Algebraic & Geometric Topology 19 (2019) 1341–1358

The topology of arrangements of ideal type

NILS AMEND
GERHARD RÖHRLE

In 1962, Fadell and Neuwirth showed that the configuration space of the braid arrangement is aspherical. Having generalized this to many real reflection groups, Brieskorn conjectured this for all finite Coxeter groups. This in turn follows from Deligne's seminal work from 1972, where he showed that the complexification of every real simplicial arrangement is a $K(\pi, 1)$–arrangement.

We study the $K(\pi, 1)$–property for a certain class of subarrangements of Weyl arrangements, the so-called arrangements of ideal type $\mathscr{A}_{\mathcal{I}}$. These stem from ideals $\mathcal{I}$ in the set of positive roots of a reduced root system. We show that the $K(\pi, 1)$–property holds for all arrangements $\mathscr{A}_{\mathcal{I}}$ if the underlying Weyl group is classical and that it extends to most of the $\mathscr{A}_{\mathcal{I}}$ if the underlying Weyl group is of exceptional type. Conjecturally this holds for all $\mathscr{A}_{\mathcal{I}}$. In general, the $\mathscr{A}_{\mathcal{I}}$ are neither simplicial nor is their complexification of fiber type.

14N20, 20F55, 52C35; 13N15

1 Introduction and results

By fundamental work of Fadell and Neuwirth [9], Brieskorn [4] and Deligne [8], all Coxeter arrangements are $K(\pi, 1)$–arrangements, ie the complements of their complexifications are aspherical spaces.

While Coxeter arrangements are well studied, their subarrangements are considerably less well understood. In this paper we study the topology of the complements of certain arrangements which are associated with ideals in the set of positive roots of a reduced root system, so-called *arrangements of ideal type* $\mathscr{A}_{\mathcal{I}}$ (Definition 1.1); see Sommers and Tymoczko [19, Section 11]. We show that a combinatorial property introduced by Röhrle [18, Condition 1.10] combined with Terao's fibration theorem [21] gives an inductive method to show that a large class of (the complexifications of) the arrangements of ideal type $\mathscr{A}_{\mathcal{I}}$ are indeed $K(\pi, 1)$–arrangements. This inductive technique was used in [18] to show that many of the arrangements $\mathscr{A}_{\mathcal{I}}$ are inductively

Published: 21 May 2019 DOI: 10.2140/agt.2019.19.1341

free. In general a subarrangement of a Weyl arrangement need not be $K(\pi, 1)$; eg see Example 2.7.

Let Φ be an irreducible, reduced root system and let Φ^+ be the set of positive roots with respect to some set of simple roots Π. An *(upper) order ideal*, or simply *ideal* for short, of Φ^+, is a subset $\mathcal{I}$ of Φ^+ satisfying the following condition: if $\alpha \in \mathcal{I}$ and $\beta \in \Phi^+$ are such that $\alpha + \beta \in \Phi^+$, then $\alpha + \beta \in \mathcal{I}$. Recall the standard partial ordering $\preceq$ on Φ, $\alpha \preceq \beta$ provided $\beta - \alpha$ is a $\mathbb{Z}_{\geq 0}$–linear combination of positive roots or $\beta = \alpha$. Then $\mathcal{I}$ is an ideal in Φ^+ if and only if, whenever $\alpha \in \mathcal{I}$ and $\beta \in \Phi^+$ with $\alpha \preceq \beta$, also $\beta \in \mathcal{I}$.

Let β be in Φ^+. Then $\beta = \sum_{\alpha \in \Pi} c_\alpha \alpha$ for $c_\alpha \in \mathbb{Z}_{\geq 0}$. The *height* of β is defined to be $\mathrm{ht}(\beta) = \sum_{\alpha \in \Pi} c_\alpha$. Let $\mathcal{I} \subseteq \Phi^+$ be an ideal and let

$$\mathcal{I}^c := \Phi^+ \setminus \mathcal{I}$$

be its complement in Φ^+.

Following [19, Section 11], we associate with an ideal $\mathcal{I}$ in Φ^+ the arrangement consisting of all hyperplanes with respect to the roots in $\mathcal{I}^c$. Let $\mathscr{A}(\Phi)$ be the *Weyl arrangement* of Φ, ie $\mathscr{A}(\Phi) = \{H_\alpha \mid \alpha \in \Phi^+\}$, where H_α is the hyperplane in the Euclidean space $V = \mathbb{R} \otimes \mathbb{Z}\Phi$ orthogonal to the root α.

Definition 1.1 [19, Section 11] Let $\mathcal{I} \subseteq \Phi^+$ be an ideal. The *arrangement of ideal type* associated with $\mathcal{I}$ is the subarrangement $\mathscr{A}_\mathcal{I}$ of $\mathscr{A}(\Phi)$ defined by

$$\mathscr{A}_\mathcal{I} := \{H_\alpha \mid \alpha \in \mathcal{I}^c\}.$$

It was shown by Sommers and Tymoczko [19, Theorem 11.1] that each $\mathscr{A}_\mathcal{I}$ is free if the root system is classical or of type G_2. The general case was settled in a uniform manner for all types by Abe, Barakat, Cuntz, Hoge and Terao [1, Theorem 1.1]. The nonzero exponents are given by the dual of the height partition of the roots in $\mathcal{I}^c$.

Note that the complement $\mathcal{I}^c$ forms a lower ideal in Φ^+. Thus in particular, in type A_n the arrangements of ideal type $\mathscr{A}_\mathcal{I}$ are graphic arrangements corresponding to chordal graphs on $n+1$ vertices. The freeness of the latter is due to Stanley [20, Proposition 2.8].

In [2, Corollary 5.15], Barakat and Cuntz showed that every Weyl arrangement $\mathscr{A}(\Phi)$ is *inductively free*. It was shown in [18] that the free subarrangements $\mathscr{A}_\mathcal{I}$ of $\mathscr{A}(\Phi)$ are also inductively free with possible exceptions only in type E_8. The remaining instances in type E_8 were settled only recently by Cuntz, Röhrle and Schauenburg [7].

Note that if $\mathcal{I} = \varnothing$, then $\mathscr{A}_\mathcal{I} = \mathscr{A}(\Phi)$ is just the reflection arrangement of Φ and so $\mathscr{A}_\varnothing$ is $K(\pi,1)$ by Deligne's result. So we may assume that $\mathcal{I} \neq \varnothing$.

Next we describe a combinatorial condition for an ideal $\mathcal{I} \subseteq \Phi^+$ from [18]. Using induction and Terao's fibration theorem [21], it allows us to show that a large class of arrangements of ideal type consists of $K(\pi,1)$ arrangements. Let Φ_0 be a (standard) parabolic subsystem of Φ and let

$$\Phi_0^c := \Phi^+ \setminus \Phi_0^+,$$

the set of positive roots in the ambient root system which do not lie in the smaller one.

