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Homotopy theory of unital algebras

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We provide an extensive study of the homotopy theory of types of algebras with units, for instance unital associative algebras or unital commutative algebras. To this purpose, we endow the Koszul dual category of curved coalgebras, where the notion of quasi-isomorphism barely makes sense, with a model category structure Quillen equivalent to that of unital algebras. To prove such a result, we use recent methods based on presentable categories. This allows us to describe the homotopy properties of unital algebras in a simpler and richer way. Moreover, we endow the various model categories with several enrichments which induce suitable models for the mapping spaces and describe the formal deformations of morphisms of algebras.

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Introduction

Among the various types of algebras, some of them include units, like the ubiquitous unital associative algebras and unital commutative algebras or the unital Batalin–Vilkovisky algebras, which arose in mathematical physics. When working with a chain complex carrying such an algebraic structure, like the de Rham algebra of differential

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manifolds, one would like to understand the properties that this algebraic data satisfies up to quasi-isomorphisms. The purpose of the present paper is to develop a framework which allows one to prove the homotopical properties carried by types of algebras with units, that is, their properties up to quasi-isomorphisms.

In order to work with types of algebras in a general way, one needs a precise notion which encodes these ones. This is achieved by the concept of an operad. Operads are generalizations of associative algebras which encode some types of algebras (associative, commutative, Lie, Batalin–Vilkovisky, …) in a way that a representation of an operad $\mathscr{P}$ is a chain complex together with a structure of algebra of the type encoded by $\mathscr{P}$.

Further, one of the most common and powerful tool to study homotopical algebra — that is to study categories with a notion of weak equivalences — is the model category structure introduced by Daniel Quillen, which makes the manipulation of weak equivalences easier by means of other maps, called cofibrations and fibrations, respectively. Hinich proved in [15] that the category of algebras over an operad carries a model structure whose weak equivalences are quasi-isomorphisms and whose fibrations are surjections. In a purely theoretical perspective, this model structure describes all the homotopical data of this category. However, the cofibrant objects are not easy to handle; they are the retracts of free algebras whose generators carry a particular filtration.

Hinich [16] embedded the category of differential graded (dg) Lie algebras into the category of dg cocommutative coalgebras. From the model structure of the category of dg Lie algebras he obtained a model structure on the category of dg cocommutative coalgebras which is Quillen equivalent to the first one. In this new model category, any object is cofibrant. Moreover, this context allows one to build an obstruction theory for the existence of the algebra structures and the algebra morphisms. So this new context of dg cocommutative coalgebras is more suitable to study the homotopy theory of dg Lie algebras than the category of dg Lie algebras itself. With a similar perspective, Lefevre and Hasegawa embedded the category of nonunital dg associative algebras into the category dg coassociative coalgebras, shown to be Quillen equivalent to the first one; see [18]. Vallette generalized these results to all types of algebras encoded by any operad satisfying a technical condition: that it is an augmented operad. Augmented operads are related to the dual notion of conilpotent cooperads by an adjunction called the operadic bar–cobar adjunction $\Omega \dashv B$. Vallette embedded the category of algebras over an augmented operad $\mathscr{P}$ into category of coalgebras over a cooperad $\mathscr{P}^{¡}$ called the Koszul dual of $\mathscr{P}$. He transferred the model structure on the category of $\mathscr{P}$–algebras to the category of $\mathscr{P}^{¡}$–coalgebras and got again a Quillen equivalence between these two model categories; see [26].

However the operads describing types algebras with units do not satisfy the technical condition to be augmented. To extend the result of Vallette to categories of algebras over any operad, one first needs to modify the operadic bar–cobar adjunction. Inspired by the work of Hirsh and Millès [17], we introduce an adjunction à la bar–cobar relating dg operads to curved conilpotent cooperads:

$$\text{curved conilpotent cooperads} \underset{B_c}{\overset{\Omega_u}{\rightleftarrows}} \text{dg operads.}$$

Moreover, any morphism of dg operads f from a cobar construction $\Omega_u\mathscr{C}$ of a curved conilpotent cooperad $\mathscr{C}$ to an operad $\mathscr{P}$ comes equipped with an adjunction $\Omega_f \dashv B_f$ relating $\mathscr{P}$–algebras to $\mathscr{C}$–coalgebras,

$$\mathscr{C}\text{–coalgebras} \underset{B_f}{\overset{\Omega_f}{\rightleftarrows}} \mathscr{P}\text{–algebras.}$$

The model structure of $\mathscr{P}$–algebras can be transferred to the category of $\mathscr{C}$–coalgebras along this adjunction.

Theorem 82 *Let $\alpha\colon \mathscr{C} \to \mathscr{P}$ be an operadic twisting morphism and let $\Omega_\alpha \dashv B_\alpha$ be the bar–cobar adjunction between $\mathscr{P}$–algebras and $\mathscr{C}$–coalgebras induced by α. There exists a model structure on the category of $\mathscr{C}$–coalgebras whose cofibrations (resp. weak equivalences) are morphisms whose image under Ω_α is a cofibration (resp. weak equivalence). With this model category structure, the adjunction $\Omega_\alpha \dashv B_\alpha$ is a Quillen adjunction.*

To prove this theorem, we use new techniques coming from category theory. Specifically, we utilize a theorem of Bayeh, Hess, Karpova, Kędziorek, Riehl and Shipley [3] involving presentable categories.

We study in detail the particular case where the morphism of operads f from $\Omega_u\mathscr{C}$ to $\mathscr{P}$ is a quasi-isomorphism, for instance if f is the identity ι of $\Omega_u\mathscr{C}$. In this case, the Quillen adjunction $\Omega_\iota \dashv B_\iota$ is a Quillen equivalence. We show that the fibrant $\mathscr{C}$–coalgebras are the images of the $\Omega_u\mathscr{C}$–algebras under the functor B_ι. So, switching from the category of $\Omega_u\mathscr{C}$–algebras to the category of $\mathscr{C}$–coalgebras by the functor B_ι amounts to introducing new morphisms between $\Omega_u\mathscr{C}$–algebras. These new morphisms can be built using obstruction methods. Moreover, any $\Omega_u\mathscr{C}$–algebra becomes cofibrant in this new context.

This article also deals with enrichments of the category of $\mathscr{P}$–algebras for any differential graded operad $\mathscr{P}$, and of the category of $\mathscr{C}$–coalgebras for any curved cooperad $\mathscr{C}$. These two categories are enriched in simplicial sets in a way that recovers the mapping spaces. Further, they are tensored, cotensored and enriched in cocommutative coalgebras. These cocommutative coalgebras encode the formal deformations of morphisms of algebras over an operad. Indeed, for any two algebras $\mathcal{A}$ and $\mathcal{A}'$ over an operad $\mathscr{P}$, the atoms of their mapping cocommutative coalgebra $\{\mathcal{A}, \mathcal{A}'\}$ — that is, the closed elements $e \in \{\mathcal{A}, \mathcal{A}'\}_0$ such that $\Delta(e) = e \otimes e$ — are exactly the morphisms of $\mathscr{P}$–algebras from $\mathcal{A}$ to $\mathcal{A}'$. Moreover, if $\mathcal{A}$ is cofibrant, the maximal coaugmented conilpotent subcoalgebra of $\{\mathcal{A}, \mathcal{A}'\}$ that contains an atom f is the bar construction of the Lie algebra that controls the formal deformations of the morphism f. In the context of nonsymmetric operads and nonsymmetric cooperads, this enrichment can be extended to all coassociative coalgebras. These coassociative coalgebras encode in single objects both the mapping spaces and the deformation of morphisms.

Finally, we apply the framework developed here to concrete operads like the operad $u\mathscr{A}s$ of unital associative algebras and the operad $u\mathscr{C}om$ of unital commutative algebras. For these two operads, the process of curved Koszul duality developed in [17] relates the curved cooperads $u\mathscr{A}s^{¡}$ and $u\mathscr{C}om^{¡}$ to the operads $u\mathscr{A}s$ and $u\mathscr{C}om$, respectively. We show that the category of $u\mathscr{A}s^{¡}$–coalgebras and the category of $u\mathscr{C}om^{¡}$–coalgebras are equivalent to the category of curved conilpotent coassociative coalgebras and the category of curved conilpotent Lie coalgebras, respectively.

Layout

The article is organized as follows. In Section 1, we recall several notions about category theory, and homological algebra. In Section 2, we recall the notions of operads, cooperads, algebras over an operad and coalgebras over a cooperad. We also prove some results, as the presentability of the category of coalgebras over a curved cooperad, that we will need in the sequel. Section 3 deals with enrichments of the category of algebras over an operad and of the category of coalgebras over a curved cooperad; specifically, we study enrichments over simplicial sets, cocommutative coalgebras and coassociative coalgebras. In Section 4, we introduce an adjunction à la bar–cobar between operads and curved cooperads related to a notion of twisting morphism. We use it to define an adjunction between $\mathscr{P}$–algebras and $\mathscr{C}$–coalgebras for a twisting morphism from a curved cooperad $\mathscr{C}$ to an operad $\mathscr{P}$. In Section 5, we recall the projective model structure on the category of algebras over an operad.

We describe models for the mapping spaces and we show that the enrichment over cocommutative coalgebras encodes deformations of morphisms. Section 6 transfers the projective model structure on $\mathscr{P}$–algebras along the previous adjunction to obtain a model structure on $\mathscr{C}$–coalgebras and a Quillen adjunction. Section 7 deals with these model structures in the case where the operad $\mathscr{P}$ is the cobar construction $\Omega_u\mathscr{C}$ of $\mathscr{C}$. In particular, the adjunction induced is a Quillen equivalence. Finally, in Section 8, we apply the formalism developed in the previous sections to study the examples of unital associative algebras and unital commutative algebras.

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Conventions and notation

- We work over a field $\mathbb{K}$. Note that no further assumption is needed when working with nonsymmetric operads. However, when dealing with homotopy results that concern all operads and cooperads and their algebras and coalgebras, we will assume the characteristic of the field to be zero.

- The category of $\mathbb{Z}$–graded $\mathbb{K}$–modules, that is, functors from the discrete category $\mathbb{Z}$ to the category of $\mathbb{K}$–vector spaces, is denoted by gMod. The category of chain complexes, that is, $\mathbb{Z}$–graded $\mathbb{K}$–modules equipped with a degree -1 square-zero map, is denoted by dgMod. They are endowed with their usual closed symmetric monoidal structures. The internal hom is denoted by $[\,,\,]$. The category of chain complexes is also endowed with its projective model structure, where the weak equivalences are the quasi-isomorphisms and where the fibrations are the degreewise surjections. The degree of a homogeneous element x of a graded $\mathbb{K}$–module or a chain complex is denoted by $|x|$.

- For any integer n, let D^n be the chain complex generated by one element in degree n and its boundary in degree $n-1$. Let S^n be the chain complex generated by a cycle in degree n.

- The category of simplicial set is denoted by sSet. It is endowed with its Kan–Quillen model structure; see Goerss and Jardine [13, I.11.3].

- A diagram of the form

$$\mathsf{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathsf{D}$$

means that the functor R is right adjoint to the functor L.

- For any graded $\mathbb{K}$–module $\mathcal{V}$ endowed with a filtration $(F_n\mathcal{V})_{n\in\mathbb{N}}$, the graded complex associated to this filtration is denoted by $G\mathcal{V}$. In other words,

$$G\mathcal{V} = \bigoplus_n G_n\mathcal{V}$$

where $G_n\mathcal{V} = F_n\mathcal{V}/F_{n-1}\mathcal{V}$. If $\mathcal{V}$ is a chain complex such that $(F_n\mathcal{V})_{n\in\mathbb{N}}$ is a filtration of chain complexes, that is, $d(F_n\mathcal{V}) \subset F_n\mathcal{V}$ for any integer n, then $G\mathcal{V}$ inherits the structure of a chain complex.

1 Preliminaries

In this first section, we recall some categorical concepts like the presentability and the notions of enrichment, tensoring and cotensoring. Moreover, we describe several notions of coalgebras, like coassociative coalgebras and cocommutative coalgebras, that have been extensively studied in [11] and [16], respectively. More specifically, the category of coassociative coalgebras admits a model structure related by a Quillen adjunction to the category of simplicial sets; the category of conilpotent cocommutative coalgebras admits a model structure Quillen equivalent to the projective model structure on Lie algebras. Finally, we describe the Sullivan polynomial algebras.

1.1 Presentable categories

Definition 1 (presentable category) Let C be a cocomplete category. An object X of C is called *compact* if for any filtered diagram $F\colon I \to \mathsf{C}$ the map

$$\operatorname{colim}(\hom_{\mathsf{C}}(X, F)) \to \hom_{\mathsf{C}}(X, \operatorname{colim} F)$$

is an isomorphism. The category C is said to be *presentable* if there exists a set of compact objects such that any object of C is the colimit of a filtered diagram involving only these compact objects.

The following proposition is a classical result of category theory:

Proposition 2 [1] *A functor $L\colon \mathsf{C} \to \mathsf{D}$ between presentable categories is a left adjoint if and only if it preserves colimits.*

1.2 Tensoring, cotensoring and enrichment

In this section, we recall the definition of tensored-cotensored-enriched category over a monoidal category. See [5] for the original reference.

Definition 3 (action, coaction) Let $(\mathsf{E}, \otimes, \mathcal{I})$ be a monoidal category and let C be a category.

- An enrichment of C over E is a bifunctor $[-,-]\colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{E}$ together with functorial morphisms

  $$\gamma_{X,Y,Z}\colon [Y,Z] \otimes [X,Y] \to [X,Z], \quad \upsilon_X\colon \mathcal{I} \to [X,X]$$

  for any objects X, Y and Z of C and which are composition and unit in terms of the commutative diagrams

  $$\begin{tikzcd}
[Y,Z] \otimes [X,Y] \otimes [V,X] \arrow[r, "\gamma_{X,Y,Z} \otimes \mathrm{Id}"] \arrow[d, "\mathrm{Id} \otimes \gamma_{V,X,Y} \otimes \mathrm{Id}"'] & {[X,Z] \otimes [V,X]} \arrow[d, "\gamma_{V,X,Z}"] \\
{[Y,Z] \otimes [V,Y]} \arrow[r, "\gamma_{V,Y,Z}"'] & {[V,Z]}
\end{tikzcd}$$

  $$\begin{tikzcd}
{[X,Y] \otimes [X,X]} \arrow[dr] & {[X,Y]} \arrow[l, "\mathrm{Id} \otimes \upsilon_X"'] \arrow[r, "\upsilon_Y \otimes \mathrm{Id}"] \arrow[d, "\mathrm{Id}"] & {[Y,Y] \otimes [X,Y]} \arrow[dl] \\
 & {[X,Y]} &
\end{tikzcd}$$

- A right action of E on C is a functor

  $$- \triangleleft -\colon \mathsf{C} \times \mathsf{E} \to \mathsf{C}$$

  together with functorial isomorphisms

  $$\begin{cases} X \triangleleft (\mathcal{A} \otimes \mathcal{B}) \simeq (X \triangleleft \mathcal{A}) \triangleleft \mathcal{B}, \\ X \triangleleft \mathcal{I} \simeq X, \end{cases}$$

for any $X \in \mathsf{C}$ and any $\mathcal{A}, \mathcal{B} \in \mathsf{E}$; these functors are compatible with the monoidal structure of E in terms of the commutative diagrams

$$\begin{tikzcd}
((X \lhd \mathcal{A}) \lhd \mathcal{B}) \lhd \mathcal{C} \arrow[r] \arrow[d] & (X \lhd (\mathcal{A} \otimes \mathcal{B})) \lhd \mathcal{C} \arrow[r] & X \lhd ((\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}) \arrow[d] \\
(X \lhd \mathcal{A}) \lhd (\mathcal{B} \otimes \mathcal{C}) \arrow[rr] & & X \lhd (\mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C}))
\end{tikzcd}$$

$$\begin{tikzcd}
(X \lhd \mathcal{I}) \lhd \mathcal{A} \arrow[rr] \arrow[dr] & & X \lhd (\mathcal{I} \otimes \mathcal{A}) \arrow[dl] \\
& X \lhd \mathcal{A} &
\end{tikzcd}$$

- A left coaction of E on C is a functor

$$\langle -, - \rangle \colon \mathsf{E}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{C}$$

together with functorial isomorphisms

$$\begin{cases} \langle \mathcal{A} \otimes \mathcal{B}, X \rangle \simeq \langle \mathcal{A} \langle \mathcal{B}, X \rangle \rangle, \\ \langle \mathcal{I}, X \rangle \simeq X, \end{cases}$$

which satisfy the commutative duals of the diagrams above.

Definition 4 (category tensored-cotensored-enriched over a monoidal category) Let E be a monoidal category and let C be a category. We say that C is *tensored-cotensored-enriched* over E if there exist three functors

$$\{-, -\} \colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{E}, \quad - \lhd - \colon \mathsf{C} \times \mathsf{E} \to \mathsf{C}, \quad \langle -, - \rangle \colon \mathsf{E}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{C},$$

together with functorial isomorphisms

$$\hom_{\mathsf{C}}(X \lhd \mathcal{A}, Y) \simeq \hom_{\mathsf{E}}(\mathcal{A}, \{X, Y\}) \simeq \hom_{\mathsf{C}}(X, \langle \mathcal{A}, Y \rangle)$$

for any $X, Y \in \mathsf{C}$, any $\mathcal{A}, \mathcal{B} \in \mathsf{E}$ and where $\mathcal{I}$ is the monoidal unit of E, such that $- \lhd -$ defines a right action of E on C.

The axioms and terminology of these notions are justified by the following proposition:

Proposition 5 *If the category C is tensored-cotensored-enriched over E, then it is enriched in the usual sense and the functor $\langle -, - \rangle$ is a left coaction in the sense of Definition 3.*

Proof Suppose that the category C is tensored-cotensored-enriched over E. On the one hand, let us define the composition relative to the enrichment $\{-, -\}$. For any objects

X and Y of C, the identity morphism of $\{X,Y\}$ defines a morphism $X \triangleleft \{X,Y\} \to Y$. So, for any objects X, Y and Z, we have a map

$$X \triangleleft (\{X,Y\} \otimes \{Y,Z\}) \simeq (X \triangleleft \{X,Y\}) \triangleleft \{Y,Z\} \to Y \triangleleft \{Y,Z\} \to Z$$

and hence a map $\{X,Y\} \otimes \{Y,Z\} \to \{X,Z\}$. Thus is defined the composition. The coherence diagrams of Definition 3 ensure us that the composition is associative and gives us a unit. On the other hand, let us show that the functor $\langle -,- \rangle$ is a left coaction. For any $X, Y \in \mathsf{C}$ and any $\mathcal{A}, \mathcal{B} \in \mathsf{E}$, we have functorial isomorphisms

$$\begin{aligned}\hom_{\mathsf{C}}(X, \langle \mathcal{A} \otimes \mathcal{B}, Y \rangle) \simeq \hom_{\mathsf{C}}(X \triangleleft (\mathcal{A} \otimes \mathcal{B}), Y) &\simeq \hom_{\mathsf{C}}((X \triangleleft \mathcal{A}) \triangleleft \mathcal{B}, Y) \\ &\simeq \hom_{\mathsf{C}}(X \triangleleft \mathcal{A}, \langle \mathcal{B}, Y \rangle) \simeq \hom_{\mathsf{C}}(X, \langle \mathcal{A} \langle \mathcal{B}, Y \rangle \rangle).\end{aligned}$$

By the Yoneda lemma, this gives us a functorial isomorphism $\langle \mathcal{A} \otimes \mathcal{B}, Y \rangle \simeq \langle \mathcal{A} \langle \mathcal{B}, Y \rangle \rangle$. This functorial isomorphism satisfies the coherence conditions of Definition 3 because the functorial isomorphism $X \triangleleft (\mathcal{A} \otimes \mathcal{B}) \simeq (X \triangleleft \mathcal{A}) \triangleleft \mathcal{B}$ satisfies the coherence conditions of the same definition. □

Proposition 6 *Let E be a presentable monoidal category and let C be a presentable category.*

- *Suppose that there exists a right action $- \triangleleft -$ of E on C and that for any $\mathcal{A} \in \mathsf{E}$ and for any $X \in \mathsf{C}$, the functors $X \triangleleft -\colon \mathsf{E} \to \mathsf{C}$ and $- \triangleleft \mathcal{A}\colon \mathsf{C} \to \mathsf{C}$ preserve colimits. Then C is tensored-cotensored-enriched over E.*
- *Suppose that there exists a left coaction $\langle -,- \rangle$ of E on C and that there exists a functor*

  $$- \triangleleft -\colon \mathsf{C} \times \mathsf{E} \to \mathsf{C}$$

  together with a functorial isomorphism

  $$\hom_{\mathsf{C}}(X \triangleleft \mathcal{A}, Y) \simeq \hom_{\mathsf{C}}(X, \langle \mathcal{A}, Y \rangle).$$

  Suppose moreover that the functor $\langle -, Y \rangle\colon \mathsf{E}^{\mathrm{op}} \to \mathsf{C}$ sends colimits in E to limits. Then C is tensored-cotensored-enriched over E.

Proof The first point is a direct consequence of Proposition 2. Let us prove the second point. Since E left coacts on C, by the same arguments as in the proof of Proposition 5 we can show that the bifunctor $- \triangleleft -$ is a right action of E on C. Moreover, since the functors $\langle -, Y \rangle$ preserve limits, any functor of the form $X \triangleleft -$ preserves colimits. The result is then a direct consequence of the first point. □

Definition 7 (homotopical enrichment) Let M be a model category and let E be a model category with a monoidal structure. We say that M is homotopically enriched over E if it is enriched over E and if for any cofibration $f\colon X \to X'$ in M and any fibration $g\colon Y \to Y'$ in M, the morphism in E

$$\{X', Y\} \to \{X', Y'\} \times_{\{X, Y'\}} \{X, Y\}$$

is a fibration. Moreover, we require this morphism to be a weak equivalence whenever f or g is a weak equivalence.

This definition implies in particular that the homotopy category $\mathrm{Ho}(\mathsf{M})$ is enriched over the monoidal category $\mathrm{Ho}(\mathsf{E})$.

1.3 Coalgebras

Definition 8 (coalgebras) A *coassociative coalgebra* $\mathscr{C} = (C, \Delta, \epsilon)$ is a chain complex C equipped with a coassociative coproduct $\Delta\colon C \to C \otimes C$ and a counit $\epsilon\colon C \to \mathbb{K}$ such that $\mathrm{Id}_C = (\mathrm{Id}_C \otimes \epsilon)\Delta = (\epsilon \otimes \mathrm{Id}_C)\Delta$. The kernel of the map ϵ is denoted by $\bar{C}$. The coalgebra $\mathscr{C}$ is called cocommutative if $\Delta = \tau\Delta$, where

$$\tau(x \otimes y) = (-1)^{|x||y|} y \otimes x.$$

A *graded* atom is a nonzero element $1 \in C$ such that $\Delta 1 = 1 \otimes 1$. In this context, let us define the map $\bar{\Delta}\colon \bar{C} \to \bar{C} \otimes \bar{C}$ by

$$\bar{\Delta}x := \Delta x - 1 \otimes x - x \otimes 1 \in \bar{C} \otimes \bar{C}.$$

A graded atom 1 is called a *dg atom* if $d1 = 0$. A *conilpotent coalgebra* $\mathscr{C} = (C, \Delta, \epsilon, 1)$ is the data of a coassociative coalgebra (C, Δ, ϵ) together with a graded atom such that, for any $x \in \bar{C}$, there exists an integer n such that

$$\bar{\Delta}^n x := (\mathrm{Id}_C^{\otimes n-1} \otimes \bar{\Delta}) \cdots (\mathrm{Id}_C \otimes \bar{\Delta})\bar{\Delta}(x) = 0.$$

A conilpotent cocommutative coalgebra $\mathscr{C}$ is said to be a *Hinich coalgebra* if 1 is a dg atom. We denote by uCog be the category of coassociative coalgebras and by uCocom the category of cocommutative coalgebras. Let $\mathsf{uNilCocom}$ (resp. $\mathsf{Hinich\text{–}cog}$) be the category whose objects are conilpotent cocommutative coalgebras (resp. Hinich coalgebras) and whose morphisms are morphisms of coalgebras.

Remark 9 The reader may be familiar with the notion of a coaugmented coalgebra. This is actually exactly the data of a coassociative coalgebra together with a dg atom.

Indeed, the data of a dg atom of a coalgebra $\mathscr{C}$ is equivalent to the data of a morphism of dg coalgebras from $\mathbb{K}$ to $\mathscr{C}$.

Any conilpotent coalgebra $\mathscr{C}$ has a canonical filtration, called the coradical filtration,

$$F_n^{\mathrm{rad}}\mathcal{C} := \mathbb{K}\cdot 1 \oplus \{x \in \overline{\mathcal{C}} \mid \overline{\Delta}^{n+1}x = 0\},$$

which is not necessarily stable under the codifferential d.

Proposition 10 *Let f be a morphism of coalgebras between two conilpotent coalgebras $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon, 1)$ and $\mathscr{D} = (\mathcal{D}, \Delta', \epsilon', 1')$. Then $f(1) = 1'$.*

Proof Let $x \in \overline{\mathcal{D}}$ be such that $f(1) = 1' + x$. Since $\Delta f(1) = (f \otimes f)\Delta(1)$, then $\overline{\Delta}x = x \otimes x$. Since there exists an integer n such that $\overline{\Delta}^n(x) = x \otimes \cdots \otimes x = 0$, then $x = 0$. □

Proposition 11 *The categories* uCog, uCocom, $\mathsf{uNilCocom}$ *and* $\mathsf{Hinich\text{–}cog}$ *are presentable. The forgetful functor from* uCog *to the category of chain complexes has a right adjoint called the cofree counital coalgebra functor. The same statement holds for the category* uCocom. *The functor $\mathscr{C} \mapsto \overline{\mathcal{C}}$ from the category* $\mathsf{Hinich\text{–}cog}$ *to the category of chain complexes has a right adjoint. The tensor product of the category of chain complexes induces closed symmetric monoidal structures on the categories* uCog *and* uCocom.

Proof The results are proven in [2, Sections 2.1, 2.2 and 2.5] for the category uCog. The methods used apply mutatis mutandis for the other categories. □

Theorem 12 [11] *The full subcategory* $\mathsf{uCog}^{\geq 0}$ *of* uCog *made up of nonnegatively graded coalgebras admits a model structure whose cofibrations are the monomorphisms and whose weak equivalences are the quasi-isomorphisms.*

The category $\mathsf{Hinich\text{–}cog}$ is related to the category of Lie-algebras by an adjunction, described in [24],

$$\mathsf{Hinich\text{–}cog} \underset{\mathcal{C}}{\overset{\mathcal{L}}{\rightleftarrows}} \mathsf{Lie\text{–}alg}.$$

Theorem 13 [16] *Suppose the characteristic of the base field $\mathbb{K}$ is zero. Then there exists a model structure on the category* $\mathsf{Hinich\text{–}cog}$ *whose cofibrations are monomorphisms and whose weak equivalences are morphisms whose image under the functor $\mathcal{L}$*

is a quasi-isomorphism. The class of weak equivalences is contained in the class of quasi-isomorphisms. Moreover, the adjunction $\mathcal{L} \dashv \mathcal{C}$ is a Quillen equivalence when the category of Lie algebras is equipped with its projective model structure whose fibrations (resp. weak equivalences) are surjections (resp. quasi-isomorphisms) (see [15]).

Definition 14 (deformation problems) Let $\mathsf{Artin\text{–}alg}$ be the category of nonpositively graded local finite-dimensional dg commutative algebras. A deformation problem is a functor from the category $\mathsf{Artin\text{–}alg}$ to the category of simplicial sets.

Lurie showed in [20] that a suitable infinity-category of deformation problems (called formal moduli problems) is equivalent to the infinity-category of Lie algebras if the characteristic of the base field $\mathbb{K}$ is zero. Therefore, it is equivalent to the infinity-category of Hinich coalgebras. In that perspective, any Hinich coalgebra $\mathscr{C}$ induces a deformation problem as follows:

$$R \mapsto \mathrm{Map}_{\mathsf{Hinich\text{–}cog}}(R^*, \mathscr{C}) \quad \text{for } R \in \mathsf{Artin\text{–}alg}.$$

Remark 15 We use Hinich's definition of a deformation problem given in [16]. We do not describe here the homotopy theory of such deformation problems nor a precise link with the work of Lurie, who uses the framework of quasicategories (see [20]). In the sequel, we will only use the fact that, for any morphism of deformation problems $f\colon X \to Y$, if $f(R)$ is a weak equivalence of simplicial sets for any algebra $R \in \mathsf{Artin\text{–}alg}$, then f is an equivalence of deformation problems.

1.4 Coalgebras and simplicial sets

In this subsection, we describe a Quillen adjunction between the category of simplicial sets and the category of coassociative coalgebras. This adjunction is part of the Dold–Kan correspondence. From a simplicial set X, one can produce a chain complex $\mathrm{DK}(X)$, called the normalized Moore complex. In degree n, $\mathrm{DK}(X)_n$ is the subvector space of $\mathbb{K} \cdot X_n$ which is the intersection of the kernels of the faces $d_0, \ldots, d_{n-1}$. The differential is $(-1)^n d_n$. Moreover, the Alexander–Whitney map makes the functor DK comonoidal. Then the diagonal map $X \to X \times X$ gives to $\mathrm{DK}(X)$ a structure of coalgebras. Thus, we have a functor DK^c from simplicial sets to the category uCog of coassociative coalgebras. This functor DK^c admits a right adjoint N defined by

$$N(\mathcal{C})_n := \hom_{\mathsf{uCog}}(\mathrm{DK}^c(\Delta[n]), \mathcal{C}).$$

Actually, we have the sequence of adjunctions

$$\mathsf{sSet} \underset{N}{\overset{\mathrm{DK}^c}{\rightleftarrows}} \mathsf{uCog}^{\geq 0} \underset{\mathrm{tr}}{\overset{\mathrm{in}}{\rightleftarrows}} \mathsf{uCog},$$

where in is the embedding of $\mathsf{uCog}^{\geq 0}$ into uCog and where tr is the truncation.

Proposition 16 *The above adjunction between* $\mathsf{uCog}^{\geq 0}$ *and* sSet *is a Quillen adjunction.*

Proof The functor DK^c carries monomorphisms to monomorphisms and weak homotopy equivalences to quasi-isomorphisms; see [13, III.2]. □

1.5 The Sullivan algebras of polynomial forms on standard simplices

Definition 17 (Sullivan polynomial algebras [25]) For any integer $n \in \mathbb{N}$, the n^{th} *algebra of polynomial forms* is the differential graded unital commutative algebra

$$\Omega_n := \mathbb{K}[t_0, \dots, t_n, dt_0, \dots, dt_n]/(\Sigma t_i = 1),$$

where the degree of t_i is zero and where $d_{\Omega_n}(t_i) = dt_i$. In particular, $\sum dt_i = 0$.

Any map of finite ordinals $\phi\colon [n] \to [m]$ defines a morphism of differential graded unital commutative algebra

$$\Omega(\phi)\colon \Omega_m \to \Omega_n, \quad t_i \mapsto \sum_{\phi(j)=i} t_j.$$

Therefore, the collection $\{\Omega_n\}_{n\in\mathbb{N}}$ defines a simplicial differential graded commutative algebra. Moreover, one can extend this construction to a contravariant functor $\Omega_\bullet$ from simplicial sets to differential graded unital commutative algebras such that $\Omega_{\Delta[n]} = \Omega_n$. This functor is part of an adjunction

$$\mathsf{sSet} \overset{\Omega_\bullet}{\rightleftarrows} \mathit{uCom}-\mathsf{alg}^{\mathrm{op}}.$$

Proposition 18 [6, Chapter 8] *When the characteristic of the field* $\mathbb{K}$ *is zero, the category* uCom–alg *of differential graded unital commutative algebras admits a projective model structure where fibrations (resp. weak equivalences) are degreewise surjections (resp. quasi-isomorphisms). In that context, the adjunction between simplicial sets and* uCom–alg *is a Quillen adjunction.*

2 Operads, cooperads, algebras and coalgebras

The purpose of this section is to recall the definitions of operads, cooperads, algebras over an operad and coalgebras over a cooperad that we will use in the sequel; we refer the reader to [19]. Moreover, we prove that the category of coalgebras over a curved cooperad is presentable.

