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Hyperbolic extensions of free groups from atoroidal ping-pong

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We prove that all atoroidal automorphisms of $\mathrm{Out}(F_N)$ act on the space of projectivized geodesic currents with generalized north–south dynamics. As an application, we produce new examples of nonvirtually cyclic, free and purely atoroidal subgroups of $\mathrm{Out}(F_N)$ such that the corresponding free group extension is hyperbolic. Moreover, these subgroups are not necessarily convex cocompact.

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1 Introduction

Let F_N be a free group of rank $N \geq 3$, and $\mathrm{Out}(F_N)$ be its outer automorphism group. Consider the short exact sequence

$$1 \to F_N \xrightarrow{\iota} \mathrm{Aut}(F_N) \xrightarrow{q} \mathrm{Out}(F_N) \to 1,$$

where ι sends an element of F_N to the corresponding inner automorphism, and q is the natural quotient map.

Given a subgroup $\Gamma < \mathrm{Out}(F_N)$, the preimage $E_\Gamma = q^{-1}(\Gamma)$ gives an extension of F_N. In fact, any extension of F_N produces an extension of this form; see Dowdall and Taylor [16, Section 2]. Motivated by a long history of investigating hyperbolic extensions of hyperbolic groups — see Bestvina and Feighn [1], Bestvina, Feighn and Handel [4], Farb and Mosher [17], Hamenstaedt [20], Kent and Leininger [26] and Mosher [31] — Dowdall and Taylor [16] initiated a systematic study of the following question:

What conditions on the group Γ guarantee that the extension group E_Γ is hyperbolic?

When the group Γ is infinite cyclic, generated by $\varphi \in \mathrm{Out}(F_N)$, combined work of Bestvina and Feighn [1] and Brinkmann [8] shows that E_Γ is hyperbolic if and only if φ is *atoroidal*, meaning that no power of φ fixes a nontrivial conjugacy class in F_N. Dowdall and Taylor [16] proved that if a finitely generated subgroup $\Gamma < \mathrm{Out}(F_N)$ is

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purely atoroidal (ie every infinite-order element is atoroidal) and the orbit map from Γ into the free factor complex is a quasi-isometric embedding, then the extension E_Γ is hyperbolic. The second condition also implies that every infinite-order element $\varphi \in \Gamma$ is *fully irreducible*, meaning that no power of φ fixes the conjugacy class of a proper free factor; see Section 2 for definitions.

So far the only known examples of hyperbolic extensions of free groups come from slight variations, or iterated applications of these two constructions, and Schottky-type subgroups generated by high powers of fully irreducible and atoroidal elements. The following subgroup alternative result allows us to produce more examples:

Theorem 1.1 *Let $\mathcal{H} < \mathrm{Out}(F_N)$ be a subgroup that contains an atoroidal element φ. Then one of the following occurs:*

(1) *There is some minimal $\mathcal{H}$–invariant free factor A of F_N such that the restriction of $\mathcal{H}$ to A is virtually cyclic in $\mathrm{Out}(A)$.*

(2) *There exists a subgroup $\Gamma \leq \mathcal{H}$ such that $\Gamma \cong \mathrm{F}_2$ and that every nontrivial element of Γ is atoroidal. Moreover, the corresponding extension group E_Γ is hyperbolic.*

Remark 1.2 Theorem 1.1 generalizes a well-known result of Bestvina, Feighn and Handel. Indeed, if the subgroup $\mathcal{H}$ is irreducible, namely no finite-index subgroup of $\mathcal{H}$ fixes a proper free factor, then $\mathcal{H}$ contains a fully irreducible element by a theorem of Handel and Mosher [21]; see also Horbez [22] for a concise and more general proof. Since $\mathcal{H}$ contains an atoroidal element, then Theorem 5.4 of Uyanik [36] implies that $\mathcal{H}$ contains an element which is both fully irreducible and atoroidal. In that case, Bestvina, Feighn and Handel [4] show that either $\mathcal{H}$ is virtually cyclic, or there is a nonabelian free subgroup Γ of $\mathcal{H}$ such that every nontrivial element of Γ is atoroidal and the corresponding free group extension is hyperbolic. A different proof of the aforementioned result of Bestvina, Feighn and Handel is given by Kapovich and Lustig [25], who additionally obtained that each nontrivial element is fully irreducible.

Remark 1.3 The subgroup $\mathcal{H}$ isn't necessarily irreducible or it doesn't have to preserve a free splitting of F_N. Theorem 1.1 gives new examples of hyperbolic extensions of free groups, which do not come from previously known constructions. In particular, they are not necessarily convex cocompact; see Dowdall, Taylor and Tiozzo [15; 16; 34].

The main ingredient in the proof of Theorem 1.1 is the following dynamical result. See Section 3.1 for definitions.

Theorem 1.4 *Let $\varphi \in \mathrm{Out}(F_N)$ be an atoroidal outer automorphism of a free group of rank $N \geq 3$. Then there exist a simplex of attraction Δ_+ and a simplex of repulsion Δ_- in $\mathbb{P}\mathrm{Curr}(F_N)$ such that φ acts on $\mathbb{P}\mathrm{Curr}(F_N)$ with generalized north–south dynamics from Δ_- to $\Delta+$.*

The space $\mathbb{P}\mathrm{Curr}(F_N)$ of *projectivized geodesic currents* contains positive multiples of conjugacy classes as a dense subset, and hence serves as a natural tool for detecting atoroidal outer automorphisms; see Section 2.5 for details. The proof of Theorem 1.4 builds on our earlier results with M Lustig about dynamics of reducible substitutions [29] and is modeled on the proof of the specific case where both φ and φ^{-1} admit absolute train track representatives as we treated in [28]. In this paper, we use completely split relative train track maps (CTs), which are particularly nice topological representatives introduced by Feighn and Handel [18]. The new key insight in the proof of Theorem 1.4 is to use the legal structure coming from the splitting units in the CT that represents $\varphi \in \mathrm{Out}(F_N)$ rather than using the classical legal structure coming from the edges.

As a byproduct of Theorem 1.4 we also obtain:

Corollary 1.5 *Let $\varphi \in \mathrm{Out}(F_N)$ be a fully irreducible and atoroidal outer automorphism. Then, for any atoroidal outer automorphism $\psi \in \mathrm{Out}(F_N)$ (not necessarily fully irreducible) which is not commensurable with φ (ie $\varphi^t \neq \psi^s$ for any s and t), there exists an exponent $M > 0$ such that, for all $n, m > M$, the subgroup $\Gamma = \langle \varphi^n, \psi^m \rangle < \mathrm{Out}(F_N)$ is purely atoroidal and the corresponding extension E_Γ is hyperbolic.*

Note that the subgroup Γ in Corollary 1.5 is irreducible, and since Γ is not purely fully irreducible the orbit map to the free factor graph is not a quasi-isometric embedding [16].

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2 Preliminaries

2.1 Graphs and graph maps

A *graph* G is a 1–dimensional cell complex, where 0–cells are called *vertices* and 1–cells are called topological edges. We denote the set of vertices by VG, and the set of topological edges by $E_{\text{top}}G$. Identifying the interior of an edge with the open interval $(0,1)$ each edge admits exactly two orientations. We denote the set of *oriented* edges by EG. Choosing an orientation on each edge splits the set EG into two disjoint sets: the set E^+G of positively oriented and the set E^-G of negatively oriented edges. Given an oriented edge $e \in EG$, the initial vertex of e is denoted by $o(e)$ and the terminal vertex of e is denoted by $t(e)$, and the edge with the opposite orientation is denoted by e^{-1}.

An *edge path* γ in G is a concatenation $\gamma = e_1 e_2 \cdots e_n$ of edges in G such that $t(e_{i-1}) = o(e_i)$ for all $i = 2, \ldots, n$. An edge path $\gamma = e_1 e_2 \cdots e_n$ is called *reduced (or tight)* if $e_i^{-1} \neq e_{i+1}$ for all $i = 1, \ldots, n-1$. A reduced edge path $\gamma = e_1 e_2 \cdots e_n$ is called cyclically reduced if $o(\gamma) = t(\gamma)$ and in addition $e_n^{-1} \neq e_1$. We call cyclically reduced edge paths *circuits*.

Given an edge path γ, we denote the reduced edge path obtained by a homotopy relative to endpoints of γ by $[\gamma]$.

2.2 Markings and topological representatives

Let $\mathrm{R_N}$ denote the rose with N pedals, which is the finite graph with one vertex and N loop edges attached to that vertex. A *marking* is a homotopy equivalence $m\colon \mathrm{R_N} \to G$ where G is a finite graph all of whose vertices are at least valence 2.

A homotopy equivalence $f\colon G \to G$ is a *(topological) graph map* if it sends vertices to vertices, and its restriction to the interior of an edge is an immersion. Let $m'\colon G \to \mathrm{R_N}$ be a homotopy inverse to the marking $m\colon \mathrm{R_N} \to G$. We say that a topological graph map is a *topological representative* of an outer automorphism $\varphi \in \mathrm{Out}(F_N)$ if the induced map satisfies $(m' \circ f \circ m)_*\colon F_N \to F_N = \varphi$.

