

# Topological complexity of unordered configuration spaces of surfaces

ANDREA BIANCHI  
DAVID RECIO-MITTER

We determine the topological complexity of unordered configuration spaces on almost all punctured surfaces (both orientable and nonorientable). We also give improved bounds for the topological complexity of unordered configuration spaces on all aspherical closed surfaces, reducing it to three possible values. The main methods used in the proofs were developed in 2015 by Grant, Lupton and Oprea to give bounds for the topological complexity of aspherical spaces. As such this paper is also part of the current effort to study the topological complexity of aspherical spaces and it presents many further examples where these methods strongly improve upon the lower bounds given by zero-divisor cup-length.

55M99, 55P20; 20J06, 55M30, 68T40

## 1 Introduction

In 2003 Farber introduced the topological complexity of a space to study the problem of robot motion planning from a topological perspective [5]. It is a numerical homotopy invariant which measures the minimal instability of every motion planner on this space. More explicitly, given a path-connected space  $X$ , the topological complexity  $\mathrm{TC}(X)$  is the sectional category of the free path fibration  $p_X: X^I \rightarrow X \times X$  (see Section 2).

Determining  $\mathrm{TC}(X)$  is in general a hard problem. For over a decade the topological complexity of many spaces has been computed and diverse tools have been developed to that end.

In this context, configuration spaces have been extensively studied because they are of special interest from the point of view of robotics. Considering the problem of moving  $n$  objects on a space  $X$  avoiding collisions naturally leads to the definition of the *ordered configuration space*  $F(X, n)$  of  $n$  distinct ordered points on  $X$  as

$$F(X, n) = \{(x_1, \dots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\}.$$

These spaces model automated guided vehicles (AGVs) moving on a factory floor — see Ghrist [11] — or flying drones trying to avoid each other in the air.

Farber and Yuzvinsky determined the topological complexity of  $F(\mathbb{R}^d, n)$  for  $d = 2$  or  $d$  odd in [10]. Later, Farber and Grant extended the results to all dimensions  $d$  in [8]. The topological complexity of ordered configuration spaces of orientable surfaces has also been computed by Cohen and Farber in [2]. Many more related results can be found in the recent survey articles by Cohen [1] and Farber [7].

In the configuration spaces  $F(X, n)$  considered above, the points of a configuration are labeled (or ordered) and the symmetric group  $\mathfrak{S}_n$  acts on  $F(X, n)$  by permuting the labels. However, in certain situations it greatly improves the efficiency to consider the points to be identical. For instance, consider a scenario in which all the AGVs perform the same tasks equally. In this case we are only interested in the positions of points in  $X$  up to permutation, in other words forgetting the labels assigned to the points. This leads to the *unordered configuration spaces*  $C(X, n) = F(X, n)/\mathfrak{S}_n$ , by definition the orbits of the symmetric group action.

As we saw above, there is a very complete picture of the topological complexity of *ordered* configuration spaces of 2-dimensional manifolds and beyond. In contrast to this, very little is known for *unordered* configuration spaces, as Cohen notes at the end of [1]. One of the main reasons for this discrepancy is that all the above results use a cohomological technique involving *zero-divisors*, which seems to be insufficient for unordered configuration spaces (at least with constant coefficients).

The results in this paper use a technique to bound the topological complexity of aspherical spaces developed in 2015 by Grant, Lupton and Oprea [12]. Being a homotopy invariant, the topological complexity of an aspherical space only depends on its fundamental group and the methods are algebraic in nature. An introduction to topological complexity of groups is given in Section 2.

The mentioned technique was already used in the recent paper [13], in which Grant and the second author computed the topological complexity of some mixed configuration spaces  $F(\mathbb{R}^2, n)/(\mathfrak{S}_{n-k} \times \mathfrak{S}_k)$  on the plane, with  $1 \leq k \leq n-1$ . These spaces are in a sense intermediate between the ordered and the unordered case and they model the situation in which there are two different types of identical AGVs. It turns out that also in the mixed case the cohomological lower bounds used in previous results are insufficient.

It has to be mentioned that the topological complexity of unordered configuration spaces of trees was computed in many cases by Scheirer in [16]. To the best of the authors' knowledge that is the only previous computation of the topological complexity of an unordered configuration space with at least three points. It is worth noting that Scheirer uses the zero-divisor cup-length lower bound, which seems to be insufficient for unordered configuration spaces of surfaces.

In this paper we determine the topological complexity of the unordered configuration spaces of all punctured surfaces (orientable and nonorientable) except the disc and the Möbius band, and narrow it down to three values for all closed aspherical surfaces (orientable and nonorientable). For the Möbius band we narrow it down to two values and for the disc we give some improved bounds and a complete answer in the case of three points. Many of the proofs extend to ordered configuration spaces (this is discussed at the end of the paper).

All results except the ones for the disc are presented in the following theorem, which follows from Theorems 4.1, 4.2, 5.1 and 5.3. In the case of the annulus the upper bound is proven by finding an explicit motion planner.

**Theorem 1.1** • *Let  $S$  be obtained from a closed surface by removing a positive number of points. If  $S$  is not the disc, the annulus or the Möbius band, then*

$$\mathrm{TC}(C(S, n)) = 2n.$$

- *Let  $S$  be a closed surface. If  $S$  is not the sphere or the projective plane, then*

$$2n \leq \mathrm{TC}(C(S, n)) \leq 2n + 2.$$

- *If  $\mathcal{A}$  denotes the annulus, then*

$$\mathrm{TC}(C(\mathcal{A}, n)) = 2n - 1.$$

- *If  $\mathcal{M}$  denotes the Möbius band, then*

$$2n - 1 \leq \mathrm{TC}(C(\mathcal{M}, n)) \leq 2n.$$

**Remark 1.2** Theorem 1.1 should be compared to the corresponding results for ordered configuration spaces of Cohen and Farber in [2]. They are consistent with the possibility that the values of the topological complexity of ordered and unordered configuration spaces of surfaces always agree. Note that in [2] the nonreduced version of topological complexity is used, which is 1 greater than the one used in this paper.

The only aspherical surface not covered by [Theorem 1.1](#) is the disc. The best estimates we found for the disc are given in the following two theorems. Note that they greatly improve over the best previously known lower bounds

$$\mathrm{TC}(C(D, n)) \geq \mathrm{cat}(C(D, n)) = n - 1$$

coming from the Lusternik–Schnirelmann category  $\mathrm{cat}(C(D, n))$  (see [\[13\]](#)).

**Theorem 1.3** *If  $D$  is the disc, then*

$$2n - 2 - \frac{1}{2}n \leq n - 1 + \mathrm{cd}([P_n, P_n]) \leq \mathrm{TC}(C(D, n)) \leq 2n - 2.$$

Here  $\mathrm{cd}$  is the cohomological dimension of a group and  $[P_n, P_n]$  is the commutator subgroup of the pure braid group of the disc (see [Section 3](#)).

We expect that  $\mathrm{cd}([P_n, P_n])$  is in fact the maximum possible, which would mean that [Theorem 1.3](#) narrows  $\mathrm{TC}(C(D, n))$  down to two possible values.

**Conjecture 1.4** The cohomological dimension of  $[P_n, P_n]$  is equal to  $n - 2$ .

The following theorem gives a potentially better lower bound (depending on the actual value of  $\mathrm{cd}([P_n, P_n])$ , which is unknown to the authors). It also tells us that asymptotically  $\mathrm{TC}(C(D, n))$  behaves like  $2n$ .

**Theorem 1.5** *If  $D$  is the disc, then*

$$2n - 2 \lfloor \sqrt{n/2} \rfloor - 3 \leq \mathrm{TC}(C(D, n)) \leq 2n - 2.$$

Finally, we compute the topological complexity of the unordered configuration space of three points on the disc by finding an explicit motion planner.

**Theorem 1.6** *If  $D$  is the disc, then*

$$\mathrm{TC}(C(D, 3)) = 3.$$

The authors are grateful to Mark Grant for many useful discussions and comments on earlier drafts of the paper, and to Gabriele Viaggi for suggesting the strategy for the proof of [Lemma 3.6](#).

## 2 Topological complexity of aspherical spaces

In this section we first define the topological complexity of a general topological space and then specialize it to aspherical spaces.

For a path-connected topological space  $X$ , let  $p_X: X^I \rightarrow X \times X$  denote the free path fibration on  $X$ , with projection  $p_X(\gamma) = (\gamma(0), \gamma(1))$ .

**Definition 2.1** The *topological complexity* of  $X$ , denoted by  $\mathrm{TC}(X)$ , is defined to be the minimal  $k$  such that  $X \times X$  admits a cover by  $k + 1$  open sets  $U_0, U_1, \dots, U_k$ , on each of which there exists a local section of  $p_X$  (that is, a continuous map  $s_i: U_i \rightarrow X^I$  such that  $p_X \circ s_i = \mathrm{incl}_i: U_i \hookrightarrow X \times X$ ).

