Algebraic & Geometric Topology 19 (2019) 1477–1524

Classifying spaces from Ore categories with Garside families

STEFAN WITZEL

We describe how an Ore category with a Garside family can be used to construct a classifying space for its fundamental group(s). The construction simultaneously generalizes Brady's classifying space for braid groups and the Stein–Farley complexes used for various relatives of Thompson's groups. It recovers the fact that Garside groups have finite classifying spaces.

We describe the categories and Garside structures underlying certain Thompson groups. The indirect product of categories is introduced and used to construct new categories and groups from known ones. As an illustration of our methods we introduce the group *braided* T and show that it is of type F_∞.

57M07; 20F36, 20F65

There are many results establishing finiteness properties of Thompson groups. The proofs typically follow the same blueprint, due to Brown, Stein and Farley, to reduce the problem to its technical core, which is then solved individually; examples are due to Brown [16], Stein [36], Farley [23], Bux, Fluch, Marschler, Witzel and Zaremsky [24; 18; 43], Martínez-Pérez, Matucci and Nucinkis [31] and Belk and Forrest [3]. This fact is well known to experts but it is not apparent when looking at the articles. The reason is that the proofs are phrased using very different language. The present article provides a uniform formalization of the common ("blueprint") part of the mentioned

Published: 21 May 2019 DOI: 10.2140/agt.2019.19.1477

proofs. The result is a theorem that reduces a statement about finiteness properties of Thompson groups to its technical core, which is about connectivity of certain complexes.

In formalizing the blueprint it is fruitful to employ the language of categories, not because any sophisticated category theory would be used, but because categories are flexible enough to model posets, monoids, complexes and other objects that occur in the constructions. A single category encodes at the same time the Thompson group (its fundamental group) as well as the complex for it to act on (a subcomplex of the realization).

In formulating the proof categorically we find that the assumptions that make it work are established concepts in the (recent) literature — see Dehornoy, Digne, Godelle, Krammer and Michel [22]; the key notions are those of an Ore category and of a Garside family (see Section 1 for definitions). An Ore category not only has the property that elements of its fundamental group can be written as a fraction of two morphisms (eg "tree diagrams"), it also gives rise to a contractible space for it to act on. A Garside family of morphisms (eg "elementary splits") is what is needed to make the Quillen trick work and reduce to the smaller Stein–Farley complex. In the abstract formulation our results apply not only to Thompson groups but also to Garside groups such as the braid groups BRAID_n and possibly to entirely different examples.

The main results are given in Section 3 in greater generality (see Observation 1.7 for the relationship between a Garside map and a Garside family).

Theorem A *Let $\mathcal{C}$ be a small right-Ore category that is factor-finite and admits a right-Garside map Δ, and let $* \in \mathrm{Ob}(\mathcal{C})$. There is a contractible simplicial complex X on which $G = \pi_1(\mathcal{C}, *)$ acts. The space is covered by the G–translates of compact subcomplexes K_x for $x \in \mathrm{Ob}(\mathcal{C})$. Every stabilizer is isomorphic to a finite-index subgroup of the automorphism group $\mathcal{C}^\times(x,x)$ for some $x \in \mathrm{Ob}(\mathcal{C})$.*

Taking $\mathcal{C}$ to be a Garside monoid and Δ to be the Garside element, one immediately recovers the known fact that Garside groups, and braid groups in particular, have finite classifying spaces; see Charney, Meier and Whittlesey [20]. In fact, if $\mathcal{C}$ is taken to be the dual braid monoid, the quotient $G \backslash X$ is precisely Brady's classifying space for BRAID_n [10].

In the case of Thompson's group F the complex in Theorem A is the Stein–Farley complex. The action is not cocompact in this case because $\mathcal{C}$ has infinitely many objects. In order to obtain cocompact actions on highly connected spaces, we employ Morse theory.

Theorem B *Let $\mathcal{C}$, Δ, $*$ be as in Theorem A and let $\rho\colon \mathrm{Ob}(\mathcal{C}) \to \mathbb{N}$ be a height function such that $\{x \in \mathrm{Ob}(\mathcal{C}) \mid \rho(x) \le n\}$ is finite for every $n \in \mathbb{N}$. Assume that*

- (STAB) *$\mathcal{C}^\times(x,x)$ is of type F_n for all x,*
- (LK) *there exists an $N \in \mathbb{N}$ such that $|E(x)|$ is $(n-1)$–connected for all x with $\rho(x) \ge N$.*

Then $\pi_1(\mathcal{C},*)$ is of type F_n.

The complexes $|E(x)|$ depend on $\mathcal{C}$ and Δ and are described in Section 3.4. Establishing condition (LK) is what we referred to as the technical core of the problem in the beginning.

Theorem B provides a general scheme for proving that an (eligible) group is of type F_∞: first describe the category, second analyze the complexes $|E(x)|$, and then apply the theorem. This scheme will be illustrated in Section 5 (describe the category) and Section 6 (analyze the complexes, apply the theorem) on the examples of Thompson's groups F, T and V, their braided versions and some other groups. To our knowledge this is the first time that Garside structures are studied in connection with Thompson groups. In the process we define the Thompson group *BT*, *braided T*, and prove (see Theorem 6.7):

Theorem C *The braided Thompson group BT is of type F_∞.*

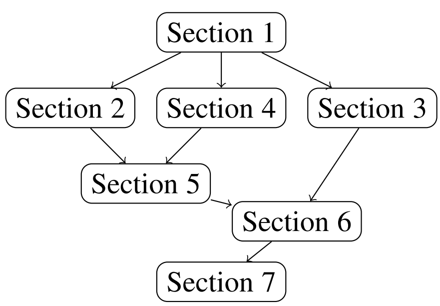
Although braided versions of V — see Dehornoy [21] and Brin [15] — and F — see Brady, Burillo, Cleary and Stein [11] — exist in the literature, our main merit is to be able to define braided T. The fact that it is F_∞ then follows from Theorem B and results from [18]. To explain the issue of defining BT we need to digress a bit (see also Remark 5.11). The category underlying Thompson's group F is the category of forests, where a morphism $m \leftarrow n$ is a rooted forest with m roots and n leaves (see Section 2). The categories underlying Thompson's groups T and V are obtained by adding in the cyclic groups $(\mathbb{Z}/n\mathbb{Z})_{n\in\mathbb{N}}$ respectively the symmetric groups $(\mathrm{SYM}_n)_{n\in\mathbb{N}}$. The categories underlying the braided groups *BF*, *BT* and *BV* are obtained from the forest category by adding in, for each n, the preimage under the map $\mathrm{BRAID}_n \to \mathrm{SYM}_n$ of the trivial group, the cyclic group and the full symmetric group, respectively.

When Brin first introduced *BV*, he avoided using categories by starting with the monoid of forests with infinitely many roots and leaves and added in the braid group on infinitely many strands $\lim_n \mathrm{BRAID}_n$. He then described which elements of the resulting infinite-strand group should belong to BV by hand. The reason that this workaround is not viable for BT, or actually even for T, is simply that the finite cyclic groups $(\mathbb{Z}/n\mathbb{Z})_{n\in\mathbb{N}}$ do not have enough inclusions into each other and therefore no limiting object (nor do their preimages in BRAID_n).

Formally, the process of adding groups to the forest monoid mentioned in the last paragraph is formation of the *indirect product* (or Zappa–Szép product) $\mathcal{F} \bowtie \mathcal{G}$, where $\mathcal{F}$ is the forest category and $\mathcal{G}$ is the category containing the groups in question. We introduce the indirect product of categories in Section 4.

The applications of Theorem B are somewhat similar to those of Thumann's results [37], so we should clarify how they compare. Basically, Theorem B applies to more general situations but has less power built in. Thumann's framework is restricted to symmetric or braided operads but the connectivity proofs from [18] verifying condition (LK) are already included. Our results apply to more general settings such as the ones discussed in Sections 5.3 and 5.5, and in particular to groups that are not of type F_∞, but leave the work of checking (LK) to the user.

The article is organized as follows. The basic notions are introduced in Section 1. The underlying structures for braid groups and Thompson's group F are described in Section 2. Section 3 contains the main construction and the proofs of Theorems A and B. The indirect product of categories is introduced in Section 4 and is used in Section 5 to construct the categories underlying Thompson's groups and their braided versions. In Section 6, Theorem B is applied to the examples from Section 5 to deduce finiteness properties, among them Theorem C. In Section 7 we briefly sketch how further Thompson groups fit into our framework. Since the results about finiteness properties and the indirect product may be of independent interest, we include the following leitfaden:



This article arose out of the introduction to the author's Habilitation thesis [40], which in addition covers Thompson groups arising from matrix groups via cloning systems; see [43] and Section 5.3. More recently our results were used in proving that for every n

there exists a simple group that is of type F_{n-1} but not of type F_n; see Skipper, Witzel and Zaremsky [35].

1 Categories generalizing monoids

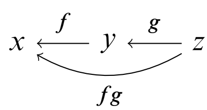
We start by collecting basic notions of categories regarding them as generalizations of monoids. Our exposition is based on [22, Chapter II], where the perspective is similar. The main difference is notational; see Remark 1.1 below.

A monoid may be regarded as (the set of morphisms) of a category with a single object. For us categories will play the role of generalized monoids where the multiplication is only partially defined. In particular, all categories in this chapter will be small. The requirement that they be locally small is important and taking them to be small is convenient; for example, it allows us to talk about morphisms of categories as maps of sets.

Let $\mathcal{C}$ be a category. Notationally, we follow [22] in denoting the set of morphisms of $\mathcal{C}$ by $\mathcal{C}$ as well (thinking of them as elements), while the objects are denoted by $\mathrm{Ob}(\mathcal{C})$. The identity at x will be denoted by 1_x. If f is a morphism from y to x, we call y the *source* and x the *target* of f. Our notation for composition is the familiar one for functions, that is, if f is a morphism from y to x and g is a morphism from z to y, then fg exists and is a morphism from z to x. If $x, y \in \mathrm{Ob}(\mathcal{C})$ then the set of morphisms from y to x is denoted by $\mathcal{C}(x, y)$, the set of morphisms from y to any object is denoted by $\mathcal{C}(-, y)$ and the set of morphisms from any object to x is denoted by $\mathcal{C}(x, -)$. This may be slightly unusual but renders the following intuitive expression valid:

$$f \in \mathcal{C}(x, y),\ g \in \mathcal{C}(y, z) \implies fg \in \mathcal{C}(x, z).$$

The corresponding diagram is



When we write an expression involving a product of morphisms, the requirement that this product exists is usually an implicit condition of the expression. Thus, $fg = h$ means that the source of f is the target of g and that the equality holds.

Remark 1.1 The net effect of the various differences in notation is that our formalism is consistent with [22], only the meaning of source/target, from/to and the direction of arrows are switched. The reason for this decision is that some of our morphisms will be group elements which we want to act from the left.

1.1 Groupoids

A morphism $f \in \mathcal{C}(x,y)$ is *invertible* if there is an *inverse*, namely a morphism $g \in \mathcal{C}(y,x)$ such that $fg = 1_x$ and $gf = 1_y$. The set of invertible morphisms in $\mathcal{C}(x,y)$ is denoted by $\mathcal{C}^\times(x,y)$ and the set of all invertible morphisms by $\mathcal{C}^\times$. A *groupoid* is a category $\mathcal{G}$ in which every morphism is invertible: $\mathcal{G} = \mathcal{G}^\times$. Just as every monoid naturally maps to a group, every category naturally maps to a groupoid; see [22, Section 3.1]:

Theorem 1.2 *For every category $\mathcal{C}$ there is a groupoid $\mathcal{G}pd(\mathcal{C})$ and a morphism $\iota\colon \mathcal{C} \to \mathcal{G}pd(\mathcal{C})$ with the following universal property: if $\varphi\colon \mathcal{C} \to \mathcal{G}$ is a morphism to a groupoid then there is a unique morphism $\widehat{\varphi}\colon \mathcal{G}pd(\mathcal{C}) \to \mathcal{G}$ such that $\varphi = \widehat{\varphi} \circ \iota$.*

The groupoid $\mathcal{G}pd(\mathcal{C})$ and the morphism ι are determined by $\mathcal{C}$ uniquely up to unique isomorphism.

We call $\mathcal{G}pd(\mathcal{C})$ the *enveloping groupoid* of $\mathcal{C}$. The morphism ι is a bijection on objects but it is not typically injective (on morphisms). One way to think about the enveloping groupoid is as the fundamental groupoid of $\mathcal{C}$:

The *nerve* of $\mathcal{C}$ is the simplicial set whose k–simplices are diagrams

$$x_0 \xleftarrow{f_1} x_1 \xleftarrow{f_2} x_2 \leftarrow \cdots \leftarrow x_{k-1} \xleftarrow{f_k} x_k$$

in $\mathcal{C}$. The i^{th} face is obtained by deleting x_i and replacing f_i and f_{i+1} by $f_i f_{i+1}$ and the j^{th} degenerate coface is obtained by introducing 1_{x_j} between f_j and f_{j+1}.

Proposition 1.3 [33, Proposition 1] *The groupoid $\mathcal{G}pd(\mathcal{C})$ is canonically isomorphic to the fundamental groupoid of the realization of the nerve of $\mathcal{C}$.*

In particular, the fundamental group of $\mathcal{C}$ in an object x is just the set of endomorphisms of $\mathcal{G}pd(\mathcal{C})$ in x: $\pi_1(\mathcal{C},x) = \mathcal{G}pd(\mathcal{C})(x,x)$.

1.2 Noetherianity conditions

If $fg = h$ then we say that f is a *left-factor* of h and that h is a *right-multiple* of f. It is a *proper left-factor* or *proper right-multiple* if g is not invertible. We say that f is a *(proper) factor* of h if $efg = h$ (and one of e and g is not invertible).

The category $\mathcal{C}$ is *Noetherian* if there is no infinite sequence $f_0, f_1, \ldots$ such that f_{i+1} is a proper factor of f_i. It is said to be *strongly Noetherian* if there exists a map $\delta\colon \mathcal{C} \to \mathbb{N}$

that satisfies $\delta(fg) \geq \delta(f) + \delta(g)$ and, for $f \in \mathcal{C}$ noninvertible, $\delta(f) \geq 1$. Clearly, a strongly Noetherian category is Noetherian. See [22, Sections II.2.3 and II.2.4] for a detailed discussion.

We call a *height function* a map $\rho\colon \mathrm{Ob}(\mathcal{C}) \to \mathbb{N}$ such that $\rho(x) = \rho(y)$ if $\mathcal{C}(x, y)$ contains an invertible morphism and $\rho(x) < \rho(y)$ if $\mathcal{C}(x, y)$ contains a noninvertible morphism. Note that the existence of a height function implies strong Noetherianity by taking $\delta(f) = \rho(y) - \rho(x)$ if $f \in \mathcal{C}(x, y)$.

We say that $\mathcal{C}$ is *factor-finite* if every morphism in $\mathcal{C}$ has only finitely many factors up pre- and postcomposition by invertibles. This condition implies strong Noetherianity (see [22, Proposition 2.48]).

1.3 Ore categories

Two elements $g, h \in \mathcal{C}(x, -)$ have a *common right-multiple* d if there exist elements $e, f \in \mathcal{C}$ with $ge = hf = d$. It is a *least common right-multiple* if every other common right-multiple is a right-multiple of d. We say that $\mathcal{C}$ *has common right-multiples* if any two elements with the same target have a common right-multiple. We say that it *has conditional least common right-multiples* if any two elements that have a common right-multiple have a least common right-multiple. We say that it *has least common right-multiples* if any two elements with the same target have a least common right-multiple. We say that $\mathcal{C}$ is *left-cancellative* if $ef = eh$ implies $f = h$ for all $e, f, g \in \mathcal{C}$. All of these notions have obvious analogues with left and right interchanged. A category is *cancellative* if it is left-cancellative and right-cancellative.

