ and $\left[\frac{2}{3},1\right]$ (we could also choose fixed shorter intervals) and is bijective with smooth inverse in $\left[\frac{1}{3},\frac{2}{3}\right]$, for example
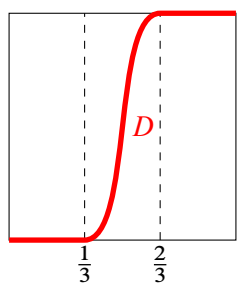

We will use this to define a deformation retract of $\gamma^{m,l}$, which we suggestively call glue. The homotopy exhibiting the deformation retract will use the following two modified functions for $\tau \in [0,1]$. Let

$$
\varsigma^s_\tau = \tau \cdot \varsigma \quad \text{and} \quad \varsigma^t_\tau = 1 + \tau \cdot (\varsigma - 1).
$$

Then for $\tau = 1$ we have that $\varsigma = \varsigma^s_1 = \varsigma^t_1$, and for $\tau = 0$ we have $\varsigma^s_0 = 0$ and $\varsigma^t_0 = 1$. Moreover, for every τ, both ς^s_τ and ς^t_τ are smooth and bijective onto their image. These give "flatter" diagonals $D_{s,\tau}$ and $D_{t,\tau}$:

---

[16] Again, the right vertical map is a weak equivalence, and it would be more convenient to take the homotopy pullback. However, the same problem appears as in the previous situation.
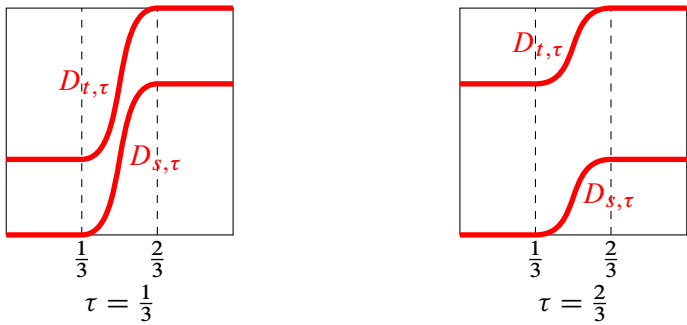
Recall from above that an element in $(\mathrm{PBord}_n^V)_{\bullet,\ldots,\bullet}^{m,l}$ is given by a pair $(M, \bar{I})$ and $(N, \bar{J})$ and a path h from the target of the former to the source of the latter, along which the interval is constant. We will use this path h to glue the embedded manifolds M and N. A similar argument works for l–simplices in $(\mathrm{PBord}_n^V)_{\bullet,\ldots,\bullet}^{m,l}$.

The 1–simplex h by definition is a submanifold P of[17] $V \times (c, b) \times |\Delta^1|_e$ such that the composition with the projection $\pi_{\{i\}} \colon P \to (c, b) \times |\Delta^1|_e$ is a submersion. We rescale the fixed smoothed diagonal D linearly to obtain a smooth diagonal $D^{c,b}$ in $(c, b) \times |\Delta^1|_e$.

Consider the preimage P_{diag} of $\pi_{\{i\}}$ of $D^{c,b}$. Since the projection $\pi_{\{i\}} \colon P \to (c, b) \times |\Delta^1|_e$ is submersive, a Morse lemma style argument shows that this preimage P_{diag} is diffeomorphic to both $D(m)(M)$ and $D(1)(N)$. Thus we glue the manifolds M and N over P_{diag} to obtain $M \cup_{P_{\mathrm{diag}}} N$. We realize it as a submanifold of $V \times \mathbb{R} \times (a, d)$ by using

- $M \cong M \times \{0\} \subset V \times \{0\} \times (a, b) \subset V \times \mathbb{R} \times (a, d)$,
- $N \cong N \times \{1\} \subset V \times \{1\} \times (c, d) \subset V \times \mathbb{R} \times (a, d)$,

and, using the coordinate in $|\Delta^1|_e \cong \mathbb{R}$,

- $P_{\mathrm{diag}} \subset V \times \mathbb{R} \times (c, b) \subset V \times \mathbb{R} \times (a, d)$.

However, note that the extra copy of $\mathbb{R}$ introduced above is not necessary: Let

$$\bar{D} = (\{0\} \times (a, c]) \cup D^{c,b} \cup (\{1\} \times [b, d)) \subset \mathbb{R} \times (a, d).$$

Then the projection onto the second coordinate induces a diffeomorphism $\bar{D} \cong (a, d)$. Thus, composing the embedding of the submanifold into $V \times \mathbb{R} \times (a, d)$ with the

---

[17]Actually, of $V \times (c, b) \times B((I_0^j \leq \cdots \leq I_{k_j}^j)_{1 \leq j \leq n, j \neq i}) \times |\Delta^1|_e$.

projection onto $V \times (a, d)$ still is an embedding,

$$M \cup_{P_{\text{diag}}} N \hookrightarrow V \times (a, d).$$

The same construction works for l–simplices: the same argument goes through with $(M, \bar{I})$ and $(N, \bar{J})$ now being l–simplices, and thus submanifolds of $V \times (a, b) \times |\Delta^l|_e$ and $V \times (c, d) \times |\Delta^l|_e$, respectively, and P a submanifold of $V \times (c, b) \times |\Delta^{l+1}|_e$. Moreover, since the shape D was chosen once and for all, this construction commutes with the spatial structure and indeed gives a map of spaces

$$\text{glue: } (\text{PBord}_n^V)_{\bullet, \ldots, \bullet}^{m,l} \to (\text{PBord}_n^V)_{\bullet, \ldots, \bullet, k_i, \bullet, \ldots, \bullet}.$$

We claim that this is a deformation retract of $\gamma^{m,l}$: Indeed, $\text{glue} \circ \gamma^{m,l}$ is the identity, since the path between the source and target in the image of $\gamma^{m,l}$ is constant. As for the other composition $\gamma^{m,l} \circ \text{glue}$, this sends a pair of elements (or l–simplices) $(M, \bar{I})$ and $(N, \bar{J})$ together with a path h from the target to the source to a pair $(\widetilde{M}, \bar{I})$ and $(\widetilde{N}, \bar{J})$ which is *not* the original one (in fact, the latter pair has a constant path $\tilde{h}$). However, there is a homotopy from $\gamma^{m,l} \circ \text{glue}$ to the identity as follows: For $\tau \in [0, 1]$, send $(M, \bar{I})$, $(N, \bar{J})$, h to the following pair: Modify the above construction by using $D_{s,\tau}$ and $D_{t,\tau}$ instead to obtain $P_{\text{diag}}^{s,\tau}$ and $P_{\text{diag}}^{t,\tau}$. Now one can glue M with $P_{\text{diag}}^{s,\tau}$ and N with $P_{\text{diag}}^{t,\tau}$ and embed each as above to obtain $(M_\tau, \bar{I})$ and $(N_\tau, \bar{J})$. A path h_τ between their target and source is given by the restriction of P to (ie the preimage of) the part between $D_{s,\tau}$ and $D_{t,\tau}$. For $\tau = 0$ this is the identity map, and for $\tau = 1$, this is exactly $\gamma^{m,l} \circ \text{glue}$.

"Rescaling": the bottom and right vertical maps in (6) are weak equivalences Both maps are part of a deformation retraction. Let us describe the right vertical map first.

The idea of "rescaling" is illustrated in the following picture for $n = 2$, $i = 1$, $l = m = 1$ and $k_2 = 2$:
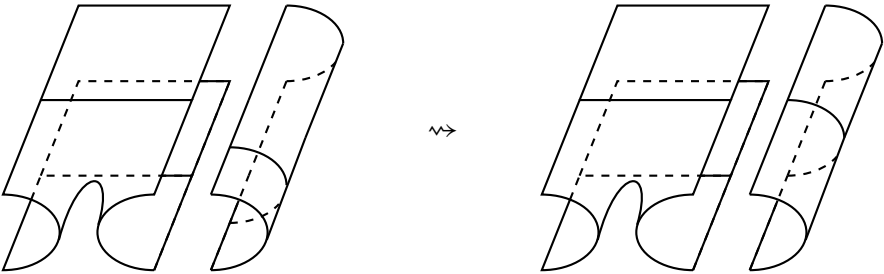
Note that we just depict the cutting lines, not the intervals around them. The rescaling is performed on the right-hand piece.

The deformation retract is given as follows: we observed above that the canonical map

$$\mathrm{Int}_\bullet \cong \mathrm{Int}_\bullet \times_{\mathrm{Int}_\bullet} \mathrm{Int}_\bullet \to \mathrm{Int}_\bullet \times_{\mathrm{Int}_\bullet}^h \mathrm{Int}_\bullet$$

levelwise has a deformation retraction. We will lift this to the desired deformation retraction.

An element (or l–simplex) in the right-hand side is given by a triple $(\underline{I}, \underline{J}, h)$, where h is a 1–simplex (or $(l+1)$–simplex) from $\underline{I}$ to $\underline{J}$, which we denote by $\underline{I} \to |\Delta^1|_e$. The latter determines a family of diffeomorphisms $B(\underline{J}) \to B(\underline{I}(s))$ and we send a triple $((M, \bar{\underline{I}}), (N, \bar{\underline{J}}), h)$ to a triple $((M, \bar{\underline{I}}), (N_s, \bar{\underline{J}}_s), h_s)$, where $(N_s, \bar{\underline{J}}_s)$ is given by the composition

$$N \subset V \times B(\bar{\underline{J}}) \to V \times B(\bar{\underline{I}}(s)).$$

We need the family of diffeomorphisms to have the following property: if for every $s \in [s, 1]$, the cardinality $|I_j(s) \cap I_{j+1}(s)|$ is 0 or 1, then $b_j(1) \mapsto b_j(s)$ and $a_{j+1}(1) \mapsto a_{j+1}(s)$. Such maps are easily defined in a piecewise linear way. However, we need them to be diffeomorphisms and vary smoothly in the parameter s, which requires smoothing. One explicit way of doing this smoothing uses flows along vector fields as in the proof of [Theorem 8.15](1).

As for the horizontal map, the rescaling in the ith direction, let $B(\underline{I}^i) = (a, b)$ and a_j^i and b_j^i the left and right endpoints of I_j^i; and $B(\underline{J}^i) = (c, d)$ and c_j^i and d_j^i the left and right endpoints of J_j^i. Similarly to above, by rescaling $(N, \bar{\underline{J}})$, we can assume that we have rescaled the embeddings and intervals so that $I_m^i = J_0^i = (a_m^i, b) = (c, d_0^i)$, and along the path this interval stays constant. This assumption implies the intervals can be "glued" (or rather, concatenated) to obtain an element in Int_{k_i}:

Similarly to above, this can be implemented using a deformation retraction of

$$\mathrm{Int}_m \times_{\mathrm{Int}_0} \mathrm{Int}_l \to \mathrm{Int}_m \times_{\mathrm{Int}_0}^h \mathrm{Int}_l,$$

which is lifted to one of the inclusion.

Step 2: globularity   We show that the degeneracy inclusion map

$$(\mathrm{PBord}_n^V)_{k_1,\ldots,k_{i-1},0,0,\ldots,0} \hookrightarrow (\mathrm{PBord}_n^V)_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n}$$

admits a deformation retraction and thus is a weak equivalence.

Consider the assignment sending a pair consisting of $t \in [0,1]$ and an l–simplex

$$\left(M \subset V \times B(\bar{I}(s)), \left((\underline{I}^\beta(s))_{1 \le \beta < i}, (a_0^i(s), b_0^i(s)), (\underline{I}^\alpha(s))_{i < \alpha \le n} \right)_{s \in |\Delta^l|_e} \right)$$

in $(\mathrm{PBord}_n^V)_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n}$ to

$$\left(M \subset V \times B(\bar{I}(s)), \left((\underline{I}^\beta(s))_{1 \le \beta < i}, (a_0^i(s), b_0^i(s)), (\underline{I}^\alpha(s,t))_{i < \alpha \le n} \right)_{(s,t) \in |\Delta^l|_e \times [0,1]} \right),$$

where for $\alpha > i$ and every $0 \le j \le k_\alpha$,

$$a_j^\alpha(s,t) = (1 - \varepsilon(t))a_j^\alpha(s) + \varepsilon(t)a_0^\alpha(s),$$
$$b_j^\alpha(s,t) = (1 - \varepsilon(t))b_j^\alpha(s) + \varepsilon(t)b_{k_\alpha}^\alpha(s).$$

for a smooth, increasing, bijective $\varepsilon \colon [0,1] \to [0,1]$ with vanishing derivative at the endpoints. This is a homotopy

$$H \colon [0,1] \times (\mathrm{PBord}_n^V)_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n} \to (\mathrm{PBord}_n^V)_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n}$$

exhibiting the deformation retract.[18] Note that $B(\bar{I}(s,t)) = B(\bar{I}(s))$ for every $t \in [0,1]$. Moreover, for $t = 0$ we have that $I_j^\alpha(s,0) = I_j^\alpha(s)$ and the l–simplex is sent to itself. For $t = 1$ we have

$$I_j^\alpha(s,1) = (a_0^\alpha(s), b_{k_\alpha}^\alpha(s)),$$

so the image lies in $(\mathrm{PBord}_n^V)_{k_1,\ldots,k_{i-1},0,0,\ldots,0}$.

It suffices to check that for every $t \in [0,1]$ the image indeed is an l–simplex in $(\mathrm{PBord}_n^V)_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n}$. Since $(M, \bar{I}(s)) \in (\mathrm{PBord}_n^V)_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n}$, this reduces to checking:

> For every $i < \alpha \le n$ and $0 \le j \le k_\alpha$, at every $x \in p_{\{\alpha\}}^{-1}(I_j^\alpha(s,t)_{s \in |\Delta^l|_e})$, the map $p_{\{\alpha,\ldots,n\}}$ is submersive.

Since in the ith direction we only have one interval, we have that

$$p_{\{i\}}^{-1}((a_0^i(s), b_0^i(s))_{s \in |\Delta^l|_e}) = M,$$

---

[18]To be precise, we take $t \in |\Delta^1|_e$ and extend the assignment so that it is constant outside $[0,1]$.

so, in particular, $p_{\{i\}}^{-1}((a_0^i(s), b_0^i(s))_{s \in |\Delta^l|_e}) \supset p_{\{\alpha\}}^{-1}(I_j^\alpha(s, t)_{s \in |\Delta^l|_e})$. Therefore, condition (3) in Definition 5.6 on (M) for i implies that $p_{\{i,\ldots,n\}}$ is a submersion in $p_{\{i\}}^{-1}((a_0^i(s), b_0^i(s))_{s \in |\Delta^l|_e}) = M \supset p_{\{\alpha\}}^{-1}(I_j^\alpha(s, t)_{s \in |\Delta^l|_e})$, so $p_{\{\alpha,\ldots,n\}}$ is submersive there as well. $\qquad\square$

Remark 5.20 It is much easier to see that the "strict" Segal condition also holds, ie that

$$(\mathrm{PBord}_n^V)_{k_1,\ldots,k_i+k_i',\ldots,k_n}$$
$$\xrightarrow{\;\sim\;} (\mathrm{PBord}_n^V)_{k_1,\ldots,k_i,\ldots,k_n} \times_{(\mathrm{PBord}_n^V)_{k_1,\ldots,0,\ldots,k_n}} (\mathrm{PBord}_n^V)_{k_1,\ldots,k_i',\ldots,k_n}.$$

An element in the right-hand side is a pair of submanifolds $M \subset V \times (a_0, b_{k_i})$ and $N \subset V \times (\tilde{a}_0, \tilde{b}_{k_i'})$ which coincide on the intersection $V \times (a_{k_i}, b_{k_i})$ together with intervals $I_0 \leq \cdots \leq I_{k_i}$ and $\tilde{I}_0 \leq \cdots \leq \tilde{I}_{k_i'}$ such that $I_{k_i} = \tilde{I}_0$. So we can glue them together to form a submanifold $M \cup N \subset V \times (a_0, \tilde{b}_{k_i'})$, and concatenate the intervals $I_0 \leq \cdots \leq I_{k_i} \leq \tilde{I}_1 \leq \cdots \leq \tilde{I}_{k_i'}$:
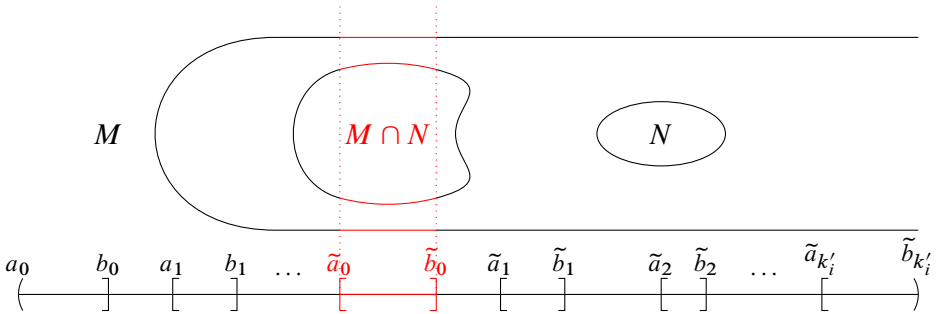
Thus, the above strict Segal map even is a homeomorphism.

Note that this construction also extends to l–simplices: we glue together submanifolds of $V \times (a_0(s), b_{k_i}(s))_{s \in |\Delta^l|_e}$ and $V \times (\tilde{a}_0(s), \tilde{b}_{k_i'}(s))_{s \in |\Delta^l|_e}$ to form one of $V \times (a_0(s), \tilde{b}_{k_i'}(s))_{s \in |\Delta^l|_e}$.

Remark 5.21 It can be checked that the maps

$$(\mathrm{PBord}_n^V)_{k_1,\ldots,k_i,\ldots,k_n} \to (\mathrm{PBord}_n^V)_{k_1,\ldots,0,\ldots,k_n}$$

are not fibrations unless $k_1 = \cdots = k_n = 0$. If they were, this together with the previous remark would have simplified the proof of the previous proposition. However, we can still conclude that PBord_n^V fits into a stricter model for (∞, n)–categories; for $n = 1$, it is a fibrant object in the model category of internal objects from [28] and Section 1.5.3

(even though it is not strongly Segal). For $n > 1$, it is a fibrant object in the model category of internal n–uple categories from [15] and Section 2.1, although it seems not to be a fibrant internal n–fold category, since the identities are not strict.

So far the definition of PBord_n^V depended on the choice of the vector space V. However, in the bordism category we would like to consider all (not necessarily compact) n–dimensional manifolds. By Whitney's embedding theorem any such manifold can be embedded into some finite-dimensional vector space V, so we need to allow big enough vector spaces.

Definition 5.22 Fix some countably infinite-dimensional vector space,[19] eg $\mathbb{R}^\infty$. Then we define PBord_n to be the homotopy colimit of n–fold Segal spaces[20]

$$\mathrm{PBord}_n = \varinjlim_{V \subset \mathbb{R}^\infty} \mathrm{PBord}_n^V = \mathrm{hocolim}_{V \subset \mathbb{R}^\infty} \mathrm{PBord}_n^V.$$

Remark 5.23 If the vector space V is r–dimensional, the n–fold Segal space PBord_n^V is also interesting in its own right. It describes codimension-r *tangles*; see also [40, Section 4.4]. For example, if $n = 1$ and $r = 2$, we obtain a Segal space of 1–dimensional tangles in $\mathbb{R}^3$. We will elaborate on this more in the next section. Moreover, we have not used that V is a vector space. Instead, one could take V to be some fixed manifold (as in eg [20]). This requires some extra care which we will not pursue here.

The last condition necessary to be a good model for the (∞, n)–category of bordisms is completeness, which PBord_n in general does not satisfy. However, we can pass to its completion Bord_n.

Definition 5.24 The (∞, n)–*category of bordisms* Bord_n is the n–fold completion $\widehat{\mathrm{PBord}}_n$ of PBord_n, which is a complete n–fold Segal space.

Remark 5.25 For $n \geq 6$, PBord_n is not complete; see the full explanation in [40, 2.2.8]. For $n = 1$ and $n = 2$, by the classification theorems of one- and two-dimensional manifolds, PBord_n is complete, and therefore $\mathrm{Bord}_n = \mathrm{PBord}_n$.

---

[19]The definition does not depend on the choice of the countably infinite-dimensional vector space; any such is the colimit over all finite-dimensional vector spaces.

[20]The identity map from the model category of n–fold simplicial spaces to the model category of n–fold Segal spaces is a left adjoint (since it is a localization) and therefore preserves homotopy colimits. Thus, the homotopy colimit can be computed in n–fold simplicial spaces.