Note that here we use the reduced version of  $\mathrm{TC}(X)$ , which is 1 less than the number of open sets in the cover.

Let  $\pi$  be a discrete group. It is well known that there exists a connected CW-complex  $K(\pi, 1)$  with

$$\pi_i(K(\pi, 1)) = \begin{cases} \pi & \text{if } i = 1, \\ 0 & \text{if } i \geq 2. \end{cases}$$

Such a space is called an *Eilenberg–Mac Lane space* for the group  $\pi$ . Furthermore,  $K(\pi, 1)$  is unique up to homotopy. Because the topological complexity  $\mathrm{TC}(X)$  is a homotopy invariant of the space  $X$  (see [5]), the following definition is sensible:

**Definition 2.2** The topological complexity of a discrete group  $\pi$  is given by

$$\mathrm{TC}(\pi) := \mathrm{TC}(K(\pi, 1)).$$

In [6] Farber posed the problem of giving an algebraic description of  $\mathrm{TC}(\pi)$ . This problem is far from being solved but some progress has been made, including the following theorem:

**Theorem 2.3** (Grant, Lupton and Oprea [12, Theorem 1.1]) *Let  $\pi$  be a discrete group and let  $A$  and  $B$  be subgroups of  $\pi$ . Suppose that  $gAg^{-1} \cap B = \{1\}$  for every  $g \in \pi$ . Then*

$$\mathrm{TC}(\pi) \geq \mathrm{cd}(A \times B).$$

It is worth noting that this theorem has recently been generalized using different methods in [9, Corollary 3.5.4].

The corresponding problem for the Lusternik–Schnirelmann category of a group has been completely answered:  $\text{cat}(\pi) = \text{cd}(\pi)$ . This classical result is due to Eilenberg and Ganea [3] for  $\text{cd}(\pi) \neq 1$ , while the remaining case follows from the later work by Stallings [17] and Swan [18].

We will also need the following standard result:

**Lemma 2.4**  $\text{TC}(\pi) \leq \text{cd}(\pi \times \pi)$ .

**Proof** This follows from the upper bound  $\text{TC}(X) \leq \text{cat}(X \times X)$  given by Farber in [5].  $\square$

### 3 The surface braid groups

In this section we introduce the surface braid groups and we recall their main properties.

**Definition 3.1** A surface  $\mathcal{S}$  is a connected closed 2–dimensional manifold possibly with a finite number of points removed, called *punctures*.

Recall from the introduction that the configuration space  $F(\mathcal{S}, n)$  admits an action by the symmetric group  $\mathfrak{S}_n$  which permutes the points in each configuration. The unordered configuration space

$$C(\mathcal{S}, n) = F(\mathcal{S}, n) / \mathfrak{S}_n$$

is by definition the orbit space of that action.

**Definition 3.2** We call  $P_n(\mathcal{S}) = \pi_1(F(\mathcal{S}, n))$  the *pure braid group* on  $n$  strands of the surface  $\mathcal{S}$ , and  $B_n(\mathcal{S}) = \pi_1(C(\mathcal{S}, n))$  the (*full*) *braid group* on  $n$  strands of  $\mathcal{S}$ . When  $\mathcal{S}$  is the disc  $D$ , we also abbreviate  $P_n = P_n(D)$  and  $B_n = B_n(D)$ .

The covering  $F(\mathcal{S}, n) \rightarrow C(\mathcal{S}, n)$  yields the short exact sequence

$$1 \rightarrow P_n(\mathcal{S}) \rightarrow B_n(\mathcal{S}) \rightarrow \mathfrak{S}_n \rightarrow 1.$$

The following theorem is due to Fadell and Neuwirth:

**Theorem 3.3** (Fadell and Neuwirth [4]) *Denote by  $\mathcal{S}_n$  the surface obtained from  $\mathcal{S}$  by removing  $n$  points. There is a locally trivial fibration*

$$(1) \quad \mathcal{S}_n \rightarrow F(\mathcal{S}, n+1) \rightarrow F(\mathcal{S}, n),$$

*where the projection map forgets the last point of the ordered configuration.*

It is well known that the only surfaces that are not aspherical are the sphere  $S^2$  and the projective plane  $\mathbb{R}P^2$ . From now on all the surfaces that we will consider are assumed to be aspherical. The reason for this is that the methods in this paper only apply to aspherical spaces.

**Corollary 3.4** *Let  $\mathcal{S}$  be an aspherical surface. From the long exact sequence of the homotopy groups applied to the Fadell–Neuwirth fibrations (1) and induction it follows that the spaces  $F(\mathcal{S}, n)$  are also aspherical. Furthermore, we get the short exact sequence*

$$(2) \quad 1 \rightarrow \pi_1(\mathcal{S}_n) \rightarrow P_{n+1}(\mathcal{S}) \rightarrow P_n(\mathcal{S}) \rightarrow 1.$$

We will need the following technical result, which we expect to be well known to the experts. However, we could not find a full proof in the literature and thus we will give a detailed proof here. The result appears as Proposition 2.2 in [15] but it relies on Lemma 3.6 below (Proposition 2.1 in [15]), which is stated there without a proof.

**Theorem 3.5** *Let  $\mathcal{S} \hookrightarrow \mathcal{T}$  be a smooth embedding of aspherical surfaces such that the induced homomorphism  $\pi_1(\mathcal{S}) \rightarrow \pi_1(\mathcal{T})$  is injective.*

*Then the corresponding inclusion  $C(\mathcal{S}, n) \hookrightarrow C(\mathcal{T}, n)$  induces an injective homomorphism  $B_n(\mathcal{S}) \rightarrow B_n(\mathcal{T})$ .*

In the proof of the theorem the following lemma will be essential. In that lemma a slightly different definition of nonclosed surface is needed, with open balls removed instead of points removed. This is the only place in which we make use of this definition. We stress that this is not an essential distinction because the configuration spaces of punctured surfaces and the configuration spaces of surfaces with boundary are homotopy equivalent.

**Lemma 3.6** *Let  $\mathcal{S} \hookrightarrow \mathcal{T}$  be a smooth embedding of aspherical surfaces, which we assume to be closed surfaces with (possibly) some open balls removed instead of points removed. Further assume that the image of  $\mathcal{S}$  lies in the interior of  $\mathcal{T}$ . Then the induced homomorphism  $\pi_1(\mathcal{S}) \rightarrow \pi_1(\mathcal{T})$  is injective if and only if no boundary component of  $\mathcal{S}$  bounds a disc in  $\mathcal{T} \setminus \mathcal{S}$ .*

**Proof** Recall that we are assuming that surfaces are path-connected. Therefore, if  $\mathcal{S}$  is closed, the embedding has to be surjective and the claim is trivial. Assume  $\mathcal{S}$  is

not closed. Because the boundary components of  $\mathcal{S}$  are smooth simple closed curves inside  $\mathcal{T}$ , they separate  $\mathcal{T}$  into  $\mathcal{S}$  on one side and a disjoint union of surfaces on the other side.

We first assume that the homomorphism  $\pi_1(\mathcal{S}) \rightarrow \pi_1(\mathcal{T})$  induced by the embedding is not injective and claim that there is a disc in  $\mathcal{T} \setminus \mathcal{S}$  bounded by a boundary component of  $\mathcal{S}$ .

A nontrivial element in the kernel of  $\pi_1(\mathcal{S}) \rightarrow \pi_1(\mathcal{T})$  can be represented by a smooth map  $f: S^1 \rightarrow \mathcal{S}$  which extends to a smooth map on the disc  $g: D \rightarrow \mathcal{T}$ . We may assume that the image of  $f$  is in the interior of  $\mathcal{S}$  and that  $g$  is transverse to  $\partial\mathcal{S}$ .

Observe that the image of  $g$  needs to have a nonempty intersection with the boundary of  $\mathcal{S}$ . Otherwise,  $g$  would yield a nullhomotopy of  $f$  inside  $\mathcal{S}$ , but by assumption  $f$  represents a nontrivial class in  $\pi_1(\mathcal{S})$ . Let  $B$  be a boundary component of  $\mathcal{S}$  which intersects the image of  $g$ .

The preimage of  $B$  in  $D$  under  $g$  is now a nonempty, smooth 1-dimensional manifold. Since  $f: S^1 \rightarrow \mathcal{S}$  doesn't intersect  $\partial\mathcal{S}$ ,  $g^{-1}(B)$  is a compact subset of the interior of  $D$ , hence it must be a closed 1-manifold.

