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Homotopical intersection theory III: Multirelative intersection problems

John R Klein
Bruce Williams

We extend some results of Hatcher and Quinn (1974) beyond the metastable range. We give a bordism-theoretic obstruction $\chi(f)$ to deforming a map $f\colon P \to N$ between manifolds simultaneously off of a collection of pairwise disjoint submanifolds $Q_1, \dots, Q_j \subset N$ under the assumption that f can be deformed off of any proper subcollection in a homotopy coherent way. In a certain range of dimensions, $\chi(f)$ is a complete obstruction to finding the desired deformation. We apply this machinery to embedding problems and to the study of linking phenomena.

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1. Introduction 1079
2. Language 1089
3. Strongly cocartesian cubes 1096
4. Proof of Theorems A and C 1101
5. Proof of Theorem 3.12 1106
6. Multiple disjunction 1112
7. The embedding tower 1113
8. Spaces of link maps 1119
References 1132

1 Introduction

1.1 Intersection problems

In [16] we considered the problem of deforming a map $f\colon P \to N$ between compact smooth manifolds off a compact smooth submanifold $Q \subset N$. This was called an *intersection problem*. We obtained an obstruction $\chi(f)$ residing in a normal bordism

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Note from JRK: Bruce Williams passed away on January 11, 2018, before the final revision of this paper was completed. Bruce was a close friend and an inspiring mentor.

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group $\Omega_0(X;\xi)$. The vanishing of the obstruction is necessary for finding such a deformation. One of the main results of [16] was that in a certain metastable range of dimensions, $\chi(f)$ is a complete obstruction to finding a homotopy from f to a map having disjoint image from Q. The goal of the current paper is to extend these ideas to the multirelative setting.

Fix a positive integer j and let

$$Q_1,\dots,Q_j \subset N$$

be a collection of pairwise disjoint, closed, smooth submanifolds of a compact, connected, smooth manifold N. Given a map $P \to N$, where P is a closed manifold, the problem we consider is that of finding a deformation of f off of the Q_i *simultaneously.* We approach this inductively, by assuming that P can be deformed off of any proper union of the Q_i in such a way that the deformations line up in a certain homotopically coherent fashion. We first explain what this precisely means.

Recall that a $(k+1)$–ad of spaces consists of a space X together with k distinguished subspaces $X_1,\dots X_k \subset X$. The notation for such data is $(X;X_1,\dots,X_k)$, but it will often be convenient to simply write X when the subspaces are understood.

Example 1.1 (1) A space Z can be considered as a constant $(k+1)$–ad, that is, $(Z;Z,\dots,Z)$.

(2) The standard $(k-1)$–simplex Δ^{k-1} together with its codimension one faces is a $(k+1)$–ad, ie $(\Delta^{k-1};d_0\Delta^{k-1},\dots,d_{k-1}\Delta^{k-1})$.

(3) If Z is a space and X is a $(k+1)$–ad, then the cartesian product $Z \times X$ is a $(k+1)$–ad in the evident way.

A map of $(k+1)$–ads $X \to Y$ is a continuous map of underlying spaces which restricts to maps $X_i \to Y_i$ for all i. We can topologize this as the subspace of the mapping space of all maps from X to Y in the compact–open topology.

Consider N together with the subspaces $N \setminus Q_1,\dots,N \setminus Q_j$ as a $(j+1)$–ad

$$(N;N \setminus Q_1,\dots,N \setminus Q_j).$$

Then a *multirelative intersection problem* is defined to be a map of $(j+1)$–ads

$$f\colon P \times \Delta^{j-1} \to N.$$

Set $Q_J = Q_1 \sqcup \cdots \sqcup Q_j$. We will consider $N \setminus Q_J$ as a constant $(j+1)$–ad; it is then a sub-ad of $(N;N \setminus Q_1,\dots,N \setminus Q_j)$. We define a *solution* to a multirelative intersection problem to be a homotopy (of maps of $(j+1)$–ads) f_t from $f = f_0$ to an ad map $f_1\colon P \times \Delta^{j-1} \to N$ which factors as

$$P \times \Delta^{j-1} \to N \setminus Q_J \xrightarrow{\subset} N$$

In particular, the image of f_1 is disjoint from Q_J.

In more modern language the problem can be reformulated as follows: Let $J = \{1, \dots, j\}$. For $S \subset J$, let

$$Q_S = \coprod_{i \in S} Q_i.$$

Then a multirelative intersection problem is equivalent to specifying a map

$$f\colon P \to \operatorname*{holim}_{S \subsetneq J}(N \setminus Q_S), \tag{1}$$

where the target is the homotopy inverse limit of the spaces $N \setminus Q_S$ as S ranges through the proper subsets of J. Explicitly, the displayed homotopy limit is given by the space of maps of $(j+1)$–ads $\Delta^{j-1} \to N$.