Condition 1.2 [18, Condition 1.10] Let $\mathcal{I} \neq \varnothing$ be an ideal in Φ^+ and let Φ_0 be a maximal parabolic subsystem of Φ such that $\Phi_0^c \cap \mathcal{I}^c \neq \varnothing$. Assume that, firstly, $\Phi_0^c \cap \mathcal{I}^c$ is linearly ordered with respect to $\preceq$, so that there is a unique root of every occurring height in $\Phi_0^c \cap \mathcal{I}^c$, and, secondly, for any $\alpha \neq \beta$ in $\Phi_0^c \cap \mathcal{I}^c$, there is a $\gamma \in \Phi_0^+$ such that α, β and γ are linearly dependent.

The instances when this condition is satisfied have been determined in [18].

Our first main result shows that Condition 1.2 entails the $K(\pi,1)$–property for the associated arrangement of ideal type $\mathscr{A}_\mathcal{I}$.

Theorem 1.3 *Let $\mathcal{I} \neq \varnothing$ be an ideal in Φ^+ and let Φ_0 be a maximal parabolic subsystem of Φ such that Condition 1.2 is satisfied. Then $\mathscr{A}_\mathcal{I}$ is $K(\pi,1)$.*

Specifically, this is the case if and only if one of the following holds:

(i) *Φ is of type A_n, B_n or C_n for $n \geq 2$ or G_2 and $\mathcal{I}$ is any ideal in Φ^+;*

(ii) *Φ is of type D_n for $n \geq 4$ and either $\mathcal{I}^c$ does not contain both $e_1 \pm e_n$ or $\mathcal{I}$ is generated by the root $e_{n-2} + e_{n-1}$;*

(iii) *Φ is of type F_4, E_6, E_7 or E_8 and $\mathcal{I}$ is as in* [18, Section 4].

In addition we use Thom's first isotopy lemma to construct explicit locally trivial fibrations in each of the remaining instances in type D_n not covered in Theorem 1.3(ii), ie when Φ is of type D_n and $\mathcal{I}^c$ does contain both $e_1 \pm e_n$. Combined with Theorem 1.3, this gives our second main result.

Theorem 1.4 *For Φ of classical type and $\mathcal{I}$ an ideal in Φ^+, we have that $\mathscr{A}_\mathcal{I}$ is $K(\pi,1)$.*

In Table 1 we present the number of all arrangements of ideal type for each exceptional type in the first row. In the second row, we list the number of all $\mathscr{A}_{\mathcal{I}}$ when $\mathcal{I}$ satisfies Condition 1.2 with respect to a suitable parabolic subsystem; see [18, Table 1]. Thus, in these instances $\mathscr{A}_{\mathcal{I}}$ is $K(\pi, 1)$, by Theorem 1.3(iii).

Φ	E_6	E_7	E_8	F_4	G_2
all $\mathscr{A}_{\mathcal{I}}$	833	4160	25080	105	8
aspherical $\mathscr{A}_{\mathcal{I}}$	771	3433	18902	85	8

Table 1: Aspherical $\mathscr{A}_{\mathcal{I}}$ for exceptional Φ from Theorem 1.3

It is evident from Table 1 that with the possible exception of a relatively small number of cases in the exceptional types, all $\mathscr{A}_{\mathcal{I}}$ are $K(\pi, 1)$. The number of possible exceptions in types F_4, E_6, E_7 and E_8 are 20, 62, 727 and 6178, respectively. Thus, Theorems 1.3 and 1.4 give strong evidence for the following conjecture:

Conjecture 1.5 *Let Φ be a reduced root system with Weyl arrangement $\mathscr{A}(\Phi)$. Then any subarrangement of ideal type $\mathscr{A}_{\mathcal{I}}$ of $\mathscr{A}(\Phi)$ is a $K(\pi, 1)$–arrangement.*

Remarks 1.6 (i) Let Φ be of type F_4 and let $\mathcal{I}$ be the ideal generated by the root 0122 of height 5. Although $\mathcal{I}$ is not covered by Theorem 1.3, it turns out that $\mathscr{A}_{\mathcal{I}}$ is simplicial (see Cuntz and Heckenberger [6]), and so $\mathscr{A}_{\mathcal{I}}$ is $K(\pi, 1)$.

(ii) Since the $\mathscr{A}_{\mathcal{I}}$ in type E_6 and type E_7 are localizations of arrangements of ideal type in type E_8, thanks to Remark 2.2, the open cases in Conjecture 1.5 reduce to the ones in type F_4 and E_8.

Remark 1.7 It is worth emphasizing that Theorems 1.3 and 1.4 provide new examples for $K(\pi, 1)$–arrangements that are neither of fiber type nor simplicial. For instance, one can check that none of the nonsupersolvable arrangements $\mathscr{A}_{\mathcal{I}}$ in type E_6 that are shown to be $K(\pi, 1)$ by Theorem 1.3 are simplicial. See also Examples 3.6.

Note that in type D_n and type B_n, some of the arrangements $\mathscr{A}_{\mathcal{I}}$ that contain the full braid arrangement of A_{n-1} as a subarrangement are shown to be $K(\pi, 1)$ by Falk and Proudfoot [10, Section 5].

For general information about arrangements, Weyl groups and root systems, we refer the reader to Bourbaki [3] and Orlik and Terao [16].

2 Preliminaries

2.1 Hyperplane arrangements

Let $V = \mathbb{C}^n$ be an n–dimensional complex vector space. A *hyperplane arrangement* is a pair $(\mathscr{A}, V)$, where $\mathscr{A}$ is a finite collection of hyperplanes in V. Usually, we simply write $\mathscr{A}$ in place of $(\mathscr{A}, V)$.

The *lattice* $L(\mathscr{A})$ of $\mathscr{A}$ is the set of subspaces of V of the form $H_1 \cap \cdots \cap H_i$, where $\{H_1, \ldots, H_i\}$ is a subset of $\mathscr{A}$. For $X \in L(\mathscr{A})$, we have two associated arrangements: firstly $\mathscr{A}_X := \{H \in \mathscr{A} \mid X \subseteq H\} \subseteq \mathscr{A}$, the *localization of* $\mathscr{A}$ *at* X, and, secondly, the *restriction of* $\mathscr{A}$ *to* X, $(\mathscr{A}^X, X)$, where $\mathscr{A}^X := \{X \cap H \mid H \in \mathscr{A} \setminus \mathscr{A}_X\}$. The lattice $L(\mathscr{A})$ is a partially ordered set by reverse inclusion: $X \leq Y$ provided $Y \subseteq X$ for $X, Y \in L(\mathscr{A})$.

Throughout, we only consider arrangements $\mathscr{A}$ such that $0 \in H$ for each H in $\mathscr{A}$. These are called *central*. In that case the *center* $T(\mathscr{A}) := \bigcap_{H \in \mathscr{A}} H$ of $\mathscr{A}$ is the unique maximal element in $L(\mathscr{A})$ with respect to the partial order. A *rank* function on $L(\mathscr{A})$ is given by $r(X) := \operatorname{codim}_V(X)$. The *rank* of $\mathscr{A}$ is defined as $r(\mathscr{A}) := r(T(\mathscr{A}))$.