Therefore, given a path-component  $C \subset D$  of  $g^{-1}(B)$ , we know that  $C$  is a smooth circle and, by the Jordan–Schoenflies curve theorem,  $C$  bounds a disc  $\tilde{D}$  in  $D$  on one side and an annulus  $A$  on the other side such that  $\partial A = C \cup \partial D$ . We can further assume, by choosing  $C$  to be *outermost* in  $D$  among the path-components of  $g^{-1}(B)$ , that there exists a collar neighborhood  $U \supset C$  in  $D$  such that  $g(U \cap A) \subseteq \mathcal{S}$ . Indeed, by transversality we have, for a small collar neighborhood  $U$ , that  $g(U \cap A)$  is contained either in  $\mathcal{S}$  or in  $\mathcal{T} \setminus \mathcal{S}$ . If  $C$  is *outermost*, the former must be the case, as under this condition there is a path in  $A$  from  $C$  to  $\partial D$  only intersecting  $g^{-1}(B)$  at the starting point, and  $g(\partial D) \subset \mathcal{S}$ .

The curve  $C$  gives an element in  $\pi_1(B) \simeq \mathbb{Z}$ . If this element is trivial then we can redefine  $g$  on  $\tilde{D}$  by a nullhomotopy living on  $B$ . After pushing the image of  $\tilde{D}$  along the collar neighborhood into the interior of  $\mathcal{S}$ , we get a replacement of  $g$  with (at least) one less connected component in  $g^{-1}(\partial\mathcal{S})$  than for the original map.

Hence, there must exist a circle  $C$  such that  $g|_C$  is a nontrivial element in  $\pi_1(B)$ , otherwise we would construct a nullhomotopy of  $f$  inside  $\mathcal{S}$  after finitely many iterations of the above procedure. Therefore, there is a power of the generator  $[B] \in \pi_1(B)$  that



vanishes in  $\pi_1(\mathcal{T})$ . Because  $\pi_1(\mathcal{T})$  is torsion-free (indeed  $\mathcal{T}$  is a finite-dimensional classifying space for  $\pi_1(\mathcal{T})$ ),  $[B]$  is already trivial in  $\pi_1(\mathcal{T})$ .

Then  $B$  is a nullhomotopic simple closed curve and it must bound a disc in  $\mathcal{T}$  by the classification of surfaces. There are two possibilities. Either this disc doesn't intersect the interior of  $\mathcal{S}$  and it is glued to the boundary component  $B$  to obtain  $\mathcal{T}$ , or  $\mathcal{S}$  is a punctured sphere and  $\mathcal{T}$  is obtained from  $\mathcal{S}$  by glueing discs onto all the path-components of  $\partial\mathcal{S}$  different from  $B$  (there is at least one other boundary component because by assumption  $\pi_1(\mathcal{S}) \rightarrow \pi_1(\mathcal{T})$  is not injective and therefore  $\mathcal{S}$  is not a disc).

We showed that if  $\pi_1(\mathcal{S}) \rightarrow \pi_1(\mathcal{T})$  is not injective, there must be a disc in  $\mathcal{T} \setminus \mathcal{S}$  bounded by boundary component of  $\mathcal{S}$ .

Conversely, assume that  $\mathcal{T} \setminus \mathcal{S}$  contains a disc  $D$  bounded by some boundary component  $B$  of  $\partial\mathcal{S}$ . Then the corresponding element  $[B] \in \pi_1(\mathcal{S})$  vanishes in  $\pi_1(\mathcal{T})$ . Therefore, the homomorphism  $\pi_1(\mathcal{S}) \rightarrow \pi_1(\mathcal{T})$  is not injective unless  $[B]$  is already trivial in  $\pi_1(\mathcal{S})$ . Again by the classification of surfaces, this can only happen if  $\mathcal{S}$  itself is a disc, but then  $\mathcal{T}$  would be a sphere, contradicting the hypothesis that  $\mathcal{T}$  is aspherical.  $\square$

**Proof of Theorem 3.5** By the commutativity of the following diagram with exact rows, it suffices to show that  $P_n(\mathcal{S}) \rightarrow P_n(\mathcal{T})$  is injective:

$$\begin{array}{ccccccc} 1 & \longrightarrow & P_n(\mathcal{S}) & \longrightarrow & B_n(\mathcal{S}) & \longrightarrow & \mathfrak{S}_n \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & P_n(\mathcal{T}) & \longrightarrow & B_n(\mathcal{T}) & \longrightarrow & \mathfrak{S}_n \longrightarrow 1 \end{array}$$

We do this by induction using the Fadell–Neuwirth fibrations.

For  $n = 1$ , the homomorphism  $\pi_1(\mathcal{S}) \rightarrow \pi_1(\mathcal{T})$  is injective by assumption.

Suppose now that  $P_{n-1}(\mathcal{S}) \rightarrow P_{n-1}(\mathcal{T})$  is injective. The embedding  $\mathcal{S} \hookrightarrow \mathcal{T}$  gives rise to an embedding  $\mathcal{S}_n \hookrightarrow \mathcal{T}_n$ , in which the  $n$  new punctures in  $\mathcal{S}_n$  are sent to the  $n$  new punctures in  $\mathcal{T}_n$ . The short exact sequences (2) give rise to the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(\mathcal{S}_{n-1}) & \longrightarrow & P_n(\mathcal{S}) & \longrightarrow & P_{n-1}(\mathcal{S}) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \pi_1(\mathcal{T}_{n-1}) & \longrightarrow & P_n(\mathcal{T}) & \longrightarrow & P_{n-1}(\mathcal{T}) \longrightarrow 1 \end{array}$$

The rows are exact and we assumed the vertical homomorphism on the right is injective. If the vertical homomorphism on the left were also injective, the vertical homomorphism in the middle would have to be injective, which would complete the induction argument. It is not hard to see that the configuration spaces of punctured surfaces (points removed) and the configuration spaces of surfaces with boundary components (open balls removed) are homotopy equivalent. Because of this we might assume that  $S$  and  $T$  are surfaces with boundary and that  $\mathcal{S}_{n-1}$  and  $\mathcal{T}_{n-1}$  are the surfaces which result from removing  $n - 1$  open balls, in order to be able to use [Lemma 3.6](#). Then the embedding  $S \hookrightarrow T$  satisfies the assumptions of [Lemma 3.6](#) if and only if  $\mathcal{S}_{n-1} \hookrightarrow \mathcal{T}_{n-1}$  satisfies them. Therefore, the injectivity of the leftmost vertical homomorphism is equivalent to the injectivity of  $\pi_1(S) \twoheadrightarrow \pi_1(T)$ , which is part of the assumptions.  $\square$

### 4 Lower bounds

**Theorem 4.1** *Let  $S$  be an aspherical surface which is not the disc, the annulus or the Möbius band. Then*

$$\mathrm{TC}(C(S, n)) \geq 2n.$$

**Proof** Let  $S$  be a surface satisfying the assumptions in the theorem. Then, with the only exception of the Klein bottle, we have  $\mathrm{rank}(H_1(S)) \geq 2$  and there are two smooth simple closed curves  $\alpha$  and  $\alpha'$  on  $S$  representing linearly independent classes of  $H_1(S)$ . We may assume that there exist tubular neighborhoods  $\mathcal{A}$  of  $\alpha$  and  $\mathcal{A}'$  of  $\alpha'$  that are annuli. If the tubular neighborhood of  $\alpha$  were a Möbius band, then we could replace  $\alpha$  by the boundary of this Möbius band.

The homomorphism  $\pi_1(\mathcal{A}) \rightarrow \pi_1(S)$  is injective, as can be checked by further projecting to  $H_1(S)$ . Similarly, the homomorphism  $\pi_1(\mathcal{A}') \rightarrow \pi_1(S)$  is injective.

For the Klein bottle  $\mathcal{K}$ , recall that the fundamental group  $\pi_1(\mathcal{K})$  has a presentation

$$\pi_1(\mathcal{K}) = \langle a, b \mid aba^{-1}b \rangle,$$

where both  $a$  and  $b$  are represented by simple closed curves  $\alpha$  and  $\beta$  in  $\mathcal{K}$ . Both subgroups  $\langle a \rangle$  and  $\langle b \rangle$  are infinite cyclic, and therefore the inclusions of collar neighborhoods  $\mathcal{A}$  of  $\alpha$  and  $\mathcal{A}'$  of  $\beta$  in  $\mathcal{K}$  are injective at the level of  $\pi_1$ ; the collar neighborhood of  $\alpha$  is a Möbius band so we replace  $\alpha$  with its *double* as above.

Hence, by [Theorem 3.5](#) the homomorphisms  $P_n(\mathcal{A}) \rightarrow P_n(S)$  and  $P_n(\mathcal{A}') \rightarrow P_n(S)$  are injective.

