Occupants in simplicial complexes

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Let M be a smooth manifold and $K \subset M$ be a simplicial complex of codimension at least 3. Functor calculus methods lead to a homotopical formula of $M \setminus K$ in terms of spaces $M \setminus T$ where T is a finite subset of K. This is a generalization of the author's previous work with Michael Weiss (Contemp. Math. 682, Amer. Math. Soc., Providence, RI (2017) 237–259), where the subset K is assumed to be a smooth submanifold of M and uses his generalization of manifold calculus adapted for simplicial complexes.

1 Introduction

Let K be a simplicial complex — that is, the geometric realization of an abstract simplicial complex. Let M be a smooth manifold with codimension $\dim M - \dim K \geq 3$. Throughout this paper we assume that K is a subset of M such that each (closed) simplex of K is smoothly embedded in M. We would like to recover the homotopy type of $M \setminus K$ from the homotopy types of the spaces $M \setminus T$ where T is a finite subset of K. The finite subset $T \subset K$ could be regarded as a finite set of occupants.

It turns out that it is possible to find such a homotopical formula, but only if we allow standard thickenings of the finite subsets $T \subset K$ and inclusions between them. We get an interesting poset regarded as a category — the configuration category $\mathrm{con}(K)$ of K. The objects of $\mathrm{con}(K)$ are pairs (T, ρ) where T is a finite subset of K and $\rho \colon T \to (0, \infty)$ is a function which assigns to each element $t \in T$ the radius $\rho(t)$ of the corresponding thickening using a standard metric on K. These pairs have to fulfill certain conditions, eg the thickenings of the elements $t \in T$ are pairwise disjoint (for a precise definition, see Section 3.1). For each object (T, ρ) in $\mathrm{con}(K)$, we get a corresponding open subset $V_K(T, \rho) \subset K$, which is the disjoint union of the open balls of radius $\rho(t)$ about the points $t \in T$. We note that for each element (T, ρ) of the configuration category, there is an inclusion

$$M \setminus K \to M \setminus V_K(T, \rho)$$

and thus a map from $M \setminus K$ into the associated homotopy limit. The following theorem is our (technical) main result:

Theorem 1.1 *If the codimension* $\dim M - \dim K$ *is at least* 3, *the canonical map*

$$M \setminus K \to \operatornamewithlimits{holim}_{(T,\rho) \in \mathrm{con}(K)} M \setminus V_K(T,\rho)$$

is a weak equivalence.

The condition on the codimension is essential, that is, the result is not true for codimension ≤ 2. A nice counterexample is given in Tillmann and Weiss [11, 1.3.3].

Theorem 1.1 is an application of manifold calculus adapted for simplicial complexes, as developed in Tillmann [10]. In this paper the configuration category $\mathrm{con}(K)$ is a convenient replacement of the category of special open subsets $\bigcup_k \mathcal{O}k(K)$ there. Recall: the objects of $\bigcup_k \mathcal{O}k(K)$ are those open subsets V of K which have finitely many components and where each component of V is stratified isotopy equivalent to the open star of some simplex σ in K (intersection of the open stars of the vertices of σ). Roughly speaking, a stratified isotopy equivalence is a simplexwise smooth isotopy equivalence.

As is to be expected from manifold calculus, there is a stronger version of our main result with restricted cardinalities (see Theorem 4.1). More precisely, the map from $M \setminus K$ into the homotopy limit over the full subcategory of $\mathrm{con}(K)$ of the set with restricted cardinality is highly connected, depending on that cardinality.

Now let M be a Riemannian manifold with boundary and let $L \subset M \setminus \partial M$ be a smooth submanifold without boundary. Using Theorem 1.1, we can prove an approximation theorem of $M \setminus L$ in some cases where no conditions on the codimension of M and L is needed. More precisely, we can recover the homotopy type of $M \setminus L$ from the homotopy types of the spaces $M \setminus T$ where T is a finite subset of L. Again, we need to allow thickenings of the finite subsets T in L and inclusions between them. Therefore, we consider the configuration category $\mathrm{con}(L)$ of L (see Section 5.3 for a precise definition). For each object (T,ρ) in $\mathrm{con}(L)$, we have again a corresponding open subset $V_L(T,\rho)$ (using the Riemannian metric), which is the union of the open balls of radius $\rho(t)$ about the points $t \in T$. The inclusions

$$M \setminus L \to M \setminus V_L(T,\rho)$$

induce a map from $M \setminus L$ into the homotopy limit taken over the category $\mathrm{con}(L)$. Assume now that L is a *smooth thickening* of a compact simplicial complex $K \subset L$,

as defined in Definition 5.2. In particular, this means that K is a retract of L weakly equivalent to it. This is our main application:

Theorem 1.2 *If the codimension* $\dim M - \dim K$ *is at least* 3, *the canonical map*

$$M \setminus L \to \operatorname*{holim}_{(T,\rho) \in \operatorname{con}(L)} M \setminus V_L(T, \rho)$$

is a weak equivalence.

In particular, we can prove an approximation theorem for the boundary of the manifold in some cases. Namely, if $M \setminus \partial M$ is a *smooth thickening* of a compact simplicial complex $K \subset M \setminus \partial M$, we get the following corollary:

Corollary 1.3 *If the codimension* $\dim M - \dim K$ *is at least* 3, *the canonical map*

$$\partial M \to \operatorname*{holim}_{(T,\rho) \in \operatorname{con}(M \setminus \partial M)} M \setminus V_{M \setminus \partial M}(T, \rho)$$

is a weak equivalence.

In this case we also have a stronger version with restricted cardinalities (see Corollary 5.11) and it generalizes one of the main results in [11]. In the absence of the calculus for simplicial complexes as developed in [10], there we had to assume the existence of a smooth disk fiber bundle $M \to L$ with fiber dimension $c \geq 3$ where L is a closed smooth submanifold of M. This condition is a special case of our *smooth thickening* condition here (see Examples 5.4).

The ideas and strategies of [11] and of the generalization here thus intersect, so we feel compelled to indicate the substantial technical issues needed to establish the generalization. The main issue is to reformulate the key definitions. We give two basic examples: First, the definition of the configuration category $\operatorname{con}(K)$ of a simplicial complex K is quite different from its analogue, the configuration category of a smooth manifold (see Remark 4.4 for a comparison). Since we will apply manifold calculus for simplicial complexes, the technical conditions introduced in [10] go into the definition of $\operatorname{con}(K)$. Using these technical conditions, it becomes clear that in order to prove the main theorem, we also have to solve new technical challenges. Second, the definition of a smooth thickening of a simplicial complex involves various technical conditions. Again we have to verify that this definition is a convenient replacement of its analogue, the smooth fiber bundle condition, in [11].

In an application we will study the following question: Let M be a smooth manifold with boundary. It is well known that the boundary ∂M can be recovered as the homotopy link of the basepoint in $M/\partial M \cong (M \setminus \partial M) \cup \infty$. Therefore, it is possible

to say that there is an action of the homeomorphism group homeo$(M \setminus \partial M)$ on the pair $(M, \partial M)$ by homotopy automorphisms, ie each homeomorphism of $M \setminus \partial M$ determines a homotopy automorphism of the pair $(M, \partial M)$. But it is also well known that there is a canonical map of topological grouplike monoids (if an explanation is needed, see Section 6)

$$\text{homeo}(M \setminus \partial M) \hookrightarrow \text{haut}_{N\text{Fin}}(\text{con}(M \setminus \partial M)),$$

where NFin is the nerve of the category of finite sets and maps between finite sets and haut$_{N\text{Fin}}(\text{con}(M \setminus \partial M))$ is the space of the homotopy automorphisms of con$(M \setminus \partial M)$ over NFin. In [14] Weiss studies the question in what cases the action of homeo$(M \setminus \partial M)$ on the pair $(M, \partial M)$ by homotopy automorphisms extends to an action of haut$_{N\text{Fin}}(\text{con}(M \setminus \partial M))$ on the pair $(M, \partial M)$ by homotopy automorphisms. This has also applications in Weiss [15]. We can generalize his result (see Theorem 6.5): the action can be extended if the condition in Theorem 1.2 is satisfied.

Our paper with Weiss [11] attracted attention in applied topology because of possible relevance in the study of sensor network problems (for an introduction from the topological point of view see Adams and Carlsson [1] and de Silva and Ghrist [9]). At the moment there is no application of the theory developed in this paper outside the smooth setting, but we give a short explanation why there are potential ones in the context of sensor networks: In [1] movable sensor networks and evasion paths are studied. More concretely, let X be a subspace of a euclidean space. Assume we have a collection of points in X, each point equipped with a sensor. Each sensor covers a neighborhood of its location, for simplicity a ball of fixed radius. Then an evasion path is a specific embedding of a one-dimensional space into X minus the sensor region, which is the space covered by the union of all sensors. The spaces involved are usually not equipped with a smooth manifold structure, so the authors explicitly ask for an extension of the Goodwillie–Weiss manifold calculus to the setting of nonmanifold spaces [1, Section 7]. In particular, the theory developed in this paper could be a relevant application of manifold calculus for simplicial complexes because complements in manifolds are studied and the sensor region can be represented as a simplicial complex.

Outline

In Section 2 we recall the basic results of manifold calculus adapted for simplicial complexes. Using Goodwillie's homotopy functor calculus, we give general criteria for when a functor is analytic or polynomial and manifold calculus can be applied.

In Section 3 we will introduce the configuration categories of a simplicial complex and a smooth manifold. The configuration category carries a continuous structure. We will take this into account when we define homotopy limits. This leads to the notion of the *continuous homotopy limit*. We prove that in cases important to us it is weakly equivalent to the ordinary (or discrete) homotopy limit.

In Section 4 we will formulate Theorem 1.1 more precisely as well as the stronger version with restricted cardinalities and compare it with the situation in [11], where K is replaced by a smooth submanifold. Then we use manifold calculus (adapted for simplicial complexes) to prove it.

In Section 5 we will define a *smooth thickening* of a simplicial complex embedded in a smooth manifold and explain how this is a generalization of a smooth disk bundle over a smooth manifold. We will prove Theorem 1.2 and its stronger version with restricted cardinalities. In Section 6 these results will be applied in our study of homotopy automorphisms of the pair $(M, \partial M)$.

Notation The category (Top) is the category of topological spaces. By a simplex S of a simplicial complex, we mean a nondegenerate closed simplex. For such a simplex S, we denote by op(S) the open simplex. For a positive integer k, we set $[k] := \{0, 1, \dots, k\}$ and $\underline{k} := \{1, \dots, k\}$.