2.2 $K(\pi, 1)$–arrangements

A member X in $L(\mathscr{A})$ is said to be *modular* provided $X + Y \in L(\mathscr{A})$ for every $Y \in L(\mathscr{A})$ [16, Corollary 2.26]. The following is an immediate consequence of Terao's work [21] (see also [16, Section 5.5]). Indeed, $\mathscr{A}$ is strictly linearly fibered (see Definition 2.3) if and only if $L(\mathscr{A})$ admits a modular element of rank $r-1$; see [21, Corollary 2.14] (see also [16, Corollary 5.112]).

Lemma 2.1 *Let $\mathscr{A}$ be a complex arrangement of rank r. Suppose that $X \in L(\mathscr{A})$ is modular of rank $r-1$. If $\mathscr{A}_X$ is $K(\pi,1)$, then so is $\mathscr{A}$.*

Remark 2.2 Thanks to an observation by Oka, if the complex arrangement $\mathscr{A}$ is $K(\pi,1)$, then so is every localization $\mathscr{A}_X$ for X in $L(\mathscr{A})$; eg see [17, Lemma 1.1].

There is a standard construction for $K(\pi,1)$–arrangements using locally trivial fibrations with $K(\pi,1)$–spaces as bases and fibers. The long exact sequence in homotopy theory then gives that $\mathcal{M}(\mathscr{A})$ is a $K(\pi,1)$–space; eg see [16, Theorem 5.9]. We recall two basic definitions due to Falk and Randell [11]; also see [16, Definitions 5.10 and 5.11].

Definition 2.3 An n–arrangement $\mathscr{A}$ is called *strictly linearly fibered* if, after a suitable linear change of coordinates, the restriction of the projection of $\mathcal{M}(\mathscr{A})$ to the first $n-1$ coordinates is a locally trivial fibration whose base space is the complement of an arrangement in $\mathbb{C}^{n-1}$ and whose fiber is the complex line $\mathbb{C}$ with finitely many points removed.

Definition 2.4 (i) The 1–arrangement $(\{0\}, \mathbb{C})$ is of *fiber type*.

(ii) For $n \geq 2$, the n–arrangement $\mathscr{A}$ is of *fiber type* if $\mathscr{A}$ is strictly linearly fibered with base $\mathcal{M}(\mathscr{B})$, where $\mathscr{B}$ is an $(n-1)$–arrangement of fiber type.

A repeated application of the homotopy exact sequence shows that a fiber-type arrangement $\mathscr{A}$ is $K(\pi, 1)$; eg see [16, Proposition 5.12].

The following important tool for proving that a given map is a locally trivial fibration is due to Thom [22]; see also [15].

Theorem 2.5 (Thom's first isotopy lemma) *Let M and P be smooth manifolds, $f\colon M \to P$ a smooth mapping and $S \subseteq M$ a closed subset which admits a Whitney stratification $\mathscr{S}$. Suppose $f|_S\colon S \to P$ is proper and $f|_X\colon X \to P$ is a submersion for each stratum $X \in \mathscr{S}$. Then $f|_S\colon S \to P$ is a locally trivial fibration and, in particular, $f|_X\colon X \to P$ is a locally trivial fibration for all $X \in \mathscr{S}$.*

Let $\mathscr{B}_n$ be the reflection arrangement of the hyperoctahedral group of type B_n. In the following example we consider a fiber-type subarrangement $\mathscr{J}_n$ of $\mathscr{B}_n$ which is used in Section 4 in the proof of Theorem 1.4.

Example 2.6 The subarrangement $\mathscr{J}_n$ of $\mathscr{B}_n$ is obtained by removing the anti-diagonals from $\mathscr{B}_n$. So $\mathscr{J}_n$ is the union of the rank n Boolean arrangement and the braid arrangement $\mathscr{A}_{n-1}$, ie $\mathscr{J}_n$ has defining polynomial

$$Q(\mathscr{J}_n) := \prod_{i=1}^{n} x_i \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

One easily checks that $\mathscr{J}_n$ is of fiber type, eg by projecting onto the first $n-1$ coordinates and using induction on n.

We observe that the fiber-type arrangement $\mathscr{J}_n$ was already used by Brieskorn in his proof of the asphericity of the Coxeter arrangement in type D_n; see [4; 11, Section 5]. Also note that $\mathscr{J}_n$ is the irreducible version of the braid arrangement of type A_n. It is isomorphic to the restriction $\mathscr{A}(A_n)^X$, where $X = \ker(x_0)$; the hyperplane $\ker x_i$ in $\mathscr{J}_n$ then corresponds to the hyperplane $\ker(x_0 - x_i)$ in $\mathscr{A}(A_n)$.

The following related example shows that in general a subarrangement of a Coxeter arrangement need not be $K(\pi, 1)$ (nor free):

Example 2.7 Let $\mathscr{B}_n$ be as above and let $\mathscr{A}_{n-1}$ be its subarrangement consisting of the braid arrangement of type A_{n-1}. Let

$$\mathscr{K}_n := \mathscr{B}_n \setminus \mathscr{A}_{n-1}$$

be the complement of $\mathscr{A}_{n-1}$ in $\mathscr{B}_n$. As opposed to the subarrangement $\mathscr{J}_n$ of $\mathscr{B}_n$ from Example 2.6, rather than removing the antidiagonal hyperplanes from $\mathscr{B}_n$, for $\mathscr{K}_n$ we remove all the diagonals instead. Thus, $\mathscr{K}_n$ has defining polynomial

$$Q(\mathscr{K}_n) = \prod_{i=1}^{n} x_i \prod_{1 \le i < j \le n} (x_i + x_j).$$

We show by induction on n that $\mathscr{K}_n$ is not $K(\pi, 1)$ for $n \ge 3$. Owing to [12, (3.12)], $\mathscr{K}_3$ is not $K(\pi, 1)$. Now suppose that $n > 3$ and that the statement holds for $\mathscr{K}_{n-1}$. Let $X := \bigcap_{i=1}^{n-1} \ker x_i$. Then one readily checks that

$$(\mathscr{K}_n)_X \cong \mathscr{K}_{n-1}.$$

It follows from our induction hypothesis and Remark 2.2 that also $\mathscr{K}_n$ fails to be $K(\pi, 1)$.

In [12, (3.12)], Falk and Randell also observe that $\mathscr{K}_3$ is not free. Accordingly, by the argument above along with [16, Theorem 4.37], we see that $\mathscr{K}_n$ is not free for all $n \ge 3$.

So, while the construction of $\mathscr{K}_n$ is quite similar to that of $\mathscr{J}_n$, its combinatorial, algebraic and topological properties differ sharply from those of $\mathscr{J}_n$.

3 Proof of Theorem 1.3

Let Φ be a reduced root system of rank n with Weyl group W and *reflection arrangement* $\mathscr{A} = \mathscr{A}(\Phi) = \mathscr{A}(W)$. Let Φ^+ be the set of positive roots with respect to some set of simple roots Π of Φ. For Π_0 a proper subset of Π, the (*standard parabolic*) *subsystem* of Φ generated by Π_0 is $\Phi_0 := \mathbb{Z}\Pi_0 \cap \Phi$; see [3, Chapter VI, Section 1.7]. Define $\Phi_0^+ := \Phi_0 \cap \Phi^+$, the set of positive roots of Φ_0 with respect to Π_0. If the rank of Φ_0 is $n-1$, then Φ_0 is said to be *maximal*.