2.1 Operads and cooperads

We recall here the definitions of operads and cooperads. We refer to [19; 17].

Definition 19 (symmetric modules) Let $\mathbb{S}$ be the groupoid whose objects are integers $n \in \mathbb{N}$ and whose morphisms are

$$\begin{cases} \hom_{\mathbb{S}}(n,m) = \varnothing & \text{if } n \neq m, \\ \hom_{\mathbb{S}}(n,n) = \mathbb{S}_n & \text{otherwise.} \end{cases}$$

A graded $\mathbb{S}$–module (resp. dg $\mathbb{S}$–module) is a presheaf on $\mathbb{S}$ valued in the category of graded $\mathbb{K}$–modules (resp. chain complexes). The name $\mathbb{S}$–module will refer both to graded $\mathbb{S}$–modules and dg $\mathbb{S}$–modules. We say that a $\mathbb{S}$–module $\mathcal{V}$ is reduced if $\mathcal{V}(0) = \{0\}$.

The category of $\mathbb{S}$–modules has a monoidal structure which is as follows: for any $\mathbb{S}$–modules $\mathcal{V}$ and $\mathcal{W}$, and for any $n \geq 1$,

$$(\mathcal{V} \circ \mathcal{W})(n) := \bigoplus_{k \geq 1} \mathcal{V}(k) \otimes_{\mathbb{S}_k} \Bigg(\bigoplus_{X_1 \sqcup \cdots \sqcup X_k = \{1,\ldots,n\}} \mathcal{W}(\# X_1) \otimes \cdots \otimes \mathcal{W}(\# X_k) \Bigg),$$

where $\# X_i$ is the cardinal of the set X_i. For $n = 0$,

$$(\mathcal{V} \circ \mathcal{W})(0) := \mathcal{V}(0) \oplus \Bigg(\bigoplus_{k \geq 1} \mathcal{V}(k) \otimes_{\mathbb{S}_k} (\mathcal{W}(0) \otimes \cdots \otimes \mathcal{W}(0)) \Bigg).$$

The monoidal unit is given by the $\mathbb{S}$–module $\mathcal{I}$ which is $\mathbb{K}$ in arity 1 and $\{0\}$ in other arities.

Notation

- For any dg $\mathbb{S}$–module $\mathcal{V}$, we will denote by $\mathcal{V}^{\mathrm{grad}}$ the underlying graded $\mathbb{S}$–module.
- Let $f\colon \mathcal{V} \to \mathcal{V}'$ and $g\colon \mathcal{W} \to \mathcal{W}'$ and $h\colon \mathcal{W} \to \mathcal{W}'$ be three morphisms of $\mathbb{S}$–modules. Then we denote by $f \circ (g; h)$ the map from $\mathcal{V} \circ \mathcal{W}$ to $\mathcal{V}' \circ \mathcal{W}'$ defined

as follows:

$$f \circ (g; h) := \sum_{i+j=n-1} f \otimes_{\mathbb{S}_n} (g^{\otimes i} \otimes h \otimes g^{\otimes j}).$$

In the case where g is the identity, we use the notation $f \circ' h$:

$$f \circ' h := f \circ (\mathrm{Id}; h).$$

- For any two graded $\mathbb{S}$–modules (resp. dg $\mathbb{S}$–modules) $\mathcal{V}$ and $\mathcal{W}$, we denote by $[\mathcal{V}, \mathcal{W}]$ the graded $\mathbb{K}$–module (resp. chain complex)

$$[\mathcal{V}, \mathcal{W}]_n := \prod_{\substack{k \geq 0 \\ l \in \mathbb{N}}} \hom_{\mathbb{K}[\mathbb{S}_n]}(\mathcal{V}(k)_l, \mathcal{W}(k)_{l+n}).$$

In that context morphisms of chain complexes from X to $[\mathcal{V}, \mathcal{W}]$ are in one-to-one correspondence with morphisms of $\mathbb{S}$–modules from the aritywise tensor product $X \otimes \mathcal{V}$ to $\mathcal{W}$.

Proposition 20 [19, Chapter 6] *If the characteristic of the field* $\mathbb{K}$ *is zero, then the operadic Künneth*

$$H(\mathcal{V} \circ \mathcal{W}) \simeq H(\mathcal{V}) \circ H(\mathcal{W})$$

holds for any dg $\mathbb{S}$*–modules* $\mathcal{V}$ *and* $\mathcal{W}$*, where* H *denotes the homology.*

Definition 21 (operads) A graded operad $\mathscr{P} = (\mathcal{P}, \gamma, 1)$ (resp. dg operad) is a monoid in the category of graded $\mathbb{S}$–modules (resp. dg $\mathbb{S}$–modules). We denote by Operad the category of dg operads.

Example 22 For any graded $\mathbb{K}$–module (resp. chain complex) $\mathcal{V}$, $\mathrm{End}_{\mathcal{V}}$ is the graded operad (resp. dg operad) defined by

$$\mathrm{End}_{\mathcal{V}}(n) := \hom(\mathcal{V}^{\otimes n}, \mathcal{V}).$$

The composition in the operad $\mathrm{End}_{\mathcal{V}}$ is given by the composition of morphisms of graded $\mathbb{K}$–modules (resp. chain complexes).

A degree k derivation d on a graded operad $\mathscr{P} = (\mathcal{P}, \gamma, 1)$ is the data of degree k maps $d\colon \mathcal{P}(n) \to \mathcal{P}(n)$ which commute with the action of $\mathbb{S}_n$ and such that

$$d\gamma = \gamma(d \circ \mathrm{Id} + \mathrm{Id} \circ' d).$$

Proposition 23 [19, Chapter 5] *The forgetful functor from operads to $\mathbb{S}$–modules has a left adjoint called the free operad functor and denoted by $\mathbb{T}$. For any $\mathbb{S}$–module $\mathcal{V}$, $\mathbb{T}\mathcal{V}$ is the $\mathbb{S}$–module made up of trees whose vertices are filled with elements of $\mathcal{V}$ with coherent arity. The composition is given by the grafting of trees.*

There is a one-to-one correspondence between the degree k derivation on the graded free operad $\mathbb{T}\mathcal{V}$ and the degree k maps from $\mathcal{V}$ to $\mathbb{T}\mathcal{V}$. Indeed, from such a map u one can produce the derivation D_u such that, for any tree T labeled by elements of $\mathcal{V}$,

$$D_u(T) := \sum_v \mathrm{Id} \otimes \cdots \otimes u(v) \otimes \cdots \otimes \mathrm{Id},$$

where the sum is taken over the vertices of the tree T.

Definition 24 (cooperads) A cooperad $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon)$ is a comonoid in the category of $\mathbb{S}$–modules. We denote by $\bar{\mathcal{C}}$ the kernel of the morphism $\epsilon : \mathcal{C} \to \mathcal{I}$. A cooperad $\mathscr{C}$ is said to be coaugmented if it is equipped with a morphism of cooperads $\mathcal{I} \to \mathscr{C}$. In this case, we denote by 1 the image of the unit of $\mathbb{K}$ into $\mathcal{C}(1)$. A coaugmented cooperad $\mathscr{C}$ is said to be conilpotent if the process of successive decomposition stabilizes in finite time for any element. A precise definition is given in [19, Section 5.8.6].

The forgetful functor from conilpotent cooperads to $\mathbb{S}$–modules which sends $\mathscr{C}$ to $\bar{\mathcal{C}}$ has a right adjoint sending $\mathcal{V}$ to the tree module $\mathbb{T}(\mathcal{V})$ with the decomposition given by the degrafting of trees. We denote it by $\mathbb{T}^c(\mathcal{V})$. We also denote by $\delta\colon \mathscr{C} \to \mathbb{T}^c(\bar{\mathcal{C}})$ the counit of the adjunction. Any conilpotent cooperad is equipped with a filtration, called the coradical filtration,

$$F_n^{\mathrm{rad}}\mathcal{C}(m) := \{p \in \mathcal{C}(m) \mid \delta(p) \in \mathbb{T}^{\leq n}(\bar{\mathcal{C}})(m)\},$$

where the symbol $\mathbb{T}^{\leq n}$ denotes the trees with at most n vertices. In particular, $F_0^{\mathrm{rad}}\mathcal{C} = \mathcal{I}$.

Notation Let $\mathcal{C}$ be coaugmented cooperad and m be an integer. We denote by Δ_m the composite map

$$\Delta_m\colon \mathcal{C} \xrightarrow{\Delta} \mathcal{C} \circ \mathcal{C} \to \mathbb{T}(\bar{\mathcal{C}}) \twoheadrightarrow \mathbb{T}^m(\bar{\mathcal{C}}).$$

A degree k coderivation on a cooperad $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon)$ is a degree k map d of $\mathbb{S}$–modules from $\mathcal{C}$ to $\mathcal{C}$ such that

$$\Delta d = (d \circ \mathrm{Id} + \mathrm{Id} \circ' d)\Delta.$$

If the cooperad is coaugmented, we also require that $d(1) = 0$. Let $\mathbb{T}^c(\mathcal{V})$ be a cofree conilpotent cooperad. There is a one-to-one correspondence between degree k coderivations on $\mathbb{T}^c(\mathcal{V})$ and degree k maps from $\overline{\mathbb{T}}(\mathcal{V})$ to $\mathcal{V}$. Indeed, such a map u is uniquely extended by the following coderivation D_u, defined on any tree T labeled by elements of $\mathcal{V}$ as follows:

$$D_u(T) := \sum_{T' \subset T} \mathrm{Id} \otimes \cdots \otimes u(T') \otimes \cdots \otimes \mathrm{Id},$$

where the sum is taken on the subtrees T' of T.

Definition 25 (curved cooperads) A curved cooperad $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon, 1, d, \theta)$ is a coaugmented graded cooperad equipped with a degree -2 map of graded $\mathbb{S}$–modules $\theta\colon \mathcal{C} \to \mathcal{I}$ and a degree -1 coderivation d such that

$$d^2 = (\theta \otimes \mathrm{Id} - \mathrm{Id} \otimes \theta)\Delta_2, \quad \theta d = 0.$$

A morphism of curved cooperads is a morphism of cooperads $\phi\colon \mathscr{C} \to \mathscr{D}$ which commutes with the coderivations and such that $\theta_{\mathscr{C}} = \theta_{\mathscr{D}}\phi$. We denote by cCoop the category of curved conilpotent cooperads.

The coradical filtration of a conilpotent cooperad has the following property with respect to the decomposition map:

Lemma 26 *Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon, 1)$ be a conilpotent cooperad. Then*

$$\Delta(F_n^{\mathrm{rad}}\mathcal{C}) \subset \sum_{p_0+\cdots+p_k \leq n} (F_{p_0}^{\mathrm{rad}}\mathcal{C})(k) \otimes_{\mathbb{S}_k} (F_{p_1}^{\mathrm{rad}}\mathcal{C} \otimes \cdots \otimes F_{p_k}^{\mathrm{rad}}\mathcal{C}).$$

Proof It suffices to prove the result for cofree cooperads. Indeed, any conilpotent cooperad $\mathscr{C}$ is equipped with a map $\delta\colon \mathcal{C} \to \mathbb{T}^c(\bar{\mathcal{C}})$ such that $F_n^{\mathrm{rad}}\mathcal{C} = \delta^{-1}(F_n^{\mathrm{rad}}\mathbb{T}^c(\bar{\mathcal{C}}))$. □

Lemma 27 *Let $\mathscr{C} = \mathbb{T}^c(\mathcal{V})$ be a cofree conilpotent graded cooperad equipped with a degree -2 map $\theta\colon \mathbb{T}(\mathcal{V})(1) \twoheadrightarrow \mathcal{V}(1) \to \mathbb{K}$. Let $\phi\colon \overline{\mathbb{T}}\mathcal{V} \to \mathcal{V}$ be a degree -1 map and let D_ϕ be the corresponding coderivation on $\mathscr{C}$. Then the triple $(\mathbb{T}^c\mathcal{V}, D_\phi, \theta)$ is a curved cooperad if and only if ϕ satisfies the equation*

$$\phi D_\phi = (\theta \otimes \pi_{\mathcal{V}} - \pi_{\mathcal{V}} \otimes \theta)\Delta_2,$$

where $\pi_{\mathcal{V}}$ is the projection $\mathbb{T}(\mathcal{V}) \to \mathcal{V}$.

Proof If $(\mathbb{T}^c\mathcal{V}, D_\phi, \theta)$ is a curved cooperad, then $\phi D_\phi = \pi_\mathcal{V} D_\phi^2 = (\theta\otimes\pi_\mathcal{V} - \pi_\mathcal{V}\otimes\theta)\Delta_2$. Conversely, suppose that $\phi D_\phi = (\theta\otimes\pi_\mathcal{V} - \pi_\mathcal{V}\otimes\theta)\Delta_2$. For any tree T labeled by elements of $\mathcal{V}$, one can prove that

$$D_\phi^2(T) = \sum_{T'\subset T} \mathrm{Id}\otimes(\phi D_\phi(T'))\otimes \mathrm{Id}.$$

Actually, it is the sum over every arity 1 vertex v of

- $\pm\theta(v)(T-v)$ if v is the bottom vertex or a top vertex;
- $\pm(\theta(v)(T-v) - \theta(v)(T-v)) = 0$ otherwise.

Hence, $(\mathbb{T}^c\mathcal{V}, D_\phi, \theta)$ is a curved cooperad. □

There exist notions of $\mathbb{N}$–modules, nonsymmetric operads, nonsymmetric cooperads and their morphisms, defined for instance in [19, Section 5.9]. We will speak about the nonsymmetric context to refer to these ones. Notice that the operadic Künneth formula holds in the nonsymmetric context without the assumption that the characteristic of the field $\mathbb{K}$ is zero.

2.2 Modules and algebras over an operad

Definition 28 (algebras over an operad) Let $\mathscr{P} = (\mathcal{P}, \gamma, 1)$ be an operad. A $\mathscr{P}$–*module* $\mathscr{A} = (\mathcal{A}, \gamma_\mathscr{A})$ is a left module in the category of $\mathbb{S}$–module, that is, an $\mathbb{S}$–module $\mathcal{A}$ equipped with a map $\gamma_\mathscr{A}\colon \mathcal{P}\circ\mathcal{A}\to\mathcal{A}$ such that the following diagrams commute:

$$\begin{tikzcd} \mathcal{P}\circ\mathcal{P}\circ\mathcal{A} \arrow[r, "\mathrm{Id}\circ\gamma_\mathscr{A}"] \arrow[d, "\gamma\circ\mathrm{Id}"'] & \mathcal{P}\circ\mathcal{A} \arrow[d, "\gamma_\mathscr{A}"] \\ \mathcal{P}\circ\mathcal{A} \arrow[r, "\gamma_\mathscr{A}"'] & \mathcal{A} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{I}\circ\mathcal{A} \arrow[r, "1\circ\mathrm{Id}"] \arrow[rr, bend right, "\mathrm{Id}"'] & \mathcal{P}\circ\mathcal{A} \arrow[r, "\gamma_\mathcal{A}"] & \mathcal{A} \end{tikzcd}$$

A morphism of $\mathscr{P}$–modules from $\mathscr{A}$ to $\mathscr{B} = (\mathcal{B}, \gamma_\mathscr{B})$ is a morphism of $\mathbb{S}$–modules $f\colon \mathcal{A}\to\mathcal{B}$ such that $\gamma_\mathscr{B}(\mathrm{Id}\circ f) = f\gamma_\mathscr{A}$. A $\mathscr{P}$–*algebra* is a $\mathscr{P}$–module $\mathcal{A}$ concentrated in arity 0. We denote by $\mathscr{P}$–alg the category of $\mathscr{P}$–algebras.

The forgetful functor from the category of $\mathscr{P}$–modules to the category of $\mathbb{S}$–modules has a left adjoint given by

$$\mathcal{V}\mapsto\mathcal{P}\circ\mathcal{V}.$$

The images of this left adjoint functor are called the free $\mathscr{P}$–modules.

Definition 29 (ideal) An *ideal* of a $\mathscr{P}$–module $\mathscr{A}$ is a sub-$\mathbb{S}$–module $\mathcal{B} \subset \mathcal{A}$ such that, for any $p \in \mathcal{P}(n)$ and $(x_i)_{i=1}^n \in \mathcal{A}(k_i)$ with $n \geq 1$,

$$\gamma_{\mathscr{A}}(p \otimes_{\mathbb{S}_n} (x_1 \otimes \cdots \otimes x_n)) \in \mathcal{B}(k_1 + \cdots k_n)$$

whenever one of the x_i is in $\mathcal{B}$ (for $n \geq 1$). Then the quotient $\mathcal{A}/\mathcal{B}$ has an induced structure of $\mathscr{P}$–module.

Definition 30 (derivation) Let $\mathscr{P}$ be a graded operad and let $\mathscr{A}$ be a $\mathscr{P}$–module. Suppose that the graded operad $\mathscr{P}$ is equipped with a degree k derivation $d_{\mathscr{P}}$. Then a *derivation* of $\mathscr{A}$ is a degree k map $d_{\mathscr{A}}$ from $\mathcal{A}$ to $\mathcal{A}$ such that

$$d_{\mathscr{A}}\gamma_{\mathscr{A}} = \gamma_{\mathscr{A}}(d_{\mathscr{P}} \circ \mathrm{Id}_{\mathscr{A}} + \mathrm{Id} \circ' d_{\mathscr{A}}).$$

Let $\mathscr{P}$ be a graded operad equipped with a degree k derivation $d_{\mathscr{P}}$. There is a one-to-one correspondence between the derivations of a free $\mathscr{P}$–module $\mathscr{A} = \mathcal{P} \circ \mathcal{V}$ and the degree k maps $\mathcal{V} \to \mathcal{P} \circ \mathcal{V}$. Indeed, any such map $u\colon \mathcal{V} \to \mathcal{P} \circ \mathcal{V}$ is uniquely extended by the derivation

$$D_u = d_{\mathscr{P}} \circ \mathrm{Id} + \mathrm{Id} \circ (i\,; u),$$

where i denotes the canonical inclusion map $\mathcal{V} \to \mathcal{P} \circ \mathcal{V}$.

2.3 Comodules and coalgebras over a cooperad

Definition 31 (comodules and coalgebras over a cooperad) Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon)$ be a cooperad. A *$\mathscr{C}$–comodule* $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ is a left $\mathscr{C}$–comodule in the category of $\mathbb{S}$–modules, that is a $\mathbb{S}$–module $\mathcal{D}$ together with a morphism $\Delta_{\mathscr{D}}\colon \mathcal{D} \to \mathcal{C} \circ \mathcal{D}$ such that the following diagrams commute:

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\Delta_{\mathscr{D}}"] \arrow[d, "\Delta_{\mathscr{D}}"'] & \mathcal{C} \circ \mathcal{D} \arrow[d, "\mathrm{Id} \circ \Delta_{\mathscr{D}}"] \\ \mathcal{C} \circ \mathcal{D} \arrow[r, "\Delta_{\mathscr{C}} \circ \mathrm{Id}"'] & \mathcal{C} \circ \mathcal{C} \circ \mathcal{D} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D} \arrow[r, "\Delta_{\mathscr{D}}"] \arrow[rr, bend right, "\mathrm{Id}"'] & \mathcal{C} \circ \mathcal{D} \arrow[r, "\epsilon \circ \mathrm{Id}"] & \mathcal{D} \end{tikzcd}$$

A *$\mathscr{C}$–coalgebra* is a $\mathscr{C}$–comodule concentrated in arity 0.

Remark 32 Our notion of $\mathscr{C}$–coalgebra actually recovers a notion sometimes called in the literature conilpotent $\mathscr{C}$–coalgebra; see [19, 5.4.8].

Let $\mathscr{C}$ be a coaugmented cooperad. Then the forgetful functor from the category of $\mathscr{C}$–comodules to the category of $\mathbb{S}$–modules has a right adjoint which sends $\mathcal{V}$ to $\mathscr{C} \circ \mathcal{V}$. The images of the right adjoint are called the cofree $\mathscr{C}$–comodules.

Definition 33 (coderivation) Let $\mathscr{C}$ be a graded cooperad and let $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ be a $\mathscr{C}$–comodule. Suppose that $\mathscr{C}$ is equipped with a degree k coderivation $d_{\mathscr{C}}$. A *coderivation* on $\mathscr{D}$ is a degree k map $d_{\mathscr{D}}$ from $\mathcal{D}$ to $\mathcal{D}$ such that

$$\Delta_{\mathscr{D}} d_{\mathscr{D}} = (d_{\mathscr{C}} \circ \mathrm{Id} + \mathrm{Id} \circ' d_{\mathscr{D}})\Delta_{\mathscr{D}}.$$

Let $\mathscr{C}$ be a cooperad equipped with a degree k coderivation and let $\mathcal{V}$ be a graded $\mathbb{K}$–module. Then there is a one-to-one correspondence between the coderivations on the $\mathscr{C}$–coalgebra $\mathscr{C} \circ \mathcal{V}$ and the degree k maps $\mathscr{C} \circ \mathcal{V} \to \mathcal{V}$. Indeed, any such map u induces the coderivation

$$D_u := (d_{\mathscr{C}} \circ \mathrm{Id}_{\mathcal{V}}) + (\mathrm{Id} \circ (\pi; u))(\Delta_{\mathscr{C}} \circ \mathrm{Id}_{\mathcal{V}}),$$

where $\pi = \epsilon \circ \mathrm{Id}\colon \mathscr{C} \circ \mathcal{V} \to \mathcal{V}$.

Definition 34 (comodules and coalgebras over a curved cooperad) Let $\mathscr{C}$ be a curved cooperad. A *$\mathscr{C}$–comodule* is a graded $\mathscr{C}^{\mathrm{grad}}$–comodule $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ together with a coderivation $d_{\mathcal{D}}$ such that

$$d_{\mathcal{D}}^2 = (\theta_{\mathscr{C}} \circ \mathrm{Id})\Delta_{\mathscr{D}}.$$

Moreover, a *$\mathscr{C}$–coalgebra* is a $\mathscr{C}$–comodule concentrated in arity 0.

Proposition 35 *Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon, 1, d, \theta)$ be a conilpotent curved cooperad and let $\mathcal{V}$ be a graded $\mathbb{S}$–module. There is a one-to-one correspondence between the degree -1 maps $\phi\colon \mathcal{C} \circ \mathcal{V} \to \mathcal{V}$ such that*

$$\phi D_\phi := \phi(\mathrm{Id} \circ (\pi; \phi))(\Delta_{\mathscr{C}} \circ \mathrm{Id}_{\mathcal{V}}) + \phi(d_{\mathcal{C}} \circ \mathrm{Id}_{\mathcal{V}}) = \theta \circ \mathrm{Id}_{\mathcal{V}}$$

and the structures of $\mathscr{C}$–comodule (where $\mathscr{C}$ is considered as a curved cooperad) *on the graded cofree comodule $\mathscr{C}^{\mathrm{grad}} \circ \mathcal{V}$.*

Proof A structure of $\mathscr{C}$–comodule on $\mathscr{C}^{\mathrm{grad}} \circ \mathcal{V}$ amounts to the data of a degree -1 coderivation D_ϕ such that $D_\phi^2 = (\theta \circ \mathrm{Id}_{\mathscr{C}} \circ \mathrm{Id}_{\mathcal{V}})(\Delta_{\mathscr{C}} \circ \mathrm{Id}_{\mathcal{V}})$. Given this equality, $\phi D_\phi = \theta \circ \mathrm{Id}_{\mathcal{V}}$. Conversely, suppose that $\phi D_\phi = \theta \circ \mathrm{Id}$. We have

$$D_\phi^2 = (d_{\mathscr{C}}^2 \circ \mathrm{Id}_{\mathcal{C}}) + (\mathrm{Id} \circ (\pi; \phi))(\Delta \circ \mathrm{Id})(d_{\mathscr{C}} \circ \mathrm{Id}_{\mathcal{V}}) + (d_{\mathscr{C}} \circ (\pi; \phi))(\Delta \circ \mathrm{Id}) \\ + (\mathrm{Id} \circ (\pi; \phi))(\Delta \circ \mathrm{Id})(\mathrm{Id} \circ (\pi; \phi))(\Delta \circ \mathrm{Id}).$$

On the one hand,

$$\begin{aligned}(\mathrm{Id}\circ(\pi;\phi))(\Delta\circ\mathrm{Id})(\mathrm{Id}\circ(\pi;\phi))(\Delta\circ\mathrm{Id})+(\mathrm{Id}\circ(\pi;\phi))(\Delta\circ\mathrm{Id})(d_{\mathscr{C}}\circ\mathrm{Id}_{\mathcal{C}})&\\ +(d_{\mathscr{C}}\circ(\pi;\phi))(\Delta\circ\mathrm{Id})&\\ =(\mathrm{Id}\circ(\pi;\phi))(\mathrm{Id}\circ' D_\phi)(\Delta\circ\mathrm{Id})&\\ =(\mathrm{Id}\circ(\pi;\phi D_\phi))(\Delta\circ\mathrm{Id})&\\ =\big((\mathrm{Id}\circ(\epsilon;\theta))\Delta\big)\circ\mathrm{Id}.&\end{aligned}$$

On the other hand,

$$(d_{\mathscr{C}}^2\circ\mathrm{Id})=((\theta\circ\mathrm{Id})\Delta)\circ\mathrm{Id}+\Big(\Big(\sum\mathrm{Id}\otimes_{\mathbb{S}}(\epsilon^{\otimes i}\otimes\theta\otimes\epsilon^{\otimes j})\Big)\Delta\Big)\circ\mathrm{Id}.$$

Hence, $D_\phi^2=((\theta\circ\mathrm{Id})\Delta)\circ\mathrm{Id}_{\mathcal{V}}$. □

Definition 36 (coradical filtration) Any $\mathscr{C}$–coalgebra $\mathscr{D}=(\mathcal{D},\Delta_{\mathscr{D}})$ over a conilpotent cooperad $\mathscr{C}$ admits a filtration called the *coradical filtration* and defined as follows:

$$F_n^{\mathrm{rad}}\mathcal{D}:=\{x\in\mathcal{D}\mid\Delta_{\mathscr{D}}(x)\in(F_n^{\mathrm{rad}}\mathcal{C})\circ\mathcal{D}\}.$$

Proposition 37 *Let $\mathscr{C}$ be a conilpotent cooperad and let $\mathscr{D}$ be a $\mathscr{C}$–coalgebra. For any integer n,*

$$\Delta_{\mathscr{D}}(F_n^{\mathrm{rad}}\mathcal{D})\subset\sum_{i_0+i_1+\cdots+i_k=n}(F_{i_0}^{\mathrm{rad}}\mathcal{C})(k)\otimes_{\mathbb{S}_k}(F_{i_1}^{\mathrm{rad}}\mathcal{D}\otimes\cdots\otimes F_{i_k}^{\mathrm{rad}}\mathcal{D}).$$

Lemma 38 *Let $\mathcal{V}$ and $\mathcal{W}$ be two graded $\mathbb{K}$–modules equipped with filtrations $(F_n\mathcal{V})_{n\in\mathbb{N}}$ and $(F_n\mathcal{W})_{n\in\mathbb{N}}$, and let $\phi\colon\mathcal{V}\to\mathcal{W}$ be an injection such that $F_n\mathcal{V}=\phi^{-1}(F_n\mathcal{W})$ for any integer n. Then there exists a map $\psi\colon\mathcal{W}\to\mathcal{V}$ such that $\psi\phi=\mathrm{Id}$ and $\psi(F_n\mathcal{W})=F_n\mathcal{V}$ for any $n\in\mathbb{N}$.*

Proof For an integer $n\geq-1$, suppose that we have built a subgraded $\mathbb{K}$–module $\mathcal{U}_n$ of $F_n\mathcal{W}$ such that $F_m\mathcal{W}=\phi(F_m\mathcal{V})\oplus(\mathcal{U}_n\cap F_m\mathcal{W})$ for any $m\leq n$. Let $\mathcal{U}'_n$ be a subgraded $\mathbb{K}$–module of $F_{n+1}\mathcal{W}$ that is an algebraic complement to $\phi(F_{n+1}\mathcal{V})\oplus\mathcal{U}_n$. Then let $\mathcal{U}_{n+1}:=\mathcal{U}_n\oplus\mathcal{U}'_n$. Finally, let $\mathcal{U}:=\operatorname{colim}\mathcal{U}_n$. We define ψ by

$$\psi=\begin{cases}\phi^{-1} & \text{on } \phi(\mathcal{V}),\\ 0 & \text{on } \mathcal{U}.\end{cases}$$

□

Proof of Proposition 37 The map $\Delta_{\mathscr{D}}\colon\mathcal{D}\to\mathcal{C}\circ\mathcal{D}$ is actually a morphism of $\mathscr{C}$–coalgebras such that $\Delta_{\mathscr{D}}^{-1}(F_n^{\mathrm{rad}}\mathcal{C}\circ\mathcal{D})=F_n^{\mathrm{rad}}\mathcal{D}$. By Lemma 38, there exists a map of graded $\mathbb{K}$–modules $\nabla\colon \mathcal{C}\circ\mathcal{D}\to\mathcal{D}$ such that $\nabla\Delta_{\mathscr{D}}=\mathrm{Id}_{\mathcal{D}}$ and $\nabla(F_n\mathcal{C}\circ\mathcal{D})=F_n\mathcal{D}$. Then the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathcal{C}\circ\mathcal{D} \arrow[d, "\Delta\circ\mathrm{Id}"] & \\
\mathcal{C}\circ\mathcal{D} \arrow[r, "\mathrm{Id}\circ\Delta"'] \arrow[rr, bend right, "\mathrm{Id}"'] & \mathcal{C}\circ\mathcal{C}\circ\mathcal{D} \arrow[r, "\mathrm{Id}\circ\nabla"'] & \mathcal{C}\circ\mathcal{D}
\end{tikzcd}$$

By Lemma 26, we know that

$$(\Delta\circ\mathrm{Id})\Delta(F_n^{\mathrm{rad}}\mathcal{D})\subset\sum_{i_0+\cdots+i_k=n}F_{i_0}^{\mathrm{rad}}\mathcal{C}(k)\otimes_{\mathbb{S}_k}(F_{i_1}^{\mathrm{rad}}\mathcal{C}\circ\mathcal{D}\otimes\cdots\otimes F_{i_k}^{\mathrm{rad}}\mathcal{C}\circ\mathcal{D}).$$

Moreover, we know that

$$(\mathrm{Id}\circ\nabla)(F_{i_0}^{\mathrm{rad}}\mathcal{C}(k)\otimes_{\mathbb{S}_k}(F_{i_1}^{\mathrm{rad}}\mathcal{C}\circ\mathcal{D}\otimes\cdots\otimes F_{i_k}^{\mathrm{rad}}\mathcal{C}\circ))\subset F_{i_0}^{\mathrm{rad}}\mathcal{C}(k)\otimes_{\mathbb{S}_k}(F_{i_1}^{\mathrm{rad}}\mathcal{D}\otimes\cdots\otimes F_{i_k}^{\mathrm{rad}}\mathcal{D}).$$

So, we have

$$\begin{aligned}\Delta(F_n^{\mathrm{rad}}\mathcal{D})&=(\mathrm{Id}\circ\nabla)(\Delta\circ\mathrm{Id})\Delta(F_n^{\mathrm{rad}}\mathcal{D})\\&\subset\sum_k\sum_{i_0+\cdots+i_k=n}F_{i_0}^{\mathrm{rad}}\mathcal{C}(k)\otimes_{\mathbb{S}_k}(F_{i_1}^{\mathrm{rad}}\mathcal{D}\otimes\cdots\otimes F_{i_k}^{\mathrm{rad}}\mathcal{D}).\end{aligned}$$ □

2.4 Presentability

This subsection deals with the presentability of the category of algebras over an operad and the presentability of the category of coalgebras over a conilpotent curved cooperad.

