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Homotopical intersection theory III: Multirelative intersection problems

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We extend some results of Hatcher and Quinn (1974) beyond the metastable range. We give a bordism-theoretic obstruction $\chi(f)$ to deforming a map $f\colon P \to N$ between manifolds simultaneously off of a collection of pairwise disjoint submanifolds $Q_1, \dots, Q_j \subset N$ under the assumption that f can be deformed off of any proper subcollection in a homotopy coherent way. In a certain range of dimensions, $\chi(f)$ is a complete obstruction to finding the desired deformation. We apply this machinery to embedding problems and to the study of linking phenomena.

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1 Introduction

1.1 Intersection problems

In [16] we considered the problem of deforming a map $f\colon P \to N$ between compact smooth manifolds off a compact smooth submanifold $Q \subset N$. This was called an *intersection problem*. We obtained an obstruction $\chi(f)$ residing in a normal bordism

Note from JRK: Bruce Williams passed away on January 11, 2018, before the final revision of this paper was completed. Bruce was a close friend and an inspiring mentor.

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group $\Omega_0(X;\xi)$. The vanishing of the obstruction is necessary for finding such a deformation. One of the main results of [16] was that in a certain metastable range of dimensions, $\chi(f)$ is a complete obstruction to finding a homotopy from f to a map having disjoint image from Q. The goal of the current paper is to extend these ideas to the multirelative setting.

Fix a positive integer j and let

$$Q_1,\dots,Q_j \subset N$$

be a collection of pairwise disjoint, closed, smooth submanifolds of a compact, connected, smooth manifold N. Given a map $P \to N$, where P is a closed manifold, the problem we consider is that of finding a deformation of f off of the Q_i *simultaneously.* We approach this inductively, by assuming that P can be deformed off of any proper union of the Q_i in such a way that the deformations line up in a certain homotopically coherent fashion. We first explain what this precisely means.

Recall that a $(k+1)$–ad of spaces consists of a space X together with k distinguished subspaces $X_1,\dots X_k \subset X$. The notation for such data is $(X;X_1,\dots,X_k)$, but it will often be convenient to simply write X when the subspaces are understood.

Example 1.1 (1) A space Z can be considered as a constant $(k+1)$–ad, that is, $(Z;Z,\dots,Z)$.

(2) The standard $(k-1)$–simplex Δ^{k-1} together with its codimension one faces is a $(k+1)$–ad, ie $(\Delta^{k-1};d_0\Delta^{k-1},\dots,d_{k-1}\Delta^{k-1})$.

(3) If Z is a space and X is a $(k+1)$–ad, then the cartesian product $Z\times X$ is a $(k+1)$–ad in the evident way.

A map of $(k+1)$–ads $X \to Y$ is a continuous map of underlying spaces which restricts to maps $X_i \to Y_i$ for all i. We can topologize this as the subspace of the mapping space of all maps from X to Y in the compact–open topology.

Consider N together with the subspaces $N\setminus Q_1,\dots,N\setminus Q_j$ as a $(j+1)$–ad

$$(N;N\setminus Q_1,\dots,N\setminus Q_j).$$

Then a *multirelative intersection problem* is defined to be a map of $(j+1)$–ads

$$f\colon P\times\Delta^{j-1}\to N.$$

Set $Q_J = Q_1\sqcup\cdots\sqcup Q_j$. We will consider $N\setminus Q_J$ as a constant $(j+1)$–ad; it is then a sub-ad of $(N;N\setminus Q_1,\dots,N\setminus Q_j)$. We define a *solution* to a multirelative intersection problem to be a homotopy (of maps of $(j+1)$–ads) f_t from $f = f_0$ to an ad map $f_1\colon P \times \Delta^{j-1} \to N$ which factors as

$$P \times \Delta^{j-1} \to N \setminus Q_J \xrightarrow{\subset} N$$

In particular, the image of f_1 is disjoint from Q_J.

In more modern language the problem can be reformulated as follows: Let $J = \{1, \dots, j\}$. For $S \subset J$, let

$$Q_S = \coprod_{i \in S} Q_i.$$

Then a multirelative intersection problem is equivalent to specifying a map

$$f\colon P \to \operatorname*{holim}_{S \subsetneq J} (N \setminus Q_S), \tag{1}$$

where the target is the homotopy inverse limit of the spaces $N \setminus Q_S$ as S ranges through the proper subsets of J. Explicitly, the displayed homotopy limit is given by the space of maps of $(j+1)$–ads $\Delta^{j-1} \to N$.

The deliberate ambiguity in our notation is for the sake of convenience: we use f to denote the map (1) as well as for the map of ads $P \times \Delta^{j-1} \to N$, as this is not likely to cause confusion (note these maps determine each other by an adjunction).

A solution then amounts to a map $\hat{f}\colon P \to N \setminus Q_J$, together with a commuting homotopy $f_t\colon P \to \operatorname{holim}_{S \subsetneq J} (N \setminus Q_S)$ with $t \in [0, 1]$, for the diagram

$$\begin{array}{ccc} & & N \setminus Q_J \\ & \overset{\hat{f}}{\nearrow} & \downarrow \\ P & \xrightarrow[f]{} & \operatorname*{holim}_{S \subsetneq J} (N \setminus Q_S) \end{array} \tag{2}$$

Given a map of $(j+1)$–ads $f\colon P \times \Delta^{j-1} \to N$ as above, we write

$$E(P, Q_\bullet)$$

for the *iterated homotopy fiber product* of $P \times \Delta^{j-1}$ and each of the Q_i over N. This is just the homotopy pullback of the diagram

$$P \times \Delta^{j-1} \times \prod_{i=1}^{j} Q_i \to \prod_{i=0}^{j} N \xleftarrow{\Delta} N,$$

where Δ is the diagonal map and the left map is the product of $f\colon P \times \Delta^{j-1} \to N$ with the inclusions of the Q_i.

Define a virtual bundle ξ over $E(P, Q_\bullet)$ as follows: Let τ_P be the tangent bundle of P, τ_N the tangent bundle of N and τ_{Q_i} the tangent bundle of Q_i; each one of these gives a bundle over $E(P, Q_\bullet)$ using the evident (projection) maps. To avoid notational clutter, we use the same notation for these pullbacks. Then we set

$$\xi := -\tau_P + \sum_{i=1}^{j} (\tau_N - \tau_{Q_i}).$$

Suppose $p = \dim P$, $q_i = \dim Q_i$ and $n = \dim N$. It will also be convenient to write

$$\mu = \min_i (n - q_i - 2) \quad \text{and} \quad \Sigma = \sum_i (n - q_i - 2).$$

In particular, the virtual rank of ξ is $2j - p + \Sigma$. The following assumption will be made throughout the paper:

Hypothesis 1.2 For $1 \le i \le j$, we have $n - q_i \ge 2$.

We briefly review the definition of bordism with coefficients in a virtual bundle. Let X be a space equipped with a finite-dimensional inner product bundle ζ of rank s. Then one has the Thom space X^ζ, which is the quotient space formed from the unit disk bundle by collapsing the unit sphere bundle to a point. For the purposes of this paper, we define $\Omega_k(X; \zeta)$ to be the k^{th} stable homotopy group $\pi_k^{\text{st}}(X^\zeta)$. By standard transversality arguments, an element of this abelian group is represented by a compact smooth submanifold $V \subset \mathbb{R}^{k+d}$, for some $d \ge 0$, together with a map $g\colon V \to X$ such that the pullback of $\zeta \oplus \epsilon^d$ along g is identified with the normal bundle of V (where ϵ^d is the trivial bundle of rank d; note that the dimension of V is necessarily $k - s$). Then bordism defines an equivalence relation on this collection and the set of equivalence classes is canonically identified with $\Omega_k(X; \zeta)$. With respect to this identification, note that the operation of disjoint union of bordism classes corresponds to the addition of stable homotopy classes. Now suppose that ζ is a virtual bundle. This means that $\zeta \oplus \epsilon^j$ comes equipped with an isomorphism to a finite-dimensional inner product bundle η for some integer $j \ge 0$. In this instance, we define $\Omega_k(X; \zeta)$ to be $\Omega_{k+j}(X; \eta)$. Our indexing convention for the bordism group differs from that of [16], but is the same as the one we used in [15].

Theorem A *Assume $j \geq 1$. Then there is an obstruction*

$$\chi(f) \in \bigoplus_{(j-1)!} \Omega_{2j-2}(E(P, Q_\bullet); \xi)$$

which vanishes if the intersection problem defined by f possesses a solution. Conversely, if

$$p \leq 1 + \mu + \Sigma$$

then the vanishing of $\chi(f)$ guarantees the existence of a solution.

Theorem A is proved using a fiberwise version of Poincaré duality together with some general results about strongly cocartesian cubes.

Remark 1.3 The $j = 1$ case ("the metastable range") of Theorem A was already considered in [16]. That work gave a homotopy-theoretic approach to the main results of the paper of Hatcher and Quinn [12] (when $j = 1$, Theorem D below amounts to the vanishing obstruction case of [12, Theorem 2.2]).

Remark 1.4 The obstruction $\chi(f)$ is defined in a homotopy-theoretic manner. Given the identification between bordism theory and the homotopy groups of a Thom spectrum, it is reasonable to ask what $\chi(f)$ means geometrically. In the $j = 1$ case such an interpretation was provided by the "index theorem" of [16, Theorem 12.1]. The $j > 1$ case is more subtle and involves iterated intersections of null-bordism data. We hope to address this in detail in another paper. Meanwhile, to leave the reader with an impression, we now sketch a geometric description of $\chi(f)$ when $j = 2$.

Let $j = 2$ and let $f\colon P \times \Delta^1 \to N$ be an intersection problem. Let b be the barycenter of Δ^1 and let D_i be the transversal intersection of $f|_{P\times b}\colon P \times b \to N$ with Q_i. By assumption, the evident maps $D_i \to E(P, Q_i)$ are null-bordant. Let $g_i\colon W_i \to E(P, Q_i)$ be a null-bordism. Compose this with the projection $E(P, Q_i) \to P$ to get maps $h_i\colon W_i \to P$. Now take the transversal intersection of the product map $h_1 \times h_2\colon W_1 \times W_2 \to P \times P$ with the diagonal of P. This produces a closed manifold W_{12} of dimension $p - 2 - \Sigma$ equipped with a map $W_{12} \to E(P, Q_\bullet)$ which is covered by the requisite bundle data. The associated bordism class coincides with the obstruction $\chi(f)$.

Remark 1.5 (large codimension) If $p \leq 1 + \Sigma$, then the bordism group of Theorem A is trivial. Consequently, f can be homotopy factorized through $N \setminus Q_J$ in this case.

If $j = 1$, this conclusion also follows from transversality, and for $j > 1$ it follows from the higher Blakers–Massey theorem applied to the j–cubical diagram $\{N \setminus Q_S\}_{S \subset J}$ (see Goodwillie [4, Theorem 2.5]).

1.1.1 Highly connected manifolds When the manifolds P and Q_i are sufficiently highly connected, the obstruction group of Theorem A admits a simpler description. Suppose that P is a–connected and Q_i is b_i–connected. Choose basepoints in $x \in P$ and $y_i \in Q_i$. Then x gives rise to a point $x' \in N$ using f. The homotopy fiber product of $E(x, y_\bullet)$ is defined and comes equipped with a map $E(x, y_\bullet) \to E(P, Q_\bullet)$. Moreover, the pullback of ξ to $E(x, y_\bullet)$ is a trivial virtual bundle of rank $2j - p + \Sigma$. Hence, the bordism groups associated with this pullback are framed bordism groups of $E(x, y_\bullet)$ shifted in degree by $2j - p + \Sigma$.

It is also straightforward to check that the map

$$E(x, y_\bullet) \to E(P, Q_\bullet)$$

is $\min(a, b_1, \dots, b_j)$–connected. It follows that the associated map of Thom spectra is k–connected, where $k = \min(a, b_1, \dots, b_j) + 2j - p + \Sigma$. In particular, the induced homomorphism of bordism groups is an isomorphism in degrees strictly less than k.

Note that $E(x, y_\bullet)$ is the space of j–tuples $(\lambda_1, \dots, \lambda_j)$ in which $\lambda_i\colon [0,1] \to N$ is a path from x' to y_i for $1 \le i \le j$. The j–fold cartesian product of loop spaces $\prod_j \Omega N$ based at x' acts on $E(x, y_\bullet)$ by path composition. After a basepoint for $E(x, y_\bullet)$ is fixed, we obtain a homotopy equivalence $E(x, y_\bullet) \simeq \prod_j \Omega N$. Consequently, we have shown:

Addendum B *Assume $p \le 1 + \Sigma + \min(a, b_1, \dots, b_j)$. Then the obstruction group appearing in Theorem A is isomorphic to the direct sum of framed bordism groups*

$$\bigoplus_{(j-1)!} \Omega^{\mathrm{fr}}_{p-2-\Sigma}\Big(\prod_j \Omega N\Big).$$

Example 1.6 Suppose $P = S^p$ and $Q_i = S^{q_i}$ are spheres. Then $a = p-1$ and $b_i = q_i - 1$. Consequently, the inequality appearing in Addendum B becomes $p \le \Sigma + \mu - j$.

Example 1.7 Suppose $p = 2 + \Sigma$ and $a, b_i \ge 1$. Then the obstruction group of Addendum B is isomorphic to $\bigoplus_{(j-1)!} \mathbb{Z}[\pi]^{\otimes j}$, with $\pi = \pi_1(N)$.

1.2 The solution space

The space of lifts solving the multirelative intersection problem (2) is defined by converting the vertical map appearing in that diagram into a fibration and then taking the space of sections of this fibration along P. The space of such lifts is called the *solution space* and is denoted by $\mathscr{L}(f)$.

For a spectrum E we let $\Omega^\infty E$ be the associated infinite loop space.

Theorem C *Assume that in the solution space $\mathscr{L}(f)$ is nonempty and is equipped with a choice of basepoint. Then there is a $(1-p+\mu+\Sigma)$–connected map*

$$\mathscr{L}(f) \to \prod_{(j-1)!} \Omega^\infty E(P, Q_\bullet)^{\xi+(1-2j)\epsilon}.$$

1.3 Families of embeddings

A variant of the multirelative intersection problem involves families of smooth embeddings. In this instance one is given a map of $(j+1)$–ads $f\colon P \times \Delta^{j-1} \to N$ which is also a $(j-1)$–parameter family of smooth embeddings from P to N. The solution of the problem in this case is to find a deformation of ad-maps, this time through an isotopy, to a $(j-1)$–parameter family of embeddings having image disjoint from Q_J.

By combining Theorem A with Theorem E of Goodwillie and Klein [6], we obtain:

Theorem D (multiple disjunction) *Assume*

$$p, q_i \le n-3 \quad \textit{and} \quad p \le 1 + \min(n-p-2, \mu) + \Sigma.$$

Then $\chi(f) = 0$ if and only if the multirelative intersection problem of embeddings has a solution.

1.4 The embedding tower

For a smooth manifold P of dimension p without boundary and a smooth manifold N of dimension n, possibly with boundary, let $E(P, N)$ denote the space of smooth embeddings. When P is closed, Weiss [30] exhibits a tower of fibrations

$$\cdots \to E_2(P, N) \to E_1(P, N)$$

and compatible maps $E(P, N) \to E_k(P, N)$. Up to homotopy, the j^{th} layer of the tower is given by the space of compactly supported global sections of a certain fibration over the configuration space $\binom{P}{j}$, the latter given by the space of subsets of P having cardinality j. The space $E_j(P, N)$ is in some sense the best approximation to $E(P, N)$

obtained from spaces of embeddings $E(U,N)$ as U ranges throughout the open subsets of P that are diffeomorphic to a disjoint union of at most j open balls. In what follows, we assume that P is compact.

If $p \le n-1$, then $E_1(P,N)$ has the homotopy type of the space of immersions of P in N. If $p \le n-3$, then the map

$$E(P,N) \to \lim_{j\to\infty} E_j(P,N)$$

is a homotopy equivalence; see Goodwillie and Weiss [8] and Goodwillie and Klein [6]. The above motivates the following question: given a point of some stage of the tower, say $E_{j-1}(P,N)$, what are the obstructions to lifting the given point to the embedding space? If $j=2$, the work of Haefliger [10], Dax [2], Salomonsen [26] and Hatcher and Quinn [12] provides answers to this question in the metastable range (for the discussion of this case in the context of the tower, see [30, Section 4]).

