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C^*–algebraic drawings of dendroidal sets

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In recent years the theory of dendroidal sets has emerged as an important framework for higher algebra. We introduce the concept of a *C^*–algebraic drawing* of a dendroidal set. It depicts a dendroidal set as an object in the category of presheaves on C^*–algebras. We show that the construction is functorial and, in fact, it is the left adjoint of a Quillen adjunction between combinatorial model categories. We use this construction to produce a bridge between the two prominent paradigms of noncommutative geometry via adjunctions of presentable ∞–categories, which is the primary motivation behind this article. As a consequence we obtain a single mechanism to construct bivariant homology theories in both paradigms. We propose a (conjectural) roadmap to harmonize algebraic and analytic (or topological) bivariant K–theory. Finally, a method to analyze graph algebras in terms of trees is sketched.

46L85, 55P48; 18D50, 46L87, 55U10

0 Introduction

Dendroidal sets provide a convenient model for ∞–operads (see Heuts, Hinich and Moerdijk [21] for a comparison with Lurie's model [31] for ∞–operads without constants). The category of dendroidal sets $\mathtt{dSet}$ was introduced by Moerdijk and Weiss [44; 45] so that (inter alia) it can serve as a receptacle for the nerve functor on the category of operads $\mathtt{Operad}$. The following commutative diagram is explanatory:

$$\begin{array}{ccc} \mathtt{Cat} & \longrightarrow & \mathtt{Operad} \\ {\scriptstyle \mathrm{N}}\downarrow & & \downarrow{\scriptstyle \mathrm{N_d}} \\ \mathtt{sSet} & \longrightarrow & \mathtt{dSet} \end{array}$$

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where the vertical arrow N (resp. $\mathrm{N_d}$) denotes the nerve (resp. dendroidal nerve) functor. Cisinski and Moerdijk [10] constructed a cofibrantly generated model structure on $\mathtt{dSet}$ such that the fibrant objects are precisely the ∞–operads [31]. Over the last decade the theory of dendroidal sets has reached an advanced stage, subsuming several aspects of the theory of operads and that of simplicial sets; see Cisinski and Moerdijk [11; 12].

For a small category $\mathcal{C}$ let $\mathcal{P}(\mathcal{C})$ denote the category of $\mathtt{Set}$–valued presheaves on $\mathcal{C}$. Let $\mathtt{SC}^*_{\mathrm{un}}$ denote the category of *nonzero* separable unital C^*–algebras equipped with unit-preserving $*$–homomorphisms. The Gelfand–Naĭmark duality implies that $\mathtt{SC}^{*\,\mathrm{op}}_{\mathrm{un}}$ can be regarded as the category of nonempty compact second countable noncommutative spaces with continuous maps. Let Ω denote the small category of trees, so that $\mathtt{dSet} := \mathcal{P}(\Omega)$ is the category of *dendroidal sets*. In this article we prove the following results:

(1) We construct a *noncommutative dendrices* functor $D\colon \Omega \to \mathtt{SC}^{*\,\mathrm{op}}_{\mathrm{un}}$.

(2) We construct an *operadic model structure* on $\mathcal{P}(\mathtt{SC}^{*\,\mathrm{op}}_{\mathrm{un}})$, an instance of Cisinksi's model structure on presheaves.

(3) We observe that the canonical adjoint pair induced by the noncommutative dendrices functor via left Kan extension

$$\mathtt{dr}\colon \mathtt{dSet} \rightleftarrows \mathcal{P}(\mathtt{SC}^{*\,\mathrm{op}}_{\mathrm{un}}) :\mathtt{dd}$$

is a Quillen pair between combinatorial model categories.

We call the image of a dendroidal set under the left adjoint functor $\mathtt{dr}\colon \mathtt{dSet} \to \mathcal{P}(\mathtt{SC}^{*\,\mathrm{op}}_{\mathrm{un}})$ the *C^*–algebraic drawing* of the dendroidal set.

These results constitute the first steps towards a bigger objective, which we briefly explain below. There are two prevalent perspectives on noncommutative geometry: analytic and algebraic. The analytic approach was pioneered by Connes [13] — see also Connes and Marcolli [14] — whereas the algebraic approach builds upon work of Drinfeld, Keller, Kontsevich, Lurie, Manin, Mahanta, Tabuada, Toën and several others [39; 25; 27; 31; 50; 34]. Table 1 compares the two approaches as of now.

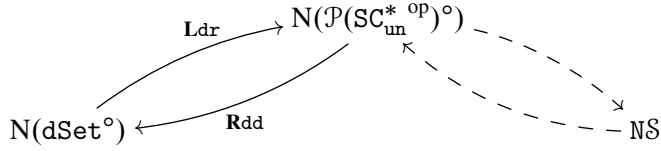
The space X above in each case must satisfy some reasonable hypotheses. The ∞–category $\mathtt{Perf}_\infty(X)$ is stable and in some contexts stability is included in the definition. This article is primarily motivated by the author's desire to reconcile the two viewpoints. In view of the disparate nature of the basic ingredients of the two paradigms, a *bridge* between the basic objects of the two worlds in the form (a zigzag of) ∞–categorical

	Analytic	Algebraic
objects	C^*–algebras	∞–categories
morphisms	$*$–homomorphisms	∞–functors
how to subsume traditional spaces	$X \mapsto C(X)$	$X \mapsto \mathtt{Perf}_\infty(X)$

Table 1: Comparison between the analytic and algebraic approaches

adjunctions subject to a reasonable requirement (explained below) seems to be a sensible target. While constructing the bridge we have resorted to ∞–categories, which reflects the state of the art.

Let $\mathtt{NS}$ denote the compactly generated ∞–category of (unpointed) noncommutative spaces, whose construction is presented in Section 3.1. The following diagram of adjunctions between presentable ∞–categories summarizes our list of results and puts them in the broader context (see also Remark 3.6):

(1)


Here $\mathrm{N}(\mathcal{M}^\circ)$ denotes the underlying ∞–category of a model category $\mathcal{M}$. The ∞–categorical adjunction $\mathbf{L}\mathtt{dr}\colon \mathrm{N}(\mathtt{dSet}^\circ) \rightleftarrows \mathrm{N}(\mathcal{P}(\mathtt{SC}^{*\,\mathrm{op}}_{\mathrm{un}})^\circ) : \mathbf{R}\mathtt{dd}$ is induced by the Quillen adjunction $\mathtt{dr}\colon \mathtt{dSet} \rightleftarrows \mathcal{P}(\mathtt{SC}^{*\,\mathrm{op}}_{\mathrm{un}}) : \mathtt{dd}$ between combinatorial model categories mentioned earlier (see also Remark 3.4). However, the dashed pair between $\mathtt{NS}$ and $\mathrm{N}(\mathcal{P}(\mathtt{SC}^{*\,\mathrm{op}}_{\mathrm{un}})^\circ)$ is merely a zigzag of adjunctions that is constructed at the level of ∞–categories. This construction actually passes through a *mixed model structure*, denoted by $\mathcal{P}(\mathtt{SC}^{*\,\mathrm{op}}_{\mathrm{un}})_{\mathrm{mix}}$, on $\mathcal{P}(\mathtt{SC}^{*\,\mathrm{op}}_{\mathrm{un}})$ which is a left Bousfield localization of the operadic model structure (see Definition 3.12). Diagram (1) is our proposed *bridge* between the two paradigms of noncommutative geometry.

0.1 Bivariant homology theories

Given any stable presentable ∞–category $\mathcal{C}$, a colimit-preserving functor

$$\mathrm{B}_{\mathcal{C}}\colon \mathrm{N}(\mathcal{P}(\mathtt{SC}^{*\,\mathrm{op}}_{\mathrm{un}})^\circ) \to \mathcal{C}$$

can be viewed as a $\mathcal{C}$–valued bivariant homology theory on $\mathrm{N}(\mathcal{P}(\mathtt{SC}^{*\,\mathrm{op}}_{\mathrm{un}})^\circ)$. For a presentable ∞–category $\mathcal{D}$, let $\mathtt{Sp}(\mathcal{D})$ denote its stabilization. The functor $\mathrm{B}_{\mathcal{C}}$ factors as $\mathrm{N}(\mathcal{P}(\mathtt{SC}^{*\,\mathrm{op}}_{\mathrm{un}})^\circ) \to \mathtt{Sp}\big(\mathrm{N}(\mathcal{P}(\mathtt{SC}^{*\,\mathrm{op}}_{\mathrm{un}})^\circ)\big) \to \mathcal{C}$.

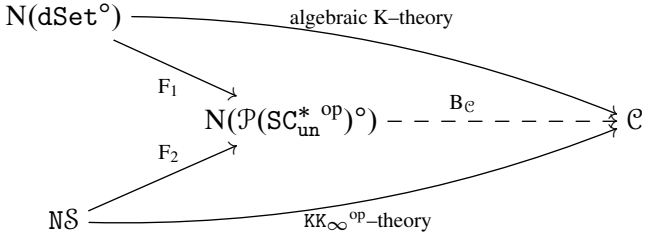
There must be a unified framework for bivariant homology theories in the two paradigms of noncommutative geometry. In order to realize this objective one must construct a functor $\mathrm{B}_{\mathcal{C}}$ that passes the following two acid tests:

(i) the composite functor $\mathrm{N}(\mathtt{dSet}^{\circ}) \to \mathrm{N}(\mathcal{P}(\mathtt{SC}^{*}_{\mathtt{un}}{}^{\mathrm{op}})^{\circ}) \xrightarrow{\mathrm{B}_{\mathcal{C}}} \mathcal{C}$ should lead to the (nonconnective version of) algebraic K–theory of ∞–operads as in Nikolaus [46], and

(ii) the composite functor $\mathtt{NS} \to \mathrm{N}(\mathcal{P}(\mathtt{SC}^{*}_{\mathtt{un}}{}^{\mathrm{op}})^{\circ}) \xrightarrow{\mathrm{B}_{\mathcal{C}}} \mathcal{C}$ should recover the opposite of the bivariant K–theory of (pointed) noncommutative spaces as in Mahanta [36] after stabilization.

Let us provide a pictorial description of our vision:

(2)



Here the functors F_1 and F_2 are furnished by those of diagram (1), so that $\mathrm{F}_1 = \mathbf{L}\mathtt{dr}$. For any $X \in \mathrm{N}(\mathtt{dSet}^{\circ})$ we require $\mathcal{C}(\mathrm{B}_{\mathcal{C}} \circ \mathrm{F}_1(\mathbf{1}), \mathrm{B}_{\mathcal{C}} \circ \mathrm{F}_1(X))$ to be the (nonconnective version of) algebraic K–theory of X, where $\mathbf{1}$ is a unit object. Moreover, for any pair $A, B \in \mathtt{NS}$ we require the equivalence of spectra

$$\mathcal{C}\big(\mathtt{Sp}(\mathrm{B}_{\mathcal{C}}) \circ \mathtt{Sp}(\mathrm{F}_2)(\Sigma^{\infty}_{+}(A)), \mathtt{Sp}(\mathrm{B}_{\mathcal{C}}) \circ \mathtt{Sp}(\mathrm{F}_2)(\Sigma^{\infty}_{+}(B))\big) \simeq \mathtt{KK}_{\infty}{}^{\mathrm{op}}(\mathrm{k}_{+}{}^{\mathrm{op}}(A), \mathrm{k}_{+}{}^{\mathrm{op}}(B)),$$

where $\mathrm{k}_{+}{}^{\mathrm{op}}$ is the composite functor $\mathtt{NS} \to \mathtt{NS}_{*} \xrightarrow{\mathrm{k}^{\mathrm{op}}} \mathtt{KK}_{\infty}{}^{\mathrm{op}}$ [36]. Varying $\mathrm{B}_{\mathcal{C}}$, one can construct new bivariant homology theories using the above mechanism in both paradigms. For more generalities on bivariant homology theories of noncommutative spaces in the setting of ∞–categories and model categories, the reader may refer to Mahanta [38] or Barnea, Joachim and Mahanta [2]. One possible application of this vision is outlined in Remark 4.9.