Theorem 39 [8, Lemma 5.2] *Let $\mathscr{P}$ be a dg-operad. Then the category $\mathscr{P}$–alg of $\mathscr{P}$–algebras is presentable.*

The essence of the last theorem is that any $\mathscr{P}$–algebra is the colimit of a filtered diagram of finitely presented $\mathscr{P}$–algebras.

Theorem 40 *Let $\mathscr{C}$ be a conilpotent curved cooperad. The category $\mathscr{C}$–cog of $\mathscr{C}$–coalgebras is presentable.*

The essence of this theorem is that any $\mathscr{C}$–coalgebra is the colimit of a filtered diagram of finite-dimensional $\mathscr{C}$–coalgebras. Since the category of $\mathscr{C}$–coalgebras does not seem to be comonadic over a known presentable category, we cannot use the same kind of arguments as in the proof of [8, Lemma 5.2].

Lemma 41 *The category $\mathscr{C}$–cog is cocomplete.*

Proof The colimit of a diagram of $\mathscr{C}$–coalgebras is its colimit in the category of graded $\mathbb{K}$–modules, together with the obvious decomposition map and coderivation map. □

Lemma 42 *For any $\mathscr{C}$–coalgebra $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ and any finite-dimensional subgraded $\mathbb{K}$–module $\mathcal{V} \subset \mathcal{C}$, there exists a finite-dimensional sub-$\mathscr{C}$–coalgebra $\mathscr{E}$ of $\mathscr{D}$ which contains $\mathcal{V}$.*

Proof Let us prove the result by induction on the coradical filtration of $\mathscr{D}$. Suppose first that $\mathcal{V} \subset F_0\mathcal{D}$. Then $\mathcal{V} + d\mathcal{V}$ is a sub-$\mathscr{C}$–coalgebra of $\mathscr{D}$. Then suppose that, for any finite-dimensional subgraded $\mathbb{K}$–module $\mathcal{W} \in F_n^{\mathrm{rad}}\mathcal{D}$, there exists a finite-dimensional sub-$\mathscr{C}$–coalgebra $\mathscr{E}$ of $F_n^{\mathrm{rad}}\mathscr{D}$ which contains $\mathcal{W}$. Consider now a finite-dimensional subgraded $\mathbb{K}$–module $\mathcal{V} \subset F_{n+1}\mathcal{D}$. By Proposition 37, for any element $x \in F_{n+1}^{\mathrm{rad}}\mathcal{D}$, $\Delta_{\mathscr{D}}(x) - 1 \otimes x \in \mathcal{C} \circ F_n^{\mathrm{rad}}\mathcal{D}$. Since we are working with conilpotent $\mathscr{C}$–coalgebras, there exists a finite-dimensional subgraded $\mathbb{K}$–module $\mathcal{V}(x)$ of $F_n^{\mathrm{rad}}\mathcal{D}$ such that $\Delta_{\mathscr{D}}(x) - 1 \otimes x \in \mathcal{C} \circ \mathcal{V}(x)$. Let $(e_i)_{i=1}^k$ be a linearly free family of elements of $\mathcal{V}$ such that $\mathcal{V} = \mathcal{V} \cap F_n^{\mathrm{rad}}\mathcal{D} \oplus \bigoplus_{i=1}^k \mathbb{K}.e_i$. By the induction hypothesis, let $\mathcal{E}$ be a finite-dimensional sub-$\mathscr{C}$–coalgebra of $\mathscr{D}$ which contains

$$\mathcal{V} \cap F_n^{\mathrm{rad}}\mathcal{D} \oplus \sum \mathcal{V}(e_i) + \mathcal{V}(d_{\mathcal{D}} e_i).$$

Then the sum

$$\mathcal{E} + \sum_i (\mathbb{K}.e_i \oplus \mathbb{K}.d_{\mathcal{D}} e_i)$$

is a finite-dimensional sub-$\mathscr{C}$–coalgebra of $\mathscr{D}$ which contains $\mathcal{V}$. □

Finally, we show that a finite-dimensional $\mathscr{C}$–coalgebra is a compact object.

Proposition 43 *A finite-dimensional $\mathscr{C}$–coalgebra is a compact object.*

We need the following technical lemma:

Lemma 44 *Let $D\colon I \to \mathscr{C}$–cog be a filtered diagram. Let $x \in D(i)$ for an object i of I. If the image of x in $\operatorname{colim} D$ is zero, then there exists an object i' of I and a map $\phi\colon i \to i'$ such that $D(\phi)(x) = 0$.*

Proof The colimit of the diagram D is the cokernel of the map

$$g\colon \bigoplus_{f\colon j\to j'} D(j) \to \bigoplus_{i\in Ob(I)} D(i)$$

such that for any morphism $f\colon j \to j'$ of I, the morphism g sends $x \in D(j)$ to $x - D(f)(x)$. Let $x \in D(i)$ whose image in $\operatorname{colim} D$ is zero. Then there exists an element $y = \sum y_f$ of $\bigoplus_{f\colon j\to j'} D(j)$ such that $g(y) = x$. Let i' be a cocone in I of the finite diagram made up of the morphisms f such that $y_f \neq 0$. Then the image in $D(i')$ of $\sum y_f$ is the same as the image in $D(i')$ of $\sum D(f)(y_f)$. Hence, the image of x in $D(i')$ is zero. □

Proof of Proposition 43 Let $D\colon I \to \mathscr{C}$–cog be a filtered diagram and let $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ be a finite-dimensional $\mathscr{C}$–coalgebra. We have to show that the canonical map

$$\operatorname{colim}(\hom_{\mathscr{C}\text{–cog}}(\mathscr{D}, D)) \to \hom_{\mathscr{C}\text{–cog}}(\mathscr{D}, \operatorname{colim} D)$$

is bijective.

- Let us first show that it is surjective. Let $f\colon \mathscr{D} \to \operatorname{colim} D$ be a map of $\mathscr{C}$–coalgebra and let $\mathcal{D}'$ be the image of f inside $\operatorname{colim} D$ which is also a sub-$\mathscr{C}$–coalgebra of $\operatorname{colim} D$. Let $\{e_a\}_{a=1}^n$ be a basis of the graded $\mathbb{K}$–module $\mathcal{D}'$. Since the diagram D is filtered, there exists an object i of I and for each a an element $x_a \in D(i)$ whose image in $\operatorname{colim} D$ is e_a. Let $\mathscr{E}$ be the smallest sub-$\mathscr{C}$–coalgebra of $D(i)$ which contains all the x_a and let $\mathscr{E}'$ be the image of $\mathscr{E}$ in $\operatorname{colim} D$. Notice that $\mathscr{E}'$ contains $\mathcal{D}'$ and that the map $\mathscr{E} \to \mathscr{E}'$ is surjective. By Lemma 44 and since $\mathscr{E}$ is finite-dimensional, there exists an object i' and a map $\phi\colon i \to i'$ such that the map $\mathscr{E}'' := D(\phi)(\mathscr{E}) \to \mathscr{E}'$ is an isomorphism of $\mathscr{C}$–coalgebras. So let $\mathscr{D}''$ be the sub-$\mathscr{C}$–coalgebra of $\mathscr{E}''$ which is the image of $\mathcal{D}'$ through the inverse isomorphism $\mathscr{E}' \to \mathscr{E}''$. Hence, the map $\mathscr{D} \to \mathscr{D}' \to \operatorname{colim} D$ factors through the map $\mathscr{D} \to \mathscr{D}' \simeq \mathscr{D}'' \to D(i')$ and so the canonical map $\operatorname{colim}(\hom_{\mathscr{C}\text{–cog}}(\mathscr{D}, D)) \to \hom_{\mathscr{C}\text{–cog}}(\mathscr{D}, \operatorname{colim} D)$ is surjective.

- Let us show that it is injective. Let

$$f \in \hom_{\mathscr{C}\text{–cog}}(\mathscr{D}, D(i)) \quad \text{and} \quad g \in \hom_{\mathscr{C}\text{–cog}}(\mathscr{D}, D(j))$$

be two maps whose images in $\hom_{\mathscr{C}\text{–cog}}(\mathscr{D}, \operatorname{colim} D)$ are the same; it is denoted by h. Since the category I is filtered, there exists an object k together with maps $\phi\colon i \to k$ and $\psi\colon j \to k$. Then $D(\phi)f(\mathscr{D}) + D(\psi)g(\mathscr{D})$ is a finite-dimensional sub-$\mathscr{C}$–coalgebra of $D(k)$ whose image in colim D is $h(\mathscr{D})$. As in the previous point (by Lemma 44), there exists a map $\zeta\colon k \to k'$ in I such that the map

$$u\colon D(\zeta)(D(\phi)f(\mathscr{D}) + D(\psi)g(\mathcal{C})) \to h(\mathscr{D})$$

is an isomorphism. Since the dimension (as a graded $\mathbb{K}$–module) of $D(\zeta)D(\phi)f(\mathscr{D})$ and the dimension of $D(\zeta)D(\psi)g(\mathscr{D})$ are both greater than the dimension of $h(\mathscr{D})$, we must have

$$D(\zeta)(D(\phi)f(\mathscr{D}) + D(\psi)g(\mathscr{D})) = D(\zeta)D(\phi)f(\mathscr{D}) = D(\zeta)D(\psi)g(\mathscr{D}).$$

In this context, we have

$$D(\zeta)D(\phi)f = u^{-1}h = D(\zeta)D(\psi)g.$$

Hence, f and g represent the same element of $\mathrm{colim}(\hom_{\mathscr{C}\text{–}\mathsf{cog}}(\mathscr{D}, D))$. □

Proof of Theorem 40 The isomorphisms classes of finite-dimensional $\mathscr{C}$–coalgebras form a set. By Proposition 43, any finite-dimensional $\mathscr{C}$–coalgebra is a compact object of the category $\mathscr{C}$–cog. Moreover, any $\mathscr{C}$–coalgebra is the colimit of the diagram of its finite-dimensional sub-$\mathscr{C}$–coalgebras (with inclusions between them); this is a filtered diagram (and even a directed set). Hence, the category $\mathscr{C}$–cog is presentable. □

3 Enrichment

This section deals with several enrichments of the category of algebras of an operad and of the category of coalgebras of a curved conilpotent cooperad. Specifically, we prove that both the category of algebras over an operad and the category of coalgebras over a curved conilpotent cooperad are tensored, cotensored and enriched over cocommutative coalgebras and enriched over simplicial sets. In the nonsymmetric context, algebras over an operad and coalgebras over a curved conilpotent cooperad are tensored, cotensored and enriched over coassociative coalgebras.

3.1 Enrichment over coassociative coalgebras and cocommutative coalgebras

We show in this subsection that the category of algebras over an operad and the category of coalgebras over a curved conilpotent cooperad are tensored-cotensored-enriched (see Definition 4) over the category uCocom of counital cocommutative coalgebras.

Moreover, in the nonsymmetric context, they are tensored-cotensored-enriched over the category uCog of coassociative coalgebras. We will use these enrichments in the sequel to describe deformations of morphisms and mapping spaces, respectively.

3.1.1 Enrichment of $\mathscr{P}$–algebras over coalgebras Let $\mathscr{P} = (\mathcal{P}, \gamma, 1)$ be a dg operad. For any counital cocommutative coalgebra $\mathscr{C} = (\mathcal{C}, \Delta_{\mathscr{C}}, \epsilon)$ and any $\mathscr{P}$–algebra $\mathscr{A} = (\mathcal{A}, \gamma_{\mathscr{A}})$, the chain complex $[\mathcal{C}, \mathcal{A}]$ has a canonical structure of $\mathscr{P}$–algebra as follows.

- For any $p \in \mathcal{P}(n)$ $(n \geq 1)$, and for any $f_1, \dots, f_n \in [\mathcal{C}, \mathcal{A}]$ and any $x = \mathcal{C}$,
$$\gamma_{[\mathcal{C},\mathcal{A}]}(p \otimes_{\mathbb{S}_n} (f_1 \otimes \cdots \otimes f_n))(x) = \gamma_{\mathscr{A}}(p \otimes -)(f_1 \otimes \cdots \otimes f_n)\Delta_{\mathscr{C}}^{n-1}(x)$$
- For any $p \in \mathcal{P}(0)$,
$$\gamma_{[\mathcal{C},\mathcal{A}]}(p) = \gamma_{\mathscr{A}}(p)\epsilon_{\mathcal{C}}$$

The chain complex $[\mathcal{C}, \mathcal{A}]$ together with its structure of $\mathscr{P}$–algebra is denoted by $[\mathscr{C}, \mathscr{A}]$.

Lemma 45 *The assignment $\mathscr{C}, \mathscr{A} \mapsto [\mathscr{C}, \mathscr{A}]$ defines a left coaction* (see Definition 3) *of the category* uCocom *of counital cocommutative coalgebras on the category* $\mathscr{P}$–alg *of $\mathscr{P}$–algebras.*

Proof The construction is functorial covariantly with respect to $\mathscr{P}$–algebras and contravariantly with respect to counital cocommutative coalgebras. Moreover, for any counital cocommutative coalgebras $\mathscr{C}$ and $\mathscr{D}$, and any $\mathscr{P}$–algebra $\mathscr{A}$ there is an isomorphism of chain complexes
$$\rho_{\mathscr{C},\mathscr{D},\mathscr{A}} \colon [\mathcal{C} \otimes \mathcal{D}, \mathcal{A}] \to [\mathcal{C}, [\mathcal{D}, \mathcal{A}]]$$
such that $\rho_{\mathscr{C},\mathscr{D},\mathscr{A}}(f)(x)(y) = f(x \otimes y)$. This is a morphism of $\mathscr{P}$–algebras which is functorial in $\mathscr{C}$, $\mathscr{D}$ and $\mathscr{A}$, and it satisfies the coherence conditions of Definition 3. □

One can define a left adjoint to the functor $[\mathscr{C}, -]$ as follows. Let $\mathscr{A} \lhd \mathscr{C}$ be the quotient of the free $\mathscr{P}$–algebra $\mathcal{P} \circ (\mathcal{A} \otimes \mathcal{C})$ by the ideal I generated by the relations
$$\gamma_{\mathscr{A}}(p \otimes_{\mathbb{S}_n} (y_1 \otimes \cdots \otimes y_n)) \otimes x \sim \sum (-1)^{\sum_{i<j} |x_{(i)}||y_j|} p \otimes_{\mathbb{S}_n} ((y_1 \otimes x_{(1)}) \otimes \cdots \otimes (y_n \otimes x_{(n)})),$$
$$\gamma_{\mathscr{A}}(p) \otimes x \sim \epsilon(x) p \quad \text{for any } p \in \mathcal{P}(0),$$
with $\Delta^{n-1}(x) = \sum x_{(1)} \otimes \cdots \otimes x_{(n)}$.

Theorem 46 *The category of $\mathscr{P}$–algebras is tensored-cotensored-enriched over the category* uCocom *of counital cocommutative coalgebras. The right action is given by the functor $- \triangleleft -$ and the left coaction is given by the functor $[-,-]$. We denote the enrichment by $\{-,-\}$.*

Proof Since the functor $[-,-]$ defines a coaction of the category of counital cocommutative coalgebras on the category of $\mathscr{P}$–algebras, since the functor $[-,\mathcal{A}]$ sends colimits to limits and since the functor $[\mathscr{C},-]$ is left adjoint to the functor $- \triangleleft \mathscr{C}$, we can conclude by Proposition 6. □

Let us describe $\{\mathscr{A},\mathscr{A}'\}$ for two $\mathscr{P}$–algebras $\mathscr{A}$ and $\mathscr{A}'$. This is the maximal subcoalgebra of the cofree cocommutative coalgebra $F([\mathcal{A},\mathcal{A}'])$ such that the following diagram commutes:

$$\begin{tikzcd}
\{\mathscr{A},\mathscr{A}'\} \arrow[rrr] \arrow[d, "{(\epsilon,\mathrm{Id},\Delta,\dots)}"'] & & & {[\mathcal{A},\mathcal{A}']} \arrow[d, "{[\gamma_{\mathcal{A}},\mathrm{Id}]}"] \\
\prod_{n\geq 0}\{\mathscr{A},\mathscr{A}'\}^{\otimes n}/\mathbb{S}_n \arrow[r] & \prod_{n\geq 0}[\mathcal{A},\mathcal{A}']^{\otimes n}/\mathbb{S}_n \arrow[r] & {[\mathcal{P}\circ\mathcal{A},\mathcal{P}\circ\mathcal{A}']} \arrow[r, "{[\mathrm{Id},\gamma_{\mathcal{A}'}]}"'] & {[\mathcal{P}\circ\mathcal{A},\mathcal{A}']}
\end{tikzcd}$$

where the map $\prod_{n\geq 0}[\mathcal{A},\mathcal{A}']^{\otimes n}/\mathbb{S}_n \to [\mathcal{P}\circ\mathcal{A},\mathcal{P}\circ\mathcal{A}']$ sends $f_1\otimes\cdots\otimes f_n$ to

$$\mathrm{Id}_{\mathcal{P}(n)} \otimes_{\mathbb{S}_n} (f_1\otimes\cdots\otimes f_n),$$

and where the map $\{\mathscr{A},\mathscr{A}'\} \to [\mathcal{A},\mathcal{A}']$ is the composition

$$\{\mathscr{A},\mathscr{A}'\} \to F([\mathcal{A},\mathcal{A}']) \to [\mathcal{A},\mathcal{A}'].$$

3.1.2 Enrichment of $\mathscr{C}$–coalgebras over coalgebras

Let $\mathscr{C} = (\mathcal{C},\Delta,\epsilon,1,d,\theta)$ be a curved conilpotent cooperad.

For any $\mathscr{C}$–coalgebra $\mathscr{D} = (\mathcal{D},\Delta_{\mathscr{D}})$ and any counital cocommutative coalgebra $\mathscr{E} = (\mathcal{E},\Delta_{\mathscr{E}},\epsilon)$, the tensor product $\mathcal{D}\otimes\mathcal{E}$ has a structure of $\mathscr{C}$–coalgebra given by

$$\mathcal{D}\otimes\mathcal{E} \xrightarrow{\bigoplus_n \Delta_n\otimes\Delta^{n-1}} \bigoplus_n (\mathcal{C}(n)\otimes_{\mathbb{S}_n}\mathcal{D}^{\otimes n})\otimes\mathcal{E}^{\otimes n} \to \bigoplus_n \mathcal{C}(n)\otimes_{\mathbb{S}_n}(\mathcal{D}\otimes\mathcal{E})^{\otimes n}.$$

Theorem 47 *The category $\mathscr{C}$–*cog *of $\mathscr{C}$–coalgebras is tensored-cotensored-enriched over the category of cocommutative counital coalgebras. The right action is given by the construction $-\otimes-$. We denote the left coaction by $\langle -,-\rangle$ and the enrichment by $\{-,-\}$.*

Proof The assignment $\mathscr{D}, \mathscr{E} \mapsto \mathscr{D} \otimes \mathscr{E}$ defines a right action of the category of counital cocommutative coalgebras on the category of $\mathscr{C}$–coalgebras. Moreover, the functor $\mathscr{D} \otimes -$ and the functor $- \otimes \mathscr{E}$ preserve colimits. We conclude by Proposition 6. $\square$

If $\mathscr{D}$ and $\mathscr{D}'$ are two $\mathscr{C}$–coalgebras, then the cocommutative counital hom coalgebra $\{\mathscr{D}, \mathscr{D}'\}$ is the final subcoalgebra of the cofree counital cocommutative coalgebra $F([\mathcal{D}, \mathcal{D}'])$ over the chain complex $[\mathcal{D}, \mathcal{D}']$ such that the following diagram, built in a similar way as its counterpart for algebras, commutes:

$$\begin{tikzcd}
\{\mathscr{D}, \mathscr{D}'\} \arrow[rrr] \arrow[d, "{(\epsilon, \mathrm{Id}, \Delta, \dots)}"'] & & & {[\mathcal{D}, \mathcal{D}']} \arrow[d] \\
\prod_{n \geq 0} \{\mathscr{D}, \mathscr{D}'\}^{\otimes n}/\mathbb{S}_n \arrow[r] & \prod_{n \geq 0} [\mathcal{D}, \mathcal{D}']^{\otimes n}/\mathbb{S}_n \arrow[r] & {[\mathcal{C} \circ \mathcal{D}, \mathcal{C} \circ \mathcal{D}']} \arrow[r] & {[\mathcal{D}, \mathcal{C} \circ \mathcal{D}']}
\end{tikzcd}$$

3.1.3 Morphisms are atoms

Proposition 48 *For any two $\mathscr{P}$–algebras $\mathscr{A}$ and $\mathscr{A}'$, the dg atoms of the cocommutative coalgebra $\{\mathscr{A}, \mathscr{A}'\}$ are the morphisms of $\mathscr{P}$–algebras from $\mathscr{A}$ to $\mathscr{A}'$. Similarly, for any two $\mathscr{C}$–coalgebras $\mathscr{D}$ and $\mathscr{D}'$, the dg atoms of the cocommutative coalgebra $\{\mathscr{D}, \mathscr{D}'\}$ are the morphisms of $\mathscr{C}$–coalgebras from $\mathscr{D}$ to $\mathscr{D}'$.*

Proof We have

$$\hom_{\mathsf{uCocom}}(\mathbb{K}, \{\mathscr{A}, \mathscr{A}'\}) \simeq \hom_{\mathscr{P}\text{–}\mathsf{alg}}(\mathscr{A} \triangleleft \mathbb{K}, \mathscr{A}') \simeq \hom_{\mathscr{P}\text{–}\mathsf{alg}}(\mathscr{A}, \mathscr{A}'). \qquad \square$$

3.1.4 Nonsymmetric context In the nonsymmetric context, we can get rid of the cocommutativity condition.

Proposition 49

- *If $\mathscr{P}$ is a nonsymmetric operad, then the category of $\mathscr{P}$–algebras is tensored-cotensored-enriched over the category uCog of counital coassociative coalgebras.*
- *If $\mathscr{C}$ is a nonsymmetric conilpotent curved cooperad, then the category of $\mathscr{C}$–coalgebras is tensored-cotensored-enriched over the category uCog of counital coassociative coalgebras.*

We denote by $\{-,-\}^{\mathrm{ns}}$ these two enrichments over counital coassociative coalgebras.

Proof The proof is similar to the proofs of Theorems 46 and 47. $\square$

The inclusion functor $\mathsf{uCocom} \hookrightarrow \mathsf{uCog}$ is a left adjoint (since it preserves colimits). Let R be its right adjoint. It sends any counital coassociative coalgebra to its final cocommutative subcoalgebra.

Proposition 50 *For any $\mathscr{P}$–algebras $\mathscr{A}$ and $\mathscr{A}'$, the cocommutative coalgebra $\{\mathscr{A}, \mathscr{A}'\}$ is the final cocommutative subcoalgebra $R(\{\mathscr{A}, \mathscr{A}'\}^{\mathrm{ns}})$ of $\{\mathscr{A}, \mathscr{A}'\}^{\mathrm{ns}}$. Similarly, for any $\mathscr{C}$–coalgebras $\mathscr{D}$ and $\mathscr{D}'$, the cocommutative coalgebra $\{\mathscr{D}, \mathscr{D}'\}$ is the final cocommutative subcoalgebra $R(\{\mathscr{D}, \mathscr{D}'\}^{\mathrm{ns}})$ of $\{\mathscr{D}, \mathscr{D}'\}^{\mathrm{ns}}$.*

Proof For any cocommutative coalgebra $\mathscr{E}$, we have

$$\begin{aligned}\hom_{\mathsf{uCocom}}(\mathscr{E}, \{\mathscr{A}, \mathscr{A}'\}) \simeq \hom_{\mathscr{P}\text{-}\mathsf{alg}}(\mathscr{A} \triangleleft \mathscr{E}, \mathscr{A}') &\simeq \hom_{\mathsf{uCog}}(\mathscr{E}, \{\mathscr{A}, \mathscr{A}'\}^{\mathrm{ns}}) \\ &\simeq \hom_{\mathsf{uCocom}}(\mathscr{E}, R(\{\mathscr{A}, \mathscr{A}'\}^{\mathrm{ns}})).\end{aligned}$$

Since these isomorphisms are functorial, $R(\{\mathscr{A}, \mathscr{A}'\}^{\mathrm{ns}})$ is isomorphic to $\{\mathscr{A}, \mathscr{A}'\}$. □

3.2 Simplicial enrichment

In this section, we recall the fact that the Sullivan polynomials forms algebras allow one to enrich the category of algebras over an operad. See for instance [16].

3.2.1 General case Let A be a differential graded unital commutative $\mathbb{K}$–algebra. The category of dg A–modules is equipped with a tensor product

$$M \otimes_A N = \mathrm{colim}(M \otimes A \otimes N \rightrightarrows M \otimes N),$$

where the two maps are given by the action of A on M and on N, respectively. The functor $A \otimes -\colon \mathsf{dgMod} \to \mathsf{dgMod}_A$ is strong symmetric monoidal. Hence, it induces several functors:

- from operads to operads enriched in A–modules,
- from cooperads to cooperads enriched in A–modules,
- from $\mathscr{P}$–algebras (in the category of $\mathbb{K}$–modules) to $A \otimes \mathscr{P}$–algebras (in the category of A–modules),
- from $\mathscr{C}$–coalgebras (in the category of $\mathbb{K}$–modules) to $A \otimes \mathscr{C}$–coalgebras (in the category of A–modules).

Applying this to the case of the Sullivan algebras of polynomial forms on standard simplices leads us to the following proposition:

Proposition 51 *Let $\mathscr{P}$ be a dg operad and let $\mathscr{C}$ be a curved conilpotent cooperad. The category of $\mathscr{P}$–algebras and the category of $\mathscr{C}$–coalgebras are enriched in simplicial sets as follows:*

$$\mathrm{HOM}(\mathscr{A},\mathscr{A}')_n := \hom_{\mathscr{P}\text{-alg}}(\mathscr{A},\Omega_n\otimes\mathscr{A}')\simeq \hom_{\Omega_n\otimes\mathscr{P}\text{-alg}}(\Omega_n\otimes\mathscr{A},\Omega_n\otimes\mathscr{A}'),$$
$$\mathrm{HOM}(\mathscr{D},\mathscr{D}')_n := \hom_{\Omega_n\otimes\mathscr{C}\text{-cog}}(\Omega_n\otimes\mathscr{D},\Omega_n\otimes\mathscr{D}').$$

Proof The only point that needs to be cleared up is the simplicial structure on $\mathrm{HOM}(\mathscr{D},\mathscr{D}')$. Let $\phi\colon [m]\to[n]$ be a map between finite ordinals. We want to define $\phi^*\colon \mathrm{HOM}(\mathscr{D},\mathscr{D}')_n\to\mathrm{HOM}(\mathscr{D},\mathscr{D}')_m$. An element of $\mathrm{HOM}(\mathscr{D},\mathscr{D}')_n$ is a morphism of graded $\mathbb{K}$–modules f from $\mathcal{D}$ to $\Omega_n\otimes\mathcal{D}'$ such that $fd_{\mathcal{D}}=(d_{\Omega_n}\otimes\mathrm{Id}_{\mathcal{D}'}+\mathrm{Id}_{\Omega_n}\otimes d_{\mathcal{D}'})$ and such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "f"] \arrow[d, "\Delta"'] & & \Omega_n\otimes\mathcal{D}' \arrow[d, "\mathrm{Id}\otimes\Delta"] \\
\mathcal{C}\circ\mathcal{D} \arrow[r, "\mathrm{Id}\circ f"'] & \mathcal{C}\circ(\Omega_n\otimes\mathcal{D}') \arrow[r] & \Omega_n\otimes(\mathcal{C}\circ\mathcal{D}')
\end{tikzcd}$$

where the map $\mathcal{C}\circ(\Omega_n\otimes\mathcal{D}')\to\Omega_n\otimes(\mathcal{C}\circ\mathcal{D}')$ is the map

$$x\otimes_{\mathbb{S}_k}((a_1\otimes x_1)\otimes\cdots\otimes(a_k\otimes x_k))$$
$$\mapsto (-1)^{|x|(\sum|a_i|)}(-1)^{\sum_{i>j}|a_i||x_j|}(a_1\cdots a_k)\otimes(x\otimes_{\mathbb{S}_k}(x_1\otimes\cdots\otimes x_k)).$$

Then $\phi^*(f)=(\Omega[\phi]\otimes\mathrm{Id})f$ where $\Omega[\phi]\colon\Omega_n\to\Omega_m$ is the structural map induced by ϕ. □

Proposition 52 *For any simplicial set X which is the colimit of a finite diagram of simplices $\Delta[n]$ and for any $\mathscr{P}$–algebras $\mathscr{A}$ and $\mathscr{A}'$, we have*

$$\hom_{\mathsf{sSet}}(X,\mathrm{HOM}(\mathscr{A},\mathscr{A}'))\simeq\hom_{\mathscr{P}\text{-alg}}(\mathscr{A},\Omega_X\otimes\mathscr{A}').$$

Proof It suffices to notice that the functor from commutative algebras to $R\otimes\mathscr{P}$–algebras $R\mapsto R\otimes\mathscr{A}'$ preserves finite limits. □

Remark 53 The enrichment of the category of $\mathscr{P}$–algebras and of the category of $\mathscr{C}$–coalgebras over simplicial sets that we described above is a part of a more general enrichment over functors from the category of unital commutative algebras to simplicial sets:

$$R\mapsto(\hom_{\mathscr{P}\text{-alg}}(\mathscr{A},\Omega_n\otimes R\otimes\mathscr{B}))_{n\in\mathbb{N}},$$
$$R\mapsto(\hom_{\Omega_n\otimes R\otimes\mathscr{C}\text{-cog}}(\Omega_n\otimes R\otimes\mathscr{D},\Omega_n\otimes R\otimes\mathscr{D}'))_{n\in\mathbb{N}}.$$

3.2.2 Nonsymmetric context In the nonsymmetric context, we can use some associative algebras instead of the commutative Sullivan algebras to define a simplicial mapping spaces. However, this does not define an enrichment any more. Let Λ_n be the linear dual of the Dold–Kan coalgebra over the standard simplex,

$$\Lambda_n := \mathrm{DK}^c(\Delta[n])^*.$$

This defines a simplicial unital associative algebra.

Further, let $\mathscr{P}$ be a nonsymmetric dg operad. For any $\mathscr{P}$–algebra $\mathscr{A} = (\mathcal{A}, \gamma_{\mathscr{A}})$, and for any associative algebra A, $A \otimes \mathcal{A}$ has a canonical structure of a $\mathscr{P}$–algebra.

Definition 54 (nonsymmetric simplicial mapping spaces of algebras over an operad) For any two $\mathscr{P}$–algebras $\mathscr{A}$ and $\mathscr{B}$, let $\mathrm{HOM}^{\mathrm{ns}}(\mathscr{A}, \mathscr{B})$ be the simplicial set

$$\mathrm{HOM}^{\mathrm{ns}}(\mathscr{A}, \mathscr{B})_n := \hom_{\mathscr{P}\text{-}\mathsf{alg}}(\mathscr{A}, \Lambda_n \otimes \mathscr{B}).$$

Let $\mathscr{C}$ be a nonsymmetric curved conilpotent cooperad. For any associative algebra A and for any two $\mathscr{C}$–coalgebras $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ and $\mathscr{E} = (\mathcal{E}, \Delta_{\mathcal{E}})$, we denote by $\hom_{A,\mathscr{C}}(\mathscr{D}, \mathscr{E})$ the set of morphisms of graded $\mathbb{K}$–modules f from $\mathcal{D}$ to $A \otimes \mathcal{E}$ which commute with the coderivations and such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "f"] \arrow[d, "\Delta"'] & & A \otimes \mathcal{E} \arrow[d, "\mathrm{Id}_A \otimes \Delta_{\mathscr{E}}"] \\
\mathcal{C} \circ_{\mathrm{ns}} \mathcal{D} \arrow[r] & \mathcal{C} \circ_{\mathrm{ns}} (A \otimes \mathcal{E}) \arrow[r] & A \otimes (\mathcal{C} \circ_{\mathrm{ns}} \mathcal{E})
\end{tikzcd}$$

Definition 55 (nonsymmetric simplicial mapping spaces of coalgebras over a curved cooperad) For any two $\mathscr{C}$–coalgebras $\mathscr{D}$ and $\mathscr{D}'$, let $\mathrm{HOM}^{\mathrm{ns}}(\mathscr{D}, \mathscr{D}')_n$ be the simplicial set

$$\mathrm{HOM}^{\mathrm{ns}}(\mathscr{D}, \mathscr{D}')_n := \hom_{\Lambda_n,\mathscr{C}}(\mathscr{D}, \mathscr{D}').$$

These simplicial sets are related to the enrichments over coassociative coalgebras that we described above.