It will be convenient to consider the following modification of this problem. Fix a basepoint of $E_1(P,N)$, ie an immersion. Let $\bar{E}_j(P,N)$ be the fiber of $E_j(P,N) \to E_1(P,N)$. Then the tower

$$\cdots \to \bar{E}_2(P,N) \to \bar{E}_1(P,N) = *$$

converges to $\bar{E}(P,N) = \mathrm{fiber}(E(P,N) \to E_1(P,N))$. Furthermore, the layers of this tower for $j>1$ coincide with the layers of the embedding tower.

Recall that $J = \{1,\dots,j\}$. In Section 7 we construct a fiberwise spectrum with Σ_j–action $\mathscr{C}_J$ over the configuration space $E_J(P) := E(J,P)$, which depends only on the data P, N and j. Let τ be the tangent bundle of $E_J(P)$ (ie restriction of the cartesian product of j copies of the tangent bundle of P). Then we can twist $\mathscr{C}_J$ by $-\tau$ to obtain a fiberwise spectrum with Σ_j–action ${}^{-\tau}\mathscr{C}_J$ over $E_J(P)$. In particular, one can speak about the equivariant homology of $E_J(P)$ with coefficients in ${}^{-\tau}\mathscr{C}_J$.

We will define an invariant

$$\mu\colon \pi_0(\bar{E}_{j-1}(P,N)) \to H_0^{\Sigma_j}(E_J(P); {}^{-\tau}\mathscr{C}_J)$$

which vanishes on the image of $\pi_0(\bar{E}_j(P,N))$.

Theorem E *Assume $j \ge 2$ and N is r–connected with $r \le n-2$. Assume additionally*

$$r \ge p-1-(j-1)(n-p-2).$$

If $x \in \bar{E}_{j-1}(P,N)$, then $\mu(x)=0$ implies that x lifts to $\bar{E}_j(P,N)$.

If N is contractible then we can take $r = n-2$. In this case the displayed inequality $r \geq p-1-(j-1)(n-p-2)$ is automatically satisfied:

Corollary F *Assume $j \geq 2$ and that N is contractible. If $\mu(x) = 0$, then $x \in \bar{E}_{j-1}(P,N)$ lifts to $\bar{E}_j(P,N)$.*

Remark 1.8 By [8], the map $\bar{E}(P,N) \to \bar{E}_j(P,N)$ is $((j+1)(n-p-2)+3-n)$–connected. Consequently, in both Theorem E and Corollary F, if $\mu(x) = 0$, then $x \in \bar{E}_{j-1}(P,N)$ will lift to $\bar{E}(P,N)$ if in addition $(j+1)(n-p-2)+3-n \geq 0$.

1.5 Link maps

Our main results can also be used to study higher-order linking phenomena. Given connected closed manifolds $P_1, \dots, P_j$ and a connected manifold N, a (j–component) *link map* is a continuous function

$$f\colon P_1 \sqcup \cdots \sqcup P_j \to N$$

such that $f(P_i) \cap f(P_k) = \varnothing$ for $i \neq k$. The space of link maps will be denoted by $\mathscr{L}(\boldsymbol{P}, N)$.[1] Fix an embedding $J \to N$, where we recall again that $J = \{1, 2, \dots, j\}$. We will also identify J with its image in N.

We define the *trivial link map* to be the link map given by sending the component P_i to $i \in J$, ie the trivial link map factors as the composition $P_1 \sqcup \cdots \sqcup P_j \to J \subset N$, where the first map is the canonical surjection from a space onto its set of components. The trivial link map equips $\mathscr{L}(\boldsymbol{P}, N)$ with a basepoint. A link map is *trivializable* if it admits a path to the trivial link map in the space of link maps.

Definition 1.9 The space of *(homotopy coherent) Brunnian link maps*

$$\mathscr{B}(\boldsymbol{P}, N)$$

is the total homotopy fiber of the j–cube of based spaces

$$S \mapsto \mathscr{L}^S(\boldsymbol{P}, N),$$

where $\mathscr{L}^S(\boldsymbol{P}, N)$ is the space of maps $f\colon P_1 \sqcup \cdots \sqcup P_j \to N$ such that for every $S \subset J$ the restriction

$$f_S := f|_{P_S}\colon \coprod_{i \in S} P_i \to N$$

is an $|S|$–component link map.

[1] The path components of $\mathscr{L}(\boldsymbol{P}, N)$ are called *link homotopy classes*. The latter is usually studied in the special case when $N = \mathbb{R}^n$ and the P_i are spheres; see Milnor [21], Massey [19] and Koschorke [18].

Since $\mathscr{B}(\boldsymbol{P}, N)$ is the homotopy fiber of the map

$$\mathscr{L}^J(\boldsymbol{P}, N) \to \operatorname*{holim}_{S \subsetneq J} \mathscr{L}^J(\boldsymbol{P}, N),$$

a point of $\mathscr{B}(\boldsymbol{P}, N)$ determines a link map $f \in \mathscr{L}^J(\boldsymbol{P}, N)$ with the property that any proper sublink map is trivializable. In particular, f satisfies the classical Brunnian condition; see Milnor [21] and Debrunner [3].

Restricting now to the case when $N = \mathbb{R}^n$, we will construct in Section 8 a *higher stable linking number map*[2]

$$\lambda \colon \mathscr{B}(\boldsymbol{P}, \mathbb{R}^n) \to \prod_{i=1}^{(j-2)!} F^{\mathrm{st}}\Big(\prod_{i=1}^{j} P_i, S^{(j-1)(n-2)+1}\Big), \tag{3}$$

where for an unbased space X and a spectrum E, $F^{\mathrm{st}}(X, E)$ denotes the function space of stable maps from X to E, ie the function space $F(X, \Omega^\infty E)$.

A result of Goodwillie and Munson in the case $j = 2$ [7, Theorem 1.1] suggests to us the following:

Conjecture G *The map λ is $(1+\Sigma')$–connected, where*

$$\Sigma' = \sum_{i=1}^{j} (n - 2p_i - 2).$$

(For variant forms of this statement see Section 8.) We submit the following evidence for Conjecture G:

Theorem H (realization of higher linking numbers) *Assume that P_i embeds in $\mathbb{R}^n$ and $n - p_i \geq 2$ for $2 \leq i \leq j$. Then the higher stable linking number map λ induces a surjection on homotopy groups in degrees $\leq 1 - \hat{p} + \Sigma$, where*

$$\hat{p} := \max_{2 \leq i \leq j} p_i \quad \text{and} \quad \Sigma = \sum_{i=1}^{j} (n - p_i - 2).$$

In the above, we do not need to assume that the embeddings are pairwise disjoint. Since $1 - \hat{p} + \Sigma \geq 1 + \Sigma'$, it follows that λ induces a surjection on homotopy groups in degrees $\leq 1 + \Sigma'$. Hence, Theorem H gives evidence for the validity of Conjecture G.

[2] For link maps of circles in three-dimensional euclidean space, it seems likely that on path components, our map coincides with Milnor's μ–invariants [21].

Further evidence is contained in Section 8. Our results on link maps overlap with those of Munson [22]. Our methods are homotopy-theoretical, whereas Munson relies on bordism and transversality. It seems likely to us that Theorem H could also be extracted from Munson's approach, possibly at the expense of a dimension.

Outline Section 2 is a breezy exposition on the basic definitions as well as the machinery used throughout the paper. Section 3 is about strongly cocartesian cubes of spaces, and the main technical results of the paper are stated there. Section 4 recasts the results of Section 3 in the setting of homotopical intersection theory to give a proof of Theorems A and C modulo the proof of Theorem 3.12. In Section 5 we prove Theorem 3.12, which is one of our main technical results. In Section 6 we combine Theorem A with [6, Theorem E] to obtain a multiple disjunction result for smooth embeddings. Section 7 contains the proof of Theorem E. In Section 8 we apply our machinery to the study of spaces of link maps.

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2 Language

2.1 Spaces

Let $\mathscr{T}$ be the category of compactly generated spaces. Then $\mathscr{T}$ is a Quillen model category in which the weak equivalences are the weak homotopy equivalences, the fibrations are the Serre fibrations and the cofibrations are the retracts of relative cell complexes [25, Chapter 2, Section 3] (a relative cell complex is a pair of spaces (Y, A) such that Y is obtained from A by attaching cells). A space X is *r–connected* if every map $S^k \to X$ for $k \le r$ is homotopic to a constant map; here S^k is the sphere of dimension k. In particular, the empty space is (-2)–connected and every nonempty space is (at least) (-1)–connected. A map $f\colon X \to Y$ is r–connected if its homotopy fiber at any basepoint is $(r-1)$–connected. An ∞–connected map is, by definition, a weak equivalence.

A commutative square of spaces

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{array} \tag{4}$$

is *r–cocartesian* if the map

$$\operatorname{hocolim}(B \leftarrow A \to C) \to D$$

is r–connected.

Dually, the square (4) is *r–cartesian* if the map

$$A \to \operatorname{hocolim}(B \to D \leftarrow C)$$

is r–connected.

Definition 2.1 Let

$$X \to Y \to Z \tag{5}$$

be maps of spaces equipped with a homotopy to a constant z. One says that (5) is a *homotopy fiber sequence in degrees $\le s$* if the induced map from X to the homotopy fiber of $Y \to Z$ is s–connected. If this condition holds for all integers s, then (5) is called a *homotopy fiber sequence*.

Dually, if the induced map from the homotopy cofiber of $X \to Y$ to Z is s–connected, then one says that (5) is a *homotopy cofiber sequence in degrees $\le s$* and a *homotopy cofiber sequence* if the condition holds for all s.

When the square (4) is ∞–cocartesian and C is contractible, $A \to B \to D$ is a homotopy cofiber sequence once a contraction $C \times [0,1] \to C$ is specified. The dual case is analogous.

2.2 Fiberwise spaces

For an object $X \in \mathscr{T}$, we let $\mathscr{T}(X)$ denote the category of spaces over X. This is the category whose objects are pairs (Y, r) such that $r\colon Y \to X$ is a map. A morphism $(Y, r) \to (Y', r')$ is a map $f\colon Y \to Y'$ such that $r' \circ f = r$. We more often than not suppress the structure map $r\colon Y \to X$ when specifying an object and write Y in place of (Y, r).

Similarly, let $\mathscr{R}(X)$ denote the category of retractive spaces over X. This has objects (Y, r, s) where $r\colon Y \to X$ and $s\colon X \to Y$ are maps such that $r \circ s$ is the identity map. A morphism $(Y, r, s) \to (Y', r', s')$ is a map $f\colon Y \to Y'$ such that $r' \circ f = r$ and $f \circ s = s'$. Again, the structure maps are usually supressed.

Note that the case $\mathscr{R}(*)$ gives the category of based spaces. We sometimes regard objects of $\mathscr{R}(X)$ as objects of $\mathscr{T}(X)$ by means of the forgetful functor. When $X = *$ we usually write $\mathscr{T}_*$ in place of $\mathscr{R}(*)$, ie the category of based spaces.

Both $\mathscr{T}(X)$ and $\mathscr{R}(X)$ have simplicial model category structures where a weak equivalence (cofibration, fibration) in each case is a morphism whose underlying map of spaces is a weak homotopy equivalence (cofibration, fibration) of spaces [25, Chapter II, page 2.8, Proposition 6]. In particular, the set of (fiberwise) homotopy classes $[Y, Z]_{\mathscr{T}(X)}$ is defined for objects Y and Z of $\mathscr{T}(X)$. Similarly, one can define homotopy classes in $\mathscr{R}(X)$. If $Y \in \mathscr{T}(X)$ is an object, let $Y^+ \in \mathscr{R}(X)$ be the object given by $Y \sqcup X$ with evident structure maps. If $Z \in \mathscr{R}(X)$ is an object, then we have $[Y^+, Z]_{\mathscr{R}(X)} = [Y, Z]_{\mathscr{T}(X)}$. As usual, when defining homotopy classes $[Y, Z]_{\mathscr{T}(X)}$, Y is replaced by a cofibrant approximation and Z is replaced by a fibrant approximation.

A morphism $Y \to Z$ in either $\mathscr{T}(X)$ or $\mathscr{R}(X)$ is said to be *j–connected* if and only if its underlying map in $\mathscr{T}$ is j–connected. An object Y is said to be j–connected if and only if the structure map $Y \to X$ is $(j+1)$–connected. A commutative square in $\mathscr{T}(X)$ or $\mathscr{R}(X)$ is *j–cocartesian* (j–cartesian) if it is so when considered in $\mathscr{T}$ (here j may be ∞).

We say an object Y of $\mathscr{T}(X)$ or $\mathscr{R}(X)$ has dimension $\le s$ if it is built up from the initial object by attaching cells of dimension at most s. In $\mathscr{T}(X)$ this means that the underlying space of Y is a cell complex of dimension at most s. In $\mathscr{R}(X)$ it means that the pair (Y, X) is a relative cell complex of dimension at most s. In either case we write $\dim Y \le s$.

A sequence of maps $A \to Y \to C$ in $\mathscr{T}(X)$ forms a *homotopy cofiber sequence* (respectively in degrees $\le r$) if it comes equipped with a homotopy from $A \to C$ to a composition of the form $A \to X \to C$ (where X is viewed as the terminal object) such that the induced map from the homotopy cofiber of $A \to Y$ (ie the homotopy colimit of $X \leftarrow A \to Y$) to C is a weak equivalence (respectively r–connected). The dual notion of homotopy fiber sequence (in degrees $\le r$) is defined analogously.

Lemma 2.2 *Suppose that $A \to Y \to C$ is a homotopy cofiber sequence of $\mathscr{T}(X)$. Assume that A is r_1–connected and C is r_2–connected. Then $A \to Y \to C$ is a homotopy fiber sequence in dimensions $\le r_1 + r_2$.*

Proof The square

$$\begin{array}{ccc} A & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & C \end{array}$$

has a preferred commuting homotopy making it ∞–cocartesian. The result follows from the Blakers–Massey theorem [11, Theorem 4.23; 4, page 309]. □

Corollary 2.3 *Assume in addition that $Z \in \mathscr{T}(X)$ is an object of dimension $\le r_1 + r_2$. Then the sequence of sets*

$$[Z, A]_{\mathscr{T}(X)} \to [Z, Y]_{\mathscr{T}(X)} \to [Z, C]_{\mathscr{T}(X)}$$

is exact.

(Explanation: The set $[Z, C]_{\mathscr{T}(X)}$ has a preferred basepoint given by $Z \to X' \to C$. Any element of $[Z, Y]_{\mathscr{T}(X)}$ which maps to the basepoint lifts back to $[Z, A]_{\mathscr{T}(X)}$.)

2.3 Fiberwise suspension

The *unreduced fiberwise suspension* of an object $Y \in \mathscr{T}(X)$ is the object of $\mathscr{R}(X)$ given by the double mapping cylinder

$$S_X Y := (X \times 0) \cup (Y \times [0,1]) \cup (X \times 1),$$

where the structure map $S_X Y \to X$ is obvious and the structure map $X \to S_X Y$ is given by $X \times 0$. This gives a functor $S_X \colon \mathscr{T}(X) \to \mathscr{R}(X)$. Similarly, $\mathscr{R}(X)$ has a *reduced fiberwise suspension* functor $\Sigma_X \colon \mathscr{R}(X) \to \mathscr{R}(X)$ defined as follows: given an object $Y \in \mathscr{R}(X)$, we take $\Sigma_X Y$ to be the pushout of the diagram $X \leftarrow S_X X \to S_X Y$. If Y is cofibrant, then the map $S_X Y \to \Sigma_X Y$ is a weak equivalence. The functor Σ_X has a right adjoint Ω_X, called the *fiberwise loop functor*.

Given objects $Y, Z \in \mathscr{R}(X)$, define

$$\{Y, Z\}_{\mathscr{R}(X)} := \operatorname*{colim}_k [\Sigma_X^k Y, \Sigma_X^k Z].$$

This is the abelian group of fiberwise stable homotopy classes from Y to Z.