Remark A knowledgeable reader might contend that *spectral triples* constitute the notion of a space in noncommutative geometry à la Connes. Let us clarify that by a *space* we really mean a *topological space*. A spectral triple (A, H, D) should be regarded as a noncommutative manifold, whose underlying topological space is determined by the C^{*}–algebra A. Therefore, our proposed bridge (1) exists in the realm of noncommutative topology.

Remark There is also a Quillen adjunction $i_!\colon \mathtt{sSet} \rightleftarrows \mathtt{dSet} : i^*$ that connects the theory of ∞–categories with that of ∞–operads. In this case the relevant model structure on $\mathtt{sSet}$ is the Joyal model structure, whose fibrant objects are ∞–categories. Via the Yoneda embedding $\mathtt{SC}^{*\,\mathrm{op}}_{\mathrm{un}} \hookrightarrow \mathcal{P}(\mathtt{SC}^{*\,\mathrm{op}}_{\mathrm{un}})$ the category $\mathtt{SC}^{*\,\mathrm{op}}_{\mathrm{un}}$ acquires a new class of weak equivalences from the operadic model structure on $\mathcal{P}(\mathtt{SC}^{*\,\mathrm{op}}_{\mathrm{un}})$ as in Theorem A.11. We call these weak equivalences on $\mathtt{SC}^{*\,\mathrm{op}}_{\mathrm{un}}$ the *weak operadic equivalences*. The associated homotopy theory is different from (the opposite of) the standard homotopy theory of C^*–algebras endowed with the C^*–homotopy equivalences. The exact difference between the two homotopy theories is not clear to the author (see Remark 3.5).

Remark The technology developed in this article works for all dendroidal sets. But from the viewpoint of topology it is preferable to restrict one's attention to *open dendroidal sets*, which model ∞–operads without constants (see Remark 3.6).

Notation and conventions Unless otherwise stated, a graph means a finite directed graph and a presheaf is considered to be $\mathtt{Set}$–valued. For the sake of definiteness we adopt the quasicategorical model for ∞–categories. An operad always means a coloured operad. We are mostly going to deal with the category of *nonzero* unital separable C^*–algebras $\mathtt{SC}^*_{\mathrm{un}}$ with unit-preserving $*$–homomorphisms (except for in Section 3.1). Including the zero C^*–algebra from the viewpoint of trees and operads does not seem appropriate.

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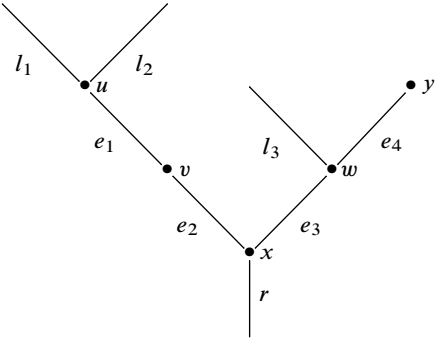
1 Dendroidal sets

We are going to assume familiarity with the theory of (coloured) operads and simplicial sets. For the uninitiated we recommend the following good sources of knowledge: [41; 8; 40; 29; 19; 6], a list that is obviously nonexhaustive. Since the article is written for topologists as well as operator algebraists, we review the theory of dendroidal sets from [51; 44; 45; 10], which is a simultaneous generalization of both operads and simplicial sets. The exposition is quite brief and necessarily not entirely self-contained.

Trees have played an important role in the theory of operads ever since its inception. We provide an informal and very concise introduction to trees. We follow the nomenclature and presentation in [44; 43]. A tree is a finite directed graph whose underlying undirected graph is connected and acyclic. The vertices will be marked by • as shown below:

(3)

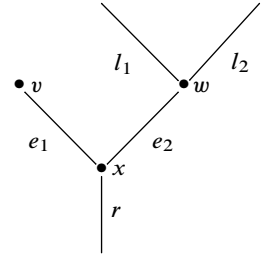


An edge that is connected to two vertices is called an *inner edge*; the rest are called *outer edges*. Amongst the outer edges, ie those that are attached to only one vertex, there is a distinguished one called the *root*; the other outer edges are called *leaves*. A *nonplanar rooted tree* is a nonempty tree with both inner and outer edges with the choice of one distinguished outer edge as the root. Henceforth, unless otherwise stated, by a tree we shall mean a nonplanar rooted tree. Such a tree will be drawn with the root at the bottom and all arrows directed from top to bottom (with arrowheads deleted) as shown above. For instance, in the above tree there are three leaves l_1, l_2 and l_3, four inner edges e_1, e_2, e_3 and e_4, and the root is r. Note that the number of inner edges as well as leaves in a tree could be zero. The simplest possible tree is

which is called the *unit tree*.

The category of simplicial sets, denoted by $\mathtt{sSet}$, is the category of $\mathtt{Set}$–valued presheaves on the category of simplices Δ, ie $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathtt{Set})$. The notion of a morphism between trees is described in Section 1.1, and this allows us to define a category Ω of trees. Then, in analogy with simplicial sets, we define dendroidal sets to be $\mathtt{dSet} = \mathrm{Fun}(\Omega^{\mathrm{op}}, \mathtt{Set})$, the category of $\mathtt{Set}$–valued presheaves on Ω. It will be clear from the definition of the objects and the morphisms of Ω that it can be viewed as a full subcategory of the category of symmetric coloured operads. There is a fully faithful functor $i\colon \Delta \hookrightarrow \Omega$, leading to an adjunction $i_!\colon \mathtt{sSet} \rightleftarrows \mathtt{dSet} : i^*$. The functor $i_!$ is fully faithful and hence the category of dendroidal sets is a generalization of that of simplicial sets. Since $\mathtt{dSet} = \mathrm{Fun}(\Omega^{\mathrm{op}}, \mathtt{Set})$, it suffices to describe the category Ω. The objects of Ω are nonplanar rooted trees as described above. Note that in a *planar* rooted tree the incoming edges at each vertex have a prescribed linear ordering, which does not exist in a nonplanar rooted tree. Hence, each such planar (resp. nonplanar) rooted tree generates a nonsymmetric (resp. symmetric) coloured operad $\Omega[T]$. The set of morphisms $\Omega(S, T)$ between two nonplanar rooted trees S and T is by definition the set of coloured operad maps between $\Omega[S]$ to $\Omega[T]$. Thus, by construction, Ω is the full subcategory of the category of symmetric coloured operads spanned by the objects of the form $\Omega[T]$. The colours of the operad $\Omega[T]$ correspond to the edges of T and a morphism between such operads is completely determined by its effect on colours. Each vertex v of a tree T with outgoing edge e and a labelling of the incoming edges $e_1, \dots, e_n$ defines an operation $v \in \Omega[T](e_1, \dots, e_n; e)$. Consider the nonplanar rooted tree T:

(4)



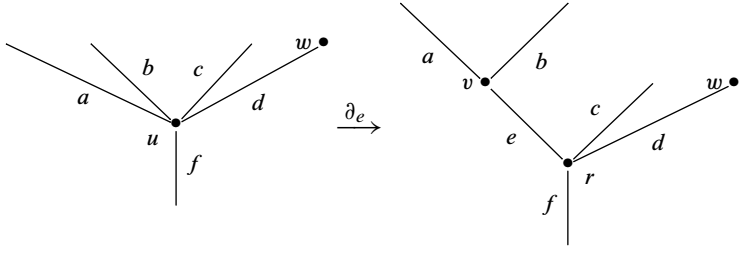
The operad $\Omega[T]$ that it generates has five colours, l_1, l_2, e_1, e_2 and r. The generating operations are $v \in \Omega[T](; e_1)$, $w \in \Omega[T](l_1, l_2; e_2)$ and $x \in \Omega[T](e_1, e_2; r)$. There are also operations that arise from the action of the symmetric group in the nonplanar case. For instance, if $\sigma \in \Sigma_2$, then $w \circ \sigma \in \Omega[T](l_2, l_1; e_2)$ is another operation. There are also the unit operations 1_{l_1}, 1_{l_2}, 1_{e_1}, 1_{e_2} and 1_r and compositions like $x \circ_2 w \in \Omega[T](e_1, l_1, l_2; r)$. We refrain from documenting a complete list of all operations and the relations they satisfy, which the reader can herself/himself reproduce

from the above diagram. Instead, we turn towards a more concrete (and pictorial) description of the morphisms in Ω that will be needed later.

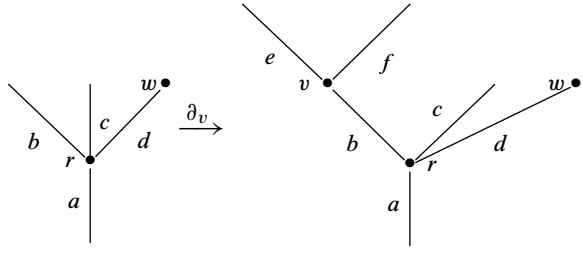
1.1 Face and degeneracy maps

We illustrate the face and degeneracy maps in Ω by examples that are taken directly from [44], where one can find a more elaborate discussion. These maps provide an explicit description of all morphisms in the category Ω, as we shall see at the end of this subsection.

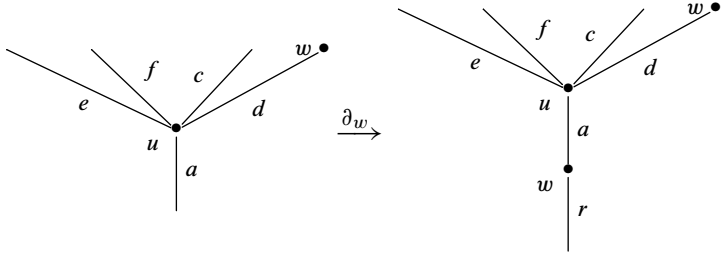
(1) If e is an inner edge in T, then one obtains an *inner face* map $\partial_e\colon T/e \to T$, where T/e is constructed by contracting the edge e as shown below:



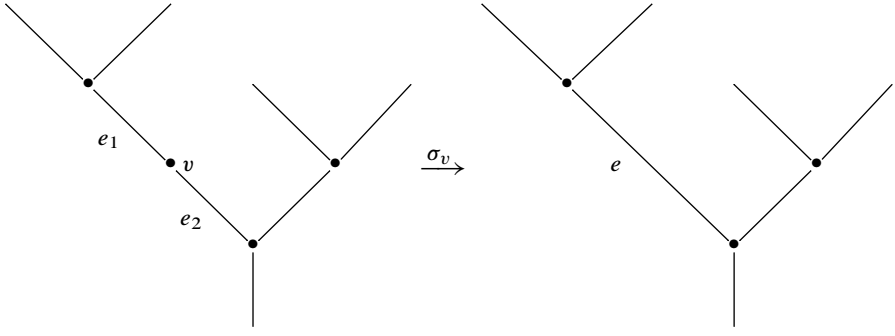
(2) If a vertex v in T has exactly one inner edge attached to it, one obtains the *outer face* map $\partial_v\colon T/v \to T$, where T/v is constructed by deleting v and all the outer edges attached to it as shown below:



It is also possible to remove the root and the vertex that it is attached to by this process, as shown below:



(3) If a vertex $v \in T$ has exactly one incoming edge, there is a tree $T\backslash v$, obtained from T by deleting the vertex v and merging the two edges e_1 and e_2 on either side of v into one new edge e. This defines the *degeneracy map* $\sigma_v\colon T \to T\backslash v$ as shown below:



The following lemma explains the importance of these maps:

Lemma 1.1 [44, Lemma 3.1] *Any arrow $f\colon S \to T$ in Ω decomposes as*

$$\begin{array}{ccc} S & \xrightarrow{f} & T \\ {\scriptstyle\sigma}\downarrow & & \uparrow{\scriptstyle\delta} \\ S' & \xrightarrow{\varphi} & T' \end{array}$$

where $\sigma\colon S \to S'$ is a composition of degeneracy maps, $\varphi\colon S' \to T'$ is an isomorphism and $\delta\colon T' \to T$ is a composition of face maps.