Acknowledgements

2 Manifold calculus adapted for simplicial complexes

In [10] we develop a generalization of manifold calculus where the smooth manifold is replaced by a simplicial complex. The main results of this paper are applications of this theory. Therefore, we introduce the constructions and main results of [10] and compare them with the homotopy functor calculus. The comparison leads to criteria which help us to apply manifold calculus (adapted to simplicial complexes).

2.1 Definitions and main results

All the constructions and results can be found in [10]. We define the category $\mathcal{O} = \mathcal{O}(K)$ as follows: the objects are the open subsets of K and the morphisms are inclusions, ie for $U, V \in \mathcal{O}$ there is exactly one morphism $U \to V$ if $U \subset V$ and there are no morphisms otherwise.

Definition 2.1 Let $U, V \in \mathcal{O}$ be open subsets and let $f_0, f_1 \colon U \to V$ be two maps such that $f_i|_{U \cap S}$ is a smooth embedding from $U \cap S$ into $V \cap S$ for all simplices S of K and $i = 0, 1$. We call f_0 and f_1 *stratified isotopic* if there is a continuous map $H \colon U \times [0, 1] \to V$ such that

$$H|_{(U \cap S) \times [0,1]} \colon (U \cap S) \times [0, 1] \to (V \cap S)$$

is a smooth isotopy from $f_0|_{U \cap S}$ to $f_1|_{U \cap S}$ for all simplices S of K. In this case we call H a *stratified isotopy* (from f_0 to f_1).

Note: for an n–dimensional simplex S, we can regard $U \cap S$ as a subspace in the euclidean space $\mathbb{R}^{n+1}$.

Definition 2.2 Let $U, V \in \mathcal{O}$ be two open subsets with $U \subset V$. The inclusion $i \colon U \to V$ is a *stratified isotopy equivalence* if there is a map $e \colon V \to U$ such that $e|_{V \cap S}$ is an embedding from $V \cap S$ into $U \cap S$ for all simplices S of K and $i \circ e$ (resp. $e \circ i$) is stratified isotopic to id_V (resp. id_U).

In the manifold calculus of Goodwillie and Weiss we consider functors which take smooth isotopy equivalences between open subsets of a fixed manifold to weak equivalences. In the version for simplicial complexes, stratified isotopy equivalences are replacing these smooth isotopy equivalences.

Definition 2.3 A contravariant functor $F \colon \mathcal{O} \to$ (Top) is *good* if:

(1) F takes stratified isotopy equivalences to weak homotopy equivalences.

(2) For every family $\{V_i\}_{i \in \mathbb{N}}$ of objects in $\mathcal{O}$ with $V_i \subset V_{i+1}$ for all $i \in \mathbb{N}$, the following canonical map is a weak homotopy equivalence:

$$F\left(\bigcup_i V_i\right) \to \operatorname*{holim}_i F(V_i).$$

Recall: For a positive integer k, let $\mathcal{P}([k])$ be the power set of $[k]$. Then a functor from $\mathcal{P}([k])$ to topological spaces is a $(k+1)$–cube of spaces.

Definition 2.4 Let χ be a cube of spaces. The *total homotopy fiber* of χ is the homotopy fiber of the canonical map

$$\chi(\varnothing) \to \operatorname*{holim}_{\varnothing \neq T \subset [k]} \chi(T).$$

If this map is a weak homotopy equivalence, we call the cube χ *(weak homotopy) cartesian*.

Now we define polynomial functors. To this end, let F be a good functor, let $V \in \mathcal{O}$ be an open subset of K and let $A_0, A_1, \ldots, A_k$ be pairwise disjoint closed subsets of V (for a positive integer k). Define a k–cube by

$$(2\text{-}1) \qquad\qquad T \mapsto F\left(V \setminus \bigcup_{i \in T} A_i\right).$$

Definition 2.5 The functor F is *polynomial of degree $\leq k$* if the k–cube defined in (2-1) is cartesian for all $V \in \mathcal{O}$ and pairwise disjoint closed subsets $A_0, A_1, \ldots, A_k$ of V.

Notation Let $x \in K$ be given and let $\mathcal{S}_x$ be the open star of the open simplex containing x, ie $\mathcal{S}_x := \bigcup_S \operatorname{op}(S)$, where the union ranges over all closed simplices S of K such that x is an element of S.

Definition 2.6 For a positive integer k, we define a full subcategory $\mathcal{O}k(K) = \mathcal{O}k$ of $\mathcal{O}$. Its objects are the open subsets $V \subset K$ with the following properties: V has at most k connected components and, for each component V_0 of V, there is an $x \in K$ such that $V_0 \subset \mathcal{S}_x$ and the inclusion $V_0 \to \mathcal{S}_x$ is a stratified isotopy equivalence. An element of $\mathcal{O}k$ (for some k) is called a *special open set*.

Theorem 2.7 *Let $F_1 \to F_2$ be a natural transformation between two k–polynomial functors. If $F_1(V) \to F_2(V)$ is a weak equivalence for all $V \in \mathcal{O}k$, it is a weak equivalence for all $V \in \mathcal{O}$.*

Let $F \colon \mathcal{O} \to (\text{Top})$ be a good functor. There is a concept of (relative) handle index in a simplicial complex [10, Section 3.1]. We can use it to define analyticity for F. To this end, let P be a compact codimension-zero subobject of K and let ρ be a fixed integer. Suppose $A_0, A_1, \ldots, A_r$ are pairwise disjoint compact codimension-zero subobjects of $K \setminus \operatorname{int}(P)$ with relative handle index $q_{A_i} \leq \rho$ (relative to P). For $T \subset [r]$, we set $A_T := \bigcup_{i \in T} A_i$ and assume $r \geq 1$.

Definition 2.8   The functor F is called ρ–*analytic with excess* c if, in these circumstances, the cube

$$T \mapsto F(\operatorname{int}(P \cup A_T)), \quad T \subset [r],$$

is $\left(c + \sum_{i=0}^{r}(\rho - q_{A_i})\right)$–cartesian for some integer c.

Theorem 2.9   [10, Theorem 3.6]   *Let* F *be a* ρ–*analytic functor with excess* c *and let* $V \in \mathcal{O}$ *be an open subset. Then the map*

$$\eta_{k-1}(V) \colon F(V) \to T_{k-1}F(V)$$

is $(c + k(\rho - \dim K))$–*connected for every* $k > 1$.

Remark 2.10   Theorem 2.9 is weaker than [10, Theorem 3.6], which uses the homotopy dimension of V [10, Definition 3.4] in order to increase the connectivity. For our purposes we do not need this stronger version.

Corollary 2.11   *Let* F *be a* ρ–*analytic functor with* $\rho > \dim K$. *For all open sets* $V \in \mathcal{O}(K)$, *the canonical map*

$$F(V) \to T_{\infty}F(V) = \operatorname*{holim}_{k} T_k F(V)$$

is a weak equivalence.

2.2   Comparison with homotopy functor calculus

In the last section we introduced a version of manifold calculus for simplicial complexes. We saw that in order to apply the approximation theorem, Theorem 2.9, we need to assume analyticity of the functor. Therefore, we should look for criteria which imply that a functor is analytic. Surprisingly, the homotopy functor calculus introduced by Goodwillie [6] helps to find such criteria.

Functor calculus investigates (covariant) *homotopy functors* from topological spaces to themselves. A functor $G \colon (\mathrm{Top}) \to (\mathrm{Top})$ is called homotopy functor if it takes weak equivalences to weak equivalences. If G is such a functor, we can compose it with a contravariant functor F from $\mathcal{O}(K)$ to (Top). The composition $G \circ F$ is a contravariant functor from $\mathcal{O}(K)$ to (Top). We will examine this composition.

Definition 2.12   A cube of spaces is called *strongly cocartesian* if each sub-2–face is a homotopy pushout.

Definition 2.13 A homotopy functor G from (Top) to itself is called *polynomial of degree $\leq k$* if it takes any strongly cocartesian $(k+1)$–cube to a weakly cartesian $(k+1)$–cube.

Let $V \in \mathcal{O}(K)$ be an open subset of K, let $A_0, A_1, \ldots, A_k$ be pairwise disjoint closed subsets of V (for a positive integer k) and let $A_T := \bigcup_{i \in T} A_i$, where T is a subset of $[k]$. The following proposition is an easy observation:

Proposition 2.14 Let $F \colon \mathcal{O}(K) \to$ (Top) *be a good (contravariant) functor (see* Definition 2.3*) such that*

$$
\begin{array}{ccc}
F(V \setminus A_{T \cap T'}) & \longrightarrow & F(V \setminus A_T) \\
\downarrow & & \downarrow \\
F(V \setminus A_{T'}) & \longrightarrow & F(V \setminus A_{T \cup T'})
\end{array}
$$

is a homotopy pushout for all $T, T' \subset [k]$ and all choices of $V, A_0, \ldots, A_k$ as above and let $G \colon$ (Top) $\to$ (Top) be a (covariant) homotopy functor. We suppose that G is k–polynomial in the sense of homotopy functor calculus (see Definition 2.13*). Then the composition $G \circ F$ is k–polynomial in the sense of manifold calculus (adapted for simplicial complexes).*

We would like to have a similar statement for analyticity.

Definition 2.15 Let ρ be an integer and let χ be a cocartesian k–cube of spaces such that the maps $\chi(\varnothing) \to \chi(\{i\})$ are k_i–connected with $k_i > \rho$ for all $i \in [k]$. A homotopy functor G is called *ρ–analytic with excess c* if the cube $G \circ \chi$ is $\big(c + \sum_{i \in [k]}(k_i - \rho)\big)$–cartesian (for all choices of χ).

Example 2.16 According to the Blakers–Massey theorem [5], for any strongly cocartesian cube χ where the map $\chi(\varnothing) \to \chi(\{i\})$ is κ_i–connected for each $i \in [k]$, the cube χ is κ–cartesian with $\kappa = 1 + \sum_{i \in [k]}(\kappa_i - 1)$. Therefore, by definition, the identity functor id: (Top) $\to$ (Top) is 1–analytic with excess 1.

Let $F \colon \mathcal{O}(K) \to$ (Top) be a good functor (see Definition 2.3). Recall that there is a concept of relative handle index in a simplicial complex [10, Section 3.1]. Let P be a compact codimension-zero subobject of K and let ρ be a fixed integer. Suppose $A_0, A_1, \ldots, A_r$ are pairwise disjoint compact codimension-zero subobjects of $K \setminus \text{int}(P)$ with relative handle index $q_{A_i} \leq \rho$ (relative to P). For $T \subset [k]$, we set $A_T := \bigcup_{i \in T} A_i$ and assume $k \geq 1$.