Lemma 1.4 *If $\mathcal{C}$ is cancellative and $f \in \mathcal{C}$ has a left-inverse or right-inverse then it is invertible.*

Proof Let $f \in \mathcal{C}(x, y)$ and assume that there is an $e \in \mathcal{C}(y, x)$ that is a left-inverse for f, that is, $ef = 1_y$. Then $fef = f$ and canceling f on the right shows that e is also a right-inverse. The other case is symmetric. □

Lemma 1.5 *Let $\mathcal{C}$ be strongly Noetherian. Then $\mathcal{C}$ has least common right-multiples if and only if it has greatest common left-factors.*

Proof Suppose that $\mathcal{C}$ has least common right-multiples and let $f, g \in \mathcal{C}(x, -)$. Let s and t be common left-factors of f and g and let r be a least common right-multiple of s and t. Then, since f and g are common right-multiples of s and t, they are right-multiples of r, meaning that r is a common left-factor. If s and t are not right-multiples of each other then $\delta(r) > \delta(s), \delta(t)$ and an induction on $\delta(r) \le \delta(f), \delta(g)$ over the common left-factors of f and g produces a greatest common left-factor. The other direction is analogous. □

We say that $\mathcal{C}$ is right/left-Ore if it is cancellative and has common right/left-multiples.

Theorem 1.6 *A category $\mathcal{C}$ that is right-Ore embeds in a groupoid $\mathcal{G}$ such that every element $h \in \mathcal{G}$ can be written as $h = fg^{-1}$ with $f, g \in \mathcal{C}$.*

The groupoid $\mathcal{G}$ in the theorem is called the *Ore localization* $\mathcal{O}re(\mathcal{C})$ of $\mathcal{C}$. Using the universal property, it is not hard to see that it coincides with the enveloping groupoid of $\mathcal{C}$.

The fundamental group of an Ore category has a particularly easy description. In general, an element of $\pi_1(\mathcal{C}, x)$ is represented by a sequence $f_0 g_1^{-1} f_1 \cdots f_{n-1} g_n^{-1}$ with $f_i, g_i \in \mathcal{C}(x_i, -)$ and $f_j, g_{j+1} \in \mathcal{C}(-, y_j)$. But if $\mathcal{C}$ has common right-multiples, then $g_1^{-1} f_1$ can be rewritten as $f_1' g_1'^{-1}$ and so the sequence can be shortened to $(f_0 f_1')(g_2 g_1')^{-1} f_2 \cdots f_{n-1} g_n^{-1}$. Iterating this argument, we find that every element of $\pi_1(\mathcal{C}, x)$ is of the form fg^{-1} with $f, g \in \mathcal{C}(x, -)$.

1.4 Presentations

We introduce presentations for categories. This is analogous to the situation for monoids and we will be brief. See [22, Section II.1.4] for details.

A (small) *precategory* $\mathcal{S}$ consists of a set of objects $\mathrm{Ob}(\mathcal{S})$ and a set of morphisms $\mathcal{S}$. As for categories, each morphism has a *source* and a *target* that are objects and it is a morphism from the source to its target. The set of morphisms from y to x is denoted by $\mathcal{S}(y, x)$. The monoidal aspects of a category are missing in a precategory: it does not have identities or a composition.

Given a precategory $\mathcal{S}$ there exists a free category $\mathcal{S}^*$ generated by $\mathcal{S}$. It has the universal property that if $\phi\colon \mathcal{S} \to \mathcal{C}$ is a morphism of precategories and $\mathcal{C}$ is a category, then ϕ uniquely factors through $\mathcal{S} \to \mathcal{S}^*$. One can construct $\mathcal{S}^*$ to have the same objects as $\mathcal{S}$ and have morphisms finite words in $\mathcal{S}$ that are composable.

A *relation* is a pair $r = s$ of morphisms in $\mathcal{S}^*$ with the same source and target. If $\phi\colon \mathcal{S}^* \to \mathcal{C}$ is a morphism, the relation *holds* in $\mathcal{C}$ if $\phi(r) = \phi(s)$. A *presentation* consists of a precategory $\mathcal{S}$ and a family of relations $\mathcal{R}$ in $\mathcal{S}^*$. The category it presents is denoted by $\langle \mathcal{S} \mid \mathcal{R} \rangle$.

It has the universal property that if $\phi\colon \mathcal{S} \to \mathcal{C}$ is a morphism of precategories and $\mathcal{C}$ is a category in which all relations in $\mathcal{R}$ hold then ϕ uniquely factors through $\mathcal{S} \to \langle \mathcal{S} \mid \mathcal{R} \rangle$. One can construct $\langle \mathcal{S} \mid \mathcal{R} \rangle$ by quotienting $\mathcal{S}^*$ by the symmetric, transitive closure of the relations.

1.5 Garside families

The following notions are at the core of [22]. We will sometimes be needing the notions with the reverse order. What in [22] is referred to as a Garside family in a left-cancellative category will be called a left-Garside family here to avoid confusion in categories that are left- and right-cancellative.

Let $\mathcal{C}$ be a left-cancellative category and let $\mathcal{S} \subseteq \mathcal{C}$ be a set of morphisms. We denote by $\mathcal{S}^\sharp$ the set $\mathcal{C}^\times \cup \mathcal{S}\mathcal{C}^\times$ of morphisms that are invertible or left-multiples of invertibles by elements of $\mathcal{S}$. We say that $\mathcal{S}^\sharp$ is *closed under (left/right-)factors* if every (left/right-)factor of an element in $\mathcal{S}^\sharp$ is again in $\mathcal{S}^\sharp$. An element $s \in \mathcal{S}$ is an *$\mathcal{S}$–head* of $f \in \mathcal{C}$ if s is a left-factor of f and every left-factor of f in $\mathcal{S}$ is a left-factor of s [22, Definition IV.1.10]. The set $\mathcal{S}$ is a *left-Garside family* if $\mathcal{S}^\sharp$ generates $\mathcal{C}$, is closed under right-factors and every noninvertible element of $\mathcal{C}$ admits and $\mathcal{S}$–head [22, Proposition IV.1.24]. If $\mathcal{S}$ is a left-Garside family then $\mathcal{C}^\times\mathcal{S} \subseteq \mathcal{S}^\sharp$, so in fact $\mathcal{S}^\sharp = \mathcal{C}^\times \cup \mathcal{C}^\times\mathcal{S}\mathcal{C}^\times$ [22, Proposition III.1.39].

All notions readily translate to right-Garside families, except that the head is called an *$\mathcal{S}$–tail* if $\mathcal{S}$ is a right-Garside family. Note that $\mathcal{S}^\sharp$ is defined as $\mathcal{C}^\times \cup \mathcal{C}^\times\mathcal{S}$ when $\mathcal{S}$ is (regarded as) a right-Garside family.

We will be interested in Garside families that are closed under factors. We describe two situations where this is the case.

Let $\mathcal{C}$ be left-cancellative and consider a map $\Delta\colon \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}$ with $\Delta(x) \in \mathcal{C}(x,-)$. We write

$$\mathrm{Div}(\Delta) = \{g \in \mathcal{C} \mid gh = \Delta(x) \text{ for some } x \in \mathrm{Ob}(\mathcal{C}),\ h \in \mathcal{C}\},$$
$$\widetilde{\mathrm{Div}}(\Delta) = \{h \in \mathcal{C} \mid gh = \Delta(x) \text{ for some } x \in \mathrm{Ob}(\mathcal{C}),\ g \in \mathcal{C}\},$$

for the families of left- and right-factors of morphisms in the image of Δ. Such a map is a *right-Garside map* if $\mathrm{Div}(\Delta)$ generates $\mathcal{C}$, if $\widetilde{\mathrm{Div}}(\Delta) \subseteq \mathrm{Div}(\Delta)$, and if, for every $g \in \mathcal{C}(x,-)$, the elements g and $\Delta(x)$ admit a greatest common left-factor. If Δ is a right-Garside map then $\mathrm{Div}(\Delta)$ is a left-Garside family closed under left-factors and thus under factors [22, Proposition V.1.20]. We note the following for future reference:

Observation 1.7 *Let $\mathcal{C}$ be a left-cancellative, factor-finite category and let Δ be a right-Garside map. Then $\mathcal{S} := \mathrm{Div}(\Delta)$ is a left-Garside family closed under factors and $\mathcal{S}(x,-)$ is finite for every $x \in \mathrm{Ob}(\mathcal{C})$.*

Let $\mathcal{C}$ be right-Ore. A right-Garside family is *strong* if for $s, t \in \mathcal{S}^\sharp$ there exist $s', t' \in \mathcal{S}^\sharp$ such that $st' = ts'$ is a least common right-multiple of s and t [22, Definition 2.29]. If $\mathcal{S}$ is a strong right-Garside family then $\mathcal{S}^\sharp$ is also closed under left-factors and thus is closed under factors [22, Proposition 1.35].

2 Fundamental examples

2.1 Thompson's group F and the category $\mathcal{F}$

Our description of Thompson's groups is not the standard one, which can be found in [19]. An element of Thompson's group F is given by a pair (T_+, T_-) of finite rooted binary trees with the same number of leaves, say n. If we add a caret to the i^{th} leaf ($1 \le i \le n$) of T_+, that is we make it into an inner vertex with two leaves below it, we obtain a tree T'_+ on $n+1$ vertices. If we also add a caret to the i^{th} leaf of T_- we obtain another tree T'_-. We want to regard (T'_+, T'_-) as equivalent to (T_+, T_-) so we take the reflexive, symmetric, transitive closure of the operation just described and write the equivalence class by $[T_+, T_-]$. Thompson's group F is the set of equivalence classes $[T_+, T_-]$.

In order to define the product of two elements $[T_+, T_-]$ and $[S_+, S_-]$, we note that we can add carets to both tree pairs to get representatives $[T'_+, T'] = [T_+, T_-]$ and $[T', T'_-] = [S_+, S_-]$, where the second tree of the first element and the first tree of the second element are the same. Therefore, multiplication is completely defined by declaring that $[T'_+, T'] \cdot [T', T'_-] = [T'_+, T'_-]$. It is easy to see that $[T, T]$ is the neutral element for any tree T and that $[T_+, T_-]^{-1} = [T_-, T_+]$.

We have defined the group F in such a way that a categorical description imposes itself; see [2]. We define $\mathcal{F}$ to be the category whose objects are positive natural numbers and whose morphisms $m \leftarrow n$ are binary forests on m roots with n leaves. Multiplication of a forest $E \in \mathcal{F}(\ell, m)$ and a forest $F \in \mathcal{F}(m, n)$ is defined by identifying the leaves of E with the roots of F and taking EF to be the resulting tree. Pictorially this corresponds to stacking the two forests on top of each other (see Figure 1).

Proposition 2.1 *The category $\mathcal{F}$ is strongly Noetherian and right-Ore. In fact, it has least common right-multiples and greatest common left-factors.*

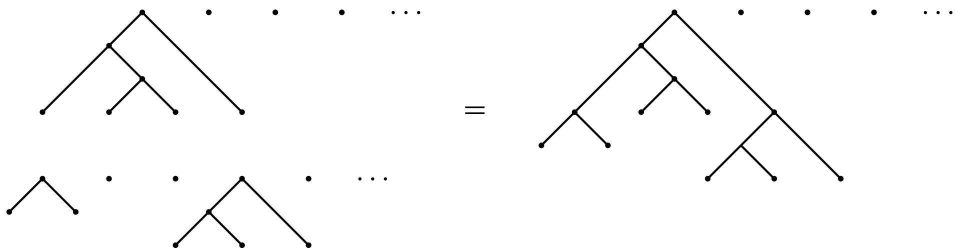


Figure 1: Multiplication of forests (taken from [42])

Proof The identity map $\rho\colon \mathbb{N} = \mathrm{Ob}(\mathcal{F}) \to \mathbb{N}$ is a height function on $\mathcal{F}$. Thus, $\mathcal{F}$ is strongly Noetherian.

The least common right-multiple of two forests in $\mathcal{F}(m, -)$ is their union (regarding both forests as subforests of the leafless binary forest on m roots). The greatest common left-factor is their intersection. Left-cancellativity means that given a forest $f \in \mathcal{F}(m, \ell)$ and a left-factor $a \in \mathcal{F}(m, n)$, the forest in $b \in \mathcal{F}(n, \ell)$ with $f = ab$ is unique. Indeed, it is the forest obtained from f by removing a and turning the leaves of a into roots. Right-cancellativity means that a is uniquely determined if $f = ab$. To see this, we identify the leaves of f with the leaves of b. Now the common predecessor in f of a set of leaves of a tree of b is a leaf of a and every leaf of a arises in that way. □

The proposition together with the remark at the end of Section 1.3 shows that every element of $\pi_1(\mathcal{F}, 1)$ is represented by fg^{-1} where $f, g \in \mathcal{F}(1, -)$ are binary trees. Cancellativity ensures that $fg^{-1} = f'g'^{-1}$ if and only if there exist h and h' such that $fh = f'h'$ and $gh = g'h'$. Comparing this description with our definition of F we see:

Proposition 2.2 *Thompson's group F is isomorphic to $\pi_1(\mathcal{F}, 1)$.*

Later on it will be convenient to have a presentation for $\mathcal{F}$. The shape of the relations will not come as a surprise to the reader familiar with Thompson's groups. A proof can be found in [40].

Proposition 2.3 *The category $\mathcal{F}$ has a presentation with morphisms $\lambda_i^n\colon n \leftarrow n+1$ for $1 \le i \le n$ as generators subject to the relations*

$$\lambda_i^n \lambda_j^{n+1} = \lambda_j^n \lambda_{i+1}^{n+1} \quad \text{for } 1 \le j < i \le n. \tag{2-1}$$

Every morphism in $\mathcal{F}(m, n)$ can be written in a unique way as $\lambda_{i_m}^m \cdots \lambda_{i_{n-1}}^{n-1}$ with $(i_j)_j$ nondecreasing.

Remark 2.4 The relations (2-1) reflect a commutation phenomenon: for any forest, adding a caret to the i^{th} leaf and then to the j^{th} leaf has the same effect as doing it the other way around. That it does not algebraically look like a commutation relation is due to the fact the index of the right one of the two leaves has changed when adding the left caret. This is inevitable in the present setup because the i^{th} leaf has no identity as a particular vertex in the infinite rooted binary tree but simultaneously represents all i^{th} leaves of trees with n leaves. A larger category in which the relations are algebraically commutation relations will appear in Section 5.5.

Note that since $\mathcal{F}$ is connected, the fundamental groups at different objects are isomorphic. This corresponds to the elementary fact that the tree pair (T_+, T_-) representing an element of F can always be chosen so that T_+ and T_- contain an arbitrary fixed subtree.

The most convenient way to exhibit a Garside family in $\mathcal{F}$ is by describing a right-Garside map: for every $n \in \mathbb{N} = \mathrm{Ob}(\mathcal{F})$ let $\Delta(n)$ be the forest where every tree is a single caret.

Proposition 2.5 *The map* $\Delta\colon \mathrm{Ob}(\mathcal{F}) \to \mathcal{F}$ *is a right-Garside map.*

Proof The family $\mathrm{Div}(\Delta)$ consists of morphisms where every forest is either a single caret or trivial. Every forest can be built of from these, for example by adding one caret at a time. This shows that $\mathrm{Div}(\Delta)$ generates $\mathcal{F}$. The family $\widetilde{\mathrm{Div}}(\Delta)$ also consists of morphisms where every forest is either a single caret or trivial with the additional condition that the total number of leaves is even and the left leaf of every caret has an odd index. In particular, $\widetilde{\mathrm{Div}}(\Delta) \subseteq \mathrm{Div}(\Delta)$. If $g \in \mathcal{F}(x, -)$ then g and $\Delta(x)$ have a greatest common left-factor by Proposition 2.1. □

With Observation 1.7 we get:

Corollary 2.6 *The category* $\mathcal{F}$ *admits a left-Garside family* $\mathcal{S}$ *that is closed under factors such that* $\mathcal{S}(x, -)$ *is finite for every* $x \in \mathcal{F}$.

Remark 2.7 The family $\mathrm{Div}(\Delta)$ is in fact a right- as well as a left-Garside family. It is strong as a right-Garside family but not as a left-Garside family.

If instead of rooted binary trees one takes rooted n–ary trees ($n \geq 2$) in the description above, one obtains the category $\mathcal{F}_n$. Everything is analogous to $\mathcal{F}$ but the new aspect that occurs for $n > 2$ is that the category is no longer connected: the number of leaves of an n–ary tree with r roots will necessarily be congruent to r modulo $n-1$; hence, there is no morphism in $\mathcal{F}_n$ connecting objects that are not congruent modulo $n-1$. As a consequence, the point at which the fundamental group is taken does matter and we obtain $n-1$ different groups for each category. It turns out, however, that the fundamental groups are in fact isomorphic independently of the basepoint [16, Proposition 4.1] and are denoted by

$$F_{n,\infty} = \pi_1(\mathcal{F}_n, 1).$$

The groups $F_{n,\infty}$ are the smallest examples of the *Higman–Thompson groups* introduced by Higman [27]. As we will see later, the fundamental groups of the different components are nonisomorphic in the categories for the larger Higman–Thompson groups.