2.4 Fiberwise smash product

Given objects $Y, Z \in \mathscr{T}(X)$, we have the fiber product $Y \times_X Z \in \mathscr{T}(X)$, which is defined as the limit of the diagram $Y \to X \leftarrow Z$. If $Y, Z \in \mathscr{R}(X)$, the fiberwise wedge (or coproduct) $Y \vee_X Z$ is the object of $\mathscr{R}(X)$ given by the pushout of the inclusions $Y \supset X \subset Z$. The *(internal fiberwise) smash product* is the object $Y \wedge_X Z$ given by the pushout of the diagram $X \leftarrow Y \vee_X Z \subset Y \times_X Z$. As is usual with most functors in the model category-theoretic setting, this construction needs to be suitably derived to have a meaningful homotopy type (in this instance Y and Z should be made fibrant and cofibrant). To avoid notational clutter, we will be intentionally sloppy: we will write the underived smash product but the reader should understand that it needs to be derived to have a sensible homotopy-theoretic meaning.

2.5 Fiberwise Thom spaces

Given an object $Y \in \mathscr{T}(X)$ and an inner product bundle ξ over Y, the *fiberwise Thom space* is the object of $\mathscr{R}(X)$ given by

$$T_X(\xi) = D(\xi) \cup_{S(\xi)} X.$$

By collapsing X to a point we obtain the usual Thom space $X^\xi := D(\xi)/S(\xi)$, which in the present notation appears as $T_*(\xi)$.

Let η be an inner product bundle over another object $Z \in \mathscr{T}(X)$. Let $p\colon Y \times_X Z \to Y$ and $q\colon Y \times_X Z \to Z$ be the projections. Then the Whitney sum $p^*\xi \oplus q^*\eta$ is an inner product bundle over $Y \times_X Z$. The following is just an unraveling of definitions (and is well known when X is a point):

Lemma 2.4 *There is a preferred isomorphism of* $\mathscr{R}(X)$,

$$T_X(p^*\xi \oplus q^*\eta) \cong T_X(\xi) \wedge_X T_X(\eta).$$

2.6 Fiberwise spectra

Using Σ_X also enables one to define spectra built from objects of $\mathscr{R}(X)$. A *fiberwise spectrum* $\mathscr{E}$ is a collection of objects $\mathscr{E}_n \in \mathscr{R}(X)$ for $n = 0, 1, \ldots$ together with morphisms $\Sigma_X \mathscr{E}_n \to \mathscr{E}_{n+1}$. Note that $\mathscr{E}$ comes equipped with a *zero section*, namely the collection of structure maps $X \to \mathscr{E}_n$ for $n \geq 0$. A morphism of fiberwise spectra is the evident thing.

If $\mathscr{E}$ is a fiberwise spectrum then the associated fiberwise infinite loop space $\Omega_X^\infty \mathscr{E}$ is an object of $\mathscr{R}(X)$. Fiberwise spectra form a model category (see eg [27]; for a more detailed treatment see [20]).

Here are two examples:

Example 2.5 (trivial fiberwise spectra) Start with an ordinary spectrum E given by based spaces $\{E_n\}_{n\geq 0}$ and structure maps $\Sigma E_n \to E_{n+1}$. Form $E_n \times X$ for $n \geq 0$. These fit into a fiberwise spectrum $E \times X$, where the structure map $\Sigma_X(E_n \times X) \to E_{n+1} \times X$ is given by noticing that $\Sigma_X(E_n \times X) \cong (\Sigma E_n) \times X$.

Example 2.6 (fiberwise suspension spectra) Start with any object $Y \in \mathscr{R}(X)$ and form the iterates $\Sigma_X^n Y$. These give a fiberwise spectrum $\Sigma_X^\infty Y$, using the identity maps for the structure maps.

We remark that the zero section of $\mathscr{E}$ gives a morphism $\Sigma_X^\infty X^+ \to \mathscr{E}$.

Given an object $Z \in \mathscr{R}(X)$ and a fiberwise spectrum $\mathscr{E}$, we define

$$\{Z, \mathscr{E}\}_{\mathscr{R}(X)} := \operatorname*{colim}_n [Z, \Omega_X^\infty \mathscr{E}]_{\mathscr{R}(X)}.$$

For example, if $\mathscr{E} = \Sigma_X^\infty Y$ is a fiberwise suspension spectrum, then $\{Z, \mathscr{E}\}_{\mathscr{R}(X)} = \{Z, Y\}_{\mathscr{R}(X)}$.

2.7 Homology and cohomology

Let $\mathscr{E}$ be a fiberwise spectrum over X (which we take to be fibrant). Then an object $Z \in \mathscr{T}(X)$ (which we take to be cofibrant) with structure map $p\colon Z \to X$ gives rise to a fiberwise spectrum over Z,

$$p^*\mathscr{E},$$

whose k^{th} space is the pullback of $\mathscr{E}_k \to X$ along p. Let $(p^*\mathscr{E})^\flat$ denote the effect of making $p^*\mathscr{E}$ cofibrant. Then for each $n \geq 0$ we have a cofibration $Z \to (p^*\mathscr{E})^\flat_n$ and as n varies the quotient spaces $(p^*\mathscr{E})^\flat_n / Z$ form a spectrum, denoted by $H_\bullet(Z; \mathscr{E})$. The *homology groups* of Z with coefficients in $\mathscr{E}$ are the homotopy groups of this spectrum.

To define cohomology we take, for each n, the space of sections of $\mathscr{E}_n \to X$ along the map $Z \to X$ (this is the same thing as the space of maps $Z \to \mathscr{E}_n$ which commute with the structure map to X). As n varies, these spaces form a spectrum $H^\bullet(Z;\mathscr{E})$. The *cohomology groups* of Z with coefficients in $\mathscr{E}$ are defined to be homotopy groups of this spectrum, ie

$$H^i(Z;\mathscr{E}) = \{Z^+, \Sigma^i_X \mathscr{E}\}_{\mathscr{R}(X)}.$$

2.8 Induction and restriction

Let $f\colon X \to Y$ be a map of spaces. Then a fiberwise spectrum $\mathscr{E}$ over Y gives rise to a fiberwise spectrum $f^*\mathscr{E}$ over X by taking base change. This operation defines a *restriction* functor from fiberwise spectra over Y to fiberwise spectra over X (the construction is homotopy-invariant when $\mathscr{E}$ is fibrant). Using f to regard X as an object of $\mathscr{T}(Y)$, we obtain a tautological identification $H^\bullet(X;\mathscr{E}) = H^\bullet(X, f^*\mathscr{E})$, where on the right side X is viewed as an object of $\mathscr{R}(X)$ using the identity.

Suppose $\mathscr{F}$ is a fiberwise spectrum over X. Then we obtain a fiberwise *pushforward* spectrum over Y, denoted $f_*\mathscr{F}$ in which $(f_*\mathscr{F})_k = (\mathscr{F}_k) \cup_f Y$ (the construction is homotopy-invariant when $\mathscr{F}$ is cofibrant). The operation $E \mapsto f_*E$ is also called *induction.* Note that $H_\bullet(X;\mathscr{F}) = H_\bullet(Y; f_*\mathscr{F})$ tautologically. Note also that (f_*, f^*) is an adjoint pair.

2.9 Poincaré duality

Let ξ be a finite-dimensional vector bundle over X. Let S^ξ denote the fiberwise one-point compactification of ξ. Then S^ξ is an object of $\mathscr{R}(X)$. More generally, if ξ is a virtual bundle, ie $\xi + \epsilon^j$ is identified with a finite-dimensional vector bundle η for some j, then we define S^ξ is this case to be a fiberwise spectrum over X given by the j–fold desuspension of S^η.

Given a fiberwise spectrum $\mathscr{E}$ over X, set

$$^{\xi}\mathscr{E} := S^\xi \wedge_X \mathscr{E}.$$

When ξ is a vector bundle, the definition of the right side is given by the fiberwise smash products in each degree, ie $S^\xi \wedge_X \mathscr{E}_k$. In the virtual bundle case one merely fiberwise desuspends $S^\eta \wedge_X \mathscr{E}$ j times.

Theorem 2.7 (Poincaré duality [14; 15, Theorem 6.2; 20, Theorem 19.6.1]) *Suppose $f\colon P \to X$ is a map in which P is a closed smooth manifold of dimension d. Let*

$-\tau_P$ be the virtual stable normal bundle given by the negation of the tangent bundle of P. Then, for any fiberwise spectrum $\mathscr{E}$ over X, there is a preferred weak equivalence of spectra

$$H^\bullet(P;\mathscr{E}) \simeq H_\bullet(P;{}^{-\tau_P}f^*\mathscr{E}).$$

Remark 2.8 More generally, if P is an open manifold then there is a weak equivalence

$$H^\bullet_{\mathrm{cs}}(P;\mathscr{E}) \simeq H_\bullet(P;{}^{-\tau_P}f^*\mathscr{E}),$$

where the left side denotes cohomology with compact supports. The latter is defined by taking the spectrum of sections of $\mathscr{E}$ which coincide with the zero section near infinity.

3 Strongly cocartesian cubes

3.1 Cubical diagrams

For a finite set J, we let 2^J be the poset of consisting of the subsets of J partially ordered by inclusion. A *J–cube* in a category $\mathscr{C}$ is a contravariant functor

$$A_\bullet\colon 2^J \to \mathscr{C}, \quad S \mapsto A_S.$$

(If J has cardinality j, we also say that $A_\bullet$ is a *j–cube*.) Since $A_\bullet$ is contravariant, the initial vertex is A_J and the terminal vertex is $A_\varnothing$. When $J=\{i\}$ we usually write $A_S = A_i$.

In what follows we will only consider J–cubes in which the target category $\mathscr{C}$ is either $\mathscr{T}(X)$ or $\mathscr{R}(X)$ for some space X, and, often enough, we shall be interested in the case when X is a point.

A *weak equivalence* of T–cubes $A_\bullet \to B_\bullet$ is a natural transformation such that $A_S \to B_S$ is a weak equivalence for each S, ie an objectwise weak equivalence. Two J–cubes are said to be *weakly equivalent* if there is a finite zigzag of weak equivalences connecting them.

Definition 3.1 [4, Definition 1.3] A J–cube $A_\bullet$ is *r–cartesian* if the map

$$A_J \to \operatorname*{holim}_{S\subsetneq J} A_S \tag{6}$$

is r–connected. Similarly, $A_\bullet$ is *r–cocartesian* if the map

$$\operatorname*{hocolim}_{S\neq\varnothing} A_S \to A_\varnothing \tag{7}$$

is r–connected. In both cases r may be ∞.

We remark that when $A_\bullet$ is a cube in which the maps $A_S \to A_T$ are based for $|S| < j$, the target of (6) inherits a basepoint. In this case, we will say that $A_\bullet$ is *almost based*.

Definition 3.2 The *total homotopy cofiber* of $A_\bullet$ is the homotopy cofiber of the map (7). If $A_\bullet$ is an almost based J–cube, then its *total homotopy fiber* is the homotopy fiber of (7) taken at the preferred basepoint.

For fixed subsets $U \subset W \subset J$, one has a (W,U)*–face* of $A_\bullet$ given by restricting $A_\bullet$ to those A_S for which $U \subset S \subset W$. This is a $(W \setminus U)$–cube and every face of $A_\bullet$ arises in this fashion. When $|W \setminus U| = k$ we also call this a k*–face* of $A_\bullet$.

Definition 3.3 [4, Definition 2.1] A J–cube $A_\bullet$ is *strongly cocartesian* if each 2–face of $A_\bullet$ is ∞–cocartesian.

In Definition 3.3, it is enough to check the condition on each 2–face meeting the initial vertex A_J (ie those (W,U)–faces in which $|W \setminus U| = 2$ and $W = J$; see loc. cit.).

Henceforth, we set

$$J := \{1, 2, \dots, j\}.$$

Example 3.4 (wedge cubes) Let $X_1, \dots, X_j$ be cofibrant based spaces. For $T \subset J$, let A_T be the wedge $\bigvee_{i \in T} X_i$ (by convention $A_\varnothing$ is a point). This defines a strongly cocartesian j–cube $A_\bullet$ whose maps are given by projections onto summands.

More generally, let $X_1, \dots, X_j \in \mathscr{R}(X)$ be cofibrant. Let A_T be the fiberwise wedge of X_i as i varies in T. Then $A_\bullet$ is strongly cocartesian.

Example 3.5 (backwards wedge cubes) With $X_1, \dots, X_j \in \mathscr{R}(X)$ as above, let B_T be the fiberwise wedge of those X_i with $i \in J \setminus T$. The maps of this cube are inclusions of summands. Then $B_\bullet$ is strongly cocartesian.

Example 3.6 (suspension) Let $A_\bullet$ be a strongly cocartesian j–cube of $\mathscr{T}(X)$. Then the j–cube $S_X A_\bullet$ given by $T \mapsto S_X A_T$ is also strongly cocartesian. Similarly, if $A_\bullet$ is a strongly cocartesian j–cube of $\mathscr{R}(X)$, then the cube of reduced fiberwise suspensions $\Sigma_X A_\bullet$ is strongly cocartesian.

Lemma 3.7 *Let $A_\bullet$ be a strongly cocartesian j–cube of connected based spaces in which $A_\varnothing$ is a point. Then the suspended j–cube $\Sigma A_\bullet$ is weakly equivalent to a wedge cube $B_\bullet$ in which $B_i = \Sigma A_i$ for $i \in J$.*

Proof The following sketch was provided to us by Tom Goodwillie. Let B_T be the wedge of ΣA_i for all $i \in T$, but write this as the wedge, over all $i \in J$, of either

- ΣA_i if $i \in T$, or
- $*$ if $i \notin T$.

Define a map $\Sigma A_T \to B_T$ as follows. First do a pinch to go from ΣA_T to the wedge of j copies of ΣA_T indexed by $i \in J$. Now map that to B_T by sending the i^{th} copy of ΣA_T to ΣA_i using the original map $A_T \to A_i$ if $i \in T$, or the constant map to a point if $i \notin T$.

The above recipe defines a map of j–cubes $\Sigma A_\bullet \to B_\bullet$. By the Whitehead theorem, it suffices to show that the map $\Sigma A_T \to B_T$ is a homology isomorphism for all $T \subset J$. Let C_T be the homotopy cofiber of this map. Then $T \mapsto C_T$ is also a strongly cocartesian j–cube. It is enough to show that C_T has trivial reduced homology. If T is a singleton, this is clear since the maps $\Sigma A_i \to B_i$ are homotopic to the identity. By a straightforward induction argument, we can assume that C_T has trivial homology for $|T| \le j-1$. We are reduced to showing that C_J has trivial homology. But the homology of C_J coincides with the homology of the total homotopy cofiber of the cube $C_\bullet$ with a degree shift by j. Since $C_\bullet$ is strongly cocartesian, the total homotopy cofiber is contractible. Hence, C_J has trivial homology. □

Given a strongly cocartesian j–cube $A_\bullet$, let $C(A_\bullet)$ denote the homotopy colimit

$$\operatorname{hocolim}\bigl(A_\varnothing \leftarrow A_J \to \operatorname*{holim}_{S\neq J} A_S\bigr). \tag{8}$$

Then $C(A_\bullet)$ is a retractive space over $A_\varnothing$. In what follows we rename

$$X := A_\varnothing.$$

Then $C(A_\bullet) \in \mathscr{R}(X)$ and one has a homotopy cofiber sequence of $\mathscr{T}(X)$

$$A_J \to \operatorname*{holim}_{S\neq J} A_S \to C(A_\bullet). \tag{9}$$

Notation 3.8 For a sequence of integers $r_1, \dots, r_j$ we write

$$\Sigma = \sum_i r_i \quad \text{and} \quad \mu = \min_i r_i .$$

If $1 \le i \le j$ and $T \subset J$, set

$$T_i := T \setminus \{i\}.$$

Hypothesis 3.9 X is 0–connected. Furthermore, for $1 \le i \le j$, the map

$$A_J \to A_{J_i}$$

is (r_i+1)–connected, where $r_i \ge 0$.

Note that $A_T \to A_{T_i}$ is also (r_i+1)–connected for all $T \subset S$, since $A_\bullet$ is strongly cocartesian. We assume Hypothesis 3.9 holds throughout the rest of this section.

Proposition 3.10 *Let $Z \in \mathscr{T}(X)$ be an object of dimension $\le 1+\mu+\Sigma$. Then the sequence*

$$[Z, A_J]_{\mathscr{T}(X)} \to \big[Z, \operatorname*{holim}_{S \neq \varnothing} A_S\big]_{\mathscr{T}(X)} \to [Z, C(A_\bullet)]_{\mathscr{T}(X)}$$

is exact.