Proposition 2.17 *Suppose that the cube*

$$T \mapsto F(\mathrm{int}(P \cup A_T)), \quad T \subset [k],$$

is strongly cocartesian and suppose that there is a positive integer δ such that the maps

$$F(\mathrm{int}(P \cup A_{[k]})) \to F(\mathrm{int}(P \cup A_{[k]\setminus\{i\}}))$$

are $(\delta - q_{A_i})$–connected. Then F is $(\delta - 1)$–analytic with excess 1 (in the sense of Definition 2.8).

Proof The idea is to apply the Blakers–Massey theorem. By assumption, the cube $T \mapsto F(\mathrm{int}(P \cup A_T))$ is strongly cocartesian. We consider the cube

$$T \mapsto \mathrm{id} \circ F(\mathrm{int}(P \cup A_T)), \quad T \subset [k].$$

By applying Example 2.16, we deduce that the cube is $\left(1 + \sum_{i \in [k]} (\delta - q_{A_i} - 1)\right)$–cartesian. □

Remark 2.18 In the last proposition we use the analyticity of the identity map in topological spaces to find a criteria for analyticity of F, where F is a good functor. More generally, the following statement holds: for a ρ–analytic functor $G: (\mathrm{Top}) \to (\mathrm{Top})$ with excess c and F as above, the composition $G \circ F$ is a $(\delta - \rho)$–analytic functor with excess c and where δ is as above.

For an additional short note on the relationship of manifold calculus (for smooth manifolds) and homotopy functor calculus, see [11, Remark 1.3.2].

3 Background

In this section we provide some background which we will need for the discussions in the next sections. We introduce the configuration category of a simplicial complex and the continuous homotopy limit.

3.1 Configuration category of a simplicial complex

We will need the configuration category of a manifold as well as the configuration category of a simplicial complex. First, we recall the Riemannian model of the configuration category of a smooth manifold. Note that there are several equivalent definitions of the configuration category of a manifold [2].

Let M be a smooth manifold without boundary of dimension m and suppose that we have fixed a Riemannian metric on M. Then the configuration category $\mathrm{con}(M)$ of M is a topological poset. The objects are pairs (T, ρ) where T is a finite subset of M and $\rho\colon T \to (0, \infty)$ is a function such that:

(1) For each $t \in T$, the exponential map $\exp_t$ is defined and regular on the compact disk of radius $\rho(t)$ about the origin in the tangent space $T_t M$.

(2) The images in M of these disks under the exponential maps $\exp_t$ are pairwise disjoint.

For such a pair (T, ρ), let $V_M(T, \rho) \subset M$ be the union of the open balls of radius $\rho(t)$ about $t \in T$. Then $V_M(T, \rho)$ is an open subset of M which is diffeomorphic to $T \times \mathbb{R}^m$. All these pairs form a topological poset $\mathrm{con}(M)$ by

$$(T, \rho) \leq (T', \rho') \iff V_M(T, \rho) \subset V_M(T', \rho').$$

This poset can also be regarded as a category. We would like to adapt this definition and introduce the configuration category $\mathrm{con}(K)$ of the simplicial complex K. Therefore, we should start with the following observation:

Remark 3.1 Let x be an element of K and let $\mathcal{S}_x$ be the open star neighborhood of x in K. The closure $K_x := \mathrm{cl}(\mathcal{S}_x)$ of $\mathcal{S}_x$ in K carries a canonical metric $d = d_x$ induced by the euclidean structure of each simplex. The precise definition is technical and can be done by distinguishing the following two cases: If two elements $y, y' \in K_x$ are in the same simplex, we can use the euclidean structure of the simplex to define $d(y, y') \in [0, \infty)$ as the distance between y and y' in the euclidean space. If they are not in the same simplex, we set

$$d(y, y') := \min_{z \in S_y \cap S_{y'}} d(y, z) + d(z, y'),$$

where S_y (resp. $S_{y'}$) is the simplex of maximal dimension which includes y (resp. y'). By definition, we can use again the euclidean structure.

We wrote d instead of d_x to avoid the index x. In fact, $d(y, y')$ is independent of the element x in K: if x and x' are two elements of K with $y, y' \in \mathcal{S}_x \cap \mathcal{S}_{x'}$, then $d_x(y, y') = d_{x'}(y, y')$.

Now we introduce the configuration category $\mathrm{con}(K)$. The objects are again pairs (T, ρ) where T is a finite subset of K and $\rho\colon T \to (0, \infty)$ is a function fulfilling the following two conditions:

(1) For each $t \in T$, there is an element $x \in K$ such that $t \in S_x$ and the open ball $B^d_{\rho(t)}(t) \subset K_x = \text{cl}(S_x)$ of radius $\rho(t)$ about t determined by the metric $d = d_x$ is a subset of the open star neighborhood S_x and the inclusion $B^d_{\rho(t)}(t) \hookrightarrow S_x$ is a stratified isotopy equivalence (see Definition 2.2). In particular, $B^d_{\rho(t)}(t) \in \mathcal{O}1$ is a special open set (see Definition 2.6).

(2) The open balls $B^d_{\rho(t)}(t) \subset K$ with origin t and radius $\rho(t)$ are pairwise disjoint.

For such a pair (T, ρ), let $V_K(T, \rho) \subset M$ be the union of the open balls $B^d_{\rho(t)}(t) \subset K$ of radius $\rho(t)$ about $t \in T$. Then $V_K(T, \rho)$ is a special open subset of K (see Definition 2.6). By analogy with the manifold case, we form the topological poset $\text{con}(K)$ by

$$(T, \rho) \leq (T', \rho') \iff V_K(T, \rho) \subset V_K(T', \rho').$$

This poset can also be regarded as a category.

Remark 3.2 Since this is a very technical notion, we feel compelled to give a short explanation why this category $\text{con}(K)$ is nonempty. Let T be a configuration in K. If we choose ϵ small enough, then the function $\rho\colon T \to (0, \infty)$ mapping all elements of T to ϵ fulfills all conditions in the definition of $\text{con}(K)$. More precisely, the inclusion of the open ball $B^d_\epsilon(t)$ about an element $t \in T$ of radius ϵ into the open star S_t of t is a stratified isotopy equivalence. If ϵ is small enough, the open balls for different elements of T are also pairwise disjoint.

Now we want to take a closer look at the configuration category $\text{con}(K)$. But note that the following results are also true for $\text{con}(M)$, the configuration category of a smooth manifold M (without boundary).

Remark 3.3 The configuration category $\text{con}(K)$ is a topological poset, ie the objects as well as the morphisms form a topological space. More generally, if $N(\text{con}(K))$ is the nerve of the category $\text{con}(K)$, then $N_r(\text{con}(K))$ is a topological space for all $r \geq 0$. This is obvious since $N_r(\text{con}(K))$ is the space of all strings

$$(T_0, \rho_0) \leq (T_1, \rho_1) \leq \cdots \leq (T_r, \rho_r),$$

where (T_i, ρ_i) for $0 \leq i \leq r$ is an element of $\text{con}(K)$.

Now we want to investigate the homotopy type of the configuration category $\text{con}(K)$ as a topological space. It is very reminiscent of the configuration spaces.

Definition 3.4 We define $C_r(K)$ to be the space of unordered configurations of r points in K: Let $F_r(K)$ be the space of ordered r–configurations of K given by

$$F_r(K) := \{(x_1, \ldots, x_r) \in K^r \mid x_i \neq x_j \text{ for all } i \neq j\}.$$

The symmetric group Σ_r acts freely on $F_r(K)$. Then

$$C_r(K) := F_r(K)/\Sigma_r$$

is the space of unordered r–configurations.

Remark 3.5 What is the relation between the configuration category and the configuration spaces? Let $r \geq 0$ be a fixed integer. We define the space $C_r^{\text{fat}}(K)$ to be the space of all pairs $(T, \rho) \in \text{con}(K)$ with $|T| = r$. Then we have a forgetful projection map

$$C_r^{\text{fat}}(K) \to C_r(K),$$

which is a fiber bundle with contractible fibers. Therefore, this map is a weak equivalence of spaces.

3.2 Continuous homotopy limit

Let $\text{con}(K)$ be the configuration category of K and let $N(\text{con}(K))$ be its nerve. We saw that $N_r(\text{con}(K))$ is a topological space for all $r \geq 0$. We are studying the functor Φ from $\text{con}(K)$ to topological spaces defined by

$$\Phi((T, \rho)) := M \setminus V_K(T, \rho)$$

and its homotopy limit

$$\underset{\text{con}(K)}{\text{holim}} \Phi = \underset{(T,\rho)\in\text{con}(K)}{\text{holim}} M \setminus V_K(T, \rho).$$

During our study of this homotopy limit, we would like to integrate the continuous structure of the nerve of $\text{con}(K)$. To this end, we will introduce the continuous homotopy limit of Φ using the topological structure of the configuration category.

We recall that the ordinary (or discrete) homotopy limit $\text{holim}_{\text{con}(K)} \Phi$ of the contravariant functor Φ is defined to be the totalization of the cosimplicial space

$$[r] \mapsto \prod_{(T_0,\rho_0)\leq\cdots\leq(T_r,\rho_r)\in N_r(\text{con}(K))} \Phi((T_r, \rho_r)).$$

By definition, the right-hand side is equal to the space of all sections from $N_r(\mathrm{con}(K))$ equipped with the discrete topology to

$$\coprod_{(T_0,\rho_0)\leq\cdots\leq(T_r,\rho_r)\in N_r(\mathrm{con}(K))} \Phi((T_r,\rho_r)).$$

Equivalently, it is equal to the space of all maps $f\colon N_r(\mathrm{con}(K))\to M$ such that

$$f((T_0,\rho_0)\leq\cdots\leq(T_r,\rho_r))\in M\setminus V_K(T_r,\rho_r),$$

where $N_r(\mathrm{con}(K))$ is again given the discrete topology. Using the continuous structure of $\mathrm{con}(K)$, we introduce the following notation:

Definition 3.6 We define $\Gamma_r(\Phi)$ as the space of all continuous maps $f\colon N_r(\mathrm{con}(K))\to M$ such that $f((T_0,\rho_0)\leq\cdots\leq(T_r,\rho_r))\in M\setminus V_K(T_r,\rho_r)$.