Remark 1.2 We have quoted the statement of Lemma 1.1 from the original source. If one carefully inspects its proof (see Lemma 2.3.2 of [43]) one notices immediately that the factorization $f = \delta \circ \varphi \circ \sigma$ is unique. Hence, the degeneracy maps and the face maps of Ω actually constitute a factorization system.

1.2 Face and degeneracy identities

These face and degeneracy maps satisfy numerous identities. We illustrate them in terms of various commuting diagrams in Ω (with the existence of certain nonobvious arrows as assertions). The interested reader is referred to [44; 43] for further details and also the discussion of a couple of special cases that we have left out (see Remark 1.3).

(I) If e and f are distinct inner edges, then $(T/e)/f = (T/f)/e$ and the following diagram commutes:

$$\begin{array}{ccc} (T/e)/f & \xrightarrow{\partial_f} & T/e \\ {\scriptstyle \partial_e}\downarrow & & \downarrow{\scriptstyle \partial_e} \\ T/f & \xrightarrow{\partial_f} & T \end{array}$$

(II) Assume T has at least three vertices and let ∂_v and ∂_w be distinct outer face maps. Then $(T/v)/w = (T/w)/v$ and the following diagram commutes:

$$\begin{array}{ccc} (T/v)/w & \xrightarrow{\partial_w} & T/v \\ {\scriptstyle \partial_v}\downarrow & & \downarrow{\scriptstyle \partial_v} \\ T/w & \xrightarrow{\partial_w} & T \end{array}$$

(III) If e is an inner edge that is not adjacent to a vertex v, then $(T/e)/v = (T/v)/e$ and the following diagram commutes:

$$\begin{array}{ccc} (T/v)/e & \xrightarrow{\partial_e} & T/v \\ {\scriptstyle \partial_v}\downarrow & & \downarrow{\scriptstyle \partial_v} \\ T/e & \xrightarrow{\partial_e} & T \end{array}$$

(IV) Let e be an inner edge that is adjacent to a vertex v and let w be the other adjacent vertex. In T/e the two vertices combine to contribute a vertex z (expressing the composition of v and w in some order). Then the outer face $\partial_z\colon (T/e)/z \to T/e$ exists if and only if the outer face $\partial_w\colon (T/v)/w \to T/v$ exists, and in this case $(T/e)/z = (T/v)/w$. Summarizing the setup, the following diagram commutes:

$$\begin{array}{ccccc} (T/v)/w & = & (T/e)/z & \xrightarrow{\partial_z} & T/e \\ {\scriptstyle \partial_w}\downarrow & & & & \downarrow{\scriptstyle \partial_e} \\ T/v & & \xrightarrow{\qquad\qquad\partial_v\qquad\qquad} & & T \end{array}$$

(V) If σ_v and σ_w are two degeneracies of T, then $(T\backslash v)\backslash w = (T\backslash w)\backslash v$ and the following diagram commutes:

$$\begin{array}{ccc} T & \xrightarrow{\sigma_v} & T\backslash v \\ {\scriptstyle \sigma_w}\downarrow & & \downarrow{\scriptstyle \sigma_w} \\ T\backslash w & \xrightarrow{\sigma_v} & (T\backslash v)\backslash w \end{array}$$

(VI) Let $\sigma_v\colon T \to T\backslash v$ be a degeneracy and $\partial\colon T' \to T$ be any face map such that T' still contains v and its two adjacent edges as a subtree. Then the following diagram commutes:

$$\begin{array}{ccc} T & \xrightarrow{\sigma_v} & T\backslash v \\ {\scriptstyle\partial}\uparrow & & \uparrow{\scriptstyle\partial} \\ T' & \xrightarrow{\sigma_v} & T'\backslash v \end{array}$$

(VII) Let $\sigma_v\colon T \to T\backslash v$ be a degeneracy map and $\partial\colon T' \to T$ be a face map induced by one of the adjacent edges to v or the removal of v (if that is possible). Then $T' = T\backslash v$ and the following diagram commutes:

$$\begin{array}{ccc} T\backslash v & \xrightarrow{\mathrm{id}_{T\backslash v}} & T\backslash v \\ & {\scriptstyle\partial}\searrow \quad \nearrow{\scriptstyle\sigma_v} & \\ & T & \end{array}$$

Remark 1.3 We have left out the following special cases of dendroidal identities:

- Outer face identities when T has fewer than three vertices.
- Predictable identities expressing the compatibility of the face and degeneracy maps with isomorphisms (see for instance Section 2.3.1 of [43]).

1.3 The model structure on `dSet`

The formalism of model categories was introduced by Quillen [48] as an abstract framework for homotopy theory. For a modern treatment the reader may refer to [24; 23]. We review the model structure on `dSet` constructed by Cisinski and Moerdijk [10] that generalizes the Joyal model structure on `sSet`.

The construction of the model structure on `dSet` exploits the Cisinski model structure on any category of presheaves [9] (see the appendix) and also a transfer principle. Typically one begins with certain desired features on the model structure based on intended applications. Keeping in mind the Joyal model structure on `sSet`, it is natural to expect that in the would-be model structure on `dSet` (certain) monomorphisms should be cofibrations, some class of objects (generalizing ∞–categories) should be fibrant, and certain morphisms (generalizing categorical equivalences) should be weak equivalences.

A monomorphism of dendroidal sets $X \to Y$ is *normal* if for any $T \in \Omega$, the action of $\mathrm{Aut}(T)$ on $Y(T) \setminus X(T)$ is free. If e is an inner edge of a tree T, then one obtains

an *inner horn inclusion* $\Lambda^e[T] \to \Omega[T]$, where $\Lambda^e[T]$ is obtained as the union of the images of all the elementary face maps apart from $\partial_e\colon T/e \to T$. A map of dendroidal sets is called an *inner anodyne extension* if it belongs to the smallest class of maps which is stable under pushouts, transfinite compositions and retracts, and which contains the inner horn inclusions. There is an adjunction $\tau_{\mathrm{d}}\colon \mathtt{dSet} \rightleftarrows \mathtt{Operad} :\mathrm{N}_{\mathrm{d}}$, where τ_{d} is called the *operadic realization* functor. The model structure on $\mathtt{dSet}$ can be described as (see Theorem 2.4 of [10]):

- the cofibrations are the *normal monomorphisms*;
- the fibrant objects are the ∞–operads;
- the fibrations between fibrant objects are the inner Kan fibrations (see [45; 10, Section 2.1]), whose image under τ_{d} is an operadic fibration, ie a fibration in the canonical model structure on operads;
- the class of weak equivalences is the smallest class $\mathtt{W}$ of maps in $\mathtt{dSet}$ satisfying
  (a) the 2-out-of-3 property;
  (b) that inner anodyne extensions are in $\mathtt{W}$;
  (c) that trivial fibrations between ∞–operads are in $\mathtt{W}$.

We omit further details but explain an additional property of this model category that is relevant for our purposes. Let κ be regular cardinal. A category $\mathcal{A}$ is said to be κ–*accessible* if there is a small category $\mathcal{C}$ such that $\mathcal{A} \cong \mathrm{Ind}_\kappa(\mathcal{C})$. A *locally* κ–*presentable* category is a κ–accessible category that, in addition, possesses all small colimits. A category is *locally presentable* if it is locally κ–presentable for some regular cardinal κ. If $\mathcal{C}$ is a small category, the category of presheaves on $\mathcal{C}$ (eg $\mathtt{dSet} = \mathrm{Fun}(\Omega^{\mathrm{op}}, \mathtt{Set})$) is locally ω–presentable (see for instance [1]). Recall that a model category is said to be *combinatorial* if it is cofibrantly generated and its underlying category is locally presentable. It is also shown in Proposition 2.6 of [10] that the model category $\mathtt{dSet}$ is combinatorial. The set of generating cofibrations I consists of the boundary inclusions of trees, ie $I = \{\partial\Omega[T] \to \Omega[T] \mid T \in \Omega\}$.

2 C^*–algebras associated with trees: noncommutative dendrices

The description of a tree presented in the previous section differs slightly from the one that one might encounter in graph theory. For instance, in the graph algebra

literature a *directed graph* $G = (E^0, E^1, r, s)$ consists of two (countable) sets E^0 and E^1 and functions $r, s\colon E^1 \to E^0$. The elements of E^0 are called the *vertices* and those of E^1 are called the *edges* of G. For an edge e, the vertex $s(e)$ is its *source* and the vertex $r(e)$ is its *range*. Thus, in a directed graph one does not have edges attached only to one vertex like the leaves or the root that we considered in the previous section. In a graph a *path of length n* is a sequence $\mu = e_1 e_2 \cdots e_n$ of edges such that $s(e_i) = r(e_{i+1})$ for all $i \leqslant i \leqslant n-1$. For such a path $\mu = e_1 e_2 \cdots e_n$ we denote by $\text{edge}(\mu) = \{e_1, e_2, \ldots, e_n\}$ the set of all edges traversed by it.

The C^*–algebra associated with a tree that we are going to describe shortly is to some extent inspired by the construction of noncommutative simplicial complexes in [16]. However, we design the C^*–algebra from the edges of the tree, since from the categorical (or operadic) viewpoint the edges are more fundamental than the vertices.

Definition 2.1 Given a set G of generators and a set R of relations, the *universal C^*–algebra*, denoted by $C^*(G, R)$, is a C^*–algebra equipped with a set map $\iota\colon G \to C^*(G, R)$ that satisfies the following universal property: for every C^*–algebra A and set map $\iota_A\colon G \to A$ such that the relations R are fulfilled inside A, there is a unique $*$–homomorphism $\theta\colon C^*(G, R) \to A$ satisfying $\theta \circ \iota = \iota_A$.

This is a subtle concept; for instance, if $G = \{x\}$ and $R = \varnothing$, then the universal C^*–algebra $C^*(G, R)$ does not exist. In other words, free (or relation-free) objects do not exist in the category of C^*–algebras. It follows from two simple facts:

(1) Every element in a C^*–algebra has a finite norm $\|\cdot\|$, ie a real number.

(2) Every $*$–homomorphism is norm-decreasing, ie $\phi\colon A \to B$ implies $\|\phi(a)\| \leqslant \|a\|$.

If $C^*(G = \{x\}, R = \varnothing)$ were to exist, then the generator x would have a finite norm $\|x\|$. Now choose any C^*–algebra A and an element $a \in A$ with $\|a\| > \|x\|$, which can evidently be done. Then it is manifestly clear that one cannot find the desired $*$–homomorphism $\iota\colon C^*(G = \{x\}, R = \varnothing) \to A$ with $\iota(x) = a$ that satisfies requirement (2) above. If the relations R put a nonstrict bound on the norm of each generator, then typically one obtains an interesting nontrivial universal C^*–algebra (although it can be trivial in certain cases).