6 Variants of Bord_n and comparison with Lurie's definition

6.1 The (∞, d)–category of n–bordisms and tangles for any $d \geq 0$

For $d \geq 0$ we define a d–fold Segal space whose top, ie d–morphisms are n–dimensional bordisms. For $d < n$ this amounts to extending the category of n–dimensional bordisms only down to $(n-d)$–dimensional objects.

Definition 6.1 Let V be a finite-dimensional $\mathbb{R}$–vector space, which we identify with some $\mathbb{R}^r$. Let $n \geq 0$ and $d = n + l \geq 0$. For every d–tuple $k_1, \ldots, k_d \geq 0$, we let $(\mathrm{PBord}_n^{l,V})_{k_1,\ldots,k_d}$ be the collection of tuples $(M, \bar{\underline{I}} = (I_0^i \leq \cdots \leq I_{k_i}^i)_{1 \leq i \leq d})$ satisfying conditions analogous to (1)–(3) in [Definition 5.1](#), ie

(1) For $1 \leq i \leq d$,
$$(I_0^i \leq \cdots \leq I_{k_i}^i) \in \mathrm{Int}_{k_i}.$$

(2) M is a closed and bounded n–dimensional submanifold of $V \times B(\bar{I})$ and the composition $\pi \colon M \hookrightarrow V \times B(\bar{I}) \twoheadrightarrow B(\bar{I})$ is a proper map.

(3) For every $S \subseteq \{1, \ldots, d\}$, let $p_S \colon M \xrightarrow{\pi} B(\bar{I}) \xrightarrow{\pi_S} \mathbb{R}^S$ be the composition of π with the projection π_S onto the S–coordinates. Then for every $1 \leq i \leq d$ and $0 \leq j_i \leq k_i$, at every $x \in p_{\{i\}}^{-1}(I_{j_i}^i)$, the map $p_{\{i,\ldots,d\}}$ is submersive.

We make $(\mathrm{PBord}_n^{l,V})_{k_1,\ldots,k_d}$ into a space similarly to $(\mathrm{PBord}_n^V)_{k_1,\ldots,k_n}$.

Proposition 6.2 $(\mathrm{PBord}_n^{l,V})_{\bullet,\cdots,\bullet}$ *is a* $d = (n+l)$*–fold Segal space.*

Proof The proof is completely analogous to the proof of [Proposition 5.19](#). $\square$

Again we take the homotopy colimit of n–fold Segal spaces, ie in the model category $s\mathcal{S}pace_{n,f}^{\mathrm{Se}}$ over all finite-dimensional vector spaces in a given infinite-dimensional vector space, say $\mathbb{R}^\infty$:
$$\mathrm{PBord}_n^l = \operatorname*{colim}_{V \subset \mathbb{R}^\infty} \mathrm{PBord}_n^{l,V}.$$

Definition 6.3 For $l \leq 0$ let $d = n + l \leq n$ and let Bord_n^l, which we will also denote by $\mathrm{Bord}_n^{(\infty,d)}$, be the d–fold completion of PBord_n^l, the (∞, d)*–category of* n*–bordisms*.

Notation 6.4 For $l \leq 0$ let $d = n + l \leq n$ and let $\mathrm{Bord}_n^{(\infty,d),V}$ be the d–fold completion of $\mathrm{PBord}_n^{l,V}$. If V is r–dimensional and $l = 0$, this is the unframed version of what Lurie calls the (∞, n)*–category of* n*–tangles* $\mathrm{Tang}_{n,n+r}^V$ in [40].

Remark 6.5 For $l > 0$, the underlying submanifold of objects of PBord_n^l, ie elements in $(\text{PBord}_n^l)_{0,\ldots,0}$, are n–dimensional manifolds M which have a submersion onto $\mathbb{R}^{n+l}$. This implies that $M = \varnothing$. Thus, the only objects are $(\varnothing, I_0^1, \ldots, I_0^{n+l})$ and $(\text{PBord}_n^l)_{0,\ldots,0} \cong \text{Int}_{0,\ldots,0}^n \simeq *$. Similarly, $(\text{PBord}_n^l)_{0,k_2,\ldots,k_{n+l}} \cong \text{Int}_{0,k_2,\ldots,k_{n+l}}^n \simeq *$. Thus, $(\text{PBord}_n^l)_{0,\bullet,\ldots,\bullet}$ is equivalent to the point viewed as a constant $(n-1)$–fold Segal space. Similarly, $(\text{PBord}_n^l)_{1_{l-1},0,\bullet,\ldots,\bullet}$ is equivalent to the point viewed as a constant $(n-l)$–fold Segal space. These will appear again later in Section 7.2.

6.2 Unbounded submanifolds, $(0,1)$ as a parameter space and cutting points

6.2.1 Unbounded submanifolds
We could have omitted the condition that M be bounded in condition (2) in Definitions 5.1 and 5.6, requiring it only to be closed. This modification leads to an n–fold simplicial space $\text{PBord}_n^{\text{unb}}$ together with a levelwise inclusion

$$\text{PBord}_n \hookrightarrow \text{PBord}_n^{\text{unb}}.$$

Recall from Remark 5.14 that for every element in $(\text{PBord}_n)_{k_1,\ldots,k_n}$, we constructed a path to its cutoff. There is a similar cutoff path for every element in $(\text{PBord}_n^{\text{unb}})_{k_1,\ldots,k_n}$ to an element whose underlying submanifold

$$\text{cut}(M) = \pi^{-1}\left(\prod_{i=1}^{n} \left(\tfrac{1}{2}b_0^i, \tfrac{1}{2}a_{k_i}^i \right) \right)$$

is bounded in the V–direction. Moreover, this construction extends to l–simplices. Altogether, it shows that the inclusion is a levelwise equivalence of n–fold simplicial spaces. Finally, since PBord_n is an n–fold Segal space, $\text{PBord}_n^{\text{unb}}$ is as well.

6.2.2 Restricting the boxing to $(0,1)^n$
Instead of basing PBord_n and PBord_n^l on Int, we could instead use $\text{Int}^{(0,1)}$ from Section 4.5. This approach leads to $(n+l)$–fold Segal spaces $\text{PBord}_n^{l,(0,1)}$ using the box rescaling maps $\rho(\bar{I})\colon B(\bar{I}) \to (0,1)^{n+l}$ and the functor ρ from Definition 4.11 and Remark 4.14. It fits exactly into a commuting diagram

$$
\begin{array}{ccc}
\text{PBord}_n^l & \xrightarrow{\ \rho\ } & \text{PBord}_n^{l,(0,1)} \\
\downarrow & & \downarrow \\
\text{Int}^{n+l} & \xrightarrow{\ \rho\ } & (\text{Int}^{(0,1)})^{\times (n+l)}
\end{array}
$$

where the vertical maps are the forgetful maps. Any simplex

$$(M \subset V \times B(\bar{I}), \bar{I} = (I_0^i \leq \cdots \leq I_{k_i}^i)_{1 \leq i \leq n+l})$$

is sent to

$$\left(M \subset V \times B(\bar{I}) \xrightarrow{\rho(\bar{I})} V \times (0,1)^{n+l}, (\rho(\bar{I})(I_0^i) \leq \cdots \leq \rho(\bar{I})(I_{k_i}^i))_{1 \leq i \leq n+l}\right).$$

On a fixed level, ie for fixed $k_1, \ldots, k_{n+l}$, there is an inclusion of spaces

$$\iota \colon (\mathrm{PBord}_n^{l,(0,1)})_{k_1,\ldots,k_{n+l}} \hookrightarrow (\mathrm{PBord}_n^l)_{k_1,\ldots,k_{n+l}}$$

and the above map is a retract of the inclusion. The $(n+l)$–fold simplicial structure needs to be modified by rescaling maps to ensure that the boxing stays $(0,1)^{n+l}$: For a morphism $g = \prod g_i$ in Δ^{n+l}, the associated morphism of spaces is $\rho \circ g \circ \iota$. Since ρ only involves rescalings, $\mathrm{PBord}_n^l \xrightarrow{\rho} \mathrm{PBord}_n^{l,(0,1)}$ is a levelwise weak equivalence, so in particular also a DK-equivalence of n–fold Segal spaces. We leave it to the reader to fill in the details.

6.2.3 Cutting points Another variant of an n–fold Segal space of bordisms can be obtained by replacing the intervals I_j^i in Definition 5.1 of PBord_n by specified "cutting points" t_j^i, which correspond to where we cut our composition of bordisms. Equivalently, we can say that in this case the intervals are replaced by intervals consisting of just one point, ie $a_j^i = b_j^i =: t_j^i$. The levels of this n–fold Segal space PBord_n^t can be made into spaces as we did for PBord_n, but we now need to add paths between 0–simplices which coincide inside the boxing of t's, ie over $[t_0^1, t_{k_1}^1] \times \cdots \times [t_0^n, t_{k_n}^n]$. However, for PBord_n^t the Segal condition is more difficult to prove, as in this case we do not specify the collar along which we glue. Since the space of collars is contractible, sending an interval I with endpoints a and b to its midpoint $t = \frac{1}{2}(a+b)$ induces an equivalence of n–fold Segal spaces from PBord_n to PBord_n^t. We will not elaborate more on this variant and leave details to the interested reader.

6.3 Comparison with Lurie's definition of bordisms

In [40], Lurie defined the n–fold Segal space of bordisms as follows:

Definition 6.6 Let V be a finite-dimensional vector space. For every n–tuple $k_1, \ldots,$ $k_n \geq 0$, let $(\mathrm{PBord}_n^{V,L})_{k_1,\ldots,k_n}$ be the collection of tuples $(M, (t_0^i \leq \cdots \leq t_{k_i}^i)_{i=1,\ldots,n})$, where:

(1) For $1 \le i \le n$,

$$t_0^i \le \cdots \le t_{k_i}^i$$

is an ordered (k_i+1)–tuple of elements in $\mathbb{R}$.

(2) M is a closed n–dimensional submanifold of $V \times \mathbb{R}^n$ and the composition $\pi: M \hookrightarrow V \times \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ is a proper map.

(3̃) For every $S \subseteq \{1, \ldots, n\}$ and for every collection $\{j_i\}_{i \in S}$, where $0 \le j_i \le k_i$, the composition $p_S: M \xrightarrow{\pi} \mathbb{R}^n \to \mathbb{R}^S$ does not have $(t_{j_i})_{i \in S}$ as a critical value.

(4̃) For every $x \in M$ such that $p_{\{i\}}(x) \in \{t_0^i, \ldots, t_{k_i}^i\}$, the map $p_{\{i+1,\ldots,n\}}$ is submersive at x.

It is endowed with a topology coming from the Whitney topology similar to what we described in Section 5.2.1. Similarly to before, let

$$\text{PBord}_n^L = \varinjlim_{V \subset \mathbb{R}^\infty} \text{PBord}_n^{V,L}.$$

Comparing this definition with Definition 5.1 and PBord_n^t from Section 6.2.3 above, our condition (3) on PBord_n^t is replaced by the two strictly weaker conditions (3̃) and (4̃) on PBord_n^L, which are both implied by (3):

Lemma 6.7 *Let M be a closed n–dimensional manifold and $\pi: M \to \mathbb{R}^n$. Moreover, for $1 \le i \le n$ let $(t_0^i \le \cdots \le t_{k_i}^i)$ be an ordered (k_i+1)–tuple of elements in $\mathbb{R}$. Denote for $S \subseteq \{1, \ldots, n\}$ the composition $M \xrightarrow{\pi} \mathbb{R}^n \twoheadrightarrow \mathbb{R}^S$ by p_S. Assume that condition (3) from Definition 5.1 holds, ie for every $1 \le i \le n$ and $0 \le j_i \le k_i$ for $x \in M$ such that $p_{\{i\}}(x) = t_{j_i}^i$, the map $p_{\{i,\ldots,n\}}$ is submersive at x. Then:*

(3̃) *For every $S \subseteq \{1, \ldots, n\}$ and for every collection $\{j_i\}_{i \in S}$, where $0 \le j_i \le k_i$, the composition $p_S: M \xrightarrow{\pi} \mathbb{R}^n \to \mathbb{R}^S$ does not have $(t_{j_i})_{i \in S}$ as a critical value.*

(4̃) *For every $x \in M$ such that $p_{\{i\}}(x) \in \{t_0^i, \ldots, t_{k_i}^i\}$, the map $p_{\{i+1,\ldots,n\}}$ is submersive at x.*

Proof Let $i_0 = \min S$. Consider the diagram

For (3̃) let $x \in p_S^{-1}((t_{j_i})_{i \in S})$. Then $p_{\{i_0\}}(x) = t_{j_{i_0}}^{i_0}$, so by assumption the map $p_{\{i_0,\dots,n\}}$ is submersive at x. Since proj is submersive, $p_S = \text{proj} \circ p_{\{i_0,\dots,n\}}$ is also submersive at x.

For (4̃) note that if $p_{\{i,\dots,n\}}$ is submersive at x then $p_{\{i+1,\dots,n\}}$ is submersive at x. □

However, Lurie's n–fold simplicial space PBord_n^L is not an n–fold Segal space, as we will see in the example below. Thus, our PBord_n^t is a corrigendum of Lurie's PBord_n^L from [40].

Example 6.8 Consider the 2–dimensional torus T in $\mathbb{R} \times \mathbb{R}^2$, embedded such that the projection onto $\mathbb{R}^2$ is an annulus, and consider the tuple $(T \subset \mathbb{R} \times \mathbb{R}^2, t_0^1, t_0^2 \leq \cdots \leq t_{k_2}^2)$, where t_0^1 is indicated in the picture of the projection plane $\mathbb{R}^2$ on the left:
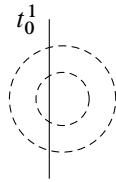
Then, because of condition (3̃), $t_0^2 \leq \cdots \leq t_{k_2}^2$ can be chosen everywhere such that any (t_0^1, t_j^2) is not a point where the vertical $(t_0^1$–)line intersects the two circles in the picture. Thus, if t_j^2 and $t_{\tilde{j}}^2$ are in two different connected components of this line minus these forbidden points, there is no path connecting this point to an element in the image of the degeneracy map. However, it satisfies the conditions (1), (2), (3̃) and (4̃) in the definition of $(\text{PBord}_2^L)_{0,k_2}$, so $(\text{PBord}_2^L)_{0,\bullet}$ is not essentially constant.

6.4 The n–uple Segal space $\text{PBord}_n^{\text{uple}}$

Consider the following interval version of condition (3̃) in Definition 6.6:

(3′) For every $S \subseteq \{1, \dots, n\}$ and for every collection $\{j_i\}_{i \in S}$, where $0 \leq j_i \leq k_i$, the composition $p_S : M \xrightarrow{\pi} \mathbb{R}^n \to \mathbb{R}^S$ does not have any critical value in $(I_{j_i})_{i \in S}$.

It ensures that the fibers $p_S^{-1}((t_{j_i}^i)_{i \in S})$ for $t_{j_i}^i \in I_{j_i}^i$ are $(n-|S|)$–dimensional smooth manifolds. This can be interpreted similarly to in Remark 5.3 as follows: Choosing $t_{j_i}^i < t_{j_i'}^i$, we picture the composed bordism as

$$\pi^{-1}\left(\prod [t_{j_i}^i, t_{j_i'}^i]\right),$$

which, by condition (3′) is a "cubical" n–dimensional bordism whose sources and targets in each direction are themselves "cubical" $(n-1)$–dimensional bordisms, and,

furthermore, its diffeomorphism class does not depend on the choice of cutting points $t^i_{j_i}$; see Section 8.1 for more details. Our condition (3) — which we saw in Lemma 6.7 to imply ($\widetilde{3}$) in the point version; the same argument shows that it implies (3′) in the case of intervals — ensures in addition the globularity condition, namely that the source and target of the n–dimensional bordism are $(n-1)$–dimensional bordisms which themselves have the same source and target. This is reflected in the essential constancy condition, namely that we have an "n–category" instead of an "n–uple category". Thus, relaxing condition (3) in Definition 5.1 to (3′), we obtain:

Definition 6.9 The n–uple Segal space $\mathrm{PBord}^{\mathrm{uple}}_n$ has levels whose elements are tuples of $(M, \bar{I} = (I^i_0 \leq \cdots \leq I^i_{k_i})_{1 \leq i \leq n})$, satisfying conditions (1) and (2) in Definition 5.1 and (3′).

Using the construction in Section 2.4.5 one can see that PBord_n indeed is the maximal n–fold Segal space underlying the n–uple Segal space $\mathrm{PBord}^{\mathrm{uple}}_n$.

Proposition 6.10 *The maximal underlying n–fold Segal space of $\mathrm{PBord}^{\mathrm{uple}}_n$ is equivalent to PBord_n.*

Proof Recall from Section 2.4.5 that the levels of the maximal underlying n–fold Segal space of a n–uple Segal space X are given by

$$\boldsymbol{R}(X)_{\vec{k}} = \mathrm{Map}^h(\Theta^{\vec{k}}, X).$$

The remaining structure arises from that of X.

The levelwise inclusion $\iota\colon \mathrm{PBord}_n \hookrightarrow \mathrm{PBord}^{\mathrm{uple}}_n$ and the weak equivalence from Lemma 2.27 give a map

$$\mathrm{PBord}_n \simeq \mathrm{Map}^h(\Delta^{\vec{k}}, \mathrm{PBord}_n)$$
$$\simeq \mathrm{Map}^h(\Theta^{\vec{k}}, \mathrm{PBord}_n) \to \mathrm{Map}^h(\Theta^{\vec{k}}, \mathrm{PBord}^{\mathrm{uple}}_n) = \boldsymbol{R}(\mathrm{PBord}^{\mathrm{uple}}_n).$$

By the Segal condition, it suffices to show that this map is an equivalence at every $(i) = (\boldsymbol{1}_i, 0, \ldots, 0) \in (\Delta^{\mathrm{op}})^k$. To compute the correct mapping spaces, we use the cofibrant replacement of $\Theta^{(i)}$ from Proposition 2.25.

Let C^i be the n–uple category, or rather its nerve, the n–fold simplicial space which is the image of the Yoneda embedding of $([1], \ldots, [1]) \in \Delta^{\times i}$ and thus is cofibrant for the projective model structure. Then

$$\mathrm{Map}^h(C^i, \mathrm{PBord}^{\mathrm{uple}}_n) = (\mathrm{PBord}^{\mathrm{uple}}_n)_{(i)}.$$

Now we proceed by induction. The first index for which there is something to show is $i = 2$. Using the explicit cofibrant replacement from Proposition 2.25, we compute that

$$\boldsymbol{R}(\mathrm{PBord}_n^{\mathrm{uple}})_{(2)} = \mathrm{Map}^h(\Theta^{(2)}, \mathrm{PBord}_n^{\mathrm{uple}})$$

is the homotopy pullback of the diagram

$$(\mathrm{PBord}_n^{\mathrm{uple}})_{(2)}$$
$$\downarrow$$
$$(\mathrm{PBord}_n^{\mathrm{uple}})_0 \times (\mathrm{PBord}_n^{\mathrm{uple}})_0 \longrightarrow (\mathrm{PBord}_n^{\mathrm{uple}})_{(1)} \times (\mathrm{PBord}_n^{\mathrm{uple}})_{(1)}$$

From this, we see that an element in $\boldsymbol{R}(\mathrm{PBord}_n^{\mathrm{uple}})_{(2)}$ is an element in $(\mathrm{PBord}_n^{\mathrm{uple}})_{(2)}$ together with a path from its source and target to elements in $(\mathrm{PBord}_n^{\mathrm{uple}})_0$. By Theorem 8.15 such a path determines a diffeomorphism between the composed bordisms associated to the start and the end (the source and target, respectively) of the path, which in turn are elements in $(\mathrm{PBord}_n^{\mathrm{uple}})_0$. Thus, they satisfy condition $(3')$ everywhere, ie the projection π is a submersion. This property is preserved by diffeomorphisms, which implies that the source and target of the element in $(\mathrm{PBord}_n^{\mathrm{uple}})_{(2)}$ also satisfy this condition. So the original element must satisfy condition (3). and thus lies in $(\mathrm{PBord}_n)_{(2)}$.