If we define $E_r^!(\Phi)$ to be the space

$$\coprod_{(T_0,\rho_0)\leq\cdots\leq(T_r,\rho_r)\in N_r(\mathrm{con}(K))} \Phi((T_r,\rho_r))$$

equipped with the subspace topology of $N_r(\mathrm{con}(K))\times M$, then the projection map $E_r^!(\Phi)\to N_r(\mathrm{con}(K))$ is a fiber bundle and $\Gamma_r(\Phi)$ is the space of all continuous sections of this fiber bundle.

Definition 3.7 The *continuous homotopy limit* $\mathrm{ctsholim}_{\mathrm{con}(K)}\Phi$ of Φ is defined to be the totalization of the cosimplicial space $[r]\mapsto\Gamma_r(\Phi)$.

Lemma 3.8 *The canonical inclusion map*

$$\mathrm{ctsholim}_{\mathrm{con}(K)}\Phi\to\mathrm{holim}_{\mathrm{con}(K)}\Phi$$

is a weak equivalence.

We skip the proof because it is equal to the proof of [11, Lemma 1.2.1]. (If we replace the manifold L appearing in [11, 1.2.1] by the simplicial complex K, then we get a proof for Lemma 3.8.)

Using this result, we can work in the following with either of these homotopy limits — the discrete homotopy limit or the continuous homotopy limit.

Remark 3.9 For an open subset U of K, let $\mathrm{con}(K)|_U$ be the full subcategory of $\mathrm{con}(K)$ such that the objects are all elements (T,ρ) in $\mathrm{con}(K)$ with $V_K(T,\rho)\subset U$.

For $r \geq 0$, let $\Gamma_r(\Phi)|_U$ be the space of all continuous maps $f \colon N_r(\operatorname{con}(K)|_U) \to M$ such that

$$f((T_0, \rho_0) \leq \cdots \leq (T_r, \rho_r)) \in M \setminus V_K(T_r, \rho_r).$$

Now we define $\operatorname{ctsholim}_{\operatorname{con}(K)|_U} \Phi$ to be the totalization of the cosimplicial space $r \mapsto \Gamma_r(\Phi)|_U$. There is a canonical inclusion map

$$\operatorname*{ctsholim}_{\operatorname{con}(K)|_U} \Phi \to \operatorname*{holim}_{\operatorname{con}(K)|_U} \Phi,$$

which is a weak equivalence. The proof is equal to that of Lemma 3.8.

Remark 3.10 The cosimplicial space $r \mapsto \Gamma_r(\Phi)|_U$ is Reedy fibrant for every open subset U of K. The verification is the same as that in [11, 1.1.3]. Recall that for a map $X \to Y$ between cosimplicial spaces which is a degreewise weak equivalence, the map of their totalizations $\operatorname{Tot}(X) \to \operatorname{Tot}(Y)$ is a weak equivalence.

4 The main theorem

We formulate the main theorem and apply manifold calculus (adapted to simplicial complexes) in order to prove it.

4.1 The formulation of the problem

We remind the reader that M is a smooth manifold and $K \subset M$ is a simplicial complex such that each (closed) simplex of K is smoothly embedded in M. For each element (T, ρ) of the configuration category $\operatorname{con}(K)$, there is an inclusion map

$$M \setminus K \to M \setminus V_K(T, \rho),$$

where $V_K(T, \rho)$ is the open subset of K corresponding to the pair (T, ρ). If we define a contravariant functor Φ from $\operatorname{con}(K)$ to topological spaces by $\Phi((T, \rho)) := M \setminus V_K(T, \rho)$, then the inclusion maps induce a canonical map

$$(4\text{-}1) \qquad\qquad M \setminus K \to \operatorname*{holim}_{\operatorname{con}(K)} \Phi.$$

We can ask if the canonical map is a weak equivalence. There is a variant with restricted cardinalities. Let $n \geq 0$ be an integer. Then we define $\operatorname{con}_{\leq n}(K)$ to be the full subcategory of $\operatorname{con}(K)$ where the objects are all elements (T, ρ) of $\operatorname{con}(K)$ with $|T| \leq n$. Again, we get a canonical map

$$(4\text{-}2) \qquad\qquad M \setminus K \to \operatorname*{holim}_{\operatorname{con}_{\leq n}(K)} \Phi$$

induced by inclusions. In this case we do not expect that this map is a weak equivalence. But, we can ask if it is highly connected. In the following theorem we use the notation $m := \dim M$ and $\kappa := \dim K$.

Theorem 4.1 *If $\kappa + 3 \leq m$, then the canonical map* (4-1) *is a weak equivalence and* (4-2) *is $(1+(n+1)(m-\kappa-2))$–connected.*

Remark 4.2 The homotopy limit appearing in (4-1) is the ordinary (or discrete) homotopy limit. By Lemma 3.8, we could also use the continuous homotopy limit and the theorem would still hold. Using similar arguments, we could also use the continuous homotopy limit in (4-2).

Remark 4.3 We assumed that the codimension of K in M is at least three. In fact, the theorem would be false without this assumption. There is a nice counterexample in codimension two [11, Remark 1.3.3].

Remark 4.4 The theorem is a generalization of [11, Theorem 1.1.1]. Let L be a compact, smooth submanifold (without boundary) of M where the codimension of L in M is at least three. We can choose a triangulation of L and get a simplicial complex K, ie $K = L$ as a topological space but the configuration categories con(L) and con(K) are quite distinct because the structure of K as a simplicial complex goes into the definition of con(K).

Let $\bigcup_k \mathcal{O}k(L)$ be the category of all special open subsets of L [12]. These are all the open subsets of L which are diffeomorphic to a disjoint union of open disks. Then we have the inclusions of categories

$$\text{con}(L) \hookrightarrow \bigcup_k \mathcal{O}k(L) \hookleftarrow \text{con}(K)$$

and we get a zigzag

$$\underset{(T,\rho)\in\text{con}(L)}{\text{holim}} \Phi((T,\rho)) \leftarrow \underset{U\in\bigcup_k \mathcal{O}k(L)}{\text{holim}} M \setminus U \rightarrow \underset{(T,\rho)\in\text{con}(K)}{\text{holim}} \Phi((T,\rho)).$$

These projection maps of homotopy limits given by inclusion of categories are both weak equivalences.

4.2 A good functor

In order to prove Theorem 4.1, we would like to apply manifold calculus (adapted to simplicial complexes). Naively, one could suggest to apply the approximation theorem

([Theorem 2.9](#)) to the contravariant functor which maps an open subset $V \subset K$ to the topological space $M \setminus V$. Unfortunately, this functor is not good because in general it does not take stratified isotopy equivalences to weak equivalences (for a counterexample, see [11, 1.3]). Therefore, we need a modification.

Definition 4.5 We define the functor F from the category $\mathcal{O}(K)$ of open subsets of K to topological spaces by

$$F(V) := \operatorname*{holim}_{C \subset V} M \setminus C,$$

where C runs over all compact subsets of V.

We will see that F is an appropriate replacement of the functor $V \mapsto M \setminus V$. The proof in the following lemma is similar to that of [11, 1.3.1]. For the sake of completeness, we will give all required arguments.

Lemma 4.6 *The functor F is good* (in the sense of [Definition 2.3](#)).

Proof First, we notice that the (co)limit axiom is fulfilled. This is obvious. In order to show that the functor takes stratified isotopy equivalences to weak homotopy equivalences, we will use the reformulation of stratified isotopy equivalences as given in [Remark 4.7](#). To this end, let V_0 and V_1 be two open subsets of K with $V_0 \subset V_1$ and let $e_t\colon V_0 \to V_1$ for $t \in [0, 1]$ be a stratified isotopy such that e_0 is the inclusion and, for each simplex S of K, e_1 is a homeomorphism such that $e_1|_S\colon S \cap V_0 \to S \cap V_1$ can be extended to a diffeomorphism (see [Remark 4.7](#)).

Let $\{C_i\}_{i \geq 0}$ be a sequence of compact subsets of V_1 such that $C_i \subset C_{i+1}$ for all $i \geq 0$ and such that, for every compact subset C of V_1, there is an element C_i of this sequence with $C \subset C_i$. By definition, the inclusion

$$\{C_i\}_{i \geq 0} \to \{C \subset V_1 \mid C \text{ compact}\}$$

is homotopy terminal. (Note that the morphisms are the inclusions of compact subsets.) Therefore, the canonical map

$$F(V_1) \to \operatorname*{holim}_{i} M \setminus C_i$$

is a weak equivalence. Now we define the compact sets $C_{t,i} := e_t(e_1^{-1}(C_i))$. Note that $C_{1,i} = C_i$. By definition, the inclusion

$$\{C_{0,i}\}_{i \geq 0} \to \{C \subset V_0 \mid C \text{ compact}\}$$

is homotopy terminal and induces a weak equivalence

$$F(V_0) \to \operatorname*{holim}_i M \setminus C_{0,i}.$$

We fix the notation

$$Y_i := \{w \colon [0,1] \to M \mid w(t) \notin M \setminus C_{t,i}\}.$$

There are evaluation maps $Y_i \to M \setminus C_{0,i}$ and $Y_i \to M \setminus C_{1,i}$. Using the isotopy extension theorem [8, 6.5], it is straightforward to find homotopy inverses. For a comment on the isotopy extension theorem for stratified spaces, see Remark 4.8. We get homotopy equivalences

$$M \setminus C_{0,i} \longleftrightarrow Y_i \longleftrightarrow M \setminus C_{1,i}.$$

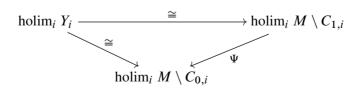Since the evaluation maps are natural, we get weak equivalences

$$\operatorname*{holim}_i M \setminus C_{0,i} \leftarrow \operatorname*{holim}_i Y_i \to \operatorname*{holim}_i M \setminus C_{1,i}$$

To summarize, we have shown that the spaces $F(V_1)$ and $F(V_0)$ are weakly equivalent. Now we have to argue that the canonical map $F(V_1) \to F(V_0)$ induced by inclusion is a weak equivalence.