Definition 2.2 Given any tree $T = (E^0, E^1)$ (viewed as a graph as described above) we define its *associated C^*–algebra* as the universal unital C^*–algebra generated by $\{q_e \mid e \in E^1\}$ satisfying

(1) $q_e \geq 0$ for all $e \in E^1$,

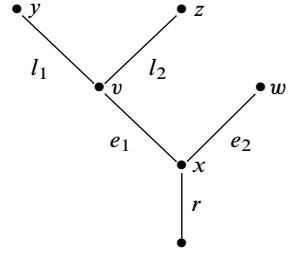
(2) $\sum_{e \in E^1} q_e = 1$, and

(3) $q_{e_1} q_{e_2} \cdots q_{e_n} = 0$ unless there is a path μ with $\{e_1, e_2, \ldots, e_n\} \subseteq \text{edge}(\mu)$ (inclusion of sets disregarding order).

Remark 2.3 Let us briefly clarify the motivation behind the relations.

- The relations (1) and (2) clearly put a bound on the norm of each generator and hence the existence of the universal C^*–algebra is clear.
- Relation (3) encodes the compositional nature of trees. It retains those terms that lie in a path (and hence bound a simplex). However, it also retains reorderings and repetitions of edges within the path because we want the canonical abelianization map to be surjective (see Remark 2.5 and Example 2.7).

Example 2.4 Note that repetitions are allowed amongst the e_i in relation (3) above. For instance, if T is



then $q_{l_2} q_{e_1} q_{e_2} = q_{l_1} q_{l_2} = q_{e_2} q_{e_1} q_{l_2} = 0$, whereas $q_r q_{e_1} q_{l_1} \neq 0$ and $q_{e_1} q_{l_2} q_{e_1} \neq 0$.

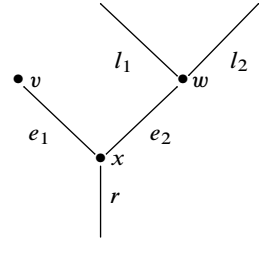
Given any nonplanar rooted tree T we construct its associated C^*–algebra $D(T)$ as follows:

(a) insert a vertex at each of the top tips of the leaves (if any) and the bottom tip of the root;

(b) construct the universal C^*–algebra of the modified tree as explained above.

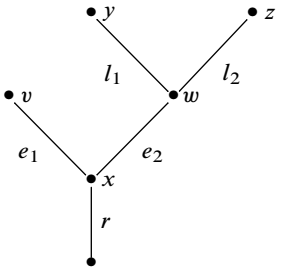
For instance, given the tree

(5)



then according to procedure (a) we modify the tree as

(6)



and then construct its universal C^*–algebra.

Remark 2.5 In the above construction we can add the relation that the generators commute, ie $q_e q_f = q_f q_e$ for all $e, f \in E^1$ to obtain a commutative C^*–algebra $D^{\mathrm{ab}}(T)$.

Definition 2.6 The C^*–algebra $D(T)$ associated with a nonplanar rooted tree T is called a *noncommutative dendrex*. Note that if $X \in \mathtt{dSet}$ and $T \in \Omega$, then $X(T)$ is viewed as the set of T–shaped dendrices in X.

Example 2.7 An object $[n] \in \Delta$ can be viewed as a linear tree L_n as

$$\leftarrow \bullet_1 \leftarrow \cdots \leftarrow \bullet_n \leftarrow$$

(drawn horizontally instead of vertically with arrowheads inserted to indicated the direction). This association $[n] \mapsto L_n$ defines a fully faithful functor $\Delta \hookrightarrow \Omega$ that produces the adjunction $\mathtt{sSet} \rightleftarrows \mathtt{dSet}$. After modification L_n produces the tree

$$\bullet_0 \leftarrow \bullet_1 \leftarrow \cdots \leftarrow \bullet_{n+1},$$

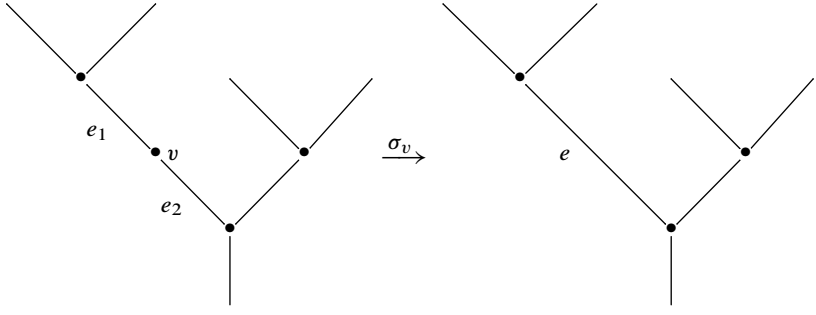
whose associated C^*–algebra is the universal unital C^*–algebra generated by $n+1$ positive generators $\{q_1, \ldots, q_{n+1}\}$ such that $\sum_{i=1}^{n} q_i = 1$. Its associated commutative C^*–algebra (see Remark 2.5) is isomorphic to $\mathrm{C}(\Delta^n)$, where Δ^n is the standard n–simplex (see Proposition 2.1 of [16]). Our choice for the noncommutative dendrex construction was guided by this consideration. Observe that $D(L_0) = \mathbb{C}$, since $[0]$ corresponds to the unit tree

whose modified tree is simply

with only one edge. This phenomenon reflects the fact that the edges of a tree correspond to the colours of its associated operad.

2.1 Functoriality

The aim of this subsection is to establish the (contravariant) functoriality of the above construction $T \mapsto D(T)$ with respect to morphisms of Ω. To this end we begin by defining the $*$–homomorphisms that the faces and degeneracies induce. If $\sigma_v\colon T \to T\backslash v$ is a degeneracy map (see Lemma 1.1) like



then define $\sigma_v^*\colon D(T\backslash v) \to D(T)$ as

$$q_f \mapsto \begin{cases} q_f & \text{if } f \neq e, \\ q_{e_1} + q_{e_2} & \text{otherwise.} \end{cases}$$

Remark 2.8 The notation employed in the definition of σ_v^* is potentially ambiguous. In the domain q_f is a generator of $D(T\backslash v)$ and in the codomain it is a generator of $D(T)$. One should ideally differentiate them by writing $q_f^{T\backslash v}$ and q_f^T (or something similar) to indicate the dependence on the tree. For notational simplicity we avoid doing this.

Lemma 2.9 *The map $\sigma_v^*\colon D(T\backslash v) \to D(T)$ is a $*$–homomorphism.*

Proof We need to verify that the set $\{\sigma_v^*(q_f) \mid f \text{ an edge in } T\backslash v\}$ satisfies the relations (1), (2) and (3) in $D(T)$ that define the universal C^*–algebra $D(T\backslash v)$.

For (1) note that q_{e_1} and q_{e_2} are both positive in $D(T)$, whence so is $q_{e_1}+q_{e_2}$. Clearly each q_f is also positive in $D(T)$. Let $E^1(T)$ be the set of edges in T. We verify (2) by computing

$$\sum_{f\in E^1(T\backslash v)} \sigma_v^*(q_f) = \sum_{f\neq e} q_f + (q_{e_1}+q_{e_2}) = \sum_{f\in E^1(T)} q_f = 1.$$

For (3) one can check by inspection that if f_1 and f_2 are two edges in $T\backslash v$ that do not lie in a path, then they cannot lie in a path in T. □

Note that every face map can be viewed as an injective map on edges (or colours of the associated operad). Thus, if $\partial_e\colon T/e \to T$ is an inner face map then define a $*$–homomorphism $\partial_e^*\colon D(T)\to D(T/e)$ as

$$q_f \mapsto \begin{cases} q_f & \text{if } f\neq e,\\ 0 & \text{otherwise.}\end{cases}$$

Similarly, if $\partial_v\colon T/v\to T$ is an outer face map then define $\partial_v^*\colon D(T)\to D(T/v)$ as

$$q_f \mapsto \begin{cases} q_f & \text{if } f \text{ has not been removed,}\\ 0 & \text{otherwise.}\end{cases}$$

Lemma 2.10 *The maps*

$$\partial_e^*\colon D(T)\to D(T/e) \quad \textit{and} \quad \partial_v^*\colon D(T)\to D(T/v)$$

are $*$–homomorphisms.

Proof One needs to again verify that the set $\{\partial_e^*(q_f) \mid f \text{ an edge in } T\}$ satisfies the relations (1), (2) and (3) in $D(T/e)$ that define the universal C^*–algebra $D(T)$. Relations (1) and (2) are clearly satisfied; for relation (3) one needs to observe that if two edges e and f in T do not lie in a path, then this property continues to hold in T/e or T/v. A similar argument is applicable to ∂_v^*. □

Remark 2.11 If $\theta\colon S\to T$ is an isomorphism in Ω then $\theta^*\colon D(T)\to D(S)$ acts on the generators as $q_e\mapsto q_{\theta^{-1}(e)}$. One can readily verify that θ^* is a unital $*$–homomorphism.

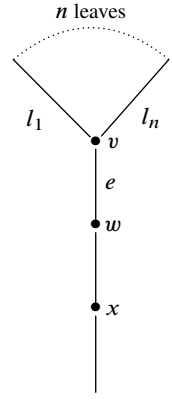
Let $\mathtt{SC}^*_{\mathtt{un}}$ denote the category of separable unital C^*–algebras with unit-preserving $*$–homomorphisms. Extending the Gelfand–Naĭmark duality, $\mathtt{SC}^{*\ \mathrm{op}}_{\mathtt{un}}$ is regarded as the category of compact Hausdorff noncommutative spaces with continuous maps.

Proposition 2.12 *The association of a noncommutative dendrex with a tree $T \mapsto D(T)$ defines a functor $D\colon \Omega \to \mathtt{SC}^{*\ \mathrm{op}}_{\mathtt{un}}$.*

Proof In view of Lemma 1.1 it suffices to show that the $*$–homomorphisms ∂_e^*, ∂_v^*, σ_v^* and θ^* satisfy the face and degeneracy identities (see Section 1.2). Note that thanks to the universal property of universal C^*–algebras we simply need to verify that various combinations of these $*$–homomorphisms governed by the identities agree on generators.

It is easy to verify that identities (I), (II), (III) and (V) are satisfied. The point is to observe that the order in which a certain number of generators are sent to 0 or sums of two other generators does not affect the final outcome.

For (IV) let us suppose that the tree around e looks like



Now $\partial_z^*\partial_e^*$ will first send q_e to 0 and then $q_{l_1}, \ldots, q_{l_n}$ to 0. On the other hand, $\partial_w^*\partial_v^*$ will first send $q_{l_1}, \ldots, q_{l_n}$ to 0 and then q_e to 0. The end result is evidently the same.

For (VI) we begin with the commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{\sigma_v} & T\backslash v \\ {\scriptstyle\partial}\uparrow & & \uparrow{\scriptstyle\partial} \\ T' & \xrightarrow{\sigma_v} & T'\backslash v \end{array}$$

Let us suppose that the face map ∂ removes edges $f_1, \ldots, f_n$. Since T' still contains v and its two adjacent edges (say e_1 and e_2), one can merge them to a new edge e. Thus, ∂^* is defined by $q_{f_i} \mapsto 0$ for $i = 1, \ldots, n$ and σ_v^* by $q_e \mapsto q_{e_1} + q_{e_2}$. Hence, it is clear

that $\partial^* \sigma_v^* = \sigma_v^* \partial^*$. The verifications of (VII) and the special cases (see Remark 1.3) are similar and omitted.