The deliberate ambiguity in our notation is for the sake of convenience: we use f to denote the map (1) as well as for the map of ads $P \times \Delta^{j-1} \to N$, as this is not likely to cause confusion (note these maps determine each other by an adjunction).

A solution then amounts to a map $\hat{f}\colon P \to N \setminus Q_J$, together with a commuting homotopy $f_t\colon P \to \operatorname{holim}_{S \subsetneq J}(N \setminus Q_S)$ with $t \in [0,1]$, for the diagram

$$\begin{tikzcd} & N \setminus Q_J \arrow[d] \\ P \arrow[ur, "\hat{f}"] \arrow[r, "f"'] & \operatorname*{holim}_{S \subsetneq J}(N \setminus Q_S) \end{tikzcd} \tag{2}$$

Given a map of $(j+1)$–ads $f\colon P \times \Delta^{j-1} \to N$ as above, we write

$$E(P, Q_\bullet)$$

for the *iterated homotopy fiber product* of $P \times \Delta^{j-1}$ and each of the Q_i over N. This is just the homotopy pullback of the diagram

$$P \times \Delta^{j-1} \times \prod_{i=1}^{j} Q_i \to \prod_{i=0}^{j} N \xleftarrow{\Delta} N,$$

where Δ is the diagonal map and the left map is the product of $f\colon P \times \Delta^{j-1} \to N$ with the inclusions of the Q_i.

Define a virtual bundle ξ over $E(P, Q_\bullet)$ as follows: Let τ_P be the tangent bundle of P, τ_N the tangent bundle of N and τ_{Q_i} the tangent bundle of Q_i; each one of these gives a bundle over $E(P, Q_\bullet)$ using the evident (projection) maps. To avoid notational clutter, we use the same notation for these pullbacks. Then we set

$$\xi := -\tau_P + \sum_{i=1}^{j} (\tau_N - \tau_{Q_i}).$$

Suppose $p = \dim P$, $q_i = \dim Q_i$ and $n = \dim N$. It will also be convenient to write

$$\mu = \min_i (n - q_i - 2) \quad \text{and} \quad \Sigma = \sum_i (n - q_i - 2).$$

In particular, the virtual rank of ξ is $2j - p + \Sigma$. The following assumption will be made throughout the paper:

Hypothesis 1.2 For $1 \le i \le j$, we have $n - q_i \ge 2$.

We briefly review the definition of bordism with coefficients in a virtual bundle. Let X be a space equipped with a finite-dimensional inner product bundle ζ of rank s. Then one has the Thom space X^ζ, which is the quotient space formed from the unit disk bundle by collapsing the unit sphere bundle to a point. For the purposes of this paper, we define $\Omega_k(X; \zeta)$ to be the k^{th} stable homotopy group $\pi_k^{\text{st}}(X^\zeta)$. By standard transversality arguments, an element of this abelian group is represented by a compact smooth submanifold $V \subset \mathbb{R}^{k+d}$, for some $d \ge 0$, together with a map $g\colon V \to X$ such that the pullback of $\zeta \oplus \epsilon^d$ along g is identified with the normal bundle of V (where ϵ^d is the trivial bundle of rank d; note that the dimension of V is necessarily $k - s$). Then bordism defines an equivalence relation on this collection and the set of equivalence classes is canonically identified with $\Omega_k(X; \zeta)$. With respect to this identification, note that the operation of disjoint union of bordism classes corresponds to the addition of stable homotopy classes. Now suppose that ζ is a virtual bundle. This means that $\zeta \oplus \epsilon^j$ comes equipped with an isomorphism to a finite-dimensional inner product bundle η for some integer $j \ge 0$. In this instance, we define $\Omega_k(X; \zeta)$ to be $\Omega_{k+j}(X; \eta)$. Our indexing convention for the bordism group differs from that of [16], but is the same as the one we used in [15].

Theorem A *Assume $j \geq 1$. Then there is an obstruction*

$$\chi(f) \in \bigoplus_{(j-1)!} \Omega_{2j-2}(E(P, Q_\bullet); \xi)$$

which vanishes if the intersection problem defined by f possesses a solution. Conversely, if

$$p \leq 1 + \mu + \Sigma$$

then the vanishing of $\chi(f)$ guarantees the existence of a solution.

Theorem A is proved using a fiberwise version of Poincaré duality together with some general results about strongly cocartesian cubes.

Remark 1.3 The $j = 1$ case ("the metastable range") of Theorem A was already considered in [16]. That work gave a homotopy-theoretic approach to the main results of the paper of Hatcher and Quinn [12] (when $j = 1$, Theorem D below amounts to the vanishing obstruction case of [12, Theorem 2.2]).