Remark 3.11 The set $[Z, C(A_\bullet)]_{\mathscr{T}(X)}$ is pointed. As in Corollary 2.3, exactness means that an element of $[Z, \operatorname{holim}_{S\neq\varnothing} A_S]_{\mathscr{T}(X)}$ pushes forward to the basepoint if and only if it lifts to an element of $[Z, A_J]_{\mathscr{T}(X)}$.

Proof The object $A_J \in \mathscr{T}(X)$ is μ–connected. The higher Blakers–Massey theorem for cubical diagrams [4, Theorem 2.5] (or see [5, Theorem 2.3]) says that $A_\bullet$ is $(1+\Sigma)$–cartesian, Consequently, $C(A_\bullet) \in \mathscr{T}(X)$ is a $(1+\Sigma)$–connected object. The conclusion now follows from Corollary 2.3. □

3.2 Identification of $C(A_\bullet)$

In the remainder of this section we identify $C(A_\bullet)$ up through dimension $1+\mu+\Sigma$.

Let

$$W_j := \bigvee_{(j-1)!} S^{2-2j} \tag{10}$$

be the wedge of $(j-1)!$ copies of the $(2-2j)$–sphere spectrum.

Let

$$\mathscr{W}_j = X \times W_j$$

be the trivial fiberwise spectrum on W_j.

Theorem 3.12 *With respect to the above assumptions, there is a preferred map*

$$C(A_\bullet) \to \Omega^\infty\Big(\mathscr{W}_j \wedge_X \bigwedge_{i\in J}{}_X\, S_X A_i\Big), \tag{11}$$

which is $(2+\mu+\Sigma)$–connected.

The proof of Theorem 3.12 is deferred to Section 5. If we combine Theorem 3.12 with Proposition 3.10, we obtain:

Corollary 3.13 *Let* $Z \in \mathscr{T}(X)$ *be an object such that* $\dim Z \le 1 + \mu + \Sigma$. *Then there is an exact sequence*

$$[Z, A_J]_{\mathscr{T}(X)} \to \big[Z, \operatorname*{holim}_{S \neq \varnothing} A_S\big]_{\mathscr{T}(X)} \to \Big\{Z^+, \mathscr{W}_j \wedge_X \bigwedge_{i \in J}{}_X\, S_X A_i\Big\}_{\mathscr{R}(X)}.$$

Remark 3.14 Corollary 3.13 is a robust generalization of a result of Barratt and Whitehead [1] and, independently, Toda [29].

3.3 The Euler class

Let $f\colon Z \to \operatorname{holim}_{S \neq J} A_S$ be a map of spaces. Then f is also a morphism of $\mathscr{T}(X)$. Using Theorem 3.12, we see that the composed map

$$Z^+ \xrightarrow{f} \operatorname*{holim}_{S \neq J} A_S \to C(A_\bullet)$$

gives rise to a fiberwise stable homotopy class

$$e(f) \in \Big\{Z^+, \mathscr{W}_j \wedge_X \bigwedge_{i \in J}{}_X\, S_X A_i\Big\}_{\mathscr{R}(X)},$$

which we call the *Euler class* of f. Equivalently, $e(f)$ resides in the cohomology group

$$H^0\Big(Z; \mathscr{W}_j \wedge_X \bigwedge_{i \in J}{}_X\, S_X A_i\Big).$$

Then, from Corollary 3.13, we deduce:

Corollary 3.15 *The Euler class* $e(f)$ *vanishes when* f *admits a homotopy factorization through* A_J. *Conversely, if* $\dim Z \le 1 + \mu + \Sigma$ *and* $e(f) = 0$, *then* f *admits a homotopy factorization through* A_J.

3.4 A special case

When $X = A_0$ is a point, the above results can be expanded upon as follows: There is a homotopy cofiber sequence of spaces

$$A_J \to \operatorname*{holim}_{S \neq \varnothing} A_S \to C(A_\bullet) \tag{12}$$

and a $(2+\mu+\Sigma)$–connected map

$$C(A_\bullet) \to \Omega^\infty\Big(W_j \wedge \bigwedge_{i \in J} SA_i\Big). \tag{13}$$

Furthermore, the space A_J is μ–connected. If we choose a basepoint in A_J then $A_\bullet$ becomes a cube of based spaces. Let $F(A_\bullet)$ be its total homotopy fiber. By the Blakers–Massey theorem applied to (12) and using the map (13), we infer:

Corollary 3.16 *There is a* $(1+\mu+\Sigma)$*–connected map*

$$F(A_\bullet) \to \Omega^\infty\Big(\Sigma^{j-1} W_j \wedge \bigwedge_{i\in J} A_i\Big) \simeq \prod_i^{(j-1)!} Q(\Sigma^{1-j} A_1 \wedge \cdots \wedge A_j).$$

Remark 3.17 The proof we give of Theorem 3.12 implies that the map of Corollary 3.16 is natural with respect to morphisms of based cubes $A_\bullet \to B_\bullet$.

4 Proof of Theorems A and C

In this section we give the proof of Theorems A and C modulo the proof of Theorem 3.12. The proof of the latter result will appear in Section 5.

Returning to the situation of Section 1, we are given pairwise disjoint, connected, closed submanifolds $Q_1, \dots, Q_j \subset N$. Let $N \setminus Q_\bullet$ denote the j–cubical diagram of $\mathscr{R}(N)$ defined by

$$S \mapsto N \setminus Q_S, \quad S \subset J.$$

Note that $N \setminus Q_\bullet$ satisfies Hypothesis 3.9 since $n - q_i \geq 2$.

Proof of Theorem A Recall that we are given a map

$$f\colon P \to \operatorname*{holim}_{S \subsetneq J} (N \setminus Q_S)$$

and we wish to identify the obstructions to deforming it into $N \setminus Q_J$. By transversality, the map $N \setminus Q_J \to N \setminus Q_{J-\{i\}}$ is $(n - q_i - 1)$–connected for $1 \leq i \leq j$. By Corollary 3.15, we infer:

Proposition 4.1 *If* $P \to \operatorname{holim}_{S\subsetneq J} N \setminus Q_S$ *admits a homotopy factorization through* $N \setminus Q_J$*, then* $e(f) = 0$*. The converse is true provided* $p \leq 1 + \mu + \Sigma$*, where* $\Sigma = \sum_i (n - q_i - 2)$ *and* $\mu_i = \min_i (n - q_i - 2)$*.*

Proof This follows from Corollary 3.15 since a closed manifold P of dimension p admits the structure of a cell complex of dimension p. □

Let ν_i be the normal bundle of Q_i in N. The tubular neighborhood theorem gives a weak equivalence of $\mathscr{R}(N)$,

$$S_N(N \setminus Q_1) \simeq D(\nu_i) \cup_{S(\nu_i)} N =: T_N(\nu_i),$$

where the right side is the fiberwise Thom space of ν_i over N.

Stably, we can identify ν_i with the virtual bundle $\xi_i := f^*\tau_N - \tau_{Q_i}$, given by the difference of tangent bundles. We write $T_N(\xi_i)$ for the associated fiberwise Thom spectrum. With these notational changes, $e(f)$ can be regarded as residing in the cohomology group

$$H^0\Big(P; \mathscr{W}_j \wedge_N \bigwedge_{i\in J}{}_N\, T_N(\xi_i)\Big). \tag{14}$$

The remainder of the proof of Theorem A will involve application of Poincaré duality (Theorem 2.7) to the cohomology group (14).

4.1 The Euler characteristic

By Poincaré duality (Theorem 2.7), $e(f)$ corresponds to a homology class

$$\chi(f) \in H_0\Big(P; {}^{-\tau_P} f^*\Big(\mathscr{W}_j \wedge_N \bigwedge_{i\in J}{}_N\, T_N(\xi_i)\Big)\Big).$$

Using the induction isomorphism (Section 2.8), the group where $\chi(f)$ resides can alternatively be written as

$$H_0\Big(N; f_*{}^{-\tau_P} f^*\Big(\mathscr{W}_j \wedge_N \bigwedge_{i\in J}{}_N\, T_N(\xi_i)\Big)\Big).$$

By definition, the latter is the stable homotopy group in degree zero of the spectrum

$$\Big(\mathscr{W}_j \wedge_N T_N(-\tau_P) \wedge_N \bigwedge_{i\in J}{}_N\, T_N(\xi_i)\Big)/N.$$

Using Lemma 2.4 in virtual form, we deduce that the fiberwise spectrum

$$\mathscr{W}_j \wedge_N T_N(-\tau_P) \wedge_N \bigwedge_{i\in J}{}_N\, T_N(\xi_i)$$

can be rewritten up to homotopy as

$$\mathscr{W}_j \wedge_N T_N(\xi),$$

where ξ is the virtual bundle over $E(P, Q_\bullet)$ that was defined in Section 1.

Recall that $\mathscr{W}_j$ is just the fiberwise wedge of $(j-1)!$ copies of the fiberwise spectrum $N \times S^{2-2j}$. From this we infer

$$(\mathscr{W}_j \wedge_N T_N(\xi))/N \simeq \bigvee_{(j-1)!} \Sigma^{2-2j} E(P, Q_\bullet)^\xi.$$

Since

$$\pi_0(\Sigma^{2-2j} E(P, Q_\bullet)^\xi) \cong \Omega_{2j-2}(E(P, Q_\bullet); \xi),$$

we have deduced that the obstruction $\chi(f)$ resides in the abelian group

$$\bigoplus_{(j-1)!} \Omega_{2j-2}(E(P, Q_\bullet); \xi).$$

By Proposition 4.1, $\chi(f)$ vanishes whenever $f\colon P \to \operatorname{holim}_{S \subsetneq J}(N \setminus Q_S)$ admits a homotopy factorization through $N \setminus Q_J$. Conversely, if $p \le 1+\mu+\Sigma$, then $\chi(f) = 0$; we have shown there is such a factorization of f. □

Proof of Theorem C Given a multirelative intersection problem f, recall that the solution space $\mathscr{L}(f)$ is the space of homotopy factorizations of f of the form

$$P \to N \setminus Q_J \to \operatorname*{holim}_{S \subsetneq J} N \setminus Q_S,$$

where we have suppressed the lifting homotopy. Consider the ∞–cocartesian square of spaces

$$\begin{array}{ccc} N \setminus Q_J & \longrightarrow & \operatorname{holim}_{S \subsetneq J} N \setminus Q_S \\ \downarrow & & \downarrow \\ N & \longrightarrow & C(N \setminus Q_\bullet) \end{array} \tag{15}$$

whose horizontal maps are $(1+\Sigma)$–connected (by the higher Blakers–Massey theorem applied to the j–cube $N \setminus Q_\bullet$ [4, Theorem 2.5]) and whose vertical maps are $(1+\mu)$–connected. By the Blakers–Massey theorem, the square is $(1+\mu+\Sigma)$–cartesian. Hence, if $\mathscr{F}$ is defined as the homotopy pullback of the diagram given by deleting $N \setminus Q_J$ from the square, then the map $N \setminus Q_J \to \mathscr{F}$ is $(1+\mu+\Sigma)$–connected.

Suppose that the given multirelative intersection problem comes equipped with a preferred solution $\hat{f}\colon P \to N \setminus Q_J$ (where again the lifting homotopy is suppressed). The solution gives a preferred null-homotopy of the composite

$$P \xrightarrow{f} \operatorname*{holim}_{S \subsetneq T} N \setminus Q_S \to C(N \setminus Q_\bullet) \tag{16}$$

as a morphism of $\mathscr{T}(N)$.

In other words, we have a map

$$\mathscr{L}(f) \to \mathscr{N}(f), \tag{17}$$

where $\mathscr{L}(f)$ is the solution space and $\mathscr{N}(f)$ is the space of null-homotopies of the composite (16). With respect to the preferred basepoint of $\mathscr{L}(f)$, this is a map of based spaces.

Furthermore, $\mathscr{N}(f)$ can be interpreted as the moduli space of homotopy factorizations of f of the form

$$P \to \mathscr{F} \to \operatorname*{holim}_{S \subsetneq T} N \setminus Q_S .$$

Since the map $N \setminus Q_J \to \mathscr{F}$ is $(1+\mu+\Sigma)$–connected, we infer by elementary obstruction theory that the map $\mathscr{L}(f) \to \mathscr{N}(f)$ is $(1-p+\mu+\Sigma)$–connected. The rest of the proof involves identifying $\mathscr{N}(f)$.

On the one hand, rather than considering null-homotopies in $\mathscr{T}(N)$, we can equivalently add a disjoint copy of N to P to get a null-homotopy in $\mathscr{R}(N)$ of the associated morphism

$$P^+ \to C(N \setminus Q_\bullet). \tag{18}$$

Then $\mathscr{N}(f)$ can be equivalently defined as the mapping space of null-homotopies of (18) in $\mathscr{R}(N)$.

On the other hand, the (derived) mapping space

$$\hom_{\mathscr{R}(N)}(\Sigma_N P^+, C(N \setminus Q_\bullet))$$

acts on the space $\mathscr{N}(f)$ (this is the fiberwise analogue of the classical fact that for a null-homotopic map of spaces $X \to Y$, the moduli space of null-homotopies, ie the space of extensions of the map to the cone on X, is a torsor over the space of maps $\Sigma X \to Y$). The orbit of the basepoint of $\mathscr{N}(f)$ with respect to this action gives a preferred weak equivalence

$$\hom_{\mathscr{R}(N)}(\Sigma_N P^+, C(N \setminus Q_\bullet)) \simeq \mathscr{N}(f).$$

Using the adjunction between Σ_N and Ω_N, we infer that for $k := 1 - p + \mu + \Sigma$ there is a preferred k–connected (weak) map

$$\mathscr{L}(f) \to \hom_{\mathscr{R}(N)}(P^+, \Omega_N C(N \setminus Q_\bullet)). \tag{19}$$

By Theorem 3.12 we also have a $(2+\mu+\Sigma)$–connected map

$$C(N \setminus Q_\bullet) \to \Omega_N^\infty(\mathscr{W}_j \wedge_N T_N(\xi)).$$

Applying to the latter the fiberwise loop functor Ω_N, then applying $\hom_{\mathscr{R}(N)}(P^+, -)$, and composing with (19) we get a $(1-p+\mu+\Sigma)$–connected (weak) map

$$\mathscr{L}(f) \to \hom_{\mathscr{R}(N)}\big(P^+, \Omega_N^{\infty+1}(\mathscr{W}_j \wedge_N T_N(\xi))\big). \tag{20}$$

By definition, the target of the map (20) is identified with the infinite loop space associated with the cohomology spectrum

$$H^\bullet(P; \Sigma_N^{-1}\mathscr{W}_j \wedge_N T_N(\xi)).$$

By the Poincaré duality argument appearing in the proof of Theorem A above, this spectrum is weakly equivalent to

$$\bigvee_{(j-1)!} E(P, Q_\bullet)^{\xi+(1-2j)\epsilon}.$$

Assembling, we have produced a $(1-p+\mu+\Sigma)$–connected (weak) map

$$\mathscr{L}(f) \to \prod_{(j-1)!} \Omega^\infty(E(P, Q_\bullet)^{\xi+(1-2j)\epsilon}).$$

This completes the proof of Theorem C. □

4.2 The euclidean case

When $N = \mathbb{R}^n$, we have a corollary to Corollary 3.16. Consider an embedding $Q_J \subset \mathbb{R}^n$, where now each Q_i is a manifold admitting a handle decomposition with handles having index at most q_i, where $n - q_i \geq 3$.

Consider the j–cubical diagram $\mathbb{R}^n \setminus Q_\bullet$. Choose a basepoint in $\mathbb{R}^n \setminus Q_J$. Then the j–cube is based and we consider its total homotopy fiber,

$$\Phi(\mathbb{R}^n \setminus Q_\bullet).$$

For $A \subset \mathbb{R}^n$ let $A^* = \mathbb{R}^n \setminus A$ denote its complement.

Corollary 4.2 *There is a $(1+\mu+\Sigma)$–connected map*

$$\Phi(\mathbb{R}^n \setminus Q_\bullet) \to \prod_{i=1}^{(j-1)!} \Omega^\infty(\Sigma^{1-j} Q_1^* \wedge \cdots \wedge Q_j^*), \tag{21}$$

where $\mu = \min_i(n - q_i - 2)$ and $\Sigma = \sum_i (n - q_i - 2)$.