For $i > 2$, again using the explicit cofibrant replacement, there is a homotopy pullback diagram

$$\begin{array}{ccc} \boldsymbol{R}(\mathrm{PBord}_n^{\mathrm{uple}})_{(i)} & \longrightarrow & \mathrm{Map}(C^1 \times \mathrm{cof}(\Theta^{(i-1)}), \mathrm{PBord}_n^{\mathrm{uple}}) \\ \downarrow \quad {}^{\ulcorner}h & & \downarrow \\ (\mathrm{PBord}_n)_0^{\mathrm{uple}} \times (\mathrm{PBord}_n)_0^{\mathrm{uple}} & \longrightarrow & (\mathrm{PBord}_n)_{(i-1)}^{\mathrm{uple}} \times (\mathrm{PBord}_n)_{(i-1)}^{\mathrm{uple}} \end{array}$$

Since there is a surjection $C^i \twoheadrightarrow C^1 \times \Theta^{(i-1)}$, elements in the top-right corner arise from elements in $\mathrm{Map}^h(C^i, \mathrm{PBord}_n^{\mathrm{uple}}) = (\mathrm{PBord}_n^{\mathrm{uple}})_{(i)}$. Moreover,

$$\mathrm{Map}(C^1 \times \mathrm{cof}(\Theta^{(i-1)}), \mathrm{PBord}_n^{\mathrm{uple}}) = \boldsymbol{R}((\mathrm{PBord}_n^{\mathrm{uple}})_{1,\bullet,\ldots,\bullet})_{(i-1)}.$$

The induction hypothesis implies that elements therein are elements in $(\mathrm{PBord}_n^{\mathrm{uple}})_{(i)}$ which satisfy condition (3) in Definition 5.1 or $i > 1$.

Thus, elements in $\boldsymbol{R}(\mathrm{PBord}_n^{\mathrm{uple}})_{(i)}$ are such elements together with a path from their source and target to elements in $(\mathrm{PBord}_n^{\mathrm{uple}})_0$. Unraveling this condition as in the base case, we see that this implies condition (3) for $i = 1$. $\qquad\square$

Example 6.11 (the torus as a composition) The difference between the n–fold and the n–uple Segal spaces can be seen when decomposing the torus, viewed as a 2–morphism in the respective n–(uple) categories. We will omit drawing the intervals outside of the torus and just draw the "cutting lines", which should be understood as actually extending to small closed intervals around them.

The torus as a 2–morphism in $\mathrm{Bord}_2^{\mathrm{uple}}$ can be decomposed simultaneously in both directions. One possible decomposition into in some sense elementary pieces is the following:
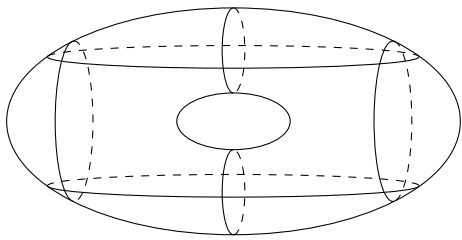
However, similar to the argument in Example 6.8, this decomposition is not a valid decomposition in Bord_2, as condition (3) in Definition 5.1 is violated.
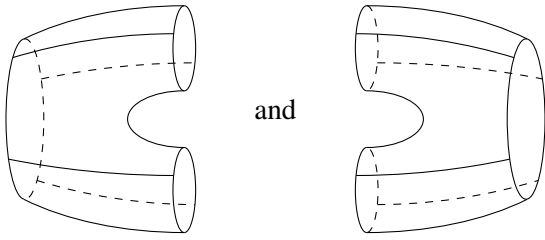
The torus as a 2–morphism in Bord_2 can only be decomposed "successively", so we first decompose it in the first direction, ie the first coordinate, eg as
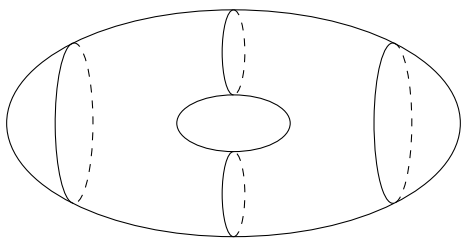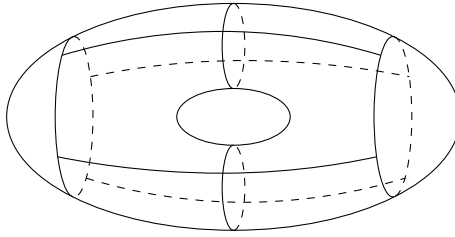

which is an element in $(\mathrm{Bord}_2)_{4,1}$ and then decompose the two middle pieces, which are the images under the compositions of face maps

$$D^1(2), D^1(3)\colon (\mathrm{Bord}_2)_{4,1} \rightrightarrows (\mathrm{Bord}_2)_{1,1}$$

as

and

Altogether, a possible decomposition of the torus into elementary pieces in Bord_2 is

This of course is also a valid decomposition in the 2–fold category $\mathrm{Bord}_2^{\mathrm{uple}}$.

7 The symmetric monoidal structure on bordisms and tangles

The (∞, n)–category Bord_n is symmetric monoidal with its symmetric monoidal structure essentially arising from taking disjoint unions. In this section we endow Bord_n with a symmetric monoidal structure in two ways. In Section 7.1 the symmetric monoidal structure arises from a Γ–object. In Section 7.2 a symmetric monoidal structure is defined using a tower of monoidal i–hybrid $(n+i)$–fold Segal spaces.

7.1 The symmetric monoidal structure arising as a Γ–object

We construct a sequence of n–fold Segal spaces $(\mathrm{PBord}_n^V \langle m \rangle)_{\bullet, \ldots, \bullet}$ which form a Γ–object in n–fold Segal spaces, which in turn endows Bord_n with a symmetric monoidal structure as defined in Section 3.1.

Definition 7.1 Let V be a finite-dimensional vector space. For every $k_1, \ldots, k_n$, let $(\mathrm{PBord}_n^V \langle m \rangle)_{k_1, \ldots, k_n}$ be the collection of tuples

$$(M_1, \ldots, M_m, (I_0^i \leq \cdots \leq I_{k_i}^i)_{i=1,\ldots,n}),$$

where each $(M_\beta, (I_0^i \leq \cdots \leq I_{k_i}^i)_{i=1,\ldots,n})$ is an element of $(\mathrm{PBord}_n^V)_{k_1,\ldots,k_n}$ and $M_1, \ldots, M_m$ are disjoint. It can be made into an n–fold simplicial space similarly to PBord_n^V. Moreover, similarly to the definition of Bord_n, we take the homotopy colimit over all $V \subset \mathbb{R}^\infty$ and complete to get an n–fold complete Segal space $\mathrm{Bord}_n[m]$.

Proposition 7.2 *Let*

$$\Gamma \to \mathcal{S}e\mathcal{S}p_n, \quad \langle m \rangle \mapsto \mathrm{PBord}_n \langle m \rangle,$$

where to a morphism $f\colon \langle m \rangle \to \langle k \rangle$ *in* Γ *we assign the map*

$$\mathrm{PBord}_n \langle m \rangle \to \mathrm{PBord}_n \langle k \rangle,$$

$$(M_1, \ldots, M_m, I's) \mapsto \left(\coprod_{\beta \in f^{-1}(1)} M_\beta, \ldots, \coprod_{\beta \in f^{-1}(k)} M_\beta, I's \right).$$

This assignment is functorial and endows Bord_n *with a symmetric monoidal structure.*

Proof By Lemma 3.7 it is enough to show that the assignment is a functor $\Gamma \to \mathcal{SeSp}_n$ with the property that for every $m \geq 0$ the map

$$\prod_{1 \leq \beta \leq n} \gamma_\beta \colon \mathrm{PBord}_n \langle m \rangle \to (\mathrm{PBord}_n \langle 1 \rangle)^m$$

is an equivalence of n–fold Segal spaces.

Functoriality follows from the functoriality of taking preimages. For $m = 0$ both sides are contractible. For $m > 0$ the map $\prod_{1 \leq \beta \leq n} \gamma_\beta$ is a levelwise inclusion and we show that levelwise it is a weak equivalence.

First, we can show that for every $k_1, \ldots, k_n$, the space $(\mathrm{PBord}_n \langle 1 \rangle)^m_{k_1, \ldots, k_n}$ is weakly equivalent to its pullback, which we will denote by P^m, along the diagonal map $\mathrm{Int}^n_{k_1, \ldots, k_n} \to (\mathrm{Int}^n_{k_1, \ldots, k_n})^m$. The argument is analogous to the proofs of the rescaling steps in the proof of the Segal condition in Proposition 5.19. Note that P^m is the subspace of those elements for which the intervals coincide, and $\prod_{1 \leq \beta \leq n} \gamma_\beta$ factors through P^m.

Now, we will exhibit an explicit deformation retraction of

$$\prod_{1 \leq \beta \leq n} \gamma_\beta \colon \mathrm{PBord}_n \langle m \rangle_{k_1, \ldots, k_n} \to P^m,$$

which shows that the two spaces are equivalent.

Consider the family of embeddings $\iota_s \colon V \to \mathbb{R} \otimes V$, $v \mapsto (s\alpha, v)$, parametrized by $s \in [0, 1]$. Note that this also induces a family of embeddings $\mathbb{R}^\infty \to \mathbb{R} \oplus \mathbb{R}^\infty \cong \mathbb{R}^\infty$.

Let $((M_1), \ldots, (M_m))$ be any k–simplex in the target P^m. We construct a k–simplex in $\mathrm{PBord}_n \langle m \rangle_{k_1, \ldots, k_n}$ together with a map $F \colon \Delta^k \times [0, 1] \to P^m$ restricting to the original one at 1 and the new k–simplex at 0. The map F is defined as follows: For fixed s, it consists of the k–simplex which is defined by composing the embedding $M_\alpha \hookrightarrow V \times B(\bar{I})$ with the embedding ι_s. This depends smoothly on s (and nothing

else). For $s > 0$, this indeed lands in $\mathrm{PBord}_n \langle m \rangle_{k_1, \ldots, k_n}$, since the images of M_i are disjoint. The construction is compatible with the simplicial structure, since ι_s did not affect the copy of $B(\bar{I})$. Altogether, this induces a strong homotopy equivalence between the above spaces. $\qquad \square$

Remark 7.3 More generally, the same construction works for $\mathrm{Bord}_n^{(\infty, d)}$ for $d \leq n$ using a sequence of d–fold Segal spaces $\mathrm{PBord}_n^l \langle m \rangle$ for $l = n - d$.

7.2 Looping, the monoidal structure and the tower

Our goal for this section is to endow Bord_n and $\mathrm{Bord}_n^{(\infty, d)}$ with symmetric monoidal structures arising from a tower of monoidal l–hybrid $(n+l)$–fold Segal spaces $\mathrm{Bord}_n^{(l)}$ for $l \geq 0$. We will prove a stronger statement first, namely that the tangle categories $\mathrm{Bord}_n^{(\infty, d), V}$ are r–monoidal if V is r–dimensional.

Recall from Section 6.1 the $(n+l)$–fold Segal spaces PBord_n^l of n–dimensional bordisms. We saw in Remark 6.5 that PBord_n^l is $(l-1)$–connected if $l > 0$. However, it does not have a discrete space of objects, 1–morphisms, $\ldots$, $(l-1)$–morphisms. For $l \leq 0$ the situation is even worse as PBord_n^l is not even connected. However, in any PBord_n^l, there is the distinguished object

$$\varnothing = (\varnothing, (0, 1), \ldots, (0, 1)),$$

and by Proposition 3.32 it suffices to prove the following theorem.

Theorem 7.4 *For* $n + l \geq k \geq 0$ *and an* $(r-1)$*–dimensional vector space* V, *there is a weak equivalence*

$$\Omega_\varnothing^k (\mathrm{PBord}_n^{l, V}) \xrightarrow{u_r} \mathrm{PBord}_n^{l-k, V \oplus \mathbb{R}},$$

which induces a weak equivalence $u \colon \Omega_\varnothing^k (\mathrm{PBord}_n^l) \to \mathrm{PBord}_n^{l-k}$.

Since looping and hybrid completion commute by Lemma 2.22 the following corollary is immediate.

Corollary 7.5 *Let* $n \geq 0$ *and* $d \leq n$, *and let* V *be an* r*–dimensional vector space.*

- *The tangle categories* Bord_n^V *and* $\mathrm{Bord}_n^{(\infty, d), V}$ *are* r*–monoidal.*
- *The bordism categories* Bord_n *and* $\mathrm{Bord}_n^{(\infty, d)}$ *are symmetric monoidal.*

We can extract the k–monoidal $(n+l)$–fold complete Segal spaces which form the tower for the symmetric monoidal structure:

Definition 7.6 For $k > 0$ and $d \geq 0$, the $(n+l)$–fold Segal space

$$\mathscr{L}^k_\varnothing(\mathrm{PBord}^{k-(n-d)}_n)$$

is $(k-1)$–connected and satisfies

$$\Omega^k_\varnothing(\mathscr{L}^k_\varnothing(\mathrm{PBord}^{k-(n-d)}_n)) \simeq \Omega^k_\varnothing(\mathrm{PBord}^{k-(n-d)}_n) \simeq \mathrm{PBord}^{-(n-d)}_n$$

by the above theorem. Its k–hybrid completion thus is a k–monoidal complete $(d+k)$–fold Segal space. The collection thereof, for $k \geq 0$, endows the complete d–fold Segal space $\mathrm{Bord}^{(\infty,d)}_n = \mathrm{Bord}^l_n$ for $d = n + l$ with a symmetric monoidal structure. For $d = n$, we obtain the symmetric monoidal structure on the complete n–fold Segal space Bord_n.

Since $\Omega^{n-d}(\mathrm{Bord}_n)$ is complete, the universal property for the completion $\mathrm{Bord}^{(\infty,d)}_n$ of PBord^{d-n}_n applied to the map $\mathrm{PBord}^{d-n}_n \simeq \Omega^{n-d}(\mathrm{PBord}_n) \to \Omega^{n-d}(\mathrm{Bord}_n)$ gives the following corollary.

Corollary 7.7 *There is a morphism of symmetric monoidal (∞, d)–categories*

$$\mathrm{Bord}^{(\infty,d)}_n \to \Omega^{n-d}(\mathrm{Bord}_n).$$

Remark 7.8 Since completion and looping do not necessarily commute, this map is not necessarily an equivalence.

Proof of Theorem 7.4 It is enough to show that

$$\Omega_\varnothing(\mathrm{PBord}^{l,\mathbb{R}^{r-1}}_n) = \mathrm{Hom}_{\mathrm{PBord}^{l,\mathbb{R}^{r-1}}_n}(\varnothing, \varnothing) \simeq \mathrm{PBord}^{l-1,\mathbb{R}^r}_n.$$

The statement for general k follows by induction.

We define a map

$$u_r \colon \Omega(\mathrm{PBord}^{l,\mathbb{R}^{r-1}}_n) \to \mathrm{PBord}^{l-1,\mathbb{R}^r}_n$$

by sending an element in $\mathrm{Hom}_{\mathrm{PBord}^{l,\mathbb{R}^{r-1}}_n}(\varnothing, \varnothing)_{k_2,\dots,k_{n+l}}$,

$$(M_l) = \left(M \subseteq \mathbb{R}^{r-1} \times (a^1_0, b^1_1) \times B(\underline{\bar{I}}), I^1_0 \leq I^1_1, \underline{\bar{I}} = (I^i_0 \leq \cdots \leq I^i_{k_i})^{n+l}_{i=2} \right)$$

$$\in (\mathrm{PBord}^l_n)_{1,k_2,\dots,k_{n+l}},$$

to

$$(M_{l-1}) = \big(M \subseteq (\underbrace{\mathbb{R}^{r-1} \times (a_0^1, b_1^1)}_{=:\widetilde{V}}) \times B(\bar{I}), \bar{I} = (I_0^i \leq \cdots \leq I_{k_i}^i)_{i=2}^{n+l}\big),$$

so it "forgets" the first specified intervals. In the above, we view $\widetilde{V} = \mathbb{R}^{r-1} \times (a_0^1, b_1^1)$ as a vector space using a diffeomorphism $(a_0^1, b_1^1) \cong \mathbb{R}$, and thus get an isomorphism $\widetilde{V} \cong \mathbb{R}^r$.

First of all, we need to check that this map is well defined, that is, that $(M_{l-1}) \in (\mathrm{PBord}_n^{l-1, \mathbb{R}^r})_{k_2, \ldots, k_{n+l}}$. The condition we need to check is the second one, ie we need to check that $M \subset \widetilde{V} \times B(\bar{I})$ is a bounded submanifold and $M \hookrightarrow \widetilde{V} \times B(\bar{I}) \twoheadrightarrow B(\bar{I})$ is proper. Since $p_1^{-1}(I_0^1) = p_1^{-1}(I_1^1) = \varnothing$, we know that M is bounded in the direction of the first coordinate, since $M = p_1^{-1}([b_0^1, a_1^1])$, and moreover, we know that $M \to B(I_0^1 \leq I_1^1, \bar{I})$ is proper. Together this implies the statements.

We claim that the homotopy fibers of this map are contractible. The homotopy fiber at a point

$$(M_1) = \big(M_1 \subseteq \mathbb{R}^r \times B(\bar{I}), \bar{I} = (I_0^i \leq \cdots \leq I_{k_i}^i)_{i=2}^{n+l}\big)$$

in the target consists of pairs of:

- A 1–simplex (M) in $(\mathrm{PBord}_n^{l-1, \mathbb{R}^r})_{k_2, \ldots, k_{n+l}}$ with endpoint (M_1).
- For the source vertex (M_0), a choice of pair of intervals which "bound" the manifold in the last coordinate in $\mathbb{R}^r$, ie an element $I_0^1 \leq I_1^1$ in Int_1 such that

$$\big(M_0 \subseteq \mathbb{R}^r \times B(\bar{I}(0)), I_0^1 \leq I_1^1, \bar{I}(0) = (I_0^i(0) \leq \cdots \leq I_{k_i}^i(0))_{i=2}^{n+l}\big)$$
$$\in \Omega\,(\mathrm{PBord}_n^{l, \mathbb{R}^{r-1}})_{k_2, \ldots, k_{n+l}}.$$

In the presence of the 1–simplex, the choice of the intervals is equivalent to the choice of such intervals for the original element (M_1), since they can be transported back and forth along the 1–simplex. Thus, the homotopy fiber is a product of the space of paths to the chosen (M_1) (which is contractible) with the space of choices of pairs of intervals as in the second item, but for the original (M_1).

We claim that this latter space is contractible as well: The fixed submanifold $M_1 \subseteq \mathbb{R}^r \times B(\bar{I})$ is a bounded and closed submanifold. Therefore, it is bounded in the $\mathbb{R}^r$–direction. Therefore, the image of the projection $p_{\{r\}} \colon M_1 \to \mathbb{R}^{\{r\}}$ is bounded and the intervals can be chosen to be any intervals which lie of either side of the convex hull of this image. The complement of the convex hull has two connected components

and each interval can be chosen arbitrarily in one of the connected components. The space of subintervals of a given interval is contractible, and therefore the homotopy fibers are contractible.   □

7.3 Comparison of the symmetric monoidal structures

Starting with the Γ–object $\mathrm{PBord}_n\langle-\rangle\colon \Gamma \to \mathcal{S}e\mathcal{S}p_n$ we can extract, as explained in Section 3.3, the lth layer of a tower for this symmetric monoidal structure on PBord_n. It is the $(n+l)$–fold Segal space $\mathrm{PBord}_n\langle-\rangle_{\leq l}$ given by precomposing $\mathrm{PBord}_n\langle-\rangle$ with the map $(\Delta^{\mathrm{op}})^l \xrightarrow{f^l} \Gamma^l \xrightarrow{\wedge} \Gamma$. We now show that it is equivalent to the layers of the tower constructed in the previous section.

Proposition 7.9 *The $(n+l)$–fold Segal spaces*

$$\mathrm{PBord}_n\langle-\rangle_{\leq l} \quad \text{and} \quad \mathcal{L}^l_{\varnothing}(\mathrm{PBord}^l_n)$$

are weakly equivalent.

Proof Recall from Section 6.2.2 the variant of bordisms which are constrained to the box $(0,1)^n$. The rescaling maps determine a weak equivalence

$$\mathrm{PBord}^l_n \xrightarrow{\rho} \mathrm{PBord}^{l,(0,1)}_n.$$

Similarly, rescaling determines a weak equivalence

$$\mathrm{PBord}_n\langle-\rangle \xrightarrow{\rho} \mathrm{PBord}^{(0,1)}_n\langle-\rangle$$

to a Γ–object in n–fold Segal spaces, where $\mathrm{PBord}^{(0,1)}_n\langle m\rangle$ is the obvious modification constraining the bordisms to the box $(0,1)^n$ and using rescaling maps as in Section 6.2.2. We will construct a weak equivalence between $\mathrm{PBord}^{(0,1)}_n\langle-\rangle_{\leq l}$ and $\mathcal{L}^l_{\varnothing}(\mathrm{PBord}^{l,(0,1)}_n)$ as the colimit of a zigzag of maps below. We need this intermediate step of constraining bordisms to the box to ensure that the maps in this zigzag are indeed functorial.