Set $X_0 := \bigcap_{\gamma \in \Phi_0^+} H_\gamma$. Then $\mathscr{A}(\Phi)_{X_0} = \mathscr{A}(\Phi_0)$. Therefore, the reflection arrangement $\mathscr{A}(W_{X_0})$ of the parabolic subgroup W_{X_0} is just $\mathscr{A}(\Phi_0)$, ie Φ_0 is the root system of W_{X_0} (see [16, Theorem 6.27, Corollary 6.28]).

Definition 3.1 Fix a standard parabolic subsystem Φ_0 of Φ. For $\mathcal{I}$ an ideal in Φ^+,

$$\mathcal{I}_0 := \mathcal{I} \cap \Phi_0^+$$

is an ideal in Φ_0^+. Thus,

$$\mathscr{A}_{\mathcal{I}_0} := \{H_\gamma \mid \gamma \in \mathcal{I}_0^c = \Phi_0^+ \setminus \mathcal{I}_0\}$$

is an arrangement of ideal type in $\mathscr{A}(\Phi_0)$, the Weyl arrangement of Φ_0.

Obviously, since $\mathcal{I}_0^c = \Phi_0^+ \setminus \mathcal{I}_0 = \mathcal{I}^c \cap \Phi_0^+ \subseteq \mathcal{I}^c$, we may view $\mathscr{A}_{\mathcal{I}_0}$ as a subarrangement of $\mathscr{A}_{\mathcal{I}}$ rather than as a subarrangement of $\mathscr{A}(\Phi_0)$. Note however, as such, $\mathscr{A}_{\mathcal{I}_0}$ is not of ideal type in $\mathscr{A}$ in general, since $\mathcal{I}_0$ need not be an ideal in Φ^+. We continue by recalling some basic facts from [18].

Lemma 3.2 [18, Lemma 3.1] *Viewing $\mathscr{A}_{\mathcal{I}_0}$ as a subarrangement of $\mathscr{A}_{\mathcal{I}}$, we have $\mathscr{A}_{\mathcal{I}_0} = (\mathscr{A}_{\mathcal{I}})_{X_0}$.*

The next observation shows that Condition 1.2 entails the presence of a modular element in $L(\mathscr{A}_{\mathcal{I}})$ of rank $r(\mathscr{A}_{\mathcal{I}}) - 1$.

Lemma 3.3 [18, Lemma 3.4] *If $\mathcal{I} \subseteq \Phi^+$ and Φ_0 satisfy Condition 1.2, then the center $Z := T((\mathscr{A}_{\mathcal{I}})_{X_0})$ of $(\mathscr{A}_{\mathcal{I}})_{X_0}$ is modular of rank $r(\mathscr{A}_{\mathcal{I}}) - 1$ in $L(\mathscr{A}_{\mathcal{I}})$.*

Observe that X_0 itself need not belong to $L(\mathscr{A}_{\mathcal{I}})$; eg see [18, Example 3.3].

Our next result shows that Condition 1.2 allows us to derive the $K(\pi, 1)$–property for $\mathscr{A}_{\mathcal{I}}$ from that of $\mathscr{A}_{\mathcal{I}_0}$. It is just a consequence of Lemma 2.1.

Corollary 3.4 *Let $\mathcal{I}$ be an ideal in Φ^+ and let Φ_0 be a maximal parabolic subsystem of Φ such that either $\Phi_0^c \cap \mathcal{I}^c = \varnothing$ or Condition 1.2 is satisfied. Then $\mathscr{A}_{\mathcal{I}_0}$ is $K(\pi, 1)$ if and only if $\mathscr{A}_{\mathcal{I}}$ is $K(\pi, 1)$.*

Proof If $\Phi_0^c \cap \mathcal{I}^c = \varnothing$, then $\mathscr{A}_{\mathcal{I}}$ is the product of the empty 1–dimensional arrangement and $\mathscr{A}_{\mathcal{I}_0}$, and so the result is clear. Otherwise, $\mathscr{A}_{\mathcal{I}_0} = (\mathscr{A}_{\mathcal{I}})_{X_0} = (\mathscr{A}_{\mathcal{I}})_Z$, by Lemmas 3.2 and 3.3. Therefore, the forward implication follows from Lemmas 2.1 and 3.3, while the reverse implication is clear by Remark 2.2. □

We note that modular elements of corank 1 were constructed in [10, Lemma 5.4] for certain subarrangements of the reflection arrangement $\mathscr{B}_n$ of the hyperoctahedral group of type B_n that contain the full braid arrangement $\mathscr{A}_{n-1}$ of type A_{n-1}.

Remark 3.5 Let Φ be of type D_n for $n \geq 4$ and let Φ_0 be the standard subsystem of Φ of type D_{n-1}. Here and in Section 4 we use the notation for the positive roots from [3, Section 4.8, Planche IV]. Then $\Phi_0^c = \{e_1 \pm e_j \mid 2 \leq j \leq n\}$. Note that Φ_0^c is not linearly ordered by $\preceq$, as $e_1 \pm e_n$ both have height $n-1$.

Suppose that $\mathcal{I} \neq \varnothing$ fails to satisfy Condition 1.2 (with respect to our fixed Φ_0). This is precisely the case when both $e_1 \pm e_n$ belong to $\mathcal{I}^c$. Then $\mathcal{I}$ consists of roots from Φ^+ each of which admits the root $e_{n-2} + e_{n-1}$ of height 3 as a summand. Otherwise, at least one of $e_1 \pm e_n$ must belong to $\mathcal{I}$, as $\mathcal{I}$ is an ideal in Φ^+. This contradicts the assumption on $\mathcal{I}$. In turn this implies that if $\mathcal{I}_0 = \Phi_0 \cap \mathcal{I}$ is nonempty and fails to satisfy Condition 1.2 with respect to the maximal rank subsystem of Φ_0 of type D_{n-2}, then $\mathcal{I}$ fails to satisfy Condition 1.2 with respect to Φ_0. For, if each root in $\mathcal{I}_0$ admits the root $e_{n-3} + e_{n-2}$ as a summand, then necessarily each root in $\mathcal{I}$ has $e_{n-2} + e_{n-1}$ as a summand.

We conclude that if $\mathcal{I}$ satisfies Condition 1.2 with respect to Φ_0, then $\mathcal{I}_0$ satisfies Condition 1.2 with respect to the subsystem of Φ_0 of type D_{n-2}.

Proof of Theorem 1.3 (i) For Φ of type A_n, B_n or C_n for $n \geq 2$, it follows from [19, Section 7] that for Φ_0 the canonical maximal rank subsystem of type A_{n-1}, B_{n-1} or C_{n-1}, respectively, each $\mathcal{I}$ satisfies Condition 1.2, because irrespective of $\mathcal{I}$, in each case Φ_0^c is linearly ordered by $\preceq$. So the result follows in this instance from induction on the rank, Corollary 3.4 and the fact that central rank 2–arrangements are $K(\pi,1)$; see [16, Proposition 5.6]. The last result also implies that for Φ of type G_2 each arrangement of ideal type is $K(\pi,1)$. The very same inductive argument shows that in all these cases each $\mathscr{A}_{\mathcal{I}}$ is actually supersolvable; see [18, Theorem 1.5], and also [13, Theorems 6.6 and 7.1], where this is proved by different means.