Let us observe that $D(T)$ is unital for every $T \in \Omega$ and the $*$–homomorphisms ∂_e^*, ∂_v^*, σ_v^* and θ^* are all unit-preserving, whence the essential image of the functor D is indeed $\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}}$.

Note that for a map $\tau\colon S \to T$ in Ω the induced map is $\tau^*\colon D(T) \to D(S)$. It remains to check that the association $\tau \mapsto \tau^*$ respects composition of morphisms. It is clear that this association preserves composition of face maps as well as composition of degeneracy maps. To complete the proof we now simply invoke Remark 1.2. □

3 Draw–dendraw adjunction and the bridge

For a small category $\mathcal{C}$ let $\mathcal{P}(\mathcal{C})$ denote the category of $\mathtt{Set}$–valued presheaves on $\mathcal{C}$, ie $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathtt{Set})$. Thus, setting $\mathcal{C} = \Omega$ we find $\mathcal{P}(\Omega) = \mathtt{dSet}$. Since $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})$ is cocomplete, using the covariant functoriality of the category of presheaves (via left Kan extension) one obtains the dashed functor below:

(7)
$$\begin{array}{ccc} \Omega & \xrightarrow{\ D\ } & \mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}} \\ \downarrow & & \downarrow \\ \mathtt{dSet} & \dashrightarrow & \mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}}) \end{array}$$

where the vertical functors are the canonical Yoneda embeddings and the top horizontal functor $D\colon \Omega \to \mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}}$ is the one constructed in the previous section (see Proposition 2.12). Let $\mathtt{dr}$ denote the dashed functor in the above diagram (7). There is an adjunction

$$\mathtt{dr}\colon \mathtt{dSet} \rightleftarrows \mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}}) :\mathtt{dd},$$

where the right adjoint $\mathtt{dd}$ is defined as $[\mathtt{dd}(Y)](T) = Y(D(T))$ for any $Y \in \mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})$.

Definition 3.1 For any $X \in \mathtt{dSet}$ the object $\mathtt{dr}(X)$ is its *C^*–algebraic drawing*. We call the functor $\mathtt{dr}$ (resp. $\mathtt{dd}$) the *draw* (resp. *dendraw*) functor.

Remark 3.2 In sheaf-theoretic notation, $\mathtt{dr} = D_!$ and $\mathtt{dd} = D^*$. The dendraw functor $\mathtt{dd}$ also admits a right adjoint $D_*\colon \mathtt{dSet} \to \mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})$, whence it preserves colimits.

Recall from Section 1.3 that the category $\mathtt{dSet}$ admits a combinatorial model structure.

Theorem 3.3 *There is a combinatorial model structure on $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})$ such that the draw–dendraw adjunction*

$$\mathtt{dr}\colon \mathtt{dSet} \rightleftarrows \mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}}) \colon \mathtt{dd}$$

becomes a Quillen adjunction.

Proof The model structure on $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})$ that we are referring to is constructed in Theorem A.11 (see the appendix). The left adjoint $\mathtt{dr}$ sends generating cofibrations in $\mathtt{dSet}$ to cofibrations in $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})$ (see Proposition A.6 below) and generating trivial cofibrations to trivial cofibrations in $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})$ (see Remark A.13 below). Now, using Lemma 2.1.20 of [24], one concludes that the draw–dendraw adjunction is actually a Quillen adjunction. □

Remark 3.4 Associated with any (combinatorial) model category $\mathcal{M}$ there is an underlying (presentable) ∞–category $\mathrm{N}(\mathcal{M}^{\circ})$ (see Definition 1.3.1 of [22]). Moreover, a Quillen adjunction between (combinatorial) model categories (like $\mathtt{dr}\colon \mathtt{dSet} \rightleftarrows \mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}}) \colon \mathtt{dd}$) induces an ∞–categorical adjunction between the underlying (presentable) ∞–categories (like $\mathbf{L}\mathtt{dr}\colon \mathrm{N}(\mathtt{dSet}^{\circ}) \rightleftarrows \mathrm{N}(\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})^{\circ}) \colon \mathbf{R}\mathtt{dd}$) — see [22, Proposition 1.5.1; 42, Theorem 2.1]. Although we are mainly interested in the ∞–categorical adjunction pair $(\mathbf{L}\mathtt{dr}, \mathbf{R}\mathtt{dd})$, it is often convenient to have at our disposal an explicit Quillen adjunction modelling it.

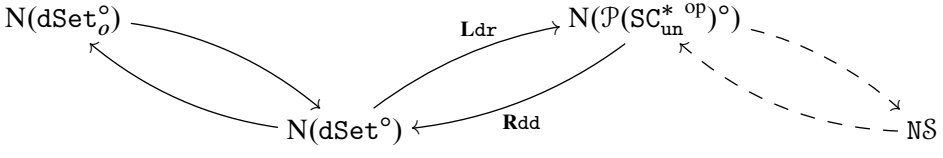
Remark 3.5 Viewing $\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}}$ inside the category of presheaves $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})$ via the Yoneda functor, we obtain a new homotopy theory for (the opposite category of) separable unital C^*–algebras, whose weak equivalences are called *weak operadic equivalences*. This new class of weak operadic equivalences is potentially interesting in its own right. The weak operadic equivalences on $\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}}$ are different from those inherited from the model structure on $\mathrm{Ind}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})$ (see [2]) via the embedding $\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}} \hookrightarrow \mathrm{Ind}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})$. These two classes of weak equivalences on $\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}}$ give rise to different homotopy theories. The class of weak operadic equivalences is not contained in the class of standard homotopy equivalences on $\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}}$ (see Remark A.12); it is not clear to the author whether the other containment holds. Those readers who prefer to stick to the category of C^*–algebras (and not venture into the category of presheaves) may try to classify the objects in it up to weak operadic equivalences.

Remark 3.6 A vertex that has no incoming edges is called a *stump*, eg in the 0–corolla

the top vertex is a stump. A tree devoid of stumps is called an *open tree*. Let Ω_o denote the full subcategory of Ω spanned by the open trees. The canonical inclusion $\Omega_o \hookrightarrow \Omega$ induces an adjunction $\mathtt{dSet}_o := \mathcal{P}(\Omega_o) \rightleftarrows \mathcal{P}(\Omega) = \mathtt{dSet}$ such that the left adjoint $\mathtt{dSet}_o \hookrightarrow \mathtt{dSet}$ is fully faithful. The objects of $\mathtt{dSet}_o$ are called *open dendroidal sets*. The category $\mathtt{dSet}_o$ inherits a combinatorial model structure via the adjunction $\mathtt{dSet}_o \rightleftarrows \mathtt{dSet}$, making it a Quillen pair (see Section 2.3 of [21]). The fully faithful functor $\mathtt{sSet} \to \mathtt{dSet}$ factors through $\mathtt{dSet}_o$. The fibrant objects of $\mathtt{dSet}_o$ are ∞–operads without constants. It was noticed by Moerdijk that our construction of the noncommutative dendrices functor does not distinguish between a leaf and an edge whose top vertex is a stump; in particular, the C^*–algebra associated with the unit tree and the 0–corolla are both $\mathbb{C}$. Thus, our draw–dendraw adjunction should be restricted to open dendroidal sets via the composite adjunction

$$\mathtt{dSet}_o \rightleftarrows \mathtt{dSet} \rightleftarrows \mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}}).$$

So far we have constructed the solid adjunctions in the following diagram of ∞–categories:



Now we define the ∞–category of noncommutative spaces $\mathtt{NS}$. Then we complete the connection between ∞–operads and noncommutative spaces via a sequence of ∞–categorical adjunctions. The dashed pair above actually represents a zigzag of adjunctions.

3.1 The rest of the bridge between $\mathtt{NS}$ and $\mathbf{N}(\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})^\circ)$

Earlier we constructed the compactly generated ∞–category of pointed noncommutative spaces generalizing the category of pointed compact noncommutative spaces (see Definition 2.13 of [37]). Let $\mathtt{SC}^{*\mathrm{op}}$ denote the opposite topological category of separable C^*–algebras with all (not necessarily unit-preserving) $*$–homomorphisms.

We view it as a topological category by endowing the morphism sets with the point-norm topology. Let $\mathtt{SC}^{*}_{\infty}{}^{\mathrm{op}}$ denote the topological nerve of $\mathtt{SC}^{*\mathrm{op}}$. It is shown in Proposition 2.7 of [37] that $\mathtt{SC}^{*}_{\infty}{}^{\mathrm{op}}$ admits finite colimits.

Definition 3.7 We set $\mathtt{NS}_{*} = \mathrm{Ind}_{\omega}(\mathtt{SC}^{*}_{\infty}{}^{\mathrm{op}})$ and call it the compactly generated ∞–category of pointed noncommutative spaces.

Similarly, there exists a compactly generated ∞–category $\mathtt{NS}$ of noncommutative (unpointed) spaces whose construction is outlined below.

Definition 3.8 Let $\mathcal{C}$ denote the opposite of the topological category of separable unital C^{*}–algebras with unit-preserving $*$–homomorphisms. We again view it as a topological category by endowing the morphism sets with the point-norm topology.

Here we have included the zero C^{*}–algebra in the topological category $\mathcal{C}$. The zero C^{*}–algebra should be viewed as the (unital) C^{*}–algebra of continuous functions on the empty space. Therefore, for every separable unital C^{*}–algebra A there is a unique unital $*$–homomorphism $A \to 0$, ie the opposite category $\mathcal{C}$ has an initial object. But the zero $*$–homomorphism $0 \to A$ is *not unital* unless $A = 0$.

Definition 3.9 Let $\mathtt{NS}^{\mathtt{fin}}$ denote the topological nerve of the topological category $\mathcal{C}$. Here it is vitally important to consider the point-norm topology on the morphism spaces while constructing the topological nerve.

One can show as in Proposition 2.7 of [37] that $\mathtt{NS}^{\mathtt{fin}}$ admits finite colimits. For the rest of this section we set $\mathrm{Ind} = \mathrm{Ind}_{\omega}$, which denotes the ∞–categorical ind–completion.

Definition 3.10 We set $\mathtt{NS} := \mathrm{Ind}(\mathtt{NS}^{\mathtt{fin}})$ and call it the compactly generated ∞–category of (unpointed) noncommutative spaces.

Remark 3.11 This ∞–categorical construction of noncommutative spaces $\mathtt{NS}$ is simple and practical. It incorporates homotopy theory and analysis in a systematic manner; the analytical aspects are contained within the world of C^{*}–algebras. More complicated topological algebras like pro-C^{*}–algebras can be viewed within this setup via the homotopy theory of diagrams of C^{*}–algebras. The mechanism is explained in our earlier work [37; 36].

There is a canonical fully faithful embedding of (topological) categories $\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}} \hookrightarrow \mathcal{C}$. This functor induces an adjunction of the corresponding categories of presheaves $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}}) \rightleftarrows \mathcal{P}(\mathcal{C})$. A map $f\colon C \to D$ in $\mathcal{C}$ is a *C^*–homotopy equivalence* if there is another map $g\colon D \to C$ and homotopies $fg \simeq \mathrm{id}_D$ and $gf \simeq \mathrm{id}_C$. The set of C^*–homotopy equivalences gives rise to a set of maps in $\mathcal{P}(\mathcal{C})$ that, finally, gives rise to another set of maps in $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})$ via the adjunction $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}}) \rightleftarrows \mathcal{P}(\mathcal{C})$.