A *filtration* for a topological representative $f\colon G \to G$ is an ascending sequence of f–invariant subgraphs $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_k = G$. The closure of $G_r \setminus G_{r-1}$ is called the r^{th} *stratum*, and is denoted by H_r.

For each stratum H_r, there is an associated *transition matrix* M_r of H_r which is a nonnegative integer square matrix. The ij^{th} entry of H_r records the number of times $[f(e_i)]$ crosses e_j or e_j^{-1}. A nonnegative square matrix M is called *irreducible* if for each i and j, there exists $k = k(i, j)$ such that $M_{ij}^k > 0$, the matrix M is called *primitive* if k can be chosen independent of i and j. The stratum H_r is called *irreducible (resp. primitive)* if and only if M_r is irreducible (resp. primitive). If M_r is irreducible then it has a unique eigenvalue $\lambda \geq 1$, called the Perron–Frobenius (PF) eigenvalue, for which the associated eigenvector is positive. We say that H_r is an *exponentially growing stratum* or EG stratum if $\lambda > 1$ and *nonexponentially growing stratum* or NEG stratum if $\lambda = 1$. We say that H_r is a *zero stratum* if M_r is the zero matrix.

2.3 Train track maps

We first set up the relevant terminology to define relative train track maps, and their strengthened versions, CTs. The standard resources for this section are [6; 5; 18].

Let $f\colon G \to G$ be a topological graph map. A *direction* at a point $v \in G$ is the germ of an initial segment of an oriented edge. The map $f\colon G \to G$ induces a natural *derivative map* Df on the set of germs, and we say that a direction is *fixed* or *periodic* if it is fixed or periodic under the derivative map. A *turn* in G is an unordered pair of directions. We say that a turn is *degenerate* if the two directions are the same, and *nondegenerate* otherwise. A turn is called *illegal* if its image under some iterate of Df is degenerate, otherwise a turn is called *legal*. An edge path $\gamma = e_1 e_2 \cdots e_k$ is called legal if each turn (e_i^{-1}, e_{i+1}) is legal. We say that, a turn is contained in a stratum H_r if both directions are contained in H_r. An edge path γ is called *r–legal* if every turn in γ that is contained in H_r is legal. If H_r is an EG stratum, and γ is a path whose endpoints are in $H_r \cap G_{r-1}$, then γ is called a *connecting path*.

Definition 2.1 A homotopy equivalence $f\colon G \to G$ representing $\varphi \in \mathrm{Out}(F_N)$ is called a *relative train track* map if for every exponentially growing stratum H_r the following hold:

- (RTT-i) Df maps the set of directions in H_r to itself.
- (RTT-ii) For each connecting path γ for H_r, $[f(\gamma)]$ is a connecting path for H_r. In particular, $[f(\gamma)]$ is nontrivial.
- (RTT-iii) If γ is r–legal, then $[f(\gamma)]$ is r–legal.

Definition 2.2 (Nielsen paths) A path ρ is a *periodic Nielsen path* if there is an exponent $k \geq 1$ such that $[f^k(\rho)] = \rho$. The minimal such k is called the *period*, and if $k = 1$ then ρ is called a *Nielsen path*. A periodic Nielsen path is called *indivisible* if it cannot be written as a concatenation of periodic Nielsen paths. We will denote the (periodic) indivisible Nielsen paths by (pINPs) INPs.

Definition 2.3 (taken and exceptional paths) A path $\gamma \in G$ is called r*–taken* by $f\colon G \to G$ if γ appears as a subpath of $f^k(e)$ for some $k \geq 1$ and for some edge $e \in H_r$ in an irreducible stratum. We will drop r and only say *taken* whenever r is irrelevant. Let e_i and e_j be linear edges as defined in Definition 2.10 below such that $f(e_i) = e_i w^{m_i}$ and $f(e_j) = e_j w^{m_j}$ for some root free Nielsen path w. Then a path of the form $e_i w^p e_j^{-1}$ for $p \in \mathbb{Z}$ is called an *exceptional path*.

Definition 2.4 (splittings and complete splittings) Let $f\colon G \to G$ be a relative train track map. A decomposition of a path γ in G into subpaths $\gamma = \gamma_1 \cdot \gamma_2 \cdots \gamma_m$ is called a *splitting* if $[f^k(\gamma)] = [f^k(\gamma_1)][f^k(\gamma_2)]\cdots[f^k(\gamma_m)]$. Namely, one can tighten the image $f^k(\gamma)$ by tightening the images of the subpaths γ_i. We use the "$\cdot$" notation for splittings.

A splitting $\gamma = \gamma_1 \cdot \gamma_2 \cdots \gamma_m$ is called a *complete splitting* if each term γ_i is one of the following:

(1) an edge in an irreducible stratum;

(2) an INP;

(3) an exceptional path;

(4) a connecting path in a zero stratum that is both maximal and taken.

The paths in the above list are called *splitting units*.

Lemma 2.5 [5; 18] *Every completely split path or circuit has a unique complete splitting.*

The properties of relative train track maps are not strong enough for our purposes. Hence, in order to study the dynamics of atoroidal outer automorphisms, we utilize *completely split train track maps* (CTs) introduced by Feighn and Handel. In what follows, rather than giving the defining properties of CTs we will list the relevant properties of CTs and cite the appropriate resources. We refer the reader to [18] for a detailed discussion of CTs. We begin with two definitions:

Definition 2.6 A subgroup $\mathrm{F} < F_N$ is called a *free factor* of F_N if there is another subgroup $\mathrm{F}' < F_N$ such that $\mathrm{F} * \mathrm{F}' = F_N$. We denote the conjugacy class of a free factor F with [F]. A *free factor system* is a collection $\mathcal{F} = \{[\mathrm{F}^1], \ldots, [\mathrm{F}^k]\}$ of conjugacy classes of free factors of F_N such that there exists $\mathrm{F}' < F_N$ (possibly trivial) with the property that $F_N = \mathrm{F}^1 * \cdots * \mathrm{F}^k * \mathrm{F}'$. There is a partial order on the set of free factor systems as follows: given two free factor systems $\mathcal{F} = \{[\mathrm{F}^1], \ldots, [\mathrm{F}^k]\}$ and $\mathcal{F}' = \{[\mathrm{F}'^1], \ldots, [\mathrm{F}'^l]\}$, we say that $\mathcal{F} \sqsubset \mathcal{F}'$ if for each $[\mathrm{F}^i] \in \mathcal{F}$ there exists $[\mathrm{F}'^j] \in \mathcal{F}'$ such that $g\,\mathrm{F}^i\,g^{-1} < F'^j$ for some $g \in F_N$.

The *free factor graph* $\mathcal{FF}(F_N)$ is the (infinite) graph whose vertices correspond to conjugacy classes of proper free factors, and there is an edge between [F] and [F′] if either $\mathrm{F} < g\,\mathrm{F}'\,g^{-1}$ or $\mathrm{F}' < g\,\mathrm{F}\,g^{-1}$ for some $g \in F_N$. By declaring the length of each edge 1, $\mathcal{FF}(F_N)$ is equipped with a path metric d, and a result of Bestvina and Feighn says that $\mathcal{FF}(F_N)$ is hyperbolic [2]. The group $\mathrm{Out}(F_N)$ acts on $\mathcal{FF}(F_N)$ with simplicial isometries and fully irreducible elements are precisely the loxodromic isometries [2].

Definition 2.7 For any marked graph G and a subgraph K of G, the fundamental group of the noncontractible components of K determines a free factor system $[\pi_1(K)] = \mathcal{F}$ of F_N. We say that K *realizes* $\mathcal{F}$. Given a nested sequence $\mathcal{C}$ of free factor systems $\mathcal{F}^1 \sqsubset \mathcal{F}^2 \sqsubset \cdots \sqsubset \mathcal{F}^n$ we say that $\mathcal{C}$ is *realized* by a relative train track map $f\colon G \to G$ if there is an f–invariant filtration $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_k = G$ such that for all $1 \le i \le n$ we have $\mathcal{F}^i = [\pi_1(G_{k(i)})]$ for some $k(i)$.

The following theorem is the main existence result about CTs:

Theorem 2.8 [18, Theorem 4.28, Lemma 4.42] *There exists a uniform constant $M = M(N) \ge 1$ such that for any $\varphi^M \in \mathrm{Out}(F_N)$ and any nested sequence $\mathcal{C}$ of φ^M–invariant free factor systems, there exists a CT $f\colon G \to G$ that represents φ^M and realizes $\mathcal{C}$.*

We now state several results about structures of paths in CTs that will be relevant in the discussion follows.

Lemma 2.9 [18, Lemma 4.21] *If $f\colon G \to G$ is a CT, then every NEG stratum H_r consists of a single edge e_i. Moreover, either e_i is fixed, or $f(e_i) = e_i \cdot u_i$, where u_i is a nontrivial, completely split circuit in G_{i-1}.*

Definition 2.10 Let $e \in G$ be an NEG edge. The edge e is called a *fixed edge* if $f(e) = e$, a *linear edge* if $f(e) = e\eta$, where η is a nontrivial Nielsen path, and a *superlinear edge* otherwise.