Remark 4.3 The target of the map (21) may also be identified with the infinite loop space associated with the wedge of $(j-1)!$ copies of the spectrum

$$\Sigma^{1-jn} D_+(Q_1 \times \cdots \times Q_j),$$

where $D_+(X) = F(X_+, S^0)$ is the Spanier–Whitehead dual of X_+.

5 Proof of Theorem 3.12

The proof of Theorem 3.12 relies on basic results arising in the calculus of the identity functor which we now summarize. Let

$$\mathbb{I}\colon \mathscr{T} \to \mathscr{T}$$

be the identity functor. By [5] one has a tower of natural transformations

$$\cdots \to P_2\mathbb{I} \to P_1\mathbb{I} \to P_0\mathbb{I} = *$$

and compatible natural transformations $\mathbb{I} \to P_j\mathbb{I}$. Furthermore, the functor $P_j\mathbb{I}$ is j–excisive in the sense that it transforms strongly cocartesian $(j+1)$–cubes into ∞–cartesian ones. In what follows, we abbreviate notation by setting $P_j := P_j\mathbb{I}$.

If Y is r–connected, then the map $Y \to P_jY$ is $(jr+1)$–connected. In particular, when $r > 0$, the map

$$Y \to \lim_{j\to\infty} P_jY$$

is a weak homotopy equivalence.

If Y is a based space, then the j^{th} layer of the tower, that is, the homotopy fiber of $P_jY \to P_{j-1}Y$, is isomorphic in the homotopy category of functors to the infinite loop space valued functor

$$Y \mapsto \Omega^\infty \mathbb{D}_jY,$$

where $\mathbb{D}_j$ takes values in spectra.

The functor $\mathbb{D}_j$ is classified by a certain spectrum with Σ_j–action, denoted by L_j, whose underlying homotopy type is that of a wedge of $(j-1)!$ copies of the $(1-j)$–sphere spectrum [13; 5, page 706]. Then

$$\mathbb{D}_jY \simeq L_j \wedge_{h\Sigma_j} Y^{[j]}, \tag{22}$$

where $Y^{[j]}$ denotes the j–fold fiberwise smash product Y. This description of $\mathbb{D}_j$ enables one to extend its domain of definition to the category of spectra, ie if A is a spectrum then $\mathbb{D}_jA$ is the spectrum $L_j \wedge_{h\Sigma_j} A^{[j]}$.

Remark 5.1 The maps of the tower $P_j Y \to P_{j-1} Y$ are principal fibrations in the sense that there is a homotopy fiber sequence

$$P_j Y \to P_{j-1} Y \to BD_j Y,$$

where $BD_j Y$ is the delooping of $D_j Y$ given by $\Omega^\infty(\Sigma \mathbb{D}_j Y)$ (see [5, page 653]).

We now consider the strongly cocartesian j–cube $A_\bullet$ of $\mathscr{R}(X)$. Assume for now that X is contractible. Without loss in generality we can replace X by the one-point space. The assignment $S \mapsto P_k A_S$ defines a j–cube, denoted by $P_k A_\bullet$. A choice of basepoint in A_J equips $A_\bullet$ with the structure of a based j–cube. Then $D_k A_\bullet$ is a j–cube of infinite loop spaces. Let

$$\mathrm{fib}(D_k(A_\bullet))$$

denote its total homotopy fiber.

Proposition 5.2 *The total homotopy fiber of $D_k A_\bullet$ is $(\mu{+}\Sigma)$–connected if $k \geq j+1$. Furthermore, when $k = j$ there is a $(1{+}\mu{+}\Sigma)$–connected map*

$$\mathrm{fib}(D_j(A_\bullet)) \to \Omega^\infty(L_j \wedge A_1 \wedge \cdots \wedge A_j).$$

Proof Suppose first that $A_\bullet$ is a wedge cube on the based spaces $X_1, \dots, X_j$. Then X_i is r_i–connected. Using (22), the total homotopy fiber of $D_k(A_\bullet)$ may be identified with the infinite loop space associated with the total homotopy fiber of the j–cube of spectra

$$S \mapsto L_k \wedge_{h\Sigma_k} X_S^{[k]}, \tag{23}$$

where X_S is the wedge of the spaces X_i for $i \in S$. Applying the binomial theorem to expand $X_S^{[k]}$, direct calculation shows that the total homotopy fiber of (23) decomposes into a wedge of terms of the form

$$L_k \wedge_{h\Sigma_{s_\bullet}} (X_1^{[s_1]} \wedge \cdots \wedge X_j^{[s_j]}), \tag{24}$$

where

- $\sum_i s_1 = k$ with $s_i \geq 1$ for all i,
- $\Sigma_{s_\bullet} := \Sigma_{s_1} \times \cdots \times \Sigma_{s_j} \subset \Sigma_k$.

If $k \geq j+1$ then there is always at least one term $s_i \geq 2$. It follows that the displayed spectrum is at least $(\mu{+}\Sigma)$–connected. Hence, the total homotopy fiber $\mathrm{fib}(D_k(A_\bullet))$ is also $(\mu{+}\Sigma)$–connected when $k \geq j+1$.

When $k = j$, we can ignore those terms in which $s_i \geq 2$ since they are highly connected: the projection away from those terms produces the $(1+\mu+\Sigma)$–connected map

$$\mathrm{fib}(D_j(A_\bullet)) \to \Omega^\infty(L_j \wedge A_1 \wedge \cdots \wedge A_j).$$

This completes the proof in the case of wedge cubes.

Turning to the general case, we use the fact that $\mathbb{D}_k$ is defined on the category of spectra. By Lemma 3.7, the j–cube of spectra $\Sigma^\infty A_\bullet$ is weakly equivalent to a wedge cube on the spectra $\Sigma^\infty A_1, \dots, \Sigma^\infty A_j$. Replacing the spaces X_i of the previous case by the spectra $\Sigma^\infty A_i$ and making the same kind of calculation, the conclusion follows. □

Corollary 5.3 *Assume that X is contractible and $k \geq j+1$. Then the $(j+1)$–cube*

$$P_k A_\bullet \to P_{k-1} A_\bullet$$

is $(1+\mu+\Sigma)$–cartesian.

Proposition 5.4 *Assume that X is contractible. Then the $(j+1)$–cube*

$$A_\bullet \to P_j A_\bullet$$

is $(1+\mu+\Sigma)$–cartesian.

Proof If $r_i \geq 1$ for all i, the result follows easily from induction, Corollary 5.3 and the convergence of the tower for the identity functor for 1–connected spaces. In the general case one must proceed differently, using the higher Blakers–Massey theorem. We are indebted to the referee for communicating the following argument.

We first recall how $Y \mapsto P_j Y$ is defined in terms of an auxiliary functor $Y \mapsto T_j Y$ as in [5, Section 1]. The latter is given by taking the homotopy limit of the functor

$$U \mapsto Y * U,$$

where $*$ means topological join and U ranges over the poset of nonempty subsets of $\{1, \dots, j+1\}$. There is an evident natural transformation $Y \to T_j Y$ and $P_j Y$ is defined to be the homotopy colimit of the diagram

$$Y \to T_j Y \to T_j^2 Y \to \cdots .$$

For the rest of the proof we set $\underline{k} = \{1, 2, \dots, k\}$ to avoid notational clutter.

We first determine how cartesian the $(j+1)$–cube $A_\bullet \to T_j A_\bullet$ is. This is the same as asking the degree to which the $(2j+1)$–cube

$$(T,U) \mapsto A_T * U$$

is cartesian, where $T \subset \underline{j}$ and $U \subset \underline{j+1}$ (note: by our conventions this functor is contravariant in the first variable and covariant in the second).

For fixed T, the $(j+1)$–cube $U \mapsto A_T * U$ is strongly cocartesian. Similarly, for fixed U, the j–cube $T \mapsto A_T * U$ is strongly cocartesian. Any pair (T,U) corresponds to a subcube whose initial term is $A_T * U$. It follows that this subcube will be ∞–cocartesian whenever $|T| \geq 2$ or $|U| \geq 2$. Consequently, there are three remaining types of pairs (T,U) to consider:

(1) $|T| = 1$ and $|U| = 0$.

(2) $|T| = 0$ and $|U| = 1$.

(3) $|T| = |U| = 1$.

By inspection, one finds for a type (1) pair that the subcube is (r_i+1)–cocartesian. Similarly, for a type (2) pair the subcube is $(\mu+1)$–cocartesian and for a type (3) pair the subcube is (r_i+2)–cocartesian.

Given a partition of $\underline{j} \sqcup \underline{j+1}$ consisting of sets of these types only, the sum of these numbers indexed over the sets of the partition is given by

$$\Sigma + j + D + (j+1-D)(\mu+1), \tag{25}$$

where D is the number of times a set of type (3) occurs in the partition. To see this, note that any such partition is determined by a choice of injections $a\colon \underline{D} \to \underline{j}$ and $b\colon \underline{D} \to \underline{j}$, in which the complement of the image of a defines the type (1) singletons of the partition and the complement of the image of b defines the singletons of type (2). Hence, the sum of the numbers for such a partition is given by

$$\sum_{i \notin a(\underline{D})} (r_i+1) + \sum_{i \notin b(\underline{D})} (\mu_i+1) + \sum_{i \in a(\underline{D})} (r_i+2),$$

which clearly coincides with the expression (25).

Observe that (25) achieves a minimum when D is at its maximal value j. It follows that the minimal value is $1+\mu+\Sigma+2j$. Since we are dealing with a $(2j+1)$–cube, we subtract $2j$ to get $1+\mu+\Sigma$, which is how cartesian the cube is by [4, Theorem 2.5]. Hence, the $(j+1)$–cube $A_\bullet \to T_j A_\bullet$ is $(1+\mu+\Sigma)$–cartesian.

The next step is to consider $T_j A_\bullet \to T^2_{j+1} A_\bullet$. For each fixed nonempty $U \subset \underline{j+1}$, the map of j–cubes

$$A_\bullet * U \to T_j(A_\bullet * U)$$

is of the kind we considered above with the number r_i increased by 1 (so Σ is increased by j) and μ increased by 1. Hence, the corresponding $(j+1)$–cube is $(1+(1+\mu)+(\Sigma+j))$–cartesian. Moreover, taking the homotopy limit over U yields the map of j–cubes $T_j A_\bullet \to T_j^2 A_\bullet$. In taking this homotopy limit the degree to which the latter is cartesian is decreased by j. We infer that $T_j A_\bullet \to T_j^2 A_\bullet$ is $(2+\mu+\Sigma)$–cartesian, which is one better than the estimate we obtained for $A_\bullet \to T_j A_\bullet$. Repeating this argument, we infer that $T_j^k A_\bullet \to T_j^{k+1} A_\bullet$ is $(k+1+\mu+\Sigma)$–cartesian for any $k \geq 0$. It follows that $A_\bullet \to P_j A_\bullet$ is $(1+\mu+\Sigma)$–cartesian. □

Proof of Theorem 3.12 The proof is a verification in two cases.

Case 1 (X is contractible) There is no loss in generality in assuming that X is a point. Equip A_J with a basepoint. Then $A_\bullet$ is a j–cube of 1–connected based spaces. Consider the commutative diagram

$$\begin{array}{ccccc} P_j A_J & \xrightarrow{a_1} & P_{j-1} A_J & \xrightarrow{a_2} & BD_j A_J \\ {\scriptstyle b_1}\downarrow & & {\scriptstyle b_2}\downarrow \simeq & & \downarrow{\scriptstyle b_3} \\ \operatorname{holim}_{S \subsetneq J} P_j A_S & \xrightarrow[a_3]{} & \operatorname{holim}_{S \subsetneq J} P_{j-1} A_S & \xrightarrow[a_4]{} & \operatorname{holim}_{S \subsetneq J} BD_j A_S \end{array}$$

in which the top and bottom rows form fibration sequences. The map b_2 is a homotopy equivalence since P_{j-1} is $(j-1)$–excisive. The map b_3 is equivalent to a principal fibration in the following sense: it may be identified with the map of infinite loop spaces arising from the map of spectra

$$\Sigma\mathbb{D}_j(A_\bullet) \to \operatorname{holim}_{S \subsetneq J} \Sigma\mathbb{D}_j(A_S)$$

associated with the j–cube $\Sigma\mathbb{D}_j(A_\bullet)$.

Set $W_j := \Sigma^{1-j} L_j$. By Proposition 5.2 there is a $(2+\mu+\Sigma)$–connected map of spectra

$$\Sigma\mathrm{fib}(\mathbb{D}_j(\Sigma^\infty A_\bullet)) \to W_j \wedge SA_1 \wedge \cdots \wedge SA_j, \tag{26}$$

where we have implicitly identified $\Sigma(L_j \wedge A_1 \wedge \cdots \wedge A_j) \simeq W_j \wedge SA_1 \wedge \cdots \wedge SA_j$ to avoid displaying the choice of basepoint. The infinite loop space associated with the source of (26) is identified with the homotopy fiber of the map b_3.

Consequently,

$$BD_j A_J \xrightarrow{b_3} \operatorname*{holim}_{S \subsetneq J} BD_j A_S \to \Omega^\infty(\Sigma W_j \wedge SA_1 \wedge \cdots \wedge SA_j)$$

is a homotopy fiber sequence in degrees $\le 2 + \mu + \Sigma$.

Hence, by Lemma 5.5 below there is a homotopy fiber sequence in degrees $\le 1+\mu+\Sigma$ of the form

$$P_j A_J \xrightarrow{b_1} \operatorname*{holim}_{S \subsetneq J} P_j A_S \to \Omega^\infty(W_j \wedge SA_1 \wedge \cdots \wedge SA_j). \tag{27}$$

According to Proposition 5.4, the square

$$\begin{array}{ccc} A_J & \longrightarrow & P_j A_J \\ \big\downarrow & & \big\downarrow b_1 \\ \operatorname*{holim}_{S \subsetneq J} A_S & \longrightarrow & \operatorname*{holim}_{S \subsetneq J} P_j A_S \end{array}$$

is $(1+\mu+\Sigma)$–cartesian. Let $\operatorname{holim}_{S \subsetneq J} A_S \to \Omega^\infty(W_j \wedge SA_1 \wedge \cdots \wedge SA_j)$ be the composition of the bottom map of the square with the second map of (27). Then

$$A_J \to \operatorname*{holim}_{S \subsetneq J} A_S \to \Omega^\infty(W_j \wedge SA_1 \wedge \cdots \wedge SA_j) \tag{28}$$

is also a homotopy fiber sequence in degrees $\le 1 + \mu + \Sigma$. By the dual Blakers–Massey theorem, we conclude that (28) is also a homotopy cofiber sequence in degrees $\le 2 + \mu + \Sigma$.

Consequently, the induced map

$$C(A_\bullet) \to \Omega^\infty(W_j \wedge SA_1 \wedge \cdots \wedge SA_j)$$

is $(2+\mu+\Sigma)$–connected.

Case 2 (X is general) Let $\widetilde{X} \to X$ be a universal principal bundle for X with topological structure group G. Then $\widetilde{X}$ is contractible. Let $\widetilde{A}_\bullet$ be the strongly cocartesian j–cube of G–spaces given by the fiber product

$$\widetilde{A}_S := \widetilde{X} \times_X A_S.$$

The terminal vertex of this cube is then contractible, and one checks that the argument in Case 1 preserves equivariance. It follows that there is a $(2+\mu+\Sigma)$–connected map of based G–spaces

$$C(\widetilde{A}_\bullet) \to \Omega^\infty(W_j \wedge S\widetilde{A}_1 \wedge \cdots \wedge S\widetilde{A}_j). \tag{29}$$

The result follows by applying the Borel construction $-\times_G \widetilde{X}$ to (29) to obtain a $(2+\mu+\Sigma)$–connected map of $\mathscr{R}(X)$,

$$C(A_\bullet) \to \Omega_X^\infty(\mathscr{W}_j \wedge_X S_X A_1 \wedge \cdots \wedge_X S_X A_j). \qquad \square$$

The section ends with an elementary result about fibrations that was used in the proof of Theorem 3.12. Let

$$\begin{array}{ccccc} F_1 & \longrightarrow & E_1 & \longrightarrow & B_1 \\ \downarrow & & \downarrow \simeq & & \downarrow \\ F_2 & \longrightarrow & E_2 & \longrightarrow & B_2 \end{array}$$

be a commutative diagram of connected spaces in which the rows are fibration sequences and the map $E_1 \to E_2$ is a homotopy equivalence. Here $B_1 \to B_2$ is a map of based spaces and the fiber over the basepoint of B_i is F_i.