Let $g \colon \mathbb{N} \to \mathbb{N}$ be a monotone injective function such that for every $i \in \mathbb{N}$ and $t \in [0,1]$, the compact set $C_{t,i}$ is a subset of $C_{1,g(i)}$. We consider the composition

$$\Psi \colon \operatorname*{holim}_i M \setminus C_{1,i} \to \operatorname*{holim}_i M \setminus C_{1,g(i)} \to \operatorname*{holim}_i M \setminus C_{0,i},$$

where the first map is induced by the inclusion $\{C_{1,g(i)}\}_i \to \{C_{1,i}\}_i$ of categories and the second map is induced by the inclusions $C_{0,i} \hookrightarrow C_{1,g(i)}$ of spaces for $i \in \mathbb{N}$. In order to verify that the composition Ψ is a weak equivalence, we consider the homotopy commutative triangle

$$
\begin{array}{ccc}
\operatorname{holim}_i Y_i & \xrightarrow{\;\;\cong\;\;} & \operatorname{holim}_i M \setminus C_{1,i} \\
& {\scriptstyle \cong}\searrow \qquad \swarrow {\scriptstyle \Psi} & \\
& \operatorname{holim}_i M \setminus C_{0,i} &
\end{array}
$$

It does not seem to be trivial that the triangle is homotopy commutative. But, by careful inspection, the definition of the homotopy limit provides a homotopy whereby the triangle is homotopy commutative. Using the same argument, we get a homotopy

commutative square

$$
\begin{array}{ccc}
F(V_1) & \longrightarrow & F(V_0) \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
\operatorname{holim}_i M \setminus C_{1,i} & \xrightarrow{\Psi} & \operatorname{holim}_i M \setminus C_{0,i}
\end{array}
$$

Since Ψ is a weak equivalence, the canonical map $F(V_1) \to F(V_0)$ is also a weak equivalence. $\qquad\square$

Remark 4.7 We need a slight reformulation of a stratified isotopy equivalence. According to Definition 2.2, an inclusion $i\colon V_0 \to V_1$ of open subsets of K is a stratified isotopy equivalence if there is a continuous map $e\colon V_1 \to V_0$ such that $e|_{V_1 \cap S}$ is a smooth embedding from $V_1 \cap S$ into $V_0 \cap S$ for all simplices S of K and if there are a stratified isotopy from $i \circ e$ to id_{V_1} and a stratified isotopy from $e \circ i$ to id_{V_0}. The following definition would also be appropriate: we could call an inclusion $i\colon V_0 \to V_1$ of open subsets of K a stratified isotopy equivalence if i is stratified isotopic to a homeomorphism $e\colon V_0 \to V_1$ such that $e|_{V_0 \cap S}$ is a diffeomorphism from $V_0 \cap S$ to $V_1 \cap S$ for all simplices S of K. (Note that S is not a manifold, so more precisely we should say: the map $e|_{V_0 \cap S}$ from $V_0 \cap S$ to $V_1 \cap S$ can be extended to a diffeomorphism using that S is canonically embedded in an euclidean space.)

Why is the second definition of stratified isotopy equivalences also appropriate? We do not know if these definitions are equivalent, but it is straightforward to verify the following claim: Let $G\colon \mathcal{O}(K) \to (\mathrm{Top})$ be a contravariant functor. Then G takes stratified isotopy equivalences as in Definition 2.2 to weak equivalences if and only if G takes stratified isotopy equivalences as in the second definition to weak equivalences.

Remark 4.8 In the proof of the last lemma we can use a continuous version of the isotopy extension theorem for stratified spaces as provided in [8, 6.5]: Let $C \subset V_0$ be a compact subset, where $V_0 \subset K$ is an open subset as above. We consider a continuous family of open topological embeddings $f_t\colon C \to K$ for $0 \le t \le 1$, with $f_0 = \operatorname{id}_C$. Then there is a continuous family of homeomorphisms $H_t\colon K \to K$ such that $H_t|_C = f_t$ and $H_0 = \operatorname{id}_K$.

We can use this theorem in the proof above as follows: Let $e_t\colon V_0 \to V_1$ for $t \in [0,1]$ be a stratified isotopy as above. In particular, e_0 is the inclusion of V_0 into V_1 and e_1 is a homeomorphism. For a positive integer i, we define $C := e_1^{-1}(C_i)$ and $f_t := e_t|_C\colon C \to V_1 \subset K$. Using the isotopy extension theorem, we get a continuous

family of homeomorphisms $H_t\colon K \to K$ such that $H_t|_C = f_t$ and $H_0 = \mathrm{id}_K$. Then a homotopy inverse of the evaluation map $Y_i \to K \setminus C_{0,i}$ given by $w \mapsto w(0)$ can be defined by $x \mapsto (t \mapsto H_t(x))$.

4.3 Proof of the main theorem

Now we prove Theorem 4.1, ie we show that the top horizontal arrow in the commutative diagram

$$
\begin{array}{ccc}
M \setminus K & \longrightarrow & \underset{(T,\rho)\in\mathrm{con}(K)}{\mathrm{holim}} M \setminus V_K(T,\rho) \\
\downarrow & & \downarrow \\
F(K) & \longrightarrow & \underset{(T,\rho)\in\mathrm{con}(K)}{\mathrm{holim}} F(V_K(T,\rho))
\end{array}
$$

is a weak equivalence. The left vertical arrow is a weak equivalence because K is a maximal element in the category (poset) of all compact subsets of K. The right vertical arrow is a weak equivalence because for every $(T,\rho) \in \mathrm{con}(K)$, the category of all compact subsets of $V_K(T,\rho)$ has a directed subcategory which is homotopy terminal. Therefore, we have to show that the bottom horizontal arrow is a weak equivalence. To this end, we will use the good properties of the functor F and manifold calculus (adapted to simplicial complexes). The bottom arrow equals the composition

$$
F(K) \to \underset{U \in \bigcup_k \mathcal{O}k(K)}{\mathrm{holim}} F(U) \to \underset{(T,\rho)\in\mathrm{con}(K)}{\mathrm{holim}} F(V_K(T,\rho)),
$$

where the first map is the canonical map and the second map is induced by the inclusion of posets

$$
\mathrm{con}(K) \to \bigcup_k \mathcal{O}k(K)
$$

given by $(T,\rho) \mapsto V_K(T,\rho)$. Therefore, the following two lemmas complete the proof. (The proof of the case with restricted cardinalities follows similar lines.)

Lemma 4.9 *The canonical projection map*

$$
\underset{U \in \bigcup_k \mathcal{O}k(K)}{\mathrm{holim}} F(U) \to \underset{(T,\rho)\in\mathrm{con}(K)}{\mathrm{holim}} F(V_K(T,\rho))
$$

induced by the inclusion $\mathrm{con}(K) \to \bigcup_k \mathcal{O}k(K)$ *is a weak equivalence.*

Proof By [3, Theorem 6.14], it remains to show that the canonical map

$$
F(U) \to \underset{(T,\rho)\in\mathrm{con}(K)|_U}{\mathrm{holim}} F(V_K(T,\rho))
$$

is a weak equivalence for all $U \in \bigcup_k \mathcal{O}k(K)$. Recall that $\mathrm{con}(K)|_U$ is the full subcategory of $\mathrm{con}(K)$ where the objects are all elements (T, ρ) in $\mathrm{con}(K)$ with $V_K(T, \rho) \subset U$. For a fixed $U \in \bigcup_k \mathcal{O}k(K)$, we choose an element $(T', \sigma) \in \mathrm{con}(K)|_U$ such that the map $F(U) \to F(V_k(T, \sigma))$ is a weak equivalence. We set $W := V_K(T', \sigma)$ and consider the commutative diagram

$$
\begin{array}{ccc}
F(U) & \longrightarrow & \underset{(T,\rho)\in\mathrm{con}(K)|_U}{\mathrm{holim}} F(V_K(T,\rho)) \\
\downarrow & & \downarrow \\
F(W) & \longrightarrow & \underset{(T,\rho)\in\mathrm{con}(K)|_W}{\mathrm{holim}} F(V_K(T,\rho))
\end{array}
$$

The bottom arrow is a weak equivalence because W is a maximal element in $\mathrm{con}(K)|_W$. In order to show that the right vertical arrow is a weak equivalence, we will consider the two homotopy limits as continuous homotopy limits. This is allowed by Remark 3.9. Then we compare the two spaces $\mathrm{con}(K)|_W$ and $\mathrm{con}(K)|_U$. By definition of their topologies, the inclusion $\mathrm{con}(K)|_W \to \mathrm{con}(K)|_U$ is a weak equivalence. Similarly, the maps of section spaces $\Gamma_r(\Phi)|_U \to \Gamma_r(\Phi)|_W$ are weak equivalences for all $r \geq 0$. So they induce a weak equivalence of continuous homotopy limits. $\square$

Lemma 4.10 *If* $\dim K + 3 \leq \dim M$, *the canonical map*

$$
F(K) \to \underset{U \in \bigcup_k \mathcal{O}k(K)}{\mathrm{holim}} F(U)
$$

is a weak equivalence.

Proof Note that we have already shown that F is good (Lemma 4.6). Let P be a smooth compact codimension-zero subobject of K and let $A_0, A_1, \ldots, A_r$ be compact codimension-zero subobjects of $K \setminus \mathrm{int}(P)$ with relative handle index q_{A_i} (relative to P). For $T \subset [r]$, we define

$$
W_T := \mathrm{int}\left(P \cup \bigcup_{i \in T} A_i \right),
$$

where $\mathrm{int}(-)$ is the interior in K. We have to show that the cube

$$
T \mapsto F(W_T), \quad T \subset [r],
$$

is strongly cocartesian and that, for every $0 \leq i \leq r$, the maps

$$
F(W_{[r]}) \to F(W_{[r]\setminus\{i\}})
$$

are $((m-1)-q_{A_i})$–connected, where m is the dimension of M. Note that W_S is the interior of a compact codimension-zero subobject of K. Therefore, instead of using the functor F, we can work with the cube

$$T \mapsto G(W_T) := M \setminus W_T.$$

Why can we use this cube? Because of the special assumption, there is a directed homotopy terminal subcategory in the category of all compact subsets of W_T. Thus, the canonical map $G(W_T) \to F(W_T)$ is a weak equivalence.