Remark 1.4 The obstruction $\chi(f)$ is defined in a homotopy-theoretic manner. Given the identification between bordism theory and the homotopy groups of a Thom spectrum, it is reasonable to ask what $\chi(f)$ means geometrically. In the $j = 1$ case such an interpretation was provided by the "index theorem" of [16, Theorem 12.1]. The $j > 1$ case is more subtle and involves iterated intersections of null-bordism data. We hope to address this in detail in another paper. Meanwhile, to leave the reader with an impression, we now sketch a geometric description of $\chi(f)$ when $j = 2$.

Let $j = 2$ and let $f\colon P \times \Delta^1 \to N$ be an intersection problem. Let b be the barycenter of Δ^1 and let D_i be the transversal intersection of $f|_{P\times b}\colon P \times b \to N$ with Q_i. By assumption, the evident maps $D_i \to E(P, Q_i)$ are null-bordant. Let $g_i\colon W_i \to E(P, Q_i)$ be a null-bordism. Compose this with the projection $E(P, Q_i) \to P$ to get maps $h_i\colon W_i \to P$. Now take the transversal intersection of the product map $h_1 \times h_2\colon W_1 \times W_2 \to P \times P$ with the diagonal of P. This produces a closed manifold W_{12} of dimension $p - 2 - \Sigma$ equipped with a map $W_{12} \to E(P, Q_\bullet)$ which is covered by the requisite bundle data. The associated bordism class coincides with the obstruction $\chi(f)$.

Remark 1.5 (large codimension) If $p \leq 1 + \Sigma$, then the bordism group of Theorem A is trivial. Consequently, f can be homotopy factorized through $N \setminus Q_J$ in this case.

If $j=1$, this conclusion also follows from transversality, and for $j>1$ it follows from the higher Blakers–Massey theorem applied to the j–cubical diagram $\{N \setminus Q_S\}_{S \subset J}$ (see Goodwillie [4, Theorem 2.5]).

1.1.1 Highly connected manifolds When the manifolds P and Q_i are sufficiently highly connected, the obstruction group of Theorem A admits a simpler description. Suppose that P is a–connected and Q_i is b_i–connected. Choose basepoints in $x \in P$ and $y_i \in Q_i$. Then x gives rise to a point $x' \in N$ using f. The homotopy fiber product of $E(x, y_\bullet)$ is defined and comes equipped with a map $E(x, y_\bullet) \to E(P, Q_\bullet)$. Moreover, the pullback of ξ to $E(x, y_\bullet)$ is a trivial virtual bundle of rank $2j - p + \Sigma$. Hence, the bordism groups associated with this pullback are framed bordism groups of $E(x, y_\bullet)$ shifted in degree by $2j - p + \Sigma$.

It is also straightforward to check that the map

$$E(x, y_\bullet) \to E(P, Q_\bullet)$$

is $\min(a, b_1, \ldots, b_j)$–connected. It follows that the associated map of Thom spectra is k–connected, where $k = \min(a, b_1, \ldots, b_j) + 2j - p + \Sigma$. In particular, the induced homomorphism of bordism groups is an isomorphism in degrees strictly less than k.

Note that $E(x, y_\bullet)$ is the space of j–tuples $(\lambda_1, \ldots, \lambda_j)$ in which $\lambda_i\colon [0,1] \to N$ is a path from x' to y_i for $1 \le i \le j$. The j–fold cartesian product of loop spaces $\prod_j \Omega N$ based at x' acts on $E(x, y_\bullet)$ by path composition. After a basepoint for $E(x, y_\bullet)$ is fixed, we obtain a homotopy equivalence $E(x, y_\bullet) \simeq \prod_j \Omega N$. Consequently, we have shown:

Addendum B *Assume $p \le 1 + \Sigma + \min(a, b_1, \ldots, b_j)$. Then the obstruction group appearing in Theorem A is isomorphic to the direct sum of framed bordism groups*

$$\bigoplus_{(j-1)!} \Omega^{\mathrm{fr}}_{p-2-\Sigma}\Big(\prod_j \Omega N\Big).$$

Example 1.6 Suppose $P = S^p$ and $Q_i = S^{q_i}$ are spheres. Then $a = p-1$ and $b_i = q_i - 1$. Consequently, the inequality appearing in Addendum B becomes $p \le \Sigma + \mu - j$.

Example 1.7 Suppose $p = 2 + \Sigma$ and $a, b_i \ge 1$. Then the obstruction group of Addendum B is isomorphic to $\bigoplus_{(j-1)!} \mathbb{Z}[\pi]^{\otimes j}$, with $\pi = \pi_1(N)$.