To shorten notation, for an ascending sequence $V_0 \subset V_1 \subset V_2 \subset \cdots$ of r–dimensional vector spaces V_r, let

$$Y_r = \mathrm{PBord}^{(0,1),V_r}_n\langle-\rangle_{\leq l} \quad \text{and} \quad X_r = \mathcal{L}^l_{\varnothing}(\mathrm{PBord}^{l,(0,1),V_r}_n).$$

We will work with $V_r = \mathbb{R}^r$ and use the standard inclusions $\mathbb{R}^r \cong \mathbb{R}^r \oplus \{0\} \hookrightarrow \mathbb{R}^{r+1}$. They induce levelwise inclusions $\iota_X\colon X_r \hookrightarrow X_{r+1}$ and $\iota_Y\colon Y_r \hookrightarrow Y_{r+1}$.

We will construct a sequence of maps

such that $f_r \circ g_r \sim \iota_Y$ and $g_{r+l} \circ f_r \sim \iota_X$. This will induce the weak equivalence on the homotopy colimits.

The first map f_r: forgetting certain intervals For $k_1, \ldots, k_n, j_1, \ldots, j_l \geq 0$, consider a general element in $(X_{r+l})_{k_1, \ldots, k_n, j_1, \ldots, j_l}$ given by

$$(M \subset V_r \times B(\bar{I}) \times B(\bar{J}), \bar{I}, \bar{J}),$$

where $\bar{I} \in \mathrm{Int}_{k_1, \ldots, k_n}^{(0,1), n}$ and $\bar{J} \in \mathrm{Int}_{j_1, \ldots, j_l}^{(0,1), l}$. Note that $B(\bar{I}) \times B(\bar{J}) = (0, 1)^n \times (0, 1)^l \subset \mathbb{R}^n \times \mathbb{R}^l$. The condition that the preimage of the projection map over the intervals in $\bar{J}$ be empty implies that M is the disjoint union of the preimages

$$M_{m_1, \ldots, m_l} = D(m_1, \ldots, m_l)(M)$$

for $1 \leq i \leq l$ and $1 \leq m_i \leq j_i$. In brief, the image under f_r will be same embedding, but tracking these disjoint manifolds, together with just the intervals $\bar{I}$.

To implement this, we order the $M_{m_1, \ldots, m_l}$ lexiographically. Finally, we relabel them from 1 to $j_1 \cdots \cdot j_l$ according to this ordering, which amounts to setting $M_s = M_{m_1, \ldots, m_l}$ for $s = m_1 + \sum_{i=2}^{l} (m_i - 1) \cdot j_1 \cdots j_{i-1}$.

The image under f_r has to be an element in

$$(Y_{r+l})_{k_1, \ldots, k_n, j_1, \ldots, j_l} = (\mathrm{PBord}_n^{V_{r+l}} \langle j_1 \cdots j_l \rangle)_{k_1, \ldots, k_n}.$$

For this, we use a fixed diffeomorphism $(0, 1) \cong \mathbb{R}$, which induces a diffeomorphism $B(\bar{J}) \cong V_l$ to transfer the vector space structure. This in turn induces an isomorphism $V_{r+l} \cong V_r \times B(\bar{J})$. Then we let the image of the general element above be

$$(M_1, \ldots, M_{j_1 \cdots j_l}, \bar{I}), \quad \text{where } M_1 \amalg \cdots \amalg M_{j_1 \cdots j_l} \subset V_{r+l} \times B(\bar{I}).$$

We remark that this assignment is functorial in $k_1, \ldots, k_n, j_1, \ldots, j_l$, as the diffeomorphism is fixed once and for all and the map just forgets certain intervals and orders the manifolds in the disjoint union in a functorial way.

The second map g_r: adding certain intervals back in Conversely, to define the map $g_r \colon Y_r \to X_r$, start with a general element in $(Y_r)_{k_1, \ldots, k_n, j_1, \ldots, j_l}$, given by

$$(M_1, \ldots, M_{j_1 \cdots j_l}, \bar{I}), \quad \text{where } M_1 \amalg \cdots \amalg M_{j_1 \cdots j_l} \subset V_r \times B(\bar{I}).$$

Set

$$M = M_1 \amalg \cdots \amalg M_{j_1 \cdots j_l},$$

and for $1 \leq i \leq l$ and $0 \leq m_i \leq j_i$, set

$$J^i_{m_i} = \left[\frac{2m_i}{2j_i + 1}, \frac{2m_i + 1}{2j_i + 1} \right] \cap (0, 1),$$

so the intervals are equidistributed among $(0, 1)$. Now we realize M as a submanifold of $V_r \times B(\bar{I}) \times B(\bar{J}) = V_r \times (0, 1)^n \times (0, 1)^l$ in the following way:

For $1 \leq s \leq j_1 \cdots j_l$ find the relabeling in the converse direction: find $m_1, \ldots, m_l$ such that $s = m_1 + \sum_{i=2}^l (m_i - 1) \cdot j_1 \cdots j_{i-1}$. For example, $m_l = \lfloor s/(j_1 \cdots j_{l-1}) \rfloor$, etc. Then, we use the inclusion of midpoints of the m_i^{th} interval

$$\left\{ \left(\frac{2m_1 + \frac{1}{2}}{2j_1 + 1}, \ldots, \frac{2m_l + \frac{1}{2}}{2j_l + 1} \right) \right\} \subset B(\bar{J})$$

and set

$$M_s \cong M_s \times \left\{ \left(\frac{2m_1 + \frac{1}{2}}{2j_1 + 1}, \ldots, \frac{2m_l + \frac{1}{2}}{2j_l + 1} \right) \right\} \subset V_r \times B(\bar{I}) \times \left\{ \left(\frac{2m_1 + \frac{1}{2}}{2j_1 + 1}, \ldots, \frac{2m_l + \frac{1}{2}}{2j_l + 1} \right) \right\}$$
$$\subset V_r \times B(\bar{I}) \times B(\bar{J}).$$

The images of the M_s by construction will be disjoint, so together we get

$$M \subset V_r \times (0, 1)^n \times (0, 1)^l = V_r \times B(\bar{I}) \times B(\bar{J}).$$

Together with the intervals $\bar{I}$ and $\bar{J}$, this gives an element in X_r.

Again, it is straightforward to check functoriality: the only choice was that of the extra intervals $\bar{J}$, and this choice was canonical because we chose them to be equidistributed in $(0, 1)$. Moreover, they are sent to each other under the face and degeneracy maps of $\text{Int}^{(0,1)}$ because of the extra rescaling step. This is the reason for constraining the bordisms to the box in this proof.

The homotopies It is easy to see that $f_r \circ g_r \sim \iota_Y$: For an element in the composition, the manifolds in the disjoint union are "spread out" over different points in $(0, 1)^l = B(\bar{J}) \subset \mathbb{R}^l$:

$$(M_1, \ldots, M_{j_1 \cdots j_l}, \bar{I}), \quad \text{where } M_1 \amalg \cdots \amalg M_{j_1 \cdots j_l} \subset V_r \times B(\bar{I}),$$

maps to

$$(M, \bar{I}, \bar{J}), \quad \text{where } M = M_1 \amalg \cdots \amalg M_{j_1 \cdots j_l} \subset V_r \times (0, 1)^n \times (0, 1)^l = V_r \times B(\bar{I}) \times B(\bar{J}),$$

which in turn maps to

$$(M_1, \ldots, M_{j_1 \cdots j_l}, \bar{I}), \quad \text{where } M_1 \amalg \cdots \amalg M_{j_1 \cdots j_l} \subset V_{r+l} \times B(\bar{I}).$$

However, for the latter element, consider the homotopy

$$H \colon [0, 1] \times \mathbb{R}^{r+l} \to \mathbb{R}^{r+l}, \quad (t, x_1, \ldots, x_r, y_1, \ldots, y_l) \mapsto (x_1, \ldots, x_r, t \cdot y_1, \ldots, t \cdot y_l).$$

For any t,

$$((H_t \times \mathrm{id}_{(0,1)^n})(M) \subset \mathbb{R}^{r+l} \times (0, 1)^n = \mathbb{R}^{r+l} \times B(\bar{I}), \bar{I})$$

still is an element in $(Y_{r+l})_{k_1, \ldots, k_n, j_1, \ldots, j_l}$, and for $t = 0$ we get back the element we started with.

A homotopy from the other composition $g_{r+l} \circ f_r$ to ι_X is also straightforward to construct. $\qquad \square$

Corollary 7.10 *The two symmetric monoidal structures on* Bord_n *constructed in the last two sections are equivalent.*

Remark 7.11 It is straightforward to get a similar result for $\mathrm{Bord}_n^{(\infty, d)}$.

8 Interpretation of bordisms as manifolds with corners and the homotopy category

In this section we compare our definitions to the definitions of higher bordisms from [34; 33; 49; 12], which are certain manifolds with corners. We first recall the definition and show that every bordism in that sense leads to an element in our space of bordisms from the previous section and vice versa. Then we prove that every path in our space of bordisms leads to diffeomorphic bordisms as manifolds with corners, and explain that the spaces indeed are the disjoint union of classifying spaces of diffeomorphisms of bordisms, as suggested in Section 5.2. Finally, we show that the homotopy category of our symmetric monoidal Segal space of bordisms recovers the usual bordism category.

## 8.1 Bordisms as manifolds with corners and embeddings thereof

For the definition and notation for $\langle k \rangle$–manifolds used in this section we refer to [34]. In brief, a $\langle k \rangle$–manifold is a manifold M with "faces"[21] together with an ordered n–tuple $(\partial_1 M, \ldots, \partial_k M)$ satisfying

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[21]For an n–dimensional manifold M with corners, for any $x \in M$ the number of zeros $c(x)$ in the coordinates of x in any chart is independent of the chart (by a chart we mean a diffeomorphism

(1) $\partial_1 M \cup \cdots \cup \partial_k M = \partial M$, and

(2) $\partial_i M \cap \partial_j M$ is a face of $\partial_i M$ and $\partial_j M$ for every $i \neq j$.

The number k indicates that the manifold has corners of codimension k, which are exactly the components of $\partial_1 M \cap \cdots \cap \partial_k M$.

Example 8.1 Consider the biangle as a manifold with corners:
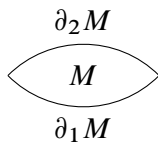
It is a manifold with faces: every point x in the interior has $c(x) = 0$, every point in $\partial_1 M \cap \partial_2 M$ has $c(x) = 2$ and is a face of both $\partial_1 M$ and $\partial_2 M$, and every other point has $c(x) = 1$ and lies either in $\partial_1 M$ or in $\partial_2 M$. Moreover, the ordered pair $(\partial_1 M, \partial_2 M)$ gives M the structure of a $\langle 2 \rangle$–manifold.

An k–bordism is a $\langle k \rangle$–manifold such that for each $\partial_i M$ we distinguish between an "incoming" and an "outgoing" part. We will see later that we can think of it as having k "directions" in which there is an "incoming" and an "outgoing" part of the boundary.

Definition 8.2 • A *(cubical) 0–bordism* is a closed manifold.

• An n–dimensional *cubical k–bordism* is an n–dimensional $\langle k \rangle$–manifold whose k–tuple of faces is denoted by $(\partial_1 M, \ldots, \partial_k M)$ together with decompositions

$$\partial_i M = \partial_{i,\mathrm{in}} M \sqcup \partial_{i,\mathrm{out}} M$$

such that $\partial_{i,\mathrm{in}} M$ and $\partial_{i,\mathrm{out}} M$ are $(n-1)$–dimensional cubical $(k-1)$–bordisms.

• An n–dimensional k–bordism is an n–dimensional cubical k–bordism such that $\partial_{i,\mathrm{in}} M$ and $\partial_{i,\mathrm{out}} M$ are trivial in the sense that there are $(n-k-1+i)$–dimensional $(i-1)$–bordisms $M_{i,\mathrm{in}}$ and $M_{i,\mathrm{out}}$ such that there are diffeomorphisms

$$\partial_{i,\mathrm{in}} M \cong M_{i,\mathrm{in}} \times [0,1]^{k-i} \quad \text{and} \quad \partial_{i,\mathrm{out}} M \cong M_{i,\mathrm{out}} \times [0,1]^{k-i}$$

for $1 \le i \le n-1$.

$x \ni U \to V \subset \mathbb{R}^n_{\ge 0}$). A connected face of M is the closure of a component of $\{x \in M : c(x) = 1\}$, ie of the $(n-1)$–dimensional part of the boundary. An n–dimensional manifold with faces is an n–dimensional manifold M with corners such that any $x \in M$ is in exactly $c(x)$ faces, ie if x is in the $(n-k)$–dimensional part of the boundary, then it lies in k different faces.

Remark 8.3 For $k = 2$ our definition of 2–bordism agrees with that in [49]. One should think of $M_{i,\text{in}}$ and $M_{i,\text{out}}$ as the i–source and i–target of M.

Example 8.4 An example of a 2–dimensional 2–bordism is illustrated in the following picture:
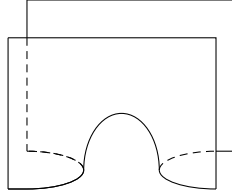
Its tuple $(\partial_1 M, \partial_2 M)$ of faces is given by the vertical and the horizontal faces, respectively.

Example 8.5 Let $M = [0, 1]^k$. It is a k–bordism with

$$\partial_{i,\text{in}} M = [0, 1]^{i-1} \times \{0\} \times [0, 1]^{k-i} \quad \text{and} \quad \partial_{i,\text{out}} M = [0, 1]^{i-1} \times \{1\} \times [0, 1]^{k-i}.$$

A $\langle k \rangle$–manifold M determines a functor

$$M \colon [1]^k \to \mathcal{T}op,$$

$$a = (a_1, \ldots, a_k) \mapsto \begin{cases} M(a) = \bigcap_{\{i : a_i = 0\}} \partial_i M & \text{if } a \neq (1, \ldots, 1), \\ M(1, \ldots, 1) = M. \end{cases}$$

Recall the following embedding theorem via "neat" embeddings for $\langle k \rangle$–manifolds.

Theorem 8.6 [34, Proposition 2.1.7] *Any compact* $\langle k \rangle$–*manifold admits a neat embedding in* $\mathbb{R}^k_+ \times \mathbb{R}^m$ *for some* m, *ie a natural transformation* $\iota \colon M \to \mathbb{R}^k_+ \times \mathbb{R}^m$ *such that*

(1) *$\iota(a)$ is an inclusion of a submanifold for all $a \in [1]^k$;*

(2) *the intersections $M(a) \cap (\mathbb{R}^m \times \mathbb{R}^k_+(b)) = M(b)$ are perpendicular for all $b < a$.*

We adapt the definition of a neat embedding for $\langle k \rangle$–manifolds for bordisms.

Definition 8.7 A *neat embedding* ι of a (cubical) k–bordism M is a natural transformation of M to $\mathbb{R}^m \times [0, 1]^k$ for some m, both viewed as functors $[1]^k \to \mathcal{T}op$, such that

(1) *$\iota(a)$ is an inclusion of a submanifold for all $a \in [1]^k$ respecting the prescribed decomposition of the faces of the bordism;*

(2) *the intersections $M(a) \cap (\mathbb{R}^m \times [0, 1]^k(b)) = M(b)$ are transverse for all $b < a$.*

To prove that any k–bordism admits a neat embedding, we use that any $\langle k \rangle$–manifold admits a compatible collaring:

Lemma 8.8 [34, Lemma 2.1.6] *For $a \in [1]^k$ we write $1 - a = (1, \ldots, 1) - a$. Any $\langle k \rangle$–manifold M admits a $\langle k \rangle$–diagram C of embeddings*

$$C(a < b) \colon \mathbb{R}_+^k (1 - a) \times M(a) \hookrightarrow \mathbb{R}_+^k (1 - b) \times M(b)$$

with the property that $C(a < b)$ restricted to $\mathbb{R}_+^k (1 - b) \times M(a)$ is the inclusion map $\mathrm{id} \times M(a < b)$.

Now the embedding theorem for $\langle k \rangle$–manifolds leads to an embedding theorem for (cubical) k–bordisms. Such embedded cubical k–bordisms appear in [12].

Theorem 8.9 *Any n–dimensional (cubical) k–bordism M admits a neat embedding into $\mathbb{R}^m \times [0, 1]^k$.*

Proof Let M be an n–dimensional k–bordism. By the above theorem, there is a (neat) embedding $\iota \colon M \hookrightarrow \mathbb{R}_+^k \times \mathbb{R}^{m'} \subset \mathbb{R}^{k+m'} = \mathbb{R}^m$ for some m' and $m = m' + k$. We will use that the product of an embedding with any smooth map still is an embedding. For this, we will construct a smooth map $h \colon M \to [0, 1]^k$ such that its product with ι is a neat embedding.

The idea for h is that the decomposition into disjoint unions of the boundary components of the k–bordism determine a decomposition of the collars as well. Starting with the lowest-dimensional corners $M(0)$, we use this decomposition to define h on each component using either the collar coordinate or one minus the collar coordinate in each coordinate direction. Then we proceed by induction on the dimension of the corner and define h successively on $M(a)$.

Recall from Example 8.5 that $[0, 1]^k$ is a k–bordism. We fix a collaring, eg the one given by the embeddings determined by diffeomorphisms

$$\mathbb{R}_+^\alpha \times [0, 1]^{k-\alpha} \cong \left[0, \tfrac{1}{6}\right)^\alpha \times [0, 1]^{k-\alpha} :$$
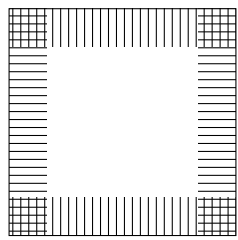
Let $a = (a_i) \in [1]^k$. Write $|a| = \sum a_i$ and $S(a) = \{i : a_i = 0\} \subset \{1, \ldots, k\}$. Note that $|S(a)| = k - |a|$.

By the above lemma, there is a collaring of the $\langle k \rangle$–manifold M. The collaring gives an embedding $C(a < 1)\colon \mathbb{R}_+^{S(a)} \times M(a) \hookrightarrow M(1-0) = M$ whose image is a neighborhood $U(a)$ of the corner $M(a)$. The decompositions $\partial_i M = \partial_{i,\mathrm{in}} M \amalg \partial_{i,\mathrm{out}} M$ give a decomposition of $M(a)$ into $2^{|S(a)|}$ disjoint components:

$$M(a) = \bigcap_{\{i : a_i \neq 0\}} \partial_i M = \bigcap_{\{i : a_i \neq 0\}} \partial_{i,\mathrm{in}} M \amalg \partial_{i,\mathrm{out}} M = \bigsqcup_{\alpha \in [1]^{|S(a)|}} M(a, \alpha),$$

ie an element $c \in M(a)$ lies in $M(a, \alpha)$ if and only if

$$\alpha_i = \begin{cases} 0 & \text{if } c \in \partial_{i,\mathrm{in}} M, \\ 1 & \text{if } c \in \partial_{i,\mathrm{out}} M. \end{cases}$$

This decomposition also determines a decomposition of $U(a)$ into $2^{k-|a|}$ disjoint components $U(a, \alpha)$ for $\alpha \in [1]^{S(a)}$ such that $U(a, \alpha)$ is the image of $\mathbb{R}_+^{S(a)} \times M(a, \alpha)$ under $C(a < 1)$. The chosen collaring of $[0, 1]^k$ induces one on $[0, 1]^{S(a)}$, which in turn determines an embedding $\iota_\alpha\colon \mathbb{R}_+^{S(a)} \hookrightarrow [0, 1]^{S(a)}$ for any particular corner $\alpha \in [1]^{S(a)} \subset [0, 1]^{S(a)}$. Note that the images of these embeddings for varying corners are disjoint. We define h_a on $U(a, \alpha)$ to be the composition

$$U(a, \alpha) \cong \mathbb{R}_+^{S(a)} \times M(a, \alpha) \xrightarrow{\mathrm{pr}} \mathbb{R}_+^{S(a)} \xrightarrow{\iota_\alpha} [0, 1]^{S(a)}.$$

For $a = 0 \in [1]^k$, ie the lowest-dimensional (ie $(n-k)$–dimensional) corners of M, the function $h_0\colon U(0) \to [0, 1]^k$ is the restriction $h|_{U(0)}$ to $U(0)$ of the desired function h.

For $a > 0$, assume h is already defined on $U(b)$ for $b < a$ with $|b| = |a| - 1$. Fix $\alpha \in [1]^{S(a)}$ and let $\beta \in [1]^{S(b)}$ be such that $\beta_i = \alpha_i$ for every $i \in S(a) \subset S(b)$. Then $U(b, \beta) \subset U(a, \alpha)$. Since $U(b, \beta)$ are disjoint for varying β and the collarings restrict compatibly, we can choose a smooth function

$$h_{\beta, \alpha}\colon U(a, \alpha) \to [0, 1]^{\{1, \ldots, k\} \setminus S(a)}$$

such that the product with h_a agrees with h on $U(b, \beta)$ for all such β. This defines a smooth map $h\colon M \to [0, 1]^k$.