2.2 Braid groups

The *braid group* on n strands, introduced by Artin [1], is the group given by the presentation

$$\text{BRAID}_n = \big\langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2, \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ if } 1 \leq i \leq n-2 \big\rangle. \tag{2-2}$$

Its elements, called *braids*, can be conveniently depicted as braid diagrams as in Figure 2, illustrating a physical interpretation as braids on n strands. The first relations are *commutation relations*; the second are *braid relations*. The group BRAID_n arise as the fundamental group of the configuration space of n unordered points in the disc and as the mapping class group of the n–punctured disc; see [7; 29] for more details.

What is known as Garside theory today arose out of Garside's study of braid groups [25]. In this classical case, the category $\mathcal{C}$ has a single object and thus is a monoid. Specifically, a *Garside monoid* is a monoid M with an element $\Delta \in M$, called a *Garside element*, such that:

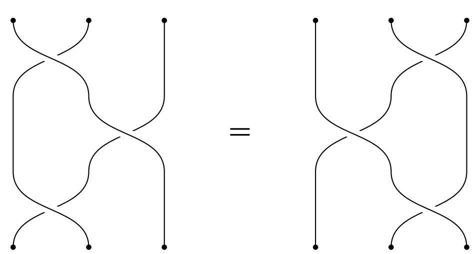


Figure 2: Diagrams illustrating the braid relation

(i) M is cancellative and has least common right- and left-multiples and greatest common right- and left-factors.

(ii) The left- and right-factors of Δ coincide, they are finite in number, and they generate M.

(iii) There is a map $\delta\colon M \to \mathbb{N}$ such that $\delta(fg) \geq \delta(f) + \delta(g)$ and $\delta(g) > 0$ if $g \neq 1$.

A *Garside group* is the group of fractions of a Garside monoid. Among the main features of Garside groups is that they have solvable word problem and conjugacy problem.

Note that a Garside monoid, regarded as a category with one object is, by definition, left- and right-Ore and strongly Noetherian. Moreover, the family of factors of Δ is a left- and right-Garside family.

To see that braid groups are in fact Garside groups, consider the *braid monoid* BRAID_n^+. It is obtained by interpreting the presentation (2-2) as a monoid presentation. It is a nontrivial consequence of Garside's work that the obvious map $\mathrm{BRAID}_n^+ \to \mathrm{BRAID}_n$ is injective, so that the braid monoid can be regarded as a subset of the braid groups. Its elements are called *positive braids* and are characterized by the property that left strands always overcrosses the right strand. The element Δ in BRAID_n^+ is the braid that performs a full half twist and is characterized by the fact that every strand crosses every other strand precisely once; see Figure 3. Its (left- or right-) factors are the braids where every strand crosses every other strand at most once. The function δ is simply the number of crossings, which is the same as length as a word in the generators. Now BRAID_n^+ is a Garside monoid with Garside element Δ; see [22, Section I.1.2, Proposition IX.1.29]. Its group of fractions is BRAID_n, which is therefore a Garside group.

It was noted by Birman, Ko and Lee [8] that there is in fact another monoid BRAID_n^{*+}, called the *dual braid monoid*, that also admits a Garside element Δ^* and has BRAID_n as its group of fractions; see also [22, Section I.1.3]. This monoid is in many ways better behaved than BRAID_n^+. Brady [10] used the dual braid monoid to construct a finite classifying space for the braid group.

Note that adding the relations σ_i^2 to the presentation (2-2) results in a presentation for the symmetric group SYM_n. In particular, there is a surjective homomorphism $\pi\colon \mathrm{BRAID}_n \to \mathrm{SYM}_n$ that takes σ_i to the transposition $s_i := (i\ i{+}1)$.

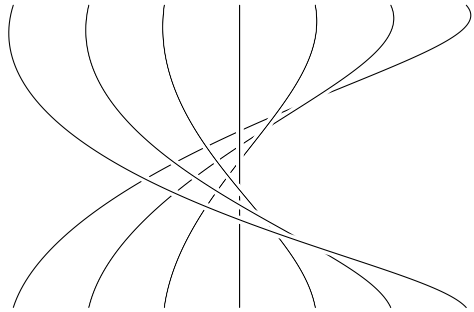


Figure 3: The element Δ in BRAID_7^+

The symmetric group is a finite Coxeter group and the braid group is its corresponding *Artin group*. For every Coxeter system (W, S) there exists an Artin group A_W obtained analogously and a morphism $\pi\colon A_W \to W$. Whenever W is finite, the Artin group A_W again contains a Garside monoid and a dual Garside monoid; see [12; 5].

3 Finiteness properties of fundamental groups of Ore categories

A classifying space for a group G is a CW complex B whose fundamental group is G and whose universal cover $X = \widetilde{B}$ is contractible. Since G acts freely on X with quotient $B = G\backslash X$, one can equivalently say that a classifying space is the quotient of a contractible CW complex by a free G–action. Our goal in this section is to construct "good" classifying spaces for fundamental groups of Ore categories. The best classifying spaces are compact ones; they have finitely many cells so we also refer to them as *finite*. If G admits a finite classifying space, we say that it is of type F. If a finite classifying space does not exist, we aim at classifying spaces with weaker finiteness properties. We start by constructing an action on a contractible space.

3.1 Contractible spaces from Ore categories with Garside families

Let $\mathcal{C}$ be a category that is right-Ore and strongly Noetherian. Let $\mathcal{S}$ be a left- or right-Garside family such that $\mathcal{S}^\sharp$ is closed under factors. Let $* \in \mathrm{Ob}(\mathcal{C})$ be a base object. Our goal is to construct a contractible space X on which $\pi_1(\mathcal{C}, *)$ acts with good finiteness properties of the stabilizers as well as the quotient. In the whole discussion $\mathcal{C}$ can be replaced by the component of $*$ in $\mathcal{C}$, so all assumptions only need to be made for objects and morphisms in that component.

We put $\mathcal{E} = \mathcal{S}^\sharp$ and recall that $\mathcal{E} = \mathcal{C}^\times \cup \mathcal{C}^\times \mathcal{S} \mathcal{C}^\times$. We call the elements of $\mathcal{E}$ *elementary*. Let $\delta\colon \mathcal{C} \to \mathbb{N}$ be a map that witnesses strong Noetherianity. Note that if $f \in \mathcal{C}(x,y)$ and $g \in \mathcal{C}^\times(-,x)$ and $h \in \mathcal{C}^\times(y,-)$ are invertible then

$$-\delta(g^{-1}) + \delta(f) - \delta(h^{-1}) \geq \delta(gfh) \geq \delta(g) + \delta(f) + \delta(h),$$

so $\delta(f) = \delta(gfh)$ and δ is invariant under pre- and postcomposition by invertibles.

We define the set $P = \mathcal{O}re(\mathcal{C})(*,-)/\mathcal{C}^\times$, that is, elements of P are equivalence classes $\overline{a}$ of elements $a \in \mathcal{O}re(\mathcal{C})(*,-)$ modulo the equivalence relation that $\overline{a} = \overline{a}'$ if there exists a $g \in \mathcal{C}^\times$ with $ag = a'$. We define a relation $\leq$ on P by declaring $\overline{a} \leq \overline{b}$ if there exists an $f \in \mathcal{C}$ with $af = b$.

Lemma 3.1 *The relation $\leq$ is a partial order on P in which any two elements have a common upper bound. In particular, the realization $|P|$ is contractible.*

Proof Note first that whether $f = a^{-1}b$ lies in $\mathcal{C}$ is independent of the representatives. Reflexivity and transitivity are clear. If $\overline{a} \leq \overline{b} \leq \overline{a}$ then there exist $f, h \in \mathcal{C}$ and $g \in \mathcal{C}^\times$ such that $af = b$ and $bh = ag$, showing that fh is a unit. In particular, f has a right-inverse and h has a left-inverse, so f and h are units by Lemma 1.4. This shows $\overline{a} = \overline{b}$.

For any $a \in \mathcal{O}re(\mathcal{C})$ there is an $f \in \mathcal{C}$ such that $af \in \mathcal{C}$. Since $\mathcal{C}$ has common right-multiples, it follows that for any two elements $a_1, a_2 \in \mathcal{O}re(\mathcal{C})$ there exist $f_1, f_2 \in \mathcal{C}$ with $a_1 f_1 = a_2 f_2$. □

We define a second, more restrictive relation $\preceq$ on P by declaring that $\overline{a} \preceq \overline{b}$ if there exists an $e \in \mathcal{E}$ with $ae = b$. Note that this relation will typically not be transitive. However, if $\overline{a} \preceq \overline{b}$ and $\overline{a} \leq \overline{c} \leq \overline{b}$ then $\overline{a} \preceq \overline{c} \preceq \overline{b}$ because $\mathcal{E}$ is closed under factors. The complex $X \subseteq |P|$ consists of those chains in $|P|$ that are chains with respect to $\preceq$. In particular, P is the vertex set of X.

Proposition 3.2 *The complex X is contractible.*

Proof Note that X is a subspace of $|P|$ containing all the vertices. One can obtain $|P|$ from X by gluing in (realizations of) intervals $[\overline{a}, \overline{b}]$ not yet contained in X. To organize the gluing, note the following: if $[\overline{c}, \overline{d}]$ is a proper subinterval of $[\overline{a}, \overline{b}]$ with $f = a^{-1}b \in \mathcal{C}$ and $h = c^{-1}d \in \mathcal{C}$, then h is a proper factor of f. To an interval $[\overline{a}, \overline{b}]$ with $f = a^{-1}b$ we assign the height $\hat{\delta}([\overline{a}, \overline{b}]) = \delta(f)$. Note that this is well defined, because any other representative f' will differ from f only by invertibles and δ is invariant under pre- and postcomposition by invertibles. Note also that proper subintervals have strictly smaller $\widehat{\delta}$–value. We can therefore glue in the intervals with increasing value of $\widehat{\delta}$ and be sure that when we glue in an interval, any proper subinterval is already glued in.

For any $n \in \mathbb{N}$ let $|P|_{\widehat{\delta}<n}$ be the subcomplex of $|P|$ consisting of X and intervals of $\widehat{\delta}$–value $< n$. If X was not contractible, there would be a sphere in X that could not be contracted in X but in $|P|$. The contraction would be compactly supported, and hence use simplices supported on finitely many simplices. It therefore suffices to show that the inclusion $X \to |P|_{\widehat{\delta}<n}$ is a homotopy equivalence for all $n \in \mathbb{N}$.

For $n = 0$ this is clear, so assume $n > 0$. Then

$$|P|_{\widehat{\delta}<n} = |P|_{\widehat{\delta}<n-1} \cup \bigcup_{\widehat{\delta}([\overline{a},\overline{b}])=n-1} |[\overline{a},\overline{b}]|.$$

The intervals that are glued in meet only in $|P|_{\widehat{\delta}<n-1}$ and they are glued in along $|[\overline{a},\overline{b})| \cup |(\overline{a},\overline{b}]|$. This is a suspension of $|(\overline{a},\overline{b})|$ and so it suffices to show that the open interval is contractible.

If $\mathcal{S}$ is a left-Garside family, every element h of $\mathcal{C}$, and every left-factor of f in particular, has an $\mathcal{S}$–head head(g). We define the map $\theta\colon [\overline{a},\overline{b}] \to [\overline{a},\overline{b}]$ by $\overline{ah} \mapsto \overline{a\,\mathrm{head}(h)}$. Note that $\theta(\overline{b}) < \overline{b}$ because otherwise $[\overline{a},\overline{b}]$ is already contained in $|P|$. Note also that $\theta(\overline{c}) > \overline{a}$ for $\overline{c} > \overline{a}$ because the head of a noninvertible is not invertible. This shows that θ restricts to a map $(\overline{a},\overline{b}) \to (\overline{a},\overline{b})$ with $\overline{c} \geq \theta(\overline{c}) \leq \theta(\overline{b})$ and we can apply [34, Section 1.5] to see that $|(\overline{a},\overline{b})|$ is contractible.

If $\mathcal{S}$ is a right-Garside family, θ is defined by $\overline{bh^{-1}} \mapsto \overline{b\,\mathrm{tail}(h)^{-1}}$. For the same reasons as above, θ restricts to a map $(\overline{a},\overline{b}) \to (\overline{a},\overline{b})$ with $\overline{c} \leq \theta(\overline{c}) \geq \theta(\overline{a})$ and we can again apply [34, Section 1.5]. □

There is an obvious action of $\pi_1(\mathcal{C},*)$ on X which is given by precomposition: if $g \in \pi_1(\mathcal{C},*) = \mathcal{O}re(\mathcal{C})(*,*)$ and $a \in \mathcal{O}re(\mathcal{C})(*,-)$ then $g\overline{a} = \overline{ga}$ and the relations $\leq$ and $\preceq$ are clearly preserved under this action.

Next we want to look at stabilizers and weak fundamental domains. These will be particularly well behaved with an additional assumption. We say that $\mathcal{S}$ is *(right-)locally finite* if for every object $x \in \mathrm{Ob}(\mathcal{C})$ the set $\mathcal{S}(x,-)$ is finite up to pre- and postcomposition by invertibles. Local finiteness of $\mathcal{S}$ does *not* imply that X is locally finite but does imply:

Observation 3.3 *Assume that $\mathcal{S}$ is locally finite. For every $\overline{a} \in P$ there are only finitely many $\overline{b} \in P$ with $\overline{a} \preceq \overline{b}$. In particular, there are only finitely many simplices for which $\overline{a}$ is $\preceq$–minimal.*

Lemma 3.4 *Every simplex-stabilizer of the action of $\pi_1(\mathcal{C},*)$ on X is isomorphic to a subgroup of $\mathcal{C}^\times(x,x)$ for some $x \in \mathrm{Ob}(\mathcal{C})$. If $\mathcal{S}$ is locally finite, the subgroup has finite index.*

Proof Let $\overline{a}$ be a vertex in X with $a \in \mathcal{O}re(\mathcal{C})(*,x)$ and suppose that $g \in \pi_1(\mathcal{C},*)$ fixes $\overline{a}$, that is, $\overline{a} = g\overline{a} = \overline{ga}$. Then $a^{-1}ga \in \mathcal{C}^\times(x,x)$. This shows that the stabilizer of $\overline{a}$ is conjugate to $\mathcal{C}^\times(x,x)$. If $\mathcal{S}$ is locally finite then Observation 3.3 implies that the stabilizer of an arbitrary simplex has finite index in a vertex stabilizer. □

Corollary 3.5 *If $\mathcal{C}^\times(x,x) = \{1_x\}$ for every object $x \in \mathrm{Ob}(\mathcal{C})$ then the action of $\pi_1(\mathcal{C},*)$ on X is free. If $\mathcal{C}^\times(x,x)$ is finite then the action is proper.*

Now let us pick, for every $x \in \mathrm{Ob}(\mathcal{C})$, a morphism $f_x \in \mathcal{O}re(\mathcal{C})(*,x)$ arbitrarily and let $K_x \subseteq X$ be the union of the realizations of the intervals $[\overline{f_x}, \overline{f_x e}]$ with $e \in \mathcal{E}(x,-)$.

Lemma 3.6 *The complex X is covered by the $\pi_1(\mathcal{C},*)$–translates of the complexes K_x for $x \in \mathrm{Ob}(\mathcal{C})$. If $\mathcal{S}$ is locally finite then each K_x is compact.*

Proof If $\sigma = \{f \prec fe_1 \prec \cdots \prec fe_k\}$ is a simplex in X with $f \in \mathcal{O}re(\mathcal{C})(*,x)$ and $e_1,\dots,e_k \in \mathcal{E}(x,-)$, then $f_x f^{-1} \in \pi_1(\mathcal{C},*)$ and $f_x f^{-1} K_x$ contains σ. The second statement is clear. □

The ideal special case is:

Corollary 3.7 *If $\mathcal{C}$ has no nonidentity invertible morphisms and has only finitely many objects and if $\mathcal{S}$ is locally finite, then $\pi_1(\mathcal{C},*)$ has a finite classifying space.*

Proof Under the assumption, the action of $\pi_1(\mathcal{C},*)$ is free by Corollary 3.5 and cocompact by Lemma 3.6. The quotient is then a finite classifying space. □

In particular, we recover the main result of [20]:

Corollary 3.8 *Every Garside group G has a finite classifying space.*

In the case of the dual braid monoid, the complex we constructed is precisely the *dual Garside complex* constructed by Brady [10].