Definition 3.12 (mixed model structure on $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})$) The left Bousfield localization of the combinatorial model category $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})$ equipped with the operadic model structure (see Theorem A.11) along the set of maps induced by the C^*–homotopy equivalences is the *mixed model structure* on $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})$. We denote the mixed model category by $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})_{\mathrm{mix}}$, which again turns out to be combinatorial.

The Bousfield localization $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}}) \to \mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})_{\mathrm{mix}}$ of combinatorial model categories induces an adjunction of underlying presentable ∞–categories $\mathrm{N}(\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})^\circ) \rightleftarrows \mathrm{N}(\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})^\circ_{\mathrm{mix}})$ that exhibits $\mathrm{N}(\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})^\circ_{\mathrm{mix}})$ as a localization of $\mathrm{N}(\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})^\circ)$. Let θ denote the composition of the functors

$$\mathcal{C} \xhookrightarrow{j} \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}}) \xrightarrow{(-)^{\mathrm{f}}} \mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})^{\mathrm{f}}_{\mathrm{mix}},$$

where j is the Yoneda embedding, $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})^{\mathrm{f}}_{\mathrm{mix}}$ is the full subcategory of (bi)fibrant objects of $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})_{\mathrm{mix}}$ and $(-)^{\mathrm{f}}$ denotes a fibrant replacement functor in the mixed model category $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})_{\mathrm{mix}}$. Let us view $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})^{\mathrm{f}}_{\mathrm{mix}}$ as a *relative category* in the sense of [4] via the weak equivalences inherited from the model category $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})_{\mathrm{mix}}$. We can also view $\mathcal{C}$ as a relative category with the C^*–homotopy equivalences as the weak equivalences.

Lemma 3.13 *The functor $\theta\colon \mathcal{C} \to \mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})^{\mathrm{f}}_{\mathrm{mix}}$ is a morphism of relative categories.*

Proof We need to verify that θ preserves weak equivalences. Our construction of the mixed model category $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})_{\mathrm{mix}}$ ensures this property (see Definition 3.12). □

For any relative category $\mathcal{A}$ we denote the underlying ∞–category by $\mathcal{A}_\infty$ (see Section 1.2 of [42]). The morphism of relative categories $\theta\colon \mathcal{C} \to \mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})^{\mathrm{f}}_{\mathrm{mix}}$ induces a morphism of underlying ∞–categories $\theta\colon \mathcal{C}_\infty \to (\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})^{\mathrm{f}}_{\mathrm{mix}})_\infty$. For any ∞–category $\mathcal{A}$ there is an ∞–category of ∞–presheaves $\mathcal{P}_\infty(\mathcal{A})$ (see [30]). Note the subtle difference in notation — for an ordinary category $\mathcal{A}$ we denote by $\mathcal{P}(\mathcal{A})$ the category of $\mathtt{Set}$–valued presheaves on $\mathcal{A}$, whereas for an ∞–category $\mathcal{A}$ we denote by $\mathcal{P}_\infty(\mathcal{A})$ the ∞–category of ∞–presheaves on $\mathcal{A}$.

Proposition 3.14 *The morphism of ∞–categories $\theta\colon \mathcal{C}_\infty \to (\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})^{\mathrm{f}}_{\mathrm{mix}})_\infty$ induces a colimit-preserving functor $\widetilde{\theta}\colon \mathcal{P}_\infty(\mathcal{C}_\infty) \to \mathrm{N}(\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})^{\circ}_{\mathrm{mix}})$.*

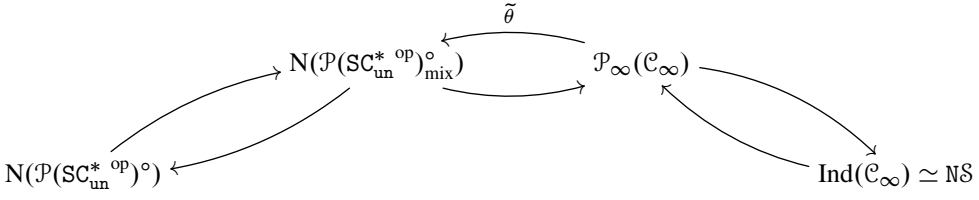
Proof The canonical inclusion $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})^{\mathrm{f}}_{\mathrm{mix}} \hookrightarrow \mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})_{\mathrm{mix}}$ induces an equivalence of underlying ∞–categories [18] (see also Lemma 2.8 of [42]). Thanks to the universal property of the category of presheaves $\mathcal{P}_\infty(-)$ in the setting of ∞–categories (see Theorem 5.1.5.6 of [30]), it suffices to show that $(\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})^{\mathrm{f}}_{\mathrm{mix}})_\infty \simeq \mathrm{N}(\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})^{\circ}_{\mathrm{mix}})$ admits small colimits. Since the model category $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})_{\mathrm{mix}}$ is combinatorial, its underlying ∞–category is presentable (see Corollary 1.5.2 of [22]), ie it is cocomplete. □

The following result is proven in Proposition 3.18 of [2] using the formalism of weak (co)fibration categories [3].

Lemma 3.15 *There is an equivalence of ∞–categories $\mathrm{Ind}(\mathcal{C}_\infty) \simeq \mathtt{NS}$.*

Remark 3.16 Actually Proposition 3.18 of [2] proves a pointed version of the above lemma. The desired result can be shown using similar methods and hence its proof is omitted.

Theorem 3.17 *There is a diagram of adjunctions of presentable ∞–categories*



Proof The presentability of each ∞–category in the above diagram is clear. Observe that $\widetilde{\theta}\colon \mathcal{P}_\infty(\mathcal{C}_\infty) \to \mathrm{N}(\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})^{\circ}_{\mathrm{mix}})$ is a colimit-preserving functor between presentable ∞–categories (see Proposition 3.14). Hence, using the adjoint functor theorem (see Corollary 5.5.2.9 of [30]) we deduce that it admits a right adjoint. The existence of the adjunction pair $\mathcal{P}_\infty(\mathcal{C}_\infty) \rightleftarrows \mathrm{Ind}(\mathcal{C}_\infty) \simeq \mathtt{NS}$ is standard (see for instance Theorem 5.5.1.1 of [30]). The adjunction $\mathrm{N}(\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})^{\circ}) \rightleftarrows \mathrm{N}(\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})^{\circ}_{\mathrm{mix}})$ has already been explained above. □

Remark 3.18 For the benefit of the reader we explain briefly the meaning and significance of this result. It is the author's perception that several results in the two paradigms

of noncommutative geometry use very similar techniques, albeit in different contexts. For example, the constructions of the bivariant K–theory category and the category of noncommutative motives are philosophically almost identical (only applied to different notions of spaces). That led to the vision of abstracting away the commonalities and providing a framework whereby results can be transferred back and forth, creating synergies (see Section 0.1). In what follows we substantiate this assertion with a few potential directions for development.

4 Prospects: commutative spaces and graph algebras

It is known how to view commutative spaces (or motives) inside their noncommutative counterparts in the algebrogeometric setting [27; 50; 7]. We briefly explain how the ∞–category of spaces (not necessarily compact) sits inside that of noncommutative spaces via a colocalization in the setting of Connes. We also highlight how noncommutative dendrices naturally interpolate between the two canonical notions of *building blocks*.

4.1 Commutative spaces via colocalization

Let $\mathcal{S}$ (resp. $\mathcal{S}_*$) denote the ∞–category of spaces (resp. pointed spaces). It is shown in Theorem 1.9(1) of [36] that there is a fully faithful ω–continuous functor $\mathcal{S}_* \hookrightarrow \mathtt{N}\mathcal{S}_*$. In the same vein one can show that there is a fully faithful ω–continuous functor $\mathcal{S} \hookrightarrow \mathtt{N}\mathcal{S}$.

Proposition 4.1 *The fully faithful ω–continuous functor $\mathcal{S}_* \hookrightarrow \mathtt{N}\mathcal{S}_*$ (as well as $\mathcal{S} \hookrightarrow \mathtt{N}\mathcal{S}$) admits a right adjoint, ie it is colimit-preserving.*

Proof Due to the Gelfand–Naĭmark correspondence there is a fully faithful functor $f\colon \mathcal{S}_*^{\mathrm{fin}} \hookrightarrow \mathtt{SC}_\infty^{*\ \mathrm{op}}$ that induces the fully faithful ω–continuous functor $\mathrm{Ind}_\omega(f)\colon \mathcal{S}_* \hookrightarrow \mathtt{N}\mathcal{S}_*$ of Theorem 1.9(1) of [36]. The functor f preserves finite colimits, whence it is right exact. Therefore, by Proposition 5.3.5.13 of [31], the functor $\mathrm{Ind}_\omega(f)$ admits a right adjoint. The proof of the corresponding assertion for $\mathcal{S} \hookrightarrow \mathtt{N}\mathcal{S}$ is similar. □

Definition 4.2 We denote the right adjoint of $\mathcal{S}_* \hookrightarrow \mathtt{N}\mathcal{S}_*$ (resp. $\mathcal{S} \hookrightarrow \mathtt{N}\mathcal{S}$) in the above Proposition 4.1 by $\mathrm{US}_*\colon \mathtt{N}\mathcal{S}_* \to \mathcal{S}_*$ (resp. $\mathrm{US}\colon \mathtt{N}\mathcal{S} \to \mathcal{S}$) and call it the *underlying pointed space* (resp. *underlying space*) functor. Since US_* and US admit fully faithful left adjoints they are colocalizations, ie they constitute the commutative (pointed) space approximation of a noncommutative (pointed) space.

Now we are going to demonstrate how noncommutative dendrices interconnect simplices and matrices. Let T_n denote the linear graph

$$\bullet_0 \xleftarrow{e_1} \bullet_1 \xleftarrow{e_2} \cdots \xleftarrow{e_n} \bullet_n,$$

whose graph algebra $C^*(T_n)$ is isomorphic to $M_{n+1}(\mathbb{C})$ (the construction of the graph algebra is explained below in Section 4.2). Let $D^{\mathrm{ab}}(T_n)$ denote the commutative unital C^*–algebra generated by requiring the generators $\{q_{e_1}, \ldots, q_{e_n}\}$ of $D(T_n)$ to commute (see Remark 2.5). There is a canonical surjective $*$–homomorphism $\pi_n\colon D(T_n) \to D^{\mathrm{ab}}(T_n)$ that is identity on the generators. It follows from Proposition 2.1 of [16] that $D^{\mathrm{ab}}(T_n)$ is isomorphic to the commutative C^*–algebra $\mathrm{C}(\Delta^n)$. There is also a canonical $*$–homomorphism $s_n\colon D(T_n) \to C^*(T_{n-1}) \cong M_n(\mathbb{C})$, sending $q_{e_i} \mapsto e_{ii}$. Note that $\sum_{i=1}^{n} e_{ii}$ is the identity matrix that is the unit in the graph algebra $C^*(T_{n-1}) \cong M_n(\mathbb{C})$. Thus, we have a zigzag of arrows

(8)
$$\begin{array}{ccccc} & & D(T_n) & & \\ & \overset{\pi_n}{\swarrow} & & \overset{s_n}{\searrow} & \\ D^{\mathrm{ab}}(T_n) \cong \mathrm{C}(\Delta^n) & & & & C^*(T_{n-1}) \cong M_n(\mathbb{C}) \end{array}$$

The set of $*$–homomorphisms $\{s_n \mid n \in \mathbb{N}\}$ defines a set of maps M in the ∞–category noncommutative spaces $\mathtt{NS}$ via the functor $j\colon \mathtt{NS}^{\mathtt{fin}} \to \mathtt{NS}$. Thus, we are going to invert the maps in M to construct the simplex–matrix-identified version of $\mathtt{NS}$. It is quite natural to consider matrix algebras as noncommutative simplices.