We claim that the product $\iota \times h\colon M \hookrightarrow \mathbb{R}^m \times [0, 1]^k$ is a neat embedding of k–bordisms. The first condition is fulfilled by construction, as h is defined so that $M(a, \alpha)$ is sent to $\alpha(c) \in [0, 1]^{S(a)}$. For the second condition note that by construction, $M(a, \alpha) = h_a^{-1}(\alpha)$, $M(b, \beta) = h_b^{-1}(\beta)$, and $h_b = h_a \times h_{\beta, \alpha}$ on $U(b, \beta)$. But on $U(b, \beta)$ the function

$h_{\beta,\alpha}$ is a projection onto the extra collar coordinate, with $M(b,\beta)$ the preimage of $0 \in \mathbb{R}_+$. Thus the intersection is transversal. $\square$

Conversely:

Proposition 8.10 *For $l \geq -n$ and $d = n + l$, any element in $(\mathrm{PBord}_n^l)_{k_1,\ldots,k_n}$ leads to a $(k_1,\ldots,k_d)$–fold composition of n–dimensional d–bordisms.*

Proof Let $(M \subset V \times B(\bar{I}), \bar{I})$ be an element in $(\mathrm{PBord}_n^l)_{k_1,\ldots,k_d}$. As usual, we use the notation $\pi\colon M \hookrightarrow V \times B(\bar{I}) \twoheadrightarrow B(\bar{I})$. Then for $(1 \leq j_i \leq k_i)_{1 \leq i \leq d}$ define

$$M_{j_1,\ldots,j_d} = \pi^{-1}\left(\prod_i [\tfrac{1}{3}(2a_{j-1}^i + b_{j-1}^i), \tfrac{1}{3}(a_j^i + 2b_j^i)]\right).$$

They are n–dimensional cubical d–bordisms since they are manifolds with corners with a decomposition of the boundary given by the preimages of the faces of the cube, similarly to Example 8.5:

$$\partial_{i_0,\mathrm{in}} M_{j_1,\ldots,j_d} = \pi^{-1}\bigg(\prod_{i < i_0} [\tfrac{1}{3}(2a_{j-1}^i + b_{j-1}^i), \tfrac{1}{3}(a_j^i + 2b_j^i)] \times \{\tfrac{1}{3}(2a_{j-1}^i + b_{j-1}^i)\} \\ \times \prod_{i > i_0} [\tfrac{1}{3}(2a_{j-1}^i + b_{j-1}^i), \tfrac{1}{3}(a_j^i + 2b_j^i)]\bigg)$$

and

$$\partial_{i_0,\mathrm{out}} M_{j_1,\ldots,j_d} = \pi^{-1}\bigg(\prod_{i < i_0} [\tfrac{1}{3}(2a_{j-1}^i + b_{j-1}^i), \tfrac{1}{3}(a_j^i + 2b_j^i)] \times \{\tfrac{1}{3}(a_j^i + 2b_j^i)\} \\ \times \prod_{i > i_0} [\tfrac{1}{3}(2a_{j-1}^i + b_{j-1}^i), \tfrac{1}{3}(a_j^i + 2b_j^i)]\bigg).$$

The triviality condition to be a d–bordism follows from condition (3) in Definition 6.1. Note that we essentially extracted the underlying k–bordisms from Remark 5.3 and Notation 5.17.

Moreover, they are composable along the faces in the sense that $\partial_{i,\mathrm{out}} M_{j_1,\ldots,j_i-1,\ldots,j_d}$ and $\partial_{i,\mathrm{in}} M_{j_1,\ldots,j_d}$ can be glued along their collar to form a new k–bordism given by

$$\pi^{-1}\bigg(\prod_{i' < i} [\tfrac{1}{3}(2a_{j-1}^i + b_{j-1}^i), \tfrac{1}{3}(a_j^i + 2b_j^i)] \times [\tfrac{1}{3}(2a_{j-2}^i + b_{j-2}^i), \tfrac{1}{3}(a_j^i + 2b_j^i)] \\ \times \prod_{i' > i} [\tfrac{1}{3}(2a_{j-1}^i + b_{j-1}^i), \tfrac{1}{3}(a_j^i + 2b_j^i)]\bigg). \quad \square$$

8.2 A time-dependent Morse lemma and spaces of bordisms

We have already seen in Remark 5.3 and Notation 5.17, and in Corollary 5.5 and Proposition 8.10, that the Morse lemma allows the interpretation of an element in $(\mathrm{PBord}_n^l)_{k_1, \ldots, k_n}$ as a composition of $k_1 \cdots k_n$ bordisms. In this section we will see that paths in that space lead to diffeomorphisms of the composed bordisms and remark on why this space should be thought of as the classifying space of these diffeomorphisms.

The following theorem is the classical Morse lemma, as can be found eg in [43].

Theorem 8.11 (Morse lemma) *Let f be a smooth proper real-valued function on a manifold M. Let $a < b$ and suppose that the interval $[a, b]$ contains no critical values of f. Then $M^a = f^{-1}((-\infty, a])$ is diffeomorphic to $M^b = f^{-1}((-\infty, b])$.*

We repeat the proof here since later in this section we will adapt it to our situation.

Proof Choose a metric on M, and consider the vector field

$$V = \frac{\nabla_y f}{|\nabla_y f|^2},$$

where ∇_y is the gradient on M. Since f has no critical value in $[a, b]$, V is defined in $f^{-1}((a - \epsilon, b + \epsilon))$ for suitable ϵ. Choose a smooth function $\widetilde{g} \colon \mathbb{R} \to \mathbb{R}$ which is 1 on $(a - \frac{1}{2}\epsilon, b + \frac{1}{2}\epsilon)$ and compactly supported in $(a - \epsilon, b + \epsilon)$. Lift $\widetilde{g}$ to a function $g \colon M \to \mathbb{R}$ by setting $g(y) = \widetilde{g}(f(y))$. Then

$$\mathcal{V} = g \frac{\nabla_y f}{|\nabla_y f|^2}$$

is a compactly supported vector field on M and hence generates a 1–parameter group of diffeomorphisms

$$\psi_t \colon M \to M.$$

Viewing $f - (a + t)$ as a function on $\mathbb{R} \times M$, $(t, y) \mapsto f(y) - (a + t)$, we find that in $f^{-1}\big((a - \frac{1}{2}\epsilon, b + \frac{1}{2}\epsilon)\big)$,

$$\partial_t (f - (a + t)) = 1 = \frac{\nabla_y f}{|\nabla_y f|^2} \cdot (f - (a + t)) = V \cdot (f - (a + t)),$$

and so the flow preserves the set

$$\{(t, y) : f(y) = a + t\}.$$

Thus, the diffeomorphism ψ_{b-a} restricts to a diffeomorphism

$$\psi_{b-a}|_{M^a} \colon M^a \to M^b. \qquad \square$$

In Lemma 3.1 in [22], Gay, Wehrheim and Woodward prove a time-dependent Morse lemma which shows that a smooth family of composed bordisms in their (ordinary) category of (connected) bordisms gives rise to a diffeomorphism which intertwines with the bordisms. We adapt this lemma to a variant which will be suitable for our situation in the higher categorical setting.

We start by defining some rescaling data to compare bordisms with different families of underlying intervals.
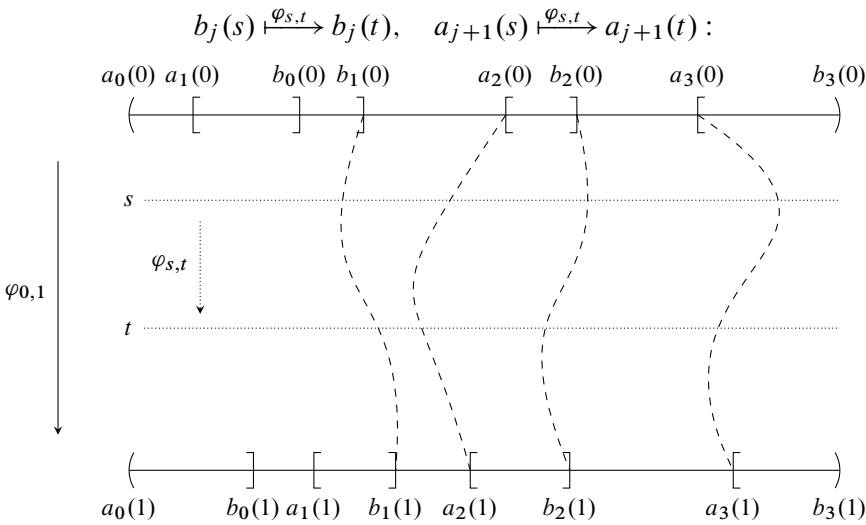
Definition 8.12 Let $(I_0(s) \leq \cdots \leq I_k(s)) \to |\Delta^l|_e$ be an l–simplex in Int_k. A smooth family of strictly monotonically increasing diffeomorphisms

$$\left(\varphi_{s,t} : (a_0(s), b_k(s)) \to (a_0(t), b_k(t))\right)_{s,t \in |\Delta^l|}$$

is said to *intertwine with the composed intervals* if the following condition is satisfied for every morphism $f : [m] \to [l]$ in the simplex category Δ: Let $|f| : |\Delta^m| \to |\Delta^l|$ be the induced map between standard simplices. For every $0 \leq j < k$ such that

- either for every $s \in |f|(|\Delta^m|)$ the intersection $I_j(s) \cap I_{j+1}(s)$ is empty,
- or for every $s \in |f|(|\Delta^m|)$ the intersection $I_j(s) \cap I_{j+1}(s)$ contains only one element,

we require that for every $s \in |f|(|\Delta^m|)$,

$$b_j(s) \xmapsto{\varphi_{s,t}} b_j(t), \quad a_{j+1}(s) \xmapsto{\varphi_{s,t}} a_{j+1}(t) :$$

Remark 8.13 It is enough to check this condition for $m \leq l$.

Definition 8.14 Let $(I_0(s) \leq \cdots \leq I_k(s)) \to [0, 1]$ be a 1–simplex in Int_k. A *rescaling datum for $\underline{I}$* is a is a smooth family of strictly monotonically increasing diffeomorphisms $\varphi_{s,t} \colon (a_0(s), b_k(s)) \to (a_0(t), b_k(t))$ for $s, t \in [0, 1]$ such that

(1) $\varphi_{s,s} = \mathrm{id}$ for every $s \in [0, 1]$,

(2) $\varphi_{s,t} = \varphi_{t,s}^{-1}$ for every $s, t \in [0, 1]$,

(3) $(\varphi_{s,t})_{s,t \in |\Delta^1|_e}$ intertwines with the composed intervals.

Theorem 8.15 Let $(M \subset \mathbb{R}^r \times B(\bar{\underline{I}}) \times |\Delta^1|_e, \bar{\underline{I}})$ be a 1–simplex in $(\mathrm{PBord}_n^l)_{k_1, \dots, k_n}$. Then:

(1) For every $1 \leq i \leq n$, there is a rescaling datum $\varphi_{s,t}^i$ for $\underline{I}^i$.

(2) There is a smooth family of diffeomorphisms

$$(\psi_{s,t} \colon M_s \to M_t)_{s,t \in [0,1]}$$

such that $\psi_{s,s} = \mathrm{id}_{M_s}$ and $\psi_{s,t} = \psi_{t,s}^{-1}$, which **intertwine with the composed bordisms** with respect to the product of the rescaling data

$$\varphi_{s,t} = (\varphi_{s,t}^i)_{i=1}^n \colon B(\bar{\underline{I}}(s)) \to B(\bar{\underline{I}}(t)).$$

By this we mean the following: denoting by π_s the composition

$$M_s \hookrightarrow V \times B(\bar{\underline{I}}(s)) \twoheadrightarrow B(\bar{\underline{I}}(s)),$$

for $1 \leq i \leq n$ and $0 \leq j_i, l_i \leq k_i$ choose any

$$t_{j_i}^i \in I_{j_i}^i(s) \quad \text{such that } \varphi_{s,t}^i(t_{j_i}^i) \in I_{j_i}^i(t),$$
$$t_{l_i}^i \in I_{l_i}^i(s) \quad \text{such that } \varphi_{s,t}^i(t_{l_i}^i) \in I_{l_i}^i(t).$$

Then $\psi_{s,t}$ restricts to a diffeomorphism

$$\pi_s^{-1}\left(\prod_{i=1}^n [t_{j_i}^i, t_{l_i}^i] \right) \xrightarrow{\psi_{s,t}} \pi_s^{-1}\left(\prod_{i=1}^n [\varphi_{s,t}(t_{j_i}^i), \varphi_{s,t}(t_{l_i}^i)] \right),$$

ie writing $B = \prod_{i=1}^n [t_{j_i}^i, t_{l_i}^i]$,

Proof The main strategy of the proof is the same as for the classical Morse lemma. Namely, we will construct a suitable vector field whose flow gives the desired diffeomorphisms. First, we fix the metric on M induced by the restriction of the standard metric on $\mathbb{R}^r \times B(\bar{I}) \times |\Delta^1|_e$. Recall from Remark 5.9 that the map $M \to |\Delta^1|_e$ exhibits M as a trivial fiber bundle, so there is a diffeomorphism $M \cong |\Delta^1|_e \times N$ as an abstract manifold. For every $s \in [0,1]$, the fiber M_s is diffeomorphic to N as an abstract manifold. We fix the metric on N induced by the diffeomorphism $N \cong M_0$, and use the notation $f_s \colon N \cong M_s \hookrightarrow V \times B(\bar{I}(s)) \twoheadrightarrow B(\bar{I}(s))$.

For Steps 1–3 assume that $l = -(n-1)$. The general case applies these arguments in each direction separately.

Step 1: disjoint intervals First assume that for all $0 \leq j \leq k$ and for every $s \in [0,1]$ we have
$$I_j(s) \cap I_{j+1}(s) = \varnothing.$$

We first define suitable vector fields V_j and W_j in neighborhoods of the preimage under f of $\bigcup_{s \in [0,1]}\{s\} \times I_j(s)$ such that their flows will preserve the preimages of the left and right endpoints of the intervals, respectively. Then we use a partition of unity to obtain a vector field $\mathcal{V}$ defined on $[0,1] \times N$ which gives rise to the desired diffeomorphisms.

Let
$$A_j = \bigcup_{s \in [0,1]} \{s\} \times f_s^{-1}(a_j(s)) \subset [0,1] \times N, \quad B_j = \bigcup_{s \in [0,1]} \{s\} \times f_s^{-1}(b_j(s)) \subset [0,1] \times N.$$

Now for $0 \leq j \leq k$ consider the vector fields
$$V_j = \left(\partial_s, \partial_s(a_j(s) - f_s)\frac{\nabla_y f_s}{|\nabla_y f_s|^2}\right), \quad W_j = \left(\partial_s, \partial_s(b_j(s) - f_s)\frac{\nabla_y f_s}{|\nabla_y f_s|^2}\right),$$
where ∇_y is the gradient on N. Since f_s has no critical value in $I_j(s)$, the vector fields V_j and W_j are defined on $f^{-1}(U_j)$, where U_j is a neighborhood of $\bigcup_{s \in [0,1]}\{s\} \times I_j(s)$. Moreover, viewing $a_j \colon (s,y) \mapsto a_j(s)$ as a function on $[0,1] \times N$,
$$V_j(f - a_j) = \partial_s(f - a_j) + \partial_s(a_j - f)\frac{\nabla_y f}{|\nabla_y f|^2}(f - a_j) = \partial_s(f - a_j) + \partial_s(a_j - f) = 0.$$

So the vector field V_j is tangent to A_j and, similarly, W_j is tangent to B_j.

We would now like to construct a vector field $\mathcal{V}$ on $[0,1] \times N$ which, for every $0 \leq j \leq k$, at A_j restricts to V_j and at B_j restricts to W_j, and such that there exists a family of functions $(c_x \colon [0,1] \to \mathbb{R})_{x \in I_j(0)}$ such that

- $c_x(0) = x$, $c_x(s) \in I_j(s)$,
- the graphs of c_x for varying x partition $\bigcup_{s \in [0,1]} \{s\} \times [a_j(s), b_j(s)]$, and
- $\mathcal{V}$ is tangent to $C_x = \bigcup_{s \in [0,1]} \{s\} \times f_s^{-1}(c_x(s))$.

We will use c_x to define $\varphi_{0,s}(x) = c_x(s)$ and $\varphi_{s,t} = \varphi_{0,t} \circ \varphi_{0,s}^{-1}$. Moreover, the diffeomorphisms $\psi_{s,t}$ will arise as the flow along $\mathcal{V}$.

Fix smooth functions $\widetilde{g}_j, \widetilde{h}_j \colon \bigcup_{s \in [0,1]} \{s\} \times B(\underline{I}(s)) \to \mathbb{R}_{\geq 0}$ which satisfy the following conditions:

(1) $\widetilde{g}_j$ and $\widetilde{h}_j$ are compactly supported in U_j.

(2) $\widetilde{g}_j = 1$ in a neighborhood of graph $a_j = \{(s, a_j(s)) : s \in [0, 1]\}$ and $\widetilde{h}_j = 1$ in a neighborhood of graph b_j.

(3) $\widetilde{g}_j + \widetilde{h}_j = 1$ in $\bigcup_{s \in [0,1]} \{s\} \times I_j(s)$, and the supports of the $\widetilde{g}_j + \widetilde{h}_j$ are disjoint.

Lift the functions $\widetilde{g}_j$ and $\widetilde{h}_j$ to functions $g_j, h_j \colon [0, 1] \times N \to \mathbb{R}$ by setting $g_j(s, y) := \widetilde{g}_j(s, f_s(y))$ and $h_j(s, y) := \widetilde{h}_j(s, f_s(y))$. Then consider the vector field on $f^{-1}(U_j)$

$$\mathcal{V}_j = \left(\partial_s, \left(g_j \partial_s(a_j) + h_j \partial_s(b_j) - \partial_s(f) \right) \frac{\nabla_y f}{|\nabla_y f|^2} \right).$$

This vector field is supported on the support of $g_j + h_j$ and thus extends to a vector field on N. Note that $\mathcal{V}_j(s, y) = V_j(s, y)$ for $(s, y) \in A_j$, and $\mathcal{V}_j(s, y) = W_j(s, y)$ for $(s, y) \in B_j$.

Now let $\mathcal{V}$ be the vector field on $[0, 1] \times N$ constructed by combining the above vector fields as follows:

(7) $$\mathcal{V} = \left(\partial_s, \sum_{0 \leq j \leq k} (g_j \partial_s(a_j) + h_j \partial_s(b_j) - \partial_s(f)) \frac{\nabla_y f_s}{|\nabla_y f_s|^2} \right).$$

Note that in $\bigcup_{s \in [0,1]} \{s\} \times f_s^{-1}(I_j(s))$, it restricts to $\mathcal{V}_j$.

In order for $\mathcal{V}$ to be tangent to C_x, the functions c_x must satisfy $\mathcal{V}_j(f - c_x) = 0$ at points in C_x. Expanding the left-hand side as

$$\mathcal{V}_j(f - c_x) = \partial_s(f - c_x) + (g_j \partial_s(a_j) + h_j \partial_s(b_j) - \partial_s(f)) \frac{\nabla f}{|\nabla f|^2}(f - c_x)$$

$$= -\partial_s(c_x) + g_j \partial_s(a_j) + h_j \partial_s(b_j)$$

leads to the ordinary differential equation with smooth coefficients on $[0, 1]$,

$$\partial_s(c_x)(s) = g_j(s, c_x(s)) \partial_s(a_j)(s) + h_j(s, c_x(s)) \partial_s(b_j)(s),$$

$$c_x(0) = x.$$

By Picard–Lindelöf, it has a unique, a priori local, solution. To see that it extends to every $s \in [0, 1]$, consider the smooth function $F: [0, 1] \times N \to \bigcup_{s \in [0,1]}\{s\} \times B(\underline{I}(s))$ defined to be π under the diffeomorphism $M \cong [0, 1] \times N$, so $F(s, y) = (s, f(s, y)) = (s, f_s(y))$. Since π is proper, so is F. Moreover, $C_x = F^{-1}(\operatorname{graph} c_x)$. For fixed x, $\operatorname{graph} c_x$ sits inside the support of $\widetilde{g}_j + \widetilde{h}_j$, for some j, and thus is compact in $\bigcup_{s \in [0,1]}\{s\} \times B(\underline{I}(s))$. Therefore, C_x lies in a compact part of $[0, 1] \times N$ and thus the local solution exists for all $s \in [0, 1]$.