We now construct a subgroup  $Z_n \subset P_n(\mathcal{A})$ . Consider  $n$  parallel, disjoint copies  $\alpha_1, \dots, \alpha_n$  of the curve  $\alpha$  inside  $\mathcal{A}$ , and let  $\mathfrak{T} \subset F(\mathcal{A}, n)$  be the subspace of ordered configurations  $(x_1, \dots, x_n)$  with  $x_i$  lying on the curve  $\alpha_i$  for all  $1 \leq i \leq n$ ; then  $\mathfrak{T}$  is an embedded  $n$ -fold torus in  $F(\mathcal{A}, n)$ , and at the level of fundamental groups we have a map  $\mathbb{Z}^n \simeq \pi_1(\mathfrak{T}) \rightarrow P_n(\mathcal{A})$ .

This map is injective: indeed the composition

$$\mathbb{Z}^n \simeq \pi_1(\mathfrak{T}) \rightarrow P_n(\mathcal{A}) = \pi_1(F(\mathcal{A}, n)) \rightarrow \pi_1(\mathcal{A}^n) \simeq \mathbb{Z}^n$$

is an isomorphism. We call  $Z_n \simeq \mathbb{Z}^n \subset P_n(\mathcal{A})$  the image of this map.

In the same way we construct an  $n$ -fold torus  $\mathfrak{T}' \subset F(\mathcal{A}', n)$  and get a subgroup  $Z'_n \subset P_n(\mathcal{A}')$  as the image of the map between fundamental groups induced by the inclusion, with  $Z'_n \simeq \mathbb{Z}^n$ .

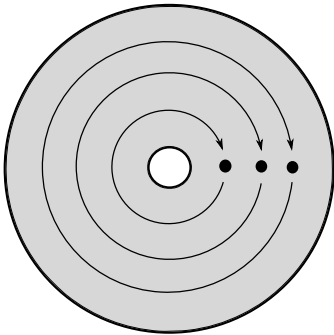


Figure 1: Braids from  $Z_n$  as seen from above

There is a homomorphism

(3)

$$P_n(S) \rightarrow \prod_{k=1}^n \pi_1(S) \rightarrow \bigoplus_{k=1}^n H_1(S),$$

under which nontrivial elements in the image of  $Z_n$  and  $Z'_n$  inside  $P_n(S)$  are mapped to elements which lie in different orbits under the action which permutes the summands in  $\bigoplus_{k=1}^n H_1(S)$ . This is because the image of each nontrivial element in  $Z_n$  will have at least one summand corresponding to a nontrivial multiple of the class in  $H_1(S)$  represented by the curve  $\alpha$ , whereas the image of each braid in  $Z'_n$  has only summands corresponding to multiples of the class represented by the curve  $\alpha'$ . Notice that for the Klein bottle it suffices that the homology class represented by  $\alpha$  is infinite cyclic, and the argument works even if the homology class represented by  $\alpha'$  has order 2.

Now we observe that conjugating an element of  $P_n(S)$  by an element of  $B_n(S)$  has the effect of permuting the summands in  $\bigoplus_{k=1}^n H_1(S)$  under the homomorphism (3). To see this first note that the homomorphism (3) consists of a sum of compositions of homomorphisms of the form

$$P_n(S) \rightarrow \pi_1(S) \rightarrow H_1(S)$$

given by forgetting all strands but one and then taking the abelianization. Given a braid  $\gamma \in B_n(S)$ , we can write  $\gamma = \delta\epsilon$ , where  $\epsilon$  is supported on a disc and  $\delta \in P_n(S)$ . Therefore, conjugating by  $\gamma$  reduces to conjugating by  $\epsilon$  and  $\delta$ . Conjugating by  $\epsilon$  permutes the order of the strands by the corresponding permutation under the canonical map  $B_n \rightarrow \mathfrak{S}_n$ . Conjugating by  $\delta$  results in a conjugation inside  $\pi_1(S)$  under the first homomorphism  $P_n(S) \rightarrow \pi_1(S)$ , but this has no effect on the abelianization.

Therefore, no nontrivial element of  $Z_n$  is conjugate to an element of  $Z'_n$  in  $B_n(S)$ . By Theorem 2.3 this implies the lower bound  $\text{TC}(B_n(S)) \geq \text{cd}(Z_n \times Z'_n) = 2n$ .  $\square$

**Theorem 4.2** *Let  $S$  be either the annulus or the Möbius band. Then*

$$\text{TC}(C(S, n)) \geq 2n - 1.$$

**Proof** In the same way as in the previous proof we can find an annulus  $\mathcal{A}$  inside  $S$  and a subgroup  $Z_n$  in  $P_n(\mathcal{A})$  isomorphic to  $\mathbb{Z}^n$ . Because  $\pi_1(S) \simeq H_1(S) \simeq \mathbb{Z}$ , this time we cannot find a second annulus inducing a linearly independent homology class, nor even a disjoint infinite cyclic subgroup of  $\pi_1(S)$ .

However, the inclusion of a disc  $D$  in  $S$  also induces a monomorphism  $P_n(D) \rightarrow P_n(S)$  and no nontrivial element in  $P_n(D)$  is conjugate to an element of  $Z_n$  inside  $B_n(S)$ .

Indeed, if we consider the map

$$P_n(S) \rightarrow \prod_{k=1}^n \pi_1(S) \rightarrow \bigoplus_{k=1}^n H_1(S),$$

we see that no nontrivial element of  $Z_n$  is mapped to zero, whereas all elements of  $P_n(D)$  are mapped to zero. As we saw in the proof of the previous theorem, conjugation inside  $B_n(S)$  results only in a permutation of the coordinates of the target group  $\bigoplus_{k=1}^n H_1(S)$ , and the stated properties are therefore invariant under conjugation.

By Theorem 2.3 we get

$$\text{TC}(C(S, n)) \geq \text{cd}(\mathbb{Z}^n \times P_n(D)) = 2n - 1. \qquad \square$$

## 5 Upper bounds

**Theorem 5.1** *If  $\mathcal{S}$  is a closed aspherical surface, then*

$$\mathrm{TC}(C(\mathcal{S}, n)) \leq 2n + 2.$$

*If  $\mathcal{S}$  is a punctured surface which is not the disc, then*

$$\mathrm{TC}(C(\mathcal{S}, n)) \leq 2n.$$

**Proof** It is well known that  $\mathrm{cd}(\pi_1(\mathcal{S})) = 2$  for closed aspherical surfaces and that  $\mathrm{cd}(\pi_1(\mathcal{S})) = 1$  for punctured surfaces (other than the disc). Using the short exact sequences (2) of Corollary 3.4, together with the fact that the cohomological dimension is subadditive under group extensions, and that  $\mathrm{cd}(B_n(\mathcal{S})) = \mathrm{cd}(P_n(\mathcal{S}))$  because  $B_n(\mathcal{S})$  is torsion-free and  $P_n(\mathcal{S})$  is a finite-index subgroup, we see that  $\mathrm{cd}(B_n(\mathcal{S})) \leq n + 1$  if  $\mathcal{S}$  is closed and  $\mathrm{cd}(B_n(\mathcal{S})) \leq n$  if  $\mathcal{S}$  has punctures and is not the disc (the two preceding inequalities are in fact equalities, but we don't need that stronger statement in this proof).

The upper bounds now follow from Lemma 2.4. □

Next we give an upper bound for the annulus which is 1 better than the one given in the previous theorem (it is in fact the optimal upper bound). For the proof we will need the following well-known technical lemma.

We defined the topological complexity in terms of the number of open sets in an open cover of  $X \times X$ , but for sufficiently nice spaces (CW-complexes for instance) there is an equivalent characterization in terms of decompositions into disjoint *Euclidean neighborhood retracts* (ENRs).

**Lemma 5.2** [6] *Let  $X$  be an ENR (for instance a finite-dimensional, locally finite CW-complex). Then the topological complexity  $\mathrm{TC}(X)$  equals the smallest integer  $k$  such that there exists a decomposition  $X \times X = E_0 \sqcup E_1 \sqcup \cdots \sqcup E_k$  into  $k + 1$  disjoint ENRs, on each of which there is a local section  $s_i: E_i \rightarrow X^I$ .*

*The existence of such a section  $s_i: E_i \rightarrow X^I$  is equivalent to the existence of a deformation of  $E_i$  into the diagonal of  $X \times X$ , ie a homotopy between the inclusion  $E_i \hookrightarrow X \times X$  and a map whose image lies entirely in the diagonal.*

**Theorem 5.3** *If  $\mathcal{A}$  is the annulus, then*

$$\mathrm{TC}(C(\mathcal{A}, n)) \leq 2n - 1.$$

The proof of [Theorem 5.3](#) occupies the rest of this section. By [Lemma 5.2](#) we need to find a decomposition of  $C_n(\mathcal{A}) \times C_n(\mathcal{A})$  into  $2n$  disjoint ENRs which can be deformed into the diagonal. Note that such deformations can equivalently be viewed as an explicit motion planner with  $2n$  different continuous rules and as such it is potentially relevant for applications.