1.2 The solution space

The space of lifts solving the multirelative intersection problem (2) is defined by converting the vertical map appearing in that diagram into a fibration and then taking the space of sections of this fibration along P. The space of such lifts is called the *solution space* and is denoted by $\mathscr{L}(f)$.

For a spectrum E we let $\Omega^\infty E$ be the associated infinite loop space.

Theorem C *Assume that in the solution space* $\mathscr{L}(f)$ *is nonempty and is equipped with a choice of basepoint. Then there is a* $(1-p+\mu+\Sigma)$*–connected map*

$$\mathscr{L}(f) \to \prod_{(j-1)!} \Omega^\infty E(P, Q_\bullet)^{\xi+(1-2j)\epsilon}.$$

1.3 Families of embeddings

A variant of the multirelative intersection problem involves families of smooth embeddings. In this instance one is given a map of $(j+1)$–ads $f\colon P\times\Delta^{j-1}\to N$ which is also a $(j-1)$–parameter family of smooth embeddings from P to N. The solution of the problem in this case is to find a deformation of ad-maps, this time through an isotopy, to a $(j-1)$–parameter family of embeddings having image disjoint from Q_J.

By combining Theorem A with Theorem E of Goodwillie and Klein [6], we obtain:

Theorem D (multiple disjunction) *Assume*

$$p, q_i \le n-3 \quad \textit{and} \quad p \le 1+\min(n-p-2,\mu)+\Sigma.$$

Then $\chi(f)=0$ *if and only if the multirelative intersection problem of embeddings has a solution.*

1.4 The embedding tower

For a smooth manifold P of dimension p without boundary and a smooth manifold N of dimension n, possibly with boundary, let $E(P,N)$ denote the space of smooth embeddings. When P is closed, Weiss [30] exhibits a tower of fibrations

$$\cdots \to E_2(P,N) \to E_1(P,N)$$

and compatible maps $E(P,N)\to E_k(P,N)$. Up to homotopy, the j^{th} layer of the tower is given by the space of compactly supported global sections of a certain fibration over the configuration space $\binom{P}{j}$, the latter given by the space of subsets of P having cardinality j. The space $E_j(P,N)$ is in some sense the best approximation to $E(P,N)$

obtained from spaces of embeddings $E(U,N)$ as U ranges throughout the open subsets of P that are diffeomorphic to a disjoint union of at most j open balls. In what follows, we assume that P is compact.

If $p \le n-1$, then $E_1(P,N)$ has the homotopy type of the space of immersions of P in N. If $p \le n-3$, then the map

$$E(P,N) \to \lim_{j\to\infty} E_j(P,N)$$

is a homotopy equivalence; see Goodwillie and Weiss [8] and Goodwillie and Klein [6]. The above motivates the following question: given a point of some stage of the tower, say $E_{j-1}(P,N)$, what are the obstructions to lifting the given point to the embedding space? If $j=2$, the work of Haefliger [10], Dax [2], Salomonsen [26] and Hatcher and Quinn [12] provides answers to this question in the metastable range (for the discussion of this case in the context of the tower, see [30, Section 4]).

It will be convenient to consider the following modification of this problem. Fix a basepoint of $E_1(P,N)$, ie an immersion. Let $\bar{E}_j(P,N)$ be the fiber of $E_j(P,N) \to E_1(P,N)$. Then the tower

$$\cdots \to \bar{E}_2(P,N) \to \bar{E}_1(P,N) = *$$

converges to $\bar{E}(P,N) = \mathrm{fiber}(E(P,N) \to E_1(P,N))$. Furthermore, the layers of this tower for $j>1$ coincide with the layers of the embedding tower.

Recall that $J=\{1,\dots,j\}$. In Section 7 we construct a fiberwise spectrum with Σ_j–action $\mathscr{C}_J$ over the configuration space $E_J(P) := E(J,P)$, which depends only on the data P, N and j. Let τ be the tangent bundle of $E_J(P)$ (ie restriction of the cartesian product of j copies of the tangent bundle of P). Then we can twist $\mathscr{C}_J$ by $-\tau$ to obtain a fiberwise spectrum with Σ_j–action ${}^{-\tau}\mathscr{C}_J$ over $E_J(P)$. In particular, one can speak about the equivariant homology of $E_J(P)$ with coefficients in ${}^{-\tau}\mathscr{C}_J$.

We will define an invariant

$$\mu\colon \pi_0(\bar{E}_{j-1}(P,N)) \to H_0^{\Sigma_j}(E_J(P); {}^{-\tau}\mathscr{C}_J)$$

which vanishes on the image of $\pi_0(\bar{E}_j(P,N))$.