(ii) Now let Φ be of type D_n for $n \geq 4$ and let Φ_0 be the standard subsystem of Φ of type D_{n-1}. We argue by induction on n. For $n = 4$, the result follows from [18, Lemma 6.1]. Indeed, each $\mathscr{A}_{\mathcal{I}}$ which satisfies the hypothesis of the theorem is already supersolvable.

Now suppose that $n \geq 5$ and that the result holds for root systems of type D of smaller rank. If $\mathcal{I}_0 = \Phi_0 \cap \mathcal{I} = \varnothing$, then $\mathscr{A}_{\mathcal{I}_0} = \mathscr{A}(D_{n-1})$. Being simplicial, the latter is $K(\pi,1)$. It follows from Corollary 3.4 that also $\mathscr{A}_{\mathcal{I}}$ is $K(\pi,1)$.

Now suppose that $\mathcal{I}_0 \neq \varnothing$. By Remark 3.5, $\mathcal{I}_0$ satisfies Condition 1.2 and so, by induction, $\mathscr{A}_{\mathcal{I}_0}$ is $K(\pi,1)$. Using Corollary 3.4 again, we conclude that $\mathscr{A}_{\mathcal{I}}$ is also $K(\pi,1)$, as desired.

Now let $\mathcal{I}$ be the ideal in Φ which is generated by $e_{n-2}+e_{n-1}$. Then one easily checks that $\mathcal{I}$ satisfies Condition 1.2 with respect to either one of the two subsystems of type A_{n-1}; see [18, Example 3.9]. So it follows from part (i) and Corollary 3.4 that $\mathscr{A}_{\mathcal{I}}$ is also $K(\pi,1)$ in this instance.

(iii) Now suppose that Φ is of type F_4, E_6, E_7 or E_8. All instances when $\mathcal{I}$ satisfies Condition 1.2 with respect to a suitably chosen maximal-rank subsystem Φ_0 are discussed in detail in [18, Section 4]. Perusing the arguments and in particular the data in Tables 6–9 in [18, Section 4], one checks that in each instance either $\mathcal{I}_0 = \varnothing$, or $\mathcal{I}_0 \neq \varnothing$ satisfies Condition 1.2 with respect to Φ_0^+. In the first instance we have $\mathscr{A}_{\mathcal{I}_0} = \mathscr{A}(\Phi_0)$, which is simplicial, and so it is $K(\pi,1)$. In the second instance, $\mathscr{A}_{\mathcal{I}_0}$ is $K(\pi,1)$ by induction. In both cases it follows from Corollary 3.4 that also $\mathscr{A}_{\mathcal{I}}$ is $K(\pi,1)$, as claimed. □

We illustrate the inductive arguments in the proof of Theorem 1.3(iii) in the following examples.

Examples 3.6 (a) Let Φ be of type E_6 and let $\mathcal{I}$ be the ideal generated by the root $\substack{00111\\0}$ of height 3. Then, according to the last entry for E_6 in [18, Table 6], $\mathcal{I}$ together with the subsystem Φ_0 of type D_5 satisfy Condition 1.2. Since $\mathcal{I}_0 = \varnothing$, $\mathscr{A}_{\mathcal{I}_0} = \mathscr{A}(\Phi_0)$ is the full reflection arrangement of type D_5, which is $K(\pi,1)$. Thus, so is $\mathscr{A}_{\mathcal{I}}$, by Corollary 3.4.

(b) Next consider Φ of type E_7 and let $\mathcal{I}$ be the ideal generated by the root $\substack{001110\\0}$ of height 3. Then according to the next to last entry for E_7 in [18, Table 6], $\mathcal{I}$ together with the subsystem Φ_0 of type E_6 satisfy Condition 1.2. Now $\mathcal{I}_0$ is just the ideal in E_6 considered in part (a). Consequently, $\mathscr{A}_{\mathcal{I}_0}$ is $K(\pi,1)$. But then so is $\mathscr{A}_{\mathcal{I}}$, again by Corollary 3.4.

(c) Finally, let Φ be of type E_8 and let $\mathcal{I}$ be the ideal generated by the root $\substack{0011100\\0}$ of height 3. Thanks to the data in the fifth row for E_8 in [18, Table 6], $\mathcal{I}$ together with the subsystem Φ_0 of type E_7 satisfy Condition 1.2. As $\mathcal{I}_0$ is the ideal in E_7 considered in part (b), we have that $\mathscr{A}_{\mathcal{I}_0}$ is $K(\pi,1)$ and so is $\mathscr{A}_{\mathcal{I}}$, thanks to Corollary 3.4.

Note that none of the three arrangements of ideal type $\mathscr{A}_{\mathcal{I}}$ considered in Examples 3.6 is supersolvable (see [13, Lemma 6.2]) and none of them is simplicial.

4 Proof of Theorem 1.4

Thanks to Theorem 1.3, Theorem 1.4 follows once the outstanding instances in type D_n not covered in Theorem 1.3(ii) are resolved. Accordingly, these are the instances when $\mathcal{I}$ consists of roots from Φ^+ each of which admits the root $e_{n-2}+e_{n-1}$ of height 3 as a summand, by Remark 3.5. In addition, by the proof of Theorem 1.3, we need not consider the case when $\mathcal{I}$ is the ideal in Φ which is generated by $e_{n-2}+e_{n-1}$. We list the different cases we need to consider below. We distinguish three different types of such ideals $\mathcal{I}$ according to their generators. In the first two instances, each $\mathcal{I}$ is generated by just a single root and by two in the third case:

(I) $0\dots01\dots1{}^1_1 = e_r+e_{n-1}$ for $1\le r<n-2$. Here r is the first position with 1 as coefficient.

(II) $0\dots01\dots12\dots12{}^1_1 = e_s+e_t$, where $1\le s<t<n-1$. Here s is the first position with a coefficient 1 and t is the first position labeled with 2.

(III) $0\dots01\dots1{}^1_1 = e_r+e_{n-1}$ for $1\le r<n-2$ and $0\dots01\dots12\dots12{}^1_1 = e_s+e_t$, where $1\le s<t<n-1$ and $r<s$. Note that the two roots are not comparable, since $r<s$.

In the following we give explicit locally trivial fibrations of the complements in each of the three cases above. First, we consider spaces that are going to serve as our bases for the locally trivial fibrations in these three instances. Recall the fiber-type subarrangement $\mathscr{J}_n$ of $\mathscr{B}_n$ from Example 2.6. In the following three lemmas, we exhibit three classes of subarrangements of $\mathscr{J}_n$ that are still of fiber type.