We now define our rescaling data essentially by following the curve c_x. Explicitly, let $\varphi_{0,s}: B(\underline{I}(0)) \to B(\underline{I}(s))$ be defined on $[a_j(0), b_j(0)]$ by sending x_0 to $c_{x_0}(s)$. Note that by construction, it sends $a_j(0)$ to $a_j(s)$ and $b_j(0)$ to $b_j(s)$. Since the solution c_x of the ODE varies smoothly with respect to the initial value x this map is a diffeomorphism. So we can define $\varphi_{s,t}: B(\underline{I}(s)) \to B(\underline{I}(t))$ on $[a_j(s), b_j(s)]$ by sending $x_s = c_{x_0}(s)$ to $c_{x_0}(t)$. We extend $\varphi_{s,t}$ to a diffeomorphism in-between these intervals in the following way. Let $\widetilde{\widetilde{g}}_j, \widetilde{\widetilde{h}}_j: [b_j(0), a_{j+1}(0)] \to \mathbb{R}$ be a partition of unity such that $\widetilde{\widetilde{g}}_j$ is strictly decreasing, $\widetilde{\widetilde{g}}_j(b_j(s)) = 1$, and $\widetilde{\widetilde{h}}_j(a_{j+1}(s)) = 1$. Then, for $x_0 \in [b_j(0), a_{j+1}(0)]$ set

$$c_{x_0}(s) = \widetilde{\widetilde{g}}_j(x_0)c_{b_j(0)}(s) + \widetilde{\widetilde{h}}_j(x_0)c_{a_{j+1}(0)}(s) \quad \text{and} \quad \varphi_{s,t}(c_{x_0}(s)) = c_{x_0}(t).$$

As mentioned above, we obtain the diffeomorphisms $\psi_{s,t}$ by flowing along the vector field $\mathcal{V}$. Since $\mathcal{V}$ is tangent to the sets $C_x = \bigcup_{s \in [0,1]}\{s\} \times f_s^{-1}(c_x(s))$ for $x \in I_0(0) \cup \cdots \cup I_k(0)$, the flow preserves C_x, and $\bigcup_{s \in [0,1]}\{s\} \times f_s^{-1}([b_j(s), a_{j+1}(s)])$ in-between. Again, this implies that the flow exists for all $s \in [0, 1]$. It is of the form $\Psi(t - s, (s, y)) = (t, \psi_{s,t}(y))$ for $0 \le s \le t \le 1$, where $(\psi_{s,t})_{s,t \in [0,1]}$ is a family of diffeomorphisms on N. We transport them under the diffeomorphism $M \cong [0, 1] \times N$ to diffeomorphisms $(\psi_{s,t}: M_s \to M_t)_{s,t \in [0,1]}$, which by construction intertwine with the composed bordisms with respect to the rescaling data $\varphi_{s,t}$.

Step 2: common endpoints   Now consider the case that for $0 \le j \le k$ we have that either for every $s \in [0, 1]$, $I_j(s) \cap I_{j+1}(s) = \varnothing$, as in the previous case, or for every $s \in [0, 1]$ we have

$$|I_j(s) \cap I_{j+1}(s)| = 1.$$

In this case, one can modify the above argument. We explain this for the case of two intervals with one common endpoint, ie $b_j(s) = a_{j+1}(s)$.

Instead of choosing smooth functions $\widetilde{g}_j, \widetilde{h}_j, \widetilde{g}_{j+1}, \widetilde{h}_{j+1}: \bigcup_{s \in [0,1]}\{s\} \times B(\underline{I}(s)) \to \mathbb{R}$ such that the supports of $\widetilde{g}_j + \widetilde{h}_j$ and $\widetilde{g}_{j+1} + \widetilde{h}_{j+1}$ are disjoint (which now is not

possible), we fix three smooth functions $\tilde{f}_j, \tilde{g}_j, \tilde{h}_j \colon \bigcup_{s \in [0,1]} \{s\} \times B(\underline{I}(s)) \to \mathbb{R}$ which satisfy the following conditions:

(1) $\tilde{f}_j$, $\tilde{g}_j$ and $\tilde{h}_j$ are compactly supported in $U_j \cup U_{j+1}$.

(2) $\tilde{f}_j = 1$ in a neighborhood of $\operatorname{graph} a_j = \{(s, a_j(s)) : s \in [0,1]\}$, $\tilde{g}_j = 1$ in a neighborhood of $\operatorname{graph} b_j = \operatorname{graph} a_{j+1}$ and $\tilde{h}_j = 1$ in a neighborhood of $\operatorname{graph} b_{j+1}$.

(3) $\tilde{f}_j + \tilde{g}_j + \tilde{h}_j = 1$ in $\bigcup_{s \in [0,1]} \{s\} \times (I_j(s) \cup I_{j+1}(s))$, and the support of the $\tilde{f}_j + \tilde{g}_j + \tilde{h}_j$ is disjoint from the sums associated to the other intervals.

Now continue the proof similarly to above.

Step 3: overlapping intervals with nonempty interior If for some $0 \le j \le k$ the intersection $I_j(s) \cap I_{j+1}(s)$ has nonempty interior for every $s \in [0,1]$, then one can do the above construction with the intervals $I_j(s)$ and $I_{j+1}(s)$ replaced by the interval $I_j(s) \cup I_{j+1}(s)$.

Step 4: partial overlaps (mixed cases) When the above cases are mixed, we can combine the cases treated so far. We will illustrate this in the case when the intervals first are disjoint and then start overlapping. The other cases are treated similarly.

Let us assume that there is an $\tilde{s}$ such that $I_j(s) \cap I_{j+1}(s) = \varnothing$ for $s < \tilde{s}$, and $I_j(s) \cap I_{j+1}(s) \ne \varnothing$ for $s \ge \tilde{s}$. In this case, $\tilde{x} = b_j(\tilde{s}) = a_{j+1}(\tilde{s})$, which is a regular value of $f_{\tilde{s}}$. Since f is smooth, there is an open ball U_j centered at $(\tilde{s}, \tilde{x})$ in $\bigcup_{s \in [0,1]} \{s\} \times B(\underline{I}(s))$ such that for $(s, x) \in U$, x is a regular value of f_s. Let $\bar{s} < \tilde{s}$ be such that for every $\bar{s} \le s \le \tilde{s}$, the set $\{s\} \times [a_j(s), b_{j+1}(s)]$ is covered by $U \cup (\{s\} \times (I_j(s) \cup I_{j+1}(s)))$. Choose s_0 and t_0 such that $\bar{s} \le s_0 < t_0$:
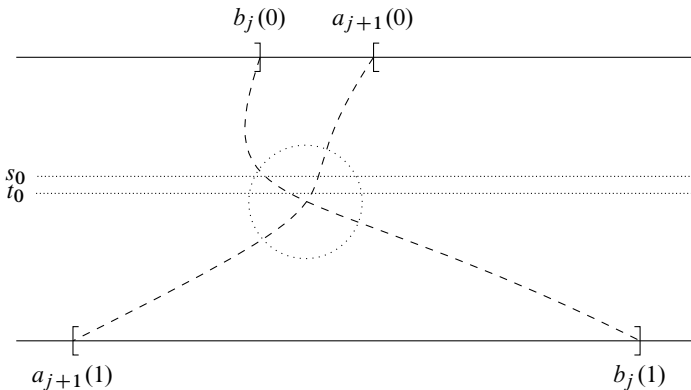
In $[0, t_0]$, we are in the situation of disjoint intervals and can use the first construction to obtain $c_x^{(1)}(s)$ and $\mathcal{V}^{(1)}(s, y)$ for $s \le t_0$.

In $[s_0, 1]$, we apply the construction from Step 1 to the intervals $I_j(s)$ and $I_{j+1}(s)$ replaced by the interval $[a_j(s), b_{j+1}(s)]$ to obtain $c_x^{(2)}(s)$ and $\mathcal{V}^{(2)}(s, y)$ for $s \geq s_0$.

Now choose a partition of unity $G, H: [0, 1] \to \mathbb{R}$ such that $G|_{[0, s_0]} = 1$, $H|_{[t_0, 1]} = 1$, and G is strictly decreasing on $[s_0, t_0]$. For $s < t$ define

$$c_x(s) = G(s)c_x^{(1)}(s) + H(s)c_x^{(2)}(s), \quad \mathcal{V}(s, y) = G(s)\mathcal{V}^{(1)}(s, y) + H(s)\mathcal{V}^{(2)}(s, y).$$

Then define $\varphi_{s,t}$ and $\psi_{s,t}$ as before.

Step 5: several directions Assume now that $l > -(n-1)$. Let

$$\pi_s: N \cong M_s \hookrightarrow V \times B(\bar{\underline{I}}(s)) \twoheadrightarrow B(\bar{\underline{I}}(s))$$

and for $1 \leq i \leq n$ denote by $(p_i)_s: N \to B(\underline{I}^i(s))$ the composition of π_s with the projection to the ith coordinate. Note that by condition (3) in Definition 5.1, the function $(p_i)_s$ does not have a critical point in $I_0^i(s) \cup \cdots \cup I_{k_i}^i(s)$.

By Steps 1–3, for each i we get a vector field of the form

$$\mathcal{V}^i = \left(\partial_s, \Pi_i(s, y) \frac{\nabla_y(p_i)_s}{|\nabla_y(p_i)_s|^2} \right)$$

for a function $\Pi_i(s, y)$; eg see (7). We combine them to obtain a new vector field on $[0, 1] \times N$ given by

$$\widetilde{\mathcal{V}} = \left(\partial_s, \sum_{i=1}^n \Pi_i(s, y) \frac{\nabla_y(p_i)_s}{|\nabla_y(p_i)_s|^2} \right).$$

For $i \neq j$ the projections $(p_i)_0$ and $(p_j)_0$ are orthogonal with respect to the metric on N and, moreover, $(p_i)_s$ and $(p_j)_s$ stay orthogonal along the path, because the change of metric on $N \cong M_s$ induced by the embedding of M_s respects orthogonality on $B(\bar{\underline{I}})$. This implies that

$$\frac{\nabla_y(p_i)_s}{|\nabla_y(p_i)_s|^2} p_j = \delta_{ij},$$

and so $\widetilde{\mathcal{V}}$ still is tangent to the respective C_x^i in each direction and thus its flow, if it exists globally, will give rise to the desired diffeomorphisms and rescaling data.

The global existence follows from the special form of the vector field. Given a point $(t, y_t) \in N$, the flow will preserve a set of the form

$$\{(s, y) : \pi_s(y_s) = (c_{x_0}^1(s), \ldots, c_{x_0}^n(s)) = (\xi_1(s), \ldots, \xi_n(s))\},$$

where the right-hand side is in the notation of Example 5.13, and $\vec{c}_{x_0}(t) = \vec{\xi}(t) = y_t$. One can show, as in the example, that this set lies in a compact part of N and thus the flow exists globally. □

We can now relate the spaces of bordisms to diffeomorphisms of bordisms in a more classical sense.

Definition 8.16 Building upon the previous section, in particular Proposition 8.10, we define a *diffeomorphism of a* $(k_1, \ldots, k_d)$–*fold composition of* n–*dimensional* d–*bordisms* to be a diffeomorphism of the composition which "intertwines with", ie restricts to, the composed bordisms.

The above theorem shows that a path in $(\mathrm{PBord}_n^l)_{k_1, \ldots, k_d}$ leads to such an intertwining diffeomorphism of the compositions at the start and at the end of the path. Actually, much more is true.

Proposition 8.17 *For fixed* $k_1, \ldots, k_d$ *and a* $(k_1, \ldots, k_d)$–*fold composition* M *of* n–*dimensional* d–*bordisms* $M_{j_1, \ldots, j_d}$, *which we denote by* $(M, (M_{j_1, \ldots, j_d}))$, *consider the group of such intertwining diffeomorphisms* $\mathrm{Diff}(M, (M_{j_1, \ldots, j_d}))$. *Then* $(\mathrm{PBord}_n^l)_{k_1, \ldots, k_n}$ *is the disjoint union of classifying spaces of* $\mathrm{Diff}(M, (M_{j_1, \ldots, j_d}))$, *where the disjoint union is taken over diffeomorphism classes.*

Proof We sketch the argument, essentially following the argument for $\mathrm{Sub}(M, \mathbb{R}^\infty)$ being a classifying space for the group of diffeomorphisms of M, and its modifications in [21; 40]: Consider the space $\mathrm{Emb}\big((M, (M_{j_1, \ldots, j_d})), \mathbb{R}^\infty \times [0, 1]^d\big)$ of neat embeddings of the composition which restricts to neat embeddings of the composed bordisms. It is nonempty by the embedding theorem for d–bordisms (Theorem 8.9) and contractible, which can be seen similarly to $\mathrm{Emb}(M, \mathbb{R}^\infty)$ being contractible. We get a principal $\mathrm{Diff}(M, (M_{j_1, \ldots, j_d}))$–bundle

$$\mathrm{Emb}\big((M, (M_{j_1, \ldots, j_d})), \mathbb{R}^\infty \times [0, 1]^d\big)$$
$$\to \mathrm{Emb}\big((M, (M_{j_1, \ldots, j_d})), \mathbb{R}^\infty \times [0, 1]^d\big)/\mathrm{Diff}(M, (M_{j_1, \ldots, j_d})).$$

The disjoint union over all diffeomorphism classes of the right-hand side is equivalent to $(\mathrm{PBord}_n^l)_{k_1, \ldots, k_n}$. □

8.3 The homotopy category $h_1(\mathrm{Bord}_n^{(\infty, 1)})$

The goal of the section is to show that there is an equivalence of symmetric monoidal categories between the homotopy category of the $(\infty, 1)$–category $\mathrm{Bord}_n^{(\infty, 1)}$ and the

usual unoriented bordism category nCob. In fact, even more is true: one can show
that there is an equivalence of symmetric monoidal bicategories between the homotopy
bicategory of the $(\infty, 2)$–category $\mathrm{Bord}_n^{(\infty,2)}$ and the unoriented bordism bicategory
defined in [49]. This will be proven in a subsequent article.

8.3.1 The symmetric monoidal structure on $h_1(\mathrm{Bord}_n^{(\infty,1)})$ The $(\infty, 1)$–category

$$\mathrm{Bord}_n^{(\infty,1)} = \mathrm{Bord}_n^{-(n-1)} \simeq \Omega^{n-1}(\mathrm{Bord}_n)$$

has a symmetric monoidal structure defined in two ways similarly to that of Bord_1.
Both induce a symmetric monoidal structure on the homotopy category $h_1(\mathrm{Bord}_n^{(\infty,1)})$.
We now make this symmetric monoidal structure more explicit for later purposes.

… coming from a Γ–object   We can either obtain the symmetric monoidal structure
as a Γ–object on $\mathrm{Bord}_n^{(\infty,1)} \simeq \Omega^{n-1}(\mathrm{Bord}_n)$ by iterating the construction of the
symmetric monoidal structure on the looping from Example 3.10 or by constructing
a functor from an assignment $[m] \mapsto \mathrm{Bord}_n^{-(n-1)}[m]$ as mentioned in Remark 7.3.
In the second case, $\mathrm{Bord}_n^{-(n-1)}[m]$ arises, similarly to $\mathrm{Bord}_n[m]$, from the spaces
$(\mathrm{PBord}_n^{V,-(n-1)}[m])_{k_1,\ldots,k_n}$, whose set of 0–simplices is the collection of tuples

$$(M_1, \ldots, M_m, (I_0 \le \cdots \le I_k)),$$

where $M_1, \ldots, M_m$ are disjoint n–dimensional submanifolds of $V \times B(\underline{I}) = (a_0, b_k)$,
and each $(M_\beta, (I_0 \le \cdots \le I_k)) \in (\mathrm{PBord}_n^{V,-(n-1)})_{k_1,\ldots,k_n}$.

We saw in Example 3.8 that a Γ–object endows the homotopy category of its underlying
Segal space with a symmetric monoidal structure. Explicitly, in the second case, it
comes from the maps

$$\mathrm{Bord}_n^{-(n-1)}\langle 1\rangle \times \mathrm{Bord}_n^{-(n-1)}\langle 1\rangle \xleftarrow[\simeq]{\gamma_1 \times \gamma_2} \mathrm{Bord}_n^{-(n-1)}\langle 2\rangle \xrightarrow{\gamma} \mathrm{Bord}_n^{-(n-1)}\langle 1\rangle,$$

$$(M_1, \underline{I}), (M_2, \underline{I}) \quad \longleftarrow\!\!\shortmid \quad (M_1, M_2, \underline{I}) \longmapsto (M_1 \amalg M_2, \underline{I}).$$

… coming from a tower   To understand the symmetric monoidal structure on the
category $h_1(\mathrm{Bord}_n^{(\infty,1)})$ coming from a symmetric monoidal structure as a tower, we
use that $\mathrm{Bord}_n^{(\infty,1)} = \mathrm{Bord}_n^{-(n-1)}$ has a symmetric monoidal structure coming from the
collection of k–hybrid $(k+1)$–fold Segal spaces given (essentially) by the k–hybrid
completion of

$$\mathscr{L}_\varnothing^k(\mathrm{PBord}_n^{k-(n-1)}).$$

This symmetric monoidal structure induces one on the homotopy category

$$h_1(\mathrm{Bord}_n^{(\infty,1)}) \simeq h_1(\mathrm{Bord}_n^{-(n-1)}).$$

Since completion is a Dwyer–Kan equivalence — see Definition 1.18 — it is enough to understand the symmetric monoidal structure on $h_1(\mathrm{PBord}_n^{-(n-1)})$.

The monoidal structure arises from composition in $\mathrm{PBord}_n^{1-(n-1)}$, the next layer of the tower $\mathrm{PBord}_n^{2-(n-1)}$ gives a braiding and the higher layers show that it is symmetric monoidal; see Section 3.3. Consider the diagram

$$(\mathrm{PBord}_n^{1-(n-1)})_{1,\bullet} \times^h_{(\mathrm{PBord}_n^{1-(n-1)})_{0,\bullet}} (\mathrm{PBord}_n^{1-(n-1)})_{1,\bullet}$$

$$\xleftarrow[\simeq]{d_0^1 \times d_2^1} (\mathrm{PBord}_n^{1-(n-1)})_{2,\bullet} \xrightarrow{d_1^1} (\mathrm{PBord}_n^{1-(n-1)})_{1,\bullet}.$$

Similarly to Remark 3.16, we find that

$$\mathscr{L}_\varnothing(\mathrm{PBord}_n^{1-(n-1)})_{1,\bullet} \simeq \Omega(\mathrm{PBord}_n^{1-(n-1)})_\bullet \simeq (\mathrm{PBord}_n^{-(n-1)})_\bullet$$

and, together with the maps above, this gives a monoidal structure

$$h_1(\mathrm{PBord}_n^{-(n-1)}) \times h_1(\mathrm{PBord}_n^{-(n-1)}) \to h_1(\mathrm{PBord}_n^{-(n-1)}).$$

We spell this structure out explicitly. Consider two objects or 1–morphisms represented by elements

$$(M) = (M \subseteq V \times B(\underline{I}), \underline{I}), \quad (N) = (N \subseteq W \times B(\widetilde{\underline{I}}), \widetilde{\underline{I}})$$

in $(\mathrm{PBord}_n^{-(n-1)})_k$ for $k = 0$ or $k = 1$. Without loss of generality we can assume that $V = W = \mathbb{R}^r$, that $(M), (N) \in (\mathrm{PBord}_n^{-(n-1),\mathbb{R}^r})_k$, and that (perhaps after rescaling) $\underline{I} = \widetilde{\underline{I}}$. Furthermore, choose $c > 0$ such that $(M), (N) \in (\mathrm{PBord}_n^{-(n-1),B(c,r)})_k$. What follows will be independent of the choice of c.