3.2 Finiteness properties

Topological finiteness properties of a group G were introduced by Wall [38; 39] and are conditions on how finite a classifying space for G can be chosen. A group is said to be *of type* F_n if it admits a classifying space B whose n–skeleton $B^{(n)}$ has finitely many cells. Equivalently a group is of type F_n if it acts freely on a contractible space X such that the action on $X^{(n)}$ is cocompact. It is clear that type F_n implies type F_m for $m < n$ and one defines the finiteness length $\phi(G)$ to be the supremal n for which G is of type F_n. If $\phi(G) = \infty$ then G is said to be of type F_∞.

In low dimensions, these properties have familiar descriptions: a group is of type F_1 if and only if it is finitely generated, and it is of type F_2 if and only if it is finitely presented.

Given a group G, in order to study its finiteness properties, one needs to let G act on a highly connected space X. If the action is free, then the low-dimensional skeleta of $G\backslash X$ are those of a classifying space. A useful result is Brown's criterion, which says that one does not have to look at free actions; see [16, Propositions 1.1, 3.1]:

Theorem 3.9 *Let G act cocompactly on an $(n-1)$–connected CW complex X. If the stabilizer of every p–cell of X is of type F_{n-p} then G is of type F_n.*

The full version of Brown's criterion also gives a way to decide that a group is not of type F_n. We formulate it here only to explain why we will not be able to apply it:

Theorem 3.10 *Let G act on an $(n-1)$–connected CW complex X and assume that the stabilizer of every p–cell of X is of type F_{n-p}. If G is of type F_n then, for every cocompact subspace Y and any basepoint $* \in Y$, there exists a cocompact subspace $Z \supseteq Y$ such that the maps $\pi_k(Y,*) \to \pi_k(Z,*)$ induced by inclusion have trivial image for $k \leq n-1$.*

Theorem 3.10 can be used to show that a group is not of type F_n if this is visible in the topology of X. On the other hand, if the stabilizers have bad finiteness properties, we cannot decide whether G has good finiteness properties or not: in that case we are looking at the wrong action.

3.3 Combinatorial Morse theory

In order to study connectivity properties of spaces and apply Brown's criterion we will be using combinatorial Morse theory as introduced by Bestvina and Brady [6]. Here we give the most basic version used in Section 3.4.

Let X be the realization of an abstract simplicial complex, regarded as a CW complex. A *Morse function* is a function $\rho\colon X^{(0)} \to \mathbb{N}$ with the property that $\rho(v) \neq \rho(w)$ if v is adjacent to w. For $n \in \mathbb{N}$ the sublevel set $X_{\rho<n}$ is defined to be the full subcomplex of X supported on vertices v with $\rho(v) < n$. The *descending link* $\mathrm{lk}^{\downarrow} v$ of a vertex v is the full subcomplex of $\mathrm{lk}\, v$ of those vertices w with $\rho(w) \leq \rho(v)$ and the *descending star* $\mathrm{st}^{\downarrow}$ is defined analogously. That ρ is a Morse function implies that the inequality $\rho(w) \leq \rho(v)$ is strict for the descending link and for the descending star is not strict only when $w = v$. In particular, the descending star is the cone over the descending link.

The goal of combinatorial Morse theory is to compare the connectivity properties of sublevel sets to each other and to those of X. The tool to do so is a basic lemma, called the Morse lemma:

Lemma 3.11 *Let ρ be a Morse function on X. Let $m \leq n \leq \infty$ and assume that for every vertex v with $m \leq \rho(v) < n$ the descending link of v is $(d-1)$–connected. Then the pair $(X_{\rho<n}, X_{\rho<m})$ is d–connected, that is, $\pi_k(X_{\rho<m} \to X_{\rho<n})$ is an isomorphism for $k < d$ and an epimorphism for $k = d$.*

Proof The basic observations are that

$$X_{\rho<m+1} = X_{\rho<m} \cup \bigcup_{\rho(v)=m} \mathrm{st}^{\downarrow} v,$$

that $\mathrm{st}^{\downarrow} v \cap \mathrm{st}^{\downarrow} v' \subseteq X_{\rho<m}$ for $\rho(v) = m = \rho(v')$, and that $\mathrm{st}^{\downarrow} v \cap X_{\rho<m} = \mathrm{lk}^{\downarrow} v$. As a consequence (using compactness of spheres) it suffices to study the extension $Y := X_{\rho<m} \cup_{\mathrm{lk}^{\downarrow} v} \mathrm{st}^{\downarrow} v$ for an individual vertex v with $\rho(v) = m$.

In this situation, $\pi_k(Y, X_{\rho<m}) \cong \pi_k(\mathrm{st}^{\downarrow} v, \mathrm{lk}^{\downarrow} v)$ for $k \leq d$. This can be seen by separately looking at π_1 and H_* (where excision holds) and applying Hurrewicz's theorem [26, Theorem 4.37]. The statement now follows from the long exact homotopy/homology sequence for the pair $(\mathrm{st}^{\downarrow} v, \mathrm{lk}^{\downarrow} v)$. □

3.4 Finiteness properties of fundamental groups of Ore categories

We take up the construction from Section 3.1. So $\mathcal{C}$ is again a right-Ore category, $\mathcal{S}$ is a left- or right-Garside family closed under factors, and $* \in \mathrm{Ob}(\mathcal{C})$ is a base object. More than requiring strong Noetherianity, we now need a height function $\rho\colon \mathrm{Ob}(\mathcal{C}) \to \mathbb{N}$.

We use these data and assumptions to provide a criterion to prove finiteness properties for the fundamental group.

We need to introduce one further space construction. It is another variant of the nerve construction. For $x \in \mathrm{Ob}(\mathcal{C})$ let $E(x)$ be the set of equivalence classes in $a \in \mathcal{E}(-,x) \setminus \mathcal{E}^\times(x,x)$ modulo the equivalence relation that $\overline{a} = \overline{a}'$ if there exists a $g \in \mathcal{C}^\times$ with $ga = a'$. We define a relation $\le$ on $E(x)$ by declaring $\overline{a} \le \overline{b}$ if there is an $f \in \mathcal{C}$ with $fa = b$. Note that if g and f as above exist, they lie in $\mathcal{E}$, so the description can be formulated purely in terms of $\mathcal{E}$. As in Lemma 3.1 one sees that $\le$ is a partial order on $E(x)$; however, it is usually not contractible.

Theorem 3.12 *Let $\mathcal{C}$ be a right-Ore category and let $* \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{S}$ be a locally finite left- or right-Garside family that is closed under factors. Let $\rho\colon \mathrm{Ob}(\mathcal{C}) \to \mathbb{N}$ be a height function such that $\{x \in \mathrm{Ob}(\mathcal{C}) \mid \rho(x) \le n\}$ is finite for every $n \in \mathbb{N}$. Assume*

(STAB) *$\mathcal{C}^\times(x,x)$ is of type F_n for all x,*

(LK) *there exists an $N \in \mathbb{N}$ such that $|E(x)|$ is $(n-1)$–connected for all x with $\rho(x) \ge N$.*

(If ρ is unbounded on the component of $*$ then it suffices if (STAB) holds for every x with $\rho(x)$ beyond a fixed bound.)

Then $\pi_1(\mathcal{C},*)$ is of type F_n.

Remark 3.13 Recall that $\mathcal{C}$ can be replaced by the component of $*$ in $\mathcal{C}$, so all assumptions need to be made only for that component.

Proof We take X to be the complex constructed in Section 3. Assume first that (STAB) holds for all $x \in \mathrm{Ob}(\mathcal{C})$.

For a vertex $\overline{a} \in X$ with $a \in \mathcal{O}re(\mathcal{C})(*,x)$ we define $\rho(\overline{a}) = \rho(x)$. This is a $\pi_1(\mathcal{C},*)$–invariant Morse function, which we think of as height. For $n \in \mathbb{N}$ we consider the subcomplex $X_{\rho<n}$ supported on vertices of height $< n$.

We want to see that every $X_{\rho<n}$ is $\pi_1(\mathcal{C},*)$–cocompact. To do so we note that $\pi_1(\mathcal{C},*)$ acts transitively on vertices $\overline{a}$ with $a \in \mathcal{O}re(\mathcal{C})(*,x)$: indeed, if $\overline{b}$ is another such then $ba^{-1} \in \pi_1(\mathcal{C},*)$ takes $\overline{a}$ to $\overline{b}$. It follows from the assumption on ρ that there are only finitely many vertices $\overline{a}$ with $\rho(\overline{a}) < n$ up to the $\pi_1(\mathcal{C},*)$–action. Cocompactness now follows from Observation 3.3.

Stabilizers are of type F_n by Lemma 3.4 because finiteness properties are inherited by finite-index subgroups.

Let N be large enough that all the $x \in \mathrm{Ob}(\mathcal{C})$ for which the nerve of $|E(x)|$ is not $(n-1)$–connected have $\rho(x) < N$. We have just seen that $\pi_1(\mathcal{C},*)$ acts on $X_{\rho<N}$ cocompactly with stabilizers of type F_n, so once we show that $X_{\rho<N}$ is $(n-1)$–connected,

we are done by Theorem 3.9. We want to apply the Morse lemma (Lemma 3.11), so let us look at the descending link of a vertex $\overline{b}$ of X, where $b \in \mathcal{C}(*, x)$. The vertices in the descending link are the $\overline{a}$ that are comparable with $\overline{b}$ and have $\rho(\overline{a}) < \rho(\overline{b})$. The condition on the height shows that a cannot be a right-multiple of b but has to be a left-factor. Thus, $a^{-1}b \in \mathcal{E}(-, x)$ and the descending link of $\overline{b}$ is the realization of $\{\overline{a} \mid a \prec b\}$. We see that the map $\mathcal{E}(-,x) \smallsetminus \mathcal{E}(x,x) \to \{\overline{a} \mid a \prec b\}$ that takes f to $\overline{af^{-1}}$ is an order-reversing surjection. The definition of $E(x)$ is made so that the induced map $E(x) \to \{\overline{a} \mid a \prec b\}$ is well defined and an order-reversing bijection. Since $|E(x)|$ is $(n-1)$–connected by assumption, this completes the proof in the case that (STAB) holds for all x.

If (STAB) only holds for x with $\rho(x) \geq M$, let $*'$ be in the component of $*$ satisfying $\rho(*') > M$. Since $\mathcal{C}$ is Ore, one sees that

$$\pi_1(\mathcal{C}, *) = \pi_1(\mathcal{C}, *') = \pi_1(\mathcal{C}_{\rho \geq M}, x_0),$$

where $\mathcal{C}_{\rho \geq M}$ is obtained from $\mathcal{C}$ by removing objects y with $\rho(y) < M$. Moreover, local finiteness of $\mathcal{S}$ implies that the complexes $E(y)$ for $\mathcal{C}$ and for $\mathcal{C}_{\rho \geq r}$ are the same for y in the component of $*'$ once $\rho(y)$ is large enough. One can therefore consider $\mathcal{C}_{\rho \geq M}$ instead of $\mathcal{C}$, with the effect that the groups $\mathcal{C}^{\times}(x,x)$ only need to be of type F_n when $\rho(x) \geq M$. □

Corollary 3.14 *Let $\mathcal{C}$, $\mathcal{S}$, ρ and $*$ be as in the theorem. If $\mathcal{C}^{\times}(x,x)$ is of type F_∞ for every x and the connectivity of $|E(x)|$ tends to infinity for $\rho(x) \to \infty$, then $\pi_1(\mathcal{C}, *)$ is of type F_∞.*

The construction of X uses two important ideas. One is the passage from $|P|$ to X, which is due to Stein; see [36, Theorem 1.5]. The other is to take P to consist of $\mathcal{C}^{\times}$–equivalence classes and goes back to [18]. Apart from these ideas the main difficulty in proving that $\pi_1(\mathcal{C}, *)$ is of type F_n lies in establishing the connectivity properties of the complexes $|E(x)|$. This problem depends individually on the concrete setup and we will see various examples later.

3.5 Example: F is of type F_∞

As a first illustration of the results in this section we reprove a result due to Brown and Geoghegan [17]:

Proposition 3.15 *Thompson's group F is of type F_∞.*

We have seen in Proposition 2.1 that $\mathcal{F}$ is right-Ore and admits a height function and by Corollary 2.6 it has a locally finite left-Garside family that is closed under factors. Moreover, $\mathcal{F}^{\times}(x,x) = \{1_x\}$ for every x, so (STAB) is satisfied as well. It only remains to verify (LK). Although things are not always as easy, we remark that this is the typical situation: property (LK) is where one actually needs to show something.

To understand the complexes $|E(n)|$ we first need to unravel the definition. Recall that a *matching* of a graph Γ is a set of edges $M \subseteq E(\Gamma)$ that are pairwise disjoint. Matchings are ordered by containment and we denote the poset of matchings by $\mathcal{M}(\Gamma)$. In fact, since every subset of a matching is again a matching, $\mathcal{M}(\Gamma)$ is (the face poset of) a simplicial complex, the *matching complex*. We denote by L_n the *linear graph* on n vertices $\{1,\dots,n\}$, so its edges are $\{i,i+1\}$ for $1 \le i < n$.

Lemma 3.16 *The poset $E_{\mathcal{F}}(n)$ is isomorphic to $\mathcal{M}(L_n)$.*

Proof Let $f \in \mathcal{E}_{\mathcal{F}}(-,n)$, so f is an element of $E_{\mathcal{F}}(n)$. We identify the roots of f with the vertices of the linear graph L_n on the vertices $\{1,\dots,n\}$. Every caret of f connects two of these roots and thus corresponds to an edge of L_n. All these edges are disjoint, so the resulting subgraph M_f of L_n is a matching. It is clear that, conversely, every matching of L_n arises in a unique way from an elementary forest.

If $h \le f$ then h is a left-multiple of f, that is, f can be obtained from h by adding carets to some roots of h that do not have carets yet. On the level of graphs this means that M_f is obtained from M_h by adding edges so that $M_h \le M_f$ in the poset of matchings. □

Remark 3.17 In particular, $E_{\mathcal{F}}(n)$ is (the face poset of) a simplicial complex. The realization as a poset is the barycentric subdivision of the realization as a simplicial complex, and in particular both are homeomorphic. So there is no harm in working with the coarser cell structure where elements of $E_{\mathcal{F}}(n)$ are simplices rather than vertices. This fact applies in most of our cases.

Matching complexes of various graphs have been studied intensely and their connectivity properties can be verified in various ways [9]. In fact, for linear and cyclic graphs the precise homotopy type is known [30, Proposition 11.16].

Rather than using the known optimal connectivity bounds we use the opportunity to introduce a criterion due to Belk and Forrest [3, Theorem 4.9] that is particularly well suited to verifying that the connectivity of the spaces $E(x)$ tends to infinity in easier cases. We need to introduce some notation.

An abstract simplicial complex X is *flag* if every set of pairwise adjacent vertices forms a simplex. A simplex σ in a simplicial flag complex is called a *k–ground* for $k \in \mathbb{N}$ if every vertex of X is connected to all but at most k vertices of σ. The complex is said to be *(n,k)–grounded* if there is an n–simplex that is a k–ground.

Theorem 3.18 [3, Theorem 4.9] *For $m,k \in \mathbb{N}$ every (mk,k)–grounded flag complex is $(m-1)$–connected.*

The reference requires $m,k \geq 1$ but it is clear that every $(0,k)$–grounded flag complex is nonempty, and every $(0,0)$–grounded flag complex is a cone and therefore contractible.