Lemma 4.1 *For $1\le r<n-1$ fixed, the n–arrangement*

$$\mathscr{J}_n(r) := \mathscr{J}_n \setminus \{\ker(x_i-x_j) \mid 1\le i\le r<j\le n\}$$

is of fiber type.

Proof We distinguish two cases: First, assume $r=1$. Then the projection

$$\pi\colon \mathbb{C}^n\to\mathbb{C}^{n-1},\quad (z_1,\dots,z_n)\mapsto(z_2,\dots,z_n),$$

induces a locally trivial fibration $\tilde{\pi}\colon \mathcal{M}(\mathscr{J}_n(r))\to\mathcal{M}(\mathscr{J}_{n-1})$ with fiber the complex plane with one point removed.

Now assume that $r>1$. Then we have $\mathscr{J}_n(r)=\mathscr{J}_r\times\mathscr{J}_{n-r}$.

Thus, in both cases, $\mathscr{J}_n(r)$ is of fiber type. $\square$

Lemma 4.2 *For $1 \le s < t < n$ fixed, the n–arrangement*

$$\mathscr{J}_n(s,t) := \mathscr{J}_n \setminus \{\ker(x_i - x_j) \mid 1 \le i \le s < j \le t\}$$

is of fiber type.

Proof As in the proof of Lemma 4.1, let $\pi\colon \mathbb{C}^n \to \mathbb{C}^{n-1}$ be the projection

$$(z_1, \dots, z_n) \mapsto (z_2, \dots, z_n).$$

First, assume $s = 1$. Then π induces a locally trivial fibration

$$\tilde{\pi}\colon \mathcal{M}(\mathscr{J}_n(1,t)) \to \mathcal{M}(\mathscr{J}_{n-1})$$

with fiber the complex plane with $n - t + 1$ points removed. So $\mathscr{J}_n(1,t)$ is of fiber type.

Now assume $s > 1$. Then π induces a locally trivial fibration

$$\tilde{\pi}\colon \mathcal{M}(\mathscr{J}_n(s,t)) \to \mathcal{M}(\mathscr{J}_{n-1}(s-1,t-1))$$

with fiber the complex plane with $n - t + s$ points removed. Thus, $\mathscr{J}_n(s,t)$ is of fiber type by induction on s. □

Lemma 4.3 *For $1 \le r < s < t < n$ fixed, the n–arrangement*

$$\mathscr{J}_n(r,s,t) := \mathscr{J}_n \setminus \{\ker(x_i - x_j) \mid 1 \le i \le r < j \le n \text{ or } r < i \le s < j \le t\}$$

is of fiber type.

Proof Take $\pi\colon \mathbb{C}^n \to \mathbb{C}^{n-1}$ to be the projection

$$(z_1, \dots, z_n) \mapsto (z_1, \dots, z_{s-1}, z_{s+1}, \dots, z_n).$$

If $s > r + 1$, this projection induces a locally trivial fibration

$$\tilde{\pi}\colon \mathcal{M}(\mathscr{J}_n(r,s,t)) \to \mathcal{M}(\mathscr{J}_{n-1}(r,s-1,t-1)).$$

If $s = r + 1$, it induces a locally trivial fibration

$$\tilde{\pi}\colon \mathcal{M}(\mathscr{J}_n(r,s,t)) \to \mathcal{M}(\mathscr{J}_{n-1}(r)).$$

In both cases the fiber is the complex plane with $n - r + s - t + 1$ points removed. Now the result follows by induction on s and Lemma 4.1. □

We observe that the identification of $\mathscr{I}_n$ with a braid arrangement mentioned in Example 2.6 yields alternative proofs of Lemmas 4.1–4.3 via Stanley's theorem [20, Proposition 2.8]. For, the subarrangement $\mathscr{I}_n(r)$ corresponds to the graphic arrangement with underlying graph the union of the complete subgraphs on the vertices $\{0, 1, \dots, r\}$ and $\{0, r+1, \dots, n\}$. Further, $\mathscr{I}_n(s,t)$ corresponds to the union of the complete subgraphs on the vertices

$$\{0, \dots, s, t+1, \dots, n\} \quad \text{and} \quad \{0, s+1, \dots, t, t+1, \dots, n\}.$$

The arrangement $\mathscr{I}_n(r,s,t)$ then corresponds to the union of complete subgraphs on the vertices $\{0, 1, \dots, r\}$, $\{0, r+1, \dots, s, t+1, \dots, n\}$ and $\{0, s+1, \dots, t, t+1, \dots, n\}$. In all cases the graph is clearly chordal, so the arrangement is of fiber type, thanks to [20, Proposition 2.8].

Now let $\mathcal{I}$ be of type (I), (II) or (III) listed above, set $\mathscr{A} = \mathscr{A}_{\mathcal{I}}$ and, in types (I)–(III), let $\mathscr{B}$ be $\mathscr{I}_{n-1}(r)$, $\mathscr{I}_{n-1}(s,t)$ or $\mathscr{I}_{n-1}(r,s,t)$, respectively. Consider the map

$$f\colon \mathcal{M}(\mathscr{A}) \to \mathcal{M}(\mathscr{B}), \quad (y_1, \dots, y_n) \mapsto (y_n^2 - y_1^2, \dots, y_n^2 - y_{n-1}^2). \tag{4.4}$$

Note that for $\mathcal{I} = \varnothing$, ie $\mathscr{A}_{\mathcal{I}} = \mathscr{A}(\Phi)$, and $\mathscr{B} = \mathscr{I}_{n-1}$, the map f was used in [4] to show asphericity in type D_n; see also [11, Section 5]. Our argument that the map f in (4.4) is a fibration over these larger bases is inspired by an argument due to Li Li (personal communication, 2006), who worked out the details of Brieskorn's approach [4].

Set $Y := \mathcal{M}(\mathscr{A})$ and $Z := \mathcal{M}(\mathscr{B})$. We can embed Y into $\mathbb{P}^n \times Z$ by the "graph" map $\iota\colon Y \to \mathbb{P}^n \times Z$ defined by

$$(y_1, \dots, y_n) \mapsto ((1 : y_1 : \dots : y_n), f(y_1, \dots, y_n))$$

and denote the image of Y by $C := \iota(Y)$. Then the map f is just $f = \pi|_C \circ \tilde{\iota}$, where $\tilde{\iota}\colon Y \to C$ is the homeomorphism induced by ι and $\pi|_C$ is the restriction of the projection $\pi\colon \mathbb{P}^n \times Z \to Z$ to C. Thus, f is a locally trivial fibration if and only if $\pi|_C$ is one.