Proposition 56 *For any two $\mathscr{P}$–algebras $\mathscr{A}$ and $\mathscr{B}$ and for any two $\mathscr{C}$–coalgebras $\mathscr{D}$ and $\mathscr{D}'$, we have isomorphisms*

$$\mathrm{HOM}^{\mathrm{ns}}(\mathscr{A}, \mathscr{B}) \simeq N(\{\mathscr{A}, \mathscr{B}\}^{\mathrm{ns}}),$$
$$\mathrm{HOM}^{\mathrm{ns}}(\mathscr{D}, \mathscr{D}') \simeq N(\{\mathscr{D}, \mathscr{D}'\}^{\mathrm{ns}}).$$

Proof The proof for $\mathscr{P}$–algebras is straightforward. Let us prove the result for the $\mathscr{C}$–coalgebras. A morphism of graded $\mathbb{K}$–modules f from $\mathcal{D}$ to $\Lambda_n \otimes \mathcal{D}'$ is equivalent to a morphism from $\mathcal{D} \otimes \mathrm{DK}^c(\Delta[n])$ to $\mathcal{D}'$. In that context, f belongs to $\hom_{\Lambda_n,\mathscr{C}}(\mathscr{D}, \mathscr{D}')$ if and only if the corresponding morphism from $\mathcal{D} \otimes \mathrm{DK}^c(\Delta[n])$ to $\mathcal{D}'$ is a morphism of $\mathscr{C}$–coalgebras. So

$$\begin{aligned} \mathrm{HOM}^{\mathrm{ns}}(\mathscr{A}, \mathscr{B})_n := \hom_{\Lambda_n,\mathscr{C}}(\mathscr{D}, \mathscr{D}') &\simeq \hom_{\mathscr{C}\text{–}\mathsf{cog}}(\mathscr{D} \otimes \mathrm{DK}^c(\Delta[n]), \mathscr{D}') \\ &\simeq \hom_{\mathsf{uCog}}(\mathrm{DK}^c(\Delta[n]), \{\mathscr{D}, \mathscr{D}'\}^{\mathrm{ns}}) \\ &\simeq \hom_{\mathsf{sSet}}(\Delta[n], N(\{\mathscr{D}, \mathscr{D}'\}^{\mathrm{ns}})). \end{aligned}$$ □

Remark 57 Beware! The construction $\mathscr{A}, \mathscr{B} \mapsto \mathrm{HOM}^{\mathrm{ns}}(\mathscr{A}, \mathscr{B})$ does not define an enrichment. This comes from the fact that the nerve functor $N\colon \mathsf{uCog} \to \mathsf{sSet}$ is not monoidal.

4 Bar–cobar adjunctions

The usual bar–cobar adjunction relates nonunital algebras to noncounital conilpotent coalgebras; see [19, Chapter 2]. It can be extended to nonunital operads and conilpotent cooperads; see [12]. Further, as a direct consequence of work of Hirsh and Millès [17], there exists an adjunction à la bar–cobar relating unital algebras with curved conilpotent coalgebras. We extend it to operads and curved conilpotent cooperads.

The bar–cobar adjunction $\Omega_u \dashv B_c$ is a tool to compute resolutions of operads. But it has other aspects: any morphism of operads from the cobar construction $\Omega_u\mathscr{C}$ of a curved conilpotent cooperad $\mathscr{C}$ to an operad $\mathscr{P}$ gives rise to a new adjunction à la bar cobar between $\mathscr{C}$–coalgebras and $\mathscr{P}$–algebras.

4.1 Operadic bar–cobar construction

The usual operadic bar–cobar adjunction (see [19, Chapter 6]) relates augmented operads to differential graded conilpotent cooperads. The bar construction $B\mathscr{P}$ of an operad $\mathscr{P}$ does use the augmentation of $\mathscr{P}$ as it is the graded cofree cooperad on the suspension of $\overline{\mathscr{P}}$. If $\mathscr{P}$ is not augmented, one can try to add an element to $\mathscr{P}$ whose boundary is the unit of $\mathscr{P}$ and try the same computation. This is the new bar construction; its output is no longer a differential graded cooperad but a curved cooperad.

The new curved bar functor B_c has also a left adjoint Ω_u whose formula looks like the usual operadic cobar functor. Again, as in [19, Chapter 6], this adjunction is related to a notion of twisting morphism.

Definition 58 (operadic bar construction) The *operadic bar construction* of a dg operad $\mathscr{P} = (\mathcal{P}, \gamma_{\mathscr{P}}, 1)$ is the curved conilpotent cooperad $B_c\mathscr{P} = (\mathbb{T}^c(s\mathcal{P} \oplus \mathbb{K}\cdot v), D, \theta)$, where $s\mathcal{P}$ is the suspension of the $\mathbb{S}$–module $\mathcal{P}$ and where v is an arity 1, degree 2 element. It is equipped with the coderivation D which extends the following map from $\overline{\mathbb{T}}(s\mathcal{P} \oplus \mathbb{K}\cdot v)$ to $s\mathcal{P} \oplus \mathbb{K}\cdot v$:

$$\mathbb{T}(s\mathcal{P} \oplus v) \to \mathbb{T}^{\leq 2}(s\mathcal{P} \oplus v) \to s\mathcal{P} \oplus v, \quad \begin{aligned} sx &\mapsto -sd_{\mathcal{P}}x, \\ sx \otimes sy &\mapsto (-1)^{|x|} s\gamma_{\mathscr{P}}(x \otimes y), \\ v &\mapsto s1. \end{aligned}$$

It has the curvature map

$$\theta\colon \overline{\mathbb{T}}(s\mathcal{P} \oplus v) \to s\mathcal{P} \oplus \mathbb{K}\cdot v \to \mathbb{K}\cdot v \to \mathbb{K}, \quad v \mapsto 1.$$

Proposition 59 *The map θ is actually a curvature for the coderivation, that is, $D^2 = (\theta \otimes \mathrm{Id} - \mathrm{Id} \otimes \theta)\Delta_2$.*

Proof Let π be the projection from $B_c\mathscr{P}$ to $s\mathcal{P}$. By Lemma 27, it suffices to prove that $\pi D^2 = (\theta \otimes \pi - \pi \otimes \theta)\Delta_2$. This is a straightforward calculation. □

Definition 60 (operadic cobar construction) The *operadic cobar construction* of a curved conilpotent cooperad $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon, 1, \theta)$ is the dg operad $\Omega_u\mathscr{C} = (\mathbb{T}s^{-1}\mathcal{C}, D)$, where D is the degree -1 derivation

$$s^{-1}x \mapsto \theta(x)1 - s^{-1}dx - \sum (-1)^{|x_{(1)}|} s^{-1}x_{(1)} \otimes s^{-1}x_{(2)},$$

where $\Delta_2(x) = \sum x_1 \otimes x_2$.

Proposition 61 *The derivation D squares to zero.*

Proof It suffices to prove the result for any element of the form $s^{-1}x$, which is a straightforward calculation. □

Definition 62 (operadic twisting morphism) Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon, 1, d, \theta)$ be a curved conilpotent cooperad and let $\mathscr{P} = (\mathcal{P}, \gamma_{\mathscr{P}}, 1_{\mathscr{P}}, d)$ be a dg operad. An *operadic twisting morphism* from $\mathscr{C}$ to $\mathscr{P}$ is a degree -1 map of $\mathbb{S}$–modules (or $\mathbb{N}$–modules in the nonsymmetric case)

$$\alpha\colon \bar{\mathcal{C}} \to \mathcal{P}$$

such that

$$\partial(\alpha)+\gamma(\alpha\otimes\alpha)\Delta_2=\Theta,$$

where $\Theta(x)=\theta(x)1_{\mathscr{P}}$ for any $x\in\mathscr{C}$. We denote by $\mathrm{Tw}(\mathscr{C},\mathscr{P})$ the set of operadic twisting morphisms from $\mathscr{C}$ to $\mathscr{P}$.

Proposition 63 *We have the functorial isomorphisms*

$$\hom_{\mathsf{Operad}}(\Omega_u\mathscr{C},\mathscr{P})\simeq\mathrm{Tw}(\mathscr{C},\mathscr{P})\simeq\hom_{\mathsf{cCoop}}(\mathscr{C},B_c\mathscr{P}).$$

Proof Proving the existence of the functorial isomorphism $\hom_{\mathsf{Operad}}(\Omega_u\mathscr{C},\mathscr{P})\simeq\mathrm{Tw}(\mathscr{C},\mathscr{P})$ is similar to the proof of [17, Theorem 3.4.1]. Let us show that we have a functorial isomorphism $\mathrm{Tw}(\mathscr{C},\mathscr{P})\simeq\hom_{\mathsf{cCoop}}(\mathscr{C},B_c\mathscr{P})$. Let $\alpha\colon\mathscr{C}\to\mathscr{P}$ be an operadic twisting morphism. We obtain a degree zero map from $\bar{\mathcal{C}}$ to $s\mathcal{P}\oplus\mathbb{K}\cdot v$ as follows:

$$\bar{\mathcal{C}}\to s\mathcal{P}\oplus\mathbb{K}\cdot v,\quad c\mapsto s\alpha(x)+\theta_{\mathscr{C}}(x).$$

This induces a morphism of graded cooperads $f_\alpha\colon\mathscr{C}\to B_c\mathscr{P}=\mathbb{T}^c(s\mathcal{P}\oplus\mathbb{K}\cdot v)$ such that $\theta_{\mathscr{C}}=\theta_{B_c\mathscr{P}}f_\alpha$. Since $\partial(\alpha)+\gamma_{\mathscr{P}}(\alpha\otimes\alpha)\Delta_2=\Theta$, the morphism f_α commutes with the coderivations and so is a morphism of curved cooperads. Conversely, from any morphism of curved cooperads f from $\mathscr{C}$ to $B_c\mathscr{P}$, one obtains a twisting morphism as follows:

$$\mathscr{C}\xrightarrow{f}B_c\mathscr{P}\twoheadrightarrow s\mathscr{P}\to\mathscr{P}.$$

The two constructions that we described are inverse one to another. □

Hence, the functors Ω_u and B_c realize an adjunction between the category of dg operads and the category of curved conilpotent cooperads,

$$\mathsf{cCoop}\underset{B_c}{\overset{\Omega_u}{\rightleftarrows}}\mathsf{Operad}.$$

4.2 Twisted products

Let $\alpha\colon\overline{\mathscr{C}}\to\mathscr{P}$ be an operadic twisting morphism.

Definition 64 (twisted $\mathscr{P}$–module) For any $\mathscr{C}$–comodule $\mathscr{D}$, let $\mathscr{P}\circ_\alpha\mathscr{D}$ be the free $\mathscr{P}^{\mathrm{grad}}$–module $\mathcal{P}\circ\mathcal{D}$ equipped with the unique derivation which extends the map

$$\mathcal{D}\to\mathcal{P}\circ\mathcal{D},\quad x\mapsto d_{\mathcal{D}}(x)-(\alpha\circ\mathrm{Id})\Delta(x).$$

Definition 65 (twisted $\mathscr{C}$–comodule) For any $\mathscr{P}$–module $\mathscr{A}$, let $\mathscr{C} \circ_\alpha \mathscr{A}$ be the cofree $\mathscr{C}^{\mathrm{grad}}$–comodule $\mathcal{C} \circ \mathcal{A}$ equipped with a unique coderivation which extends the map

$$\mathcal{C} \circ \mathcal{A} \to \mathcal{A}, \quad x \mapsto (d_{\mathcal{A}}(\epsilon_{\mathscr{C}} \circ \mathrm{Id}) + \gamma_{\mathscr{A}}(\alpha \circ \mathrm{Id}))(x).$$

Proposition 66 *The derivation of $\mathscr{P} \circ_\alpha \mathscr{D}$ squares to zero. Hence, $\mathscr{P} \circ_\alpha \mathscr{D}$ is a dg $\mathscr{P}$–module. Similarly, the coderivation of $\mathscr{C} \circ_\alpha \mathscr{A}$ squares to $(\theta \circ \mathrm{Id})\Delta$. Hence, $\mathscr{C} \circ_\alpha \mathscr{A}$ is a $\mathscr{C}$–comodule.*

Proof To prove the first point, it suffices to show that $\pi D^2 = 0$, which is a straightforward calculation. To prove the second point, it suffices to show that $\pi D^2 = (\theta \circ \mathrm{Id})$, which is a straightforward calculation. □

Definition 67 (twisting morphism relative to an operadic twisting morphism) For any $\mathscr{C}$–comodule $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ and any $\mathscr{P}$–module $\mathscr{A} = (\mathcal{A}, \gamma_{\mathscr{A}})$ an α–twisting morphism from $\mathscr{D}$ to $\mathscr{A}$ is a degree 0 map $\phi\colon \mathcal{D} \to \mathcal{A}$ such that

$$\partial(\phi) + \gamma_{\mathcal{A}}(\alpha \circ \phi)\Delta_{\mathcal{C}} = 0.$$

We denote by $\mathrm{Tw}_\alpha(\mathscr{D}, \mathscr{A})$ the set of α–twisting morphisms from $\mathscr{D}$ to $\mathscr{A}$.

Proposition 68 *There are functorial isomorphisms*

$$\hom_{\mathscr{P}\text{–mod}}(\mathscr{P} \circ_\alpha \mathscr{D}, \mathscr{A}) \simeq \mathrm{Tw}_\alpha(\mathscr{D}, \mathscr{A}) \simeq \hom_{\mathscr{C}\text{–comod}}(\mathscr{D}, \mathscr{C} \circ_\alpha \mathscr{A})$$

for any $\mathscr{C}$–comodule $\mathscr{D}$ and any $\mathscr{P}$–module $\mathscr{A}$.

Proof The proof is similar to [19, Proposition 11.3.2]. □

4.3 Bar–cobar adjunction for algebras over an operad and coalgebras over a cooperad

Following [19, Chapter 11], we call the previous functors the bar construction for $\mathscr{P}$–algebras and the cobar construction for $\mathscr{C}$–coalgebras, respectively.

Definition 69 (bar construction and cobar construction relatives to an operadic twisting morphism) Let $\alpha\colon \overline{\mathscr{C}} \to \mathscr{P}$ be an operadic twisting morphism. The α–bar construction is the functor from $\mathscr{P}$–algebras to $\mathscr{C}$–coalgebras defined by

$$B_\alpha \mathscr{A} := \mathscr{C} \circ_\alpha \mathscr{A}.$$

The α–cobar construction is the functor from $\mathscr{C}$–coalgebras to $\mathscr{P}$–algebras defined by

$$\Omega_\alpha \mathscr{D} := \mathscr{P} \circ_\alpha \mathscr{D}.$$

We already know, by Proposition 68, that Ω_α is left adjoint to B_α. Moreover, this adjunction is enriched over cocommutative coalgebras and simplicial sets.

Proposition 70 *The functors Ω_α and B_α induce functorial isomorphisms of counital cocommutative coalgebras and of simplicial sets*

$$\{\Omega_\alpha \mathscr{D}, \mathscr{A}\} \simeq \{\mathcal{C}, B_\alpha \mathscr{A}\}, \quad \mathrm{HOM}(\Omega_\alpha \mathscr{D}, \mathscr{A}) \simeq \mathrm{HOM}(\mathscr{D}, B_\alpha \mathscr{A})$$

for any $\mathscr{C}$–coalgebra $\mathscr{D}$ and any $\mathscr{P}$–algebra $\mathscr{A}$;

Lemma 71 *We have a functorial isomorphism*

$$\mathrm{Tw}_\alpha(\mathscr{D} \otimes \mathscr{E}, \mathscr{A}) \simeq \mathrm{Tw}_\alpha(\mathscr{D}, [\mathscr{E}, \mathscr{A}])$$

for any $\mathscr{C}$–coalgebra $\mathscr{D}$, any $\mathscr{P}$–algebra $\mathscr{A}$ and any counital cocommutative coalgebra $\mathscr{E}$.

Proof The set of morphisms of graded $\mathbb{K}$–modules from $\mathcal{D} \otimes \mathcal{E}$ to $\mathcal{A}$ is in bijection with the set of morphisms of graded $\mathbb{K}$–modules from $\mathcal{D}$ to $[\mathcal{E}, \mathcal{A}]$. This bijection and its inverse preserve α–twisting morphisms. □

Lemma 72 *We have a functorial isomorphism*

$$\mathrm{Tw}_\alpha(\mathscr{D}, R \otimes \mathscr{A}) \simeq \hom_{R\otimes\mathscr{C}\text{–cog}}(R \otimes \mathscr{D}, R \otimes B_\alpha \mathscr{A})$$

for any $\mathscr{C}$–coalgebra $\mathscr{D}$, any $\mathscr{P}$–algebra $\mathscr{A}$ and any dg unital commutative algebra R.

Proof Let us first denote by π the map from $R \otimes (\mathcal{C} \circ \mathcal{A})$ to $R \otimes \mathcal{A}$ defined by the formula

$$\pi = \mathrm{Id} \otimes (\epsilon \circ \mathrm{Id}).$$

As we have already seen, a morphism in $\hom_{R\otimes\mathscr{C}\text{–cog}}(R \otimes \mathscr{D}, R \otimes B_\alpha \mathscr{A})$ is equivalent to the data of a map $f\colon \mathcal{D} \to R \otimes (\mathcal{C} \circ \mathcal{A})$ which satisfies some conditions; see the proof of Proposition 51. On the one hand, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "f"] \arrow[d, "\Delta"'] & & R \otimes (\mathcal{C} \circ \mathcal{A}) \arrow[d, "\mathrm{Id}\otimes\Delta"] \\
\mathcal{C} \circ \mathcal{D} \arrow[r, "\mathrm{Id}\circ f"'] & \mathcal{C} \circ (R \otimes (\mathcal{C} \circ \mathcal{A})) \arrow[r] & R \otimes (\mathcal{C} \circ \mathcal{C} \circ \mathcal{A})
\end{tikzcd}$$

This implies that f is the composition

$$\mathcal{D} \xrightarrow{\Delta} \mathcal{C} \circ \mathcal{D} \xrightarrow{\mathrm{Id} \circ f} \mathcal{C} \circ (R \otimes (\mathcal{C} \circ \mathcal{A})) \to R \otimes (\mathcal{C} \circ \mathcal{C} \circ \mathcal{A}) \xrightarrow{\mathrm{Id} \otimes (\mathrm{Id} \circ \epsilon \circ \mathrm{Id})} R \otimes (\mathcal{C} \circ \mathcal{A}).$$

Then, exchanging the action of ϵ with the exchange between $\mathcal{C}$ and R, we see that f is the composition

$$\mathcal{D} \xrightarrow{\Delta} \mathcal{C} \circ \mathcal{D} \xrightarrow{\mathrm{Id} \circ (\pi f)} \mathcal{C} \circ (R \otimes \mathcal{A}) \to R \otimes (\mathcal{C} \circ \mathcal{A}).$$

On the other hand, $\partial(f) = 0$. Given the relation between f and πf just above, a straightforward calculation shows that this is equivalent to the fact that πf is a twisting morphism. $\square$

Proof of Proposition 70 On the one hand, for any cocommutative coalgebra $\mathscr{E}$, we have

$$\begin{aligned} \hom_{\mathsf{uCocom}}(\mathscr{E}, \{\Omega_\alpha \mathscr{D}, \mathscr{A}\}) &\simeq \hom_{\mathscr{P}\text{-}\mathsf{alg}}(\Omega_\alpha \mathscr{D}, [\mathscr{E}, \mathscr{A}]) \\ &\simeq \mathrm{Tw}_\alpha(\mathscr{D}, [\mathscr{E}, \mathscr{A}]) \simeq \mathrm{Tw}_\alpha(\mathscr{D} \otimes \mathscr{E}, \mathscr{A}) \\ &\simeq \hom_{\mathscr{C}\text{-}\mathsf{cog}}(\mathscr{D} \otimes \mathscr{E}, B_\alpha \mathscr{A}) \\ &\simeq \hom_{\mathsf{uCocom}}(\mathscr{E}, \{\mathscr{D}, B_\alpha \mathscr{A}\}). \end{aligned}$$

On the other hand, by Lemma 72 and Proposition 68 we have functorial isomorphisms

$$\hom_{\mathscr{P}\text{-}\mathsf{alg}}(\Omega_\alpha \mathscr{D}, R \otimes \mathscr{A}) \simeq \mathrm{Tw}_\alpha(\mathscr{D}, R \otimes \mathscr{A}) \simeq \hom_{R \otimes \mathscr{C}\text{-}\mathsf{cog}}(R \otimes \mathscr{D}, R \otimes B_\alpha \mathscr{A})$$

for any dg unital commutative algebra R. Taking $R = \Omega_n$ gives us a natural isomorphism of simplicial sets $\mathrm{HOM}(\Omega_\alpha \mathscr{D}, \mathscr{A}) \simeq \mathrm{HOM}(\mathscr{D}, B_\alpha \mathscr{A})$. $\square$

5 Homotopy theory of algebras over an operad

In this section, we recall a result of Hinich, stating that for any dg operad $\mathscr{P}$, the category of $\mathscr{P}$–algebras admits a projective model structure whose weak equivalences are quasi-isomorphisms (see [15; 4]). Moreover, we show that the simplicial enrichment of the category of $\mathscr{P}$–algebras that we described above gives models for the mapping spaces. Finally, we show that the enrichment over cocommutative coalgebras introduced in Section 3 encodes deformation of morphisms of $\mathscr{P}$–algebras.

5.1 Model structure on algebras over an operad

We recall here results about model structures on the category of algebras over an operad.

Definition 73 (right induced model structures) Consider the adjunction

$$\mathsf{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathsf{D}$$

Suppose that C admits a cofibrantly generated model structure. We say that D admits a model structure *right induced* by the adjunction $L \dashv R$ if it admits a model structure whose weak equivalences (resp. fibrations) are the morphisms f such that $R(f)$ is a weak equivalence (resp. a fibration) and whose generating cofibrations (resp. generating acyclic cofibrations) are the images under L of the generating cofibrations (resp. generating acyclic cofibrations) of C.

Definition 74 (admissible operad) An operad $\mathscr{P}$ is said to be *admissible* if the category of $\mathscr{P}$–algebras admits a projective model structure, that is, a model structure right induced by the adjunction

$$\mathsf{dgMod} \overset{\mathscr{P}\circ -}{\rightleftarrows} \mathscr{P}\text{–}\mathsf{alg}$$

whose right adjoint is the forgetful functor.

Theorem 75 [15] *Any nonsymmetric operad is admissible. When the characteristic of the field $\mathbb{K}$ is zero, any operad is admissible.*

5.2 Mapping spaces

The simplicial enrichments of the category of $\mathscr{P}$–algebras described above give us models for the mapping spaces.

Proposition 76 *Suppose that the characteristic of the field $\mathbb{K}$ is zero. Let $\mathscr{P}$ be a dg operad. The assignment $\mathscr{A}, \mathscr{A}' \mapsto \mathrm{HOM}(\mathscr{A}, \mathscr{A}')$ defines a homotopical enrichment of the category of $\mathscr{P}$–algebras over the category of simplicial sets. Moreover, for any cofibrant $\mathscr{P}$–algebra $\mathscr{A}$ and any $\mathscr{P}$–algebra $\mathscr{A}'$, the simplicial set $\mathrm{HOM}(\mathscr{A}, \mathscr{A}')$ is a model of the mapping space $\mathrm{Map}(\mathscr{A}, \mathscr{A}')$.*

Remark 77 The characteristic zero assumption is not necessary in the nonsymmetric context.

Proof Let $f\colon \mathscr{A} \to \mathscr{A}'$ and $g\colon \mathscr{B} \to \mathscr{B}'$ be a cofibration and a fibration of $\mathscr{P}$–algebras, respectively. Let $h\colon X \to Y$ be a monomorphism of simplicial sets which is a generating cofibration or acyclic cofibration for the Kan–Quillen model structure.

Then X and Y are colimits of finite diagrams made up of simplices $\Delta[n]$. Consider a square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{HOM}(\mathscr{A}', \mathscr{B}) \arrow[d] \\ Y \arrow[r] & \mathrm{HOM}(\mathscr{A}', \mathscr{B}') \times_{\mathrm{HOM}(\mathscr{A}, \mathscr{B}')} \mathrm{HOM}(\mathscr{A}, \mathscr{B}) \end{tikzcd}$$

By Proposition 52, it induces the square

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \Omega_Y \otimes \mathscr{B} \arrow[d] \\ \mathcal{A}' \arrow[r] & \Omega_Y \otimes \mathscr{B}' \times_{\Omega_X \otimes \mathscr{B}'} \Omega_X \otimes \mathscr{B} \end{tikzcd}$$

which has a lifting whenever f, g or h is a weak equivalence; indeed, by Proposition 18, the map $\Omega_Y \to \Omega_X$ is a fibration and it is an acyclic fibration whenever h is an acyclic cofibration. Further, to prove that $\mathrm{HOM}(\mathscr{A}, \mathscr{A}')$ is a model of the mapping space $\mathrm{Map}(\mathscr{A}, \mathscr{A}')$, it suffices to notice that $\{\Omega_n \otimes \mathscr{A}'\}_{n \in \mathbb{N}}$ is a Reedy fibrant resolution of the constant simplicial $\mathscr{P}$–algebra $\mathscr{A}'$. □

5.3 Deformation theory of morphisms of algebras over an operad

We know that the category of $\mathscr{P}$–algebras is enriched over the category uCocom of cocommutative coalgebras. In this subsection, we show that for any $\mathscr{P}$–algebras $\mathscr{A}$ and $\mathscr{B}$, the cocommutative coalgebra $\{\mathscr{A}, \mathscr{B}\}$ encodes the deformation theory of morphisms from $\mathscr{A}$ to $\mathscr{B}$. We suppose in this subsection that the field $\mathbb{K}$ is algebraically closed.

Any morphism of $\mathscr{P}$–algebras $f\colon \mathscr{A} \to \mathscr{B}$ defines a deformation problem $\mathrm{Def}(f)$.

$$\mathsf{Artin\text{–}alg} \to \mathsf{sSet},$$
$$R \mapsto \mathrm{Map}(\mathscr{A}, \mathscr{B} \otimes R) \times^h_{\mathrm{Map}(\mathscr{A}, \mathscr{B})} \{f\} \simeq \mathrm{HOM}(\mathscr{A}, \mathscr{B} \otimes R) \times_{\mathrm{HOM}(\mathscr{A}, \mathscr{B})} \{f\}.$$

The following theorem is a direct consequence of a result by Chuang, Lazarev and Mannan [7, Theorem 2.9]. It is proven in the appendix.

Theorem 78 *Suppose that the base field $\mathbb{K}$ is algebraically closed and that its characteristic is zero. Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon)$ be a dg cocommutative coalgebra and let A be its set of graded atoms. There exists a unique decomposition $\mathscr{C} \simeq \bigoplus_{a \in A} \mathscr{C}_a$, where $\mathscr{C}_a$ is a subcoalgebra of $\mathscr{C}$ which contains a and which belongs to the category* $\mathsf{uNilCocom}$. *Moreover, a morphism of dg cocommutative coalgebras $f\colon \bigoplus_{a \in A} \mathscr{C}_a \to \bigoplus_{b \in B} \mathscr{D}_b$ is the data of a function $\phi\colon A \to B$ and of a morphism $f_a\colon \mathscr{C}_a \to \mathscr{D}_{\phi(a)}$ for any $a \in A$.*

We know from Proposition 48 that a morphism f of $\mathscr{P}$–algebras from $\mathscr{A}$ to $\mathscr{B}$ is a dg atom of the cocommutative coalgebra $\{\mathscr{A}, \mathscr{B}\}$. Applying Theorem 78 to the cocommutative coalgebra $\{\mathscr{A}, \mathscr{B}\}$, we obtain the conilpotent cocommutative coalgebra $\{\mathscr{A}, \mathscr{B}\}_f$. This is in particular a Hinich coalgebra which encodes a deformation problem $R \mapsto \mathrm{Map}(R^*, \{\mathscr{A}, \mathscr{B}\}_f)$. We show in the next proposition that this deformation problem is $\mathrm{Def}(f)$.

Theorem 79 *Suppose that $\mathscr{A}$ is a cofibrant $\mathscr{P}$–algebra. Then the deformation problem induced by the conilpotent cocommutative coalgebra $\{\mathscr{A}, \mathscr{B}\}_f$ is* $\mathrm{Def}(f)$.

Lemma 80 *If $\mathscr{A}$ is a cofibrant $\mathscr{P}$–algebra, the simplicial Hinich coalgebra*

$$\{\mathscr{A}, \Omega_n \otimes \mathscr{B}\}_f$$

is a Reedy fibrant replacement of the constant simplicial Hinich coalgebra $\{\mathscr{A}, \mathscr{B}\}_f$.