Using Theorem 3.18 we verify:

Lemma 3.19 *For every $n \in \mathbb{N}$ let Γ_n be a subgraph of K_n containing L_n. The connectivity of $\mathcal{M}(\Gamma_n)$ goes to infinity as n goes to infinity.*

Proof Consider the matchings of L_n that use only the edges $\{2i-1, 2i\}$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. They form an $\bigl(\lfloor \frac{n}{2} \rfloor - 1\bigr)$–simplex σ in $\mathcal{M}(\Gamma_n)$. If $v = \{j,k\}$ is any edge of Γ_n, so a vertex of $\mathcal{M}(\Gamma_n)$, then there are at most 2 vertices of σ that v is not connected to: one is $\{j-1, j\}$ or $\{j, j+1\}$, the other is $\{k-1, k\}$ or $\{k, k+1\}$. This shows that $\mathcal{M}(\Gamma_n)$ is $\bigl(\lfloor \frac{n}{2} \rfloor - 1, 2\bigr)$–grounded, so by Theorem 3.18 it is $\bigl(\lfloor \frac{n}{4} \rfloor - 1\bigr)$–connected. □

Proof of Proposition 3.15 We want to apply Corollary 3.14. The only thing left to check is condition (LK). This follows from Lemmas 3.16 and 3.19. □

4 The indirect product of two categories

The construction introduced in this section will help us to produce more interesting examples. It is usually called the Zappa–Szép product in the literature of groups and monoids; see [14]. The Zappa–Szép product naturally generalizes the semidirect product in the same way as the semidirect product generalizes the direct product. We think that such a basic construction should have a simpler name and therefore call it the *indirect product*.

For motivation, let M be a monoid (or group) whose multiplication we denote by $\circ$ and suppose that M decomposes uniquely as $M = A \circ B$. By this we mean that A and B are submonoids of M such that every element $m \in M$ can be written in a unique way as $m = a' \circ b'$ with $a' \in A$ and $b' \in B$. In particular, if $b \in B$ and $a \in A$, the product $m = b \circ a$ can be rewritten as $b \circ a = a' \circ b'$. This allows us to formally define maps $B \times A \to A$, $(b,a) \mapsto b \cdot a := a'$, and $B \times A \to B$, $(b,a) \mapsto b^a := b'$, so that

$$b \circ a = (b \cdot a) \circ b^a.$$

These maps turn out to be actions of monoids on sets. If both actions are trivial then M is a direct product, if one of the actions is trivial then M is a semidirect product, and in general it is an indirect product.

We therefore start by introducing the appropriate notion of actions of categories.

4.1 Actions

Let $\mathcal{C}$ be a category and let $(X_m)_{m \in \mathrm{Ob}(\mathcal{C})}$ be a family of sets, one for each object of $\mathcal{C}$. We say that a left action of $\mathcal{C}$ on $(X_m)_m$ is a family of maps

$$\mathcal{C}(n,m) \times X_m \to X_n, \quad (f,s) \mapsto f \cdot s,$$

satisfying $1_m \cdot s = s$ for all $m \in \mathrm{Ob}(\mathcal{C})$ and $s \in X_m$ and $fg \cdot s = f \cdot (g \cdot s)$ whenever fg is defined. A right action is defined analogously. An action is said to be *injective* if $f \cdot x = f \cdot y$ implies $x = y$. Note that actions of groupoids are always injective.

In our examples the family $(X_m)_m$ itself will consist of morphisms of a category with the same objects as $\mathcal{C}$. We have to bear in mind, however, that the action is on these as sets and does not preserve products.

4.2 The indirect product

Let $\mathcal{C}$ be a category and let $\mathcal{F}$ and $\mathcal{G}$ be subcategories. We say that $\mathcal{C}$ is an *internal indirect product* $\mathcal{F} \bowtie \mathcal{G}$ if every $h \in \mathcal{C}$ can be written in a unique way as $h = fg$ with $f \in \mathcal{F}$ and $g \in \mathcal{G}$. Note that this means in particular that $\mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathcal{F}) = \mathrm{Ob}(\mathcal{G})$. Given elements $f \in \mathcal{F}(x,-)$ and $g \in \mathcal{G}(-,x)$ there exist then unique elements $f' \in \mathcal{F}$ and $g' \in \mathcal{G}$ such that $gf = f'g'$; see Figure 4 (left). In this situation we define $g \cdot f$ to be f' and g^f to be g'.

The following properties are readily verified — see Figure 4 (center and right) — the last four hold whenever one of the sides is defined:

(IP1) $1_x \cdot f = f$ for $f \in \mathcal{F}(x,-)$.

(IP2) $g^{1_y} = g$ for $g \in \mathcal{G}(-,y)$.

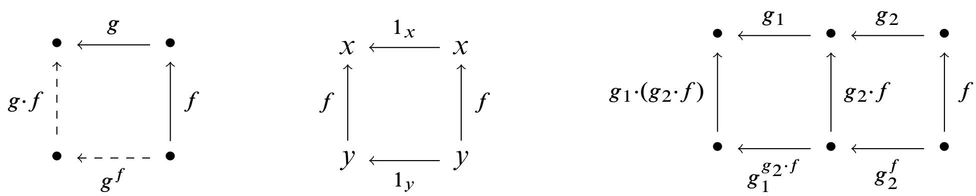


Figure 4: The indirect product

(IP3) $(g_1 g_2) \cdot f = g_1 \cdot (g_2 \cdot f)$.

(IP4) $g^{f_1 f_2} = (g^{f_1})^{f_2}$.

(IP5) $1_x^f = 1_y$ for $f \in \mathcal{F}(x, y)$.

(IP6) $g \cdot 1_y = 1_z$ for $g \in \mathcal{G}(z, x)$.

(IP7) $(g_1 g_2)^f = g_1^{(g_2 \cdot f)} g_2^f$.

(IP8) $g \cdot (f_1 f_2) = (g \cdot f_1)(g_2^{f_1} \cdot f_2)$.

The first four relations say that the map $(g, f) \mapsto g \cdot f$ is an left action of $\mathcal{G}$ on the sets $(\mathcal{F}(x, -))_x$ and that $(g, f) \mapsto g^f$ is a right action of $\mathcal{F}$ on the sets $(\mathcal{G}(-, y))_y$. The next two relations say that identity elements are taken to identity elements, while the last two are cocycle conditions. We call actions satisfying (IP1)–(IP8) *indirect product actions*.

Now assume that conversely categories $\mathcal{F}$ and $\mathcal{G}$ with $\mathrm{Ob}(\mathcal{F}) = \mathrm{Ob}(\mathcal{G})$ are given together with indirect product actions of $\mathcal{F}$ and $\mathcal{G}$ on each other. Then the *external indirect product* $\mathcal{C} = \mathcal{F} \bowtie \mathcal{G}$ is defined to have objects $\mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathcal{F}) = \mathrm{Ob}(\mathcal{G})$ and morphisms

$$\mathcal{C} = \bigcup_{x \in \mathrm{Ob}(\mathcal{C})} \{(f, g) \mid f \in \mathcal{F}(-, x), g \in \mathcal{G}(x, -)\}.$$

Composition is defined by

$$(f_1, g_1)(f_2, g_2) = (f_1(g_1 \cdot f_2), g_1^{f_2} g_2). \tag{4-1}$$

Lemma 4.1 *The external indirect product $\mathcal{F} \bowtie \mathcal{G}$ is well defined. It is naturally isomorphic to the internal indirect product of the copies of $\mathcal{F}$ and $\mathcal{G}$ inside $\mathcal{F} \bowtie \mathcal{G}$.*

Proof That the identity morphisms $(1_x, 1_x)$ behave as they should is easily seen using relations (IP1), (IP2), (IP5) and (IP6). To check associativity we verify the four

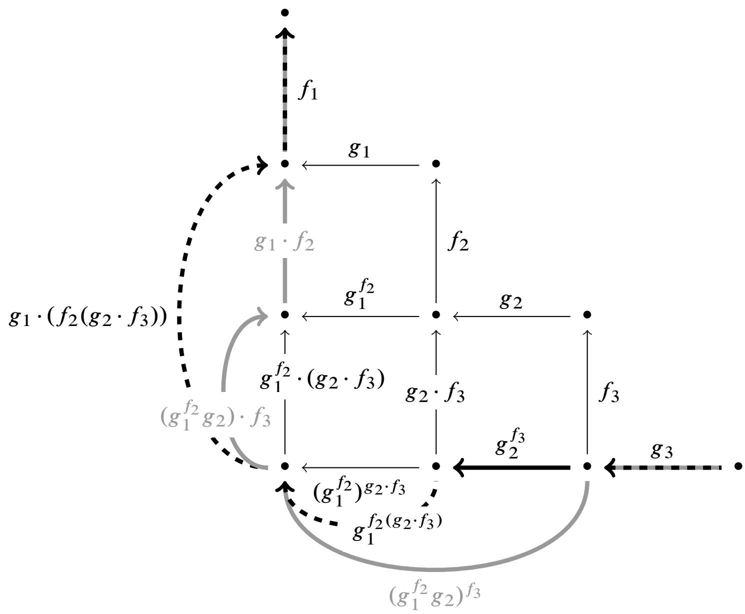


Figure 5: Associativity in $\mathcal{F} \bowtie \mathcal{G}$. The thick dashed and gray paths are the components of $(f_1, g_1)((f_2, g_2)(f_3, g_3))$ and $((f_1, g_1)(f_2, g_2))(f_3, g_3)$, respectively.

equations

$$g_1^{f_2(g_2 \cdot f_3)} \overset{\text{(IP4)}}{=} (g_1^{f_2})^{g_1 \cdot f_3}, \tag{4-2}$$

$$(g_1^{f_2} g_2) \cdot f_3 \overset{\text{(IP3)}}{=} g_1^{f_2} \cdot (g_2 \cdot f_3), \tag{4-3}$$

$$g_1^{f_2(g_2 \cdot f_3)} g_2^{f_3} \overset{\text{(4-2)}}{=} (g_1^{f_2})^{g_1 \cdot f_3} g_2^{f_3} \overset{\text{(IP7)}}{=} (g_1^{f_2} g_2)^{f_3}, \tag{4-4}$$

$$(g_1 \cdot f_2)((g_1^{f_2} g_2) \cdot f_3) \overset{\text{(4-3)}}{=} (g_1 \cdot f_2)(g_1^{f_2} \cdot (g_2 \cdot f_3)) \overset{\text{(IP8)}}{=} g_1 \cdot (f_2(g_2 \cdot f_3)); \tag{4-5}$$

see Figure 5.

The categories $\mathcal{F}$ and $\mathcal{G}$ naturally embed into the external indirect product $\mathcal{F} \bowtie \mathcal{G}$ as $f \mapsto (f, 1_y)$ for $f \in \mathcal{F}(-, y)$ and $g \mapsto (1_x, g)$ for $g \in \mathcal{G}(x, -)$. Any morphism of $\mathcal{F} \bowtie \mathcal{G}$ decomposes as $(f, g) = (f, 1_y)(1_y, g)$ and it is clear from (4-1) that the respective actions on each other are the ones used to define $\mathcal{F} \bowtie \mathcal{G}$. □

If the action of $\mathcal{G}$ on $\mathcal{F}$ is trivial then the indirect product is a *semidirect product* $\mathcal{F} \ltimes \mathcal{G}$. Similarly, if the action of $\mathcal{F}$ on $\mathcal{G}$ is trivial then it is a semidirect product $\mathcal{F} \rtimes \mathcal{G}$. Finally, if both actions are trivial then the indirect product is in fact a *direct product* $\mathcal{F} \times \mathcal{G}$.

We close the section by collecting facts that ensure that an indirect product is Ore.

Lemma 4.2 *If $\mathcal{F}$ and $\mathcal{G}$ are right-cancellative and the action of $\mathcal{F}$ on $\mathcal{G}$ is injective then $\mathcal{F} \bowtie \mathcal{G}$ is right-cancellative. Symmetrically, if $\mathcal{F}$ and $\mathcal{G}$ are left-cancellative and the action of $\mathcal{G}$ on $\mathcal{F}$ is injective, then $\mathcal{F} \bowtie \mathcal{G}$ is left-cancellative.*

Proof If $f_1 g_1 f g = f_2 g_2 f g$ then $f_1(g_1 \cdot f) = f_2(g_2 \cdot f)$ and $g_1^f g = g_2^f g$. Since $\mathcal{G}$ is right-cancellative the latter equation shows that $g_1^f = g_2^f$ and injectivity of the action then implies $g_1 = g_2$. Putting this in the former equation and using right-cancellativity of $\mathcal{F}$ gives $f_1 = f_2$. □

Observation 4.3 *Let $\mathcal{F}$ have common right-multiples and let $\mathcal{G}$ be a groupoid. Then $\mathcal{F} \bowtie \mathcal{G}$ has common right-multiples.*

Proof Let $fg \in \mathcal{F} \bowtie \mathcal{G}$ with $f \in \mathcal{F}$ and $g \in \mathcal{G}$. Since $\mathcal{G}$ is a groupoid, f is both a left-factor and a right-multiple of fg. It follows that common right-multiples exist in $\mathcal{F} \bowtie \mathcal{G}$ because they exist in $\mathcal{F}$. □

Observation 4.4 *Let $\mathcal{F}$ have no nontrivial invertible morphisms and let $\mathcal{G}$ be a groupoid. Then $(\mathcal{F} \bowtie \mathcal{G})^\times = \mathcal{G}$.*

Proposition 4.5 *Let $\mathcal{C} = \mathcal{F} \bowtie \mathcal{G}$, where $\mathcal{F}$ has no nontrivial invertibles and $\mathcal{G}$ is a discrete groupoid.*

(i) *If $\mathcal{F}$ is right-Ore and the action of $\mathcal{F}$ on $\mathcal{G}$ is injective, then $\mathcal{C}$ is right-Ore.*

(ii) *If $\mathcal{F}$ is strongly Noetherian then so is $\mathcal{C}$.*

(iii) *If ρ is a height function on $\mathcal{F}$ then it is a height function on $\mathcal{C}$.*

(iv) *If $\mathcal{S}$ is a left-Garside family in $\mathcal{F}$ then it is a left-Garside family in $\mathcal{C}$.*

(v) *If $\mathcal{S}$ is a right-Garside family in $\mathcal{F}$ then $\mathcal{S}\mathcal{G}$ is a right-Garside family in $\mathcal{C}$.*

Proof Property (i) follows from Lemma 4.2 and Observation 4.3. Properties (ii) and (iv) follow from the fact that for $f \in \mathcal{F}$ and $g \in \mathcal{G}$ the morphisms f and fg are right-multiples by invertibles of each other. Property (iii) follows from $\mathcal{G}$ being discrete (ie every morphism being an endomorphism). Toward (v), it is clear that every right-factor of $\mathcal{S}\mathcal{G}$ is contained in $\mathcal{S}\mathcal{G}$. Moreover, if t is an $\mathcal{S}$–tail for f then tg is a $\mathcal{S}$–tail for fg. □

5 Examples: categories constructed by indirect products

In this section we show how the indirect product can be used to construct new groups. The basic examples are Thompson's groups T and V as well as the braided Thompson groups, which all arise as fundamental groups of categories of the form $\mathcal{F} \bowtie \mathcal{G}$ where $\mathcal{G}$ is an appropriate groupoid. More generally, the groups studied in joint work with Zaremsky [43] are essentially by definition groups that can be obtained in this form. Later we also describe other groups obtained via indirect products.

We will sometimes draw pictures to motivate our definition. In these pictures the up direction always corresponds to left in our notation and down corresponds to right. This is especially relevant for group elements. For example, a permutation $X \leftarrow X$, $g(x) \leftarrow\!\shortmid x$, will be depicted by connecting the point x at the bottom to the point $g(x)$ at the top.

5.1 Thompson's groups T and V

In this section we introduce Thompson's groups T and V as fundamental groups of categories $\mathcal{T}$ and $\mathcal{V}$. The categories will be obtained from $\mathcal{F}$ as indirect products with groupoids and we start by introducing these.

We define $\mathcal{G}_T$ and $\mathcal{G}_V$ to be groupoids whose objects are positive natural numbers with $\mathcal{G}_T(m,n) = \varnothing$ for $m \neq n$. We put $\mathcal{G}_T(n,n) = \mathbb{Z}/n\mathbb{Z}$ and $\mathcal{G}_V(n,n) = \mathrm{SYM}_n$. We want to define $\mathcal{T} = \mathcal{F} \bowtie \mathcal{G}_T$ and $\mathcal{V} = \mathcal{F} \bowtie \mathcal{G}_V$ and have to specify the actions that define these indirect products. That is, given a forest $f \in \mathcal{F}(m,n)$ and a group element $g \in \mathcal{G}(m,m)$ we need to specify how the product gf should be written as $(g \cdot f)g^f$ with $g \cdot f \in \mathcal{F}(m,n)$ and $g^f \in \mathcal{G}(n,n)$ (for $\mathcal{G}$ one of $\mathcal{G}_T$ and $\mathcal{G}_V$).