Theorem E *Assume $j \ge 2$ and N is r–connected with $r \le n-2$. Assume additionally*

$$r \ge p-1-(j-1)(n-p-2).$$

If $x \in \bar{E}_{j-1}(P,N)$, then $\mu(x)=0$ implies that x lifts to $\bar{E}_j(P,N)$.

If N is contractible then we can take $r = n-2$. In this case the displayed inequality $r \geq p-1-(j-1)(n-p-2)$ is automatically satisfied:

Corollary F *Assume $j \geq 2$ and that N is contractible. If $\mu(x) = 0$, then $x \in \bar{E}_{j-1}(P, N)$ lifts to $\bar{E}_j(P, N)$.*

Remark 1.8 By [8], the map $\bar{E}(P, N) \to \bar{E}_j(P, N)$ is $((j+1)(n-p-2)+3-n)$–connected. Consequently, in both Theorem E and Corollary F, if $\mu(x) = 0$, then $x \in \bar{E}_{j-1}(P, N)$ will lift to $\bar{E}(P, N)$ if in addition $(j+1)(n-p-2)+3-n \geq 0$.

1.5 Link maps

Our main results can also be used to study higher-order linking phenomena. Given connected closed manifolds $P_1, \ldots, P_j$ and a connected manifold N, a (j–component) *link map* is a continuous function

$$f\colon P_1 \sqcup \cdots \sqcup P_j \to N$$

such that $f(P_i) \cap f(P_k) = \varnothing$ for $i \neq k$. The space of link maps will be denoted by $\mathscr{L}(\boldsymbol{P}, N)$.[1] Fix an embedding $J \to N$, where we recall again that $J = \{1, 2, \ldots, j\}$. We will also identify J with its image in N.

We define the *trivial link map* to be the link map given by sending the component P_i to $i \in J$, ie the trivial link map factors as the composition $P_1 \sqcup \cdots \sqcup P_j \to J \subset N$, where the first map is the canonical surjection from a space onto its set of components. The trivial link map equips $\mathscr{L}(\boldsymbol{P}, N)$ with a basepoint. A link map is *trivializable* if it admits a path to the trivial link map in the space of link maps.

Definition 1.9 The space of *(homotopy coherent) Brunnian link maps*

$$\mathscr{B}(\boldsymbol{P}, N)$$

is the total homotopy fiber of the j–cube of based spaces

$$S \mapsto \mathscr{L}^S(\boldsymbol{P}, N),$$

where $\mathscr{L}^S(\boldsymbol{P}, N)$ is the space of maps $f\colon P_1 \sqcup \cdots \sqcup P_j \to N$ such that for every $S \subset J$ the restriction

$$f_S := f|_{P_S}\colon \bigsqcup_{i \in S} P_i \to N$$

is an $|S|$–component link map.

---

[1] The path components of $\mathscr{L}(\boldsymbol{P}, N)$ are called *link homotopy classes*. The latter is usually studied in the special case when $N = \mathbb{R}^n$ and the P_i are spheres; see Milnor [21], Massey [19] and Koschorke [18].

Since $\mathscr{B}(\boldsymbol{P},N)$ is the homotopy fiber of the map

$$\mathscr{L}^J(\boldsymbol{P},N) \to \operatorname*{holim}_{S\subsetneq J} \mathscr{L}^J(\boldsymbol{P},N),$$

a point of $\mathscr{B}(\boldsymbol{P},N)$ determines a link map $f \in \mathscr{L}^J(\boldsymbol{P},N)$ with the property that any proper sublink map is trivializable. In particular, f satisfies the classical Brunnian condition; see Milnor [21] and Debrunner [3].

Restricting now to the case when $N=\mathbb{R}^n$, we will construct in Section 8 a *higher stable linking number map*[2]

$$\lambda\colon \mathscr{B}(\boldsymbol{P},\mathbb{R}^n) \to \prod_{i=1}^{(j-2)!} F^{\mathrm{st}}\Big(\prod_{i=1}^{j} P_i, S^{(j-1)(n-2)+1}\Big), \tag{3}$$

where for an unbased space X and a spectrum E, $F^{\mathrm{st}}(X,E)$ denotes the function space of stable maps from X to E, ie the function space $F(X,\Omega^\infty E)$.

A result of Goodwillie and Munson in the case $j=2$ [7, Theorem 1.1] suggests to us the following:

Conjecture G *The map λ is $(1+\Sigma')$–connected, where*

$$\Sigma' = \sum_{i=1}^{j}(n-2p_i-2).$$

(For variant forms of this statement see Section 8.) We submit the following evidence for Conjecture G:

Theorem H (realization of higher linking numbers) *Assume that P_i embeds in $\mathbb{R}^n$ and $n-p_i\ge 2$ for $2\le i\le j$. Then the higher stable linking number map λ induces a surjection on homotopy groups in degrees $\le 1-\hat{p}+\Sigma$, where*

$$\hat{p} := \max_{2\le i\le j} p_i \quad \textit{and} \quad \Sigma = \sum_{i=1}^{j}(n-p_i-2).$$

In the above, we do not need to assume that the embeddings are pairwise disjoint. Since $1-\hat{p}+\Sigma \ge 1+\Sigma'$, it follows that λ induces a surjection on homotopy groups in degrees $\le 1+\Sigma'$. Hence, Theorem H gives evidence for the validity of Conjecture G.