Lemma 2.11 (properties of CTs [18, Definition 4.7, Lemma 4.13, Lemma 4.15, Corollary 4.19, Lemma 4.25]) (1) *For each edge e in an irreducible stratum, $f(e)$ is completely split. For each taken connecting path γ in a zero stratum, $[f(\gamma)]$ is completely split.*

(2) *For each filtration element G_r, H_r is a zero stratum if and only if H_r is a contractible component of G_r. In particular, there are only finitely many reduced connecting paths that are contained in some zero stratum.*

(3) *Every periodic indivisible Nielsen path (INP) has period one.*

(4) *The endpoints of all INPs are vertices. The terminal endpoint of each NEG edge is fixed.*

(5) *If γ is a circuit or an edge-path, then $[f^k(\gamma)]$ is completely split for all sufficiently large k.*

(6) *Each zero stratum H_i is enveloped by an EG stratum H_r, each edge in H_i is r–taken, and each vertex in H_i is contained in H_r and has link contained in $H_i \cup H_r$.*

(7) *If H_r is an EG stratum, then there is at most one indivisible Nielsen path ρ_r of height r that intersects H_r nontrivially. The initial edges of ρ_r and ρ_r^{-1} are distinct edges in H_r.*

(8) *If H_r is a zero stratum or an NEG superlinear stratum, then no Nielsen path crosses an edge of H_r.*

2.4 CTs representing atoroidal automorphisms

Given an atoroidal outer automorphism $\varphi \in \mathrm{Out}(F_N)$, let $f\colon G \to G$ be a CT with filtration $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_k = G$ that represents a suitable power of φ as given by Theorem 2.8. Observe that for such a CT, there are no exceptional paths in the complete splitting of $[f^n(e)]$ for any $e \in \Gamma$ as there are no linear edges in Γ (since it requires a closed Nielsen path). The following is an easy consequence of the definitions:

Fact 2.12 *Let $f\colon G \to G$ be a CT that represents an atoroidal outer automorphism. Then every Nielsen path is a legal concatenation of INPs and fixed edges.*

Definition 2.13 We call a splitting unit σ *expanding* if $\|[f^n(\sigma)]\| \to \infty$ as $n \to \infty$. If $f\colon G \to G$ is a CT that represents an atoroidal outer automorphism, then an expanding splitting unit is one of the following three types:

(1) an edge in an EG stratum;

(2) a superlinear edge in an NEG stratum;

(3) a maximal connecting path γ in a zero stratum such that the complete splitting of $[f^k(\gamma)]$ contains at least one of the above two types for some $k \geq 1$.

2.5 Geodesic currents

Let ∂F_N denote the Gromov boundary of F_N. Let $\partial^2 F_N$ be the double boundary, ie $\partial^2 F_N = \partial F_N \times \partial F_N \setminus \Delta$, where Δ denotes the diagonal. Let $\iota\colon \partial^2 F_N \to \partial^2 F_N$ be the *flip* map given by $\iota(\zeta_1, \zeta_2) = (\zeta_2, \zeta_1)$.

The group F_N acts on itself by left multiplication, which induces an action of F_N on ∂F_N and hence on $\partial^2 F_N$. A *geodesic current* on F_N is a locally finite (positive) Borel measure on $\partial^2 F_N$ which is both F_N–invariant and flip-invariant.

The space of geodesic currents on F_N is denoted by $\mathrm{Curr}(F_N)$, and endowed with the weak-$*$ topology it is a metrizable topological space [7]. The space of *projectivized geodesic currents* $\mathbb{P}\mathrm{Curr}(F_N)$ is the quotient of $\mathrm{Curr}(F_N)$, where two currents are equivalent if they are positive scalar multiples of each other. The space $\mathbb{P}\mathrm{Curr}(F_N)$ is compact; see [24].

Both $\mathrm{Aut}(F_N)$ and $\mathrm{Out}(F_N)$ act on the space of currents by homeomorphisms, and these actions descend to well-defined actions on $\mathbb{P}\mathrm{Curr}(F_N)$.

Let $g \in F_N$ be an element which is not a proper power. We define the *counting current* η_g corresponding to g as follows: for any Borel set $S \subset \partial^2 F_N$ the value $\eta_g(S)$ is the number of F_N–translates of $(g^{-\infty}, g^{\infty})$ or of $(g^{\infty}, g^{-\infty})$ that are contained in S. For any nontrivial element $h \in F_N$ we write $h = g^k$, where g is not a proper power, and set $\eta_h := k\eta_g$. A *rational current* is a nonnegative real multiple of a counting current. The set of rational currents forms a dense subset of $\mathrm{Curr}(F_N)$; see [23; 24; 30].

3 Dynamics of atoroidal automorphisms

3.1 North–south dynamics

Let X be a compact metric space, and G be a group acting on X by homeomorphisms. We say that $g \in G$ acts on X with *(uniform) north–south dynamics* if the action of g

on X has two distinct fixed points x_- and x_+ and, for any open neighborhood $U_\pm$ of $x_\pm$ and a compact set $K_\pm \subset X \setminus x_\mp$, there exists $M > 0$ such that

$$g^{\pm n} K \subset U_\pm$$

for all $n \geq M$.

North–south dynamics is a strong form of stability for the action of a group on a compact metric space, and allows one to deduce various structural results about the group itself. For example, a fully irreducible outer automorphism $\varphi \in \mathrm{Out}(F_N)$ acts on the closure $\overline{\mathrm{CV}}$ of the projectivized outer space with north–south dynamics [27]. Similarly, if φ is both fully irreducible and atoroidal, then φ acts on $\mathbb{P}\mathrm{Curr}(F_N)$ with north–south dynamics [30]; see also [35]. On the other hand, an atoroidal outer automorphism does not act on $\mathbb{P}\mathrm{Curr}(F_N)$ with classical north–south dynamics. Existence of invariant free factors makes them dynamically more complicated but, as we show below, they still exhibit a strong form of stability.

Definition 3.1 (generalized north–south dynamics) Let X be a compact metric space, and G be a group acting on X by homeomorphisms. We say that an element $g \in G$ acts on X with generalized north–south dynamics if the action of g on X has two invariant disjoint sets Δ_-, and Δ_+ (ie $g\Delta_- = \Delta_-$ and $g\Delta_+ = \Delta_+$) and, for any open neighborhood $U_\pm$ of $\Delta_\pm$ and a compact set $K_\pm \subset \mathbb{P}\mathrm{Curr}(F_N) \setminus \Delta_\mp$, there exists $M > 0$ such that

$$g^{\pm n} K_\pm \subset U_\pm$$

for all $n \geq M$.

We restate Theorem 1.4 from the introduction for the benefit of the reader, the proof of which is given at the end of this section.

Theorem 1.4 *Let $\varphi \in \mathrm{Out}(F_N)$ be an atoroidal outer automorphism of a free group of rank $N \geq 3$. Then there exist a simplex of attraction Δ_+ and a simplex of repulsion Δ_- in $\mathbb{P}\mathrm{Curr}(F_N)$ such that φ acts on $\mathbb{P}\mathrm{Curr}(F_N)$ with generalized north–south dynamics from Δ_- to $\Delta+$.*

The rest of this section is modeled on our earlier paper [28] with Lustig, and utilizes the dynamics of reducible substitutions as treated in [29]. In what follows we explain the subtleties that arise in this new setting carefully, while referring to [28] for arguments that follow by straightforward modifications from the old setting.

3.2 Symbolic dynamics and CTs

In this section we recall the relevant definitions in symbolic dynamics and results from our earlier paper [29], that allows us to describe the simplex of attraction and simplex of repulsion in Theorem 1.4 explicitly.

Let $A = \{a_1, \dots, a_n\}$ be a finite alphabet, and A^* denote the set of all finite words in A. A *substitution* $\xi\colon A \to A^*$ is a rule that assigns to each letter $a \in A$ a nonempty word w in A^*. A substitution induces a map, which we also denote by ξ, on the set of infinite words $A^{\mathbb{N}}$ by concatenation:

$$\xi\colon A^{\mathbb{N}} \to A^{\mathbb{N}}, \quad x_1 x_2 \cdots \mapsto \xi(x_1)\xi(x_2)\cdots.$$

Given a substitution $\xi\colon A \to A^*$ there is an associated transition matrix M_ξ, where $\{M_\xi\}_{ij}$ is the number of occurrences of a_j in $\xi(a_i)$. A substitution ξ is called *irreducible* if for all $1 \le i, j \le n$, there exists an exponent $k = k(i, j) \ge 1$ such that the letter a_i appears in the word $\xi^k(a_j)$. The substitution ξ is called *primitive* if k can be chosen independently. In what follows, up to passing to powers and rearranging the letters, we will assume that each transition matrix is a lower diagonal block matrix where each diagonal block is either primitive, or has bounded entries for all M^t for all $t \ge 1$ [29, Lemma 3.1]. We refer the reader to [32; 29] for a detailed account of substitutions.