Let $i, j \in [r]$ be two distinct elements. In order to show that the cube induced by G is strongly cocartesian, we need to investigate if the canonical map from the homotopy pushout of

$$G(W_{[r]\setminus\{i\}}) \leftarrow G(W_{[r]}) \to G(W_{[r]\setminus\{j\}})$$

to $G(W_{r\setminus\{i,j\}})$ is a weak equivalence. But this can easily be seen. In fact, using the assumptions that all A_i are pairwise disjoint, we can find a copy of $G(W_{r\setminus\{i,j\}})$ in the homotopy pushout which is a retract of the homotopy pushout. Likewise, it is not difficult to check that for a fixed $i \in [r]$, the map

$$G(W_{[r]}) \to G(W_{[r]\setminus\{i\}})$$

is $(m-q_{A_i}-1)$–connected since the target is homotopy equivalent to the source with attached cells of dimension $\geq m - q_{A_i}$. $\qquad\square$

5 Occupants in the interior of a manifold

In this section, let M be a manifold with boundary and let L be a smooth submanifold without boundary. We discuss Theorem 1.2, where the homotopy type of $M \setminus L$ is recovered from the homotopy types of the spaces $M \setminus T$ with $T \subset L$ finite. To this end, we give the definition of a *smooth thickening* of a simplicial complex (in M) and discuss first observations and examples. Then we prove the tube lemma, Lemma 5.6, which we will need in order to prove Theorem 1.2.

5.1 Smooth thickenings of a simplicial complex

We consider the following situation: Let M be a manifold with boundary. Let $L \subset M \setminus \partial M$ be a smooth submanifold without boundary of dimension l.

Definition 5.1 Let $K \subset L$ be a simplicial complex. We say that $p: L \to K$ is a *nice projection map* if the following conditions hold:

(1) $p|_K = \mathrm{id}_K$.

(2) The open set $p^{-1}(V_K(T, \rho)) \subset L$ is diffeomorphic to $T \times \mathbb{R}^l$ for every element (T, ρ) of the configuration category $\mathrm{con}(K)$ of K.

Definition 5.2 We say that L is a *smooth thickening of K in M* if each (closed) simplex of K is smoothly embedded in L and if there exists a nice projection map $p: L \to K$ such that the inclusion $M \setminus p^{-1}(V) \to M \setminus V$ is a weak equivalence for all open sets $V \in \mathcal{O}(K)$.

Definition 5.3 If $M \setminus \partial M$ is a smooth thickening of K in M, then we just say that M (which is a manifold with boundary) is a *smooth thickening of K*.

Examples 5.4 (1) The definition of smooth thickening weakens the strong condition in [11, 2.1.1] in the following sense: Let L be a smooth closed manifold and let $p: M \to L$ be a smooth disk bundle, ie a smooth fiber bundle where each fiber is diffeomorphic to a (closed) disk D^r of fixed dimension $r \geq 0$. Then L can be considered as a subset of M by using the zero section of p. We can choose a triangulation of L and then L is a smooth thickening of its triangulation in M.

(2) We consider the 1–dimensional simplicial complex K with four vertices $\{a, b, c, d\}$ and 1–simplices $\{\{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}\}$, ie we have two triangles which coincide in exactly one simplex, namely $\{b, c\}$. Now it is an easy exercise to build up a compact manifold M of dimension $m = 2$ such that the interior $M \setminus \partial M$ is a smooth thickening of K in M, ie M is a smooth thickening of K. We ought to consider M as a manifold with four 0–handles and five 1–handles. This example can easily be generalized to all dimensions $m \geq 2$ and/or to an one-dimensional simplicial complex which consists of more than two triangles.

Lemma 5.5 *We assume that* $\dim K + 3 \leq m$ *and that L is a smooth thickening of K in M. Let $p: L \to K$ be a nice projection map. Then the canonical map*

$$M \setminus L \to \underset{(T, \rho) \in \mathrm{con}(K)}{\mathrm{holim}} M \setminus p^{-1}(V_K(T, \rho))$$

is a weak equivalence.

Proof We consider the five homotopy equivalences

$$M \setminus L \simeq M \setminus K$$
$$\simeq M \setminus (K \cup \partial M)$$
$$\simeq \operatorname*{holim}_{(T,\rho)} M \setminus (V_K(T, \rho) \cup \partial M)$$
$$\simeq \operatorname*{holim}_{(T,\rho)} M \setminus V_K(T, \rho)$$
$$\simeq \operatorname*{holim}_{(T,\rho)} M \setminus p^{-1}(V_K(T, \rho)),$$

where the three homotopy limits are taken over all (T, ρ) in $\mathrm{con}(K)$. By definition of smooth thickenings in M, the first equivalence can be verified, as well as the fifth equivalence. By Theorem 4.1, the third map is a weak equivalence. The second and the fourth map are weak equivalences since $M \cong M \setminus \partial M$. □

5.2 Tube lemma

Now we adapt the results of [11, 2.2] for a nice projection map. Note that for the following lemma we do not have to require that the codimension be at least three. It could also be zero.

Lemma 5.6 *Let L be a smooth manifold without boundary and let K be a compact simplicial complex K. Let $p\colon L \to K$ be a nice projection map (see Definition 5.1). Then the canonical map*

$$(5\text{-}1) \qquad\qquad \operatorname*{hocolim}_{(T,\rho)\in\mathrm{con}(K)} C_n\big(p^{-1}(V_K(T, \rho))\big) \to C_n(L)$$

is a weak equivalence.

Proof We are going to show that the map is a microfibration with contractible fibers. Then the lemma will follow [13, Lemma 2.2]. Let T be an element of the configuration space $C_n(L)$. The fiber of the map (5-1) over the configuration T is identified with the classifying space of the poset of all $(T, \rho) \in \mathrm{con}(K)$ with $T \in p^{-1}(V_K(T, \rho))$, ie $p(T) \in V_K(T, \rho)$. The inclusion of the directed poset

$$\left\{(T, \rho) \in \mathrm{con}(K) \mid \exists n \in \mathbb{N} \; \forall t \in T \; \rho(t) = \frac{1}{n}\right\}$$

into the above described poset is a homotopy initial functor. (We consider the posets as categories.) Therefore, the fiber is contractible.

Now we verify the lifting condition. We start with an observation: The projection map and the map (5-1) determine an injective, continuous map

$$\operatorname*{hocolim}_{(T,\rho)\in\mathrm{con}(K)} C_n\big(p^{-1}(V_K(T,\rho))\big) \to |N\mathrm{con}(K)| \times C_n(L).$$

(This map is not an embedding, ie a homeomorphism onto its image. See also Remark 5.7.) We call this map $g = (g_1, g_2)$.

Let Z be a compact CW–space. We consider the diagram

$$
\begin{array}{ccc}
Z & \longrightarrow & \operatorname*{hocolim}_{(T,\rho)\in\mathrm{con}(K)} C_n\big(p^{-1}(V_K(T,\rho))\big) \\
\downarrow & & \downarrow \\
Z \times I & \longrightarrow & C_n(L)
\end{array}
$$

We call the upper horizontal map f and we can consider it as a pair of maps $f = (f_1, f_2)$ if we define $f_i := g_i \circ f$ for $i = 1, 2$. We call the bottom horizontal map h. The right vertical arrow is equal to g_2. We can define a small lift

$$H\colon Z \times [0,\epsilon] \to \operatorname*{hocolim}_{(T,\rho)\in\mathrm{con}(K)} C_k\big(p^{-1}(V_K(T,\rho))\big)$$

by $H := (f_1, h)$.

How can we describe the map H? Let $z \in Z$ be given. By the formula $H := (f_1, h)$, the map

$$\{z\} \times [0,\epsilon] \xrightarrow{H} \operatorname*{hocolim}_{(T,\rho)\in\mathrm{con}(K)} C_n\big(p^{-1}(V_K(T,\rho))\big) \xrightarrow{g_1} N\mathrm{con}(K)$$

is constant; more precisely, $g_1 \circ H(\{z\} \times [0,\epsilon]) = \{f_1(z)\}$.

How can we find an $\epsilon > 0$ such that H is well defined? Let S be an r–simplex of $|N\mathrm{con}(K)|$, let E be the corresponding open simplex and let $(T_0, \rho_0) \le \cdots \le (T_r, \rho_r)$ be the corresponding element in $N_r\mathrm{con}(K)$. We define

$$Z_S := f_1^{-1}(S) = f^{-1}(g_1^{-1}(S)) \subset Z,$$
$$Z_E := f_1^{-1}(E) = f^{-1}(g_1^{-1}(E)) \subset Z.$$

We take a close look at the map

$$f_2|_{Z_S}\colon Z_S \xrightarrow{f} \operatorname*{hocolim}_{(T,\rho)\in\mathrm{con}(K)} C_n\big(p^{-1}(V_K(T,\rho))\big) \xrightarrow{g_2} C_n(L).$$

First, we note that $f_2(Z_E) \subset C_n\big(p^{-1}(V_K(T_0, \rho_0))\big)$ by definition. By definition (of smooth thickening), $p^{-1}(V_K(T_j, \rho_j))$ is a special open set for every $0 \le j \le r$. In the

spirit of Remark 5.7, we conclude that $f_2(Z_S)$ is also a subset of $C_n\big(p^{-1}(V_K(T_0, \rho_0))\big)$. (For an easier example of this argument, see [11, 2.2.1].) Since $f_2(Z_S) = h(Z_S \times \{0\})$ is compact, there is an $\epsilon_S > 0$ with

$$h(Z_S \times [0, \epsilon_S]) \subset C_n\big(p^{-1}(V_K(T_0, \rho_0))\big).$$

The image of Z is contained in a finite union of open cells of $|N\mathrm{con}(K)|$. Therefore, there is a finite number of simplices S such that Z_S is nonempty. We can define ϵ to be the minimum of all ϵ_S, where the minimum ranges over all simplices S such that Z_S is nonempty. $\qquad\square$

Remark 5.7   Let $U \in \mathbb{R}^n$ be a bounded open subset. Then the mapping cylinder of the inclusion $U \to \mathbb{R}^n$ is not homeomorphic to a subspace of $\mathbb{R}^{n+1}$. The quotient topology equips the mapping cylinder with a different structure. In fact, it is not metrizable [11, 2.2.2].

Corollary 5.8   *The canonical map*

$$\operatorname*{hocolim}_{(T,\rho)\in\mathrm{con}(K)} N_0\mathrm{con}\big(p^{-1}(V_K(T, \rho))\big) \to N_0\mathrm{con}(L)$$

determined by the inclusions is a weak equivalence.