Proof Let $g\colon X \to Y$ be a monomorphism of simplicial sets which are finite colimits of standard simplices $\Delta[n]$. Let $h\colon \mathscr{C}_1 \to \mathscr{C}_2$ be a monomorphism of Hinich coalgebras. Consider the square

$$\begin{tikzcd}
\mathscr{C}_1 \arrow[r] \arrow[d] & \{\mathscr{A}, \Omega_Y \otimes \mathscr{B}\}_f \arrow[d, "\{\mathscr{A},\Omega[g]\otimes\mathscr{B}\}"] \\
\mathscr{C}_2 \arrow[r] & \{\mathscr{A}, \Omega_X \otimes \mathscr{B}\}_f
\end{tikzcd}$$

Any morphism of cocommutative coalgebras from a conilpotent cocommutative coalgebra $\mathscr{C}$ to $\{\mathscr{A}, \mathscr{B}\}$ such that the atom of $\mathscr{C}$ targets the atom f of $\{\mathscr{A}, \mathscr{B}\}$ is a morphism from $\mathscr{C}$ to $\{\mathscr{A}, \mathscr{B}\}_f$. So, lifting the previous square amounts to lifting the square of $\mathscr{P}$–algebras

$$\begin{tikzcd}
\varnothing \arrow[rr] \arrow[d] & & {[\mathscr{C}_2, \Omega_Y \otimes \mathscr{B}]} \arrow[d] \\
\mathscr{A} \arrow[r] & {[\mathscr{C}_1, \Omega_Y \otimes \mathscr{B}] \times_{[\mathscr{C}_1, \Omega_X \otimes \mathscr{B}]} [\mathscr{C}_2, \Omega_Y \otimes \mathscr{B}]} &
\end{tikzcd}$$

This is possible whenever, g or h is a weak equivalence, since any weak equivalence of Hinich coalgebras is in particular a quasi-isomorphism. So, in particular, any face map $\{\mathscr{A}, \Omega_{n+1} \otimes \mathscr{B}\} \to \{\mathscr{A}, \Omega_n \otimes \mathscr{B}\}$ is an acyclic fibration of Hinich coalgebras and, for any integer $n \in \mathbb{N}$, the morphism $\{\mathscr{A}, \Omega_n \otimes \mathscr{B}\} \to \{\mathscr{A}, \Omega_{\partial\Delta[n]} \otimes \mathscr{B}\}$ is a fibration. $\square$

Proof of Theorem 79 By Lemma 80, the deformation problem induced by the Hinich coalgebra $\{\mathscr{A},\mathscr{B}\}_f$ is equivalent to the deformation problem

$$R \in \mathsf{Artin\text{–}alg} \mapsto (\hom_{\mathsf{Hinich\text{–}cog}}(R^*, \{\mathscr{A}, \Omega_n \otimes \mathscr{B}\}_f))_{n\in\mathbb{N}}.$$

We have

$$\begin{aligned}\hom_{\mathsf{Hinich\text{–}cog}}(R^*, \{\mathscr{A}, \Omega_n \otimes \mathscr{B}\}_f)&\\ &\simeq \hom_{\mathsf{uCocom}}(R^*, \{\mathscr{A}, \Omega_n \otimes \mathscr{B}\}) \times_{\hom_{\mathsf{uCocom}}(\mathbb{K},\{\mathscr{A},\Omega_n\otimes\mathscr{B}\})} \{f\}\\ &\simeq \hom_{\mathscr{P}\text{–}\mathsf{alg}}(\mathscr{A} \lhd R^*, \Omega_n \otimes \mathscr{B}) \times_{\hom_{\mathscr{P}\text{–}\mathsf{alg}}(\mathscr{A},\Omega_n\otimes\mathscr{B})} \{f\}\\ &\simeq \hom_{\mathscr{P}\text{–}\mathsf{alg}}(\mathscr{A}, R\otimes\Omega_n \otimes \mathscr{B}) \times_{\hom_{\mathscr{P}\text{–}\mathsf{alg}}(\mathscr{A},\Omega_n\otimes\mathscr{B})} \{f\}.\end{aligned}$$

Since the simplicial sets

$$(\hom_{\mathscr{P}\text{–}\mathsf{alg}}(\mathscr{A}, R\otimes\Omega_n \otimes \mathscr{B}))_{n\in\mathbb{N}} \quad\text{and}\quad (\hom_{\mathscr{P}\text{–}\mathsf{alg}}(\mathscr{A}, \Omega_n \otimes \mathscr{B}))_{n\in\mathbb{N}}$$

are Kan complexes and models of $\mathrm{Map}(\mathscr{A}, R\otimes\mathscr{B})$ and $\mathrm{Map}(\mathscr{A},\mathscr{B})$, respectively, and since the map between them is a fibration, the simplicial set

$$(\hom_{\mathscr{P}\text{–}\mathsf{alg}}(\mathscr{A}, R\otimes\Omega_n \otimes \mathscr{B}) \times_{\hom_{\mathscr{P}\text{–}\mathsf{alg}}(\mathscr{A},\Omega_n\otimes\mathscr{B})} \{f\})_{n\in\mathbb{N}}$$

is a model of the homotopy pullback $\mathrm{Map}(\mathscr{A}, R\otimes\mathscr{B}) \times^h_{\mathrm{Map}(\mathscr{A},\mathscr{B})} \{f\}$. □

6 Model structures on coalgebras over a cooperad

In this section, we show that, for any operadic twisting morphism $\alpha\colon \mathscr{C} \to \mathscr{P}$, the projective model structure on the category of $\mathscr{P}$–algebras can be transferred through the cobar construction functor Ω_α to the category of $\mathscr{C}$–coalgebras. This result is in the vein of similar results by Hinich [16], Lefevre and Hasegawa [18], Vallette [26] and Positselski [23]. However, we use a new method for the proof that uses the presentability of the category of algebras over an operad and of the category of coalgebras over a curved conilpotent cooperad; specifically, we use a theorem proved by Bayeh, Hess, Karpova, Kedziorek, Riehl and Shipley [3; 14].

6.1 Model structure induced by a twisting morphism

Definition 81 (left induced model structures) Consider the adjunction

$$\mathsf{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathsf{D}$$

Suppose that D admits a model structure. We say that C admits a model structure left induced by the adjunction $L \dashv R$ if it admits a model structure whose weak equivalences (resp. cofibrations) are the morphisms f such that $L(f)$ is a weak equivalence (resp. a cofibration).

Here is the main theorem of the present article:

Theorem 82 *Let $\mathscr{P}$ be a dg operad, let $\mathscr{C}$ be a curved conilpotent cooperad and let α be an operadic twisting morphism between them. Suppose that the characteristic of the base field $\mathbb{K}$ is zero. We know that the category of $\mathscr{P}$–algebras admits a projective model structure. Then the category of $\mathscr{C}$–coalgebras admits a model structure left induced by the adjunction $\Omega_\alpha \dashv B_\alpha$. We call it the α–model structure. In the nonsymmetric context, we can drop the assumption that the characteristic of the field $\mathbb{K}$ is zero.*

To prove this theorem, we will use the following result:

Theorem 83 [3, Theorem 2.23; 14, Theorem 2.2.1] *Consider an adjunction*

$$\mathsf{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathsf{M}$$

between presentable categories. Suppose that M is endowed with a cofibrantly generated model structure. Then there exists a left induced model structure on C if the morphisms which have the right lifting property with respect to left induced cofibrations are left induced weak equivalences. In particular, this is true if the category C has a cofibrant replacement functor, and if any cofibrant object has a cylinder.

From now on, a *weak equivalence* (resp. *cofibration*) of $\mathscr{C}$–coalgebras is a morphism whose image under Ω_α is a weak equivalence (resp. cofibration). An *acyclic cofibration* is a morphism which is both a cofibration and a weak equivalence. A *fibration* is a morphism which has the right lifting property with respect to all acyclic cofibrations and an *acyclic fibration* is a morphism which is both a fibration and a weak equivalence. Here is the proof.

Proof of Theorem 82 Proposition 84 ensures us that the cofibrations of the category of $\mathscr{C}$–coalgebras are the monomorphisms. Hence, any object is cofibrant. Then Proposition 90 provides us with a cylinder for any object if the characteristic of $\mathbb{K}$ is zero. In the nonsymmetric context, Proposition 93 provides us with a cylinder. We conclude by Theorem 83. □

6.2 Cofibrations

Proposition 84 *The class of cofibrations of $\mathscr{C}$–coalgebras is the class of monomorphisms.*

Lemma 85 *Let $f\colon \mathscr{D} \to \mathscr{E}$ be a monomorphism of $\mathscr{C}$–coalgebras such that $\Delta(\mathcal{E}) \subset \mathcal{C} \circ f(\mathcal{D})$. Then f is a cofibration.*

Proof We can decompose the graded $\mathbb{K}$–module $\mathcal{E}$ as $\mathcal{E} = \mathcal{D} \oplus \mathcal{F}$. The coderivation $d_{\mathcal{E}}$ corresponds then to the matrix

$$\begin{pmatrix} d_{\mathcal{D}} & \phi \\ 0 & d_{\mathcal{F}} \end{pmatrix}$$

Consider the diagram of $\mathscr{P}$–algebras

$$\begin{tikzcd} \mathcal{P} \circ (s^{-1}\mathcal{F}) \arrow[r] \arrow[d, hook] & \Omega_\alpha \mathcal{D} \\ \mathcal{P} \circ (s^{-1}\mathcal{F} \oplus \mathcal{F}) & \end{tikzcd}$$

where the horizontal map sends $s^{-1}x$ to $\phi(x) + (\alpha \circ \mathrm{Id})\Delta(x)$. The fact that it commutes with derivations is given by the fact that the derivation of $\Omega_\alpha \mathcal{E}$ squares to zero. Moreover, $s^{-1}\mathcal{F} \oplus \mathcal{F}$ is endowed with the differential $d(s^{-1}x + y) = -s^{-1}d_{\mathcal{F}}x + s^{-1}y + d_{\mathcal{F}}y$. The vertical map is a cofibration since it is the image under the left Quillen functor $\mathcal{P} \circ (-)$ of a cofibration, and f is the pushout of this vertical map along the horizontal map. Hence, f is a cofibration. □

Proof of Proposition 84 Let $f\colon \mathscr{D} \to \mathscr{E}$ be a cofibration. Then $\Omega_\alpha(f)$ is a monomorphism. Since the following square is commutative, f is a monomorphism:

$$\begin{tikzcd} \Omega_\alpha \mathscr{D} \arrow[r] & \Omega_\alpha \mathscr{E} \\ \mathscr{D} \arrow[r] \arrow[u] & \mathscr{E} \arrow[u] \end{tikzcd}$$

Conversely, if f is a monomorphism, then, it can be decomposed into the transfinite composition of the maps $f_n = \mathscr{D} + F^{\mathrm{rad}}_{n-1}\mathscr{E} \to \mathscr{D} + F^{\mathrm{rad}}_{n}\mathscr{E}$. Since the maps f_n satisfy the conditions of Lemma 85, they are cofibrations. So f is a cofibration. □

6.3 Filtered quasi-isomorphism

Definition 86 (filtered quasi-isomorphism) Let $\mathscr{D}$ and $\mathscr{E}$ be two $\mathscr{C}$–coalgebras. A morphism of $\mathscr{C}$–coalgebras f from $\mathscr{D}$ to $\mathscr{E}$ is said to be a *filtered quasi-isomorphism* if the induced morphisms between the graded complexes relative to the coradical filtrations are quasi-isomorphisms, that is, if for any integer n, the morphism from $G_n^{\mathrm{rad}}\mathscr{D}$ to $G_n^{\mathrm{rad}}\mathscr{E}$ is a quasi-isomorphism.

Proposition 87 *If the characteristic of $\mathbb{K}$ is zero, a filtered quasi-isomorphism is a weak equivalence of $\mathscr{C}$–coalgebras. The characteristic zero assumption is not necessary in the nonsymmetric context.*

We will use the following classical result:

Theorem 88 [21, Theorem XI.3.4] *Let $f\colon A \to B$ be a map of filtered chain complexes. Suppose that the filtrations are bounded below and exhaustive. If, for any integer n, the map $G_n A \to G_n B$ is a quasi-isomorphism, then f is a quasi-isomorphism.*

Proof of Proposition 87 Consider the filtration on $\Omega_\alpha\mathscr{D}$ (resp. $\Omega_\alpha\mathscr{E}$)

$$F_n\Omega_\alpha\mathscr{D} = \mathcal{P}(0) \oplus \sum_{\substack{k\geq 1\\ p_1+\cdots+p_k=n}} \mathcal{P}(k) \otimes_{\mathbb{S}_k} (F_{p_1}^{\mathrm{rad}}\mathcal{D} \otimes \cdots \otimes F_{p_k}^{\mathrm{rad}}\mathcal{D}).$$

It is clear that $\Omega_\alpha(f)$ sends $F_n\Omega_\alpha\mathscr{D}$ to $F_n\Omega_\alpha\mathscr{E}$ for any integer n. Moreover, we have

$$G_n\Omega_\alpha\mathscr{D} = \sum_{\substack{k\geq 1\\ p_1+\cdots+p_k=n}} \mathcal{P}(k) \otimes_{\mathbb{S}_k} (G_{p_1}^{\mathrm{rad}}\mathcal{D} \otimes \cdots \otimes G_{p_k}^{\mathrm{rad}}\mathcal{D}).$$

Then, by the operadic Künneth formula, $G_n(\Omega_\alpha(f))\colon G_n\Omega_\alpha\mathscr{D} \to G_n\Omega_\alpha\mathscr{E}$ is a quasi-isomorphism for any $n \in \mathbb{N}$. Hence, by Theorem 88, $\Omega_u(f)$ is a quasi-isomorphism. $\square$

Remark 89 The coradical filtration is not the only filtration whose notion of filtered quasi-isomorphism gives us weak equivalences. An exhaustive filtration $(F_n\mathcal{D})_{n\in\mathbb{N}}$ is called *admissible* if

$$\begin{aligned}
\Delta(F_n\mathcal{D}) &\subset \sum_{p_0+p_1+\cdots+p_k=n} F_0^{\mathrm{rad}}\mathcal{C} \otimes_{\mathbb{S}_k} (F_{p_1}\mathcal{D} \otimes \cdots \otimes F_{p_k}\mathcal{D}),\\
d(F_n\mathcal{D}) &\subset F_n\mathcal{D},\\
d^2(F_n\mathcal{D}) &\subset F_{n-1}\mathcal{D}.
\end{aligned}$$

Using similar arguments as in the proof just above, we can prove that a filtered quasi-isomorphism with respect to two admissible filtrations is a weak equivalence.

6.4 Cylinder object

Proposition 90 *Let $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ be a $\mathscr{C}$–coalgebra. Let $\mathscr{A} = (\mathcal{A}, \gamma_{\mathscr{A}})$ be a cylinder of $\Omega_\alpha(\mathscr{D})$ such that the structural map $p\colon \mathscr{A} \to \Omega_\alpha(\mathscr{D})$ is an acyclic fibration. The diagram*

$$\begin{tikzcd} B_\alpha\Omega_\alpha(\mathscr{D}\oplus\mathscr{D}) \arrow[r] & B_\alpha(\mathscr{A}) \arrow[r, "B_\alpha p"] & B_\alpha\Omega_\alpha(\mathscr{D}) \\ \mathscr{D}\oplus\mathscr{D} \arrow[r] \arrow[u] & \mathscr{E} \arrow[r] \arrow[u] & \mathscr{D} \arrow[u] \end{tikzcd}$$

where $\mathscr{E} := B_\alpha(\mathscr{A}) \times_{B_\alpha(\Omega_\alpha\mathscr{D})} \mathscr{D}$ provides us with a cylinder $\mathscr{E} = (\mathcal{E}, \Delta_{\mathscr{E}})$ for the $\mathscr{C}$–coalgebra $\mathscr{D}$.

Lemma 91 *The pullback $\mathscr{E}$ is the final subgraded $\mathscr{C}^{\mathrm{grad}}$–coalgebra of $B_\alpha\mathscr{A}$ whose image in $B_\alpha\Omega_\alpha\mathscr{D}$ is in the image of the morphism $\mathscr{D} \to B_\alpha\Omega_\alpha\mathscr{D}$.*

Proof Let $\mathscr{F} = (\mathcal{F}, \Delta_{\mathscr{F}})$ be the final subgraded $\mathscr{C}^{\mathrm{grad}}$–coalgebra of $B_\alpha\mathscr{A}$ whose image in $B_\alpha\Omega_\alpha\mathscr{D}$ is in the image of the morphism $\mathscr{D} \to B_\alpha\Omega_\alpha\mathscr{D}$. Proving that $\mathscr{F}$ is the underlying $\mathscr{C}^{\mathrm{grad}}$–coalgebra of $\mathscr{E}$ amounts to proving that $\mathcal{F}$ is stable under the coderivation D of $B_\alpha\mathscr{A}$. We prove it by induction on the coradical filtration of $\mathcal{F}$. First, by the maximality property of $\mathscr{F}$, $F_0^{\mathrm{rad}}\mathcal{F}$ is stable under D. Then suppose that $F_n^{\mathrm{rad}}\mathcal{F}$ is stable under D for an integer $n \geq 0$. Let x be an element of $F_{n+1}^{\mathrm{rad}}\mathcal{F}$. On the one hand, $B_\alpha(p)D(x) = D(B_\alpha(p)(x))$. Since $B_\alpha(p)(x)$ is in the image of $\mathscr{D}$ and since this image is stable under the coderivation of $B_\alpha\Omega_\alpha\mathscr{D}$, then $B_\alpha(p)D(x)$ is in the image of $\mathscr{D}$. On the other hand, we have

$$\Delta(D(x)) = (d_{\mathcal{C}} \circ \mathrm{Id} + \mathrm{Id} \circ' D)\Delta(x).$$

So, since $\Delta(x) - 1_{\mathscr{C}} \otimes x \in \mathcal{C} \circ (F_n^{\mathrm{rad}}\mathcal{F})$, and since $F_n^{\mathrm{rad}}\mathcal{F}$ is stable under D by the inductivity assumption,

$$\Delta(D(x)) - 1_{\mathscr{C}} \otimes D(x) = (d_{\mathcal{C}} \circ \mathrm{Id} + \mathrm{Id} \circ' D)(\Delta(x) - 1_{\mathscr{C}} \otimes x) \in \mathscr{C} \circ (F_n^{\mathrm{rad}}\mathcal{F}).$$

By these two points, $\mathcal{F} + \mathbb{K} \cdot D(x)$ is a subgraded $\mathscr{C}^{\mathrm{grad}}$–coalgebra of $B_\alpha(\mathscr{A})$ whose image in $B_\alpha\Omega_\alpha\mathscr{D}$ is in the image of $\mathscr{D}$. By the maximality property of $\mathscr{F}$, then, $D(x) \in \mathcal{F}$. So, $F_{n+1}^{\mathrm{rad}}\mathcal{F}$ is stable under D. Hence, by induction $\mathcal{F}$ is stable under D. □

To prove Proposition 90, we will show that the pullback map $\mathscr{E} \to \mathscr{D}$ is a filtered quasi-isomorphism. Since $\Omega_\alpha \mathscr{D}$ is a cofibrant $\mathscr{P}$–algebra, there exists a right inverse $q\colon \Omega_\alpha \mathscr{D} \to \mathscr{A}$ to the acyclic fibration $p\colon \mathscr{A} \to \Omega_\alpha \mathscr{D}$. Then let us decompose $\mathcal{A}$ as $\mathcal{A} = \Omega_\alpha \mathscr{D} \oplus K$. The chain complex K is acyclic. So let $h\colon K \to K$ be a degree 1 map such that $\partial(h) = \mathrm{Id}_K$. It can be extended to a map

$$B_\alpha \mathscr{A} \twoheadrightarrow \mathcal{A} \twoheadrightarrow K \to \mathcal{A}, \quad x \mapsto h(x).$$

The zero map is a coderivation on the graded cooperad $\mathscr{C}^{\mathrm{grad}}$. Then let D_h be the degree 1 coderivation of $(B_\alpha \mathscr{A})^{\mathrm{grad}}$ relative to the zero coderivation on $\mathscr{C}^{\mathrm{grad}}$ whose projection on $\mathcal{A}$ is h. In other words, $D_h = \mathrm{Id}_{\mathscr{C}} \circ' h$.

Lemma 92 *The sub-$\mathscr{C}$–coalgebra $\mathscr{E}$ of $B_\alpha \mathscr{A}$ is stable under D_h.*

Proof By Lemma 91, it suffices to prove that the final subgraded $\mathscr{C}^{\mathrm{grad}}$–coalgebra of $B_\alpha \mathscr{A}$ whose image in $B_\alpha \Omega_\alpha \mathscr{D}$ lies inside $\mathscr{D}$ is stable under D_h. To that purpose, we use the same arguments as in the proof of Lemma 91 and the fact that $B_\alpha(p) D_h = 0$. □

Proof of Proposition 90 Since the map $\mathscr{D} \oplus \mathscr{D} \to \mathscr{E}$ is a monomorphism and so a cofibration, it suffices to show that the map $\mathscr{E} \to \mathscr{D}$ is a weak equivalence. We show that it is a filtered quasi-isomorphism. Let $n \in \mathbb{N}$; let us show that the map $G_n \mathcal{E} \to G_n \mathcal{D}$ is a quasi-isomorphism. To that purpose, consider the filtration on $B_\alpha \mathscr{A}$

$$F'_k B_\alpha \mathscr{A} := \sum_{i \le k} \mathcal{C} \otimes_{\mathbb{S}} (K^{\otimes i} \otimes (\Omega_\alpha \mathscr{D})^{\otimes j}).$$

This filtration is stable under the coderivations d and D_h and it induces a filtration on $G_n^{\mathrm{rad}} \mathcal{E}$. It is clear that the morphism $G'_0 G_n^{\mathrm{rad}} \mathcal{E} \to G_n^{\mathrm{rad}} \mathcal{D}$ is an isomorphism. Moreover, for any integer $k \ge 1$, $\partial(D_h) = k.\mathrm{Id}$ on $G'_k G_n^{\mathrm{rad}} \mathcal{E}$. Since the characteristic of $\mathbb{K}$ is zero, $G'_k G_n^{\mathrm{rad}} \mathcal{E}$ is acyclic. By Theorem 88, the map $G_n \mathcal{E} \to G_n \mathcal{D}$ is a quasi-isomorphism. □

Proposition 93 *In the nonsymmetric context, $\mathscr{D} \otimes \mathrm{DK}^c(\Delta[1])$ provides us with a cylinder for the $\mathscr{C}$–coalgebra $\mathscr{D}$.*

Proof Since $G_n^{\mathrm{rad}}(\mathcal{D} \otimes \mathrm{DK}^c(\Delta[1])) = G_n^{\mathrm{rad}}(\mathcal{D}) \otimes \mathrm{DK}^c(\Delta[1])$ and since the map $\mathrm{DK}^c(\Delta[1]) \to \mathbb{K}$ is a quasi-isomorphism, $\mathcal{D} \otimes \mathrm{DK}^c(\Delta[1]) \to \mathcal{D}$ is a filtered quasi-isomorphism and so a weak equivalence. □

6.5 Enrichment in coalgebras the nonsymmetric context

Proposition 94 *In the nonsymmetric context, the assignment $\mathscr{D}, \mathscr{D}' \mapsto \{\mathscr{D}, \mathscr{D}'\}^{\mathrm{ns}}$ defines a homotopical enrichment of the category of $\mathscr{C}$–coalgebras together with its α–model structure over the category of counital coassociative coalgebras.*

Proof Let $f\colon \mathscr{D} \to \mathscr{D}'$ be a cofibration of $\mathscr{C}$–coalgebras, let $g\colon \mathscr{E} \to \mathscr{E}'$ be a fibration of $\mathscr{C}$–coalgebras and let $h\colon X \to Y$ be a cofibration (ie a monomorphism) of counital coassociative coalgebras. Consider the square

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] & \{\mathscr{D}', \mathscr{E}\}^{\mathrm{ns}} \arrow[d] \\
Y \arrow[r] & \{\mathscr{D}', \mathscr{E}'\} \times_{\{\mathscr{D}, \mathscr{E}'\}} \{\mathscr{D}, \mathscr{E}\}
\end{tikzcd}
$$

It induces a square

$$
\begin{tikzcd}
\mathscr{D}' \otimes X \amalg_{\mathscr{D} \otimes X} \mathscr{D} \otimes Y \arrow[r] \arrow[d] & \mathscr{E} \arrow[d] \\
\mathscr{D}' \otimes Y \arrow[r] & \mathscr{E}'
\end{tikzcd}
$$

The left vertical map is a monomorphism and so a cofibration.

- If the morphism $g\colon \mathscr{E} \to \mathscr{E}'$ is an acyclic fibration, then the square has a lifting.
- Suppose that the morphism $h\colon X \to Y$ is an acyclic cofibration. Then the morphism $\mathscr{D} \otimes X \to \mathscr{D} \otimes Y$ is a filtered quasi-isomorphism and a cofibration, so it is an acyclic cofibration. Hence, its pushout $\mathscr{D}' \otimes X \to \mathscr{D}' \otimes X \amalg_{\mathscr{D} \otimes X} \mathscr{D} \otimes Y$ is also an acyclic cofibration. Moreover, the map $\mathscr{D}' \otimes X \to \mathscr{D}' \otimes Y$ is a filtered quasi-isomorphism and so a weak equivalence. So, by the 2-out-of-3 rule, the morphism $\mathscr{D}' \otimes X \amalg_{\mathscr{D} \otimes X} \mathscr{D} \otimes Y \to \mathscr{D}' \otimes Y$ is a weak equivalence. Since it is a cofibration, it is an acyclic cofibration and the square has a lifting.
- Suppose that the morphism $f\colon \mathscr{D} \to \mathscr{D}'$ is an acyclic cofibration. Then the morphism $\mathscr{D} \otimes X \to \mathscr{D}' \otimes X$ is an acyclic cofibration. This is a consequence of the fact that $\Omega_\alpha(\mathscr{D} \otimes X) = (\Omega_\alpha \mathscr{D}) \triangleleft X$, and that for any fibration of $\mathscr{P}$–algebras $\mathscr{A} \to \mathscr{A}'$, the morphism $[X, \mathscr{A}] \to [X, \mathscr{A}']$ is also a fibration. Then the same arguments as in the previous point show us that the morphism $\mathscr{D}' \otimes X \amalg_{\mathscr{D} \otimes X} \mathscr{D} \otimes Y \to \mathscr{D}' \otimes Y$ is an acyclic cofibration and so the square has a lifting. □

6.6 Changing operads and cooperads

In this subsection, we explore how the left induced model structure on coalgebras over a curved conilpotent cooperad is modified when we change the underlying operadic twisting morphism. This is inspired by [8], where a similar study is done in the context of augmented dg operads and dg conilpotent cooperads.

Recall first that a morphism of dg operads $f\colon \mathscr{P} \to \mathscr{Q}$ induces an adjunction between their categories of algebras

$$\mathscr{P}\text{–alg} \underset{f^*}{\overset{f_!}{\rightleftarrows}} \mathscr{Q}\text{–alg}$$

whose right adjoint f^* sends a $\mathscr{Q}$–algebra $\mathscr{A}$ to the same underlying chain complex. This adjunction is a Quillen adjunction with respect to the projective model structures; see [4]. Similar things happen for coalgebras over curved conilpotent cooperads.

Proposition 95 *Let $f\colon \mathscr{C} \to \mathscr{D}$ be a morphism of curved conilpotent cooperads. It induces an adjunction between their categories of coalgebras,*

$$\mathscr{C}\text{–cog} \underset{f^!}{\overset{f_*}{\rightleftarrows}} \mathscr{D}\text{–cog},$$

whose left adjoint f_* sends a $\mathscr{C}$–coalgebra $\mathscr{E}$ to the same underlying graded $\mathbb{K}$–module.

Proof Let $\mathscr{E} = (\mathcal{E}, \Delta, d)$ be a $\mathscr{C}$–coalgebra. It has a structure of $\mathscr{D}$–coalgebra defined by the composite map

$$\mathcal{E} \xrightarrow{\Delta} \mathscr{C} \circ \mathcal{E} \xrightarrow{f \circ \mathrm{Id}} \mathscr{D} \circ \mathcal{E}.$$

This defines the functor f_*. Since it preserves colimits and since the category of $\mathscr{C}$–coalgebras and the category of $\mathscr{D}$–coalgebras are presentable, f_* has a right adjoint by Proposition 2. □

Further, let us fix a dg operad $\mathscr{P}$. The canonical operadic twisting morphism $\pi\colon B_c\mathscr{P} \to \mathscr{P}$ is universal in the sense that any operadic twisting morphism α from a curved conilpotent cooperad $\mathscr{C}$ to $\mathscr{P}$ is equivalent to a morphism of curved cooperads f from $\mathscr{C}$ to $B_c\mathscr{P}$; then $\alpha = \pi f$. In that context, the cobar functor Ω_α can be decomposed as $\Omega_\alpha = \Omega_\pi f_*$, and the α–model structure on the category of $\mathscr{C}$–coalgebras is the model structure left induced by the π–model structure on the category of $B_c\mathscr{P}$–coalgebras.

On the other hand, let us fix a curved conilpotent cooperad $\mathscr{C}$. The canonical operadic twisting morphism $\iota\colon \mathscr{C} \to \Omega_u\mathscr{C}$ is universal in the sense that any operadic twisting morphism $\alpha\colon \mathscr{C} \to \mathscr{P}$ is equivalent to the data of a morphism of operads f from $\Omega_u \mathscr{C}$ to $\mathscr{P}$; then $\alpha = f\iota$. A direct consequence of the following proposition is that the model structure on $\mathscr{C}$–coalgebras induced by the universal operadic twisting morphism $\iota\colon \mathscr{C} \to \Omega_u \mathscr{C}$ is universal in the sense that any α–model structure is a left Bousfield localization of this ι–model structure.

Proposition 96 *Let $\alpha\colon \mathscr{C} \to \mathscr{P}$ be an operadic twisting morphism and let $f\colon \mathscr{P} \to \mathscr{Q}$ be a morphism of dg operads. The $(f\alpha)$–model structure on the category of $\mathscr{C}$–coalgebras is the left Bousfield localization of the α–model structure with respect to $(f\alpha)$–weak equivalences. Moreover, if the Quillen adjunction $f_! \dashv f^*$ is a Quillen equivalence, the $(f\alpha)$–model structure coincides with the α–model structure.*

Proof The cofibrations of the α–model structure and the cofibrations of the $(f\alpha)$–model structure are both the monomorphisms. Moreover, the functor $f_!$ is a left Quillen adjoint functor. So, for any α–weak equivalence g, since $\Omega_\alpha(g)$ is a weak equivalence between cofibrant objects, $\Omega_{(f\alpha)}(g) = f_!\Omega_\alpha(g)$ is a weak equivalence. So the α–weak equivalences are in particular $(f\alpha)$–weak equivalences. So is proven the fact that the $(f\alpha)$–model structure is a left Bousfield localization of the α–model structure. Suppose now that the adjunction $f_! \dashv f^*$ is a Quillen equivalence. Then, for any $\mathscr{C}$–coalgebra $\mathscr{E}$, the morphism

$$\Omega_\alpha \mathscr{E} \to f^* f_! \Omega_\alpha \mathscr{E} = f^* \Omega_{f\alpha} \mathscr{E}$$

is a quasi-isomorphism. Since the functor f^* is the identity on the underlying chain complexes, the commutative square

$$\begin{tikzcd}
f^*\Omega_{(f\alpha)}\mathscr{E} \arrow[r, "f^*\Omega_{(f\alpha)}(g)"] & f^*\Omega_{(f\alpha)}\mathscr{E}' \\
\Omega_\alpha \mathscr{E} \arrow[u] \arrow[r, "\Omega_\alpha(g)"'] & \Omega_\alpha \mathscr{E}' \arrow[u]
\end{tikzcd}$$

ensures that a morphism $g\colon \mathscr{E} \to \mathscr{E}'$ of $\mathscr{C}$–coalgebras is a α–weak equivalence if and only if it is an $(f\alpha)$–weak equivalence. □

7 The universal model structure

In the previous section, we studied model structures on categories of coalgebras over a curved conilpotent cooperad which are induced by an operadic twisting morphism α.

In this section, we investigate the particular case where the operadic twisting morphism is the universal twisting morphism $\iota\colon \mathscr{C} \to \Omega_u\mathscr{C}$ for any curved conilpotent cooperad $\mathscr{C}$. This model structure is universal in the sense that, for any operadic twisting morphism $\alpha\colon \mathscr{C} \to \mathscr{P}$, the α–model structure on the category of $\mathscr{C}$–coalgebras is obtained from the ι–model structure by Bousfield localization. We will show that the adjunction $\Omega_\iota \dashv B_\iota$ is a Quillen equivalence, that the fibrant $\mathscr{C}$–coalgebras in the ι–model structure are the images of the $\Omega_u\mathscr{C}$–algebras under the functor B_ι, and we will describe the cofibrations, the weak equivalences and the fibrations between them. Moreover, we will prove that the enrichment of $\mathscr{C}$–coalgebras over simplicial sets that we described above computes the mapping spaces expected by the model structure.

We suppose here that the characteristic of the field $\mathbb{K}$ is zero. This assumption is not necessary in the nonsymmetric context.