Given a nonprimitive substitution we consider maximal invariant *subalphabets*

$$0 = A_0 \sqsubset A_1 \sqsubset A_2 \sqsubset \cdots \sqsubset A_n = A$$

and call $A_{i+1} \setminus A_i$ the i^{th} stratum in analogy with train tracks terminology [29, proof of Proposition 3.5].

Given two words w_1 and w_2 in A^*, let $|w_1|_{w_2}$ denote the number of occurrences of the word w_2 in w_1. The following is a slight variation of Theorem 1.2 and Corollary 1.3 of [29], a detailed proof of which is given in [29, Proposition 5.4, Case 1].

Proposition 3.2 [29] *Let ξ be a substitution on a finite alphabet A. Then there exists a positive power $\zeta = \xi^s$ such that for any nonempty word $w \in A^*$ and any letter $a_i \in A$, the limit frequency*

$$\lim_{t\to\infty} \frac{|\zeta^t(a_i)|_w}{|\zeta^t(a_i)|}$$

exists. Furthermore, if a_i is in a primitive stratum H_i, where the Perron–Frobenius eigenvalue of H_i is strictly bigger than those of the dependent strata, then the limit frequencies are independent of the chosen letter.

The next proposition shows how one can extract dynamical information from CTs by interpreting them as substitutions and invoking Proposition 3.2. Similar ideas were also used in our earlier work [35; 28] in the setting of train tracks and [19] in the CT setting for studying dynamics of *relative* outer automorphisms.

For any two reduced edge paths γ and γ' in a graph G, define

$$\langle \gamma, \gamma' \rangle := |\gamma'|_{\gamma} + |\gamma'|_{\gamma^{-1}}.$$

Proposition 3.3 *Let $f\colon G \to G$ be a CT that represents an atoroidal outer automorphism $\varphi \in \mathrm{Out}(F_N)$. For any splitting unit σ and any reduced edge path γ in G, the limit*

$$\sigma_{\gamma} := \lim_{n\to\infty} \frac{\langle \gamma, f^t(\sigma)\rangle}{|f^t(\sigma)|}$$

exists. Moreover, for any expanding splitting unit σ, the set of values

$$\{\sigma_{\gamma} \mid \gamma \text{ is a reduced edge path in } G\}$$

defines a geodesic current μ_{σ} on F_N.

Proof If the splitting unit is not expanding then there is a definite bound on the length $\|[f^n(\sigma)]\|$ for all $n \geq 1$. Hence, the image $[f^n(\sigma)]$ becomes periodic after sufficiently many iterations. Since every periodic Nielsen path has period one, the sequence of paths $[f^n(\sigma)]$ becomes eventually fixed, and the claim follows. For the remaining part of the proof we assume that σ is an expanding splitting unit and will prove the claim by induction on the height of the stratum. Let $r = 1$. Since φ is atoroidal, H_1 is necessarily an EG stratum, and the restriction of f to $G_1 = H_1$ is an absolute train track map. Hence, the result follows from [35, Proposition 2.4 and Lemma 3.7]. Now assume that the claim holds for $r \leq k-1$. There are three cases to consider.

First suppose that H_k is an EG stratum. A splitting unit of height k is either an edge $e \in H_k$, or an INP intersecting H_k. Since an INP is not expanding we just need to prove the claim for an edge $e \in H_k$. Let A be the alphabet whose letters consist of edges in irreducible strata that are in G_k, INPs contained in G_{k-1}, and maximal, taken connecting paths in a zero stratum that are in G_{k-1}. The fact that this alphabet is finite follows from the properties of the CT map that represents an atoroidal outer automorphism. Let $\zeta\colon A^* \to A^*$ be the substitution induced by the CT $f\colon G \to G$ on the alphabet A using the following rule: $\zeta(\sigma) = [f(\sigma)]$. For each "letter" in the above alphabet, the image is completely split and hence a reduced "word" in this alphabet.

Hence, the above formula is a substitution, and Proposition 3.2 gives the required convergence.

The latter claim that the set of values $\{\sigma_\gamma\}_{\gamma\in\mathcal{P}G}$ defines a unique geodesic current is easy to check. They satisfy Kirchhoff conditions, ie

(1) $0 \le \sigma_\gamma \le 2 < \infty$,

(2) $\sigma_\gamma = \sigma_{\gamma^{-1}}$,

(3) $\sigma_\gamma = \sum_{a\in A} \sigma_{a\gamma} = \sum_{a\in A} \sigma_{\gamma a}$,

as in [29, Proposition 3.13; 35, Lemma 3.7], and by the Kolmogorov measure extension theorem the result follows.

Now assume that H_k is an NEG stratum. Since σ is expanding it is necessarily a superlinear edge e. By properties of CTs, $f(e) = e\cdot u$, where u is a circuit in G_{k-1} such that u is completely split and the turn (u, u^{-1}) is legal. We can similarly define a substitution as in the EG case, where the alphabet consists of the edge e, and splitting units appearing in u, and all of its iterates. The frequency convergence for the corresponding substitution is now given by Proposition 3.2.

Finally, if H_k is a zero stratum, then σ is a maximal connecting taken path, whose image $[f(\sigma)]$ is completely split, and has height $\le k-1$. Hence, the claim follows by induction. □

Remark 3.4 Proposition 3.2 together with the arguments in the proof of Proposition 3.3 reveals that, for an EG stratum H_r where the PF–eigenvalue is strictly greater than those of the dependent strata, the currents μ_e are independent of the edge e chosen from H_r. Furthermore, combined with [29, Proposition 5.4], we have that for any other expanding splitting unit σ, the current μ_σ is a linear combination of currents coming from edges in EG strata.

Definition 3.5 Given a CT map $f\colon G\to G$ that represents an atoroidal outer automorphism $\varphi\in \mathrm{Out}(F_N)$, we define the *simplex of attraction* as the projective class of nonnegative linear combinations of currents obtained from Proposition 3.3. We define the *simplex of repulsion* similarly, using a CT map that represents φ^{-1}.

3.3 Goodness and legal structure

Lemma 3.6 (bounded cancellation lemma [11]) *Let $f\colon G\to G$ be a topological graph map. There exists a constant C_f such that for any reduced path $\rho=\rho_1\rho_2$ in G*

one has

$$\|[f(\rho)]\| \geq \|[f(\rho_1)]\| + \|[f(\rho_2)]\| - 2C_f.$$

That is, at most C_f terminal edges of $[f(\rho_1)]$ are canceled with C_f initial edges of $[f(\rho_2)]$ when we concatenate them to obtain $[f(\rho)]$.

Definition 3.7 (goodness) Let γ be a reduced edge path in G and $\gamma = \gamma_1 \cdot \gamma_2 \cdots \gamma_m$ be a splitting of γ into edge paths γ_i. Define $\mathfrak{g}_{\mathrm{CT}}(\gamma)$ to be the proportion of the sum of the lengths of the γ_i that have a complete splitting to the total length of γ. Define *goodness* of γ, denoted by $\mathfrak{g}(\gamma)$, as the supremum of $\mathfrak{g}_{\mathrm{CT}}(\gamma)$ over all splittings of γ into edge paths. Since there are only finitely many decompositions of an edge path into subedge paths, the value $\mathfrak{g}(\gamma)$ is realized for some splitting of γ. We will call the splitting for which $\mathfrak{g}(\gamma)$ is realized the *maximal edge splitting* of γ. The subpaths that are part of a complete splitting in the maximal edge splitting will be called *good*. The subpaths in the maximal edge splitting which do not admit complete splittings will be called *bad*.

Let $w \in F_N$ be a conjugacy class in F_N, and γ_w be the unique circuit in G that represents $w \in F_N$. We define the *goodness* of the conjugacy class w as $\mathfrak{g}(w) := \mathfrak{g}(\gamma_w)$.

Remark 3.8 The properties of CTs — see Lemma 2.11(1) and (5) — imply that forward images of good paths are always good, and forward images of bad paths are eventually good.

Proposition 3.9 *Let $f\colon G \to G$ be a CT representing an atoroidal outer automorphism $\varphi \in \mathrm{Out}(F_N)$. There exists $s > 0$ such that for any completely split edge path σ such that $|\sigma|$ is sufficiently big,*

$$\frac{\text{total length of expanding splitting units in } \sigma}{|\sigma|} \geq s.$$

Proof Let σ be a completely split edge path, and consider its complete splitting. By properties of CTs (Lemma 2.11(6)) each maximal connecting path in a zero stratum is necessarily followed by an edge in an EG stratum. Since zero strata are precisely the contractible components, there is an upper bound for the length of any maximal connecting path in a zero stratum, say Z_0. Since φ is atoroidal, there is also an upper bound for the length of any path that is a concatenation of INPs and fixed edges, say Z_1.