Definition 4.3 The accessible localization $L_M\colon \mathtt{NS} \to M^{-1}\mathtt{NS} =: \mathtt{NS}^{\mathtt{SM}}$, which admits a fully faithful right adjoint, is defined to be the ∞–category of *simplex–matrix-identified noncommutative spaces*.

Remark 4.4 Since $\mathtt{NS}$ is a presentable ∞–category, so is $\mathtt{NS}^{\mathtt{SM}}$.

Remark 4.5 The composite functor $\mathtt{NS}^{\mathtt{SM}} \hookrightarrow \mathtt{NS} \xrightarrow{\mathrm{US}} \mathtt{S}$ defines the underlying space functor on $\mathtt{NS}^{\mathtt{SM}}$. The subcategory of simplex–matrix-identified noncommutative spaces $\mathtt{NS}^{\mathtt{SM}}$ is a tractable part of the entire ∞–category of noncommutative spaces $\mathtt{NS}$ and it would be nice to explore it further.

Remark 4.6 Let $\mathtt{CW}^{\mathtt{fin}}$ denote the category of finite CW complexes. The geometric realization functor $|\cdot|\colon \mathtt{sSet} \to \mathrm{Ind}(\mathtt{CW}^{\mathtt{fin}})$ preserves (tensor) products and detects weak equivalences, whose counterpart in the world of dendroidal sets has been treated in [20; 5]. It is plausible (and desirable) that one could modify the functor

$\mathtt{dr}\colon \mathtt{dSet} \to \mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})$ to produce yet another C^*–*algebraic* or *noncommutative* geometric realization of dendroidal sets that fits into the commutative diagram

$$\begin{array}{ccc} \mathtt{sSet} & \xrightarrow{|\cdot|} & \mathrm{Ind}(\mathtt{CW}^{\mathtt{fin}}) \\ \downarrow & & \downarrow \\ \mathtt{dSet} & \xrightarrow{\ ?\ } & \mathrm{Ind}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}}) \subset \mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}}) \end{array}$$

We leave it as an open problem.

4.2 Graph algebras

There is a vast literature on graph algebras (or graph C^*–algebras) with several interesting results relating structural aspects of the graph algebra (like simplicity) to purely graph-theoretic properties. We encourage the interested reader to consult for instance [49].

Let E be a finite *directed* graph and let $\mathcal{H}$ be a fixed separable Hilbert space. A *Cuntz–Krieger E–family* $\{S, P\}$ *on* $\mathcal{H}$ (abbreviated as CK E–family) consists of a set $P = \{P_v \mid v \in E^0\}$ of mutually orthogonal projections on $\mathcal{H}$ and a set $S = \{S_e \mid e \in E^1\}$ of partial isometries on $\mathcal{H}$ such that

(CK1) $S_e^* S_e = P_{s(e)}$ for all $e \in E^1$; and

(CK2) $P_v = \sum_{\{e \in E^1 : r(e) = v\}} S_e S_e^*$ provided $\{e \in E^1 : r(e) = v\} \neq \varnothing$.

The graph algebra of E, denoted by $C^*(E)$, is by definition the universal C^*–algebra generated by $\{S, P\}$ subject to relations (CK1) and (CK2). It is known that $C^*(E)$ is unital if and only if the set of vertices E^0 is finite (see Proposition 1.4 of [28]).

Remark 4.7 Some authors prefer to write the relations (CK1) and (CK2) differently, viz the roles of r and s are interchanged. We have adopted the convention from [49]. The advantage of this viewpoint is that juxtaposition of edges in a path corresponds to composition of partial isometries on the Hilbert space $\mathcal{H}$.

Example 4.8 The graph algebra corresponding to the graph $\circlearrowleft \bullet \circlearrowright$ is the Cuntz algebra $\mathcal{O}_2$.

The left Quillen functor $\mathtt{dr}\colon \mathtt{dSet} \to \mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})$ is obtained by the left Kan extension of $\Omega \xrightarrow{D} \mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}} \to \mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})$ along $\Omega \to \mathtt{dSet}$. Explicitly it is given by the formula

$$[\mathtt{dr}(X)](A) = \operatorname*{colim}_{f\colon D(T) \to A} X(T),$$

where the colimit is taken over the comma category $(D \downarrow A)$. The Quillen adjunction descends to an adjunction of homotopy categories

$$\mathbf{L}\mathtt{dr}\colon \mathrm{Ho}(\mathtt{dSet}) \rightleftarrows \mathrm{Ho}(\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathtt{un}})) :\mathbf{R}\mathtt{dd},$$

after taking the total derived functors of $\mathtt{dr}$ and $\mathtt{dd}$ ($\mathbf{L}\mathtt{dr}$ and $\mathbf{R}\mathtt{dd}$, respectively).

The composite $\mathbf{L}\mathtt{dr}\circ\mathbf{R}\mathtt{dd}$ defines a comonad on $\mathrm{Ho}(\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathtt{un}}))$. Viewing any separable unital C^*–algebra A inside $\mathrm{Ho}(\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathtt{un}}))$ via the Yoneda functor, we may consider the map given by the counit of the adjunction $\mathbf{L}\mathtt{dr}\circ\mathbf{R}\mathtt{dd}(A) \to \mathrm{Id}(A)$. It is presumably not an isomorphism; nevertheless, one should consider its comonadic resolution. If A is a graph algebra, this resolution can be viewed as a *resolution of the underlying graph by trees*. It would be nice to classify C^*–algebras up to this dendroidal invariant.

Remark 4.9 In the world of C^*–algebras a celebrated result of Kirchberg asserts that topological K–theory acts as a complete invariant on the subcategory of so-called *stable Kirchberg algebras* that satisfy UCT [26]. It was shown in [35; 15] that for such C^*–algebras (in fact for a larger subcategory of C^*–algebras) algebraic K–theory is naturally isomorphic to topological K–theory (see Theorem 2.4 and Remark 1 of [35]). If the vision outlined in the introduction can be realized, viz if one can show that algebraic K–theory and KK–theory can be recovered from diagram (2), then the above-mentioned construction would provide a *higher invariant* that has the potential to act as a complete invariant on a bigger subcategory than that of stable Kirchberg algebras satisfying UCT. Observe that topological K–theory is also the primary classification tool for graph algebras. It would be actually more prudent to analyze this construction for a graph algebra at the level of underlying ∞–categories (and not at the level of homotopy categories), possibly after passing to the stabilization.

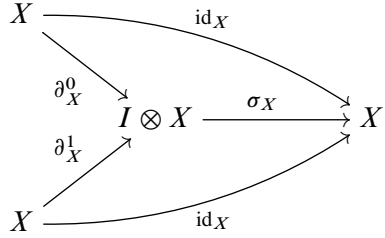
Appendix The model structure on $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathtt{un}})$

For any small category $\mathcal{C}$ there is a *Cisinski model structure* on $\mathcal{P}(\mathcal{C})$ [9], whose construction is described below. A *functorial cylinder object* is an endofunctor

$$I \otimes (-)\colon \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C})$$

such that for every $X \in \mathcal{P}(\mathcal{C})$ there are natural morphisms ∂^0_X, ∂^1_X and σ_X that satisfy:

(1) The following diagram commutes:



(2) The canonical morphism $X \amalg X \to I \otimes X$ induced by ∂^0_X and ∂^1_X is a monomorphism.

The choice of a functorial cylinder object $\mathcal{J} = (I \otimes (-), \partial^0_{(-)}, \partial^1_{(-)}, \sigma_{(-)})$ constitutes an *elementary homotopical datum* if $\mathcal{J}$ satisfies the following two additional conditions:

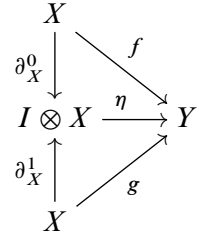
(i) the functor $I \otimes (-)$ commutes with small colimits, and

(ii) for every monomorphism $j\colon K \to L$ in $\mathcal{P}(\mathcal{C})$ for $e = 0, 1$, the diagram

$$\begin{array}{ccc} K & \xrightarrow{j} & L \\ {\scriptstyle \partial^e_K}\downarrow & & \downarrow{\scriptstyle \partial^e_L} \\ I \otimes K & \xrightarrow{I \otimes j} & I \otimes L \end{array}$$

is a pullback square.

Using the functorial cylinder object $\mathcal{J}$, one can define an *elementary $\mathcal{J}$–homotopy* between two maps in $\mathcal{P}(\mathcal{C})$, viz two maps $f, g\colon X \to Y$ are elementary $\mathcal{J}$–homotopic if there is a map $\eta\colon I \otimes X \to Y$ making the following diagram commute:



Let $\mathrm{Ho}_{\mathcal{J}}\mathcal{P}(\mathcal{C})$ denote the category whose objects are those of $\mathcal{P}(\mathcal{C})$ and whose morphisms are the elementary $\mathcal{J}$–homotopy classes of morphisms of $\mathcal{P}(\mathcal{C})$.

Definition A.1 There is a canonical functor $\mathcal{P}(\mathcal{C}) \to \mathrm{Ho}_{\mathcal{J}}\mathcal{P}(\mathcal{C})$ and the morphisms that descend to isomorphisms under this functor are called *$\mathcal{J}$–homotopy equivalences*. This notion obviously depends on the choice of $\mathcal{J}$.

The model structure on $\mathcal{P}(\mathcal{C})$ depends on another choice, viz a class $\mathtt{An}$ of *anodyne extensions*. For a class M of maps of $\mathcal{P}(\mathcal{C})$ we denote by $\mathrm{llp}(M)$ (resp. $\mathrm{rlp}(M)$) the class of maps that satisfy the left (resp. right) lifting property with respect to M. For any cartesian square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

in $\mathcal{P}(\mathcal{C})$ with $Y \to W$ and $Z \to W$ monomorphisms, the canonical map $Y \amalg_X Z \to W$ is also a monomorphism. For brevity this monomorphism is suggestively written as $Y \cup Z \to W$.

Definition A.2 Let $\mathcal{J}$ be an elementary homotopy datum on $\mathcal{P}(\mathcal{C})$. Then a *class of anodyne extensions* $\mathtt{An}$ *relative to* $\mathcal{J}$ is a class of morphisms in $\mathcal{P}(\mathcal{C})$ such that

(a) $\mathtt{An} = \mathrm{llp}(\mathrm{rlp}(M))$ for a small set of maps M,

(b) for any monomorphism $K \to L$ and $e = 0, 1$, the induced map $I \otimes K \cup \{e\} \otimes L \to I \otimes L$ belongs to $\mathtt{An}$, and

(c) if $K \to L$ belongs to $\mathtt{An}$, then so does $I \otimes K \cup \partial I \otimes L \to I \otimes L$, where $\partial I \otimes L = L \amalg L$.

Remark A.3 It is shown in Proposition 1.3.13 of [9] that for any small set S of monomorphisms of $\mathcal{P}(\mathcal{C})$ there is a smallest class of anodyne extensions relative to $\mathcal{J}$ that is generated by S. This class of morphisms is denoted by $\mathtt{An}_{\mathcal{J}}(S)$.