Proof   We remind the reader that for an open set $U \subset K$, we defined $\mathrm{con}(U)$ to be the full subcategory of $\mathrm{con}(K)$ with all objects (T, ρ) such that $V_K(T, \rho)$ is a subset of U. There is a commutative square

$$
\begin{array}{ccc}
\displaystyle\operatorname*{hocolim}_{(T,\rho)\in\mathrm{con}(K)} N_0\mathrm{con}\big(p^{-1}(V_K(T, \rho))\big) & \longrightarrow & N_0\mathrm{con}(L) \\
\downarrow & & \downarrow \\
\displaystyle\operatorname*{hocolim}_{(T,\rho)\in\mathrm{con}(K)} \coprod_n C_n\big(p^{-1}(V_K(T, \rho))\big) & \longrightarrow & \coprod_n C_n(L)
\end{array}
$$

where the vertical arrows are weak equivalences (the left one is induced by a natural transformation). Therefore, we only have to verify that the bottom map is a weak equivalence. But this follows from the fact that the homotopy colimit commutes with disjoint union. $\qquad\square$

Corollary 5.9   *For every $r \geq 0$, the canonical map*

$$\operatorname*{hocolim}_{(T,\rho)\in\mathrm{con}(K)} N_r\mathrm{con}\big(p^{-1}(V_K(T, \rho))\big) \to N_r\mathrm{con}(L)$$

induced by the inclusions is a weak equivalence.

Proof We consider the commutative square

$$\begin{array}{ccc}
\operatorname*{hocolim}_{(T,\rho)\in\operatorname{con}(K)} N_r\mathrm{con}\big(p^{-1}(V_K(T,\rho))\big) & \longrightarrow & N_r\mathrm{con}(L) \\
\downarrow & & \downarrow \\
\operatorname*{hocolim}_{(T,\rho)\in\operatorname{con}(K)} N_0\mathrm{con}\big(p^{-1}(V_K(T,\rho))\big) & \longrightarrow & N_0\mathrm{con}(L)
\end{array}$$

Here the vertical arrows are given by the ultimate target operator and the horizontal arrows are the canonical maps induced by the inclusions. We can check that this is a (strict) pullback square and that the right vertical arrow is a fibration. Since (Top) is a proper model category [7, 13.1.11] and the bottom arrow is a weak equivalence, we conclude that the upper arrow is also a weak equivalence. □

5.3 Boundary recovered

Let M be a manifold with boundary ∂M and let L be a smooth submanifold without boundary. We recover the homotopy type of $M \setminus L$ from the homotopy types of the spaces $M \setminus T$ where T is a finite subset of L. Again, we need to allow thickenings of the finite subsets T and inclusions between them. We recall that for each object (T, ρ) in the configuration category $\mathrm{con}(L)$ of L, there is a corresponding open subset $V_L(T, \rho)$ in L. We can define a contravariant functor ψ from $\mathrm{con}(L)$ to the category of topological spaces by $\psi((T, \rho)) := M \setminus V_L(T, \rho)$. We get a canonical map

$$(5\text{-}2) \qquad M \setminus L \to \operatorname*{holim}_{(T,\rho)\in\operatorname{con}(L)} M \setminus V_L(T,\rho),$$

induced by the inclusions $M \setminus L \to M \setminus V_L(T, \rho)$. We can ask if this map is a weak equivalence. There is also a variant with restricted cardinalities. Let $\mathrm{con}_{\leq n}(L)$ be the full subcategory of $\mathrm{con}(L)$ where the objects are all pairs $(T, \rho) \in \mathrm{con}(L)$ with $|T| \leq n$. Again, we get a canonical map

$$(5\text{-}3) \qquad M \setminus L \to \operatorname*{holim}_{(T,\rho)\in\operatorname{con}_{\leq n}(L)} M \setminus V_L(T,\rho),$$

induced by inclusions. We can ask whether this map is highly connected and whether there is a lower bound for the connectivity. The following theorem, where we use again the notation $\kappa := \dim K$ and $m := \dim M$, answers these questions.

Theorem 5.10 *The canonical map* (5-2) *is a weak equivalence if the following condition holds: there is a compact simplicial complex $K \subset M$ of dimension κ with $\kappa + 3 \leq m$ such that L is a smooth thickening of K in M (see Definition 5.2). In this case, the canonical map* (5-3) *is* $(1+(n+1)(m-\kappa-2))$–*connected.*

Corollary 5.11 *The canonical map*

$$\partial M \to \underset{(T,\rho)\in\mathrm{con}_{\leq n}(M\setminus\partial M)}{\mathrm{holim}} M \setminus V_{M\setminus\partial M}(T,\rho)$$

is a weak equivalence if the following condition holds: there is a compact simplicial complex $K \subset M$ of dimension κ with $\kappa + 3 \leq m$ such that M is a smooth thickening of K (see Definition 5.3). In this case, the canonical map

$$\partial M \to \underset{(T,\rho)\in\mathrm{con}_{\leq n}(M\setminus\partial M)}{\mathrm{holim}} M \setminus V_{M\setminus\partial M}(T,\rho)$$

is $(1+(n+1)(m-\kappa-2))$–connected.

Remark 5.12   In (5-2) and (5-3), the discrete (or ordinary) homotopy limit can be replaced by the continuous homotopy limit without changing the (weak) homotopy type. This can be justified with arguments which are provided in [11, 1.2] (and in Section 2.2).

Remark 5.13   This corollary is a generalization of [11, Theorem 2.1.1]; compare Example 5.4(2). It can be applied in the proof of [14, Theorems 5.2.1 and 5.3.1], whereby we get a weaker condition in these theorems (this will extensively be studied in Section 6).

In order to prove that (5-2) is a weak equivalence, we consider the following diagram, where all arrows are the canonical maps and $p\colon L \to K$ is a nice projection map:

$$
\begin{array}{ccc}
M \setminus L & \xrightarrow{\hspace{3cm}} & \underset{(T,\rho)\in\mathrm{con}(K)}{\mathrm{holim}} M \setminus p^{-1}(V_K(T,\rho)) \\
\downarrow & & \downarrow \\
\underset{(T',\sigma)\in\mathrm{con}(L)}{\mathrm{holim}} \psi(T',\sigma) & \longrightarrow & \underset{(T,\rho)\in\mathrm{con}(K)}{\mathrm{holim}} \underset{\substack{(T',\sigma)\in\mathrm{con}(L) \\ p(V_L(T',\sigma))\subset V_K(T,\rho)}}{\mathrm{holim}} \psi(T',\sigma)
\end{array}
$$

It commutes because both compositions factorize through the ordinary limit and the two maps through the ordinary limit are clearly the same. In Lemma 5.5 we have already shown that the upper horizontal arrow is a weak equivalence. Therefore, the first part of the theorem follows from the next two lemmas.

Lemma 5.14 *The right vertical arrow is a weak equivalence.*

Proof   Let $(T,\rho) \in \mathrm{con}(K)$ be fixed. Since the map under investigation is induced by a natural transformation, it suffices to show that the map

$$M \setminus p^{-1}(V_K(T,\rho)) = M \setminus U \to \underset{(T',\sigma)\in\mathrm{con}(U)}{\mathrm{holim}} \psi(T',\sigma)$$

is a weak equivalence, where, for simplicity, U is defined to be the open set

$$U := p^{-1}(V_K(T, \rho)) \subset L.$$

Note that by definition, the open set U is diffeomorphic to $T \times \mathbb{R}^l$. We consider the composition of maps

$$M \setminus U \to \underset{(T',\sigma)\in\mathrm{con}(U)}{\mathrm{holim}} \psi(T', \sigma) \to \underset{(T',\sigma)\in\mathrm{con}(U)}{\mathrm{holim}} F(V_K(T', \sigma)),$$

where F is the functor from the category $\mathcal{O}(U)$ of open subsets of U to topological spaces given by $F(W) := \mathrm{holim}_{C \subset W} M \setminus C$, where C runs through the compact subsets of W. Note that the category of all compact subsets of $V_K(T', \sigma)$ has a directed subcategory which is homotopy terminal. Therefore, the canonical map $\psi(T', \sigma) \to F(V_K(T', \sigma))$ is a weak equivalence for every $(T', \sigma) \in \mathrm{con}(U)$. Using the homotopy invariance of the homotopy limit, the second map is a weak equivalence. So, in order to prove that the first map is a weak equivalence, we have to show that the composition is a weak equivalence. To this end, we consider another composition

$$M \setminus U \to F(U) \to \underset{W \in \bigcup_k \mathcal{O}k(U)}{\mathrm{holim}} F(W) \to \underset{(T',\sigma)\in\mathrm{con}(U)}{\mathrm{holim}} F(V_K(T', \sigma)).$$

First of all, we note that the two compositions give the same map since both compositions factorize through the ordinary limit and the two maps through the ordinary limit are clearly the same. The first map in this composition is a weak equivalence because the category of all compact subsets of U has a directed subcategory which is homotopy terminal. The third map is a weak equivalence by an argument which we have seen in Lemma 4.9. The second map is a weak equivalence because the open set U is a maximal element in $\bigcup_k \mathcal{O}k(U)$. $\qquad\square$

Lemma 5.15 *The bottom horizontal arrow is a weak equivalence.*

Proof If replace the homotopy limit by the continuous homotopy limit, the source is the totalization of the cosimplicial space $[r] \mapsto \Gamma_r(\Psi)$, where $\Gamma_r(\Psi)$ is the space of all sections from $N_r\mathcal{P}(L)$ to $E^!(\Psi)$. (All notation is introduced in Section 3.2.) If replace the second homotopy limit in the target by the continuous homotopy limit (compare Remark 3.9), the target is isomorphic to the totalization of the cosimplicial space $[r] \mapsto \widetilde{\Gamma}_r(\Psi)$, where $\widetilde{\Gamma}_r(\Psi)$ is the space of all sections from

$$\underset{(T,\rho)\in\mathrm{con}(K)}{\mathrm{hocolim}} N_r\mathrm{con}\big(p^{-1}(V_K(T, \rho))\big)$$

to $E^!(\Psi)$. The bottom horizontal arrow in the above diagram is induced by composition with the map in Corollary 5.9,

$$\operatorname*{hocolim}_{(T,\rho)\in\operatorname{con}(K)} N_r \operatorname{con}\big(p^{-1}(V_K(T,\rho))\big) \to N_r \operatorname{con}(L) \to E^!(\Psi).$$

Using Corollary 5.9, this map is a weak equivalence. $\square$

Now we investigate the case with restricted cardinalities. To this end, we fix $n \geq 0$. Let j be an integer with $0 \leq j \leq n$ be given. There is the following modification of the tube lemma, Lemma 5.6. The canonical map

$$\operatorname*{hocolim}_{(T,\rho)\in\operatorname{con}_{\leq n}(K)} C_j\big(p^{-1}(V_K(T,\rho))\big) \to C_j(L)$$

is a weak equivalence. The proof is the same: The projection map is a microfibration with contractible fibers. Why do we need that $j \leq n$? In the proof of Lemma 5.6 we introduced a homotopy initial subposet, in order to show that the fibers are contractible. In the restricted case, this poset is defined if and only if $j \leq n$.