Let $Z = \max\{Z_0, Z_1\}$. From these two observations it follows that for any completely split edge path of length $\geq 2Z+1$, we have

$$\frac{\text{total length of expanding splitting units in } \sigma}{|\sigma|} \geq \frac{\text{total length of EG or superlinear edges in } \sigma}{|\sigma|} \geq \frac{1}{2Z+1}$$

□

Convention/Remark 3.10 The values Z_0, Z_1 and hence Z are valid for all powers of f. From now on, we will replace φ, and hence f, with a power (which we will still denote by f) so that each expanding splitting unit grows at least by a factor of $2(2Z+1)$.

Definition 3.11 (short and long good paths) In light of Proposition 3.9 we will call a good segment γ a *long good segment* if $|\gamma| \geq 2Z+1$ and *short good segment* if $|\gamma| \leq 2Z$.

Lemma 3.12 *Let C_f be the bounded cancellation constant and $C := \max\{C_f, 2Z+1\}$. Let $\gamma = \gamma_1\gamma_2$ be an edge path such that γ_1 and γ_2 are completely split. Then any edge that is C away from the turn $\{\gamma_1^{-1}, \gamma_2\}$ is good.*

Proof Since any completely split path of length $\geq 2Z+1$ grows at least by a factor of 2, the bounded cancellation lemma dictates that reducing $f(\gamma_1\gamma_2)$ will not result in any cancellation at edges C away from the concatenation point; hence, the claim follows. □

Lemma 3.13 *For any edge path γ the total length of bad subpaths in $[f^k(\gamma)]$ is uniformly bounded by $|\gamma| 2C$.*

Proof This is an easy consequence of Lemma 3.12. □

We first show that, up to passing to further powers, the goodness is *monotone*.

Lemma 3.14 *Let $f\colon G \to G$ be a CT representing an atoroidal outer automorphism $\varphi \in \mathrm{Out}(F_N)$. There exists an exponent $t' \geq 1$ such that for any circuit γ with $1 > \mathfrak{g}(\gamma) > 0$ and for all $t \geq t'$, one has*

$$\mathfrak{g}([f^t(\gamma)]) > \mathfrak{g}(\gamma).$$

Proof Note, by definition, the total length of good subpaths in γ is $\mathfrak{g}(\gamma)|\gamma|$. Under iteration of f, each good segment remains good and the length of each good segment is nondecreasing. Therefore, the total length of good segments in $[f^k(\gamma)]$ is $\geq \mathfrak{g}(\gamma)|\gamma|$.

Let t' be an exponent such that for each edge path β of length $\leq 2C+1$, the edge path $[f^t(\gamma)]$ is completely split for all $t \geq t'$. Therefore, for any bad segment β such that $|\beta| \leq 2C+1$, the path $[f^t(\beta)]$ is completely split, and hence contains no bad edges. For any bad segment β of length $\geq 2C+1$, divide β into subsegments β_i of length $2C+1$, with the exception of the last segment being of length $\leq 2C+1$. By the choice of t', each $[f^t(\beta_i)]$ is completely split, where the turns at concatenation points are possibly illegal. The bounded cancellation lemma dictates that total length of bad segments decreases by at least the number of subsegments, and the conclusion of the lemma follows. □

Convention/Remark 3.15 In what follows, we pass to a further power of φ and f so that each expanding splitting unit grows at least by a factor of $2(2Z+1)$ and the goodness function is monotone. We furthermore consider the bounded cancellation constant for this new power, but we continue to use f and C_f.

The following is one of the key technical lemmas in this paper. It allows us to get convergence estimates while dealing with forward iterations of CTs.

Lemma 3.16 *Let $\delta > 0$ and $\epsilon > 0$ be given. There exists an exponent $m_+ = m_+(\delta, \epsilon)$ such that for all circuits γ with $\mathfrak{g}(\gamma) > \delta$, we have $\mathfrak{g}([f^m(\gamma)]) > 1-\epsilon$ for $m \geq m_+$.*

Proof Let γ be a cyclically reduced edge path such that $\mathfrak{g}(\gamma) = \delta > 0$. First consider the splitting of γ into maximal good segments a_i and maximal bad segments b_i. There are two cases to consider:

Case 1 First assume that

$$\frac{\text{total length of long good segments in } \gamma}{\text{total length of good segments in } \gamma} \geq \frac{1}{4Z+1}$$

This gives that

$$\frac{\text{total length of expanding splitting units in } \gamma}{\text{total length of good segments in } \gamma} \geq \frac{1}{(2Z+1)(4Z+1)}.$$

Note that by Lemma 3.13 the total length of bad segments in $[f^k(\gamma)]$ is uniformly bounded by $(1-\mathfrak{g}(\gamma))|\gamma|C$. On the other hand, the assumption above together with

Convention/Remark 3.15 implies that

$$\text{total length of good segments in } [f^k(\gamma)] \geq \mathfrak{g}(\gamma)|\gamma|\frac{1}{(2Z+1)(4Z+1)}(2Z+1)^k 2^k.$$

Therefore,

$$\begin{aligned}\mathfrak{g}([f^k(\gamma)]) &\geq \frac{\mathfrak{g}(\gamma)|\gamma|\frac{1}{(2Z+1)(4Z+1)}(2Z+1)^k 2^k}{(1-\mathfrak{g}(\gamma))|\gamma|C+\mathfrak{g}(\gamma)|\gamma|\frac{1}{(2Z+1)(4Z+1)}(2Z+1)^k 2^k} \\ &= \frac{\mathfrak{g}(\gamma)\frac{1}{4Z+1}(2Z+1)^{k-1}2^k}{(1-\mathfrak{g}(\gamma))C+\mathfrak{g}(\gamma)\frac{1}{4Z+1}(2Z+1)^{k-1}2^k},\end{aligned}$$

which converges to 1 as $k \to \infty$; hence, the conclusion of Lemma 3.14 follows for big enough k, say $k = m_+$.

Case 2 Otherwise, we have

$$\frac{\text{total length of long good segments in } \gamma}{\text{total length of good segments in } \gamma} < \frac{1}{4Z+1}.$$

Equivalently,

$$\frac{\text{total length of short good segments in } \gamma}{\text{total length of long good segments in } \gamma} \geq 4Z. \tag{3-1}$$

We now subdivide the path γ into subpaths as follows. Consider the maximal edge splitting of γ. First subpath starts at a good edge, and it stops after tracing a total length of $2Z+1$ good segments end at a vertex such that the next edge is good. The second subpath starts at where the first path stops, and traces a total length of $2Z+1$ good segments, and stops at a vertex such that the next edge is good. We inductively form subpaths $\gamma_1, \gamma_2, \dots$ so that each of them contains good segments of length $2Z+1$, with the possible exception of the last subpath. Note that by construction, $\gamma_i \cdot \gamma_2 \cdots \gamma_s$ is a splitting of γ.

Observe that (3-1) implies that

$$\frac{\#\{\gamma_i \text{ containing bad segments}\}}{\#\{\gamma_i \text{ which are completely good}\}} \geq 4Z,$$

which, in turn, implies

$$\#\{\gamma_i \text{ containing bad segments}\} \geq \frac{s4Z}{4Z+1},$$

where s is the total number of subpaths in γ in the above subdivision.

Since

$$\text{total length of good segments in } \gamma \leq \frac{1-\mathfrak{g}(\gamma)}{\mathfrak{g}(\gamma)} s(2Z+1),$$

each γ_i above that contains a bad segment contains

$$\frac{1-\mathfrak{g}(\gamma)}{\mathfrak{g}(\gamma)} s(2Z+1) \frac{4Z+1}{s(4Z)} = \frac{(2Z+1)(4Z+1)}{4Z} \frac{(1-\mathfrak{g}(\gamma))}{\mathfrak{g}(\gamma)}$$

bad edges on *average*.

Therefore, for each γ with $\mathfrak{g}(\gamma) \geq \delta$, at least half of the subpaths contain bad segments of total length

$$\leq \frac{(2Z+1)(4Z+1)}{2Z} \frac{(1-\delta)}{\delta} =: C_b$$

Let $t_b > 0$ be an exponent such that for all edge paths γ with $|\gamma| \leq C_b$, the path $[f^k(\gamma)]$ is completely split for $k \geq t_b$. Therefore, at least half of the subsegments in the subdivision will be mapped to long good segments, and the result follows from Case 1. □

Lemma 3.17 *Let U a neighborhood of the simplex of attraction and a positive number $\delta_+ > 0$ be given. Then there exists an exponent $N = N(\delta, U)$ such that for any $w \in F_N$ with $\mathfrak{g}(w) > \delta$,*

$$(\varphi^N)^n(\eta_w) \in U$$

for all $n \geq 1$.

Proof We first apply a power of f so that for every conjugacy class w with $\mathfrak{g}(w) > \delta$, we have $\mathfrak{g}(\varphi(w)) > 1-\epsilon$ for small $\epsilon > 0$. The rest of the proof is nearly identical to the proof of Lemma 6.1 in [28], where edges are replaced by expanding splitting units. □

Lemma 3.18 *Let $f\colon G \to G$ be a CT that represents an atoroidal outer automorphism. Given $0 < \delta < 1$, there exists an exponent T such that, for any element $w \in F_N$, and for all $t \geq T$ either*

$$\mathfrak{g}(\varphi^t(w)) \geq \delta$$

or

$$\text{total length of bad segments in } f^t(\gamma_w) \leq \tfrac{1}{2} \text{ total length of bad segments in } \gamma_w,$$

where γ_w is the unique circuit in G representing w.