---

[2] For link maps of circles in three-dimensional euclidean space, it seems likely that on path components, our map coincides with Milnor's μ–invariants [21].

Further evidence is contained in Section 8. Our results on link maps overlap with those of Munson [22]. Our methods are homotopy-theoretical, whereas Munson relies on bordism and transversality. It seems likely to us that Theorem H could also be extracted from Munson's approach, possibly at the expense of a dimension.

Outline Section 2 is a breezy exposition on the basic definitions as well as the machinery used throughout the paper. Section 3 is about strongly cocartesian cubes of spaces, and the main technical results of the paper are stated there. Section 4 recasts the results of Section 3 in the setting of homotopical intersection theory to give a proof of Theorems A and C modulo the proof of Theorem 3.12. In Section 5 we prove Theorem 3.12, which is one of our main technical results. In Section 6 we combine Theorem A with [6, Theorem E] to obtain a multiple disjunction result for smooth embeddings. Section 7 contains the proof of Theorem E. In Section 8 we apply our machinery to the study of spaces of link maps.

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2 Language

2.1 Spaces

Let $\mathscr{T}$ be the category of compactly generated spaces. Then $\mathscr{T}$ is a Quillen model category in which the weak equivalences are the weak homotopy equivalences, the fibrations are the Serre fibrations and the cofibrations are the retracts of relative cell complexes [25, Chapter 2, Section 3] (a relative cell complex is a pair of spaces (Y, A) such that Y is obtained from A by attaching cells). A space X is *r–connected* if every map $S^k \to X$ for $k \le r$ is homotopic to a constant map; here S^k is the sphere of dimension k. In particular, the empty space is (-2)–connected and every nonempty space is (at least) (-1)–connected. A map $f\colon X \to Y$ is r–connected if its homotopy fiber at any basepoint is $(r-1)$–connected. An ∞–connected map is, by definition, a weak equivalence.

A commutative square of spaces

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & D \end{tikzcd} \tag{4}$$

is *r–cocartesian* if the map

$$\operatorname{hocolim}(B \leftarrow A \to C) \to D$$

is r–connected.

Dually, the square (4) is *r–cartesian* if the map

$$A \to \operatorname{hocolim}(B \to D \leftarrow C)$$

is r–connected.

Definition 2.1 Let

$$X \to Y \to Z \tag{5}$$

be maps of spaces equipped with a homotopy to a constant z. One says that (5) is a *homotopy fiber sequence in degrees $\le s$* if the induced map from X to the homotopy fiber of $Y \to Z$ is s–connected. If this condition holds for all integers s, then (5) is called a *homotopy fiber sequence*.

Dually, if the induced map from the homotopy cofiber of $X \to Y$ to Z is s–connected, then one says that (5) is a *homotopy cofiber sequence in degrees $\le s$* and a *homotopy cofiber sequence* if the condition holds for all s.

When the square (4) is ∞–cocartesian and C is contractible, $A \to B \to D$ is a homotopy cofiber sequence once a contraction $C \times [0,1] \to C$ is specified. The dual case is analogous.

2.2 Fiberwise spaces

For an object $X \in \mathscr{T}$, we let $\mathscr{T}(X)$ denote the category of spaces over X. This is the category whose objects are pairs (Y, r) such that $r\colon Y \to X$ is a map. A morphism $(Y, r) \to (Y', r')$ is a map $f\colon Y \to Y'$ such that $r' \circ f = r$. We more often than not suppress the structure map $r\colon Y \to X$ when specifying an object and write Y in place of (Y, r).

Similarly, let $\mathscr{R}(X)$ denote the category of retractive spaces over X. This has objects (Y, r, s) where $r\colon Y \to X$ and $s\colon X \to Y$ are maps such that $r \circ s$ is the identity map. A morphism $(Y, r, s) \to (Y', r', s')$ is a map $f\colon Y \to Y'$ such that $r' \circ f = r$ and $f \circ s = s'$. Again, the structure maps are usually supressed.

Note that the case $\mathscr{R}(*)$ gives the category of based spaces. We sometimes regard objects of $\mathscr{R}(X)$ as objects of $\mathscr{T}(X)$ by means of the forgetful functor. When $X = *$ we usually write $\mathscr{T}_*$ in place of $\mathscr{R}(*)$, ie the category of based spaces.