Since $\mathcal{G}_T$ is contained in $\mathcal{G}_V$, it would suffice to only define the actions for $\mathcal{G}_V$, but we look at the simpler case of $\mathcal{G}_T$ first.

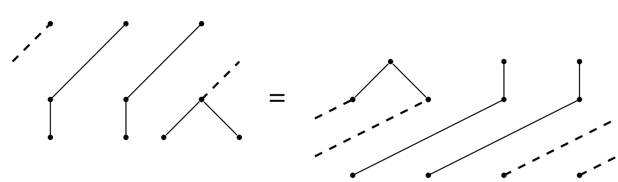


Figure 6: Defining $\mathcal{F} \bowtie \mathcal{G}_T$. The picture shows how to write gf as $(g \cdot f)g^f$ in the case where f is the caret $\lambda_3^3 \in \mathcal{F}(3,4)$ and $g = 1 + \mathbb{Z}/3\mathbb{Z} \in \mathcal{G}_T(3,3)$. The dashed strand gets doubled under the action of f. As a result, $g \cdot f = \lambda_1^3 \in \mathcal{F}(3,4)$ and $g^f = 2 + \mathbb{Z}/4\mathbb{Z} \in \mathcal{G}_T(4,4)$.

We need to rewrite a cyclic permutation followed by a tree as a tree followed by a cyclic permutation. This is illustrated in Figure 6. For $f \in \mathcal{F}(m,n)$ and $g = \ell + \mathbb{Z}/m\mathbb{Z} \in \mathcal{G}_T(m,m)$ the forest $g \cdot f$ is just f with the trees rotated by ℓ to the right. The definition of g^f is more subtle: looking at the figure we see that we have to define it to be $k + \mathbb{Z}/n\mathbb{Z}$, where k is the number of leaves of the first ℓ trees of $g \cdot f$, or equivalently, to be the number leaves of the last ℓ trees of f. Note that this number does not depend on the chosen representative ℓ: if we replace ℓ by $\ell + m$, instead of k we get $k+n$, because we counted every leaf once more. If k_ℓ denotes the number of leaves of the last ℓ trees of f, the sequence $(k_\ell)_{0 \le \ell < m}$ is strictly increasing. This shows:

Observation 5.1 *The action of $\mathcal{F}$ on $\mathcal{G}_T$ is injective.*

Lemma 5.2 *The actions of $\mathcal{F}$ and $\mathcal{G}_T$ on each other are indirect product actions.*

Proof Conditions (IP1), (IP2), (IP3), (IP4), (IP5) and (IP6) are clear.

The condition (IP7) in our setting follows from the fact that the last $k + \ell$ trees of f are the last ℓ trees of f plus the last k trees of $(\ell + m\mathbb{Z}) \cdot f$. Condition (IP8) can be verified by drawing a picture. □

The lemma allows us to define $\mathcal{T} = \mathcal{F} \bowtie \mathcal{G}_T$. Combining Observation 5.1 with Proposition 2.1 and Corollary 2.6 and applying Proposition 4.5 we find:

Corollary 5.3 *The category $\mathcal{T}$ is right-Ore and admits a height function and a left-Garside family $\mathcal{S}$ that is closed under factors such that $\mathcal{S}(x,-)/\mathcal{S}^\times$ is finite for every x.*

The fundamental group $\pi_1(\mathcal{T},1)$ is *Thompson's group T*.

Now we want to define the actions of $\mathcal{F}$ and $\mathcal{G}_V$ on each other. So let $f \in \mathcal{F}(m,n)$ and let $g \in \mathcal{G}_V(m,m)$. The action of $\mathcal{G}_V$ on $\mathcal{F}$ is again as expected: the forest $f' = (g \cdot f)$ is given by the relationship that the $g(j)^{\text{th}}$ tree of f' is the j^{th} tree of f. The permutation $g' = g^f \in \mathcal{G}_V(n,n)$ has the following description. Identify $\{1,\dots,n\}$ with the leaves of f and with the leaves of $(g \cdot f)$. If i is the k^{th} leaf of the j^{th} tree of f then $g'(i)$ is the k^{th} leaf of the $g(j)^{\text{th}}$ tree of $g \cdot f$; see Figure 6.

At this point it becomes clear that working with the actions as described above is virtually impossible. To obtain a more explicit algebraic description, we make use of the presentation of $\mathcal{F}$. Property (IP4) tells us that we know how any element of $\mathcal{F}$ acts as soon as we know how the generators act and property (IP8) tells us that we know how $\mathcal{G}_V$ acts on any element once we know how it acts on the generators of $\mathcal{F}$. It therefore suffices to specify both actions for generators of $\mathcal{F}$. Checking well-definedness then means to check various conditions coming from the relations in $\mathcal{F}$.

So now we consider $g \in \mathcal{G}_V(m,m)$ and $\lambda_i^m \in \mathcal{F}(m,m+1)$ and define the actions on each other. We start again with the easy case,

$$g \cdot \lambda_i = \lambda_{g(i)}. \tag{5-1}$$

Working out g^{λ_i} we have to distinguish four cases depending on the position of a point relative to i and relative to $g(i)$:

$$g^{\lambda_i}(j) = \begin{cases} g(j) & \text{if } j \le i,\ g(j) \le g(i), \\ g(j-1) & \text{if } j > i,\ g(j-1) \le g(i), \\ g(j)+1 & \text{if } j \le i,\ g(j) > g(i), \\ g(j-1)+1 & \text{if } j > i,\ g(j-1) > g(i). \end{cases} \tag{5-2}$$

Since $i = j$ if and only if $g(i) = g(j)$, the inequalities in the second and third case can be taken to be strict.

Lemma 5.4 *The formulas* (5-1) *and* (5-2) *define well-defined indirect product actions of $\mathcal{F}$ and $\mathcal{G}_V$ on each other.*

Proof The conditions that involve only the action of $\mathcal{G}_V$, namely (IP1), (IP3) and (IP6), are clear. Condition (IP2) is defined to hold. Verifying conditions (IP5) and (IP7) on the λ_i is straightforward, although in the second case tedious.

Conditions (IP4) and (IP8) should also be defined to hold, but in order for this to be well defined, we need to check them on relations. That is, we need to verify that

$$(g^{\lambda_i})^{\lambda_j} = g^{\lambda_i \lambda_j} = g^{\lambda_j \lambda_{i+1}} = (g^{\lambda_j})^{\lambda_{i+1}}$$

and

$$(g \cdot \lambda_i)(g_2^{\lambda_i} \cdot \lambda_j) = g \cdot (\lambda_i \lambda_j) = g \cdot (\lambda_j \lambda_{i+1}) = (g \cdot \lambda_j)(g_2^{\lambda_j} \cdot \lambda_{i+1})$$

for $j < i$. These are again not difficult but tedious and we skip them here. See [43, Example 2.9] for a detailed verification. □

Thus, we can define $\mathcal{V} = \mathcal{F} \bowtie \mathcal{G}_V$.

Lemma 5.5 *The action of $\mathcal{F}$ on $\mathcal{G}_V$ defined by* (5-2) *is injective.*

Proof Since by definition $g^{\lambda_{i_1}\cdots\lambda_{i_n}} = (\cdots(g^{\lambda_{i_1}})\cdots)^{\lambda_{i_n}}$, we only need to check that the map $g \mapsto g^{\lambda_i}$ defined in (5-1) is injective. But g can be recovered from g^{λ_i} as follows. Let $\tau_i, \pi_i\colon \mathbb{N} \to \mathbb{N}$ be given by

$$\tau_i(j) := \begin{cases} j & \text{if } j \le i, \\ j+1 & \text{if } j > i, \end{cases} \qquad \pi_i(j) := \begin{cases} j & \text{if } j \le i, \\ j-1 & \text{if } j > i. \end{cases}$$

Then $g(j) = \pi_i(g^{\lambda_i}(\tau_i(j)))$. □

Proposition 2.1, Corollary 2.6 and Proposition 4.5 now imply:

Corollary 5.6 *The category $\mathcal{V}$ is right-Ore and admits a height function and a left-Garside family $\mathcal{S}$ that is closed under factors such that $\mathcal{S}(x,-)/\mathcal{S}^\times$ is finite for every x.*

The fundamental group $\pi_1(\mathcal{V}, 1)$ is *Thompson's group V.*

5.2 The braided Thompson groups

The group BV, called *braided V*, was introduced independently by Brin [15] and Dehornoy [21]. We describe it using our framework, which is similar to Brin's approach.

To define the categories underlying the braided Thompson groups, we define the groupoid $\mathcal{G}_{BV}$ to have as objects natural numbers, and to have morphisms $\mathcal{G}_{BV}(m,n) = \varnothing$ for $m \neq n$, and $\mathcal{G}_{BV}(n,n) = \mathrm{BRAID}_n$. Note that the morphisms $\pi\colon \mathrm{BRAID}_n \to \mathrm{SYM}_n$ define a morphism $\mathcal{G}_{BV} \to \mathcal{G}_V$, which we denote by π as well. We want to define a indirect product $\mathcal{F} \bowtie \mathcal{G}_{BV}$ and need to define actions of $\mathcal{F}$ and $\mathcal{G}_{BV}$ on each other. Our guiding picture is Figure 7.

We define the action of $\mathcal{G}_{BV}$ on $\mathcal{F}$ simply as the action of $\mathcal{G}_V$ composed with π. In particular, $\sigma_i \cdot \lambda_i = \lambda_{i+1}$, $\sigma_i \cdot \lambda_{i+1} = \lambda_i$ and $\sigma_i \cdot \lambda_j = \lambda_j$ for $j \neq i, i+1$. The action

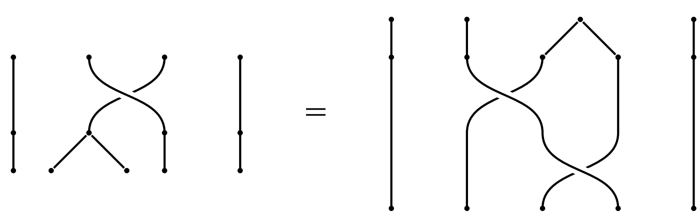


Figure 7: Defining $\mathcal{F} \bowtie \mathcal{G}_{BV}$

of $\mathcal{F}$ on $\mathcal{G}_{BV}$ we only define for generators acting on generators by

$$\sigma_i^{\lambda_j} := \begin{cases} \sigma_{i+1} & \text{if } j < i, \\ \sigma_i\sigma_{i+1} & \text{if } j = i, \\ \sigma_{i+1}\sigma_i & \text{if } j = i+1. \\ \sigma_i & j > i+1. \end{cases}$$

Lemma 5.7 *The formulas above define well-defined indirect product actions of $\mathcal{F}$ and $\mathcal{G}_{BV}$ on each other.*

In the proof we will use the fact that there is a set-theoretic splitting $\iota\colon \mathrm{SYM}_n \to \mathrm{BRAID}_n$ that takes a reduced word $w(s_1,\dots,s_{n-1})$ to the braid $w(\sigma_1,\dots,\sigma_{n-1})$. This map is not multiplicative but if β is a positive word (meaning involving no inverses) of length at most 3 in the σ_i then $\iota\pi(\beta) = \beta$.

Proof As in the proof of Lemma 5.4 most conditions hold by definition but we need to check well-definedness on relations. Namely,

$$(\sigma_i\sigma_{i+1}\sigma_i)\cdot\lambda_k = \sigma_i\cdot(\sigma_{i+1}\cdot(\sigma_i\cdot\lambda_k)) = \sigma_{i+1}\cdot(\sigma_i\cdot(\sigma_{i+1}\cdot\lambda_k)) = (\sigma_{i+1}\sigma_i\sigma_{i+1})\cdot\lambda_k, \tag{5-3}$$

$$(\sigma_i\sigma_{i+1}\sigma_i)^{\lambda_k} = \sigma_i^{(\sigma_{i+1}\sigma_i)\cdot\lambda_k}\sigma_{i+1}^{\sigma_i\cdot\lambda_k}\sigma_i^{\lambda_k} = \sigma_{i+1}^{(\sigma_i\sigma_{i+1})\cdot\lambda_k}\sigma_i^{\sigma_{i+1}\cdot\lambda_k}\sigma_{i+1}^{\lambda_k} = (\sigma_{i+1}\sigma_i\sigma_{i+1})^{\lambda_k}, \tag{5-4}$$

$$(\sigma_i\sigma_j)\cdot\lambda_k = \sigma_i\cdot(\sigma_j\cdot\lambda_k) = \sigma_i\cdot(\sigma_j\cdot\lambda_k) = (\sigma_j\sigma_i)\cdot\lambda_k, \tag{5-5}$$

$$(\sigma_i\sigma_j)^{\lambda_k} = \sigma_i^{\sigma_j\cdot\lambda_k}\sigma_j^{\lambda_k} = \sigma_j^{\sigma_i\cdot\lambda_k}\sigma_i^{\lambda_k} = (\sigma_j\sigma_i)^{\lambda_k}, \tag{5-6}$$

$$\sigma_i\cdot(\lambda_\ell\lambda_k) = (\sigma_i\cdot\lambda_\ell)(\sigma_i^{\lambda_\ell}\cdot\lambda_k) = (\sigma_i\cdot\lambda_k)(\sigma_i^{\lambda_k}\cdot\lambda_{\ell+1}) = \sigma_i\cdot(\lambda_k\lambda_{\ell+1}), \tag{5-7}$$

$$\sigma_i^{\lambda_\ell\lambda_k} = (\sigma_i^{\lambda_\ell})^{\lambda_k} = (\sigma_i^{\lambda_k})^{\lambda_{\ell+1}} = \sigma_i^{\lambda_k\lambda_{\ell+1}} \tag{5-8}$$

for $i-j \geq 2$, $\ell > k$.

Relations (5-3) and (5-5) follow from Lemma 5.4. For the remaining relations note that $\pi(\beta^{\lambda_k}) = \pi(\beta)^{\lambda_k}$. Now (5-7) follows from Lemma 5.4 as well because

$$\pi(\sigma_i^{\lambda_\ell})\cdot\lambda_k = \sigma_i^{\lambda_\ell}\cdot\lambda_k \quad\text{and}\quad \pi(\sigma_i^{\lambda_k})\cdot\lambda_{\ell+1} = \sigma_i^{\lambda_k}\cdot\lambda_{\ell+1}. \tag{5-9}$$

Relation (5-8) follows from Lemma 5.4 by noting that both sides are positive words of length at most 3 and applying ι.

We verify (5-4) by distinguishing cases. The cases $k < i$ and $k > i+2$ are clear. If $k = i+1$ then the left-hand side equals $(\sigma_{i+1}\sigma_i)\sigma_{i+2}(\sigma_{i+1}\sigma_i)$ and the right-hand side equals $(\sigma_{i+1}\sigma_{i+2})\sigma_i(\sigma_{i+1}\sigma_{i+2})$. Both are equivalent through two braid relations with intermediate commutator relations. The cases $k = i$ and $k = i+2$ are symmetric and we only verify $k = i$. The left-hand side equals $\sigma_i(\sigma_{i+1}\sigma_{i+2})(\sigma_i\sigma_{i+1})$ while the right-hand side equals $(\sigma_{i+1}\sigma_{i+2})(\sigma_i\sigma_{i+1})\sigma_{i+2}$. Again these are equivalent through two braid relations with intermediate commutator relations.

Relation (5-6) is left to the reader. □

For future reference we record (5-9), which in the presence of Lemma 5.7 can be formulated as:

Observation 5.8 *The morphism $\pi\colon \mathcal{G}_{BV} \to \mathcal{G}_V$ is equivariant with respect to the $\mathcal{F}$–action in the sense that*

$$\pi(\beta^f) = \pi(\beta)^f$$

for $\beta \in \mathcal{G}_{BV}$ and $f \in \mathcal{F}$.