Theorem A.4 [9, Théorème 1.3.22] *Let $\mathcal{J}$ be an elementary homotopy datum on $\mathcal{P}(\mathcal{C})$ and $\mathtt{An}_{\mathcal{J}}(S)$ be a class of anodyne extensions relative to $\mathcal{J}$ that is generated by a small set S of monomorphisms. Then there is a combinatorial model structure on $\mathcal{P}(\mathcal{C})$ satisfying*

(1) *the cofibrations are the monomorphisms,*

(2) *$X \in \mathcal{P}(\mathcal{C})$ is fibrant if the map $X \to \star$, where $\star$ is the terminal object, satisfies the right lifting property with respect to all anodyne extensions $\mathtt{An}_{\mathcal{J}}(S)$, and*

(3) *a map $f\colon X \to Y$ is a weak equivalence if for all fibrant objects Z the induced map $f^*\colon \mathrm{Ho}_{\mathcal{J}}\mathcal{P}(\mathcal{C})(Y, Z) \to \mathrm{Ho}_{\mathcal{J}}\mathcal{P}(\mathcal{C})(X, Z)$ is bijective.*

Remark A.5 The Cisinksi model structure on $\mathcal{P}(\mathcal{C})$ admits a functorial fibrant replacement. A set of generating cofibrations can be chosen to be those monomorphisms whose codomains are quotients of representable presheaves (see Proposition 1.2.27 of [9]). Every object of $\mathcal{P}(\mathcal{C})$ is cofibrant and its homotopy category is equivalent to the full subcategory of $\mathrm{Ho}_{\mathcal{J}}\mathcal{P}(\mathcal{C})$ spanned by the fibrant objects (see 1.3.23 of [9]). Moreover, a morphism between two fibrant objects is a weak equivalence if and only if it is a $\mathcal{J}$–homotopy equivalence.

Proposition A.6 *The functor* $\mathtt{dr}\colon \mathtt{dSet} \to \mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathtt{un}})$ *preserves cofibrations.*

Proof The set of generating cofibrations in $\mathtt{dSet}$ is $\{\partial\Omega[T] \to \Omega[T] \mid T \in \Omega\}$. Each face map $\partial\colon T' \to T$ of trees induces a monomorphism of representable presheaves, whose image is specified by the datum of this monomorphism of representable presheaves (see Chapter IV of [32]). For any tree T the boundary inclusion $\partial\Omega[T] \to \Omega[T]$ is obtained as a union of the images of such face maps. We know that $\mathtt{dr}$ sends the representable presheaf of T to that of $D(T)$. Each face map $\partial\colon T' \to T$ in Ω induces a surjective $*$–homomorphism $\partial^*\colon D(T) \to D(T')$ in $\mathtt{SC}^*_{\mathtt{un}}$ (see Section 2.1). It induces a monomorphism in $\mathtt{SC}^{*\ \mathrm{op}}_{\mathtt{un}}$ and the Yoneda embedding preserves monomorphisms, whence $\mathtt{dr}(\partial)\colon \mathtt{SC}^{*\ \mathrm{op}}_{\mathtt{un}}(-, D(T')) \to \mathtt{SC}^{*\ \mathrm{op}}_{\mathtt{un}}(-, D(T))$ is a monomorphism in $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathtt{un}})$. It follows from the universal property of the noncommutative dendrices construction that $\mathtt{dr}$ sends the generating cofibrations of $\mathtt{dSet}$ to monomorphisms of $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathtt{un}})$. Note that the cofibrations of $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathtt{un}})$ are precisely the monomorphisms, whence Lemma 2.1.20 of [24] shows that $\mathtt{dr}$ preserves cofibrations. □

Remark A.7 It is clear that the above proposition does not depend on the choice of $\mathcal{J}$.

For the choice of the elementary homotopy datum we have a few possibilities at our disposal.

Example A.8 [9, Example 1.3.9] Let $\mathcal{C}$ be any small category. For an object $C \in \mathcal{C}$ let us denote the representable presheaf of C in $\mathcal{P}(\mathcal{C})$ by h_C. Let $\mathcal{L}$ denote the presheaf that associates with every $C \in \mathcal{C}$ the set $\mathcal{L}(C) = \{\text{subobjects of } h_C\}$. For every map $u\colon C \to D$ in $\mathcal{C}$ the map $\mathcal{L}(D) \to \mathcal{L}(C)$ is induced by pullback along u. The presheaf $\mathcal{L}$ turns out to be a subobject classifier, ie $\mathcal{P}(\mathcal{C})(X, \mathcal{L}) \simeq \{\text{subobjects of the presheaf } X\}$. If $\star$ is the final object of $\mathcal{P}(\mathcal{C})$, then it has exactly two subobjects $\star \hookrightarrow \star$ and $\varnothing \hookrightarrow \star$, where $\varnothing$ denotes the initial object of $\mathcal{P}(\mathcal{C})$. These define two morphisms $\lambda_0, \lambda_1\colon \star \to \mathcal{L}$. The tuple $(\mathcal{L}, \lambda_0, \lambda_1)$ gives rise to an elementary homotopy datum by setting $I \otimes X = \mathcal{L} \times X$, $\partial^e_X = \lambda_e \times \mathrm{id}_X$ for $e = 0, 1$ and $\sigma_X = \mathrm{pr}_2\colon \mathcal{L} \times X \to X$. This

elementary homotopy datum is called the *Lawvere cylinder*, and exists in any category of presheaves like $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$.

Example A.9 For any *nonzero* separable unital C^*–algebra A there is a sequence of two $*$–homomorphisms $A \xrightarrow{\iota} A[0,1] := \mathrm{C}([0,1], A) \xrightarrow{\mathrm{ev}_t} A$ for any $t \in [0,1]$ (natural in A), whose composition is the identity $*$–homomorphism on A. Here $\iota(a)$ is the constant a–valued function on $[0,1]$ for every $a \in A$ and ev_t is the evaluation at $t \in [0,1]$. For $A = \mathbb{C}$, after reversing the arrows and passing to the representable presheaves in $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$ we get the square

$$\begin{array}{ccc} \varnothing & \longrightarrow & h_{\mathbb{C}} \\ \downarrow & & \downarrow{\scriptstyle \partial^1 = \mathrm{ev}_1^*} \\ h_{\mathbb{C}} & \xrightarrow{\partial^0 = \mathrm{ev}_0^*} & h_{\mathrm{C}([0,1])} \end{array} \tag{9}$$

where $\varnothing$ is the initial object (empty presheaf) of $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$. Note that $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$ are $\mathtt{Set}$–valued covariant functors on $\mathtt{SC}^{*}_{\mathrm{un}}$ and we do not notationally distinguish between objects in a category and in its opposite. For every $A \in \mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}}$ we find that the diagram

$$\begin{array}{ccc} \varnothing & \longrightarrow & h_{\mathbb{C}}(A) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1^*} \\ h_{\mathbb{C}}(A) & \xrightarrow{\mathrm{ev}_0^*} & h_{\mathrm{C}([0,1])}(A) \end{array}$$

is a pullback square in $\mathtt{Set}$. Indeed, $h_{\mathbb{C}}(A) = \mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}}(A, \mathbb{C}) = \{\mathbf{1}_A\}$, where $\mathbf{1}_A$ is the unique unital $*$–homomorphism $\mathbb{C} \to A$ and $(\mathbf{1}_A \circ \mathrm{ev}_t^*)(f) = f(t)\mathbf{1}_A$ for $t = 0, 1$ and for every $f \in \mathbb{C}[0,1] = \mathrm{C}([0,1], \mathbb{C})$. In this argument it is crucial that A is a *nonzero* separable unital C^*–algebra. Since limits are computed objectwise in $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$ we conclude that diagram (9) is a pullback square. It follows from Example 1.3.8 of [9] that

$$\mathcal{J} = (I \times X,\ \partial^0 \times \mathrm{id}_X,\ \partial^1 \times \mathrm{id}_X,\ \mathrm{pr}_X \colon I \times X \to X)$$

defines an elementary homotopy datum.

Example A.10 (continuous cylinder) Consider again the sequence of $*$–homomorphisms $A \xrightarrow{\iota} A[0,1] \xrightarrow{\mathrm{ev}_t} A$ (natural in A), whose composition is the identity $*$–homomorphism on A. Given any representable object h_A we set $I \otimes h_A = h_{A[0,1]}$ and extend the cylinder construction to all objects of $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$ by commuting with colimits, ie if $X \cong \mathrm{colim}_i\, h_{A_i}$, then we set $I \otimes X \cong \mathrm{colim}_i\, h_{A_i[0,1]}$.

We choose the elementary homotopy datum of Example A.8 since it is the most canonical choice for the Cisinski model structure on any presheaf category. Subsequently we are

going to localize our model structure based on our requirements. Let X be a set of generating trivial cofibrations of $\mathtt{dSet}$ and set $S = \mathtt{dr}(X)$. By Proposition A.6, S is a set of monomorphisms of $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathtt{un}})$ that generates a class of anodyne extensions $\mathtt{An}_{\mathcal{J}}(S)$ relative to $\mathcal{J}$ (see Remark A.3). As a consequence of Theorem A.4 we obtain:

Theorem A.11 (operadic model structure) *With the choice of the elementary homotopy datum $\mathcal{J}$ of Example A.8 and the class of anodyne extensions $\mathtt{An}_{\mathcal{J}}(S)$ relative to $\mathcal{J}$ described above, $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathtt{un}})$ acquires the structure of a combinatorial model category.*

Remark A.12 The Lawvere cylinder is different from the continuous cylinder of Example A.10. Hence, the evaluation map $A[0,1] \xrightarrow{\mathrm{ev}_t} A$ is not a weak equivalence in the operadic model structure; it roughly mirrors the Joyal model structure on the category of simplicial sets, in which $\Delta^1 \to \Delta^0$ is not a weak equivalence.

Remark A.13 It is shown in Lemma 1.3.31 of [9] that every anodyne extension is a weak equivalence. Since $\mathtt{dr}(X) = S \subset \mathtt{An}_{\mathcal{J}}(S)$, where X is the set of generating trivial cofibrations of $\mathtt{dSet}$, we observe that, by construction, the functor $\mathtt{dr}$ sends generating trivial cofibrations of $\mathtt{dSet}$ to trivial cofibrations of $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathtt{un}})$.

Remark A.14 The construction of the Cisinski model structure can be profitably used in other contexts. For instance, one can start with a small category $\mathcal{A}$ of topological algebras (Banach, Fréchet or locally convex) with some mild hypotheses. Then one can simply start with the minimal model structure on $\mathcal{P}(\mathcal{A}^{\mathrm{op}})$ by choosing the Lawvere cylinder (see Example A.8) for the elementary homotopy datum $\mathcal{J}$ and $\mathtt{An}_{\mathcal{J}}(\varnothing)$ for the class of anodyne extensions. Now one can localize this combinatorial model category by inverting a small set of morphisms like differentiable homotopy equivalences between the representable objects in $\mathcal{P}(\mathcal{A}^{\mathrm{op}})$. This would produce an unstable model category to start with that can be (∞–categorically) stabilized and localized further according to one's requirements; for instance, one can aim for a stable ∞–category whose morphism groups model the Cuntz kk–groups for locally convex algebras [17]. Østvær developed his homotopy theory of C^*–algebras adopting a similar strategy in the setting of cubical set-valued presheaves on the category of separable C^*–algebras [47] but we do not expect a Quillen equivalence between his unstable model category for *cubical C^*–spaces* and $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathtt{un}})$ equipped with the operadic model structure as in Theorem A.11. This is because the evaluation map $A[0,1] \xrightarrow{\mathrm{ev}_t} A$ of the continuous cylinder construction (see Example A.10) is not a weak equivalence in the operadic model structure. One final observation — all the ingredients needed to develop a Waldhausen K–theory of noncommutative spaces are now at our disposal.

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