7.1 Quillen equivalence

Theorem 97 *The adjunction $\Omega_\iota \dashv B_\iota$ relating $\mathscr{C}$–coalgebras to $\Omega_u\mathscr{C}$–algebras is a Quillen equivalence.*

Proof Let us show that for any $\Omega_u\mathscr{C}$–algebra $\mathscr{A} = (\mathcal{A}, \gamma_{\mathscr{A}})$, the map $\Omega_\iota B_\iota \mathscr{A} = \Omega_u\mathscr{C} \circ_\iota \mathscr{C} \circ_\iota \mathscr{A} \to \mathscr{A}$ is a quasi-isomorphism. The coradical filtration of $\mathscr{C}$ induces a filtration on $\Omega_u\mathscr{C}$,

$$F_0\Omega_u\mathscr{C} := \mathbb{K}.1,$$

$$F_n\Omega_u\mathscr{C} := \mathbb{K}.1 \oplus \sum_{\substack{i_1+\ldots+i_k=n\\ k\geq 1}} s^{-1}F^{\mathrm{rad}}_{i_1}\overline{\mathcal{C}} \otimes \cdots \otimes s^{-1}F^{\mathrm{rad}}_{i_k}\overline{\mathcal{C}} \quad \text{for } n \geq 1.$$

It induces a filtration on $\Omega_u\mathscr{C} \circ_\iota \mathscr{C}$ and on $\Omega_u\mathscr{C} \circ_\iota \mathscr{C} \circ_\iota \mathscr{A}$,

$$F_n(\Omega_u\mathscr{C} \circ_\iota \mathscr{C}) := F_n\Omega_u\mathscr{C}(0) \oplus \sum_{\substack{i_0+\cdots+i_k=n\\ k\geq 1}} F_{i_0}(\Omega_u\mathscr{C})(k) \otimes_{\mathbb{S}_k} (F^{\mathrm{rad}}_{i_1}\mathcal{C} \otimes \cdots \otimes F^{\mathrm{rad}}_{i_k}\mathcal{C}),$$

$$F_n(\Omega_u\mathscr{C} \circ_\iota \mathscr{C} \circ_\iota \mathscr{A}) := F_n(\Omega_u\mathscr{C} \circ_\iota \mathscr{C}) \circ \mathcal{A}.$$

Then $G(\Omega_u\mathscr{C} \circ_\iota \mathscr{C}) = \Omega_u(G\mathscr{C}) \circ_{G\iota} G\mathscr{C}$. By [19, Lemma 6.5.14], the map

$$\Omega_u(G\mathscr{C}) \circ_{G\iota} G\mathscr{C} \to \mathcal{I}$$

is a quasi-isomorphism. So, the map

$$G(\Omega_u\mathscr{C} \circ_\iota \mathscr{C} \circ_\iota \mathcal{A}) \to G\mathcal{A}$$

is a quasi-isomorphism (here $G\mathcal{A}$ is the graded complex corresponding to the constant filtration $F_n\mathcal{A} = \mathcal{A}$). Hence, by Theorem 88, the map $\Omega_u\mathscr{C} \circ_\iota \mathscr{C} \circ_\iota \mathscr{A} \to \mathscr{A}$ is a quasi-isomorphism. Since the model structure on $\mathscr{C}$–coalgebras is transferred from the model structure on $\Omega_u\mathscr{C}$–algebras, then the fact that the counit $\Omega_\iota B_\iota\mathscr{A} \to \mathscr{A}$ is a weak equivalence for any algebra $\mathscr{A}$ ensures us that the Quillen adjunction $\Omega_\iota \dashv B_\iota$ is a Quillen equivalence. □

7.2 Fibrant objects

The purpose of this subsection is to describe the fibrant objects of the ι–model structure.

Definition 98 (quasicofree $\mathscr{C}$–coalgebras) A $\mathscr{C}$–coalgebra is said to be *quasicofree* if its underlying $\mathscr{C}^{\mathrm{grad}}$–coalgebra is cofree, that is, isomorphic to a coalgebra of the form $\mathscr{C}^{\mathrm{grad}} \circ \mathcal{V}$. A morphism of quasicofree $\mathscr{C}$–coalgebras $F\colon \mathscr{C} \circ \mathcal{V} \to \mathscr{C} \circ \mathcal{W}$ (together with choices of cogenerators $\mathcal{V}$ and $\mathcal{W}$) is said to be *strict* if there exists a map $f\colon \mathcal{V} \to \mathcal{W}$ such that $F = \mathrm{Id} \circ f$.

Proposition 99 *The functor B_ι is an embedding of the category of $\Omega_u\mathscr{C}$–algebras into the category of $\mathscr{C}$–coalgebras whose essential image is spanned by quasicofree $\mathscr{C}$–coalgebras. Moreover, a morphism of $\mathscr{C}$–coalgebras $B_\iota\mathscr{A} = \mathscr{C} \circ_\iota \mathscr{A} \to B_\iota\mathscr{A}' = \mathscr{C} \circ_\iota \mathscr{A}'$ is in the image of B_ι if and only if it is strict.*

Proof It is straightforward to prove that the functor B_ι is faithful and conservative. Moreover, it is clear that the images of the functor B_ι are in particular quasicofree $\mathscr{C}$–coalgebras and strict morphisms. Conversely, let $\mathscr{D} := \mathscr{C} \circ \mathcal{A}$ be a quasicofree $\mathscr{C}$–coalgebra. Its coderivation extends the degree -1 map $d_{\mathcal{A}} \oplus \gamma\colon \mathcal{A} \oplus \overline{\mathcal{C}} \circ \mathcal{A} \to \mathcal{A}$. The map γ gives us a degree -1 map from $\overline{\mathcal{C}}$ to the operad $\mathrm{End}_{\mathcal{A}}$, that we denote by α. The coderivation which extends $d_{\mathcal{A}} \oplus \gamma$ squares to $(\theta \circ \mathrm{Id})\Delta$, so α is a twisting morphism and so induces a morphism of operads from $\Omega_u\mathscr{C}$ to $\mathrm{End}_{\mathcal{A}}$, which is an $\Omega_u\mathscr{C}$–algebra structure on $\mathcal{A}$. Then $\mathscr{D} \simeq B_\iota\mathcal{A}$. Further, let $F = \mathrm{Id} \circ f$ be a strict morphism from $B_\iota\mathscr{A}$ to $B_\iota\mathscr{B}$. Since F commutes with the coderivations, f is a morphism of $\Omega_u\mathscr{C}$–algebras. □

Theorem 100 *The fibrant $\mathscr{C}$–coalgebras in the ι–model structure are the quasicofree $\mathscr{C}$–coalgebras* (and so the objects in the essential image of the functor B_ι).

Proof Let $\mathscr{D}$ be a fibrant object. Since the morphism $\mathscr{D} \to B_\iota \Omega_\iota \mathscr{D}$ is an acyclic cofibration, the following square has a lifting:

$$\begin{tikzcd} \mathscr{D} \arrow[r, "\mathrm{Id}"] \arrow[d] & \mathscr{D} \arrow[d] \\ B_\iota \Omega_\iota \mathscr{D} \arrow[r] & * \end{tikzcd}$$

Hence, $\mathscr{D}$ is a retract of a quasicofree $\mathscr{C}$–coalgebra. By Lemma 101, it is a quasicofree $\mathscr{C}$–coalgebra. Conversely, a quasicofree $\mathscr{C}$–coalgebra is fibrant since it is isomorphic to the image under B_ι of an $\Omega_u \mathscr{C}$–algebra which is fibrant. □

Lemma 101 *A retract of a cofree graded $\mathscr{C}^{\mathrm{grad}}$–coalgebra is a cofree graded $\mathscr{C}^{\mathrm{grad}}$–coalgebra.*

Proof Let $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ be a graded $\mathscr{C}^{\mathrm{grad}}$–coalgebra which is a retract of $\mathscr{C} \circ \mathcal{V}$. On the one hand, the following diagram is a retract, that is, the compositions of the horizontal maps give the identity on the bottom and on the top:

$$\begin{tikzcd} G_n^{\mathrm{rad}} \mathcal{D} \arrow[r] \arrow[d] & G_n^{\mathrm{rad}} (\mathcal{C} \circ \mathcal{V}) \arrow[r] \arrow[d] & G_n^{\mathrm{rad}} \mathcal{D} \arrow[d] \\ (G_n^{\mathrm{rad}} \mathcal{C}) \circ F_0^{\mathrm{rad}} \mathcal{D} \arrow[r] & (G_n^{\mathrm{rad}} \mathcal{C}) \circ F_0^{\mathrm{rad}} (\mathcal{C} \circ \mathcal{V}) \arrow[r] & (G_n^{\mathrm{rad}} \mathcal{C}) \circ F_0^{\mathrm{rad}} \mathcal{D} \end{tikzcd}$$

Since the middle vertical map is an isomorphism, all the vertical maps are isomorphisms. On the other hand, the map $\epsilon \circ \mathrm{Id}\colon \mathcal{C} \circ \mathcal{V} \to \mathcal{V} = F_0^{\mathrm{rad}} \mathcal{C} \circ \mathcal{V}$ gives us a map $\mathcal{D} \to F_0 \mathcal{D}$ and hence a morphism of graded $\mathscr{C}$–coalgebras $f\colon \mathscr{D} \to \mathscr{C} \circ F_0 \mathcal{D}$. Let us show that f is an isomorphism. It is clear that the map $F_0 \mathcal{D} \to F_0 (\mathcal{C} \circ F_0 \mathcal{D})$ is an isomorphism. For any integer $n \geq 1$, the following diagram is commutative:

$$\begin{tikzcd} G_n(\mathcal{D}) \arrow[r, "f"] \arrow[d, "\Delta"'] & G_n(\mathcal{C} \circ F_0 \mathcal{D}) \arrow[d, "\Delta"] \\ (G_n \mathcal{C}) \circ F_0 \mathcal{D} \arrow[r, "\mathrm{Id} \circ f"'] & (G_n \mathcal{C}) \circ F_0 (\mathcal{C} \circ F_0 \mathcal{D}) = (G_n \mathcal{C}) \circ F_0 \mathcal{D} \end{tikzcd}$$

Since the vertical maps are isomorphisms and since the bottom horizontal map is an isomorphism, the top horizontal map is also an isomorphism. Hence, the map $Gf\colon G\mathcal{D} \to G(\mathcal{C} \circ F_0 \mathcal{D})$ is an isomorphism. By Theorem 88, f is an isomorphism. □

7.3 Cofibrations, fibrations and weak equivalences between fibrant objects

We show here that cofibrations, weak equivalences and fibrations between fibrant $\mathscr{C}$–coalgebras are easily characterized.

Proposition 102 *Let $\mathscr{A} = (\mathcal{A}, \gamma_{\mathscr{A}})$ and $\mathscr{B} = (\mathcal{B}, \gamma_{\mathscr{B}})$ be two $\Omega_u\mathscr{C}$–algebras and let $F\colon B_\iota\mathscr{A} \to B_\iota\mathscr{B}$ be a morphism between their bar constructions. We denote by $f\colon B_\iota\mathscr{A} \to \mathcal{B}$ its projection $f = \pi_{\mathcal{B}}F$ on $\mathcal{B}$.*

- *The morphism F is a cofibration if and only if the restriction $f|_{\mathcal{A}}$ is a monomorphism.*
- *The morphism F is a weak equivalence if and only $f|_{\mathcal{A}}$ is a quasi-isomorphism.*
- *The morphism F is a fibration if and only if $f|_{\mathcal{A}}$ is an epimorphism.*

Lemma 103 *The morphism of chain complexes $\mathcal{A} \to \Omega_\iota B_\iota\mathscr{A}$ which is the restriction to $\mathcal{A}$ of the canonical morphism $B_\iota\mathscr{A} \to B_\iota\Omega_\iota B_\iota\mathscr{A}$ is a quasi-isomorphism.*

Proof It is a right inverse of the canonical morphism of $\Omega_u\mathscr{C}$–algebras $\Omega_\iota B_\iota\mathscr{A} \to \mathscr{A}$, which is a quasi-isomorphism. $\square$

Proof of Proposition 102 Note first that $f|_{\mathcal{A}} = F|_{\mathcal{A}}$.

- Suppose that F is a cofibration, ie a monomorphism. Then its restriction $F|_{\mathcal{A}}$ is also a monomorphism. Conversely, suppose that the map $f|_{\mathcal{A}}$ is a monomorphism. We can prove by induction that, for any integer n, the map $F\colon F_n^{\mathrm{rad}} B_\iota\mathscr{A} \to F_n^{\mathrm{rad}} B_\iota\mathscr{B}$ is a monomorphism.

- By Lemma 103, the maps $\mathcal{A} \to \Omega_\iota B_\iota\mathscr{A}$ and $\mathcal{B} \to \Omega_\iota B_\iota\mathscr{B}$ are quasi-isomorphisms. Consider the diagram

$$\begin{tikzcd} \Omega_\iota B_\iota\mathscr{A} \arrow[r, "\Omega_\iota F"] & \Omega_\iota B_\iota\mathscr{B} \\ \mathcal{A} \arrow[u] \arrow[r, "f|_{\mathcal{A}}"'] & \mathcal{B} \arrow[u] \end{tikzcd}$$

It ensures that $f|_{\mathcal{A}}$ is a quasi-isomorphism if and only if $\Omega_\iota F$ is a quasi-isomorphism, that is, if and only if F is a weak equivalence.

- Suppose that F is a fibration. Notice first that any chain complex can be considered as a $\mathscr{C}$–coalgebra whose decomposition is given by the map with $\Delta x = 1_{\mathscr{C}} \otimes x$ (it is a coalgebra since $\Delta_{\mathscr{C}}(1_{\mathscr{C}}) \otimes x = 1_{\mathscr{C}} \otimes 1_{\mathscr{C}} \otimes x = 1_{\mathscr{C}} \otimes \Delta x$; the commutation with the derivations and the curvature condition are straightforward to check). Then any square of $\mathscr{C}$–coalgebras as follows has a lifting:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & B_\iota \mathscr{A} \arrow[d] \\ D^n \arrow[r] & B_\iota \mathscr{B} \end{tikzcd}$$

This ensures that the map $f|_{\mathcal{A}}$ is an epimorphism. Conversely, suppose that $f|_{\mathcal{A}}$ is an epimorphism. By Lemma 104, there exists an isomorphism $G\colon B_\iota \mathscr{A}' \to B_\iota \mathscr{A}$ such that FG is in the image of the functor B_ι. If we denote by g the map from $\mathcal{A}'$ to $\mathcal{A}$ which underlies G, then g is an isomorphism by Lemma 105. Then fg is a fibration of $\Omega_u \mathscr{C}$–algebras and so $FG = B_\iota(fg)$ is a fibration. Since G is an isomorphism, F is a fibration. □

Lemma 104 *Let $F\colon B_\iota \mathscr{A} \to B_\iota \mathscr{B}$ be a morphism of $\mathscr{C}$–coalgebras such that the underlying morphism $f\colon \mathcal{A} \to \mathcal{B}$ is surjective. Then there exists an $\Omega_u \mathscr{C}$–algebra $\mathscr{A}'$ and an isomorphism of $\mathscr{C}$–coalgebras $G\colon B_\iota \mathscr{A}' \to B_\iota \mathscr{A}$ such that FG is a strict morphism, that is, in the image of the functor B_ι.*

Proof We build an isomorphism of graded $\mathscr{C}^{\mathrm{grad}}$–coalgebras $G\colon \mathscr{C} \circ \mathcal{A} \to \mathscr{C} \circ \mathcal{A}$ such that FG is a strict morphism, that is, of the form $\mathrm{Id}_{\mathscr{C}} \circ h$. To that purpose we define inductively maps $g_n\colon F_n^{\mathrm{rad}} \mathscr{C} \circ \mathcal{A} \to \mathcal{A}$ such that g_{n-1} is the restriction of g_n to $F_{n-1}^{\mathrm{rad}} \mathscr{C} \circ \mathcal{A}$ and such that we have the equality between maps from $F_n^{\mathrm{rad}} \mathscr{C} \circ \mathcal{A}$ to $\mathcal{A}$

$$fg_n + f(\mathrm{Id} \circ g_{n-1})(\bar{\Delta} \circ \mathrm{Id}) = f\pi_{\mathcal{A}}, \tag{1}$$

where $\pi_{\mathcal{A}} = \epsilon \circ \mathrm{Id}$ is the projection of $\mathscr{C} \circ \mathcal{A}$ on $\mathcal{A}$. First, let us choose $g_0 = \mathrm{Id}_{\mathcal{A}}$. Then suppose that we have built g_n satisfying (1). The map $f\colon \mathcal{A} \to \mathcal{B}$ and the injection of $F_n^{\mathrm{rad}} \mathscr{C} \circ \mathcal{A}$ into $F_{n+1}^{\mathrm{rad}} \mathscr{C} \circ \mathcal{A}$ give us the square

$$\begin{tikzcd} \hom_{\mathsf{gMod}}(F_{n+1}^{\mathrm{rad}} \mathscr{C} \circ \mathcal{A}, \mathcal{A}) \arrow[r] \arrow[d] & \hom_{\mathsf{gMod}}(F_{n+1}^{\mathrm{rad}} \mathscr{C} \circ \mathcal{A}, \mathcal{B}) \arrow[d] \\ \hom_{\mathsf{gMod}}(F_{n}^{\mathrm{rad}} \mathscr{C} \circ \mathcal{A}, \mathcal{A}) \arrow[r] & \hom_{\mathsf{gMod}}(F_{n}^{\mathrm{rad}} \mathscr{C} \circ \mathcal{A}, \mathcal{B}) \end{tikzcd}$$

The following map is surjective:

$$\hom_{\mathsf{gMod}}(F_{n+1}^{\mathrm{rad}} \mathscr{C} \circ \mathcal{A}, \mathcal{A}) \\ \to \hom_{\mathsf{gMod}}(F_{n}^{\mathrm{rad}} \mathscr{C} \circ \mathcal{A}, \mathcal{A}) \times_{\hom_{\mathsf{gMod}}(F_{n}^{\mathrm{rad}} \mathscr{C} \circ \mathcal{A}, \mathcal{B})} \hom_{\mathsf{gMod}}(F_{n+1}^{\mathrm{rad}} \mathscr{C} \circ \mathcal{A}, \mathcal{B}).$$

So there exists an element of $\hom_{\mathsf{gMod}}(F^{\mathrm{rad}}_{n+1}\mathscr{C}\circ\mathcal{A},\mathcal{A})$ whose image under this map is the pair $(g_n, f\pi_{\mathcal{A}} - f_{n+1}(\mathrm{Id}\circ g_n)(\overline{\Delta}\circ\mathrm{Id}))$. We can choose this element to be g_{n+1}. Thus, let g be the map from $\mathscr{C}\circ\mathcal{A}$ to $\mathcal{A}$ whose restriction to $F^{\mathrm{rad}}_n\mathscr{C}\circ\mathcal{A}$ is g_n for any n. Let G be the map of graded $\mathscr{C}^{\mathrm{grad}}$–coalgebras which extends g. By Lemma 105, the map G is an isomorphism. Let us transfer the coderivation of $B_\iota\mathscr{A}$ to $\mathscr{C}\circ\mathcal{A}$ along the isomorphism G. This gives us a new $\Omega_u\mathscr{C}$–algebra structure on the chain complex $\mathcal{A}$, which we denote by $\mathscr{A}'$. Finally, the morphism FG is the image under the functor B_ι of the morphism of $\Omega_u\mathscr{C}$–algebras $fg_0\colon \mathscr{A}'\to\mathscr{B}'$. □

Lemma 105 *Let $F\colon \mathscr{D}=\mathscr{C}\circ\mathcal{V}\to\mathscr{E}=\mathscr{C}\circ\mathcal{W}$ be a morphism of quasicofree $\mathscr{C}$–coalgebras. Then F is an isomorphism if and only if its underlying map $f\colon\mathcal{V}\to\mathcal{W}$ is an isomorphism.*

Proof Suppose first that F is an isomorphism with inverse G. Let us denote by $g\colon\mathcal{W}\to\mathcal{V}$ the map underlying G. Then the map g is inverse to f and so f is an isomorphism. Conversely, suppose that f is an isomorphism. A straightforward induction shows that F is both injective and surjective. □

7.4 Mapping spaces and deformation theory

Proposition 106 *For any cofibrant $\mathscr{C}$–coalgebra $\mathscr{D}$ and any fibrant $\mathscr{C}$–coalgebra $\mathscr{E}$, the simplicial set $\mathrm{HOM}(\mathscr{D},\mathcal{E})$ is a Kan complex and is a model for the mapping space $\mathrm{Map}(\mathcal{C},\mathcal{D})$ expected by the ι–model structure.*

Proof Any fibrant $\mathscr{C}$–coalgebra $\mathscr{E}$ is isomorphic to the image under B_ι of an $\Omega_u\mathscr{C}$–algebra $\mathscr{A}$. So we have

$$\mathrm{HOM}(\mathscr{D},\mathscr{E})\simeq\mathrm{HOM}(\mathscr{D},B_\iota\mathscr{A})\simeq\mathrm{HOM}(\Omega_\iota\mathscr{D},\mathscr{A})\simeq\mathrm{Map}(\Omega_\iota\mathscr{D},\mathscr{A})\simeq\mathrm{Map}(\mathscr{D},B_\iota\mathscr{A}).$$

Further, we know from Proposition 76 that $\mathrm{HOM}(\Omega_\iota\mathscr{D},\mathscr{A})$ is a Kan complex. □

Corollary 107 *Let $\alpha\colon\mathscr{C}\to\mathscr{P}$ be an operadic twisting morphism. Let us endow the category of $\mathscr{C}$–coalgebras with the α–model structure. For any cofibrant $\mathscr{C}$–coalgebra $\mathscr{D}$ and any fibrant $\mathscr{C}$–coalgebra $\mathscr{E}$, the simplicial set $\mathrm{HOM}(\mathscr{D},\mathscr{E})$ is a Kan complex and is a model for the mapping space $\mathrm{Map}(\mathscr{D},\mathscr{E})$.*

Proof It suffices to notice that fibrations and acyclic fibrations in the α–model structure are in particular fibrations and acyclic fibrations in the ι–model structure. Then we can conclude by Proposition 106. □

Let $f\colon \mathscr{D} \to B_\iota \mathscr{A}$ be a morphism of $\mathscr{C}$–coalgebras. We know from Proposition 48 that it is a dg atom of the cocommutative coalgebra $\{\mathscr{D}, B_\iota \mathscr{A}\}$. Consider the Hinich coalgebra $\{\mathscr{D}, B_\iota \mathscr{A}\}_f$ that appears from the decomposition described in Theorem 78.

Proposition 108 *The deformation problem induced by $\{\mathscr{D}, B_\iota \mathscr{A}\}_f$ is equivalent to the deformation problem*

$$R \in \mathsf{Artin–alg} \mapsto (\hom_{R\otimes\Omega_n\otimes\mathscr{C}\text{–cog}}(R\otimes\Omega_n\otimes\mathscr{C}, R\otimes\Omega_n\otimes B_\iota\mathscr{A}))_{n\in\mathbb{N}}.$$

Proof This is a direct consequence of Proposition 70 and Theorem 79. □

7.5 Algebras of the operad $\Omega_u \mathscr{C}$

We have shown above that the adjunction $\Omega_\iota \dashv B_\iota$ is a Quillen equivalence. Moreover, in Proposition 99, we have shown that fibrant $\mathscr{C}$–coalgebras are $\Omega_u\mathscr{C}$–algebras. So switching from the model category of $\Omega_u\mathscr{C}$–algebras to the model category of $\mathscr{C}$–coalgebras amounts to add new morphisms between any two $\Omega_u\mathscr{C}$–algebras. The weak equivalences and the fibrations of $\Omega_u\mathscr{C}$–algebras remain weak equivalences and fibrations, respectively, under this embedding but, in the category of $\mathscr{C}$–coalgebras, any monomorphism is a cofibration. In particular, any object is cofibrant. Subsequently, $\mathscr{C}$–coalgebras provide a convenient framework to study the homotopy theory of $\Omega_u\mathscr{C}$–algebras. For instance, the following proposition provides a tool to decide whether or not two $\Omega_u\mathscr{C}$–algebras are equivalent.

Proposition 109 *Let $\mathscr{A}$ and $\mathscr{B}$ be two $\Omega_u\mathscr{C}$–algebras. There exists a chain of weak equivalences of $\Omega_u\mathscr{C}$–algebras between $\mathscr{A}$ and $\mathscr{B}$*

$$\mathscr{A} = \mathscr{A}_0 \xrightarrow{\sim} \mathscr{A}_1 \xleftarrow{\sim} \cdots \xrightarrow{\sim} \mathscr{A}_{n-1} \xleftarrow{\sim} \mathscr{A}_n = \mathscr{B}$$

if and only if there exists a weak equivalence of $\mathscr{C}$–coalgebras between $B_\iota\mathscr{A}$ and $B_\iota\mathscr{B}$.

Proof Suppose that there exists a chain of weak equivalences from $\mathscr{A}$ to $\mathscr{B}$. Then there exists a chain of weak equivalences between $B_\iota\mathscr{A}$ and $B_\iota\mathscr{B}$. Moreover, the objects of this chain are fibrant and cofibrant. So any morphism of this chain has a homotopical inverse. So there exists a weak equivalence from $B_\iota\mathscr{A}$ to $B_\iota\mathscr{B}$. Conversely, consider a weak equivalence F from $B_\iota\mathscr{A}$ to $B_\iota\mathscr{B}$. Then the following chain of weak equivalences of $\Omega_u\mathscr{C}$–algebras links $\mathscr{A}$ to $\mathscr{B}$:

$$\mathscr{A} \xleftarrow{\sim} \Omega_\iota B_\iota \mathscr{A} \xrightarrow{\Omega_\iota(F)} \Omega_\iota B_\iota \mathscr{B} \xrightarrow{\sim} \mathscr{B}.$$ □

7.6 Koszul morphisms

In this subsection, we study the operadic twisting morphisms $\alpha\colon \mathscr{C} \to \mathscr{P}$ such that the α–model structure on the category of $\mathscr{C}$–coalgebras coincides with the universal ι–model structure that we described above. Let $\alpha\colon \mathscr{C} \to \mathscr{P}$ be an operadic twisting morphism. We denote by $\phi\colon \Omega_u(\mathscr{C}) \to \mathscr{P}$ the morphism of operads induced by α.

Theorem 110 *The following assertions are equivalent:*

(1) *The adjunction*

$$\Omega_u(\mathscr{C})\text{–alg} \underset{\phi^*}{\overset{\phi_!}{\rightleftarrows}} \mathscr{P}\text{–alg}$$

is a Quillen equivalence.

(2) *The morphism of operads $\phi\colon \Omega_u(\mathscr{C}) \to \mathscr{P}$ is a quasi-isomorphism.*

(3) *The α–model structure coincides with the ι–model structure and $\Omega_\alpha \dashv B_\alpha$ is a Quillen equivalence.*

(4) *For any $\mathscr{P}$–algebra $\mathscr{A}$, the map $\mathscr{P} \circ_\alpha \mathscr{C} \circ_\alpha \mathscr{A} \to \mathscr{A}$ is a quasi-isomorphism, and for any $\mathscr{C}$–coalgebra $\mathscr{D}$, the morphism $\mathscr{D} \to \mathscr{C} \circ_\alpha \mathscr{P} \circ_\alpha \mathscr{D}$ is a ι–equivalence* (it is the case if, for instance, it is a filtered quasi-isomorphism).

(5) *The morphisms of $\mathbb{S}$–modules $\Omega_u(\mathscr{C}) \circ_\iota \mathscr{C} \circ_\iota \Omega_u(\mathscr{C}) \to \mathscr{P} \circ_\alpha \mathscr{C} \circ_\alpha \mathscr{P}$ and $\mathscr{P} \circ_\alpha \mathscr{C} \circ_\alpha \mathscr{P} \to \mathscr{P}$ are quasi-isomorphisms.*

Lemma 111 *Let $f\colon \mathcal{V} \to \mathcal{V}'$ be a morphism of dg $\mathbb{S}$–modules. Suppose that, for any chain complex $\mathcal{W}$* (that is, an $\mathbb{S}$–module concentrated in arity zero), *the morphism $\mathcal{V} \circ \mathcal{W} \to \mathcal{V}' \circ \mathcal{W}$ is a quasi-isomorphism. Then f is a quasi-isomorphism.*

Proof By the operadic Künneth formula, for any graded $\mathbb{K}$–module $\mathcal{W}$, the map $H(\mathcal{V}) \circ \mathcal{W} \to H(\mathcal{V}') \circ \mathcal{W}$ is an isomorphism. So, for any integer n, the map

$$f_n\colon H(\mathcal{V})(n) \otimes_{\mathbb{S}_n} \mathbb{K}^n \to H(\mathcal{V}')(n) \otimes_{\mathbb{S}_n} \mathbb{K}^n$$

is an isomorphism. Let $(e_i)_{i=1}^n$ be a basis of $\mathbb{K}^n$. The map

$$p \in H(\mathcal{V})(n) \mapsto p \otimes (e_1 \otimes \cdots \otimes e_n) \mapsto f_n(p) \otimes (e_1 \otimes \cdots \otimes e_n) \mapsto f_n(p) \in H(\mathcal{V}')(n)$$

is an isomorphism. So, the morphism $H(\mathcal{V}) \to H(\mathcal{V}')$ is an isomorphism. □

Proof of Theorem 110 • Let us first prove the equivalence between (1) and (2). Suppose (2). Let $\mathscr{A}$ be a cofibrant $\Omega_u\mathscr{C}$–algebra and let $\mathscr{B}$ be a $\mathscr{P}$–algebra. Consider a map $f\colon \phi_!(\mathscr{A}) \to \mathscr{B}$ and its adjoint map $g\colon \mathscr{A} \to \phi^*(\mathscr{B})$. The following diagram of $\Omega_u\mathscr{C}$–algebras is commutative:

$$\begin{tikzcd}
\Omega_\iota B_\iota \mathscr{A} \arrow[r] \arrow[d] & \phi^*\phi_!\Omega_\iota B_\iota\mathscr{A} \arrow[d] & \\
\mathscr{A} \arrow[r] \arrow[rr, bend right, "g"'] & \phi^*\phi_!\mathscr{A} \arrow[r, "\phi^*(f)"] & \phi^*\mathscr{B}
\end{tikzcd}$$

The left vertical map is a quasi-isomorphism. Since a left Quillen functor preserves weak equivalences between cofibrant objects and since ϕ^* preserves quasi-isomorphisms, the right vertical map is a quasi-isomorphism. Further, $\phi_!\Omega_\iota B_\iota\mathscr{A}$ is actually $\Omega_\alpha B_\iota\mathscr{A}$. Since the morphism ϕ is a quasi-isomorphism, the map $\Omega_\iota B_\iota\mathscr{A} \to \phi^*\Omega_\alpha B_\iota\mathscr{A}$ is a filtered quasi-isomorphism with respect to the filtrations

$$F_n\Omega_\iota B_\iota\mathscr{A} = \bigoplus_k \Omega_u\mathscr{C}(k) \otimes_{\mathbb{S}_n} \Bigg(\sum_{i_1+\cdots+i_k=n} F^{\mathrm{rad}}_{i_1} B_\iota\mathscr{A} \otimes \cdots \otimes F^{\mathrm{rad}}_{i_k} B_\iota\mathscr{A} \Bigg),$$
$$F_n\phi^*\Omega_\alpha B_\iota\mathscr{A} = \bigoplus_k \mathcal{P}(k) \otimes_{\mathbb{S}_n} \Bigg(\sum_{i_1+\cdots+i_k=n} F^{\mathrm{rad}}_{i_1} B_\iota\mathscr{A} \otimes \cdots \otimes F^{\mathrm{rad}}_{i_k} B_\iota\mathscr{A} \Bigg).$$

Indeed, the resulting map on the graded object $G(\Omega_\iota B_\iota\mathscr{A}) = \Omega_u\mathscr{C} \circ G(B_\iota\mathscr{A})$ is actually $\phi \circ \mathrm{Id}_{G(B_\iota\mathscr{A})}$. So the map $\Omega_\iota B_\iota\mathscr{A} \to \phi^*\Omega_\alpha B_\iota\mathscr{A}$ is a quasi-isomorphism. So, by the 2-out-of-3 rule, the map $\mathscr{A} \to \phi^*\phi_!\mathscr{A}$ is a quasi-isomorphism. Hence, f is a quasi-isomorphism if and only if $\phi^*(f)$ is a quasi-isomorphism, if and only if g is a quasi-isomorphism. So assertion (1) is true. Conversely, suppose (1). Then, for any chain complex (considered as a $\mathscr{C}$–coalgebra) $\mathcal{V}$, the map $\Omega_\iota\mathcal{V} \to \Omega_\alpha\mathcal{V}$ is a quasi-isomorphism. Since the coaction of $\mathscr{C}$ on $\mathcal{V}$ is trivial, we have canonical isomorphisms of chain complexes

$$\Omega_\iota\mathcal{V} \simeq \Omega_u\mathscr{C} \circ \mathcal{V}, \quad \Omega_\alpha\mathcal{V} \simeq \mathcal{P} \circ \mathcal{V}.$$

We can thus apply Lemma 111 which shows that (2) is true.

• Suppose (1) and let us show (3). By Proposition 96, the α–model structure coincides with the ι–model structure. Moreover, since the adjunctions $\phi_! \dashv \phi^*$ and $\Omega_\iota \dashv B_\iota$ are both Quillen equivalences, the adjunction $\phi_!\Omega_\iota \dashv B_\iota\phi^*$, which is $\Omega_\alpha \dashv B_\alpha$, is a Quillen equivalence.