We define the category $\mathcal{BV}$ to be $\mathcal{F} \bowtie \mathcal{G}_{BV}$ with the above indirect product actions.

Lemma 5.9 *The action of $\mathcal{G}_{BV}$ on $\mathcal{F}$ is injective.*

Proof We only need to check that $\beta \mapsto \beta^{\lambda_i}$ is injective. But β can be recovered from β^{λ_i} by removing the $(i+1)^{\text{st}}$ strand. □

Corollary 5.10 *The category $\mathcal{BV}$ is right-Ore.*

The fundamental group $\pi_1(\mathcal{BV}, 1)$ is the *braided Thompson group BV*.

It is now easy to define braided versions of T and F. We let $\mathcal{G}_{BT}$ and $\mathcal{G}_{BF}$ be the inverse image under $\pi\colon \mathcal{G}_{BV} \to \mathcal{G}_V$ of $\mathcal{G}_T$ and $\mathcal{G}_F$, respectively. Both of these act on $\mathcal{F}$ by restricting the action of $\mathcal{G}_{BV}$, which is the same as to say that they act through π.

The action of $\mathcal{F}$ of $\mathcal{G}_{BV}$ leaves $\mathcal{G}_{BT}$ and $\mathcal{G}_{BF}$ invariant and restricts to actions on these, thanks to Observation 5.8: we know from Section 5.1 that $\mathcal{F}$ leaves $\mathcal{G}_T$ invariant and it is axiomatically required that it leaves the trivial groupoid invariant. Hence, if $\beta \in \mathcal{G}_{BT}$ and $f \in \mathcal{F}$ then $\pi(\beta^f) = \pi(\beta)^f \in \mathcal{G}_T$, so that $\beta^f \in \mathcal{G}_{BT}$, and an analogous reasoning applies for $\beta \in \mathcal{G}_{BF}$.

As a consequence we can define the categories $\mathcal{BT} = \mathcal{F} \bowtie \mathcal{G}_{BT}$ and $\mathcal{BF} = \mathcal{F} \bowtie \mathcal{G}_{BF}$, which are right-Ore. The group $BF = \pi_1(\mathcal{BF}, 1)$ is called *braided* F and was first introduced in [11]. We call the group $BT = \pi_1(\mathcal{BT}, 1)$ *braided* T.

Remark 5.11 The group BT was not introduced before for the following technical reason. Instead of our category $\mathcal{BV}$, Brin [15] used a monoid that can be thought of as a category with a single object ω which represents countably infinitely many strands. This is possible because splitting one of countably infinitely many strands leads to countably infinitely many strands and because braid groups BRAID_n are contained in a braid group $\varinjlim \mathrm{BRAID}_n$ on infinitely many strands. A practical downside of that approach is that the group of fractions of that monoid is too big, so one needs to describe which elements should be elements of BV. A formal downside is that groups like BT or even T cannot be described because $\mathbb{Z}/n\mathbb{Z}$ is not contained in $\mathbb{Z}/(n+1)\mathbb{Z}$, so that the needed limit does not exist.

Despite this formal problem, the main topological ingredient to establishing the finiteness properties of BT has been verified in [18, Section 3.4].

Remark 5.12 Since braid groups are themselves groups of fractions, one can also obtain BV as the fundamental group of the category $\mathcal{F} \bowtie \mathcal{G}^+_{BV}$, where $\mathcal{G}^+_{BV}(n,n)$ is the monoid of positive (or dual positive) braids rather than the full braid group (and analogous statements hold for BF and BT). This possibility has been noted by several people; see for example the last paragraph of Section 3.1 in [28]. When applying Theorem 3.12, condition (STAB) would become trivial, so verifying condition (LK) will presumably be accordingly harder.

5.3 Groups arising from cloning systems

In [43] Zaremsky and the author have defined (filtered) cloning systems to be the data needed to define indirect product actions of $\mathcal{F}$ and a groupoid on each other. Thus, the groups considered there are by definition fundamental groups of categories $\mathcal{F} \bowtie \mathcal{G}$, where $\mathcal{G}$ is a groupoid. However, the approach follows Brin [15] to construct the groups as subgroups of an indirect product of monoids $\mathcal{F}_\infty \bowtie \mathcal{G}_\infty$. As a consequence it has to deal with technical complications such as the notion of being *properly graded*, as well as practical shortcomings such as being unable to construct (braided) T.

Our categorical approach removes the necessity that the groups $(G_n)_n$ fit into a directed system of groups and therefore the whole discussion goes through without that assumption. Thus, a *cloning system* is given by a sequence $(G_n)_{n\in\mathbb{N}}$ of groups, a sequence $(\rho_n)_{n\in\mathbb{N}}\colon G_n \to S_n$ of morphisms and a family of maps $(\kappa^n_k)_{k\le n}\colon G_n \to G_{n+1}$ such that the following hold for all $k \le n$, $k < \ell$ and $g,h \in G_n$:

(CS1) **Cloning a product** $(gh)\kappa^n_k = (g)\kappa^n_{\rho(h)k}(h)\kappa^n_k$.

(CS2) **Product of clonings** $\kappa^n_\ell \circ \kappa^{n+1}_k = \kappa^n_k \circ \kappa^{n+1}_{\ell+1}$.

(CS3) **Compatibility** $\rho_{n+1}((g)\kappa^n_k)(i) = (\rho_n(g))\varsigma^n_k(i)$ for all $i \neq k, k+1$.

Here ς^n_k describes the action of $\mathcal{F}$ on $\mathcal{G}_V$, so that $(g)\varsigma^n_k(j) = g^{\lambda_k}(j)$ as in (5-1).

Given a cloning system, a groupoid $\mathcal{G}$ is defined by setting $\mathcal{G}(m,n) = \varnothing$ if $m \neq n$ and setting $\mathcal{G}(n,n) = G_n$. Indirect product actions of $\mathcal{F}$ and $\mathcal{G}$ on each other are defined by $g \cdot \lambda^n_k = \lambda^{n+1}_{\rho_n(g)k}$ and $g^{\lambda^n_k} = (g)\kappa^n_k$ for $g \in G_n$. The axioms (CS1), (CS2) and (CS3) ensure that these indeed define indirect product actions.

5.4 The Higman–Thompson groups

In total analogy to Section 5.1 one can define $\mathcal{T}_n = \mathcal{F}_n \bowtie \mathcal{G}_T$ and $\mathcal{V}_n = \mathcal{F}_n \bowtie \mathcal{G}_V$. As mentioned in Section 2 the category $\mathcal{F}_n$ is not connected for $n > 2$ and neither are the categories $\mathcal{T}_n$ and $\mathcal{V}_n$. Thus, it makes sense to define the groups

$$T_{n,r} = \pi_1(\mathcal{T}_n, r), \quad V_{n,r} = \pi_1(\mathcal{V}_n, r)$$

and, unlike the situation of $\mathcal{F}_n$, these groups are generally nonisomorphic for different r; see [27; 32] for a precise statement concerning the $V_{n,r}$. They are the remaining *Higman–Thompson groups*.

5.5 Groups from graph rewriting systems

We now look at indirect products that do not involve $\mathcal{F}$. The corresponding groups have been introduced and described in some detail in [3]. In this section, when we talk about graphs we will take their edges to be directed and allow multiple edges and loops. In particular, every edge has an initial and a terminal vertex. The edge set of a graph G is denoted by $E(G)$ and the vertex set by $V(G)$.

An *edge replacement rule* $e \to R$ consists of a single directed edge e and a finite graph R that contains the two vertices of e (but not e itself). If G is any graph and ε is an edge of G, the edge replacement rule can be *applied to* G *at* ε by removing ε and adding in a copy of R while identifying the initial/terminal vertex of ε with the initial/terminal vertex of e in R. The resulting graph is denoted by $G \lhd \varepsilon$. If δ is another edge of G, then it is also an edge of $G \lhd \varepsilon$ and so the replacement rule can be applied to $G \lhd \varepsilon$ at δ. We regard $G \lhd \varepsilon \lhd \delta$ and $G \lhd \delta \lhd \varepsilon$ as the same graph.

The vagueness inherent in the last sentence can be remedied by declaring that a graph obtained from G by applying the edge replacement rule (possibly many times) has as edges words in $E(G) \times E(R)^*$ and as vertices words in $V(G) \cup (E(G) \times E(R)^* \times V(R))$. For example, the graph $G \lhd \varepsilon \lhd \delta$ would have edges $\zeta \in E(G) \smallsetminus \{\varepsilon, \delta\}$ as well as $\varepsilon\xi$ and $\delta\xi$ for $\xi \in E(R)$ and vertices $v \in V(G)$ as well as εw and δw for $w \in V(R)$.

For every edge replacement rule $e \to R$ we define a category $\mathcal{R}_{e\to R}$ whose objects are finite graphs. In order for the category to be small we will take the graphs to have vertices and edges coming from a fixed countable set, which in addition is closed under attaching words in $E(R)$ and $V(R)$. The category is presented by having generators

$$\lambda_\varepsilon^G \in \mathcal{R}_{e\to R}(G, G \lhd \varepsilon) \quad \text{for } G \text{ a graph and } \varepsilon \text{ an edge of } G$$

subject to the relations

$$\lambda_\delta^G \lambda_\varepsilon^{G\lhd\delta} = \lambda_\varepsilon^G \lambda_\delta^{G\lhd\varepsilon} \quad \text{for } G \text{ a graph and } \delta \text{ and } \varepsilon \text{ distinct edges of } G. \tag{5-10}$$

Lemma 5.13 *For any edge replacement rule $e \to R$ the category $\mathcal{R}_{e\to R}$ is right-Ore.*

Proof Thanks to the relations (5-10) a morphism $\lambda_{\varepsilon_1} \cdots \lambda_{\varepsilon_k}$ in $\mathcal{R}_{e\to R}$ is uniquely determined by its source, its target and the set $\{\varepsilon_1, \dots, \varepsilon_k\}$. The claim now follows by taking differences and unions of these sets of edges. □

As in previous sections, the second ingredient will be a groupoid. Its definition does not depend on the edge replacement rule, except possibly for the foundational issues of choosing universal sets of vertices and edges. We define $\mathcal{G}_{\text{graph}}$ to have as objects finite graphs and as morphisms isomorphisms of graphs.

We define actions of $\mathcal{R}_{e\to R}$ and $\mathcal{G}_{\text{graph}}$ on each other as follows. If $g\colon G \to G'$ is an isomorphism of graphs and $\varepsilon \in E(G)$ is an edge, then

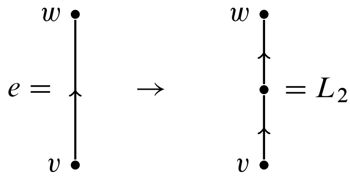
$$g \cdot \lambda_\varepsilon^G = \lambda_{g(\varepsilon)}^{G'}$$

and g^{λ_ε} is the isomorphism $G \lhd \varepsilon \to G' \lhd g(\varepsilon)$ that takes δ to $g(\delta)$ for $\delta \in E(G) \smallsetminus \{\varepsilon\}$ and that takes $\varepsilon\zeta$ to $g(\varepsilon)\zeta$ for $\zeta \in V(R) \cup E(R)$. The following is easy to verify:

Observation 5.14 *The actions of $\mathcal{R}_{e\to R}$ and $\mathcal{G}_{\text{graph}}$ on each other defined above are well-defined indirect product actions. The action of $\mathcal{R}_{e\to R}$ on $\mathcal{G}_{\text{graph}}$ is injective.*

As a consequence we obtain a right-Ore category $\mathcal{RG}_{e\to R} := \mathcal{R}_{e\to R} \bowtie \mathcal{G}_{\text{graph}}$ and for every finite graph G we obtain a potential group $\pi_1(\mathcal{RG}_{e\to R}, G)$.

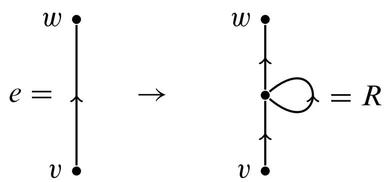
Example 5.15 If we consider the edge replacement rule



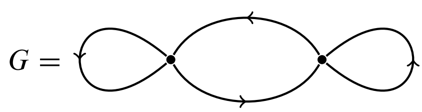
and take L_1 to be the graph consisting of a single edge, then $\pi_1(\mathcal{RG}_{e\to L_2}, L_1)$ is isomorphic to F. Similarly, if C_1 is the graph consisting of a single loop then $\pi_1(\mathcal{RG}_{e\to L_2}, C_1)$ is isomorphic to T. Finally, V arises as $\pi_1(\mathcal{RG}_{e\to D_2}, L_1)$, where the rule $e \to D_2$ replaces an edge by two disconnected edges.

Various fundamental groups of categories arising from graph rewriting systems are described in [3]. Here we will only mention the Basilica–Thompson group, introduced by them in [4].

We consider the replacement rule



and the graph



The *Basilica–Thompson group* is $T_B := \pi_1(\mathcal{RG}_{e\to R}, G)$.

6 Examples: finiteness properties

In this section we give various examples of applications of Theorem 3.12 and Corollary 3.14. In most cases these finiteness properties are known and the proofs involve proving that certain complexes are highly connected. We will see that these complexes always coincide with the complexes $|E(x)|$. As a consequence the connectivity statement from the literature together with Theorem 3.12 gives the result.

6.1 Finiteness properties of Thompson's groups

We start with the categories $\mathcal{T}$ and $\mathcal{V}$. The conditions needed to apply the results from Section 3 have been verified in Corollaries 5.3 and 5.6.

In order to apply Corollary 3.14 two more things are left to verify: that automorphism groups are of type F_∞ and that the connectivity of the simplicial complexes $|E(n)|$ goes to infinity with n. The groups $\mathcal{F}(n,n) = \{1\}$, $\mathcal{T}(n,n) = \mathbb{Z}/n\mathbb{Z}$ and $\mathcal{V}(n,n) = \mathrm{SYM}_n$ are all finite and therefore of type F_∞.

In order to describe the complexes $E(n)$, we need to talk about further graphs. The *cyclic graph* is denoted by C_n, it has the same edges as L_n and additionally $\{1,n\}$. The *complete graph* K_n has all edges $\{i,j\}$ for $1 \le i < j \le n$. We describe the complexes $E(n)$ in the case of $\mathcal{V}$ and leave $\mathcal{T}$ to the reader.

Lemma 6.1 *The poset $E_{\mathcal{T}}(n)$ is isomorphic to $\mathcal{M}(C_n)$.*

Lemma 6.2 *There is a poset morphism $E_{\mathcal{V}}(n) \to \mathcal{M}(K_n)$ whose fibers over k–simplices are k–spheres.*

Proof Every element of $(\mathcal{E} \bowtie \mathcal{G}_V)(-,n)$ can be written as a product fg of an elementary forest $f \in \mathcal{E}(-,n)$ and a permutation $g \in \mathcal{G}_V(n,n)$. By definition the vertices of $E(n)$ are these products modulo multiplication by permutations from the left. As in Lemma 3.16 an elementary forest can be interpreted as a matching on L_n. Under this correspondence, the group $\mathcal{G}_V(n,n) = \mathrm{SYM}_n$ acts on the vertices of L_n and the permutations from the left act on components of the matching. Thus, elements of $(\mathcal{E} \bowtie \mathcal{G}_V)(-,n)$ can be described by matchings on the linear graph on $g^{-1}(1), \dots, g^{-1}(n)$ modulo reordering the components of the matching.

The possibility of reordering the vertices of the matching means that any two elements of $\{1,\dots,n\}$ can be connected and so we obtain a map $|E(n)| \to \mathcal{M}(K_n)$ to the matching complex of the *complete* graph on $\{1,\dots,n\}$. This map is clearly surjective.

It is not injective because in $E(n)$ the order of two matched vertices matters while in $\mathcal{M}(K_n)$ it does not. For example, λ_i and $\lambda_i(i\ i{+}1)$ map to the same vertex in $\mathcal{M}(K_n)$. As a result the fiber over a k–simplex is a join of $k+1$ many 0–spheres, ie a k–sphere. □

The fact that the morphism in Lemma 6.2 is not an isomorphism means that we have to do one extra step, namely to apply the following result by Quillen [33, Theorem 9.1].

Rather than giving the general formulation for posets we restrict to face posets of (n–skeleta of) simplicial complexes, to save us some notation.