### 5.1 Decomposition of $C_n(\mathcal{A}) \times C_n(\mathcal{A})$

The annulus can be identified with a product  $\mathcal{A} = S^1 \times \mathbb{R}$  of a circle and the real line. The projection map  $p: \mathcal{A} \rightarrow S^1$  induces a map

$$p_n: C_n(\mathcal{A}) \rightarrow \text{Sym}_n(S^1),$$

where the latter space is the  $n$ -fold symmetric power of  $S^1$ , defined as the quotient of  $(S^1)^{\times n}$  by the action of  $\mathfrak{S}_n$  on the coordinates.

For a given pair of configurations  $(x, y) \in C_n(\mathcal{A}) \times C_n(\mathcal{A})$  we interpret  $p_n(x)$  and  $p_n(y)$  as finite subsets of  $S^1$ , ie we forget the multiplicities of points in  $S^1$ . The cardinality  $\deg(x, y) = |p_n(x) \cup p_n(y)|$  of the union of those subsets will be called the *degree* of the pair.

Notice that  $\deg(x, y)$  is at least 1 and at most  $2n$ . This yields a decomposition of  $C_n(\mathcal{A}) \times C_n(\mathcal{A})$  into  $2n$  disjoint subspaces  $L_k = \deg^{-1}(k)$ , corresponding to the different values of  $\deg$ ; see [Figure 2](#). Furthermore,  $L_k$  is a smooth embedded manifold and in particular an ENR.

### 5.2 Local motion planners

Given a pair  $(x, y) \in L_k$ , the union  $p_n(x) \cup p_n(y)$  contains exactly  $k$  distinct points  $q_1, \dots, q_k \in S^1$ , ordered cyclically on  $S^1$  in the clockwise direction. We need to

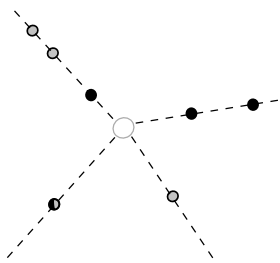


Figure 2: A pair of configurations in  $L_4$ , with one double point

introduce some notation. Let  $n_{x,i}$  be the number of points in  $x$  mapped to  $q_i$  under  $p$  and let  $n_{y,i}$  be the number of points in  $y$  mapped to  $q_i$  under  $p$ . Finally, let  $\delta_i = n_{x,i} - n_{y,i}$  be the difference between those two numbers.

The following map is continuous and well defined:

$$\psi_k: L_k \rightarrow \left\{ (m_i)_i \in \mathbb{Z}^k \mid \sum_{i=1}^k m_i = 0, \sum_{i=1}^k |m_i| \leq 2n \right\} / (12 \dots k), \quad (x, y) \mapsto [(\delta_i)_i].$$

Here  $(12 \dots k) \in \mathfrak{S}_k$  is the *long cycle*, permuting the components  $m_i$ .

Because the preimages of different  $[(\delta_i)_i]$  are topologically disjoint, we can define the local section of the free path fibration over  $L_k$  separately on each preimage.

Given a pair of configurations  $(x, y) \in L_k$  lying in the preimage  $\psi_k^{-1}([(\delta_i)_i])$ , we need to construct a path between them, continuously over  $L_k$ .

If  $\delta_i = 0$  for all  $i$ , we will simply move the points of  $x$  onto the points of  $y$  on each fiber of  $p$  by linear interpolation inside the fibers.

On the other hand, if there exists an  $i$  such that  $\delta_i \neq 0$ , first we need to construct a path from  $x$  to  $\tilde{x}$  such that  $\deg(\tilde{x}, y) = \tilde{k}$  for some  $\tilde{k} \leq k$ , and such that  $(\tilde{x}, y) \in \psi_{\tilde{k}}^{-1}([(0)_i])$ ; then we concatenate this path with the fiberwise linear interpolation used above. The path from  $x$  to  $\tilde{x}$  will consist in an iteration of one particular deformation which we describe in the following and which is illustrated in [Figure 3](#).

### 5.3 First step

Let  $(x, y) \in L_k$  as above and let  $x$  consist of the points  $x_{i,l} \in \mathcal{A}$  for  $1 \leq i \leq k$  and  $1 \leq l \leq n_{x,i}$ , where for each  $i$  the points  $x_{i,l}$  are exactly those lying over  $q_i \in S^1$  and the indices are chosen according to the order of the points on the fiber  $p^{-1}(q_i) \simeq \mathbb{R}$ .

We are going to deform  $x$  into another configuration, denoted by  $x^{(1)}$ .

Whenever  $\delta_i > 0$ , we move the  $\delta_i$  top points of  $x$  in  $p^{-1}(q_i)$  clockwise until they reach  $p^{-1}(q_{i+1})$ , on top of all points of  $x$  already in  $p^{-1}(q_{i+1})$  (if any). More precisely, we move the points  $x_{i,l}$  for  $n_{y,i} + 1 \leq l \leq n_{x,i}$  to  $p^{-1}(q_{i+1})$  so as to keep their order and their pairwise distances, and such that  $x_{i,n_{y,i}+1}$  reaches the position  $1 + \max\{0, x_{i+1,n_{x,i+1}}\}$  inside the fiber  $p^{-1}(q_{i+1}) \simeq \mathbb{R}$ . We move these points by linear interpolation along the interval  $[q_i, q_{i+1}] \subset S^1$  and along  $\mathbb{R}$ . We do this simultaneously for all  $i$  for which  $\delta_i > 0$ . Note that the indices are considered modulo  $k$ . This is shown in [Figure 3](#).

It is clear from the construction that this deformation is continuous within  $\psi_k^{-1}([(\delta_i)_i])$ .

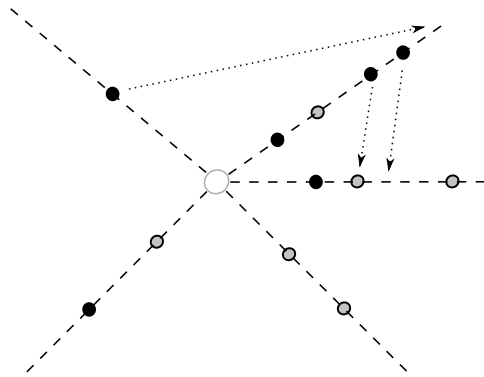


Figure 3: One iteration of the motion planner on the annulus. Notice that the positions of the gray points on a given fiber are disregarded when moving black points towards it because the points exist in two separate spaces.

5.4 Iterations of the first step

We started with a pair of configurations  $(x, y) \in L_k$  and in the previous subsection we constructed a deformation of  $x$  into  $x^{(1)}$ . Clearly  $k_1 = \deg(x^{(1)}, y) \leq k$ . We can now repeat the process starting with the pair  $(x^{(1)}, y)$  to get a new configuration  $x^{(2)}$ , again without changing  $y$ . Iterating this, we get a sequence of configurations  $x^{(j)}$  and a sequence of degrees  $k_j = \deg(x^{(j)}, y)$  which is weakly decreasing.

If this algorithm terminates after  $T$  steps, then it gives us a path from  $(x, y) \in L_k$  to  $(x^{(T)}, y) \in \psi_k^{-1}((0)_i)$ . Furthermore, because each iteration is continuous it yields a continuous deformation of  $L_k$  into  $\psi_k^{-1}((0)_i)$ , which completes the proof.

To see that the algorithm does indeed terminate, note that there exists an  $N \in \mathbb{N}$  such that  $k_j = k_N$  for all  $j \geq N$ . After  $k_N$  further iterations we have that  $\delta_i^{(N+k_N)} = 0$  for all  $1 \leq i \leq k_N$  and we are done. This follows from the following three facts, which are easy to check:

- (1) For all  $j > N$ , if  $\delta_i^{(j)} > 0$ , then  $\delta_{i-1}^{(j-1)} > 0$ .
- (2) For all  $j \geq N$ , if  $\delta_i^{(j)} \geq 0$ , then  $\delta_i^{(j+1)} \geq 0$ .
- (3) For all  $j \geq N$  we have  $\sum_i \delta_i^{(j)} = 0$ .

Indeed, if  $\delta_i^{(N+k_N)} \neq 0$  for some  $i$ , we may assume that  $\delta_i^{(N+k_N)} > 0$  because of (3). By (1), this would imply that  $\delta_{i-l}^{(N+k_N-l)} > 0$  for all  $0 \leq l \leq k_N - 1$  and therefore, by (2),  $\delta_{i-l}^{(N+k_N)} \geq 0$  for all  $0 \leq l \leq k_N - 1$ .



This would mean that  $\sum_{l=1}^{k_N-1} \delta_{i-l}^{(N+k_N)} > 0$ , which contradicts (3). This completes the proof of [Theorem 5.3](#).  $\square$

## 6 Proof of Theorems 1.3 and 1.5

**Proof of Theorem 1.3** Notice that the pure braid group on the disc  $P_n$  is isomorphic to the pure braid group  $P_{n-1}(\mathcal{A})$  on the annulus with one less strand in the following way. Every braid in  $P_n$  can be chosen such that the last strand does not move and that strand is identified with the central hole in the annulus.