Lemma 5.5 *Assume in addition that the map $B_1 \to B_2$ sits in a homotopy fiber sequence $B_1 \to B_2 \to B_3$ in degrees $\le s$. Then the map $F_1 \to F_2$ sits in a homotopy fiber sequence $F_1 \to F_2 \to \Omega B_3$ in degrees $\le s-1$.*

Proof Equip B_3 with the basepoint from B_2. The composition $E_1 \to E_2 \to B_2 \to B_3$ is null-homotopic. Hence, $E_2 \to B_2 \to B_3$ is also null-homotopic. Let $E_2 \to PB_3$ be adjoint to a null-homotopy, where PB_3 is the based path space. Then the diagram

$$\begin{array}{ccccc} E_1 & \longrightarrow & E_2 & \longrightarrow & PB_3 \\ \downarrow & & \downarrow & & \downarrow \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 \end{array}$$

commutes. The result follows by taking fibers vertically. $\square$

6 Multiple disjunction

Let P, $Q_1, \dots, Q_j$ and N be as in Section 1. Let

$$E(P, N)$$

denote the space of smooth embeddings from P to N. Then $S \mapsto E(P, N \setminus Q_S)$ forms a j–cube of spaces, denoted by $E(P, N \setminus Q_\bullet)$. The natural transformation from embeddings to functions

$$E(P, N \setminus Q_\bullet) \to F(P, N \setminus Q_\bullet) \tag{30}$$

is a map of j–cubes. One of the main results of [6] is:

Theorem 6.1 [6, Theorem E] *Assume* $p, q_i \le n-3$. *Then the* $(j+1)$*–cube* (30) *is* $(n-2p-1+\Sigma)$*–cartesian.*

Proof of Theorem D Let

$$f \in \operatorname*{holim}_{S \neq J} \mathrm{emb}(P, N \setminus Q_S)$$

be any point. Then f is represented by a map of $(j+1)$–ads

$$\Delta^{j-1} \to E(P, N),$$

where the i^{th} face of Δ^{j-1} is constrained to map into the subspace $E(P, N \setminus Q_{i+1})$ for $i = 0, 1, \ldots, j-1$. Note that by forgetting information, we may also regard f as a map $P \to \mathrm{holim}_{S \neq J} N \setminus Q_S$, and therefore we have an associated multirelative intersection problem. Consequently, Theorem D follows by combining Theorem 6.1 with Theorem A. □

Remark 6.2 Theorem D is a multirelative version of [12, Theorem 2.2].

7 The embedding tower

In [16, Section 13; 15], we described an invariant $\mu(f)$ which was shown to be a complete obstruction to regularly homotoping an immersion $f\colon P \to N$ to an embedding in the metastable range. The goal of this section is to generalize this result beyond the metastable range when N is highly connected.

7.1 Construction of the embedding tower

Let P be a smooth manifold of dimension p without boundary and let N be a smooth manifold of dimension n. We let $E(P, N)$ denote the space of embeddings of P in N, defined as the geometric realization of the simplicial set whose k–simplices are the smooth families of embeddings from P to N that are parametrized by the standard k–simplex.

Assume P is compact. Let $\mathscr{O}_j := \mathscr{O}_j(P)$ be the partially ordered set whose elements are open subsets $U \subset P$ such that U is diffeomorphic to $\mathbb{R}^p \times T$, where T is a set of cardinality at most j. A morphism $U \to V$ is given by an inclusion of subsets. The j^{th} stage of the Goodwillie–Weiss embedding tower is defined by

$$E_j(P, N) := \operatorname*{holim}_{U \in \mathscr{O}_j} E(U, N).$$

The inclusion $\mathscr{O}_{j-1} \to \mathscr{O}_j$ induces a map $E_j(P,N) \to E_{j-1}(P,N)$. The map $E(P,N) \to E_j(P,N)$ is given by restricting embeddings to elements of $\mathscr{O}_j$.

If $p \le n-1$ then $E_1(P,N)$ is homotopy equivalent to $I(P,N)$, the space of immersions from P to N (by a reformulation of Smale–Hirsch theory). Hence, a basepoint of $E_1(P,N)$ amounts to selecting an immersion $P \to N$ up to contractible choice. In what follows, we fix such a basepoint and define

$$\bar{E}_j(P,N) := \text{fiber}(E_j(P,N) \to E_1(P,N)).$$

It follows that the square

$$\begin{array}{ccc} \bar{E}_j(P,N) & \longrightarrow & E_j(P,N) \\ \downarrow & & \downarrow \\ \bar{E}_{j-1}(P,N) & \longrightarrow & E_{j-1}(P,N) \end{array} \tag{31}$$

is homotopy cartesian. Furthermore, the tower $\{\bar{E}_j(P,N)\}$ is the manifold calculus tower associated with the functor $U \mapsto \bar{E}(U,N)$, where U varies throughout the open subsets of P. Call this the *reduced embedding tower*. Note that $\bar{E}_1(P,N)$ is the one-point space.

7.2 Configuration spaces

For a set J of cardinality j, set

$$E_J(N) := E(J,N).$$

If we equip J with a total ordering, then $E_J(N)$ is the configuration space of finite ordered subsets of N of cardinality j. A choice of embedding $J \to N$ equips $E_J(N)$ with a basepoint. To each $T \subset U \subset J$ there is a projection map $E_U(N) \to E_T(N)$. These assemble into a j–cube of based spaces $E_\bullet(N)$.

Lemma 7.1 *The j–cube $E_\bullet(N)$ is $((j-1)(n-2)+1)$–cartesian.*

Proof The j–cube $E_\bullet(N)$ can be written as a map of $(j-1)$–cubes

$$E_{S\cup 1}(N) \to E_S(N),$$

where $S \subset J_1 := \{2,\dots,j\}$. The displayed map is a fibration whose fiber at the basepoint is the based space $N \setminus S$. These form a strongly cocartesian $(j-1)$–cube $N_\bullet$, all of whose maps are $(n-1)$–connected. Then $N_\bullet$ is $((j-1)(n-2)+1)$–cartesian by the higher Blakers–Massey theorem. □

7.3 The unstable obstruction

For $j \geq 2$ let $\binom{P}{j}$ denote the configuration space of subsets $S \subset P$ of cardinality j. Over this space we consider two fibrations. The first fibration

$$E \to \binom{P}{j}$$

has fiber over $S \in \binom{P}{j}$ given by the configuration space $E_S(N)$.

The second fibration

$$D \to \binom{P}{j}$$

has fiber over S given by $\operatorname{holim}_{T \subsetneq S} E_T(N)$.

Then one has an evident map of fibrations

$$E \to D. \tag{32}$$

A point $x \in E_{j-1}(P, N)$ determines a section $t = t(x)$ of $D \to \binom{P}{j}$. It also determines a partial section $s = s(x)$ of $E \to \binom{P}{j}$ along an open collar of the boundary of a compactification of $\binom{P}{j}$. The sections agree with respect to the map (32).

The following is essentially just a reformulation of Weiss's description of the layers of the embedding tower.

Lemma 7.2 *Assume $j \geq 2$. The homotopy fiber of $\bar{E}_j(P, N) \to \bar{E}_{j-2}(P, N)$ taken at x is homotopy equivalent to the space of sections of $E \to \binom{P}{j}$ which are compatible with t and which coincide with s near infinity. In particular, x lifts to a point of $E_j(P, N)$ if and only if this section space is nonempty.*

Remark 7.3 Another formulation of the lemma is that the square

$$\begin{array}{ccc} \bar{E}_j(P, N) & \longrightarrow & \Gamma(E) \\ \downarrow & & \downarrow \\ \bar{E}_{j-1}(P, N) & \longrightarrow & \Gamma_\infty(E) \times_{\Gamma_\infty(D)} \Gamma(D) \end{array}$$

is ∞–cartesian, where Γ denotes the space of sections and Γ_∞ denotes the space of germs of sections near infinity.

Proof of Lemma 7.2 Given x, define a third fibration

$$F \to \binom{P}{j}$$

whose fiber at S is the total homotopy fiber of the cube $T \mapsto E_T(N)$ for $T \subset S$. Denote this fiber by $\Phi_S(N;x)$. It is an unbased space. Note that $\Phi_S(N;x)$ is well defined since when $T \subsetneq S$, each of the spaces $E(T,N)$ is based using x.

Moreover, x gives a partial section of this fibration at infinity. Weiss shows that the space of compactly supported sections of this fibration (ie the space of sections agreeing with the partial section near infinity) coincides with the homotopy fiber of $\bar{E}_j(P,N) \to \bar{E}_{j-1}(P,N)$ at x. The latter space is homotopy equivalent to the space in the statement of the lemma. □

7.4 A cohomological obstruction

If we suspend the fibers of $D \to \binom{P}{j}$, then the obstruction to finding a compactly supported section lies in a spectrum cohomology group. If certain dimensional restrictions are present, then nothing is lost in suspending.

When X is an unbased space, we define its suspension spectrum be the homotopy fiber of the map of spectra $\Sigma^\infty X_+ \to S^0$ that is induced by the map from X to the one-point space. By slight abuse in notation, denote the homotopy fiber by $\Sigma^\infty X$.

Definition 7.4 Let

$$\mathscr{D} \to \binom{P}{j}$$

be the fiberwise spectrum whose fiber at S given by $\Sigma^\infty \Phi_S(N;x)$. This comes equipped with a section near infinity. Note that $\mathscr{D}$ depends on the choice of x.

The total obstruction $e(x)$ to finding a compactly supported section of $\mathscr{D}$ lies in π_{-1} in the spectrum of compactly supported sections, that is,

$$e(x) \in H^{-1}_{\mathrm{cs}}\Big(\binom{P}{j}; \mathscr{D}\Big).$$

Lemma 7.5 *If $x \in E_{j-1}(P,N)$ lifts to $E_j(P,N)$, then $e(x)$ vanishes. The converse is true provided that $2(j-1)(n-2) - jp + 1 \geq 0$.*

Proof The "if" part is clear. For the converse, one observes that the map $\Phi_S(N;x) \to \Omega^\infty\Sigma^\infty\Phi_S(N;x)$ is $(2(j-1)(n-2)+1)$–connected using the Freudenthal suspension theorem and fact that $\Phi_S(N;x)$ is $((j-1)(n-2))$–connected by Lemma 7.1. It follows that the map of compactly supported section spaces is $(2(j-1)(n-2)-jp+1)$–connected. □

7.5 Highly connected manifolds

When N is highly connected, the obstruction to lifting simplifies considerably.

Definition 7.6 For $S \subset \binom{P}{j}$ let

$$C_S(N)$$

denote the mapping cone of the map

$$E_S(N) \to \operatorname*{holim}_{T \subsetneq S} E_T(N).$$

Remark 7.7 In contrast with $\Phi_S(N;x)$, the space $C_S(N)$ doesn't depend on x and it has a preferred basepoint.

Lemma 7.8 *Assume $j \geq 2$ and N is r–connected, where $r \leq n-2$. Then the square*

$$\begin{array}{ccc} E_S(N) & \longrightarrow & \operatorname{holim}_{T \subsetneq S} E_T(N) \\ \downarrow & & \downarrow \\ C & \longrightarrow & C_S(N) \end{array}$$

is $((j-1)(n-2)+r+1)$–cartesian, where C is the cone on $E_S(N)$.

Proof By definition, the square is ∞–cocartesian. Furthermore, the map $E_S(N) \to \operatorname{holim}_{T \subsetneq S} E_T(N)$ is $((j-1)(n-2)+1)$–connected by Lemma 7.1.

Since N is r–connected and $r \leq n-2$, it follows that $E_S(N)$ is r–connected. Hence, the left vertical map is $(r+1)$–connected. The conclusion now follows from the Blakers–Massey theorem. □

Let

$$\mathscr{C} \to \binom{P}{j} \tag{33}$$

be the fiberwise spectrum whose fiber at S is $\Sigma^\infty C_S(N)$. This fiberwise spectrum doesn't depend on x.

The section t induces another section of (33); call it t'. The latter section is homotopic to the zero section near infinity. Then an obstruction to lifting $x \in E_{j-1}(P,N)$ to $E_j(P,N)$ is given by the associated compactly supported spectrum cohomology class of t':

$$e'(x) \in H^0_{\mathrm{cs}}\Big(\binom{P}{j};\mathscr{C}\Big).$$

Lemma 7.9 *Assume $j \geq 2$ and N is r–connected with $r \leq n-2$. If x lifts to an element of $E_j(P,N)$, then $e'(x)$ vanishes. Furthermore, the converse holds if $r \geq p-1-(j-1)(n-p-2)$.*

Proof The proof uses the commutative square

$$\begin{array}{ccc} \Sigma\Phi_S(N;x) & \longrightarrow & C_S(N) \\ \downarrow & & \downarrow \\ \Omega^\infty\Sigma^\infty\Phi_S(N;x) & \longrightarrow & \Omega^\infty\Sigma^\infty C_S(N) \end{array}$$

Since $C_S(N)$ and $\Sigma\Phi_S(N;x)$ are $((j-1)(n-2)+1)$–connected (by Lemma 7.1), the vertical maps are $(2(j-1)(n-2)+3)$–connected by the Freudenthal suspension theorem.

By Lemma 7.8, the horizontal maps are $((j-1)(n-2)+r+2)$–connected. Hence, the composite

$$\Sigma\Phi_S(N;x) \to C_S(N) \to \Omega^\infty\Sigma^\infty C_S(N)$$

is $((j-1)(n-2)+r+2)$–connected. By elementary obstruction theory the obstructions $e'(x)$ and $e(x)$ contain the same information when $jp < (j-1)(n-2)+r+2$, that is, when $r \geq p-1-(j-1)(n-p-2)$. □

Corollary 7.10 *Assume $j \geq 2$. If N is contractible, then x lifts to an element of $E_j(P,N)$ if and only if $e'(x) = 0$.*

Proof In this case we can take $r = n-2$. Then the inequality of Lemma 7.9 becomes $n-2 \geq p-1-(j-1)(n-p-2)$, which is automatically satisfied because $p \leq n-3$. □

7.5.1 Equivariant reformulation Set $J := \{1,\dots,j\}$. Then the map $E_J(P) \to \binom{P}{j}$ which assigns to an embedding its image is a regular covering space with structure group Σ_j, where the latter acts on $E_J(P)$ via the automorphisms of J.

The pullback of $\mathscr{C} \to \binom{P}{j}$ along $E_J(P) \to \binom{P}{j}$ coincides with the fiberwise spectrum with Σ_j–action

$$E_J(P) \times \mathscr{C}_J \to E_J(P), \tag{34}$$

where $\mathscr{C}_J := \Sigma^\infty C_J(N)$ is a spectrum with Σ_j–action (recall that $C_J(N)$ is the total homotopy cofiber of the j–cube $E_\bullet(N)$; the action of Σ_j arises from the evident action of Σ_j on the cube). Note that Σ_j acts diagonally on $E_J(P) \times \mathscr{C}_J$. When considered unequivariantly, (34) is a trivial fiberwise spectrum.

Then the obstruction $e'(x)$ may be interpreted as an element of the equivariant cohomology group

$$H^0_{\mathrm{cs},\Sigma_j}(E_J(P); \mathscr{C}_J),$$

or, alternatively, as an element of the function space of compactly supported Σ_j–equivariant stable maps from $E_J(P)$ to $\mathscr{C}_J$.

7.5.2 The homological invariant By Poincaré duality, there is an equivalence of spectra

$$H^0_{\mathrm{cs},\Sigma_j}(E_J(P); \mathscr{C}_J) \simeq H_0^{\Sigma_j}(E_J(P); {}^{-\tau}\mathscr{C}_J),$$

where ${}^{-\tau}\mathscr{C}_J$ is the twist of $\mathscr{C}_J$ by the inverse of the tangent bundle of $E_J(P)$ (the latter is just the restriction of the product of j copies of the tangent bundle of P).

Definition 7.11 Let

$$\mu(x) \in H_0^{\Sigma_j}(E_J(P); {}^{-\tau}\mathscr{C}_J)$$

be the class that corresponds to $e'(x)$ via the Poincaré duality isomorphism.