Proof Let γ_w be the unique circuit in G that represents $w \in F_N$. Consider the splitting of γ into maximal good segments a_i and maximal bad segments b_i. Recall that $C = \max\{C_f, 2Z+1\}$. Let us call a bad segment b_i a *long bad segment* if $|b_i| > 10C$, and a *short bad segment* otherwise.

There are two cases to consider:

Case 1 First assume that

$$\frac{\text{total length of short bad segments in } \gamma_w}{\text{total length of bad segments in } \gamma_w} \geq \frac{1}{10}.$$

Since every maximal bad segment is followed by at least one good segment, we have

$$\text{total length of good segments in } \gamma_w \geq \frac{1}{10C} \text{ total length of short bad segments in } \gamma_w$$

and hence

$$\text{total length of good segments in } \gamma_w \geq \frac{1}{100C} \text{ total length of bad segments in } \gamma_w.$$

Therefore,

$$\mathfrak{g}(\gamma_w) \geq \frac{1}{100C+1}.$$

Now, invoking Lemma 3.16, there is an exponent T_1 such that

$$\mathfrak{g}(\varphi^t(w)) \geq \delta$$

for all $t \geq T_1$, which is clearly independent of the conjugacy class w.

Case 2 Now assume, on the other hand, that

$$\frac{\text{total length of long bad segments in } \gamma_w}{\text{total length of bad segments in } \gamma_w} \geq \frac{9}{10}. \tag{3-2}$$

Let T_2 be an exponent such that for all edge paths γ with $|\gamma| < 10C$, $[f^t(\gamma)]$ is completely split for all $t \geq T_2$. Then, for any long bad segment b, the bounded cancellation lemma implies that

$$\text{total length of bad segments in } [f^t(b)] \leq \tfrac{1}{5} \text{ total length of bad segments in } b.$$

Together with (3-2), we get

$$\text{total length of bad segments in } f^t(\gamma_w) \leq \tfrac{9}{50} \text{ total length of bad segments in } \gamma_w$$

for all $t \geq T_2$. Now set $T = \max\{T_1, T_2\}$, and the lemma follows. □

Lemma 3.19 *Let $h\colon G' \to G'$ be a CT that represents $\varphi^{-1} \in \mathrm{Out}(F_N)$. Define $\mathfrak{g}'(\gamma')$ for $\gamma' \in G'$, and $\mathfrak{g}'(w)$ for $w \in F_N$ analogously. Then, given $0 < \delta < 1$, there is an exponent $T > 0$ such that, up to replacing f and h with powers, for any element $w \in F_N$ either*

$$\mathfrak{g}(\varphi^t(w)) \geq \delta \quad \text{or} \quad \mathfrak{g}'(\varphi^{-t}(w)) \geq \delta$$

for all $t \geq T$.

Proof Let $h\colon G' \to G'$ be a CT that represents $\varphi^{-1} \in \mathrm{Out}(F_N)$ and $\mathfrak{g}'$ be the corresponding goodness function, and we pass to appropriate powers according to Convention/Remark 3.15. The proof is now nearly identical to that of Proposition 4.20 of [28], where the number of illegal turns is replaced by the total length of bad segments. □

Proposition 3.20 [28, Proposition 3.3] *Let $f\colon X \to X$ be a homeomorphism of a compact metrizable space X. Let $Y \subset X$ be a dense subset of X, and let Δ_+ and Δ_- be two f–invariant sets in X that are disjoint. Assume that the following criterion holds:*

For every neighborhood U of Δ_+ and every neighborhood V of Δ_- there exists an integer $m_0 \geq 1$ such that, for any $m \geq m_0$ and any $y \in Y$, one has either $f^m(y) \in U$ or $f^{-m}(y) \in V$.

Then f^2 has generalized uniform north–south dynamics from Δ_- to Δ_+.

Proposition 3.21 [28, Proposition 3.4] *Let $f\colon X \to X$ be a homeomorphism of a compact space X, and let Δ_+ and Δ_- be disjoint f–invariant sets. Assume that some power f^p with $p \geq 1$ has generalized uniform north–south dynamics from Δ_- to Δ_+.*

Then the map f, too, has generalized uniform north–south dynamics from Δ_- to Δ_+.

Proof of Theorem 1.4 The theorem now follows from a combination of Lemmas 3.17 and 3.19 and Propositions 3.20 and 3.21. □

4 Hyperbolic extensions of free groups

In this section we use the dynamics of atoroidal outer automorphisms to prove Theorem 1.1 from the introduction, which allows us to construct new examples of hyperbolic extensions of free groups.

In what follows we will utilize theory of laminations on free groups which appear as supports of currents on F_N. We refer the reader to [4; 5; 9; 12; 13; 14; 18; 21] for detailed discussions. A *lamination* is a closed subset of $\partial^2 F_N$ which is F_N–invariant, and flip-invariant. We say that a free factor F *carries* a lamination Λ if all lines in Λ are contained in ∂^2 F.

Convention 4.1 Throughout this section we assume that we pass to the finite-index characteristic subgroup $\mathrm{IA}_N(\mathbb{Z}_3)$ of $\mathrm{Out}(F_N)$, as in Handel–Mosher subgroup decomposition theory [21], so that for each outer automorphism every periodic conjugacy class is fixed, and every periodic free factor system is invariant.

Let $\mathcal{H}$ be a subgroup of $\mathrm{Out}(F_N)$ and $\mathcal{F}^1 \sqsubset \mathcal{F}^2 \sqsubset \cdots \sqsubset \mathcal{F}^n = F_N$ be a *maximal* $\mathcal{H}$–invariant filtration of F_N by free factor systems, meaning that if $\mathcal{H}(\mathcal{A}) = \mathcal{A}$ for some $\mathcal{F}^i \sqsubset \mathcal{A} \sqsubset \mathcal{F}^{i+1}$, then either $\mathcal{A} = \mathcal{F}^i$ or $\mathcal{A} = \mathcal{F}^{i+1}$. Let $\varphi \in \mathcal{H}$ be an atoroidal outer automorphism. Consider a (possibly trivial) refinement $\mathcal{A}_1 \sqsubset \mathcal{A}_2 \sqsubset \cdots \sqsubset \mathcal{A}_m = F_N$ of $\mathcal{F}^1 \sqsubset \mathcal{F}^2 \sqsubset \cdots \sqsubset \mathcal{F}^n = F_N$ which is a maximal invariant filtration for φ.

If $\mathcal{H}$ fixes the conjugacy class of a free factor F of F_N, we will call the image of $\mathcal{H}$ in Out(F) under the natural homomorphism Stab(F) $\to$ Out(F) *the restriction* of $\mathcal{H}$ to F and denote it by $\mathcal{H}|_{\mathrm{F}}$.

We say that an $\mathcal{H}$–invariant free factor F is *minimal* if $\mathcal{H}$ does not fix the conjugacy class of any proper free factor of F. Similar definition holds for φ by considering the cyclic subgroup $\langle\varphi\rangle$. Observe that for $\varphi \in \mathcal{H}$, each minimal φ–invariant free factor F^i_φ is contained in a unique minimal $\mathcal{H}$–invariant free factor $\mathrm{F}^i_{\mathcal{H}}$.

Definition 4.2 Let φ and ψ be two atoroidal outer automorphisms with attracting and repelling simplices $\Delta_\pm(\varphi)$ and $\Delta_\pm(\psi)$ given by Theorem 1.4. We say that φ and ψ are independent if $\Delta_\pm(\varphi) \cap \Delta_\pm(\psi) = \varnothing$.

Lemma 4.3 *Let $\varphi \in \mathcal{H}$ be an atoroidal outer automorphism. Suppose that the restriction of $\mathcal{H}$ to F_i is not virtually cyclic for each minimal $\mathcal{H}$–invariant free factor F_i of F_N. Then $\mathcal{H}$ contains two independent atoroidal outer automorphisms.*

Proof Let $\{\mathrm{F}^i\}_{i=1}^s$ be the set of all minimal $\mathcal{H}$–invariant free factors. For each $i = 1, \dots, s$ the restriction of $\mathcal{H}$ to F^i is irreducible; hence, by [21, Theorem A; 22, Theorem 0.1], $\mathcal{H}$ contains an element θ_i whose restriction to F^i is fully irreducible, and since $\mathcal{H}$ is not geometric (since it contains an atoroidal element), we can choose θ_i in a way that its restriction to F^i is both fully irreducible and atoroidal [36, Theorem 5.4].