Consider the subgroup  $Z_{n-1} \cong \mathbb{Z}^{n-1} \leq P_{n-1}(\mathcal{A}) \cong P_n$  given by the braids in  $P_{n-1}(\mathcal{A})$  in which all strands move in concentric circles around the central hole.

Recall that, for the abelianization,  $P_n^{\text{ab}} \cong \mathbb{Z}^{\binom{n}{2}}$ . The abelianization homomorphism is given by the collection over all unordered pairs  $\{i, j\}$  of the maps  $\psi_{i,j}: P_n \rightarrow P_2 \cong \mathbb{Z}$  forgetting all strands except the  $i^{\text{th}}$  and the  $j^{\text{th}}$  (measuring the linking number between the strands  $i$  and  $j$ ).

Conjugating by an element  $g \in B_n$  is compatible, under the abelianization, with the induced permutation of the components  $\psi_{i,j}$  of  $P_n^{\text{ab}}$  coming from the canonical permutation in  $\mathfrak{S}_n$  associated to  $g$ .

In light of the above, it is clear that the commutator subgroup  $[P_n, P_n]$  (the kernel of the abelianization homomorphism) is not only normal in  $P_n$  but also in  $B_n$ . Furthermore, it also follows that  $Z_{n-1}$  is mapped injectively under the abelianization homomorphism and thus has a trivial intersection with  $[P_n, P_n]$ . Taken together this implies that the conjugates of a nontrivial element of  $[P_n, P_n]$  cannot lie in  $Z_{n-1}$ .

The lower bound now follows from [Theorem 2.3](#) together with [Lemma 6.1](#) below.

Finally, the upper bound follows from [Lemma 2.4](#) and  $\text{cd}(B_n) = n - 1$ , which can be shown using the Fadell–Neuwirth fibrations as for the other aspherical surfaces.  $\square$

**Proof of Theorem 1.5** Let  $(p_1, \dots, p_n) \in F(D, n)$  denote an ordered configuration of  $n$  points in the disc  $D$  and let  $1 \leq k \leq n$ , to be chosen suitably later. Recall that based loops in  $F(D, n)$  represent braids in the pure braid group  $P_n$  and let  $A \subset P_n$  consist of those pure braids represented by loops in which the points  $p_1, p_2, \dots, p_k$  are fixed *in the middle* and  $p_{k+1}, \dots, p_n$  independently rotate around this cluster in concentric orbits. Clearly we have  $A \cong \mathbb{Z}^{n-k}$ .

We now write  $n = mk + r$  for appropriate  $m \geq 0$  and  $1 \leq r \leq k$ . Notice that  $r$  is assumed to be positive.

Divide the points  $p_1, \dots, p_n$  into  $m$  clusters of  $k$  points each plus an additional cluster of  $r$  points. Let  $B$  be the subgroup of  $P_n$  in which points of the same cluster interact freely and such that moreover the  $m + 1$  clusters are allowed to move around each other, so long as they don't mix and their trajectories describe an element in  $[P_{m+1}, P_{m+1}]$ .

More formally, let  $E_2(m + 1)$  be the space of ordered configurations of  $m + 1$  little discs  $D_1, \dots, D_{m+1}$  inside the disc  $D$ . Each disc  $D_i$  is uniquely determined by its center and its (positive) radius and the little discs are required to have disjoint interiors (see [14] for an introduction to the operad of little cubes). There is a map

$$E_2(m + 1) \times F(D, k) \times \cdots \times F(D, k) \times F(D, r) \rightarrow F(D, n)$$

given by embedding each configuration of  $k$  or  $r$  points into the corresponding disc  $D_i$ , using the only positive rescaling of  $D$  onto  $D_i$ . Because  $E_2(m + 1)$  is also a classifying space for  $P_{m+1}$ , there is a homomorphism on fundamental groups

$$\gamma\colon P_{m+1} \times P_k \times \cdots \times P_k \times P_r \rightarrow P_n.$$

To show that  $\gamma$  is injective, let  $\rho$  be the product of the following  $m + 2$  maps:

- One map  $P_n \rightarrow P_{m+1}$  given by forgetting all strands but a chosen one in each cluster, so that exactly  $m + 1$  strands remain.
- The maps  $P_n \rightarrow P_k$  and  $P_n \rightarrow P_r$  given by forgetting all strands outside a given cluster.

It is easy to see that  $\rho$  is a retraction of  $\gamma$  and that therefore  $\gamma$  is injective. The subgroup  $B \subset P_n$  is defined to be the image of the restriction of  $\gamma$  to  $[P_{m+1}, P_{m+1}] \times (P_k)^m \times P_r$ .

Next we need to check that  $A$  and  $B$  satisfy the assumptions of Theorem 2.3 as subgroups of  $B_n$ , ie  $gAg^{-1} \cap B = \{1\}$  for all  $g \in B_n$ . For this we will use the abelianization of the pure braid group  $P_n$ . As we saw in the proof of Theorem 1.3, the abelianization detects the pairwise linking numbers between the braids and conjugation by  $g \in B_n$  permutes those numbers by the induced permutation.

The following property of an element  $\sigma \in P_n$  is invariant under conjugation by each  $g \in B_n$ :

There exists an index  $1 \leq j \leq n$  and  $k$  other indices  $i_1, \dots, i_k$  such that  $\psi_{j,i_l}(\sigma) \neq 0$  for all  $1 \leq l \leq k$ .

Let  $\alpha \in A$  be a nontrivial braid. In such a braid there is at least one point  $p_j$ , for  $k+1 \leq j \leq n$ , which rotates a nonzero number of times around the points  $p_1, \dots, p_k$ . Therefore, the numbers  $\psi_{l,j}(\alpha)$  are all nonzero (and equal to each other) for  $1 \leq l \leq k$ .

However, no braid  $\beta \in B$  has the property above. Indeed,  $\psi_{i,j}(\beta)$  can be nonzero only if  $p_i$  and  $p_j$  are in the same cluster, and every cluster contains at most  $k$  points.

Hence, we get that, for each  $1 \leq k \leq n$ ,

$$\begin{aligned} \text{TC}(B_n) &\geq \text{cd}(A \times B) \\ &= n - k + m(k-1) + r - 1 + \text{cd}([P_{m+1}, P_{m+1}]) \\ &\geq 2n - k - m - 1 + \frac{1}{2}m - 1 \quad (\text{by Lemma 6.1}) \\ &= 2n - k - \frac{1}{2}m - \frac{3}{2}. \end{aligned}$$

Choosing  $k = \lfloor \sqrt{n/2} \rfloor$ , the inequality

$$n = mk + r \geq mk + 1$$

implies that

$$m \leq \frac{n-1}{k}$$

and so

$$m \leq \left\lfloor \frac{n-1}{k} \right\rfloor \leq 2k + 4$$

by the choice of  $k$ . Therefore,

$$\text{TC}(B_n) \geq 2n - 2 \lfloor \sqrt{n/2} \rfloor - 3 - \frac{1}{2}$$

and since  $\text{TC}(B_n)$  is an integer we can drop the term  $\frac{1}{2}$ . □

**Lemma 6.1** *Let  $[P_n, P_n]$  be the commutator subgroup of the pure braid group  $P_n$ . Then*

$$\text{cd}([P_n, P_n]) \geq \frac{1}{2}(n-2).$$

**Proof** Like in the previous proof, let  $E_2(3)$  denote the space of ordered configurations of three little discs  $D_1$ ,  $D_2$  and  $D_3$  inside a disc  $D$ . There exists a map

$$E_2(3) \times F(D, 3) \rightarrow F(D, 5),$$

given by embedding the configurations in  $F(D, 3)$  into the first disc  $D_1$  (after the appropriate rescaling) and by mapping the other two little discs to their center points.

Iterating this construction  $k-1$  times results in the map

$$\underbrace{E_2(3) \times E_2(3) \times \cdots \times E_2(3)}_{k-1} \times F(D, 3) \rightarrow F(D, 2k+1).$$

On fundamental groups this yields a homomorphism

$$(4) \qquad P_3^k \rightarrow P_{2k+1}.$$

Similarly to the previous proof, this homomorphism is injective. By construction the images of the different  $P_3$  factors commute with each other.

Let  $\mathbb{Z}$  be an infinite cyclic subgroup of  $[P_3, P_3]$ . The image of the homomorphism (4) restricted to  $\mathbb{Z}^k \leq P_3^k$  is isomorphic to  $\mathbb{Z}^k$  by the above observations and it is a subgroup of  $[P_{2k+1}, P_{2k+1}]$ . By the well-known properties of cohomological dimension,

$$\text{cd}([P_{2k+1}, P_{2k+1}]) \geq \text{cd}(\mathbb{Z}^k) = k.$$

This proves the claim for  $n = 2k + 1$  odd. For  $n$  even the claim immediately follows from  $P_{n-1} \leq P_n$ . □

## 7 Motion planner for the disc

Let  $D$  be the disc. In this section we are going to give an explicit motion planner which will imply that  $\text{TC}(C(D, 3)) = 3$ , as stated in [Theorem 1.6](#). Observe that a motion planner on a subset of  $X \times X$  is the same as a deformation into the diagonal.