Both $\mathscr{T}(X)$ and $\mathscr{R}(X)$ have simplicial model category structures where a weak equivalence (cofibration, fibration) in each case is a morphism whose underlying map of spaces is a weak homotopy equivalence (cofibration, fibration) of spaces [25, Chapter II, page 2.8, Proposition 6]. In particular, the set of (fiberwise) homotopy classes $[Y, Z]_{\mathscr{T}(X)}$ is defined for objects Y and Z of $\mathscr{T}(X)$. Similarly, one can define homotopy classes in $\mathscr{R}(X)$. If $Y \in \mathscr{T}(X)$ is an object, let $Y^+ \in \mathscr{R}(X)$ be the object given by $Y \sqcup X$ with evident structure maps. If $Z \in \mathscr{R}(X)$ is an object, then we have $[Y^+, Z]_{\mathscr{R}(X)} = [Y, Z]_{\mathscr{T}(X)}$. As usual, when defining homotopy classes $[Y, Z]_{\mathscr{T}(X)}$, Y is replaced by a cofibrant approximation and Z is replaced by a fibrant approximation.

A morphism $Y \to Z$ in either $\mathscr{T}(X)$ or $\mathscr{R}(X)$ is said to be j*–connected* if and only if its underlying map in $\mathscr{T}$ is j–connected. An object Y is said to be j–connected if and only if the structure map $Y \to X$ is $(j+1)$–connected. A commutative square in $\mathscr{T}(X)$ or $\mathscr{R}(X)$ is j*–cocartesian* (j–cartesian) if it is so when considered in $\mathscr{T}$ (here j may be ∞).

We say an object Y of $\mathscr{T}(X)$ or $\mathscr{R}(X)$ has dimension $\le s$ if it is built up from the initial object by attaching cells of dimension at most s. In $\mathscr{T}(X)$ this means that the underlying space of Y is a cell complex of dimension at most s. In $\mathscr{R}(X)$ it means that the pair (Y, X) is a relative cell complex of dimension at most s. In either case we write $\dim Y \le s$.

A sequence of maps $A \to Y \to C$ in $\mathscr{T}(X)$ forms a *homotopy cofiber sequence* (respectively in degrees $\le r$) if it comes equipped with a homotopy from $A \to C$ to a composition of the form $A \to X \to C$ (where X is viewed as the terminal object) such that the induced map from the homotopy cofiber of $A \to Y$ (ie the homotopy colimit of $X \leftarrow A \to Y$) to C is a weak equivalence (respectively r–connected). The dual notion of homotopy fiber sequence (in degrees $\le r$) is defined analogously.

Lemma 2.2 *Suppose that $A \to Y \to C$ is a homotopy cofiber sequence of $\mathscr{T}(X)$. Assume that A is r_1–connected and C is r_2–connected. Then $A \to Y \to C$ is a homotopy fiber sequence in dimensions $\le r_1 + r_2$.*

Proof The square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & C \end{tikzcd}$$

has a preferred commuting homotopy making it ∞–cocartesian. The result follows from the Blakers–Massey theorem [11, Theorem 4.23; 4, page 309]. □

Corollary 2.3 *Assume in addition that $Z \in \mathscr{T}(X)$ is an object of dimension $\le r_1 + r_2$. Then the sequence of sets*

$$[Z, A]_{\mathscr{T}(X)} \to [Z, Y]_{\mathscr{T}(X)} \to [Z, C]_{\mathscr{T}(X)}$$

is exact.

(Explanation: The set $[Z, C]_{\mathscr{T}(X)}$ has a preferred basepoint given by $Z \to X' \to C$. Any element of $[Z, Y]_{\mathscr{T}(X)}$ which maps to the basepoint lifts back to $[Z, A]_{\mathscr{T}(X)}$.)

2.3 Fiberwise suspension

The *unreduced fiberwise suspension* of an object $Y \in \mathscr{T}(X)$ is the object of $\mathscr{R}(X)$ given by the double mapping cylinder

$$S_X Y := (X \times 0) \cup (Y \times [0, 1]) \cup (X \times 1),$$

where the structure map $S_X Y \to X$ is obvious and the structure map $X \to S_X Y$ is given by $X \times 0$. This gives a functor $S_X \colon \mathscr{T}(X) \to \mathscr{R}(X)$. Similarly, $\mathscr{R}(X)$ has a *reduced fiberwise suspension* functor $\Sigma_X \colon \mathscr{R}(X) \to \mathscr{R}(X)$ defined as follows: given an object $Y \in \mathscr{R}(X)$, we take $\Sigma_X Y$ to be the pushout of the diagram $X \leftarrow S_X X \to S_X Y$. If Y is cofibrant, then the map $S_X Y \to \Sigma_X Y$ is a weak equivalence. The functor Σ_X has a right adjoint Ω_X, called the *fiberwise loop functor*.