Proof of Theorem E The procedure described above defines a function

$$\mu\colon \pi_0(\bar{E}_{j-1}(P,N)) \to H_0^{\Sigma_j}(E_J(P); {}^{-\tau}\mathscr{C}_J)$$

such that $\mu(x) = 0$ when x lifts to $\pi_0(\bar{E}_j(P,N))$ By Lemma 7.9 the converse is true provided $r \geq p - 1 - (j-1)(n-p-2)$. □

8 Spaces of link maps

Given manifolds $P_1, \dots, P_j$ of dimension $\dim P_i = p_i$ and a connected n–manifold N without boundary, a *link map* is a continuous map

$$f\colon P_1 \sqcup \cdots \sqcup P_j \to N$$

such that $f(P_i) \cap f(P_k) = \varnothing$ for $i \neq k$. We will typically assume that P_i is connected and boundaryless. Set $\boldsymbol{P} := \langle P_1, \dots, P_j \rangle$ and write

$$\mathscr{L}(\boldsymbol{P}, N)$$

for the space of link maps in the compact–open topology.

Recall that $J = \{1, 2, \dots, j\}$. For a subset $S \subset J$, set

$$P_S := \coprod_{i \in S} P_i \quad \text{and} \quad P^{(S)} := \prod_{i \in S} P_i.$$

Then, to each $S \subset J$, we have a space

$$\mathscr{L}^S(\boldsymbol{P}, N)$$

whose points are the maps

$$f\colon P_J \to N$$

such that $f(P_i) \cap f(P_k) = \varnothing$ for each pair of distinct elements $i, k \in S$. Note that $\mathscr{L}^J(\boldsymbol{P}, N) = \mathscr{L}(\boldsymbol{P}, N)$ is the space of link maps and if $|S| \leq 1$ then $\mathscr{L}^S(\boldsymbol{P}, N) = F(P_J, N)$ is the function space of maps with no constraint. The assignment

$$S \mapsto \mathscr{L}^S(\boldsymbol{P}, N)$$

is contravariant and defines a j–cube of spaces, which we denote by

$$\mathscr{L}^\bullet(\boldsymbol{P}, N).$$

Remark 8.1 There is a related j–cube

$$\mathscr{L}(\boldsymbol{P}_\bullet, N)$$

whose value at $S \subset J$ is the space of link maps $f\colon P_S \to N$. Then the evident map of j–cubes

$$\mathscr{L}^\bullet(\boldsymbol{P}, N) \to \mathscr{L}(\boldsymbol{P}_\bullet, N)$$

is ∞–cartesian because for each S we have a homotopy fiber sequence

$$F(P_{(J \setminus S)}, N) \to \mathscr{L}^S(\boldsymbol{P}, N) \to \mathscr{L}(\boldsymbol{P}_S, N),$$

and the j–cube $S \mapsto F(P_{(J \setminus S)}, N)$ is ∞–cartesian if $j > 1$.

8.1 Homotopy coherent Brunnian links

Henceforth, we fix an embedding

$$J \subset N$$

and identify J with its image. Let $c\colon \bigsqcup_i P_i \to N$ be the link map which sends P_i to i. Call c the *trivial link map*. Then c equips $\mathscr{L}^\bullet(\boldsymbol{P}, N)$ with the structure of a j–cube of based spaces. If $n \geq 2$, then the component of the basepoint is independent of the choice of embedding $J \subset N$.

Remark 8.2 Milnor [21] considers the case of link maps $f\colon \bigsqcup_{i=1}^{j} P_i \to N$ in euclidean space $N = \mathbb{R}^3$ in which each P_i is a circle S^1. Milnor defines f to be "trivial" if there is an extension of f to a link map $\bigsqcup_i D^2 \to \mathbb{R}^3$. Note that f is trivial in Milnor's sense if and only if f is homotopic through link maps to the trivial link map c.

Definition 8.3 The space of *homotopy coherent Brunnian link maps*

$$\mathscr{B}(\boldsymbol{P}, N)$$

is the total homotopy fiber of the j–cube of based spaces $\mathscr{L}^\bullet(\boldsymbol{P}, N)$.

Remarks 8.4 By Remark 8.1, an equivalent definition up to homotopy of $\mathscr{B}(\boldsymbol{P}, N)$ is given by taking the total homotopy fiber of the j–cube $\mathscr{L}(\boldsymbol{P}_\bullet, N)$.

A point of $\mathscr{B}(\boldsymbol{P}, N)$ is given by data consisting of a link map $f\colon P_J \to N$ together with a homotopy coherent set of rules which to each $S \subsetneq J$ associates a path from the associated point of $\mathscr{L}^S(\boldsymbol{P}, N)$ to the basepoint.

By contrast, Milnor [21, Section 5] defines a link map $f\colon \bigsqcup_{i=1}^{j} S^1 \to \mathbb{R}^3$ to be *almost trivial* if every proper sublink map of f is trivial.[3] If $j \geq 4$ then this notion of Brunnian fails to be homotopy coherent. Thus, a homotopy coherent Brunnian link map gives an almost trivial link map but not conversely.

Note that there is an evident map

$$\mathscr{B}(\boldsymbol{P}, N) \to \mathrm{fiber}\Big(\mathscr{L}^J(\boldsymbol{P}, N) \to \prod_{i=1}^{j} \mathscr{L}^{J_i}(\boldsymbol{P}, N)\Big),$$

[3] Subsequent authors call Milnor's notion of almost trivial link map a Brunnian link map. The earliest reference employing this language seems to be [3].

where $J_i = J \setminus \{i\}$, $\boldsymbol{P} = \langle S^1, \dots, S^1\rangle$ and $N = \mathbb{R}^3$. However, if $j \geq 4$, this map is not a weak equivalence. Milnor's almost trivial link maps are those link maps whose components are in the image of the displayed homotopy fiber.

Terminology 8.5 As we only consider homotopy coherent Brunnian link maps in this paper, we henceforth refer to $\mathscr{B}(\boldsymbol{P}, N)$ simply as the space of *Brunnian link maps*, despite the different usage of this term in the literature.

8.2 The invariants

For each $S \subset J$, one has a map

$$\mathscr{L}^S(\boldsymbol{P}, N) \to F(P^{(J)}, E_S(N)), \tag{35}$$

where the target is the function space of maps $P^{(J)} \to E_S(N)$. One defines (35) by mapping a link map f to the map

$$(x_1, \dots, x_j) \mapsto \prod_{i \in S} f(x_i).$$

Remark 8.6 When $S = J$, the map (35) is Koschorke's *κ–invariant* $\mathscr{L}(\boldsymbol{P}, N) \to F(P^{(J)}, E_J(N))$.

If we let S vary, (35) defines a map of j–cubes of based spaces

$$\mathscr{L}^\bullet(\boldsymbol{P}, N) \to F(P^{(J)}, E_\bullet(N)). \tag{36}$$

Remark 8.7 For $S \subset J$, let $N^J(S)$ be the space of j–tuples $x \in N^J$ such that the image of x under the projection $N^J \to N^S$ lies in the subspace $E_S(N) \subset N^S$ (here $N^S := F(S, N)$). In other words, there is a pullback diagram

$$\begin{array}{ccc} N^J(S) & \longrightarrow & N^J \\ \downarrow & & \downarrow \\ E_S(N) & \longrightarrow & N^S \end{array}$$

The collection $\{N^J(S)\}_{S \subset J}$ forms both a stratification of N^J and a j–cube of based spaces.

The operation $S \mapsto F(P^{(J)}, N^J(S))$ is a j–cube of based spaces, which we denote by $F(P^{(J)}, N^J(\bullet))$. Then we have a commutative diagram of j–cubes

$$(37)\qquad \begin{array}{ccc} \mathscr{L}^{\bullet}(\boldsymbol{P}, N) & \longrightarrow & F(P^{(J)}, N^J(\bullet)) \\ \big\downarrow & & \big\downarrow \\ \mathscr{L}(\boldsymbol{P}_{\bullet}, N) & \longrightarrow & F(P^{(J)}, E_{\bullet}(N)) \end{array}$$

in which the vertical maps form ∞–cartesian $(j+1)$–cubes (even more is true if N happens to be contractible: in this case the vertical maps are objectwise weak equivalences of j–cubes). The map (36) is just the composition of the maps in diagram (37).

The top horizontal map of diagram (37) can be viewed as a kind of *coassembly map* which records the passage from global to local linking data. More precisely, set $\boldsymbol{J} := \langle 1, 2, \dots, j\rangle$, where we think of $i \in J$ as a manifold of dimension zero. Then, by definition,

$$N^J(S) = \mathscr{L}^S(\boldsymbol{J}, N),$$

and the top horizontal map of (37) associates to $f\colon \coprod_i P_i \to N$ the map which sends a j–tuple $(x_1, \dots, x_j) \in P^{(J)}$ to the composed map $\coprod_i x_i \subset \coprod_i P_i \to N$.

One has a similar description of the bottom horizontal map by reinterpreting the configuration space $E_S(N)$ as the space of link maps $\mathscr{L}(\boldsymbol{S}, N)$.

Definition 8.8 Let

$$\Phi E_{\bullet}(N)$$

be the total homotopy fiber of the j–cube $E_{\bullet}(N)$ taken with respect to the given embedding $J \to N$. (Alternatively, $\Phi E_{\bullet}(N)$ can be defined as the total homotopy fiber of the $(j-1)$–cube $N_{\bullet}$ appearing in the proof of Lemma 7.1.)

Then the map of j–cubes (36) induces a map of total homotopy fibers

$$(38)\qquad \ell\colon \mathscr{B}(\boldsymbol{P}, N) \to F(P^{(J)}, \Phi E_{\bullet}(N)),$$

called the *higher unstable linking number map*.

Remark 8.9 Let $\mathscr{O}_{\boldsymbol{P}}$ be the partially ordered set given by $\boldsymbol{U} = \langle U_1, \dots, U_j\rangle$, in which U_i is an open set in P_i, and $\boldsymbol{U} \le \boldsymbol{U}'$ if and only if $U_i \subset U_i'$ for all i. Then

$$\boldsymbol{U} \mapsto \mathscr{B}(\boldsymbol{U}, N)$$

defines a contravariant functor $\mathscr{O}_{\boldsymbol{P}} \to \mathscr{T}_*$. Its multilinearization in the sense of Weiss's manifold calculus coincides up to homotopy with the higher unstable linking number map ℓ (see [23; 22]).

Conjecture 8.10 *The map* ℓ (see (38)) *is* $(1+\Sigma')$*–connected, where*

$$\Sigma' = \sum_i (n - 2p_i - 2).$$

Remark 8.11 The $j = 2$ case of Conjecture 8.10 is known in the affirmative: it is the main result of [7].

8.2.1 The euclidean case, stabilization Assume $N = \mathbb{R}^n$. Then $\Phi(E_\bullet(\mathbb{R}^n))$ coincides with the total homotopy fiber of the based $(j-1)$–cube

$$S \mapsto \mathbb{R}^n \setminus S$$

for $S \subset J_1$ (see the proof of Lemma 7.1). By this identification and Corollary 4.2 applied to $Q_i := \{i\} \subset \mathbb{R}^n$, we infer there is a $(j(n-2)+1)$–connected map

$$\Phi E_\bullet(\mathbb{R}^n) \to \prod_{i=1}^{(j-2)!} \Omega^\infty S^{(j-1)(n-2)+1}. \tag{39}$$

Applying the functor $F(P^{(J)}, -)$ to (39), one obtains a map of function spaces

$$F(P^{(J)}, \Phi E_\bullet(\mathbb{R}^n)) \to \prod_{i=1}^{(j-2)!} F^{\mathrm{st}}(P^{(J)}, S^{(j-1)(n-2)+1}) \tag{40}$$

which is $(1+\Sigma)$–connected, where $\Sigma = \sum_{i=1}^{j} (n - p_i - 2)$. The composition of (38) with (40) defines the *higher stable linking number map*

$$\lambda\colon \mathscr{B}(\boldsymbol{P}, \mathbb{R}^n) \to \prod_{i=1}^{(j-2)!} F^{\mathrm{st}}(P^{(J)}, S^{(j-1)(n-2)+1}). \tag{41}$$

A version of (41) also appears in the work of Munson [22]. Note that [22, Corollary 1.2] gives a connectivity estimate one less than ours (see [22, Remark 3.6]).

Example 8.12 Let $n = j = 3$ and $P_i = S^1$ for $i = 1, 2, 3$. Then the higher stable linking number map λ is of the form

$$\mathscr{B}(\boldsymbol{P}, \mathbb{R}^3) \to F^{\mathrm{st}}((S^1)^{\times 3}, S^3).$$

Taking path components gives a function $\pi_0(\mathscr{B}(S^1_\bullet, \mathbb{R}^3)) \to \mathbb{Z}$. This can be described as the rule which assigns to a three-component Brunnian link in $\mathbb{R}^3$ a certain Massey product in the link complement [24].

Since $1 + \Sigma \geq 1 + \Sigma'$, we infer that Conjecture 8.10 with $N = \mathbb{R}^n$ is equivalent to the following:

Conjecture 8.13 *The higher stable linking number map* λ (see (41)) *is* $(1{+}\Sigma')$–*connected.*

8.3 Evidence for Conjecture 8.13

In this subsection we prove Theorem H, which we submit as evidence for Conjecture 8.13.

As above, $P_1, \dots, P_j$ are closed manifolds, but now we suppose that each P_i embeds in $\mathbb{R}^n$. In what follows, we don't require the P_i to be pairwise disjoint and we will not need to assume that P_1 is a submanifold of $\mathbb{R}^n$.

Recall the fixed embedding $J \subset \mathbb{R}^n$. Choose n–balls $B(i)$ containing $i \in J \setminus 1$ and assume that the collection $\{B(i)\}$ is pairwise disjoint. Choose an embedding $P_i \subset B(i)$ for $i \neq 1$. Using the inclusions $B(i) \subset \mathbb{R}^n$, we obtain an embedding

$$P_2 \sqcup \cdots \sqcup P_j \subset \mathbb{R}^n.$$

Consider the $(j-1)$–cube of function spaces

$$S \mapsto F(P_1, \mathbb{R}^n \setminus P_S), \quad S \subset J_1.$$

This is a based cube, where the basepoint of $F(P_1, \mathbb{R}^n \setminus P_S)$ is the constant map having value $1 \in \mathbb{R}^n \setminus P_S$. Consequently, the total homotopy fiber of this cube is given by

$$F(P_1, \Phi(\mathbb{R}^n \setminus P_\bullet)), \tag{42}$$

where now the convention is that $\mathbb{R}^n \setminus P_\bullet$ is the $(j-1)$–cube given by $\mathbb{R}^n \setminus P_S$ in which S ranges through subsets of J_1.

For $S \subset J_1$, consider the commutative diagram

$$\begin{array}{ccccc} F(P_1, \mathbb{R}^n \setminus P_S) & \longrightarrow & \mathscr{L}^{S \sqcup 1}(\boldsymbol{P}, \mathbb{R}^n) & \longrightarrow & F(P^{(J)}, E_{S \sqcup 1}(\mathbb{R}^n)) \\ \downarrow{\scriptstyle a_S} & & \downarrow{\scriptstyle b_S} & & \downarrow{\scriptstyle c_S} \\ F(P_1, \mathbb{R}^n) & \longrightarrow & \mathscr{L}^{S}(\boldsymbol{P}, \mathbb{R}^n) & \longrightarrow & F(P^{(J)}, E_S(\mathbb{R}^n)) \end{array}$$

As S varies, each of the vertical maps assembles to a morphism of $(j-1)$–cubes, ie each gives a j–cube $a_\bullet$, $b_\bullet$ and $c_\bullet$, respectively. The j–cube $b_\bullet$ is just $\mathscr{L}^\bullet(\boldsymbol{P},\mathbb{R}^n)$. Similarly, $c_\bullet$ is the j–cube $F(P^{(J)}, E_\bullet(\mathbb{R}^n))$. If we consider $a_\bullet$ as a map of $(j-1)$–cubes, then its target is the constant $(j-1)$–cube on the contractible space $F(P_1,\mathbb{R}^n)$; in particular, the target of $a_\bullet$ is ∞–cartesian. Hence, the total homotopy fiber $\Phi(a_\bullet)$ is identified with the total homotopy fiber of the source of $a_\bullet$, and the latter coincides with $F(P_1, \Phi(\mathbb{R}^n \setminus P_\bullet))$, ie the source of the map (42). Consequently, taking the total homotopy fibers of $a_\bullet$, $b_\bullet$ and $c_\bullet$ and composing with the map (40) results in a commutative diagram

(43)
$$\begin{array}{ccccc} F(P_1, \Phi(\mathbb{R}^n \setminus P_\bullet)) & \longrightarrow & \mathscr{B}(\boldsymbol{P},\mathbb{R}^n) & \xrightarrow{\ \ell\ } & F(P^{(J)}, \Phi E_\bullet(\mathbb{R}^n)) \\ & & & \searrow^{\lambda} & \downarrow \\ & & & & F^{\mathrm{st}}(P^{(J)}, S^{(j-1)(n-2)+1}) \end{array}$$

such that the right vertical map is $(1+\Sigma)$–connected (see (40)).