**Proof of Theorem 1.6** The lower bound follows from [Theorem 1.3](#) because

$$\text{cd}([P_3, P_3] \times Z_2) = 1 + 2 = 3.$$

We will work with the space  $C_3 = C(\mathbb{C}, 3) \simeq C(D, 3)$  for the remainder of the proof.

To show  $\text{TC}(C_3) \leq 3$ , it suffices to find a decomposition of  $C_3 \times C_3$  into four disjoint ENRs such that each of them can be deformed to the diagonal, by [Lemma 5.2](#).

In the next subsections we will first decompose  $C_3 \times C_3$  into four disjoint ENRs  $E_0, E_1, E_2$  and  $E_3$  and discuss some geometric properties of these; then we will describe a motion planner on each  $E_i$ .

### 7.1 Decomposition of $C_3 \times C_3$

First we need a notion of *orientation* for configurations in  $C_3$ . For this we define a function  $\Delta: C_3 \rightarrow \mathbb{C}^*$  by

$$\Delta(\{z_1, z_2, z_3\}) = (z_1 - z_2)^2(z_2 - z_3)^2(z_3 - z_1)^2,$$

and let  $\delta = \Delta/|\Delta|: C_3 \rightarrow S^1$  be its normalization.

We say that two configurations  $x, y \in C_3$  are *cooriented* if  $\delta(x) = \delta(y)$ . Let  $P \subset C_3 \times C_3$  denote the closed subspace of pairs  $(x, y)$  for which  $x$  and  $y$  are cooriented and let  $N = C_3 \times C_3 \setminus P$  denote its complement.

The Lie group  $S^1$  on  $C_3$  by rotations about the origin. Given a configuration  $x \in C_3$  and an element  $\theta \in S^1$ ,

$$(5) \quad \delta(\theta \cdot x) = \theta^6 \delta(x).$$

Let  $L \subset C_3$  consist of those configurations for which all three points are on a line and let  $T = C_3 \setminus L$  be its complement. The points in a configuration in  $T$  form a nondegenerate triangle;  $L \subset C_3$  is closed and  $T \subset C_3$  is open.

We define a deformation retraction of  $L$  onto the subspace  $L_R$  containing configurations of three aligned points, one at the origin and two on the unit circle and opposite to each other. Note that  $L_R$  is homeomorphic to a circle and is invariant under rotation. Given a configuration in  $L$ , we translate it so that the central point ends up at the origin and then slide the two outer points along the line which goes through all three points until they are both at distance 1 from the origin. This defines a deformation retraction  $r_L: L \rightarrow L_R$ . The deformation preserves  $\delta$ , because the direction determined by any two points in the configuration remains the same throughout the deformation.

Similarly we define a deformation retraction of  $T$  onto the subspace  $T_R$  containing configurations of three points on the unit circle that form an equilateral triangle. Note that  $T_R$  is also homeomorphic to a circle and invariant under rotation. Given a configuration in  $T$ , we translate it until the center of mass coincides with the origin. Then we slide all three points simultaneously along the lines going through the origin until the points land in the unit circle. Finally, we rotate the points until they are at equal distance from each other on the unit circle.

More precisely, let  $X, Y$  and  $Z$  be a configuration of three points on the unit circle, appearing in this order clockwise. Consider the lengths of the arcs  $XY, YZ$  and  $ZX$ . If the arcs are all of the same length, then we are done. If there is precisely one arc of minimal length, say  $XY$ , then we can slide  $X$  and  $Y$  at the same speed along the unit circle, gradually increasing the length of  $XY$  and decreasing both  $YZ$  and  $ZX$ , until the length of  $XY$  becomes equal to at least one of the other two arcs. Therefore, we may assume that there are exactly two arcs of minimal length. In this case there is one arc, say  $YZ$ , which is strictly longer than the other two arcs. Slide both  $Y$  and  $Z$  at the same speed along the unit circle, gradually decreasing the length of  $YZ$  and increasing the lengths of  $XY$  and  $ZX$ , until all three arcs are equal. See [Figure 4](#).

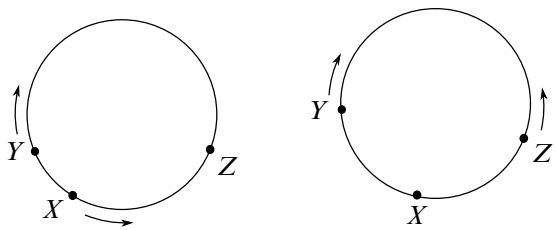


Figure 4: First step (left) and second step (right)

Additionally, we make sure that the above deformation preserves  $\delta(x)$  by constantly rotating the configuration  $x$  about the origin during the whole process to compensate for the potential change of  $\delta(x)$ . More precisely, let  $H: T \times [0, 1] \rightarrow C_3$  be the homotopy described above, with  $H(\cdot, 1) \in T_R$ , and consider the function  $\bar{\delta}: T \times [0, 1] \rightarrow S^1$  defined by

$$\bar{\delta}(x, t) = \delta(H(x, t)) / \delta(x).$$

Then  $\bar{\delta}(\cdot, 0): T \rightarrow S^1$  is the constant function 1 and it admits a lift to the universal covering  $\mathbb{R} \rightarrow S^1$ , namely the constant function 0. We can then extend this lift to all positive times, obtaining a map  $\tilde{\delta}: T \times [0, 1] \rightarrow \mathbb{R}$ . Now let  $\tilde{\rho}: T \times [0, 1] \rightarrow \mathbb{R}$  be given by

$$\tilde{\rho}(x, t) = \frac{1}{6} \tilde{\delta}(x, t)$$

and denote by  $\rho: T \times [0, 1] \rightarrow S^1$  its projection onto  $S^1$  along the universal covering map  $\mathbb{R} \rightarrow S^1$ .

Finally, consider the homotopy  $\bar{H}: T \times [0, 1] \rightarrow C_3$  given by

$$\bar{H}(x, t) = (\rho(x, t))^{-1} \cdot H(x, t).$$

Then  $\bar{H}$  is a deformation retraction of  $T$  onto  $T_R$  preserving  $\delta$  at all times; this follows easily from the construction and from formula (5).

Write  $r_T = \bar{H}(\cdot, 1): T \rightarrow T_R$ .

We are now ready to construct the decomposition into disjoint ENRs as follows:

- $E_0 = P \cap (L \times L).$
- $E_1 = N \cap (L \times L) \sqcup P \cap (T \times L \sqcup L \times T).$
- $E_2 = N \cap (T \times L \sqcup L \times T) \sqcup P \cap (T \times T).$
- $E_3 = N \cap (T \times T).$

Note that the subspaces  $E_i$  are semialgebraic sets and therefore ENRs.

Furthermore, the disjoint unions above are topological, ie they form disconnected components inside each  $E_i$ . This follows from the fact that the disjoint components are relatively open inside each  $E_i$ . For example,  $N \cap (L \times L)$  and  $P \cap (T \times L)$  are the intersections of  $E_1$  with the open sets  $N$  and  $T \times C_3$ , respectively, and  $N \cap (T \times L)$  is the intersection of  $E_2$  with the open set  $N \cap (T \times C_3)$ .

7.2 Local motion planners

We show now that each  $E_i$  deformation retracts onto a disjoint union of circles. First we notice that for  $A, B \in \{L, T\}$  the intersection  $N \cap (A \times B)$  can be deformed to  $P \cap (A \times B)$ . Given a pair  $(x, y) \in N \cap (A \times B)$ , rotate  $x$  clockwise about the origin until  $x$  and  $y$  are cooriented. This can be done continuously thanks to formula (5).

The subspaces  $P \cap (L \times L)$  and  $P \cap (T \times T)$  deformation retract to  $P \cap (L_R \times L_R)$  and  $P \cap (T_R \times T_R)$ , respectively, because the retractions  $r_L$  and  $r_T$  commute with  $\delta$ .

The subspaces  $P \cap (L_R \times L_R)$  and  $P \cap (T_R \times T_R)$  in turn consist of a disjoint union of three circles and a disjoint union of two circles, respectively, where each circle is an orbit under the diagonal action of  $S^1$  on  $C_3 \times C_3$ . Precisely one orbit in  $P \cap (L_R \times L_R)$  and one orbit in  $P \cap (T_R \times T_R)$  already lie in the diagonal of  $C_3 \times C_3$ . The remaining orbits consist of pairs of lines or pairs of triangles which are at a given angle from each other ( $\frac{\pi}{3}$  or  $\frac{2\pi}{3}$  in the case of lines and  $\frac{\pi}{3}$  in the case of triangles, to be precise). See Figures 5 and 6. They can be deformed into the diagonal by rotating the first configuration in every pair clockwise about the origin until it is equal to the second configuration in that pair.