- Suppose (3) and let us show (4). Since $\Omega_\alpha \dashv B_\alpha$ is a Quillen equivalence, $\Omega_\alpha B_\alpha \mathscr{A} \to \mathscr{A}$ is a quasi-isomorphism for any $\mathscr{P}$–algebra $\mathscr{A}$ and $\mathscr{D} \to B_\alpha \Omega_\alpha \mathscr{D}$ is an α–weak equivalence for any $\mathscr{C}$–coalgebra $\mathscr{D}$. Since the α–model structure coincides with the ι–model structure, $\mathscr{D} \to B_\alpha \Omega_\alpha \mathscr{D}$ is a ι–weak equivalence. So (4) is true.

- Suppose (4) and let us show (5). For any $\mathscr{P}$–algebra $\mathscr{A}$, the morphism $\Omega_\alpha B_\alpha \mathscr{A} \to \mathscr{A}$ is a quasi-isomorphism. In particular, this is true for any free $\mathscr{P}$–algebras. So, for any chain complex $\mathcal{V}$, the map $\Omega_\alpha B_\alpha(\mathscr{P} \circ \mathcal{V}) \to \mathscr{P} \circ \mathcal{V}$ is a quasi-isomorphism. This map is actually the morphism of chain complexes

$$(\mathscr{P} \circ_\alpha \mathscr{C} \circ_\alpha \mathscr{P}) \circ \mathcal{V} \to \mathscr{P} \circ \mathcal{V}.$$

Using Lemma 111, we conclude that the map $\mathscr{P} \circ_\alpha \mathscr{C} \circ_\alpha \mathscr{P} \to \mathscr{P}$ is a quasi-isomorphism. Moreover, for any $\mathscr{P}$–algebra $\mathscr{A}$, the following diagram commutes:

$$\begin{tikzcd}
\Omega_u(\mathscr{C}) \circ_\iota \mathscr{C} \circ_\iota \mathscr{A} \arrow[r] \arrow[dr] & \mathscr{P} \circ_\alpha \mathscr{C} \circ_\alpha \mathscr{A} \arrow[d] \\
 & \mathscr{A}
\end{tikzcd}$$

Since the composite map and the vertical map are quasi-isomorphisms (because $\Omega_\iota \dashv B_\iota$ and $\Omega_\alpha \dashv B_\alpha$ are Quillen equivalences), by the 2-out-of-3 rule the horizontal map is a quasi-isomorphism. Applying this to free $\mathscr{P}$–algebras and using Lemma 111, we conclude that the map $\Omega_u(\mathscr{C}) \circ_\iota \mathscr{C} \circ_\alpha \mathscr{P} \to \mathscr{P} \circ_\alpha \mathscr{C} \circ_\alpha \mathscr{P}$ is a quasi-isomorphism. Further, for any $\mathscr{C}$–coalgebra $\mathscr{D}$ the following diagram commutes:

$$\begin{tikzcd}
\mathscr{D} \arrow[r] \arrow[dr] & \mathscr{C} \circ_\iota \Omega_u(\mathscr{C}) \circ_\iota \mathscr{D} \arrow[d] \\
 & \mathscr{C} \circ_\alpha \mathscr{P} \circ_\alpha \mathscr{D}
\end{tikzcd}$$

By the 2-out-of-3 rule, the vertical map is a ι–weak equivalence. So the map

$$\Omega_u(\mathscr{C}) \circ_\iota \mathscr{C} \circ_\iota \Omega_u(\mathscr{C}) \circ_\iota \mathscr{D} \to \Omega_u(\mathscr{C}) \circ_\iota \mathscr{C} \circ_\alpha \mathscr{P} \circ_\alpha \mathscr{D}$$

is a quasi-isomorphism. Applying this for $\mathscr{C}$–coalgebras which are just chain complexes and using Lemma 111, we obtain that the map $\Omega_u(\mathscr{C}) \circ_\iota \mathscr{C} \circ_\iota \Omega_u(\mathscr{C}) \to \Omega_u(\mathscr{C}) \circ_\iota \mathscr{C} \circ_\alpha \mathscr{P}$ is a quasi-isomorphism.

- Suppose (5) and let us show (2). Using the previous point in the case $\mathscr{P} = \Omega_u \mathscr{C}$ gives us the fact that the map of dg $\mathbb{S}$–modules

$$\Omega_u(\mathscr{C}) \circ_\iota \mathscr{C} \circ_\iota \Omega_u(\mathscr{C}) \to \Omega_u(\mathscr{C})$$

is an aritywise quasi-isomorphism. Then the following square of $\mathbb{S}$–modules is commutative:

$$\begin{tikzcd}
\Omega_u(\mathscr{C}) \circ_\iota \mathscr{C} \circ_\iota \Omega_u(\mathscr{C}) \arrow[r] \arrow[d] & \Omega_u(\mathscr{C}) \arrow[d] \\
\mathscr{P} \circ_\alpha \mathscr{C} \circ_\alpha \mathscr{P} \arrow[r] & \mathscr{P}
\end{tikzcd}$$

Since the left vertical map and the horizontal maps are quasi-isomorphisms, the right vertical map is also a quasi-isomorphism. □

Definition 112 (Koszul morphisms) An operadic twisting morphism $\alpha\colon \mathscr{C} \to \mathscr{P}$ satisfying the properties of Theorem 110 is called a *Koszul morphism*.

In the next section, we will explore Koszul duality, which is a method to produce Koszul morphisms from a presentation of an operad.

8 Examples

The purpose of this section is to apply the general framework described in the previous sections to the case of common nonaugmented operads like the operads $u\mathcal{A}s$ and $u\mathcal{C}om$, whose algebras are the unital associative algebras and the unital commutative algebras, respectively. So, for any of these operads $\mathscr{P}$, one looks after a curved conilpotent cooperad $\mathscr{C}$ together with an operadic twisting morphism α from $\mathscr{C}$ to $\mathscr{P}$ such that the induced morphism of operads from $\Omega_u\mathscr{C}$ to $\mathscr{P}$ is a quasi-isomorphism; that is, α is a Koszul morphism. One can use the universal twisting morphism $B_c\mathscr{P} \to \mathscr{P}$. However, the bar construction is always very big. Instead, one usually tries to produce a subcooperad of $B_c\mathscr{P}$ whose cobar construction will be a resolution of $\mathscr{P}$. The Koszul duality theory is a way to produce such a subcooperad when the operad $\mathscr{P}$ has a quadratic presentation or a quadratic-linear presentation. This construction has been extended to quadratic-linear-constant presentations by Hirsh and Millès in [17], generalizing to operads the curved Koszul duality of algebras developed by Polishchuk and Positselski [22].

8.1 Koszul duality

Koszul duality is a way to build a cooperad $\mathscr{P}^{\text{¡}}$ together with a canonical operadic twisting morphism from $\mathscr{P}^{\text{¡}}$ to $\mathscr{P}$ out of an operad $\mathscr{P}$ which has a "nice enough" presentation $\mathscr{P} = \mathbb{T}(\mathcal{V})/(\mathcal{R})$. Here, we present the construction of Hirsh and Millès in [17].

Let $\mathscr{P}$ be a graded operad equipped with a presentation $\mathscr{P} = \mathbb{T}(\mathcal{V})/(\mathcal{R})$, where $\mathcal{V}$ is a graded $\mathbb{S}$–module and where $(\mathcal{R})$ is the operadic ideal generated by a subgraded $\mathbb{S}$–module $\mathcal{R}$ of $\mathbb{T}^{\leq 2}(\mathcal{V})$ such that

$$\mathcal{R} \cap (\mathcal{I} \oplus \mathcal{V}) = \{0\},$$
$$(\mathcal{R}) \cap \mathbb{T}^{\leq 2}(\mathcal{V}) = \mathcal{R}.$$

We denote by $q\mathcal{R}$ the projection of $\mathcal{R} \subset \mathbb{T}^{\leq 2}(\mathcal{V})$ onto $\mathbb{T}^2(\mathcal{V})$ along $\mathcal{I} \oplus \mathcal{V}$. Moreover, let $q\mathscr{P}$ be the operad

$$q\mathscr{P} := \mathbb{T}(\mathcal{V})/(q\mathcal{R}).$$

This is a quadratic operad. The condition $\mathcal{R} \cap (\mathcal{I} \oplus \mathcal{V}) = \{0\}$ induces a function $\phi = (\phi_0, \phi_1)\colon q\mathcal{R} \to \mathcal{I} \oplus \mathcal{V}$.

Definition 113 (curved cooperad Koszul dual of an operad [17, Section 4.1]) The Koszul dual cooperad $\mathscr{P}^{\text{¡}}$ of $\mathscr{P}$ associated to the presentation $\mathscr{P} = \mathbb{T}(\mathcal{V})/(\mathcal{R})$ is the following curved conilpotent cooperad. The underlying graded cooperad is the final graded subcooperad of $\mathbb{T}^c(s\mathcal{V})$ such that the composition

$$\mathscr{P}^{\text{¡}} \to \mathbb{T}^c(s\mathcal{V}) \to \mathbb{T}^2(s\mathcal{V})/s^2 q\mathcal{R}$$

is zero. It is equipped with the unique coderivation which extends the map

$$\mathscr{P}^{\text{¡}} \twoheadrightarrow s^2 q\mathcal{R} \to s\mathcal{V}, \quad sx \otimes sy \mapsto (-1)^{|x|} s\phi_1(x \otimes y).$$

Its curvature is the degree -2 map

$$\theta\colon \mathscr{P}^{\text{¡}} \twoheadrightarrow s^2 q\mathcal{R} \to \mathbb{K}, \quad sx \otimes sy \mapsto (-1)^{|x|} s\phi_0(x \otimes y).$$

Moreover, the map

$$\kappa\colon \mathscr{P}^{\text{¡}} \twoheadrightarrow s\mathcal{V} \to \mathcal{V} \hookrightarrow \mathscr{P},$$

is an operadic twisting morphism which induces both a morphism of operads $\Omega_u \mathscr{P}^{\text{¡}} \to \mathscr{P}$ and a morphism of curved conilpotent cooperads $\mathscr{P}^{\text{¡}} \to B_c \mathscr{P}$.

Remark 114 The coherence of the above definition is proven in [17, Section 4.1].

Definition 115 (Koszul operad) The operad $\mathscr{P}$ (together with the presentation $\mathscr{P} = \mathbb{T}(\mathcal{V})/(\mathcal{R})$) is said to be Koszul if the twisting morphism $\kappa\colon \mathscr{P}^{¡} \to \mathscr{P}$ is Koszul, that is, if the map $\Omega_c \mathscr{P}^{¡} \to \mathscr{P}$ is a quasi-isomorphism.

The following theorem is a powerful tool to show that an operad is Koszul:

Theorem 116 [17, Theorem 4.3.1] *Suppose that the canonical morphism*

$$q\mathscr{P} \circ_\kappa q\mathscr{P}^{¡} \to \mathcal{I}$$

is a quasi-isomorphism. Then $\mathscr{P}$ is Koszul.

8.2 Coalgebras over a Koszul dual

In this subsection, we describe the category of $\mathscr{P}^{¡}$–coalgebras, where $\mathscr{P}^{¡}$ is the Koszul dual of the "quadratic-linear-homogeneous operad" $\mathscr{P}$ defined above. We will need the following definition:

Definition 117 (precoradical filtration) Let $\mathcal{W}$ be a graded $\mathbb{S}$–module and let $\mathcal{C}$ be a graded $\mathbb{K}$–module equipped with a map $\Delta^{(1)}\colon \mathcal{C} \to \mathcal{W} \circ \mathcal{C}$. We define $(F_n^{\mathrm{prad}}\mathcal{C})_{n\in\mathbb{N}}$ to be the following (nonnecessarily exhaustive) filtration on $\mathcal{C}$, called the precoradical filtration:

$$F_0^{\mathrm{prad}}(\mathcal{C}) := \ker(\Delta^{(1)}),$$

$$F_n^{\mathrm{prad}}(\mathcal{C}) := (\Delta^{(1)})^{-1}\Bigg(\mathcal{W}(0) \oplus \sum_{\substack{i_1+\cdots+i_k=n-1\\ k\geq 1}} \mathcal{W}(k) \otimes_{\mathbb{S}_k} (F_{i_1}^{\mathrm{prad}}\mathcal{C} \otimes \cdots \otimes F_{i_k}^{\mathrm{prad}}\mathcal{C})\Bigg) \quad \text{if } n \geq 1.$$

Lemma 118 *Consider a cofree graded conilpotent cooperad $\mathbb{T}^c(\mathcal{W})$. The category of graded coalgebras over $\mathbb{T}^c(\mathcal{W})$ is equivalent to the category of graded $\mathbb{K}$–modules $\mathcal{C}$ equipped with a map $\Delta^{(1)}\colon \mathcal{C} \to \mathcal{W} \circ \mathcal{C}$ such that the precoradical filtration $(F_n^{\mathrm{prad}}\mathcal{C})_{n\in\mathbb{N}}$ is exhaustive. Moreover, under this equivalence, the coradical filtration coincides with the precoradical filtration.*

Proof Let $\mathcal{C}$ be a graded $\mathbb{K}$–module with a map $\Delta^{(1)}\colon \mathcal{C} \to \mathcal{W} \circ \mathcal{C}$ such that the precoradical filtration $(F_n^{\mathrm{prad}}\mathcal{C})_{n\in\mathbb{N}}$ is exhaustive. Then let us define $\Delta_{\mathcal{C}}\colon \mathcal{C} \to \mathbb{T}(\mathcal{W}) \circ \mathcal{C}$ by induction as follows:

$$\begin{cases} \Delta_{\mathcal{C}}(x) := 1 \otimes x & \text{if } x \in F_0^{\mathrm{prad}}\mathcal{C}, \\ \Delta_{\mathcal{C}}(x) := 1 \otimes x + (\mathrm{Id} \circ \Delta_{\mathcal{C}})\Delta^{(1)}(x) & \text{if } x \in F_n^{\mathrm{prad}}\mathcal{C}. \end{cases}$$

This defines a structure of $\mathbb{T}^c(\mathcal{W})$–coalgebra on $\mathcal{C}$. Conversely, let $(\mathcal{C}, \Delta)$ be a graded $\mathbb{T}^c(\mathcal{W})$–coalgebra. We obtain a map $\Delta^{(1)}\colon \mathcal{C} \to \mathcal{W} \circ \mathcal{C}$ by composing Δ with the projection of $\mathbb{T}^c(\mathcal{W})$ onto $\mathcal{W}$. Then the construction we just described recovers Δ from $\Delta^{(1)}$. □

Theorem 119 *Suppose that the characteristic of the field* $\mathbb{K}$ *is zero* (this assumption is not necessary in the nonsymmetric context). *The category of* $\mathscr{P}^{¡}$*–coalgebras is equivalent to the category of graded* $\mathbb{T}^c(s\mathcal{V})$*–coalgebras* (that is graded $\mathbb{K}$–modules $\mathcal{C}$ equipped with a map $\Delta^{(1)}\colon \mathcal{C} \to s\mathcal{V} \circ \mathcal{C}$ such that the precoradical filtration $(F_n^{\mathrm{prad}}\mathcal{C})_{n\in\mathbb{N}}$ is exhaustive) *such that the composite map*

$$\mathcal{C} \xrightarrow{\Delta_2 = (\mathrm{Id} \circ' \Delta^{(1)})\Delta^{(1)}} \mathbb{T}^2(s\mathcal{V}) \circ \mathcal{C} \twoheadrightarrow (\mathbb{T}^2(s\mathcal{V})/s^2 q\mathcal{R}) \circ \mathcal{C}$$

is zero, together with a degree -1 *map* $d_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ *such that*

$$d_{\mathcal{C}}^2 = (\theta \circ \mathrm{Id})\Delta_2, \quad \Delta^{(1)} d_{\mathcal{C}} = (d_{\mathscr{P}^{¡}} \circ \mathrm{Id})\Delta_2 + (\mathrm{Id} \circ' d_{\mathcal{C}})\Delta^{(1)}.$$

Proof Let $\mathcal{C}$ be a graded $\mathbb{T}^c(s\mathcal{V})$–coalgebra together with a degree -1 map $d_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ satisfying the conditions of Theorem 119. For any $x \in \mathcal{C}$, let $\mathcal{C}(x)$ be a finite-dimensional sub-$\mathbb{T}^c(s\mathcal{V})$–coalgebra of $\mathcal{C}$ which contains x. By Lemma 120, the map $\Delta_{\mathcal{C}(x)}\colon \mathcal{C}(x) \to \mathbb{T}^c(s\mathcal{V}) \circ \mathcal{C}(x)$ factorizes through a unique map $\mathcal{C}(x) \to \mathscr{P}^{¡} \circ \mathcal{C}(x)$. Hence, $\mathcal{C}$ has a structure of graded $(\mathscr{P}^{¡})^{\mathrm{grad}}$–coalgebra. Moreover, we can prove by induction on the coradical filtration of $\mathcal{C}$ that $d_{\mathcal{C}}$ is a coderivation. □

Lemma 120 *Let* $\mathcal{C}(x)$ *be the graded* $\mathbb{T}^c(s\mathcal{V})$*–coalgebra defined in the proof of* Theorem 119*. Then* $\mathcal{C}(x)$ *is a graded* $\mathscr{P}^{¡}$*–coalgebra.*

Proof Remember that $\mathcal{C}(x)$ is a finite-dimensional subgraded $\mathbb{T}^c(s\mathcal{V})$–coalgebra of $\mathcal{C}$. Let $(e_i)_{i=1}^m$ be a basis of $\mathcal{C}(x)$. Then, for any $i \in \{1, \dots, m\}$, let $p_{i,0} \in \overline{\mathbb{T}}(s\mathcal{V})(0)$, and for any integer $k \geq 1$ and for any nondecreasing function s from $\{1, \dots, k\}$ to $\{1, \dots, m\}$, let $p_{i,k,s} \in \overline{\mathbb{T}}(s\mathcal{V})(k)$ be such that

$$\Delta(e_i) = 1 \otimes e_i + p_{i,0} + \sum_{k=0}^{\infty} \sum_{s} p_{i,k,s} \otimes_{\mathbb{S}_k} (e_{s(1)} \otimes \cdots \otimes e_{s(k)}).$$

For any nondecreasing function s from $\{1, \dots, k\}$ to $\{1, \dots, m\}$ and for any $\sigma \in \mathbb{S}_k$, let $\epsilon(s, \sigma)$ be the element of $\mathbb{Z}/2\mathbb{Z}$ such that the structural action of σ on $\mathcal{C}^{\otimes k}$ sends $e_{s(1)} \otimes \cdots \otimes e_{s(k)}$ to $(-1)^{\epsilon(s,\sigma)} e_{s\sigma^{-1}(1)} \otimes \cdots \otimes e_{s\sigma^{-1}(k)}$. Further, let $\mathrm{Inv}(s)$ be the subgroup of $\mathbb{S}_k$ of permutations σ such that $s = s\sigma^{-1}$. Then we can choose $p_{i,k,s}$

such that $p^\sigma_{i,k,s} = (-1)^{\epsilon(s,\sigma)} p_{i,k,s}$ for any $\sigma \in \mathrm{Inv}(s)$. Indeed, if it is not the case, we can replace $p_{i,k,s}$ by

$$\frac{1}{\#\,\mathrm{Inv}(s)} \sum_{\sigma \in \mathrm{Inv}(s)} (-1)^{\epsilon(s,\sigma)} p^\sigma_{i,k,s}.$$

Let $\mathscr{D}$ be the subgraded $\mathbb{S}$–module of $\mathbb{T}(s\mathcal{V})$ generated by 1 and the elements $p_{i,k,s}$. Since $(\Delta \circ \mathrm{Id})\Delta(e_i) = (\mathrm{Id} \circ \Delta)\Delta(e_i)$ for any i, there exists an element of $q_{i,k,s} \in (\mathscr{D} \circ \mathscr{D})(k)$ such that

$$\Delta(p_{i,k,s}) \otimes_{\mathbb{S}_k} (e_{s(1)} \otimes \cdots \otimes e_{s(k)}) = q_{i,k,s} \otimes_{\mathbb{S}_k} (e_{s(1)} \otimes \cdots \otimes e_{s(k)}).$$

Since $p^\sigma_{i,k,s} = (-1)^{\epsilon(s,\sigma)} p_{i,k,s}$ for any $\sigma \in \mathrm{Inv}(s)$,

$$\Delta(p_{i,k,s}) = \frac{1}{\#\,\mathrm{Inv}(s)} \sum_{\sigma \in \mathrm{Inv}(s)} (-1)^{\epsilon(s,\sigma)} q^\sigma_{i,k,s}.$$

So, $\Delta(p_{i,k,s}) \in \mathscr{D} \circ \mathscr{D}$. Hence, $\mathscr{D}$ is a subgraded cooperad of $\mathbb{T}^c(s\mathcal{V})$. Moreover, for any i, $(\pi \circ \mathrm{Id})\Delta(e_i) = 0$, where π is the projection of $\mathbb{T}(s\mathcal{V})$ onto $\mathbb{T}^2(s\mathcal{V})/s^2q\mathcal{R}$. So, $\pi(p_{i,k,s}) = 0$ for any 3–tuple (i,k,s) and $\pi(p_{i,0}) = 0$ for any i; so $\pi|_{\mathscr{D}} = 0$. Hence, $\mathscr{D} \subset \mathscr{P}^{¡}$. □

8.3 Unital associative algebras up to homotopy

Notation Let $\mathcal{V}$ and $\mathcal{W}$ be two $\mathbb{N}$–modules, and n, p, $i_1, \dots, i_p$ natural integers such that $i_1 + \cdots + i_p = n$. We will usually denote the image of an element

$$x \otimes y_1 \otimes \cdots \otimes y_p \in \mathcal{V}(p) \otimes \mathcal{W}(i_1) \otimes \cdots \otimes \mathcal{W}(i_p)$$

under the inclusion

$$\mathcal{V}(p) \otimes \mathcal{W}(i_1) \otimes \cdots \otimes \mathcal{W}(i_p) \to (\mathcal{V} \circ_{\mathrm{ns}} \mathcal{W})(n)$$

by $x \otimes_{\mathrm{ns}} (y_1 \otimes \cdots \otimes y_p)$.

8.3.1 A presentation of the operad $u\mathcal{A}s$ Let $u\mathcal{A}s$ be the nonsymmetric operad defined by the presentation $u\mathcal{A}s := \mathbb{T}(\mathbb{K}\cdot\mu \oplus \mathbb{K}\cdot\xi)/(\mathcal{R})$, where μ is an arity 2 element and ξ is an arity 0 element. The nonsymmetric module $\mathcal{R} \subset \mathcal{I} \oplus \mathbb{T}^2(\mathbb{K}\cdot\mu \oplus \mathbb{K}\cdot\xi)$ is made up of the relations

$$\begin{cases} \mu \otimes_{\mathrm{ns}} (\xi \otimes 1) - 1, \\ \mu \otimes_{\mathrm{ns}} (1 \otimes \xi) - 1, \\ \mu \otimes_{\mathrm{ns}} (\mu \otimes 1) - \mu \otimes_{\mathrm{ns}} (1 \otimes \mu). \end{cases}$$

Remark 121 Here, the symbol ns stands for the composition product of nonsymmetric modules.

Given this presentation, the Koszul dual $u\mathcal{As}^{¡}$ is a nonsymmetric curved conilpotent cooperad whose underlying graded cooperad is the final subcooperad of $\mathbb{T}^c(\mathbb{K}\cdot s\mu \oplus \mathbb{K}\cdot \xi)$ such that

$$u\mathcal{As}^{¡} \cap \mathbb{T}^2(\mathbb{K}\cdot s\mu) = \mathbb{K}\cdot (s\mu \otimes_{\mathrm{ns}} (s\mu \otimes 1) - s\mu \otimes_{\mathrm{ns}} (1 \otimes s\mu)).$$

The coderivation of $u\mathcal{As}^{¡}$ is zero and the curvature is given by

$$\theta(s\mu \otimes_{\mathrm{ns}} (s\xi \otimes 1)) = \theta(s\mu \otimes_{\mathrm{ns}} (1 \otimes s\xi)) = -1.$$

Remark 122 The Koszul dual curved cooperad $u\mathcal{As}^{¡}$ of the operad $u\mathcal{As}$ is described in detail in [17].

8.3.2 Coalgebras over $u\mathcal{As}^{¡}$

Proposition 123 *The endofunctor of the category of graded $\mathbb{K}$–modules $\mathcal{V} \mapsto s\mathcal{V}$ induces an equivalence between the category of $u\mathcal{As}^{¡}$–coalgebras and the category of noncounital curved conilpotent coassociative coalgebras.*

Proof The proof relies on the same arguments as the proof of Proposition 129, which will be detailed. □

Remark 124 The map $\mathcal{V} \to s\mathcal{V}$ also induces an equivalence between graded $(u\mathcal{As}^{¡})^{\mathrm{grad}}$–coalgebras and graded noncounital conilpotent coassociative coalgebras $\mathscr{C}$ equipped with a degree -2 map $\mathscr{C} \to \mathbb{K}$. Moreover, this equivalence sends a cofree graded $(u\mathcal{As}^{¡})^{\mathrm{grad}}$–coalgebra $u\mathcal{As}^{¡} \circ \mathcal{V}$ to the cofree conilpotent coalgebra $\overline{\mathbb{T}}(\mathcal{V} \oplus \mathbb{K}\cdot v)$, where $|v| = 2$ with the degree -2 map

$$\overline{\mathbb{T}}(\mathcal{V} \oplus \mathbb{K}\cdot v) \twoheadrightarrow \mathbb{K}\cdot v \to \mathbb{K}, \quad v \mapsto 1.$$

Notation We denote the category of curved conilpotent coassociative coalgebras by cCog. Moreover, we denote the operad $\Omega_u u\mathcal{As}^{¡}$ by $u\mathscr{A}_\infty$.

8.3.3 The bar–cobar adjunction and $u\mathscr{A}_\infty$–algebras On the one hand, there exists an adjunction relating $u\mathcal{As}$–algebras to $u\mathcal{As}^{¡}$–coalgebras which is induced by the operadic twisting morphism $\alpha\colon u\mathcal{As}^{¡} \to u\mathcal{As}$. On the other hand, the category of $u\mathcal{As}^{¡}$–coalgebras is equivalent to the category cCog of curved conilpotent coalgebras. Thus, we obtain a bar–cobar adjunction between unital associative algebras and curved conilpotent coalgebras which is the restriction to arity 1 of the operadic bar–cobar adjunction described in Section 4.1 (with the exception that we can consider noncounital coalgebras instead of coaugmented counital coalgebras). For this reason, we denote this adjunction using the same symbols as in the operadic context, that is, $\Omega_u \dashv B_c$. So we have

$$\Omega_u \mathscr{C} := \overline{\mathbb{T}}(s^{-1}\mathscr{C}), \quad B_c \mathscr{A} := \overline{\mathbb{T}}(s\mathscr{A} \oplus \mathbb{K} \cdot v)$$

for any curved conilpotent coalgebra $\mathscr{C}$ and for any unital algebra $\mathscr{A}$. The derivation of $\Omega_u(\mathscr{C})$ and the coderivation of $B_c(\mathscr{A})$ are defined as in Section 4.1.

The adjunction $\Omega_u \dashv B_c$ is part of a larger picture,

$$\mathsf{cCog} \underset{B_\iota}{\overset{\Omega_\iota}{\rightleftarrows}} u\mathscr{A}_\infty\text{–}\mathsf{alg} \underset{\phi^*}{\overset{\phi_!}{\rightleftarrows}} u\mathcal{A}s\text{–}\mathsf{alg},$$

where the adjunction $\phi_! \dashv \phi^*$ is induced by the morphism of operads $\phi\colon u\mathscr{A}_\infty \to u\mathcal{A}s$ and where $\Omega_u = \phi_! \Omega_\iota$ and $B_c = B_\iota \phi^*$. We know that a $u\mathscr{A}_\infty$–algebra $\mathscr{A} = (\mathcal{A}, \gamma)$ is the data of a chain complex $\mathcal{A}$ together with a coderivation on the cofree graded $(u\mathcal{A}s^{¡})^{\mathrm{grad}}$–coalgebra $u\mathcal{A}s^{¡} \circ \mathcal{A}$, so that it becomes a $u\mathcal{A}s$–coalgebra. Equivalently, it is the data of a chain complex together with a coderivation on the cofree conilpotent coassociative coalgebra $\overline{\mathbb{T}}(s\mathcal{A} \oplus \mathbb{K} \cdot v)$, so that it becomes a curved conilpotent coalgebra whose curvature θ is given by

$$\overline{\mathbb{T}}(s\mathcal{A} \oplus \mathbb{K} \cdot v) \twoheadrightarrow \mathbb{K} \cdot v \to \mathbb{K}, \quad v \mapsto 1.$$

By Lemma 27, this is equivalent to a degree -1 map

$$\gamma\colon \overline{\mathbb{T}}(s\mathcal{A} \oplus \mathbb{K} \cdot v) \to \mathcal{A},$$

such that, for any $x_1, \cdots, x_n \in (s\mathcal{A} \oplus \mathbb{K} \cdot v)$,

$$\sum_{0 \le i \le j \le n} (-1)^{|x_1| + \cdots + |x_{i-1}|} \gamma(x_1 \otimes \cdots \otimes \gamma(x_i \otimes \cdots \otimes x_j) \otimes \cdots x_n) = \begin{cases} 0 & \text{if } n \neq 2, \\ \theta(x_1)x_2 - \theta(x_2)x_1 & \text{if } n = 2. \end{cases}$$

In particular, we have the following:

- A degree zero product
$$\gamma_2\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}.$$
- A degree 1 map
$$\gamma_3\colon \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \to \mathcal{A},$$
whose boundary is the associator of γ_2, that is,
$$\partial(\gamma_3) = \gamma_2(\mathrm{Id} \otimes \gamma_2) - \gamma_2(\gamma_2 \otimes \mathrm{Id}).$$

- An element $1_{\mathscr{A}}$ defined by $\gamma(v) = s1_{\mathscr{A}}$.
- Maps $\gamma_{1,l}\colon \mathcal{A} \to \mathcal{A}$ and $\gamma_{1,r}\colon \mathcal{A} \to \mathcal{A}$ of degree 1 which make $1_{\mathscr{A}}$ a unit up to homotopy, that is,

$$\partial(\gamma_{1,l}) = \gamma_2(1_{\mathscr{A}} \otimes \mathrm{Id}) - \mathrm{Id},$$
$$\partial(\gamma_{1,l}) = \gamma_2(\mathrm{Id} \otimes 1_{\mathscr{A}}) - \mathrm{Id}.$$

8.3.4 The Koszul property and the infinity category of $u\mathscr{A}_\infty$–algebras

Proposition 125 [17, Theorem 6.1.8] *The operad $u\mathcal{A}s$ is Koszul.*

Remark 126 The model structure on curved conilpotent coalgebras that we get by transfer along the adjunction $\Omega_u \dashv B_c$ is the model structure that Positselski described in [23].

There are several ways to describe the infinity-category of $u\mathcal{A}s$–algebras:

- One can take the Dwyer–Kan simplicial localization of the category of $u\mathcal{A}s$–algebras with respect to quasi-isomorphisms as described in [10; 9].
- One can take the simplicial category whose objects are cofibrant-fibrant $u\mathcal{A}s$–algebras and whose spaces of morphisms are

$$\mathrm{Map}(\mathscr{A}, \mathscr{B})_n := \mathrm{HOM}_{u\mathcal{A}s-\mathsf{alg}}(\mathscr{A}, \mathscr{B}).$$

- One can also take the simplicial category whose objects are all $u\mathcal{A}s$–algebras and whose spaces of morphisms are

$$\mathrm{Map}(\mathscr{A}, \mathscr{B}) := \mathrm{HOM}_{u\mathcal{A}s-\mathsf{alg}}(\Omega_u B_c \mathscr{A}, \mathscr{B}) \simeq \mathrm{HOM}_{\mathsf{cCog}}(B_c \mathscr{A}, B_c \mathscr{B}).$$

8.4 Unital commutative algebras up to homotopy

In this section, we assume that the characteristic of the base field $\mathbb{K}$ is zero.