Using this observation, the proof of the restricted case follows similar lines. In particular, there is a commutative diagram

$$
\begin{array}{ccc}
M \setminus L & \longrightarrow & \operatorname*{holim}_{(T,\rho)\in\operatorname{con}_{\leq n}(K)} M \setminus p^{-1}(V_K(T,\rho)) \\
\downarrow & & \downarrow \\
\operatorname*{holim}_{(T',\sigma)\in\operatorname{con}_{\leq n}(L)} \psi(T',\sigma) & \longrightarrow & \operatorname*{holim}_{(T,\rho)\in\operatorname{con}_{\leq n}(K)} \operatorname*{holim}_{\substack{(T',\sigma)\in\operatorname{con}_{\leq n}(L) \\ p(V_L(T',\sigma))\subset V_K(T,\rho)}} \psi(T',\sigma)
\end{array}
$$

By Theorem 4.1 (and Lemma 5.5), the top horizontal map is $(1+(n+1)(m-k-2))$–connected. Using a modification of Corollary 5.9, the bottom horizontal arrow is a weak equivalence. In order to justify that the right vertical arrow is a weak equivalence, we can use arguments which we have seen in Lemma 5.14.

6 Homotopy automorphisms

Let M be a smooth, compact manifold with boundary.

Definition 6.1 We define the *homotopy link* $\operatorname{holink}(M/\partial M, *)$ of the basepoint in $M/\partial M$ to be the space of paths $\gamma\colon [0,1] \to M/\partial M$ which satisfy the condition

$\gamma^{-1}(\{*\}) = \{0\}$. The topology is the compact–open topology. We define the map

$$q_M \colon \operatorname{holink}(M/\partial M, *) \to M \setminus \partial M$$

by $\gamma \mapsto \gamma(1)$.

Remark 6.2 It is well known that the map q_M is a good homotopical substitute for the inclusion map $\partial M \hookrightarrow M$: if we define Z_M to be the space of paths $\gamma \colon [0, 1] \to M$ which satisfy the condition $\gamma^{-1}(\partial M) = \{0\}$ (with the compact–open topology), we get a homotopy commutative diagram

$$
\begin{array}{ccc}
\operatorname{holink}(M/\partial M, \star) & \xrightarrow{\;\; q_M \;\;} & M \setminus \partial M \\[2pt]
\simeq \big\uparrow & & \big\downarrow \simeq \\[2pt]
Z_M & \xrightarrow{\;\cong\;} \partial M \hookrightarrow & M
\end{array}
$$

Let $\operatorname{homeo}(M)$ be the homeomorphism group of M. Evidently, there is a canonical action of $\operatorname{homeo}(M)$ on the complete diagram. This action extends to an action of the homeomorphism group $\operatorname{homeo}(M \setminus \partial M)$ on q_M. But, unfortunately, the action does not extend to an action of the homeomorphism group $\operatorname{homeo}(M \setminus \partial M)$ on the inclusion map $\partial M \hookrightarrow M$. We are interested in this extension. That is why we introduced the homotopical substitute q_M.

Definition 6.3 Let c be an object in a model category $\mathcal{C}$. We define $\operatorname{haut}(c)$ to be the *space of derived homotopy automorphisms* of c in $\mathcal{C}$, ie $\operatorname{haut}(c)$ is the union of the homotopy invertible path components of the derived mapping space $\mathbb{R}\operatorname{map}(c, c)$. With composition, $\operatorname{haut}(c)$ is a grouplike topological or simplicial monoid. (For a suitable definition of simplicial mapping spaces, we follow [4].)

Note that the map q_M can be regarded as a functor from the totally ordered set $\{0, 1\}$ to the category of topological spaces. The category of such functors has well-known standard model category structures. If we choose one of them, we can study the space of derived homotopy automorphisms $\operatorname{haut}(q_M)$ of q_M. In particular, since $\operatorname{homeo}(M \setminus \partial M)$ acts on q_M, each homeomorphism of $M \setminus \partial M$ determines a (derived) homotopy automorphism of q_M. Therefore, we get a map

$$(6\text{-}1) \qquad\qquad B\operatorname{homeo}(M \setminus \partial M) \to B\operatorname{haut}(q_M)$$

of classifying spaces.

Let Fin be the category of finite sets and maps between them. The nerve $N\mathrm{Fin}$ is a simplicial set. We introduced the Riemannian model of the configuration category $\mathrm{con}(M \setminus \partial M)$. The nerve of this category is a simplicial space over $N\mathrm{Fin}$.

Definition 6.4 Let X be a simplicial space over $N\mathrm{Fin}$. We define $\mathrm{haut}_{N\mathrm{Fin}}(X)$ to be the *space of derived homotopy automorphisms* of X over $N\mathrm{Fin}$, ie $\mathrm{haut}(X)$ is the union of the homotopy invertible path components of the derived mapping space $\mathbb{R}\mathrm{map}_{N\mathrm{Fin}}(X, X)$ of X over $N\mathrm{Fin}$. (If an introduction to derived mapping spaces of simplicial spaces is needed, we refer the reader to [14, Section 3].) With composition, $\mathrm{haut}_{N\mathrm{Fin}}(X)$ is a grouplike topological or simplicial monoid.

If we use the particle model [2, Section 3.1; 14, Section 1] of the configuration category $\mathrm{con}(M \setminus \partial M)$, it is easy to see that each homeomorphism of $M \setminus \partial M$ determines a (derived) homotopy automorphism of the nerve of $\mathrm{con}(M \setminus \partial M)$ over $N\mathrm{Fin}$.

Particle model In this model, the space of objects of the configuration category $\mathrm{con}(M \setminus \partial M)$ is

$$\coprod_{k \geq 0} \mathrm{emb}(\underline{k}, M \setminus \partial M).$$

A morphism from $f \in \mathrm{emb}(\underline{k}, M \setminus \partial M)$ to $g \in \mathrm{emb}(\underline{l}, M \setminus \partial M)$ is a map $v \colon \underline{k} \to \underline{l}$ and a homotopy

$$(\gamma_t)_{t \in [0,a]} \colon \underline{k} \to M \setminus \partial M$$

from f to gv which satisfies the *stickiness condition*: if $\gamma_s(b_1) = \gamma_s(b_2)$ for $s \in [0, a]$ and $b_1, b_2 \in \underline{k}$, then $\gamma_t(b_1) = \gamma_t(b_2)$ for all $t \in [s, a]$. Therefore, the space of morphisms of the configuration category $\mathrm{con}(M \setminus \partial M)$ in the particle model is

$$\coprod_{k,l \geq 0,\, v \colon \underline{k} \to \underline{l}} \Lambda(v).$$

Here $\Lambda(v)$ is the space of all triples (f, g, γ) where $f \in \mathrm{emb}(\underline{k}, M \setminus \partial M)$, $g \in \mathrm{emb}(\underline{l}, M \setminus \partial M)$ and γ is a homotopy from f to gv which satisfies the stickiness condition. The Riemannian model of the configuration category and the particle model are equivalent [2, Section 3.2].

Using the particle model of the configuration category $\mathrm{con}(M \setminus \partial M)$, there is an inclusion of topological grouplike monoids from $\mathrm{homeo}(M \setminus \partial M)$ to $\mathrm{haut}_{N\mathrm{Fin}}(\mathrm{con}(M \setminus \partial M))$. We get a map of classifying spaces

$$(6\text{-}2) \qquad\qquad B\mathrm{homeo}(M \setminus \partial M) \to B\mathrm{haut}_{N\mathrm{Fin}}(\mathrm{con}(M \setminus \partial M)).$$

Now we can ask whether the map (6-1) has a factorization through the map (6-2).

Theorem 6.5 *We assume that the following condition holds: there is a compact simplicial complex $K \subset M$ of dimension κ with $\kappa + 3 \leq m$ such that M is a smooth thickening (see Definition 5.3) of K. Then the broken arrow in the homotopy commutative diagram*

$$
\begin{array}{ccc}
B\mathrm{homeo}(M \setminus \partial M) & \xrightarrow{\text{(6-1)}} & B\mathrm{haut}(q_M) \\
\| & & \Big\uparrow \\
B\mathrm{homeo}(M \setminus \partial M) & \xrightarrow{\text{(6-2)}} & B\mathrm{haut}_{N\mathrm{Fin}}(\mathrm{con}(M \setminus \partial M))
\end{array}
$$

can be supplied.

Using Corollary 5.11, the proof is equal to that of [14, Theorem 5.2.1]. There is also a variant with restricted cardinalities. Following [14, 5.3], we need a Postnikov decomposition of the map q_M. It is well known that for each integer $a \geq 0$, there is a decomposition

$$\partial M \to \wp_a \partial M \to M$$

of the inclusion map $\partial M \hookrightarrow M$ such that the homotopy groups of $\wp_a \partial M$ are zero in dimension $\geq a + 2$ and equal to the homotopy groups of ∂M in dimension $\leq a + 1$. ($\wp_a \partial M$ is obtained from ∂M, as a space over M, by killing the relative homotopy groups of $\partial M \to M$ in dimensions $\geq a + 2$.) By analogy with this construction, there is a decomposition

$$\mathrm{holink}(M/\partial M, *) \to \wp_a(q_M) \to M \setminus \partial M$$

of the map q_M, where $\wp_a(q_M)$ has the same properties as $\wp_a \partial M$.

Theorem 6.6 *We assume that the following condition holds: there is a compact simplicial complex $K \subset M$ of dimension κ with $\kappa + 3 \leq m$ such that M is a smooth thickening (see Definition 5.3) of K. Then the broken arrow in the homotopy commutative diagram*

$$
\begin{array}{ccc}
B\mathrm{homeo}(M \setminus \partial M) & \xrightarrow{\text{action}} & B\mathrm{haut}(\wp_{(j+1)(m-\kappa-2)}(q_M)) \\
\| & & \Big\uparrow \\
B\mathrm{homeo}(M \setminus \partial M) & \xrightarrow{\text{action}} & B\mathrm{haut}_{N\mathrm{Fin}}(\mathrm{con}_{\leq j}(M \setminus \partial M))
\end{array}
$$

can be supplied. Here the two action maps are the maps (6-1) and (6-2) applied to the restricted case.

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