Similarly, the space  $P \cap (L \times T)$  can be deformed to  $P \cap (L_R \times T_R)$ , which consists of one single orbit under the diagonal  $S^1$ -action; see Figure 7. Specifically, it contains pairs of configurations  $(x, y)$ , where the points in  $y$  form an equilateral triangle centered at the origin and the points in  $x$  lie on a line parallel to one of the sides of said triangle and are symmetrically distributed around the origin. We move the point in  $y$  opposite to the side parallel to  $x$  to the origin and the other two points in  $y$  to the

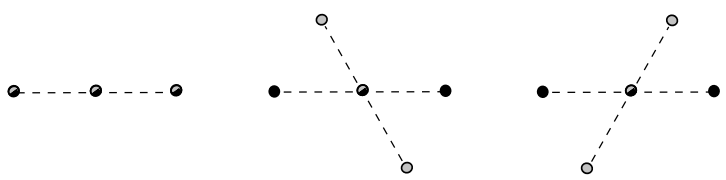


Figure 5: Path-components of  $P \cap (L_R \times L_R)$  (up to rotation)

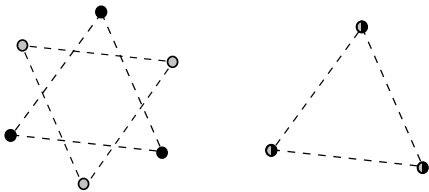


Figure 6: Path-components of  $P \cap (T_R \times T_R)$  (up to rotation)

corresponding outer points in  $x$ . The pair  $(x, x)$  is obviously in the diagonal and so we are done.

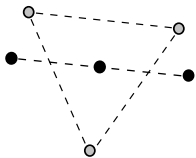


Figure 7: The subset  $P \cap (T_R \times L_R)$  (up to rotation)

This completes the proof because the deformation can be defined separately on the different disconnected components of each  $E_i$ . □

## 8 Conclusions

The results in this paper can be viewed equivalently as finding the values for the topological complexity of either full braid groups of surfaces or unordered configuration spaces of surfaces, since, for aspherical surfaces  $\mathcal{S}$ ,

$$\mathrm{TC}(C(\mathcal{S}, n)) = \mathrm{TC}(B_n(\mathcal{S})).$$

All the results except the ones which rely on finding explicit motion planners (or equivalently deformations into the diagonal) extend to finite-index subgroups of  $B_n(\mathcal{S})$  with the same proofs. To be precise, the results which generalize to finite-index subgroups are the ones given in Theorems 1.3, 1.5, 4.1, 4.2 and 5.1.

In particular, those results apply to the pure braid groups  $P_n(\mathcal{S})$  and the mixed braid groups from [13]. Observe that for aspherical surfaces  $\mathcal{S}$  the topological complexity  $\mathrm{TC}(P_n(\mathcal{S}))$  of the pure braid groups of  $\mathcal{S}$  is the same as the topological complexity  $\mathrm{TC}(F(\mathcal{S}, n))$  of the ordered configuration spaces of  $\mathcal{S}$ .

Thus, the methods in this paper yield an alternative proof for some of the results given by Cohen and Farber in [2], in particular the topological complexity of ordered configuration spaces for all nonclosed orientable surfaces (for the ordered configuration



spaces of the disc one can use a slightly modified version of the proof of [Theorem 5.3](#) to find explicit motion planners). Furthermore, it extends their results to all nonclosed nonorientable surfaces except the Möbius band.

It is worth noting that the results in this paper taken together with the results in [\[2\]](#) are consistent with the possibility that the topological complexities of the ordered and the unordered configuration spaces of a surface coincide for all surfaces.

The only remaining aspherical surface for which the gap between the lower bound and the upper bound for the topological complexity of its unordered configuration spaces is still arbitrarily large is, perhaps surprisingly, the disc.

If it is in fact true that  $\text{cd}([P_n, P_n]) = n - 2$ , then [Theorem 1.3](#) would imply

$$\text{TC}(C(D, n)) \geq 2n - 3.$$

If additionally the upper bound for  $n = 3$  given in [Theorem 1.6](#) generalized to higher  $n$ , this would completely determine  $\text{TC}(C(D, n))$ . We make the following:

**Conjecture 8.1** If  $D$  is the disc, then

$$\text{TC}(C(D, n)) = \text{TC}(B_n) = 2n - 3.$$

## References

- [1] **D C Cohen**, *Topological complexity of classical configuration spaces and related objects*, from “Topological complexity and related topics” (M Grant, G Lupton, L Vandembroucq, editors), Contemp. Math. 702, Amer. Math. Soc., Providence, RI (2018) 41–60 [MR](#)
- [2] **D C Cohen**, **M Farber**, *Topological complexity of collision-free motion planning on surfaces*, Compos. Math. 147 (2011) 649–660 [MR](#)
- [3] **S Eilenberg**, **T Ganea**, *On the Lusternik–Schnirelmann category of abstract groups*, Ann. of Math. 65 (1957) 517–518 [MR](#)
- [4] **E Fadell**, **L Neuwirth**, *Configuration spaces*, Math. Scand. 10 (1962) 111–118 [MR](#)
- [5] **M Farber**, *Topological complexity of motion planning*, Discrete Comput. Geom. 29 (2003) 211–221 [MR](#)
- [6] **M Farber**, *Topology of robot motion planning*, from “Morse theoretic methods in nonlinear analysis and in symplectic topology” (P Biran, O Cornea, F Lalonde, editors), NATO Sci. Ser. II Math. Phys. Chem. 217, Springer (2006) 185–230 [MR](#)
- [7] **M Farber**, *Configuration spaces and robot motion planning algorithms*, from “Combinatorial and toric homotopy” (A Darby, J Grbić, Z Lü, J Wu, editors), Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap. 35, World Sci., Hackensack, NJ (2018) 263–303 [MR](#)

- [8] **M Farber, M Grant**, *Topological complexity of configuration spaces*, Proc. Amer. Math. Soc. 137 (2009) 1841–1847 [MR](#)
- [9] **M Farber, M Grant, G Lupton, J Oprea**, *Bredon cohomology and robot motion planning*, preprint (2018) [arXiv](#)
- [10] **M Farber, S Yuzvinsky**, *Topological robotics: subspace arrangements and collision free motion planning*, from “Geometry, topology, and mathematical physics” (V M Buchstaber, I M Krichever, editors), Amer. Math. Soc. Transl. Ser. 2 212, Amer. Math. Soc., Providence, RI (2004) 145–156 [MR](#)
- [11] **R Ghrist**, *Configuration spaces and braid groups on graphs in robotics*, from “Knots, braids, and mapping class groups: papers dedicated to Joan S Birman” (J Gilman, W W Menasco, X-S Lin, editors), AMS/IP Stud. Adv. Math. 24, Amer. Math. Soc., Providence, RI (2001) 29–40 [MR](#)
- [12] **M Grant, G Lupton, J Oprea**, *New lower bounds for the topological complexity of aspherical spaces*, Topology Appl. 189 (2015) 78–91 [MR](#)
- [13] **M Grant, D Recio-Mitter**, *Topological complexity of subgroups of Artin’s braid groups*, from “Topological complexity and related topics” (M Grant, G Lupton, L Vandembroucq, editors), Contemp. Math. 702, Amer. Math. Soc., Providence, RI (2018) 165–176 [MR](#)
- [14] **J P May**, *The geometry of iterated loop spaces*, Lectures Notes in Mathematics 271, Springer (1972) [MR](#)
- [15] **L Paris, D Rolfsen**, *Geometric subgroups of surface braid groups*, Ann. Inst. Fourier (Grenoble) 49 (1999) 417–472 [MR](#)
- [16] **S Scheirer**, *Topological complexity of  $n$  points on a tree*, Algebr. Geom. Topol. 18 (2018) 839–876 [MR](#)
- [17] **J R Stallings**, *On torsion-free groups with infinitely many ends*, Ann. of Math. 88 (1968) 312–334 [MR](#)
- [18] **R G Swan**, *Groups of cohomological dimension one*, J. Algebra 12 (1969) 585–610 [MR](#)

Mathematics Institute, University of Bonn  
Bonn, Germany

Institute of Mathematics, University of Aberdeen  
Aberdeen, United Kingdom

Current address: Department of Mathematics, Lehigh University  
Bethlehem, PA, United States

[bianchi@math.uni-bonn.de](mailto:bianchi@math.uni-bonn.de), [dar318@lehigh.edu](mailto:dar318@lehigh.edu)

Received: 23 February 2018

Revised: 21 October 2018