Under the map $\ell_r(c) \colon \mathrm{PBord}_n^{-(n-1),B(c,r)} \to \Omega(\mathrm{PBord}_n^{1-(n-1),\mathbb{R}^{r-1}})$ from Theorem 7.4, (M) and (N) are sent to

$$(M_1) = \big(M \subseteq \mathbb{R}^{r-1} \times (-(c+1), c+1) \times B(\underline{I}), (-(c+1), -c] \leq [c, c+1), \underline{I}\big),$$

$$(N_1) = \big(N \subseteq \mathbb{R}^{r-1} \times (-(c+1), c+1) \times B(\underline{I}), (-(c+1), -c] \leq [c, c+1), \underline{I}\big).$$

Now we can use the gluing procedure as in the proof of the Segal condition for PBord_n in Proposition 5.19. In this case, the sources and targets of (M_1) and (N_1) are all empty, so the construction of the glued element is as follows: We choose a path from (N_1) to another element (N_2) by moving the first coordinate in the box so that the pair

$((M), (N_2))$ lies in

$$(\mathrm{PBord}_n^{1-(n-1)})_{1,\bullet} \times_{(\mathrm{PBord}_n^{1-(n-1)})_{0,\bullet}} (\mathrm{PBord}_n^{1-(n-1)})_{1,\bullet},$$

ie so that

$$(N_2) = \big(N \subseteq \mathbb{R}^{r-1} \times (c, 3c+2) \times B(\underline{I}), (c, c+1] \leq [3c+1, 3c+2), \underline{I}\big).$$

Since $d_1^1((M_1)) = d_0^1((N_1)) = \varnothing$ we have that M and N_2 are disjoint as submanifolds of $\mathbb{R}^{r-1} \times (-(c+1), 3c+2) \times B(\widetilde{I})$. So the glued element is

$$\big(M \amalg N \subset V_{d-1} \times (-(c+1), 3c+2) \times B(\underline{I}),$$
$$(-(c+1), -c] \leq [c, c+1] \leq [3c+1, 3c+2), \underline{I}\big).$$

The third face map d_1^1 sends it to

$$\big(M \amalg N \subset V_{d-1} \times (-(c+1), 3c+2) \times B(\underline{I}), (-(c+1), -c] \leq [3c+1, 3c+2), \underline{I}\big),$$

which by $u_r \colon \Omega(\mathrm{PBord}_n^{1-(n-1),\mathbb{R}^r}) \to \mathrm{PBord}_n^{-(n-1)}$ is sent to

$$(M \amalg N \subset \mathbb{R}^r \times B(\underline{I}), \underline{I}).$$

8.3.2 The homotopy category and nCob Our higher categories of bordisms give back the ordinary categories of n–bordisms, as we see in the main proposition in this section. First, let us briefly recall the definition of the symmetric monoidal category of bordisms. A good reference for the details and subtleties is eg [32].

Definition 8.18 The symmetric monoidal category of n–dimensional bordisms nCob is defined as follows:

- Objects are closed $(n-1)$–dimensional smooth manifolds.
- A morphism from M to N is a diffeomorphism class of n–dimensional cobordisms from M to N, where an n–*dimensional bordism* from M to N is a smooth manifold Σ with boundary, together with a specified diffeomorphism $\partial\Sigma \cong M \amalg N$.
- Composition of morphisms $\Sigma_1 \colon M_0 \to M_1$ and $\Sigma_2 \colon M_1 \to M_2$ is given by the diffeomorphism class of the gluing $\Sigma_1 \amalg_{M_1} \Sigma_2$.
- The identity morphism on M is the diffeomorphism class of the cylinder $M \times [0, 1]$ viewed as a morphism from M to M.
- The symmetric monoidal structure is given by taking disjoint unions of objects and morphisms.

Remark 8.19 An n–dimensional bordism Σ from M to N is exactly an n–dimensional 1–bordism Σ as in Definition 8.2 with $\partial_{\mathrm{in}}\Sigma = M$ and $\partial_{\mathrm{out}}\Sigma = N$.

Proposition 8.20 *There is an equivalence of symmetric monoidal categories between the homotopy category of the $(\infty, 1)$–category $\mathrm{Bord}_n^{(\infty,1)}$ and the category of n–bordisms*

$$h_1(\mathrm{Bord}_n^{(\infty,1)}) \simeq n\mathrm{Cob}.$$

Proof We first show that there is an equivalence of categories $h_1(\mathrm{Bord}_n^{(\infty,1)}) \simeq n\mathrm{Cob}$ and then show that it respects the symmetric monoidal structures.

Rezk's completion functor is a Dwyer–Kan equivalence of Segal spaces, and thus by definition induces an equivalence of the homotopy categories. So it is enough to show that

$$h_1(\mathrm{PBord}_n^{-(n-1)}) \simeq n\mathrm{Cob}.$$

We define a functor

$$F\colon h_1(\mathrm{PBord}_n^{-(n-1)}) \to n\mathrm{Cob}$$

and show that it is essentially surjective and fully faithful.

Definition of the functor By definition, an object in $h_1(\mathrm{PBord}_n^{-(n-1)})$ is an element $(M) = (M \subset V \times (a,b), I = (a,b)) \in (\mathrm{PBord}_n^{-(n-1)})_0$. Since $\pi\colon M \to (a,b)$ is submersive and proper, in particular $\frac{1}{2}(a+b)$ is a regular value of π and $\pi^{-1}\big(\frac{1}{2}(a+b)\big)$ is a closed $(n-1)$–dimensional manifold. We define

$$F((M)) = \pi^{-1}\big(\tfrac{1}{2}(a + b)\big).$$

A morphism in $h_1(\mathrm{PBord}_n^{-(n-1)})$ is an element in $\pi_0((\mathrm{PBord}_n^{-(n-1)})_1)$, represented by

$$(N) = \big(N \subset V \times (a_0, b_1), I_0 = (a_0, b_1] \leq I_1 = [a_1, b_1)\big) \in (\mathrm{PBord}_n^{-(n-1)})_1.$$

We let F send (N) to the isomorphism class of

$$\bar{N} = \pi^{-1}\big(\big[\tfrac{1}{3}(2a_0 + b_0), \tfrac{1}{3}(a_1 + 2b_1)\big]\big).$$

This is an n–dimensional manifold with boundary

$$\pi^{-1}\big(\tfrac{1}{3}(2a_0 + b_0)\big) \sqcup \pi^{-1}\big(\tfrac{1}{3}(a_1 + 2b_1)\big).$$

Since π only has regular values in I_0 and I_1, the Morse lemma gives diffeomorphisms

$$\pi^{-1}\big(\tfrac{1}{3}(2a_0+b_0)\big) \cong \pi^{-1}\big(\tfrac{1}{2}(a_0+b_0)\big) \quad \text{and} \quad \pi^{-1}\big(\tfrac{1}{3}(a_1+2b_1)\big) \cong \pi^{-1}\big(\tfrac{1}{2}(a_1+b_1)\big).$$

Thus, $F((N))$ is an n–dimensional bordism from the image of the source $F(d_0(N))$ to the image of the target $F(d_1(N))$.

We need to check that this assignment is well defined, ie independent of the choice of the representative of the isomorphism class. Any two representatives (N_0) and (N_1) are connected by a path in $(\mathrm{PBord}_n^{-(n-1)})_1$. From this path we can obtain another one which has "shorter" intervals, namely just by shrinking them to $\left(a_0(s), \frac{1}{3}(2a_0(s)+b_0(s))\right]$ and $\left[\frac{1}{3}(a_1(s) + 2b_1(s)), b_1(s)\right)$. Now Theorem 8.15 gives a diffeomorphism $\psi_{0,1}\colon N_0 \to N_1$ which restricts to a diffeomorphism $\bar\psi_{0,1}\colon \bar N_0 \to \bar N_1$.

The Morse lemma implies that any image of the degeneracy map in $(\mathrm{PBord}_n^{-(n-1)})_1$ is sent to an identity morphism in $n\mathrm{Cob}$ and that F behaves well with composition.

The functor is an equivalence of categories Whitney's embedding theorem shows that F is essentially surjective. Moreover, it is injective on morphisms: Let $\iota_0\colon N_0 \hookrightarrow V \times B(\underline{I})$ and $\iota_1\colon N_1 \hookrightarrow W \times B(\underline{\tilde I})$ be embeddings which are representatives of two 1–morphisms $(N_0 \subset V \times B(\underline{I}), \underline{I})$ and $(N_1 \subset W \times B(\underline{\tilde I}), \underline{\tilde I})$ which have diffeomorphic images. Without loss of generality we can assume that $V = W$ and $\underline{I} = \underline{\tilde I}$. Then there is a diffeomorphism $\psi\colon \bar N_0 \to \bar N_1$, which can be extended to the rest of the collars, ie we get a diffeomorphism $\psi\colon N_0 \to N_1$. Since $\mathrm{Emb}(N_0, \mathbb{R}^\infty \times B(\underline{I}))$ is contractible, there is a path from ι_0 to $\iota_1 \circ \psi$, which induces a 1–simplex $(N \subset V \times B(\underline{I}) \times [0, 1], B(\underline{I}))$ in $(\mathrm{PBord}_n^{-(n-1)})_1$ such that the fiber at $s = 0$ is $(N_0 \subset V \times B(\underline{I}), \underline{I})$ and the fiber at 1 is $(\mathrm{im}(\iota_1 \circ \psi)(N_0) = N_1 \subset V \times B(\underline{I}), \underline{I})$.

It remains to show that F is full. In the case $n = 1, 2$ this is easy to show, as we have a classification theorem for 1– and 2–dimensional manifolds with boundary. In the 1–dimensional case it is enough to show that an open line, the circle and the half-circle — once as a bordism from two points to the empty set and once vice versa — lie in the image of the map, which is straightforward. In the 2–dimensional case, the pair-of-pants decomposition tells us how to embed the manifold.

For general n we need to find a suitable embedding of our bordism. Theorem 8.9 provides one for much more general k–bordisms, but as there is a much simpler argument for $k = 1$, we provide it here.

We first embed the manifold with boundary into $\mathbb{R}^+ \times \mathbb{R}^{2n}$ using Laures' embedding, Theorem 8.6, for manifolds with boundary. Then the boundary of the half-space is $\partial(\mathbb{R}^+ \times \mathbb{R}^{2n}) = \mathbb{R}^{2n}$. We want to transform this embedding into an embedding into $(0, 1) \times \mathbb{R}^{2n}$ so that the incoming boundary is sent into $\{\epsilon\} \times \mathbb{R}^{2n}$ and the outgoing boundary is sent into $\{1-\epsilon\} \times \mathbb{R}^{2n}$.

We first show that the boundary components can be separated by a hyperplane in $\mathbb{R}^{2n}$. The boundary components are compact, so they can be embedded into (large enough) balls B^{2n}. By perhaps first applying a suitable "stretching" transformation, one can assume that these balls do not intersect. Now, since $2n > 1$ we have that the configuration space of these balls $\pi_0(\mathrm{Conf}(B^{2n}, \mathbb{R}^{2n})) \cong *$ is contractible. Hence, there is a transformation to a configuration in which the boundary components are separated by a hyperplane, without loss of generality given by the equation $\{x_1 = 0\} \subset \mathbb{R}^{2n}$.

Consider the restriction of the (holomorphic) logarithm function with branch cut $-i\mathbb{R}^+$ to $(\mathbb{R}^+ \times \mathbb{R}) \setminus \{(0,0)\} \cong \mathbb{H} \setminus 0 \subseteq \mathbb{C}$. It is a homeomorphism to $\{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq \pi\}$. We can apply $\log \times \mathrm{id}_{\mathbb{R}^{2n-1}}$ to $(\mathbb{R}^+ \times \mathbb{R}_{x_1}) \times \mathbb{R}^{2n-1}$ and, composing this with a suitable rescaling, obtain an embedding into $(\epsilon, 1 - \epsilon) \times \mathbb{R}^{2n}$. Now choose a collaring of the bordism to extend the embedding to $(0, 1) \times \mathbb{R}^{2n}$.

The functor is a symmetric monoidal equivalence In the case of the structure coming from a Γ–object, one can, similarly to the previous paragraph, define an equivalence of categories

$$F[m] \colon h_1 \mathrm{Bord}_n^{-(n-1)}[m] \to n\mathrm{Cob}^m .$$

Then one can easily check that the following diagram commutes:

$$
\begin{array}{ccccc}
h_1\mathrm{Bord}_n^{-(n-1)}[1] \times h_1\mathrm{Bord}_n^{-(n-1)}[1] & \xleftarrow{\;\simeq\;} & h_1\mathrm{Bord}_n^{-(n-1)}[2] & \longrightarrow & h_1\mathrm{Bord}_n^{-(n-1)}[1] \\
\downarrow{\scriptstyle F \times F} & & \downarrow{\scriptstyle F[2]} & & \downarrow{\scriptstyle F} \\
n\mathrm{Cob} \times n\mathrm{Cob} & =\!=\!=\!= & n\mathrm{Cob} \times n\mathrm{Cob} & \xrightarrow{\;\amalg\;} & n\mathrm{Cob}
\end{array}
$$

Thus, we have a functor of Γ–objects in categories. Finally, there is an equivalence of categories between Γ–objects in categories and symmetric monoidal categories, which is a direct consequence of Mac Lane's coherence theorem for symmetric monoidal categories [42].

For the case of the structure coming from a tower, we explicitly saw that the symmetric monoidal structure on $h_1(\mathrm{Bord}_n^{(\infty,1)})$ sends two objects or 1–morphisms determined by

$$(M) = (M \subseteq V_d \times B(\underline{I}), \underline{I}), \quad (N) = (N \subseteq V_d \times B(\underline{I}), \underline{I})$$

to

$$(M \amalg N) = (M \amalg N \hookrightarrow V \times B(\underline{I}), \underline{I}),$$

where the images of M and N lie in disjoint "heights" in the v_1–direction in V_d. Thus, under the functor F the element $(M \amalg N)$ is sent to $F((M)) \amalg F((N))$.

Finally, in both cases, any element represented by $(\varnothing, \underline{I})$ is sent to $\varnothing$. $\square$

9 Bordisms with additional structure: orientations and framings

In the study of fully extended topological field theories, one is particularly interested in manifolds with extra structure, especially that of a framing. In this section we explain how to define the (∞, n)–category of structured n–bordisms, in particular for the structure of an orientation or a framing.

9.1 Structured manifolds

We first recall the definition of structured manifolds and the topology on their morphism spaces making them into a topological category. In the next subsection we will see that the smooth singular simplices on these topological spaces essentially will give rise to the spatial structure of the levels of the n–fold Segal space of structured bordisms similarly to the construction in Section 5.2.

Throughout this subsection, let M be an n–dimensional smooth manifold.

Definition 9.1 Let X be a topological space and $E \to X$ a topological n–dimensional vector bundle which corresponds to a (homotopy class of) map(s) $e \colon X \to B\mathrm{GL}(\mathbb{R}^n)$ from X to the classifying space of the topological group $\mathrm{GL}(\mathbb{R}^n)$. More generally, we could also consider a map $e \colon X \to B\mathrm{Homeo}(\mathbb{R}^n)$ to the classifying space of the topological group of homeomorphisms of $\mathbb{R}^n$, but for our purposes vector bundles are enough. An (X, E)–*structure* or, equivalently, an (X, e)–*structure* on an n–dimensional manifold M consists of the following data:

   (1) a map $f \colon M \to X$, and

   (2) an isomorphism of vector bundles

$$\mathrm{triv} \colon TM \cong f^*(E).$$

Denote the set of (X, E)–structured n–dimensional manifolds by $\mathrm{Man}_n^{(X,E)}$.

An interesting class of such structures arises from topological groups with a morphism to $O(n)$.

Definition 9.2 Let G be a topological group together with a continuous homomorphism $e \colon G \to O(n)$, which induces $e \colon BG \to B\mathrm{GL}(\mathbb{R}^n)$. As usual, let $BG = EG/G$ be the classifying space of G, where EG is the total space of its universal bundle, which is a weakly contractible space on which G acts freely. Then consider the vector bundle $E = (\mathbb{R}^n \times EG)/G$ on BG. A (BG, E)–structure or, equivalently, a (BG, e)–structure on an n–dimensional manifold M is called a G–*structure on M*. The set of G–structured n–dimensional manifolds is denoted by Man_n^G.

For us, the most important examples will be the following three examples.

Example 9.3 If G is the trivial group, then $X = BG = *$ and E is trivial. Then a G–structure on M is a trivialization of TM, ie a framing.

Example 9.4 Let $G = O(n)$ and $e = \mathrm{id}_{O(n)}$. Then, since the inclusion $O(n) \to \mathrm{Diff}(\mathbb{R}^n)$ is a deformation retract, an $O(n)$–structured manifold is just a smooth manifold.

Example 9.5 Let $G = \mathrm{SO}(n)$ and $e \colon \mathrm{SO}(n) \to O(n)$ be the inclusion. Then an $\mathrm{SO}(n)$–structured manifold is an oriented manifold.

Definition 9.6 Let M and N be (X, E)–structured manifolds. Then let the space of morphisms from M to N be

$$\mathrm{Map}^{(X,E)}(M, N) = \mathrm{Emb}(M, N) \times^h_{\mathrm{Map}_{/B\mathrm{Homeo}(\mathbb{R}^n)}(M,N)} \mathrm{Map}_{/X}(M, N).$$

Taking (singular or differentiable) simplices leads to a space, ie a simplicial set of morphisms from M to N. Thus, we get a topological (or simplicial) category $\mathcal{M}an_n^{(X,E)}$ of (X, E)–structured manifolds. Disjoint union gives $\mathcal{M}an_n^{(X,E)}$ a symmetric monoidal structure.

Remark 9.7 For $G = O(n)$ we recover $\mathrm{Emb}(M, N)$, and for $G = \mathrm{SO}(n)$ the space of orientations on a manifold is discrete, so an element in $\mathrm{Map}^{\mathrm{SO}(n)}(M, N)$ is an orientation-preserving map.

If G is the trivial group, we saw above that a G–structure is a framing. In this case, the above homotopy fiber product reduces to

$$\mathrm{Map}^{(X,E)}(M, N) = \mathrm{Emb}(M, N) \times^h_{\mathrm{Map}_{GL(d)}(\mathrm{Fr}(TM),\mathrm{Fr}(TN))} \mathrm{Map}(M, N).$$

Thus, a framed embedding is a pair (f, h), where $f \colon M \to N$ lies in $\mathrm{Emb}(M, N)$ and h is a homotopy between the trivialization of TM induced by the framing of M and that induced by the pullback of the framing on N.

9.2 The (∞, n)–category of structured bordisms

Fix a type of structure given by the pair (X, E). In this subsection we define the n–fold (complete) Segal space of (X, E)–structured bordisms $\mathrm{Bord}_n^{(X,E)}$.

Compared to Definition 5.1, we add an (X, E)–structure to the data of an element in a level set.

Definition 9.8 Let V be a finite-dimensional vector space. For every n–tuple k_1, ..., $k_n \geq 0$, let $(\mathrm{PBord}_n^{(X,E),V})_{k_1,\ldots,k_n}$ be the collection of tuples

$$(M, f, \mathrm{triv}, (I_0^i \leq \cdots \leq I_{k_i}^i)_{i=1,\ldots,n}),$$

where

(1) $(M, (I_0^i \leq \cdots \leq I_{k_i}^i)_{i=1}^n)$ is an element in the set $(\mathrm{PBord}_n^V)_{k_1,\ldots,k_n}$, and

(2) (f, triv) is an (X, E)–structure on the (abstract) manifold M.

Remark 9.9 There is a forgetful map

$$U \colon (\mathrm{PBord}_n^{(X,E),V})_{k_1,\ldots,k_n} \to (\mathrm{PBord}_n^V)_{k_1,\ldots,k_n}$$

forgetting the (X, E)–structure.

Definition 9.10 An l–simplex of $(\mathrm{PBord}_n^{(X,E),V})_{k_1,\ldots,k_n}$ consists of tuples

$$\left(M, f, \mathrm{triv}, \bar{I}(s) = (I_0^i(s) \leq \cdots \leq I_{k_i}^i(s))_{s \in |\Delta^l|_e}\right)$$

such that:

(1) $\bar{I} = (I_0^i \leq \cdots \leq I_{k_i}^i)_{1 \leq i \leq n} \to |\Delta^l|_e$ is an l–simplex in $\mathrm{Int}_{k_1,\ldots,k_n}^n$.

(2) M is a closed and bounded $(n+l)$–dimensional submanifold of $V \times B(\bar{I})$ such that[22]

   (a) the composition $\pi \colon M \hookrightarrow V \times B(\bar{I}) \twoheadrightarrow B(\bar{I})$ of the inclusion with the projection is proper,

   (b) its composition with the projection onto $|\Delta^l|_e$ is a submersion $\pi_l \colon M \to |\Delta^l|_e$,

   (c) $(f, \mathrm{triv}) \colon \ker\big(D\pi_l \colon TM \to T|\Delta^l|_e\big) \longrightarrow f^*E$ is a fiberwise linear isomorphism.

---

[22]Recall from Section 4.4 that $B(\bar{I})$ denotes the total space of $B(\bar{I}) \to |\Delta^l|_e$ and is the subspace $\bigcup_{s \in |\Delta^l|_e} B(\bar{I}(s)) \times \{s\}$ of $\mathbb{R}^n \times |\Delta^l|_e$.

(3) For every $S \subseteq \{1, \ldots, n\}$, let $p_S \colon M \xrightarrow{\pi} B(\bar{I}) \subseteq \mathbb{R}^n \times |\Delta^l|_e \xrightarrow{\pi_S} \mathbb{R}^S \times |\Delta^l|_e$ be the composition of π with the projection π_S onto the S–coordinates. Then for every $1 \leq i \leq n$ and $0 \leq j_i \leq k_i$, at every $x \in p_{\{i\}}^{-1}(I_{j_i}^i(s) \times \{s\})$, the map $p_{\{i, \ldots, n\}}$ is submersive.