Theorem 6.3 *Let $n \in \mathbb{N}$ and let $f\colon X \to Y$ be a simplicial map. Assume that Y is $(n-1)$–connected and that for every k–simplex σ of Y the link $\operatorname{lk}\sigma$ is $(n-\dim\sigma-2)$–connected and the fiber $|f^{-1}(\sigma)|$ is $(k-1)$–connected. Then X is $(n-1)$–connected.*

Theorem 6.4 *Thompson's groups T and V are of type F_∞.*

Proof Using Corollary 3.14 we need to show that the connectivity of the complexes $|E(n)|$ goes to infinity as n goes to infinity. We work with the simplicial complexes $E(n)$ instead. In the case of T the complexes are matching complexes by Lemma 6.1 whose connectivity goes to infinity by Lemma 3.19. In the case of V the complexes map to matching complexes with good fibers by Lemma 6.2. Noting that the link of a k–simplex in $\mathcal{M}(K_n)$ is isomorphic to $\mathcal{M}(K_{n-2(k+1)})$, we can apply Theorem 6.3 to see that the connectivity of $E_{\mathcal{V}}$ goes to infinity as well. □

The proof for the Higman–Thompson groups is completely analogous.

6.2 Finiteness properties of braided Thompson groups

We have already seen that $\mathcal{BF}$, $\mathcal{BT}$ and $\mathcal{BV}$ are right-Ore. That they admit a height function and a left-Garside family follows via Proposition 4.5, just as it did for $\mathcal{T}$ and $\mathcal{V}$. The braid groups $\mathcal{BV}^\times(n,n) = \mathcal{G}_{BV}(n,n)$ are of type F by Corollary 3.8 (and hence of type F_∞). Consequently the finite-index subgroups of pure braids $\mathcal{BF}^\times(n,n)$ and of cyclically permuting braids $\mathcal{BT}^\times(n,n)$ are of type F as well.

It remains to understand the complexes $|E(n)|$. For that purpose, we will want to think of braid groups as mapping class groups. Let D be a closed disc with n punctures $p_1,\dots,p_n$, which we can think of as distinguished points in the interior of D. The mapping class group of the n–punctured disc is

$$\mathrm{Homeo}^+(D \setminus \{p_1,\dots,p_n\}, \partial D)/\mathrm{Homeo}_0^+(D \setminus \{p_1,\dots,p_n\}, \partial D),$$

where $\mathrm{Homeo}^+(D \setminus \{p_1,\dots,p_n\}, \partial D)$ is the group of orientation-preserving homeomorphisms of $D \setminus \{p_1,\dots,p_n\}$ that fix ∂D and $\mathrm{Homeo}_0^+(D \setminus \{p_1,\dots,p_n\}, \partial D)$ is the subgroup of homeomorphisms that are isotopic to the identity. It is well known that the mapping class group of the n–punctured disc is isomorphic to the braid group; see for example [29].

With this description in place, we can start to look at the complexes $|E(n)|$. Let $fg \in E(n)$ with $f \in \mathcal{E}(-,n)$ and $g \in \mathcal{G}_{BV}(n,n)$. Regard the n punctures $p_1,\dots,p_n$ as the vertices of an L_n embedded into D. As we have seen before, f corresponds to a matching M_f on L_n, which we now regard as a disjoint selection of the fixed arcs connecting pairs of adjacent punctures. The element g, regarded as a mapping class, acts on M_f and we obtain a set $M_f g$ of disjoint arcs connecting some pairs of punctures. Such a collection of arcs is called an *arc matching* in [18]. Note that if $f \in \mathcal{E}(k,n)$, so that the arc matching consists of $n-k$ arcs, then removing the arcs from the punctured disc results in a k–punctured disc. The action of $\mathcal{G}_{BV}(k,k)$ from the left is just the action of the mapping class group of that k–punctured disc and in particular does nothing to M_f.

For a subgraph Γ of K_n the *arc matching complex* $\mathcal{MA}(\Gamma)$ is the simplicial complex whose k–simplices are sets of pairwise disjoint arcs connecting punctures with the condition that an arc can only connect two punctures if they are connected by an edge in Γ.

Proposition 6.5 *There exist surjective morphisms of simplicial complexes*

(i) $E_{\mathcal{BF}}(n) \to \mathcal{MA}(L_n)$,

(ii) $E_{\mathcal{BT}}(n) \to \mathcal{MA}(C_n)$,

(iii) $E_{\mathcal{BV}}(n) \to \mathcal{MA}(K_n)$,

whose fiber over any k–simplex is the join of k countable infinite discrete sets.

Proof The product $fg \in E(n)$ is taken to the arc matching $M_f g$ as described above. Since $\mathcal{G}_{BF}(n,n)$ takes every puncture to itself, the map (i) maps onto $\mathcal{MA}(L_n)$. Similarly, since $\mathcal{G}_{BT}(n,n)$ cyclically permutes the punctures, the map (ii) maps into $\mathcal{MA}(C_n)$. Surjectivity is clear.

To describe the fibers consider a disc D' containing p_i and p_{i+1} but none of the other punctures and let β be a braid that is arbitrary inside D' but trivial outside. Then $\lambda_i\beta$ maps to the same arc ($=$ vertex of $\mathcal{MA}(K_n)$) irrespective of β. Thus, the fiber over this vertex is the mapping class group of $D' \setminus \{p_i, p_{i+1}\}$ in the case of $\mathcal{BV}$ and is the pure braid group of $D' \setminus \{p_i, p_{i+1}\}$ in the cases of $\mathcal{BF}$ and $\mathcal{BT}$. In either case it is a countable infinite discrete set. □

The connectivity properties of arc matching complexes have been studied in [18]. We summarize Theorem 3.8, Corollary 3.11 and the remark in Section 3.4 from there in the following theorem. It applies to arc matching complexes not only on disks but on arbitrary surfaces with (possibly empty) boundary.

Theorem 6.6 (i) *$\mathcal{MA}(K_n)$ is $(\nu(n)-1)$–connected,*

(ii) *$\mathcal{MA}(C_n)$ is $(\eta(n-1)-1)$–connected,*

(iii) *$\mathcal{MA}(L_n)$ is $(\eta(n)-1)$–connected,*

where $\nu(n) = \lfloor \frac{n-1}{3} \rfloor$ and $\eta(n) = \lfloor \frac{n-1}{4} \rfloor$.

Theorem 6.7 *The braided Thompson groups BF, BT and BV are of type F_∞.*

Proof We want to apply Corollary 3.14. By Proposition 6.5 the complexes $E(n)$ map onto arc matching complexes and we want to apply Theorem 6.3. To do so, we need to observe that the link of a $(k+1)$–simplex on an arc matching complex on a surface with n punctures is an arc matching complex with $n-2k$ punctures, where the k arcs connecting two punctures have been turned into boundary components. Putting these results together shows that the connectivity properties of $E(n)$ go to infinity with n by Theorem 6.6. □

6.3 Absence of finiteness properties

Theorem 3.12 gives a way to prove that certain groups are of type F_n. If the group is not of type F_n, one of the hypotheses fails. We will now discuss to what extent the construction is (un)helpful in proving that the group is not of type F_n, depending on which hypothesis fails.

In the first case the groups $\mathcal{C}^\times(x,x)$ are not of type F_n (even for $\rho(x)$ large). In this case the general part of Brown's criterion, Theorem 3.10, cannot be applied. Thus, the whole construction from Section 3.4 is useless for showing that $\pi_1(\mathcal{G}_\mathcal{C},*)$ is not of type F_n. An example of this case are the groups $\mathcal{T}(B_*(\mathcal{O}_S))$ treated in [43, Theorem 8.12]. The proof redoes part of the proof that the groups $\mathcal{C}^\times(x,x)$, which are the groups in $B_n(\mathcal{O}_S)$ in this case, are not of type $F_n = F_{|S|}$.

In the second case the complexes $E(x)$ are not (even asymptotically) $(n-1)$–connected. In this case Brown's criterion, Theorem 3.10, can in principle be applied, but not by using just Morse theory. An example of this case is the Basilica–Thompson group from Section 5.5, which is not finitely presented [41], so $n = 2$. A morphism in $\mathcal{RG}_{e\to R} = \mathcal{R}_{e\to R} \bowtie \mathcal{G}_{\mathrm{graph}}$ is declared to be elementary if there are edges $\{e_1,\dots,e_k\}$

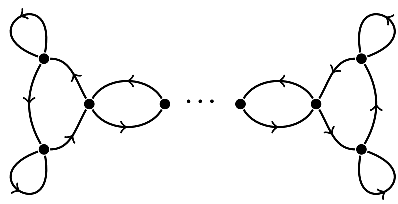


Figure 8: Arbitrarily large graphs H with $E(H)$ not simply connected

of G such that $f = \lambda_{e_1} \cdots \lambda_{e_k}$. The function $\rho \colon \mathcal{R}_{e\to R} \to \mathbb{N}$ is the number of edges of a graph. The basepoint $*$ is the Basilica graph G.

The connectivity assumption of Theorem 3.12 is violated because the $\mathcal{R}\mathcal{G}_{e\to R}$–component of G contains graphs H with arbitrarily many edges for which $E(H)$ is not simply connected. Examples of such graphs are illustrated in Figure 8. In these examples $E(H)$ has four vertices: two vertices v_{ll}, v_{ul} corresponding to the loops on the left and two vertices v_{lr} and v_{ur} corresponding to the loops on the right. The left vertices are connected to the right vertices but not to each other and neither are the right vertices. Thus, $E(H)$ is a circle $v_{\mathrm{ll}}, v_{\mathrm{lr}}, v_{\mathrm{ul}}, v_{\mathrm{ur}}$ and is not simply connected.

Looking into the proof of Theorem 3.12 we can compare directly what the non-simple connectedness of $E(H)$ tells us and what is needed to apply Brown's criterion (Theorem 3.10) in order to prove that the group is not of type F_n. To apply Theorem 3.10, one needs to show that for every m there is an arbitrarily large n such that, passing from $X_{\rho<m}$ to $X_{\rho<n+1}$, a nontrivial 1–sphere in $X_{\rho<m}$ is filled in. The assumption that $E(H)$ is not simply connected for $\rho(H) = n$ translates via the Morse argument to the statement that when passing from $X_{\rho<n}$ to $X_{\rho<n+1}$ either a nontrivial 1–sphere in $X_{\rho<n}$ is filled in, or a nontrivial 2–sphere is created. The proof in [41] that the Basilica–Thompson group T_B is not finitely presented therefore needs to rule out the second possibility and also show that the 1–sphere that is filled in was nontrivial already in $X_{\rho<m}$.

7 Sketch of further examples

In this final section we sketch two further examples of categories associated to Thompson groups that fit in our framework. This is aimed mainly at experts who already know the groups and we will be brief.

7.1 Brin–Thompson groups

The higher-dimensional versions of V, denoted by sV for $s \geq 1$, were introduced by Brin [13]. If $C = \{0,1\}^\omega$ denotes Cantor space, a morphism in $\mathcal{V}(m,n)$ can be interpreted as a homeomorphism (subject to conditions)

$$\{1,\dots,m\} \times C \leftarrow \{1,\dots,n\} \times C$$

that represents subdividing m copies of C into n copies. The category $s\mathcal{V}$ similarly consists of homeomorphisms

$$\{1,\dots,m\} \times C^s \leftarrow \{1,\dots,n\} \times C^s$$

that represent subdividing m copies of C^s into n copies. See Figure 9 for an example illustrating composition. The Brin–Thompson groups are the groups $sV = \pi_1(s\mathcal{V}, 1)$.

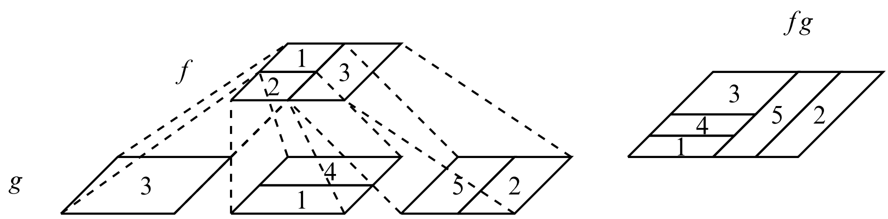


Figure 9: Composition of two morphisms in $2\mathcal{V}$

If one wants to obtain a presentation for $s\mathcal{V}$ whose objects are the natural numbers, one always needs to pick an order for the copies of C^s and the order is changed under relations. The presentation will therefore involve $\mathcal{G}_V$ from the start. Besides that, we take generators

$$\lambda^n_{a,i} \in s\mathcal{V}(n, n+1) \quad \text{for } 1 \leq a \leq s,\ 0 \leq i < n$$

representing the subdivision of the $(i+1)^{\text{st}}$ of n copies of C^s in the a^{th} direction. For each direction these satisfy the familiar relations

$$\lambda^n_{a,i}\lambda^{n+1}_{a,j} = \lambda^n_{a,j}\lambda^{n+1}_{a,i+1} \quad \text{for } 1 \leq a \leq s,\ 0 \leq j < i < n. \tag{7-1}$$

In addition, for two distinct directions we have the relations

$$\lambda^n_{a,i}\lambda^{n+1}_{b,i+1}\lambda^{n+2}_{b,i} = \lambda^n_{b,i}\lambda^{n+1}_{a,i+1}\lambda^{n+2}_{a,i}s_{i+1} \quad \text{for } 1 \leq a < b \leq s,\ 0 \leq i < n \tag{7-2}$$

(recall that s_{i+1} is the transposition $(i+1\ i+2)$).

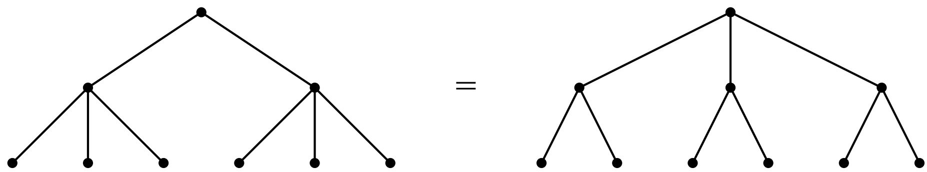


Figure 10: A relation in $\mathcal{F}_{2,3}$, also depicted in [36, page 485]

We claim without proof that $s\mathcal{V}$ has the presentation

$$s\mathcal{V} = \langle \mathcal{G}_V, \lambda^n_{a,i} \mid \text{relations in } \mathcal{G}_V, \text{(7-1)}, \text{(7-2)} \rangle.$$

To apply Theorem B to this setup one needs to verify condition (LK). This verification is the essence of [24].

7.2 Stein–Thompson groups

The idea underlying the Stein–Thompson groups is to allow carets with different arity chosen from a finite set $S = \{n_1, \dots, n_k\}$; see [36]. Thus, the underlying category $\mathcal{F}_S$ may be thought of as generated by $\mathcal{F}_{n_1}, \dots, \mathcal{F}_{n_k}$. There are number-theoretic relations, however. For instance, a tree that has a full layer of n_1–carets followed by a full layer of n_2–carets is the same as one with a full layer of n_2–carets followed by a full layer of n_1–carets; see Figure 10. We refrain from writing down a presentation but we should point out that the perspective taken in [36] is fairly close to ours. This does not include the F_∞–proof as Stein's space is carefully tailored to provide more precise homological information.

The categories $\mathcal{T}_S = \mathcal{F}_S \bowtie \mathcal{G}_T$ and $\mathcal{V}_S = \mathcal{F}_S \bowtie \mathcal{G}_V$ arise as indirect products in a straightforward manner.

It is clear that the categories are right-Ore and admit a height function.

A Garside family consists of the family of forests $\mathcal{S}$ where along any path from root to leaf at most one n_i–caret is met for any i. The maximal such tree $\Delta(x) \in \mathcal{F}_S(x, -)$ is the one that has a full layer of n_i–carets for each i. Any two elements in $\mathcal{S}(x, -)$ have a least common right multiple by [36, Proposition 1.2]. This together with the height function implies the existence of $\mathcal{S}$–heads.

The rest of the proof that the groups are of type F_∞ is completely analogous to that for Thompson's groups and the Higman–Thompson groups in Section 5.1.

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Faculty of Mathematics, Bielefeld University
Bielefeld, Germany

switzel@math.uni-bielefeld.de

Received: 27 April 2018 Revised: 1 November 2018