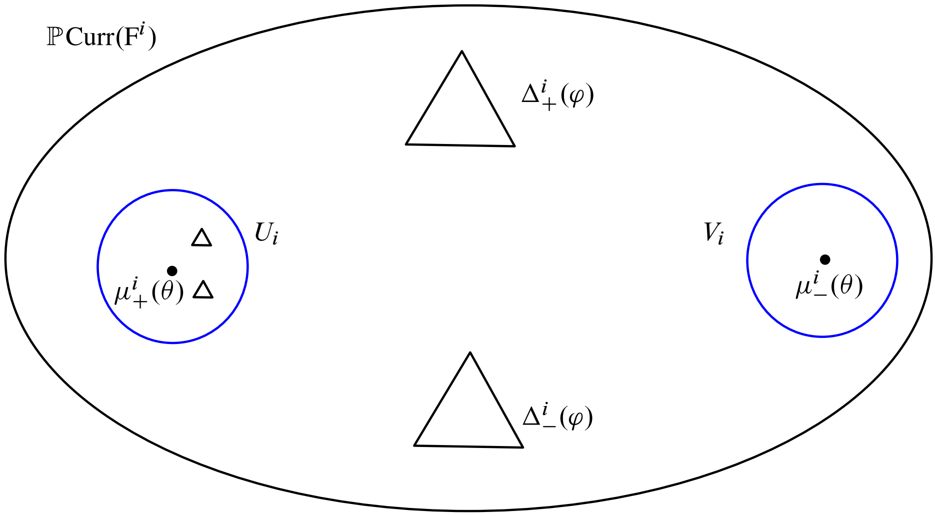

Figure 1: Dynamics on $\mathbb{P}\mathrm{Curr}(\mathrm{F}^i)$

Since fully irreducible and atoroidal elements are precisely the loxodromic isometries of the *cosurface graph* [15], invoking [10, Theorem 5.1] we can find a single $\theta \in \mathcal{H}$ such that for each $i = 1, \dots, s$, the restriction $\theta|_{\mathrm{F}^i}$ is fully irreducible and atoroidal. Recall that each fully irreducible and atoroidal outer automorphism acts on the space of projectivized geodesic currents with uniform north–south dynamics [30; 35]. Let $[\mu^i_-(\theta)] \in \mathbb{P}\mathrm{Curr}(\mathrm{F}^i)$ and $[\mu^i_+(\theta)] \in \mathbb{P}\mathrm{Curr}(\mathrm{F}^i)$ denote the unstable and stable currents for the restriction $\theta|_{\mathrm{F}^i}$. Since the stabilizer of the set $\{[\mu^i_-(\theta)], [\mu^i_+(\theta)]\}$ is virtually cyclic in $\mathrm{Out}(\mathrm{F}^i)$ [25], using the assumption on $\mathcal{H}$ we can furthermore assume that for the above $\theta \in \mathcal{H}$ it holds that $\theta|_{\mathrm{F}^i}$ and $\varphi|_{\mathrm{F}^i}$ are independent.

Hence, we can find $M > 0$ large enough that $\theta^M(\Delta^i_\pm(\varphi)) \cap \Delta^i_\pm(\varphi) = \varnothing$, where $\Delta^i_\pm(\varphi)$ are the attracting and repelling simplices of $\varphi|_{\mathrm{F}^i}$ in $\mathbb{P}\mathrm{Curr}(\mathrm{F}^i)$.

More precisely, choose open neighborhoods U_i and V_i of $[\mu^i_+(\theta)]$ and $[\mu^i_-(\theta)]$ in $\mathbb{P}\mathrm{Curr}(\mathrm{F}^i)$ which are disjoint from $\Delta^i_\pm(\varphi)$. Pick $M > 0$ such that $\theta^m(\mathbb{P}\mathrm{Curr}(\mathrm{F}^i) \setminus V_i) \subset U_i$ for all $m \geq M$; in particular, $\theta^M(\Delta^i_\pm) \subset U_i$. See Figure 1. In fact, we choose M that works for all minimal $\mathcal{H}$–invariant free factors for suitable open neighborhoods of attracting simplices as there are only finitely many minimal $\mathcal{H}$–invariant free factors.

Now consider the automorphism $\eta = \theta^M \varphi \theta^{-M}$, which is atoroidal since being atoroidal is invariant under conjugacy. Furthermore $\Delta^i_+(\eta) = \theta^M(\Delta^i_+(\varphi))$ and $\Delta^i_-(\eta) = \theta^M(\Delta^i_-(\varphi))$ in $\mathbb{P}\mathrm{Curr}(\mathrm{F}^i)$, and $\Delta_+(\eta) = \theta^M(\Delta_+(\varphi))$ and $\Delta_-(\eta) = \theta^M(\Delta_-(\varphi))$ in $\mathbb{P}\mathrm{Curr}(F_N)$.

Claim *η and φ are independent.*

Let $[\mu_+^k]$ be an extremal point in the attracting simplex $\Delta_+(\varphi)$. We first want to show that $\theta^M([\mu_+^k]) \neq [\mu]$ for any point $[\mu] \in \Delta_\pm(\varphi)$.

Notice that by Proposition 3.3 the point $[\mu_+^k]$ corresponds to some EG stratum H_k in the CT map $f\colon G \to G$ that represents φ.

Let F_φ^k be the unique (minimal) free factor carrying $\mathrm{supp}(\mu_+^k)$ (this support is the *attracting lamination* corresponding to the EG stratum H_k in the sense of Bestvina, Feighn and Handel [5]), and consider a minimal φ–invariant free factor $\mathrm{F}_\varphi^i \sqsubset \mathrm{F}_\varphi^k$. The free factor F_φ^i is contained in some minimal $\mathcal{H}$–invariant free factor F^i as above.

Let $[\mu_+^i] \in \Delta_+$ be the unique geodesic current whose support is carried by F_φ^i. We first observe that by definition $\mathrm{supp}(\mu_+^i) \subset \mathrm{supp}(\mu_+^k)$. Second, the subgroup $\mathcal{H}$ and so the element θ^M preserves F^i; therefore, $\mathrm{supp}(\theta^M \mu_+^i)$ is carried by F^i.

Suppose, for the sake of contradiction, that $\theta^M([\mu_+^k]) = [\mu]$. In that case we have $\mathrm{supp}(\theta^M([\mu_+^k])) = \mathrm{supp}([\mu])$; hence,

$$\mathrm{supp}(\theta^M([\mu_+^i])) \subset \mathrm{supp}(\theta^M([\mu_+^k])) = \mathrm{supp}([\mu]).$$

Only sublaminations of $\mathrm{supp}([\mu])$ that are carried by F^i could possibly come from supports of extremal points of $\Delta_\pm^i(\varphi)$, and since $(\theta^M \Delta_+^i) \cap \Delta_\pm^i = \varnothing$, the above inclusion is not possible; hence, we get a contradiction.

Since the support of any point in $\Delta_+(\eta)$ is a union of supports of extremal points, we get $\Delta_+(\eta) \cap \Delta_\pm(\varphi) = \varnothing$. A symmetric argument finishes the proof. □

We will prove the hyperbolicity of the extension using a classical argument of Bestvina, Feighn and Handel [4] which originates in the work of Mosher [31] as interpreted by Kapovich and Lustig [25].

Proposition 4.4 *Let $\varphi, \psi \in \mathcal{H}$ be two independent atoroidal outer automorphisms. Then there exist $M, N > 0$ such that for any $\mu \in \mathrm{Curr}(F_N)$ and for all $n \geq N$ and $m \geq M$, for at least three out of four elements α in $\{\varphi^n, \varphi^{-n}, \psi^m, \psi^{-m}\}$,*

$$|\alpha\mu|_G \geq 2|\mu|_G.$$

Proof Let U be a sufficiently small open neighborhood of $\Delta_+(\varphi)$, and $M_0 > 0$ be such that for any $\mu \in \mathrm{Curr}(F_N)$ such that $[\mu] \in \Delta_+(\varphi)$ it holds that $|\varphi^n(\mu)|_G \geq 2|\mu|_G$

for all $n \geq M_0$. This can be done, because of the topology of the space of currents, and the fact that for each extremal point $[\mu_+]$ of $\Delta_+(\varphi)$, $\varphi(\mu_+) = \lambda\mu_+$ for some $\lambda > 1$. For the corresponding statement in the fully irreducible case see [25, Lemma 4.12].

We also choose a small neighborhood V of $\Delta_-(\varphi)$, and M_1 so that for each $\mu \in \mathrm{Curr}(F_N)$ such that $[\mu] \in \Delta_-(\varphi)$ it holds that $|\varphi^{-n}(\mu)|_G \geq 2|\mu|_G$ for all $n \geq M$. Let $M' = \max\{M_0, M_1\}$.

Similarly, we choose neighborhoods U' and V' of $\Delta_+(\psi)$ and $\Delta_-(\psi)$, respectively, and a corresponding $N' > 0$.

By Theorem 1.4, there exists an exponent M_+ such that

$$\varphi^n(\mathbb{P}\mathrm{Curr}(F_N) \setminus V) \subset U$$

and

$$\varphi^{-n}(\mathbb{P}\mathrm{Curr}(F_N) \setminus U) \subset V$$

for all $n \geq M_+$.