8.4.1 A presentation of the operad $u\mathcal{C}om$ Let $u\mathcal{C}om$ be the operad defined by the presentation $u\mathcal{C}om := \mathbb{T}(\mathbb{K}\cdot\mu \oplus \mathbb{K}\cdot\xi)/(\mathcal{R})$, where μ is an arity 2 element such that $\mu^{(1,2)} = \mu$ and ξ is an arity 0 element. The $\mathbb{S}$–module $\mathcal{R} \subset \mathcal{I} \oplus \mathbb{T}^2(\mathbb{K}\cdot\mu \oplus \mathbb{K}\cdot\xi)$ is generated by the elements

$$\mu \otimes_{\mathbb{S}_2} (\mu \otimes 1) - \mu \otimes_{\mathbb{S}_2} (1 \otimes \mu), \quad \mu \otimes_{\mathbb{S}_2} (\xi \otimes 1) - 1.$$

Remark 127 • Since the action of $\mathbb{S}_2$ on μ is trivial, we have

$$\mu \otimes_{\mathbb{S}_2} (1 \otimes \mu) = (\mu \otimes_{\mathbb{S}_2} (\mu \otimes 1))^{(132)}.$$

- The element $\mu \otimes_{\mathbb{S}_2} (\mu \otimes 1) - \mu \otimes_{\mathbb{S}_2} (1 \otimes \mu)$ is a generator of the $\mathbb{S}_3$–module $\mathcal{R}(3)$. However, it is not a generator of $\mathcal{R}(3)$ as a $\mathbb{K}$–module; one needs to add the element $\mu \otimes_{\mathbb{S}_2} (\mu \otimes 1) - (\mu \otimes_{\mathbb{S}_2} (\mu \otimes 1))^{(2,3)}$.

Given this presentation, the Koszul dual $u\mathcal{C}om^{\text{!`}}$ is a curved conilpotent cooperad whose underlying graded cooperad is the final subcooperad of $\mathbb{T}^c(\mathbb{K} \cdot s\mu \oplus \mathbb{K} \cdot \xi)$ such that

$$u\mathcal{A}s^{\text{!`}}(3) \cap \mathbb{T}^2(\mathbb{K} \cdot s\mu)(3) = \mathbb{K}[\mathbb{S}_3] \cdot (s\mu \otimes_{\mathbb{S}_2} (s\mu \otimes 1) - s\mu \otimes_{\mathbb{S}_2} (1 \otimes s\mu)).$$

The coderivation of $u\mathcal{C}om^{\text{!`}}$ is zero and the curvature is given by

$$\theta(s\mu \otimes_{\mathbb{S}_2} (s\xi \otimes 1)) = -1.$$

Notation We denote by $u\mathcal{C}om_\infty$ the operad $\Omega_u u\mathcal{C}om^{\text{!`}}$.

8.4.2 Coalgebras over $u\mathcal{C}om^{\text{!`}}$ We will show that the category of $u\mathcal{C}om^{\text{!`}}$–coalgebras is equivalent to the category of curved conilpotent Lie coalgebras.

Definition 128 (curved Lie coalgebra) A *curved Lie coalgebra* $\mathscr{C} = (C, \delta, d, \theta)$ is a graded $\mathbb{K}$–module C equipped with an antisymmetric map $\delta\colon C \to C \otimes C$ such that

$$(\delta \otimes \mathrm{Id})\delta = (\mathrm{Id} \otimes \delta)\delta + (\mathrm{Id} \otimes \tau)(\delta \otimes \mathrm{Id})\delta,$$

where τ is the exchange map $\tau(x \otimes y) = (-1)^{|x||y|} y \otimes x$. It is also equipped with a degree -1 map $d\colon C \to C$ which is a coderivation, that is,

$$\delta d = (d \otimes \mathrm{Id} + \mathrm{Id} \otimes d)\delta,$$

and with a degree -2 map $\theta\colon C \to \mathbb{K}$ which is a curvature, that is,

$$d^2 = (\theta \otimes \mathrm{Id} - \mathrm{Id} \otimes \theta)\delta.$$

A curved Lie coalgebra C is said to be *conilpotent* if for any $x \in C$, there exists an integer n such that the element

$$(\mathrm{Id} \otimes \cdots \otimes \delta \otimes \cdots \otimes \mathrm{Id}) \cdots \delta(x)$$

is zero whenever δ appears n times. We denote by $\mathsf{cLieCog}$ the category of curved conilpotent Lie coalgebras.

Proposition 129 *The endofunctor of the category of graded $\mathbb{K}$–modules $\mathcal{V} \mapsto s\mathcal{V}$ induces an equivalence between the category of $u\mathcal{C}om^{\text{¡}}$–coalgebras and the category* $\mathsf{cLieCog}$ *of curved conilpotent Lie coalgebras.*

Lemma 130 *The category of $u\mathcal{C}om^{\text{¡}}$–coalgebras is equivalent to the category whose objects are graded $\mathbb{K}$–modules $\mathcal{C}$ equipped with three maps:*

- *A degree -1 map $\delta\colon \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ which is symmetric is the sense that $\tau\delta = \delta'$, which satisfy the equation*
$$(\delta \otimes \mathrm{Id})\delta(x) + ((\delta \otimes \mathrm{Id})\delta(x))^{(2,3)} + ((\delta \otimes \mathrm{Id})\delta(x))^{(1,3)} = 0,$$
and such that for any $x \in \mathcal{C}$, there exists an integer n such that the element
$$(\mathrm{Id} \otimes \cdots \otimes \delta \otimes \cdots \otimes \mathrm{Id}) \cdots (\mathrm{Id} \otimes \delta)\delta(x)$$
is zero whenever δ appears at least n times.
- *A degree -1 map $\theta\colon \mathcal{C} \to \mathbb{K}$.*
- *A degree -1 map $d\colon \mathcal{C} \to \mathcal{C}$ such that $\theta d = 0$, such that $\delta d = -(d \otimes \mathrm{Id} + \mathrm{Id} \otimes d)\delta$ and such that $d^2 = -(\theta \otimes \mathrm{Id} + \mathrm{Id} \otimes \theta)\delta = -2(\theta \otimes \mathrm{Id})\delta$.*

The morphisms of this category are the morphisms of graded $\mathbb{K}$–modules which commute with these structure maps.

Proof We apply Theorem 119. A graded $u\mathcal{C}om^{\text{¡}}$–coalgebra is a graded $\mathbb{K}$–module $\mathcal{C}$ equipped with maps
$$\begin{aligned}
&\delta'\colon \mathcal{C} \to (\mathbb{K} \cdot s\mu) \otimes_{\mathbb{S}_2} (\mathcal{C} \otimes \mathcal{C}),\\
&\theta'\colon \mathcal{C} \to \mathbb{K} \cdot s\xi,\\
&d'\colon \mathcal{C} \to \mathcal{C},
\end{aligned}$$

such that the corresponding precoradical filtration is exhaustive, such that

(2) $$(\mathrm{Id}_{s\mu} \circ' \delta')\delta'(\mathcal{C}) \subset s^2\mathcal{R}(3) \otimes_{\mathbb{S}_3} \mathcal{C}^{\otimes 3}$$

and such that
$$\begin{aligned}
&\delta' d' = (\mathrm{Id} \circ' d')\delta',\\
&\theta' d' = 0,\\
&d'^2 = (\theta_{u\mathcal{C}om^{\text{¡}}} \circ \mathrm{Id})(\mathrm{Id} \circ' \delta')\delta'.
\end{aligned}$$

These maps induce new maps

$$\begin{aligned} \delta\colon \mathcal{C} &\xrightarrow{\delta'} (\mathbb{K}\cdot s\mu)\otimes_{\mathbb{S}_2}(\mathcal{C}\otimes\mathcal{C})\to\mathcal{C}\otimes\mathcal{C},\\ \theta\colon \mathcal{C} &\xrightarrow{\theta'} \mathbb{K}\cdot s\xi\to\mathbb{K},\\ d=d'\colon \mathcal{C}&\to\mathcal{C}, \end{aligned}$$

where the degree -1 map $(\mathbb{K}\cdot s\mu)\otimes_{\mathbb{S}_2}(\mathcal{C}\otimes\mathcal{C})\to\mathcal{C}\otimes\mathcal{C}$ sends $s\mu\otimes_{\mathbb{S}_2}(x\otimes y)$ to $\frac{1}{2}(x\otimes y+(-1)^{|x||y|}y\otimes x)$. Then, for any $x\in\mathcal{C}$,

$$\delta'(x)=s\mu\otimes_{\mathbb{S}_2}\delta(x).$$

We know from [19, Section 7.6.3] that the $\mathbb{K}$–module $\mathbb{T}(s\mu)(3)$ has three generators ν_{I}, ν_{II} and ν_{III}, which are obtained from the composite $s\mu\otimes_{\mathbb{S}_2}(s\mu\otimes 1)$ by applying the permutations $\mathrm{Id}\in\mathbb{S}_3$, $(2,3)$ and $(1,3)$, respectively. Moreover, $s^2\mathcal{R}(3)$ is spanned by $\nu_{\mathrm{I}}-\nu_{\mathrm{II}}$ and $\nu_{\mathrm{I}}-\nu_{\mathrm{III}}$. Further, $\mathbb{K}\cdot(\nu_{\mathrm{I}}+\nu_{\mathrm{II}}+\nu_{\mathrm{III}})$ is a complementary sub-$\mathbb{K}[\mathbb{S}_3]$–module of $s^2\mathcal{R}(3)$ in $\mathbb{T}(s\mu)(3)$. Let us denote by π the projection of $\mathbb{T}(s\mu)(3)$ onto $\mathbb{K}\cdot(\nu_{\mathrm{I}}+\nu_{\mathrm{II}}+\nu_{\mathrm{III}})$ along $s^2\mathcal{R}(3)$. Since the action of the group $\mathbb{S}_2$ on $s\mu$ is trivial, we have, for any $x\in\mathcal{C}$,

$$(\mathrm{Id}_{s\mu}\circ'\delta')\delta'(x)=2\nu_{\mathrm{I}}\otimes_{\mathbb{S}_3}((\delta\otimes\mathrm{Id})\delta(x)).$$

Then

$$(\pi\circ\mathrm{Id})(\mathrm{Id}_{s\mu}\circ'\delta')\delta'(x)=\tfrac{2}{3}(\nu_{\mathrm{I}}+\nu_{\mathrm{II}}+\nu_{\mathrm{III}})\otimes_{\mathbb{S}_3}((\delta\otimes\mathrm{Id})\delta(x)).$$

The above condition (2) is equivalent to the fact that $(\nu_{\mathrm{I}}+\nu_{\mathrm{II}}+\nu_{\mathrm{III}})\otimes_{\mathbb{S}_3}((\delta\otimes\mathrm{Id})\delta(x))$ is zero, which is equivalent to

$$(\delta\otimes\mathrm{Id})\delta(x)+((\delta\otimes\mathrm{Id})\delta(x))^{(2,3)}+((\delta\otimes\mathrm{Id})\delta(x))^{(1,3)}=0.$$

The other conditions are equivalent to

$$\begin{aligned} \delta d&=-(d\otimes\mathrm{Id}+\mathrm{Id}\otimes d)\delta,\\ \theta d&=0,\\ d^2&=-(\theta\otimes\mathrm{Id}+\mathrm{Id}\otimes\theta)\delta. \end{aligned}$$

Conversely, from the maps δ, θ and d, one can reconstruct δ', θ' and d' in the obvious way. □

Proof of Proposition 129 We show that the category described in Lemma 130 is equivalent to the category of curved conilpotent Lie coalgebras. Let $\mathscr{C}=(\mathcal{C},\delta,\theta,d)$ be a curved conilpotent Lie coalgebra. Then we can define the maps (δ',θ',d') on $s^{-1}\mathcal{C}$, where δ' is the composite

$$s^{-1}\mathcal{C} \simeq \mathbb{K}\cdot s^{-1} \otimes \mathcal{C} \to \mathbb{K}\cdot s^{-1} \otimes \mathbb{K}\cdot s^{-1} \otimes \mathcal{C} \otimes \mathcal{C} \simeq \mathbb{K}\cdot s^{-1} \otimes \mathcal{C} \otimes \mathbb{K}\cdot s^{-1} \otimes \mathcal{C},$$
$$s^{-1} \otimes x \mapsto s^{-1} \otimes s^{-1} \otimes \delta(x),$$

and where $\theta'(s^{-1}x) = \theta(x)$ and $d'(s^{-1}x) = -s^{-1}dx$ for any $x \in \mathcal{C}$. It is straightforward to prove that these maps satisfy the conditions of Lemma 130. Conversely, from a graded $\mathbb{K}$–module $\mathcal{D}$ and maps (δ, θ, d) as in Lemma 130, one can build a structure of curved conilpotent Lie coalgebra (δ', θ', d') on $s\mathcal{D}$, where δ' is the composite

$$s\mathcal{D} \simeq \mathbb{K}\cdot s \otimes \mathcal{D} \to \mathbb{K}\cdot s \otimes \mathbb{K}\cdot s \otimes \mathcal{D} \otimes \mathcal{D} \simeq \mathbb{K}\cdot s \otimes \mathcal{D} \otimes \mathbb{K}\cdot s \otimes \mathcal{D},$$
$$s \otimes x \mapsto -s \otimes s \otimes \delta(x),$$

and where $\theta'(sx) = \theta(x)$ and $d'(sx) = -sdx$ for any $x \in \mathcal{D}$. It is again straightforward to prove that these maps define actually a structure of curved conilpotent Lie coalgebra. Moreover, these two constructions are inverse to one another. □

8.4.3 The bar–cobar adjunction If we compose the bar–cobar adjunction between $u\mathcal{C}om$–algebras and $u\mathcal{C}om^{¡}$–coalgebras with the equivalence between $u\mathcal{C}om^{¡}$–coalgebras and curved conilpotent Lie coalgebras, then we obtain an adjunction $\Omega_C \dashv B_L$ between unital commutative algebras and curved conilpotent Lie coalgebras, which is as follows.

Definition 131 (curved Lie bar construction) Let $\mathscr{A} = (\mathcal{A}, \gamma_{\mathscr{A}}, 1)$ be a unital commutative algebra. Its *curved Lie bar construction* $B_L(\mathscr{A})$ is the following curved conilpotent Lie coalgebra. The underlying graded Lie coalgebra of $B_L(\mathscr{A})$ is

$$B_L(\mathcal{A}) := \mathcal{L}ie^c \circ (s\mathcal{A} \oplus \mathbb{K}\cdot v),$$

where $\mathcal{L}ie^c$ denotes the Lie cooperad which is the linear dual of the Lie operad and where $|v| = 2$. The coderivation of $B_L(\mathscr{A})$ extends the map

$$\mathcal{L}ie^c(s\mathcal{A} \oplus \mathbb{K}v) \twoheadrightarrow s\mathcal{A} \wedge s\mathcal{A} \oplus s\mathcal{A} \oplus \mathbb{K}v \to s\mathcal{A}, \qquad \begin{aligned} sx \wedge sy &\mapsto (-1)^{|x|} s\gamma_{\mathcal{A}}(x \otimes y), \\ v &\mapsto s1, \\ sx &\mapsto -sdx. \end{aligned}$$

The curvature is the map

$$\mathcal{L}ie^c(s\mathcal{A} \oplus \mathbb{K}\cdot v) \twoheadrightarrow \mathbb{K}\cdot v \to \mathbb{K}, \quad v \mapsto 1.$$

Definition 132 (unital commutative cobar construction) Let $\mathscr{C} = (\mathcal{C}, \delta, d_{\mathcal{C}}, \theta)$ be a curved Lie coalgebra. Its *unital commutative cobar construction* $\Omega_C(\mathscr{C})$ is the free unital commutative algebra

$$\Omega_C\mathcal{C} := S(s^{-1}\mathcal{C}) := \bigoplus_{n\in\mathbb{N}} (s^{-1}\mathcal{C})^{\otimes n}/\mathbb{S}_n,$$

whose coderivation extends the map

$$s^{-1}\mathcal{C} \to S(s^{-1}\mathcal{C}), \quad s^{-1}x \mapsto \theta(x)1 - s^{-1}d_{\mathcal{C}}x - \sum (-1)^{|x_1|} s^{-1}x_1 \otimes_{\mathbb{S}_2} s^{-1}x_2,$$

where $\sum x_1 \wedge x_2 = \delta(x)$.

Definition 133 (twisting morphisms) A twisting morphism from a curved conilpotent Lie coalgebra $\mathscr{C}$ to a unital commutative algebra $\mathscr{A}$ is a degree -1 map $\alpha\colon \mathcal{C} \to \mathcal{A}$ such that

$$\partial\alpha + \gamma_{\mathscr{A}}(\alpha\otimes\alpha)\delta_{\mathscr{C}} = \theta(-)1_{\mathscr{A}}.$$

We denote by $\mathrm{Tw}_L(\mathscr{C},\mathscr{A})$ the set of twisting morphisms from $\mathscr{C}$ to $\mathscr{A}$.

Proposition 134 *We have functorial isomorphisms*

$$\hom_{u\mathit{Com}-\mathsf{alg}}(\Omega_C\mathscr{C},\mathscr{A}) \simeq \mathrm{Tw}_L(\mathscr{C},\mathscr{A}) \simeq \hom_{\mathsf{cLieCog}}(\mathscr{C}, B_L\mathscr{A})$$

for any unital commutative algebra $\mathscr{A}$ and any curved conilpotent Lie coalgebra $\mathscr{C}$.

Proof The proof uses the same arguments as the proof of Proposition 63. □

The adjunction $\Omega_C \dashv B_L$ is part of a larger picture,

$$\mathsf{cLieCog} \underset{B_\iota}{\overset{\Omega_\iota}{\rightleftarrows}} u\mathit{Com}_\infty\text{–}\mathsf{alg} \underset{\psi^*}{\overset{\psi_!}{\rightleftarrows}} u\mathit{Com}\text{–}\mathsf{alg},$$

where the adjunction $\psi_! \dashv \psi^*$ is induced by the morphism of operads $\psi\colon u\mathit{Com}_\infty \to u\mathit{Com}$ and where $\Omega_C = \psi_!\Omega_\iota$ and $B_L = B_\iota\psi^*$. We know that a $u\mathit{Com}_\infty$–algebra $\mathscr{A} = (\mathcal{A},\gamma)$ is the data of a chain complex $\mathcal{A}$ together with a degree -1 map

$$\gamma\colon \mathcal{L}ie^c(s\mathcal{A}\oplus\mathbb{K}\cdot v) \to s\mathcal{A}$$

such that the coderivation of the curved Lie coalgebra $\mathcal{L}ie^c(s\mathcal{A}\oplus\mathbb{K}\cdot v)$ which extends γ squares to $(\theta\otimes\mathrm{Id})\delta$, where θ is given by

$$\mathcal{L}ie^c(s\mathcal{A}\oplus\mathbb{K}\cdot v) \twoheadrightarrow \mathbb{K}\cdot v \to \mathbb{K}, \quad v \mapsto 1.$$

In particular, we have the following:

- A degree zero symmetric product
$$\gamma_2\colon \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}.$$
- A degree 1 map
$$\gamma_{\mathrm{I,II}}\colon \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$$
whose boundary is the associator of γ_2, that is,
$$\partial(\gamma_{\mathrm{I,II}})=\gamma_2(\mathrm{Id}\otimes\gamma_2)-\gamma_2(\gamma_2\otimes\mathrm{Id}).$$
- A degree 1 map
$$\gamma_{\mathrm{I,III}}\colon \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$$
whose boundary is
$$\partial(\gamma_{\mathrm{I,III}})=\gamma_2(\mathrm{Id}\otimes\gamma_2)-\gamma_2(\mathrm{Id}\otimes\gamma_2)(\tau\otimes\mathrm{Id}).$$
- An element $1_{\mathscr{A}}$ defined by $\gamma(v)=s1_{\mathscr{A}}$.
- A degree 1 map $\gamma_u\colon \mathcal{A}\to\mathcal{A}$ which makes $1_{\mathscr{A}}$ a unit up to homotopy:
$$\partial(\gamma_u)=\gamma_2(1_{\mathscr{A}}\otimes\mathrm{Id})-\mathrm{Id}.$$

8.4.4 The Koszul property and the infinity category of $uCom_\infty$–algebras

Theorem 135 *The operad $uCom$ is Koszul.*

Proof We know from [17] that $quCom^{¡}\simeq Com^{¡}\circ(\mathcal{I}\oplus\mathbb{K}\cdot s\xi)$. So, we have
$$quCom\circ quCom^{¡}\simeq\mathbb{K}\cdot\xi\oplus Com\circ Com^{¡}\circ(\mathcal{I}\oplus\mathbb{K}\cdot s\xi).$$
We can filter $quCom\circ_\kappa quCom^{¡}$ by the number of ξ and $s\xi$ appearing in the trees. Then the induced graded complex have the form
$$G(quCom\circ_\kappa quCom^{¡})\simeq\mathbb{K}\cdot\xi\oplus(Com\circ_\kappa Com^{¡})\circ(\mathcal{I}\oplus\mathbb{K}\cdot s\xi).$$
We already know by [19, Theorems 7.4.6 and 13.1.7] that the canonical morphism $Com\circ_\kappa Com^{¡}\to\mathcal{I}$ is a weak equivalence. Then the map $G(quCom\circ_\kappa quCom^{¡})\to\mathcal{I}$ may be decomposed as follows:
$$G(quCom\circ_\kappa quCom^{¡})\simeq\mathbb{K}\cdot\xi\oplus(Com\circ_\kappa Com^{¡})\circ(\mathcal{I}\oplus\mathbb{K}\cdot s\xi)\to\mathcal{I}\oplus\mathbb{K}\cdot\xi\oplus\mathbb{K}\cdot s\xi\to\mathcal{I}.$$
All the maps of this composition are quasi-isomorphisms. So, by Theorem 88, the canonical map $quCom\circ_\kappa quCom^{¡}\to\mathcal{I}$ is a quasi-isomorphism. We conclude by Theorem 116. □

There are several ways to describe the infinity category of $uCom$–algebras:

- One can take the Dwyer–Kan simplicial localization of the category of $uCom$–algebras with respect to quasi-isomorphisms as described in [10; 9].
- One can take the simplicial category whose objects are cofibrant-fibrant $uCom$–algebras and whose spaces of morphisms are
$$\mathrm{Map}(\mathscr{A},\mathscr{B})_n := \mathrm{HOM}_{uCom-\mathsf{alg}}(\mathscr{A},\mathscr{B}).$$
- One can also take the simplicial category whose objects are all $uCom$–algebras and whose spaces of morphisms are
$$\mathrm{Map}(\mathscr{A},\mathscr{B}) := \mathrm{HOM}_{uCom-\mathsf{alg}}(\Omega_C B_L \mathscr{A},\mathscr{B}) \simeq \mathrm{HOM}_{\mathsf{cLieCog}}(B_L\mathscr{A}, B_L\mathscr{B}).$$

Appendix

The purpose of this appendix is to describe the category of dg counital cocommutative coalgebras over an algebraically closed field of characteristic zero in the vein of [7]. In the sequel, dg counital cocommutative coalgebras are simply called cocommutative coalgebras. We suppose that the base field $\mathbb{K}$ is algebraically closed field and of characteristic zero.

Remark 136 The characteristic zero assumption is needed in [7, Theorem 2.9].

We know that the linear dual of a cocommutative coalgebra is a commutative algebra. Moreover, for any cocommutative coalgebra $\mathscr{C}$, the subcoalgebras of $\mathscr{C}$ are in correspondence with the ideals of $\mathscr{C}^*$.

Definition 137 (orthogonal ideals and subcoalgebras) Let $\mathscr{D} = (\mathcal{D},\Delta,\epsilon)$ be a subcoalgebra of $\mathscr{C}$. The orthogonal of $\mathscr{D}$ is the subchain complex
$$\mathscr{D}^{\perp} := \{f \in \mathscr{C}^* \mid f(x) = 0 \text{ for all } x \in \mathcal{D}, \} \subset \mathscr{C}^*,$$
which is an ideal of $\mathscr{C}^*$. Let I be an ideal of the commutative algebra $\mathscr{C}^*$. The orthogonal of I is the subchain complex $I^{\perp} := \{x \in \mathscr{C} \mid f(x) = 0 \text{ for all } f \in I\} \subset \mathscr{C}$, which is a subcoalgebra of $\mathscr{C}$.

Definition 138 (pseudocompact algebras) A pseudocompact algebra is a dg unital commutative algebra $\mathscr{A}$ together with a set $\{I_u\}_{u\in U}$ of ideals of finite codimension, which is stable under finite intersections and such that

$$\mathscr{A} \simeq \lim \mathscr{A}/I_u.$$

A morphism of pseudocompacts algebras from $(\mathscr{A}, \{I_u\}_{u\in U})$ to $(\mathscr{B}, \{J_v\}_{v\in V})$ is a morphism of algebras $f\colon \mathscr{A} \to \mathscr{B}$ which is continuous with respect to the induced topologies, and is such that for any $v \in V$, there exists a $u \in U$ such that the composite morphism $\mathscr{A} \to \mathscr{B} \to \mathscr{B}/J_v$ factors through $\mathscr{A} \to \mathscr{A}/I_u$. A pseudocompact algebra $\mathscr{A}$ is called local if its underlying graded algebra is local.

Proposition 139 *The linear dual of a cocommutative coalgebra is a pseudocompact algebra. Moreover, the linear dual functor is an antiequivalence between the category of cocommutative coalgebras and the category of pseudocompact algebras.*

Proof It is clear that linear duality induces an antiequivalence between finite-dimensional cocommutative coalgebras and finite-dimensional commutative algebras. The rest is a consequence of the following Proposition 140. □

Proposition 140 [11] *Let $\mathscr{C}$ be a cocommutative coalgebra and let x be an element of $\mathscr{C}$. There exists a finite-dimensional subcoalgebra of $\mathscr{C}$ which contains x. Then $\mathscr{C}$ is the colimit of the filtered diagram of its finite-dimensional subcoalgebras.*

Chuang, Lazarev and Mannan showed that any pseudocompact algebra can be decomposed into a product of local pseudocompact algebras.

Theorem 141 [7, Theorem 2.9] *Any pseudocompact algebra $\mathscr{A}$ is isomorphic to the product of local pseudocompact algebras $\mathscr{A} \simeq \prod_{i\in I} \mathscr{A}_i$. Moreover, a morphism of products of local pseudocompact algebras $f\colon \prod_{i\in I} \mathscr{A}_i \to \prod_{j\in J} \mathscr{B}_j$ is the data of a function $\phi\colon J \to I$ and a morphism $f_j\colon \mathscr{A}_{\phi(j)} \to \mathscr{B}_j$ for any $j \in J$, where* $\pi_j f = f_j \pi_{\phi(j)}$ (here π_j and $\pi_{\phi(j)}$ denote the projection of $\prod_{j\in J} \mathscr{B}_j$ onto $\mathscr{B}_j$ and the projection of $\prod_{i\in I} \mathscr{A}_i$ onto $\mathscr{A}_{\phi(j)}$, respectively).

We show that local pseudocompact algebras are linear duals of conilpotent cocommutative coalgebras.

Definition 142 (irreducible coalgebras) A nonzero graded cocommutative coalgebra is said to be *irreducible* if any two nonzero subcoalgebras have a nonzero intersection.

Proposition 143 *A graded cocommutative coalgebra is irreducible if and only if its dual algebra is local.*

Proof Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon)$ be a graded cocommutative coalgebra. We first suppose that it is irreducible. Let M_1 and M_2 be two maximal ideals of the commutative algebra $\mathscr{C}^*$. Since $\mathscr{C}$ is irreducible, the subcoalgebras $M_1^\perp$ and $M_2^\perp$ have a nonzero intersection. So $M_1 + M_2 \subset (M_1^\perp \cap M_2^\perp)^\perp$ is a proper ideal. Since M_1 and M_2 are maximal ideals, $M_1 = M_1 + M_2 = M_2$. So $\mathscr{C}^*$ is local. Conversely, suppose that $\mathscr{C}^*$ is local. We denote by M its maximal ideal. By Lemma 144, M is the kernel of an augmentation $\mathscr{C}^* \to \mathbb{K}$. By the antiequivalence between pseudocompact algebras and cocommutative coalgebras, we obtain a morphism of coalgebras $\mathbb{K} \to \mathscr{C}$, that is, an atom a of $\mathscr{C}$. For any nonzero subcoalgebra $\mathscr{D}$ of $\mathscr{C}$, the orthogonal $\mathscr{D}^\perp$ is contained in M. Thus, $\mathbb{K} \cdot a = M^\perp \subset (\mathscr{D}^\perp)^\perp = \mathscr{D}$. So any nonzero subcoalgebra of $\mathscr{C}$ contains a. Subsequently, $\mathscr{C}$ is irreducible. □

Lemma 144 *Let $\mathscr{A}$ be a graded local pseudocompact algebra. Then the maximal ideal M of $\mathscr{A}$ is the kernel of an augmentation $\mathscr{A} \to \mathbb{K}$.*

Proof Since $\mathscr{A} = (\mathcal{A}, \gamma_{\mathscr{A}}, 1)$ is the inverse limit of finite-dimensional algebras and since M is maximal, M is the kernel of a surjection $\mathscr{A} \to \mathscr{B}$, where $\mathscr{B} = (\mathcal{B}, \gamma_{\mathcal{B}}, 1)$ is a finite-dimensional commutative algebra. Since M is maximal, any nonzero element of $\mathcal{B}$ is invertible. Since the elements in nonzero degrees are nilpotent, $\mathcal{B}$ is concentrated in degree zero. So $\mathcal{B}$ is a finite-dimensional field extension of $\mathbb{K}$. Finally, $\mathcal{B} \simeq \mathbb{K}$ because $\mathbb{K}$ is an algebraically closed field. □

Corollary 145 *A graded cocommutative coalgebra is irreducible if and only if it contains a single atom.*

Proof It is a direct consequence of Proposition 143. □

Proposition 146 *Irreducible graded cocommutative coalgebras are conilpotent graded cocommutative coalgebras.*

Proof Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon)$ be an irreducible graded cocommutative coalgebra. Let x be an element of $\mathcal{C}$ and let $\mathscr{D} = (\mathcal{D}, \Delta, \epsilon)$ be a finite-dimensional subcoalgebra of $\mathscr{C}$ which contains x. The commutative algebra $\mathscr{D}^*$ is local; its maximal ideal is $M := \overline{\mathcal{D}}^*$. Then $\mathscr{D}_0^*$ is also local with maximal ideal M_0. By Nakayama's lemma, M_0 is nilpotent. So, M is nilpotent and so $\mathscr{D}$ is a conilpotent cocommutative coalgebra. □

Corollary 147 *The antiequivalence between the category of pseudocompact algebras and the category* uCocom *of cocommutative coalgebras restricts to an antiequivalence* *between the category of local pseudocompact algebras and the category* uNilCocom *of conilpotent cocommutative coalgebras.*

Proof It is a direct consequence of Propositions 143 and 146. □

Theorem 78 *Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon)$ be a dg cocommutative coalgebra over an algebraically closed field of characteristic zero and let A be its set of graded atoms. There exists a unique decomposition $\mathscr{C} \simeq \bigoplus_{a \in A} \mathscr{C}_a$, where $\mathscr{C}_a$ is a subcoalgebra of $\mathscr{C}$ which contains a and which belongs to the category* uNilCocom*. Moreover, a morphism of dg cocommutative coalgebras $f \colon \bigoplus_{a \in A} \mathscr{C}_a \to \bigoplus_{b \in B} \mathscr{D}_b$ is the data of a function $\phi \colon A \to B$ and of a morphism $f_a \colon \mathscr{C}_a \to \mathscr{D}_{\phi(a)}$ for any $a \in A$.*

Proof The only point that needs to be cleared up is that, in the decomposition $\mathscr{C} = \bigoplus_{i \in I} \mathscr{C}_i$, the set I is isomorphic to the set of graded atoms of $\mathscr{C}$. A graded atom of $\mathscr{C}$ is a morphism of graded cocommutative coalgebras from $\mathbb{K}$ to $\mathscr{C}$, that is, a morphism of graded pseudocompact algebras from $\prod_{i \in I} \mathscr{C}_i^*$ to $\mathbb{K}$. So it is the choice of an element of I. □

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