Now let S_i be the hypersurface in $\mathbb{C}^n \times Z \subset \mathbb{P}^n \times Z$ defined by $z_i = y_n^2 - y_i^2$, so that $C = S_1 \cap \dots \cap S_{n-1}$. For $z = (z_1, \dots, z_{n-1}) \in Z$, let

$$(S_i)_z := S_i \cap (\mathbb{C}^n \times \{z\}) \subset \mathbb{P}^n \times \{z\} \quad \text{and} \quad C_z := (S_1)_z \cap \dots \cap (S_{n-1})_z,$$

ie C_z is the fiber of $\pi|_C$ over z. Moreover, let $\overline{C}$ and $\overline{C}_z$ denote the projective closures of C and C_z in $\mathbb{P}^n \times Z$, respectively. Then

$$\overline{C} = \overline{S}_1 \cap \dots \cap \overline{S}_{n-1} \quad \text{and} \quad \overline{C}_z = \overline{(S_1)}_z \cap \dots \cap \overline{(S_{n-1})}_z,$$

where $\bar{S}_i$ is the hypersurface in $\mathbb{P}^n \times Z$ given by $z_i y_0^2 = y_n^2 - y_i^2$ and for $z = (z_1, \dots, z_{n-1}) \in Z$,

$$\overline{(S_i)}_z := \bar{S}_i \cap (\mathbb{P}^n \times \{z\}).$$

Since $\bar{S}_i$ is defined by $y_n^2 - y_i^2 = z_i y_0$ for all $1 \le i \le n-1$ and the points at infinity are given by setting $y_0 = 0$, we get that $\bar{C}_z$ has the following 2^{n-1} points at infinity:

$$((0:\pm 1:\dots:\pm 1:1), (z_1, \dots, z_{n-1})).$$

Lemma 4.5 *For each $z \in Z$, the projective closure $\bar{C}_z$ of C_z is a smooth curve.*

Proof The $\overline{(S_i)}_z$ intersect transversally, which can be seen by looking at the Jacobian $J = (\partial f_j / \partial t_i(y))$ of the polynomials given by

$$f_i \colon \bar{Y} \to \mathbb{C}, \quad (t_0 : t_1 : \dots : t_n) \mapsto t_n^2 - t_i^2 - z_i t_0^2,$$

where $\bar{Y}$ is the projective closure of Y in $\mathbb{P}^n$. □

Moreover, we have the following:

Lemma 4.6 *For each $z \in Z$, $\bar{C}_z$ is connected.*

Proof Every point in $\bar{C}_z$ satisfies the equations

$$\frac{y_n^2 - y_1^2}{z_1} = \dots = \frac{y_n^2 - y_{n-1}^2}{z_{n-1}} = y_0^2.$$

First take U_n to be the subset of $\bar{C}_z$ consisting of points $((y_0:\dots:y_n), (z_1, \dots, z_{n-1}))$ with $y_n \neq 0$. Thus, considering the change of coordinates $x_i := y_i / y_n$ and fixing some $1 \le j \le n-1$, we get that

$$x_i^2 = g_i^j(x_j) \quad \text{for all} \quad 1 \le i \le n-1 \qquad \text{and} \qquad x_0^2 = g_0^j(x_j),$$

where $g_i^j(x) = (z_i/z_j)x^2 + (z_j - z_i)/z_j$ and $g_0^j(x) = -(1/z_j)x^2 + 1/z_j$. Let α_0 and α_1 be the two branches of $y = x^2$. Then, for any point $p \in U_n$ there are indices $k_i \in \{0, 1\}$ such that

$$p = \Big(\big(\alpha_{k_0}(g_0^j(x_j)) : \dots : \alpha_{k_{j-1}}(g_{j-1}^j(x_j)) : x_j : \alpha_{k_{j+1}}(g_{j+1}^j(x_j)) : \dots : \alpha_{k_{n-1}}(g_{n-1}^j(x_j)) : 1 \big), (z_1, \dots, z_{n-1}) \Big).$$

So, by choosing an appropriate path in $\mathbb{C}$, we may path-connect p to one of the points at infinity $((0:\pm 1:\dots:\pm 1:1), (z_1, \dots, z_{n-1}))$. As $1 \le j \le n-1$ is arbitrary and $g_i^j(x) = g_i^j(-x)$, any point $p \in U_n$ is path-connected to the point $((0:1:\dots:1), (z_1, \dots, z_{n-1}))$.

Now take U_1 to be the subset of $\overline{C}_z$ consisting of points $((y_0 : \cdots : y_n), (z_1, \ldots, z_{n-1}))$ with $y_1 \neq 0$ and observe that $U_1 \cup U_n = \overline{C}_z$. By a similar argument as the one above, for any point $q \in U_1$ there are indices $k_i \in \{0, 1\}$ such that

$$q = \big((\alpha_{k_0}(h_0(x_n)) : 1 : \alpha_{k_2}(h_2(x_n)) : \cdots : \alpha_{k_{n-1}}(h_{n-1}(x_n)) : x_n\big), (z_1, \ldots, z_{n-1})\big),$$

where $h_0(x) = (1/z_1)x_n^2 - 1/z_1$, $h_i(x) = ((z_1 - z_i)/z_1)x^2 + z_i/z_1$ and $x_i = y_i/y_1$. Now we can again choose a path in $\mathbb{C}$ that connects q to one of the points at infinity $((0 : \pm 1 : \cdots : \pm 1 : 1), (z_1, \ldots, z_{n-1}))$. Thus, $\overline{C}_z$ is connected. □

Note that this also proves that C_z is connected: as two points in C_z are connected by a path through finitely many points at infinity and $\overline{C}_z$ is locally homeomorphic to $\mathbb{C}$, we can alter the path around each of the points at infinity to get a path that completely lies inside C_z.

The above lemmas prove the following:

Corollary 4.7 *For each $z \in Z$, the curve $\overline{C}_z$ is a connected Riemann surface and C_z is a connected Riemann surface with 2^{n-1} puncture points.*

Theorem 4.8 *The map f defined in* (4.4) *is a locally trivial fibration.*

Proof Set $D = \overline{C} \setminus C$, the intersection of $\overline{C}$ with the infinity hyperplane. Then $\mathscr{S} = \{C, D\}$ is a Whitney stratification of $\overline{C}$: It is obviously locally finite and satisfies the condition of the frontier and, as C is open and D its boundary, $\mathscr{S}$ trivially satisfies Whitney condition B. The intersection of D with a fiber $\mathbb{P}^n \times \{z\}$ of the projection π is just the set of the 2^{n-1} points $((0 : \pm 1 : \cdots : \pm 1 : 1), (z_1, \ldots, z_{n-1}))$, which we can think of locally as 2^{n-1} sections of π. Thus, $\pi|_D$ is locally homeomorphic and therefore it is a submersion. The map $\pi|_C$ is a submersion as well, which can be seen by considering the Jacobian again. Moreover, $\pi|_{\overline{C}}$ is proper, as $\overline{C}$ is a closed subset of $\mathbb{P}^n \times Z$ and π is proper. Now, using Thom's first isotopy lemma, Theorem 2.5, $\pi|_{\overline{C}}$ is a locally trivial fibration and, in particular, $f = \pi|_C \circ \tilde{\imath}$ is a fibration as well. □

This proves the following:

Theorem 4.9 *If $\mathcal{I}$ is of type* (I), (II) *or* (III), *then $\mathscr{A}_{\mathcal{I}}$ is $K(\pi, 1)$.*

Proof Consider the map $f \colon Y \to Z$ from (4.4). Clearly, the fiber $f^{-1}(z)$ is homeomorphic to C_z, so by Corollary 4.7 it is a connected Riemann surface with 2^{n-1}

puncture points. Thus, by the uniformization theorem, it is a $K(\pi,1)$–space. By Lemmas 4.1, 4.2 and 4.3, Z is a $K(\pi,1)$–space as well. This proves the theorem. □

This concludes the proof of Theorem 1.4. Note that none of the arrangements of ideal type $\mathscr{A}_{\mathcal{I}}$ of types (I)–(III) considered here are supersolvable (see [13, Lemma 6.2]) and none of them are simplicial. So these families of $\mathscr{A}_{\mathcal{I}}$ also provide new classes of $K(\pi,1)$–arrangements.