Similarly as for PBord$_n$ the levels can be given a spatial structure with the above l–simplices and then the collection of levels can be made into a complete n–fold Segal space Bord$_n^{(X,E)}$.

Moreover, Bord$_n^{(X,E)}$ has a symmetric monoidal structure given by (X, E)–structured versions of the Γ–object and of the tower giving Bord$_n$ a symmetric monoidal structure.

9.3 Example: objects in Bord$_2^{\text{fr}}$ are 2–dualizable

In dimension one, a framing is the same as an orientation. Thus the first interesting case is the two-dimensional one. In this case, the existence of a framing is a rather strong condition. However, we will see that any object in Bord$_2^{\text{fr}}$ is 2–dualizable. Being 2–dualizable means that it is dualizable with evaluation and coevaluation maps themselves have adjoints, see [40].

Consider an object in Bord$_2^{\text{fr}}$, which, since in this case Bord$_2^{\text{fr}} = $ PBord$_2^{\text{fr}}$ by Remark 5.25, is an element of the form

$$\left(M \subseteq V \times (a^1, b^1) \times (a^2, b^2), F, (a^1, b^1), (a^2, b^2) \right),$$
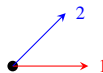
where F is a framing of M. By the submersivity condition (3) in the definition (Definition 5.1) of PBord$_2$, M is a disjoint union of manifolds which are diffeomorphic to $(0, 1)^2$. Thus, it suffices to consider an element of the form
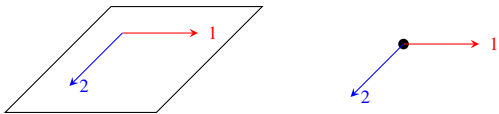
$$((0, 1)^2 \subseteq (0, 1)^2, F, (0, 1), (0, 1)),$$

where F is a framing of $(0, 1)^2$. Depict this element by
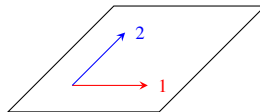
One should think of this as a point together with a 2–framing,

We claim that its dual is the same underlying unstructured manifold together with the opposite framing
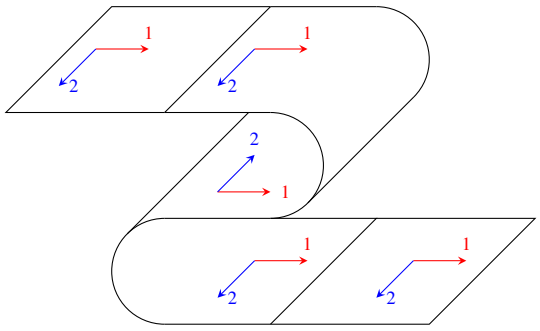
An evaluation 1–morphism $\mathrm{ev}_{\overset{2}{\underset{1}{\nearrow}}}$ between them is given by the element in $(\mathrm{Bord}_2^{\mathrm{fr}})_{1,0}$ which is a strip, ie $(0,1)^2$, with the framing given by slowly rotating the framing by 180°, and is embedded into $\mathbb{R} \times (0,1)^2$ by folding it over once as depicted on the right:
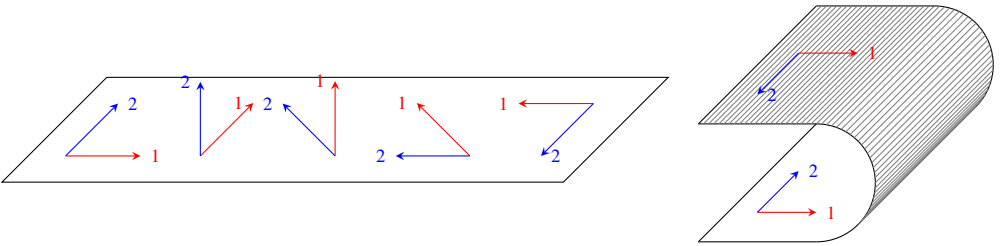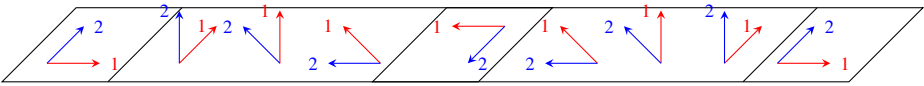
A coevaluation $\mathrm{coev}_{\overset{2}{\underset{1}{\nearrow}}}$ is given similarly by rotating the framing along the strip in the other direction, by $-180°$.

The composition

is connected by a path to the flat strip with the following framing, given by pulling at the ends of the strip to flatten it:

This strip is homotopic to the same strip with the trivial framing. Thus the composition is connected by a path to the identity and thus is the identity in the homotopy category.
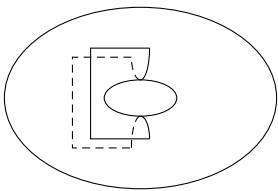
Similarly,

$$(\text{ev}_{\nearrow^2_1} \otimes \text{id}_{\nearrow^2_1}) \circ (\text{id}_{\nearrow^2_1} \otimes \text{coev}_{\nearrow^2_1}) \simeq \text{id}_{\nearrow^2_1}.$$

In the above construction, we used $\text{ev}_{\nearrow^2_1}$ and $\text{coev}_{\nearrow^2_1}$, which arose from strips with framing rotating by $\pm 180°$. A similar argument holds if you use for the evaluation any strip with the framing rotating by $\alpha\pi$ for any odd integer α and for the coevaluation rotation by $\beta\pi$ for any odd β. Denoting these by $\text{ev}(\alpha)$ and $\text{coev}(\beta)$, they will be adjoints to each other if $\alpha + \beta = 2$.
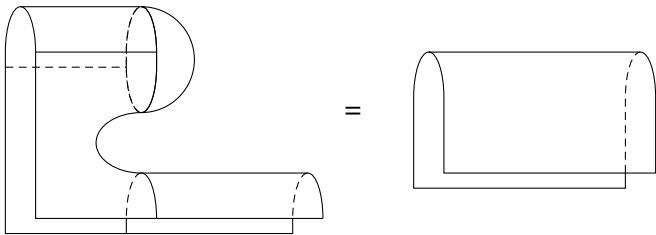
The counit of the adjunction is given by the cap with the framing coming from the trivial framing on the (flat) disk:
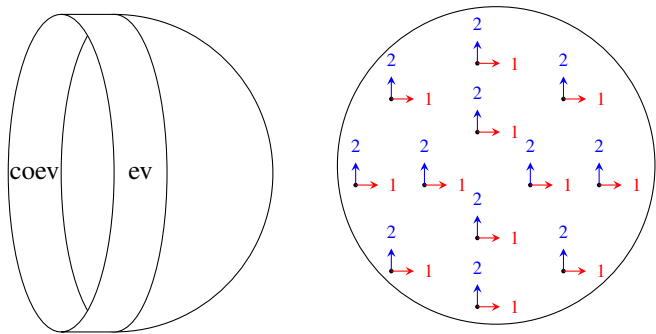
Similarly, the unit of the adjunction is given by a saddle with the framing coming from the one of the torus which turns by 2π along each of the fundamental loops:

Then the following 2–bordism is also framed and exhibits the adjunction:

Remark 9.11 One can use a similar, but much longer, argument to show that objects in $\mathrm{Bord}_n^{\mathrm{fr}}$ are in fact n–dualizable.

10 Fully extended topological field theories

Now that we have a good definition of a symmetric monoidal (∞, n)–category of bordisms modeled as a symmetric monoidal complete n–fold Segal space, we can define fully extended topological field theories à la Lurie.

10.1 Definition

Definition 10.1 A *fully extended unoriented n–dimensional topological field theory* is a symmetric monoidal functor of (∞, n)–categories with source Bord_n.

Remark 10.2 Consider a fully extended unoriented n–dimensional topological field theory

$$Z \colon \mathrm{Bord}_n \to \mathcal{C},$$

where $\mathcal{C}$ is a symmetric monoidal complete n–fold Segal space. We have seen in Corollary 7.7 and Section 8 that there is a map

$$n\mathrm{Cob} \simeq h_1(\mathrm{Bord}_n^{(\infty,1)}) \to h_1(\Omega^{n-1}(\mathrm{Bord}_n)).$$

Precomposition of $\Omega_\varnothing^{n-1}(Z)$ with this map induces a symmetric monoidal functor

$$n\mathrm{Cob} \to h_1(\Omega^{n-1}(\mathrm{Bord}_n)) \to h_1(\Omega_{Z(*)}^{n-1}(\mathcal{C})),$$

ie an ordinary n–dimensional topological field theory.

Additional structure Recall from the previous section that there are variants of Bord_n which require that the underlying manifolds of their elements be endowed with some additional structure, eg an orientation or a framing. These variants lead to the following definitions.

Definition 10.3 Fix a type of structure given by the pair (X, E). A *fully extended n–dimensional (X, E)–topological field theory* is a symmetric monoidal functor of (∞, n)–categories with source $\mathrm{Bord}_n^{(X,E)}$.

In particular, the most interesting cases are the following:

Definition 10.4 A *fully extended n–dimensional framed topological field theory* is a symmetric monoidal functor of (∞, n)–categories with source $\mathrm{Bord}_n^{\mathrm{fr}}$.

Definition 10.5 A *fully extended n–dimensional oriented topological field theory* is a symmetric monoidal functor of (∞, n)–categories with source $\mathrm{Bord}_n^{\mathrm{or}}$.

10.2 *n*–TFT yields *k*–TFT

Every fully extended n–dimensional (unoriented, oriented, framed) TFT yields a fully extended k–dimensional (unoriented, oriented, framed) TFT for any $k \leq n$ by truncation from Section 2.4.1.

Note that for $k < n$, we have a map of k–fold Segal spaces

$$\mathrm{PBord}_k \to \tau_k(\mathrm{PBord}_n) = (\mathrm{PBord}_n)_{\bullet_k, \mathbf{0}_{n-k}}$$

induced by sending $\left(M \hookrightarrow V \times B(\underline{\bar{I}}), \underline{\bar{I}} = (I_0^i \leq \cdots \leq I_{j_k}^i)_{i=1}^k \right) \in (\mathrm{PBord}_k)_{j_1, \ldots, j_k}$ to

$$\left(M \times (0,1)^{n-k} \hookrightarrow V \times (0,1)^{n-k} \times B(\underline{\bar{I}}), \underline{\bar{I}}, (\mathbf{0}, \mathbf{1})_{n-k} \right).$$

The completion map $\mathrm{PBord}_n \to \mathrm{Bord}_n$ induces a map on the truncations. Precomposition with the above map yields a map of (in general noncomplete) n–fold Segal spaces

$$\mathrm{PBord}_k \to \tau_k(\mathrm{PBord}_n) \to \tau_k(\mathrm{Bord}_n).$$

Recall from Section 2.4.1 that since $\tau_k(\mathrm{Bord}_n)$ is complete, by the universal property of the completion we obtain a map $\mathrm{Bord}_k \to \tau_k(\mathrm{Bord}_n)$, which is compatible with the symmetric monoidal structure (for both approaches).

Remark 10.6 This map is usually not an equivalence, since completion does not commute with truncation. Moreover, if we equip the bordisms with an orientation or a framing, the image of PBord_k in $\tau_k(\mathrm{PBord}_n)$ consists of those n–oriented or n–framed bordisms whose orientation or framing is a stabilization of a k–orientation or k–framing.

We conclude that any fully extended n–dimensional (unoriented, oriented, framed) TFT with values in a complete n–fold Segal space $\mathcal{C}$, $\mathrm{Bord}_n \to \mathcal{C}$ leads to a k–dimensional (unoriented, n–oriented, n–framed) TFT given by the composition

$$\mathrm{Bord}_k \to \tau_k(\mathrm{Bord}_n) \to \tau_k(\mathcal{C})$$

with values in the complete k–fold Segal space $\tau_k(\mathcal{C})$.

References

[1] **L Abrams**, *Two-dimensional topological quantum field theories and Frobenius algebras*, J. Knot Theory Ramifications 5 (1996) 569–587 MR

[2] **J C Baez**, **J Dolan**, *Higher-dimensional algebra and topological quantum field theory*, J. Math. Phys. 36 (1995) 6073–6105 MR

[3] **C Barwick**, (∞, n)–Cat *as a closed model category*, PhD thesis, University of Pennsylvania (2005) MR Available at `https://search.proquest.com/docview/305445747`

[4] **C Barwick**, **D M Kan**, *Partial model categories and their simplicial nerves*, preprint (2011) arXiv

[5] **C Barwick**, **D M Kan**, *Relative categories: another model for the homotopy theory of homotopy theories*, Indag. Math. 23 (2012) 42–68 MR

[6] **C Barwick**, **C Schommer-Pries**, *On the unicity of the homotopy theory of higher categories*, preprint (2011) arXiv

[7] **J E Bergner**, *Three models for the homotopy theory of homotopy theories*, Topology 46 (2007) 397–436 MR

[8] **J E Bergner**, *A survey of* $(\infty, 1)$*–categories*, from "Towards higher categories" (J C Baez, J P May, editors), IMA Vol. Math. Appl. 152, Springer (2010) 69–83 MR

[9] **J E Bergner**, *Models for* (∞, n)*–categories and the cobordism hypothesis*, from "Mathematical foundations of quantum field theory and perturbative string theory" (H Sati, U Schreiber, editors), Proc. Sympos. Pure Math. 83, Amer. Math. Soc., Providence, RI (2011) 17–30 MR

[10] **J E Bergner**, **C Rezk**, *Comparison of models for* (∞, n)*–categories, I*, Geom. Topol. 17 (2013) 2163–2202 MR

[11] **J E Bergner**, **C Rezk**, *Comparison of models for* (∞, n)*–categories, II*, preprint (2014) arXiv

[12] **M Bökstedt**, **I Madsen**, *The cobordism category and Waldhausen's K–theory*, from "An alpine expedition through algebraic topology" (C Ausoni, K Hess, B Johnson, W Lück, J Scherer, editors), Contemp. Math. 617, Amer. Math. Soc., Providence, RI (2014) 39–80 MR

[13] **D Calaque**, **C Scheimbauer**, *Factorization homology as a fully extended topological field theory*, in preparation

[14] **J A Campbell**, *Derived Koszul duality and topological Hochschild homology*, preprint (2014) arXiv

[15] **G Caviglia**, **G Horel**, *Rigidification of higher categorical structures*, Algebr. Geom. Topol. 16 (2016) 3533–3562 MR

[16] **W G Dwyer**, **D M Kan**, *Function complexes in homotopical algebra*, Topology 19 (1980) 427–440 MR

[17] **W G Dwyer**, **D M Kan**, **J H Smith**, *Homotopy commutative diagrams and their realizations*, J. Pure Appl. Algebra 57 (1989) 5–24 MR

[18] **C Ehresmann**, *Catégories structurées*, Ann. Sci. École Norm. Sup. 80 (1963) 349–426 MR

[19] **S Galatius**, *Stable homology of automorphism groups of free groups*, Ann. of Math. 173 (2011) 705–768 MR

[20] **S Galatius**, **O Randal-Williams**, *Monoids of moduli spaces of manifolds*, Geom. Topol. 14 (2010) 1243–1302 MR

[21] **S Galatius**, **U Tillmann**, **I Madsen**, **M Weiss**, *The homotopy type of the cobordism category*, Acta Math. 202 (2009) 195–239 MR

[22] **D Gay**, **K Wehrheim**, **C Woodward**, *Connected cerf theory* (2012) Available at https://math.berkeley.edu/~katrin/papers/cerf.pdf

[23] **D Gepner**, **R Haugseng**, *Enriched* ∞*–categories via non-symmetric* ∞*–operads*, Adv. Math. 279 (2015) 575–716 MR

[24] **A Grothendieck**, *Pursuing stacks*, unpublished manuscript (1983) Available at https://thescrivener.github.io/PursuingStacks

[25] **R Haugseng**, *Iterated spans and classical topological field theories*, Math. Z. 289 (2018) 1427–1488 MR

[26] **P S Hirschhorn**, *Model categories and their localizations*, Mathematical Surveys and Monographs 99, Amer. Math. Soc., Providence, RI (2003) MR

[27] **A Hirschowitz**, **C Simpson**, *Descente pour les* n*–champs*, preprint (1998) arXiv

[28] **G Horel**, *A model structure on internal categories in simplicial sets*, Theory Appl. Categ. 30 (2015) 704–750 MR

[29] **T Johnson-Freyd**, **C Scheimbauer**, *(Op)lax natural transformations, twisted quantum field theories, and "even higher" Morita categories*, Adv. Math. 307 (2017) 147–223 MR

[30] **A Joyal**, *Disks, duality, and Theta-categories*, preprint (1997) Available at https://ncatlab.org/nlab/files/JoyalThetaCategories.pdf

[31] **A Joyal**, *Notes on quasicategories*, unpublished manuscript (2008) Available at http://www.math.uchicago.edu/~may/IMA/Joyal.pdf

[32] **J Kock**, *Frobenius algebras and 2D topological quantum field theories*, London Mathematical Society Student Texts 59, Cambridge Univ. Press (2004) MR

[33] **A D Lauda**, **H Pfeiffer**, *Open-closed strings: two-dimensional extended TQFTs and Frobenius algebras*, Topology Appl. 155 (2008) 623–666 MR

[34] **G Laures**, *On cobordism of manifolds with corners*, Trans. Amer. Math. Soc. 352 (2000) 5667–5688 MR

[35] **J M Lee**, *Introduction to smooth manifolds*, 2nd edition, Graduate Texts in Mathematics 218, Springer (2013) MR

[36] **Z L Low**, *Cocycles in categories of fibrant objects*, preprint (2015) arXiv

[37] **Z L Low**, **A Mazel-Gee**, *From fractions to complete Segal spaces*, Homology Homotopy Appl. 17 (2015) 321–338 MR

[38] **J Lurie**, *Higher topos theory*, Annals of Mathematics Studies 170, Princeton Univ. Press (2009) MR

[39] **J Lurie**, $(\infty, 2)$–*Categories and the Goodwillie calculus, I*, preprint (2009) arXiv

[40] **J Lurie**, *On the classification of topological field theories*, from "Current developments in mathematics, 2008" (D Jerison, B Mazur, T Mrowka, W Schmid, R Stanley, S-T Yau, editors), Int., Somerville, MA (2009) 129–280 MR

[41] **J Lurie**, *Higher algebra*, book project (2017) Available at `https://tinyurl.com/Lurie-Higher-alg`

[42] **S Mac Lane**, *Natural associativity and commutativity*, Rice Univ. Studies 49 (1963) 28–46 MR

[43] **J Milnor**, *Morse theory*, Annals of Mathematics Studies 51, Princeton Univ. Press (1963) MR

[44] **C Rezk**, *Fibrations and homotopy colimits of simplicial sheaves*, preprint (1998) arXiv

[45] **C Rezk**, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math. Soc. 353 (2001) 973–1007 MR

[46] **C Rezk**, *A Cartesian presentation of weak n–categories*, Geom. Topol. 14 (2010) 521–571 MR

[47] **E Riehl**, *Category theory in context*, Dover, Mineola, NY (2016)

[48] **C Scheimbauer**, *Factorization homology as a fully extended topological field theory*, PhD thesis, ETH Zürich (2014) Available at `https://people.maths.ox.ac.uk/scheimbauer/ScheimbauerThesis.pdf`

[49] **C J Schommer-Pries**, *The classification of two-dimensional extended topological field theories*, PhD thesis, University of California, Berkeley (2009) MR Available at `https://search.proquest.com/docview/304843359`

[50] **G Segal**, *Categories and cohomology theories*, Topology 13 (1974) 293–312 MR

[51] **C Simpson**, *On the Breen–Baez–Dolan stabilization hypothesis for Tamsamani's weak n–categories*, preprint (1998) arXiv

[52] **B Toen**, *Dualité de tannaka supérieure, I: Structures monoidales*, preprint, Max Planck Institut für Mathematik (2000) Available at `https://tinyurl.com/Toen-Dualite`

[53]   **B Toën**, *Vers une axiomatisation de la théorie des catégories supérieures*, *K*–Theory 34 (2005) 233–263   MR

[54]   **B Toen**, **G Vezzosi**, *Segal topoi and stacks over Segal categories*, preprint (2002)   arXiv

[55]   **B Toën**, **G Vezzosi**, *Homotopical algebraic geometry, I: Topos theory*, Adv. Math. 193 (2005) 257–372   MR

[56]   **B Toën**, **G Vezzosi**, *Caractères de Chern, traces équivariantes et géométrie algébrique dérivée*, Selecta Math. 21 (2015) 449–554   MR

[57]   **Y Zhao**, *Extended topological field theories and the cobordism hypothesis*, Master's thesis, Université Paris XI (Paris-Sud) and Università degli studi di Padova (2013) Available at http://algant.eu/documents/theses/zhao.pdf

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