Similarly, there exists an exponent N_+ such that

$$\psi^n(\mathbb{P}\mathrm{Curr}(F_N) \setminus V') \subset U'$$

and

$$\psi^{-n}(\mathbb{P}\mathrm{Curr}(F_N) \setminus U') \subset V'$$

for all $n \geq N_+$.

Now set $M = M' + M_+$ and $N = N' + N_+$. Let $\mu \in V$. Then the choice of M and N guarantees that $|\varphi^{-n}\mu|_G \geq 2|\mu|_G$, $|\psi^m\mu|_G \geq 2|\mu|_G$ and $|\psi^{-m}\mu|_G \geq 2|\mu|_G$. The other cases can be proved similarly; hence, the proposition follows. □

Proof of Theorem 1.1 Let F^i be a minimal, nontrivial $\mathcal{H}$–invariant free factor, and let φ, η and θ be as in Lemma 4.3. Since θ is fully irreducible, for large M the free factors F^i_φ and $\theta^M F^i_\varphi$ *fill* the free factor F^i. Under this assumption, based on work of Bestvina and Feighn [3], Taylor [33, Theorem 1.3] proved that for some $K > 0$, the group $\langle \varphi^K, \eta^K \rangle$ is isomorphic to a free group of rank 2. (He proves much more but we don't need that here.)

The fact that the corresponding free group extension is hyperbolic now follows from Proposition 4.4 and the Bestvina–Feighn combination theorem; see the proof of [4, Theorem 5.2]. □

We finish the paper with Corollary 1.5 from the introduction:

Corollary 1.5 *Let $\varphi \in \mathrm{Out}(F_N)$ be a fully irreducible and atoroidal outer automorphism. Then, for any atoroidal outer automorphism $\psi \in \mathrm{Out}(F_N)$ (not necessarily fully irreducible) which is not commensurable with φ, there exists an exponent $M > 0$ such that for all $n, m > M$, the subgroup $\Gamma = \langle \varphi^n, \psi^m \rangle < \mathrm{Out}(F_N)$ is purely atoroidal and the corresponding free extension E_Γ is hyperbolic.*

Proof Let φ be as above, and let $[\mu_+(\varphi)]$ and $[\mu_-(\varphi)]$ be the corresponding stable and unstable currents in $\mathbb{P}\mathrm{Curr}(F_N)$. Since ψ is not commensurable with φ, the attracting simplex $\Delta_+(\psi)$, the repelling simplex $\Delta_-(\psi)$, and the stable and unstable currents $[\mu_+(\varphi)]$ and $[\mu_-(\varphi)]$ are all disjoint.

Choose disjoint open neighborhoods of these sets, and choose high enough powers of φ and ψ so that there is a uniform north–south dynamics, which is guaranteed by Theorem 1.4. Then Proposition 4.4, together with Bestvina–Feighn combination theorem, gives the required result. □

References

[1] **M Bestvina**, **M Feighn**, *A combination theorem for negatively curved groups*, J. Differential Geom. 35 (1992) 85–101 MR

[2] **M Bestvina**, **M Feighn**, *Hyperbolicity of the complex of free factors*, Adv. Math. 256 (2014) 104–155 MR

[3] **M Bestvina**, **M Feighn**, *Subfactor projections*, J. Topol. 7 (2014) 771–804 MR

[4] **M Bestvina**, **M Feighn**, **M Handel**, *Laminations, trees, and irreducible automorphisms of free groups*, Geom. Funct. Anal. 7 (1997) 215–244 MR

[5] **M Bestvina**, **M Feighn**, **M Handel**, *The Tits alternative for $\mathrm{Out}(F_n)$, I: Dynamics of exponentially-growing automorphisms*, Ann. of Math. 151 (2000) 517–623 MR

[6] **M Bestvina**, **M Handel**, *Train tracks and automorphisms of free groups*, Ann. of Math. 135 (1992) 1–51 MR

[7] **F Bonahon**, *Geodesic currents on negatively curved groups*, from "Arboreal group theory" (R C Alperin, editor), Math. Sci. Res. Inst. Publ. 19, Springer (1991) 143–168 MR

[8] **P Brinkmann**, *Hyperbolic automorphisms of free groups*, Geom. Funct. Anal. 10 (2000) 1071–1089 MR

[9] **M Clay**, **A Pettet**, *Relative twisting in outer space*, J. Topol. Anal. 4 (2012) 173–201 MR

[10] **M Clay**, **C Uyanik**, *Simultaneous construction of hyperbolic isometries*, Pacific J. Math. 294 (2018) 71–88 MR

[11] **D Cooper**, *Automorphisms of free groups have finitely generated fixed point sets*, J. Algebra 111 (1987) 453–456 MR

[12] **T Coulbois**, **A Hilion**, **M Lustig**, *$\mathbb{R}$–trees and laminations for free groups, I: Algebraic laminations*, J. Lond. Math. Soc. 78 (2008) 723–736 MR

[13] **T Coulbois**, **A Hilion**, **M Lustig**, *$\mathbb{R}$–trees and laminations for free groups, II: The dual lamination of an $\mathbb{R}$–tree*, J. Lond. Math. Soc. 78 (2008) 737–754 MR

[14] **T Coulbois**, **A Hilion**, **M Lustig**, *$\mathbb{R}$–trees and laminations for free groups, III: Currents and dual $\mathbb{R}$–tree metrics*, J. Lond. Math. Soc. 78 (2008) 755–766 MR

[15] **S Dowdall**, **S J Taylor**, *The co-surface graph and the geometry of hyperbolic free group extensions*, J. Topol. 10 (2017) 447–482 MR

[16] **S Dowdall**, **S J Taylor**, *Hyperbolic extensions of free groups*, Geom. Topol. 22 (2018) 517–570 MR

[17] **B Farb**, **L Mosher**, *Convex cocompact subgroups of mapping class groups*, Geom. Topol. 6 (2002) 91–152 MR

[18] **M Feighn**, **M Handel**, *The recognition theorem for $\mathrm{Out}(F_n)$*, Groups Geom. Dyn. 5 (2011) 39–106 MR

[19] **R Gupta**, *Relative currents*, Conform. Geom. Dyn. 21 (2017) 319–352 MR

[20] **U Hamenstaedt**, *Word hyperbolic extensions of surface groups*, preprint (2005) arXiv

[21] **M Handel**, **L Mosher**, *Subgroup decomposition in $\mathrm{Out}(F_n)$: research announcement and introduction*, preprint (2013) arXiv

[22] **C Horbez**, *A short proof of Handel and Mosher's alternative for subgroups of $\mathrm{Out}(F_N)$*, Groups Geom. Dyn. 10 (2016) 709–721 MR

[23] **I Kapovich**, *The frequency space of a free group*, Internat. J. Algebra Comput. 15 (2005) 939–969 MR

[24] **I Kapovich**, *Currents on free groups*, from "Topological and asymptotic aspects of group theory" (R Grigorchuk, M Mihalik, M Sapir, Z Šuniḱ, editors), Contemp. Math. 394, Amer. Math. Soc., Providence, RI (2006) 149–176 MR

[25] **I Kapovich**, **M Lustig**, *Ping-pong and outer space*, J. Topol. Anal. 2 (2010) 173–201 MR

[26] **R P Kent, IV**, **C J Leininger**, *Shadows of mapping class groups: capturing convex cocompactness*, Geom. Funct. Anal. 18 (2008) 1270–1325 MR

[27] **G Levitt**, **M Lustig**, *Irreducible automorphisms of F_n have north-south dynamics on compactified outer space*, J. Inst. Math. Jussieu 2 (2003) 59–72 MR

[28] **M Lustig**, **C Uyanik**, *North–south dynamics of hyperbolic free group automorphisms on the space of currents*, J. Topol. Anal. (online publication November 2017)

[29] **M Lustig**, **C Uyanik**, *Perron–Frobenius theory and frequency convergence for reducible substitutions*, Discrete Contin. Dyn. Syst. 37 (2017) 355–385 MR

[30] **R Martin**, *Non-uniquely ergodic foliations of thin-type, measured currents and automorphisms of free groups*, PhD thesis, University of California, Los Angeles (1995) MR Available at `https://search.proquest.com/docview/304185823`

[31] **L Mosher**, *A hyperbolic-by-hyperbolic hyperbolic group*, Proc. Amer. Math. Soc. 125 (1997) 3447–3455 MR

[32] **M Queffélec**, *Substitution dynamical systems—spectral analysis*, 2nd edition, Lecture Notes in Mathematics 1294, Springer (2010) MR

[33] **S J Taylor**, *A note on subfactor projections*, Algebr. Geom. Topol. 14 (2014) 805–821 MR

[34] **S J Taylor**, **G Tiozzo**, *Random extensions of free groups and surface groups are hyperbolic*, Int. Math. Res. Not. 2016 (2016) 294–310 MR

[35] **C Uyanik**, *Dynamics of hyperbolic iwips*, Conform. Geom. Dyn. 18 (2014) 192–216 MR

[36] **C Uyanik**, *Generalized north-south dynamics on the space of geodesic currents*, Geom. Dedicata 177 (2015) 129–148 MR

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