Remark 8.14 In the above, we've neglected to mention that the map of cubes $a_\bullet \to b_\bullet$ isn't basepoint-preserving. This means that the map doesn't define a map of total homotopy fibers in an obvious way.

However, the map is easily seen to be basepoint-preserving up to a preferred path (the path is defined by the radial deformation retraction of each ball $B(i)$ onto its center i). It is this preferred path that enables us to define the map from the total homotopy fiber of $a_\bullet$ to the total homotopy fiber of $b_\bullet$, which is the leftmost map in (43).

Claim 8.15 *The horizontal composite*

(44)
$$F(P_1, \Phi(\mathbb{R}^n \setminus P_\bullet)) \to F(P^{(J)}, \Phi E_\bullet(\mathbb{R}^n))$$

of diagram (43) *is* $(1-\hat{p}+\Sigma)$*–connected.*

The claim, proved below, gives evidence for the validity of Conjecture 8.13: it implies that ℓ is a retraction on homotopy in degrees $\le 1-\hat{p}+\Sigma$ (the same is true for λ since the vertical map of (43) is $(1+\Sigma)$–connected). Furthermore, we have $1-\hat{p}+\Sigma \ge 1+\Sigma'$, so λ will be a retraction in degrees $\le 1+\Sigma'$. Consequently, the proof of Theorem H has been reduced to verification of the claim.

Proof of Claim 8.15 For $S \subset J_1$, consider the pullback diagram

$$\begin{array}{ccc} \mathscr{E}_S & \longrightarrow & E_{S \sqcup 1}(\mathbb{R}^n) \\ \downarrow & & \downarrow \\ P^{(J_1)} & \longrightarrow & E_S(\mathbb{R}^n) \end{array}$$

where the right vertical map is given by projection and the bottom horizontal map is the projection $P^{(J_1)} \to P^{(S)}$ followed by the inclusion $P^{(S)} \subset E_S(\mathbb{R}^n)$. Observe that the fiber of $\mathscr{E}_S \to P^{(J_1)}$ at a point $(x_2, \dots, x_j)$ is given by $\mathbb{R}^n \setminus \{x_i\}_{i \in S}$.

The map $P^{(J_1)} \to E_S(\mathbb{R}^n)$ factors through the contractible space $B^{(J_1)} := \prod_i B(i)$, so the fibration $\mathscr{E}_S \to P^{(J_1)}$ is trivializable. Let $\Gamma(\mathscr{E}_S)$ be the space of sections of $\mathscr{E}_S \to P^{(J_1)}$. Define a map

$$\mathbb{R}^n \setminus P_S \to \Gamma(\mathscr{E}_S)$$

by sending a point $z \in \mathbb{R}^n \setminus P_S$ to the section given by $(x_2, \dots, x_j) \mapsto z$. This makes sense since z also lies in $\mathbb{R}^n \setminus \{x_i\}_{i \in S}$.

As S varies we obtain a map of J_1–cubes

$$\mathbb{R}^n \setminus P_\bullet \to \Gamma(\mathscr{E}_\bullet), \tag{45}$$

and applying the functor $F(P_1, -)$ to the induced map of total homotopy fibers of (45) yields the map of the claim.

Hence, it suffices to prove that (45) is $(1+\mu_2+\Sigma_2)$–cartesian, where

$$\mu_2 := \min_{2 \le i \le j} (n - p_i - 2), \quad \Sigma_2 := \sum_{i=2}^{j} (n - p_i - 2), \tag{46}$$

since $F(P_1, -)$ reduces connectivity by p_1 and

$$1 + \mu_2 + \Sigma_2 - p_1 = 1 - \hat{p} + \Sigma.$$

We will explain the proof when $2 \le j \le 3$. The remaining cases are analogous to the case $j = 3$ and we will leave them for the reader to verify.

When $j = 2$, it is readily checked that the statement to be proved amounts to the assertion that the map

$$\mathbb{R}^n \setminus P_2 \to F(P_2, S^{n-1})$$

given by

$$z \mapsto \left(x \mapsto \frac{x - z}{|x - z|} \right)$$

is $(1+2(n-p_2-2))$–connected. This follows from the commutative diagram

$$\begin{array}{ccc} \mathbb{R}^n \setminus P_2 & \longrightarrow & F(P_2, S^{n-1}) \\ \downarrow & & \downarrow \\ \Omega^\infty\Sigma^\infty(\mathbb{R}^n \setminus P_2) & \longrightarrow & F^{\mathrm{st}}(P_2, S^{n-1}) \end{array}$$

where the left vertical map is $(1+2(n-p_2-2))$–connected by the Freudenthal suspension theorem, the right vertical map is $(1-p+2(n-2))$–connected, also by the Freudenthal suspension theorem, and the lower horizontal map is a homotopy equivalence by Spanier–Whitehead duality.

When $j=3$ one proceeds as follows: We think of the square $\mathbb{R}^n \setminus P_S$ for $S \subset \{2,3\}$ as defining an isotopy functor $\phi\colon \mathscr{O}_{P_2} \times \mathscr{O}_{P_3} \to T_*$ which assigns to an open set $U \subset P_2$ and an open set $V \subset P_3$ the total homotopy fiber of the square

$$\begin{array}{ccc} U^* \cap V^* & \longrightarrow & V^* \\ \downarrow & & \downarrow \\ U^* & \longrightarrow & \mathbb{R}^n \end{array}$$

where A^* denotes the complement of $A \subset \mathbb{R}^n$. Similarly, one has an isotopy functor $\phi^\sharp\colon \mathscr{O}_{P_2} \times \mathscr{O}_{P_3} \to T_*$ associated with the total homotopy fiber of the square $S \mapsto \Gamma(\mathscr{E}_S)$. In fact, the latter is easy to identify: it is given by

$$(U,V) \mapsto F(U \times V, S^{n-1} \flat S^{n-1}),$$

where $S^{n-1} \flat S^{n-1}$ is the total homotopy fiber of the wedge square on S^{n-1}. The natural map

$$\phi(U,V) \to \phi^\sharp(U,V) \tag{47}$$

is a kind of bilinearization (or coassembly) in the sense that

- its value when U and V are open balls is a homotopy equivalence;
- $\phi^\sharp(U,V)$ is linear in each variable in the sense of isotopy calculus.

Furthermore, (47) is initial with respect to these properties. On the other hand, Corollary 3.16 (see Corollary 4.2 and Remark 3.17) defines a natural transformation

$$\phi(U,V) \to \Omega^\infty\Sigma^\infty(S^{-1} \wedge U^* \wedge V^*) \tag{48}$$

whose connectivity can be described as follows: If U is a tubular neighborhood of a closed manifold of dimension k_1 and V is a tubular neighborhood of a closed manifold of dimension k_2, then (48) is $\bigl(1+\min(n-k_1-2, n-k_2-2)+\sum(n-k_i-2)\bigr)$–connected. In particular, it is $(3n-5)$–connected when U and V are balls.

The functor $(U,V)\mapsto\Omega^\infty(S^{-1}\wedge U^*\wedge V^*)$ is also bilinear. In fact, by Spanier–Whitehead duality it is naturally equivalent to the functor ψ given by

$$(U,V)\mapsto F(U\times V,\Omega^\infty\Sigma^\infty(S^{2n-3})).$$

As $\phi\to\phi^\sharp$ is initial in the homotopy category of functors, there is a natural transformation

$$\phi^\sharp\to\psi \tag{49}$$

that yields a factorization $\phi\to\phi^\sharp\to\psi$. Clearly, (49) is induced by a map of spaces $S^{n-1}\flat S^{n-1}\to\Omega^\infty\Sigma^\infty(S^{2n-3})$. Furthermore, it is automatic that the map $\phi^\sharp(U,V)\to\psi(U,V)$ is $(3n-5)$–connected when U and V are balls.

It follows that the map $\phi^\sharp(P_2,P_3)\to\psi(P_2,P_3)$ is $(3n-5-p_2-p_3)$–connected. As $3n-5-p_2-p_3$ is strictly larger than $1+\mu_2+\Sigma_2$, it follows that the map $\phi(P_2,P_3)\to\phi^\sharp(P_2,P_3)$ is $(1+\mu_2+\Sigma_2)$–connected, as was to be shown. □

Example 8.16 Let $\boldsymbol{P}=\langle S^1,\dots,S^1\rangle$ be an ordered j–tuple of circles and let $n=3$. By Theorem H,

$$\pi_0(\lambda)\colon \pi_0(\mathscr{B}(\boldsymbol{P},\mathbb{R}^3))\to\prod_{i=1}^{(j-2)!}\mathbb{Z}$$

is surjective. We conjecture that $\pi_0(\lambda)$ coincides with Milnor's μ–invariants [21, Section 5] on the set of (classical) Brunnian link maps.

8.4 Postscript: the two-component case

When $j=2$ there is some additional evidence for Conjecture 8.13 with the numerical improvements suggested by Theorem H. Let $\boldsymbol{P}=\langle P,Q\rangle$, with $p:=\dim P$ and $q:=\dim Q$. In this situation, λ is the classical stable linking pairing

$$\mathscr{L}(\langle P,Q\rangle,\mathbb{R}^n)\to F^{\mathrm{st}}(P\times Q,S^{n-1}), \tag{50}$$

which associates to a link map $f\sqcup g\colon P\sqcup Q\to\mathbb{R}^n$ the map

$$(x,y)\mapsto\frac{f(x)-g(y)}{|f(x)-g(y)|}.$$

On path components the above gives a function of pointed sets

$$\alpha\colon \pi_0(\mathscr{L}(\langle P, Q\rangle, \mathbb{R}^n)) \to \{P_+ \wedge Q_+, S^{n-1}\}, \tag{51}$$

where we have identified the set of path components of $F^{\mathrm{st}}(P \times Q, S^{n-1})$ with the abelian group of stable homotopy classes of based maps $P_+ \wedge Q_+ \to S^{n-1}$.

Suppose A and B are pointed sets. We denote the basepoint in each case by $*$. A basepoint-preserving map $h\colon A \to B$ is said to be *weakly injective* if there are no nontrivial solutions to the equation $h(x) = *$. If h is a homomorphism of groups, then weak injectivity implies injectivity (compare [9, Lemma 1.1]).

Proposition 8.17 *Assume that $Q \subset \mathbb{R}^n$ is a submanifold of codimension ≥ 3. Then the function α is a surjection on path components if $2n-2q-p-3 \geq 0$. Furthermore, if $2n-2q-p-3 > 0$ then α is weakly injective.*

Remarks 8.18 (1) Proposition 8.17 gives a better estimate than [7], but at the expense of an additional hypothesis on Q.

(2) The number $2n-2q-p-3$ may be rewritten in the form $1-q+\Sigma$, where $\Sigma = (n-p-2)+(n-q-2)$. This is the number of Theorem H when $j=2$. Hence, only weak injectivity needs to be verified.

(3) Proposition 8.17 suggests that the connectivity estimate of Conjecture 8.13 might be improved to $1-\hat{p}+\Sigma$ under the additional assumption that $P_2, \dots, P_j \subset \mathbb{R}^n$ are submanifolds of codimension ≥ 3.

(4) Proposition 8.17 delivers more information in the spherical case $P = S^p$ and $Q = S^q$ with $q \leq n-3$. Then $\pi_0(\mathscr{L}(\langle S^p, S^q\rangle, \mathbb{R}^n))$ possesses a group structure (see [28; 17, page 765]) and the function α becomes a homomorphism. Consequently, weak injectivity implies injectivity and we recover [28, page 190]. We infer that Proposition 8.17 implies that α is an isomorphism when $2n-2q-p-3 > 0$. According to [9, Theorem 1.1], in the spherical case α is actually an isomorphism if $3n-2q-2p-4 > 0$ and $p, q \geq 1$.

Proof of Proposition 8.17 As pointed out above, we only need to verify the last part of the statement. Let

$$x := f \sqcup g \in \mathscr{L}(\langle P, Q\rangle, \mathbb{R}^n)$$

be any point. We can assume without loss in generality that $f\colon P \to \mathbb{R}^n$ is a smooth map. We first show how to find a path in $\mathscr{L}(\langle P, Q\rangle, \mathbb{R}^n)$ from x to $x' = (f, h)$ in which h is a smooth embedding. It then suffices to prove that if the stable linking number of x' is trivial then the map $f\colon P \to \mathbb{R}^n \setminus h(Q)$ is null-homotopic.

Consider the commutative square

$$\begin{array}{ccc} E(Q,\mathbb{R}^n \setminus f(P)) & \longrightarrow & F(Q,\mathbb{R}^n \setminus f(P)) \\ \downarrow & & \downarrow \\ E(Q,\mathbb{R}^n) & \longrightarrow & F(Q,\mathbb{R}^n) \end{array}$$

in which $E(-,-)$ denotes the space of embeddings. By Lemma 8.19 below, the square is $(2n{-}2q{-}p{-}3)$–cartesian. If particular, if we use the preferred basepoint of $E(Q,\mathbb{R}^n)$, it follows that, when $2n-2q-p-3 \geq 0$, we can find an isotopy of the submanifold $Q \subset \mathbb{R}^n$ to an embedding $h\colon Q \to \mathbb{R}^n \setminus f(P)$ such that the underlying map of this embedding is homotopic to the map $g\colon Q \to \mathbb{R}^n \setminus f(P)$. Then $x' = (f,h) \in \mathscr{L}(\langle P,Q\rangle,\mathbb{R}^n)$ is in the same path component as x.

But, as we've seen above, the composition

$$F(P,\mathbb{R}^n \setminus h(Q)) \to \mathscr{L}(\langle P,Q\rangle,\mathbb{R}^n) \to F^{\mathrm{st}}(P \times Q, S^{n-1})$$

is $(2n{-}2q{-}p{-}3)$–connected. In particular, if $2n-2q-p-3>0$ then the triviality of the stable linking number of x' implies that the map $P \to \mathbb{R}^n \setminus h(Q)$ is null-homotopic. □

The following result was used in the proof of Proposition 8.17:

Lemma 8.19 *Assume N is a connected smooth n–manifold, and let P and Q be closed smooth manifolds of dimensions p and q. Assume $q \leq n-3$. Let $f\colon P \to N$ be a smooth map. Then the square*

$$\begin{array}{ccc} E(Q,N \setminus f(P)) & \longrightarrow & F(Q,N \setminus f(P)) \\ \downarrow & & \downarrow \\ E(Q,N) & \longrightarrow & F(Q,N) \end{array}$$

is $(2n{-}2q{-}p{-}3)$–cartesian.

Remark 8.20 When f is an embedding, this amounts to the $j=2$ case of [6, Theorem E].

Proof sketch The argument was communicated to us by Tom Goodwillie. If we replace embeddings with immersions, then the analogous diagram is ∞–cartesian by Smale–Hirsch theory (in this instance we only need to assume $q \le n-1$). Hence, it suffices to show that the square

$$\begin{array}{ccc} E(Q, N \setminus f(P)) & \longrightarrow & I(Q, N \setminus f(P)) \\ \downarrow & & \downarrow \\ E(Q,N) & \longrightarrow & I(Q,N) \end{array}$$

is $(2n-2q-p-3)$–cartesian, where $I(-,-)$ denotes the space of immersions.

The proof then proceeds by comparing the homotopy fibers of the horizontal maps of the square. The map $N \setminus f(P) \to N$ is $(n-p-1)$–connected by transversality. If $q \le n-3$, then the Goodwillie–Weiss embedding calculus applied to the embedding spaces $E(Q, N \setminus f(P))$ and $E(Q,N)$ gives towers for these homotopy fibers, where the first nontrivial layer is in degree $j \ge 2$. The homotopy-theoretic model for these layers provided by [30] implies that the map of the j^{th} layers is $(2n-2q-p-3)$–connected for all j. The conclusion then follows from the five lemma. □

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