Remarks 4.10 (i) If $\mathscr{A}$ is strictly linearly fibered over $\mathscr{B}$, then there always exists a section of the associated fibration of the complements $\mathcal{M}(\mathscr{A}) \to \mathcal{M}(\mathscr{B})$, eg see [5, Corollary 1.1.6]. As a consequence, by the splitting lemma, $\pi_1(\mathcal{M}(\mathscr{A}))$ is a semidirect product of $\pi_1(\mathcal{M}(\mathscr{B}))$ acting on the fundamental group of the fiber. In particular, this applies to each of the cases considered in Theorem 1.3.

(ii) One can also construct a cross-section to the fibration $f\colon Y \to Z$ used in the proof of Theorem 1.4 as follows: Let

$$y_n = y_n(z_1,\dots,z_{n-1}) = \sqrt{|z_1| + \dots + |z_{n-1}|}.$$

Now, for all $(z_1,\dots,z_{n-1}) \in Z$, for all $1 \le i \le n-1$ the real part of $y_i^2 = y_n^2 - z_i$ is positive. Thus, choosing a branch α of the square root, we can define $y_i = \alpha(y_n^2 - z_i)$ continuously, yielding a cross-section $s\colon Z \to Y$. This section was initially constructed by Falk and Randell in [11, Section 5] in the case $\mathscr{A}$ is the full reflection arrangement of type D_n, which is strictly linearly fibered over $\mathscr{B} = \mathscr{J}_{n-1}$; see Example 2.6. See also [14, Section 1.1] for a locally trivial fibration in this case with a slightly different section.

As $f \circ s = \mathrm{id}_Z$, the short exact sequence of fundamental groups splits. Thus, by the splitting lemma we see that $\pi_1(Y)$ is a semidirect product of $\pi_1(Z)$ acting on $\pi_1(C_z)$, where C_z is the fiber over $z \in Z$ as above.

Acknowledgements We are grateful to Li Li for helpful discussions concerning his argument (personal communication, 2006) and to Graham Denham and Michael Falk for providing the reference [12, (3.12)], where the non-$K(\pi,1)$–arrangement $\mathscr{K}_3$ used in Example 2.7 seems to appear first in the literature. We would also like to thank Daniel Cohen for raising the question about the existence of the sections of the fibrations discussed in Remarks 4.10. Thanks are also due to the anonymous referee for making numerous suggestions improving the paper.

The research of this work was supported by DFG-grant RO 1072/16-1.

References

[1] **T Abe**, **M Barakat**, **M Cuntz**, **T Hoge**, **H Terao**, *The freeness of ideal subarrangements of Weyl arrangements*, J. Eur. Math. Soc. 18 (2016) 1339–1348 MR

[2] **M Barakat**, **M Cuntz**, *Coxeter and crystallographic arrangements are inductively free*, Adv. Math. 229 (2012) 691–709 MR

[3] **N Bourbaki**, *Groupes et algèbres de Lie, Chapitre I: Algèbres de Lie*, 2nd edition, Actualités Scientifiques et Industrielles 1285, Hermann, Paris (1971) MR

[4] **E Brieskorn**, *Sur les groupes de tresses (d'après V I Arnold)*, from "Séminaire Bourbaki, 1971/1972", Lecture Notes in Math. 317, Springer (1973) Exposé 401, 21–44 MR

[5] **D C Cohen**, *Monodromy of fiber-type arrangements and orbit configuration spaces*, Forum Math. 13 (2001) 505–530 MR

[6] **M Cuntz**, **I Heckenberger**, *Finite Weyl groupoids*, J. Reine Angew. Math. 702 (2015) 77–108 MR

[7] **M Cuntz**, **G Röhrle**, **A Schauenburg**, *Arrangements of ideal type are inductively free*, Internat. J. Algebra Comput. (online publication February 2019)

[8] **P Deligne**, *Les immeubles des groupes de tresses généralisés*, Invent. Math. 17 (1972) 273–302 MR

[9] **E Fadell**, **L Neuwirth**, *Configuration spaces*, Math. Scand. 10 (1962) 111–118 MR

[10] **M J Falk**, **N J Proudfoot**, *Parallel connections and bundles of arrangements*, Topology Appl. 118 (2002) 65–83 MR

[11] **M Falk**, **R Randell**, *The lower central series of a fiber-type arrangement*, Invent. Math. 82 (1985) 77–88 MR

[12] **M Falk**, **R Randell**, *On the homotopy theory of arrangements*, from "Complex analytic singularities" (T Suwa, P Wagreich, editors), Adv. Stud. Pure Math. 8, North-Holland, Amsterdam (1987) 101–124 MR

[13] **A Hultman**, *Supersolvability and the Koszul property of root ideal arrangements*, Proc. Amer. Math. Soc. 144 (2016) 1401–1413 MR

[14] **A Leibman**, **D Markushevich**, *The monodromy of the Brieskorn bundle*, from "Geometric topology" (C Gordon, Y Moriah, B Wajnryb, editors), Contemp. Math. 164, Amer. Math. Soc., Providence, RI (1994) 91–117 MR

[15] **J N Mather**, *Stratifications and mappings*, from "Dynamical systems" (M M Peixoto, editor), Academic, New York (1973) 195–232 MR

[16] **P Orlik**, **H Terao**, *Arrangements of hyperplanes*, Grundl. Math. Wissen. 300, Springer (1992) MR

[17] **L Paris**, *The Deligne complex of a real arrangement of hyperplanes*, Nagoya Math. J. 131 (1993) 39–65 MR

[18] **G Röhrle**, *Arrangements of ideal type*, J. Algebra 484 (2017) 126–167 MR

[19] **E Sommers**, **J Tymoczko**, *Exponents for B–stable ideals*, Trans. Amer. Math. Soc. 358 (2006) 3493–3509 MR

[20] **R P Stanley**, *Supersolvable lattices*, Algebra Universalis 2 (1972) 197–217 MR

[21] **H Terao**, *Modular elements of lattices and topological fibration*, Adv. in Math. 62 (1986) 135–154 MR

[22] **R Thom**, *Ensembles et morphismes stratifiés*, Bull. Amer. Math. Soc. 75 (1969) 240–284 MR

Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Fakultät für Mathematik und Physik, Gottfried Wilhelm Leibniz Universität Hannover
Hannover, Germany

Fakultät für Mathematik, Ruhr-Universität Bochum
Bochum, Germany

amend@math.uni-hannover.de, gerhard.roehrle@rub.de

Received: 30 January 2018 Revised: 21 August 2018