Given objects $Y, Z \in \mathscr{R}(X)$, define

$$\{Y, Z\}_{\mathscr{R}(X)} := \operatorname*{colim}_k [\Sigma_X^k Y, \Sigma_X^k Z].$$

This is the abelian group of fiberwise stable homotopy classes from Y to Z.

2.4 Fiberwise smash product

Given objects $Y, Z \in \mathscr{T}(X)$, we have the fiber product $Y \times_X Z \in \mathscr{T}(X)$, which is defined as the limit of the diagram $Y \to X \leftarrow Z$. If $Y, Z \in \mathscr{R}(X)$, the fiberwise wedge (or coproduct) $Y \vee_X Z$ is the object of $\mathscr{R}(X)$ given by the pushout of the inclusions $Y \supset X \subset Z$. The *(internal fiberwise) smash product* is the object $Y \wedge_X Z$ given by the pushout of the diagram $X \leftarrow Y \vee_X Z \subset Y \times_X Z$. As is usual with most functors in the model category-theoretic setting, this construction needs to be suitably derived to have a meaningful homotopy type (in this instance Y and Z should be made fibrant and cofibrant). To avoid notational clutter, we will be intentionally sloppy: we will write the underived smash product but the reader should understand that it needs to be derived to have a sensible homotopy-theoretic meaning.

2.5 Fiberwise Thom spaces

Given an object $Y \in \mathscr{T}(X)$ and an inner product bundle ξ over Y, the *fiberwise Thom space* is the object of $\mathscr{R}(X)$ given by

$$T_X(\xi) = D(\xi) \cup_{S(\xi)} X.$$

By collapsing X to a point we obtain the usual Thom space $X^{\xi} := D(\xi)/S(\xi)$, which in the present notation appears as $T_*(\xi)$.

Let η be an inner product bundle over another object $Z \in \mathscr{T}(X)$. Let $p\colon Y \times_X Z \to Y$ and $q\colon Y \times_X Z \to Z$ be the projections. Then the Whitney sum $p^*\xi \oplus q^*\eta$ is an inner product bundle over $Y \times_X Z$. The following is just an unraveling of definitions (and is well known when X is a point):

Lemma 2.4 *There is a preferred isomorphism of $\mathscr{R}(X)$,*

$$T_X(p^*\xi \oplus q^*\eta) \cong T_X(\xi) \wedge_X T_X(\eta).$$

2.6 Fiberwise spectra

Using Σ_X also enables one to define spectra built from objects of $\mathscr{R}(X)$. A *fiberwise spectrum* $\mathscr{E}$ is a collection of objects $\mathscr{E}_n \in \mathscr{R}(X)$ for $n = 0, 1, \dots$ together with morphisms $\Sigma_X \mathscr{E}_n \to \mathscr{E}_{n+1}$. Note that $\mathscr{E}$ comes equipped with a *zero section*, namely the collection of structure maps $X \to \mathscr{E}_n$ for $n \geq 0$. A morphism of fiberwise spectra is the evident thing.

If $\mathscr{E}$ is a fiberwise spectrum then the associated fiberwise infinite loop space $\Omega_X^\infty \mathscr{E}$ is an object of $\mathscr{R}(X)$. Fiberwise spectra form a model category (see eg [27]; for a more detailed treatment see [20]).

Here are two examples:

Example 2.5 (trivial fiberwise spectra) Start with an ordinary spectrum E given by based spaces $\{E_n\}_{n \geq 0}$ and structure maps $\Sigma E_n \to E_{n+1}$. Form $E_n \times X$ for $n \geq 0$. These fit into a fiberwise spectrum $E \times X$, where the structure map $\Sigma_X(E_n \times X) \to E_{n+1} \times X$ is given by noticing that $\Sigma_X(E_n \times X) \cong (\Sigma E_n) \times X$.

Example 2.6 (fiberwise suspension spectra) Start with any object $Y \in \mathscr{R}(X)$ and form the iterates $\Sigma_X^n Y$. These give a fiberwise spectrum $\Sigma_X^\infty Y$, using the identity maps for the structure maps.

We remark that the zero section of $\mathscr{E}$ gives a morphism $\Sigma_X^\infty X^+ \to \mathscr{E}$.

Given an object $Z \in \mathscr{R}(X)$ and a fiberwise spectrum $\mathscr{E}$, we define

$$\{Z, \mathscr{E}\}_{\mathscr{R}(X)} := \operatorname*{colim}_n [Z, \Omega_X^\infty \mathscr{E}]_{\mathscr{R}(X)}.$$

For example, if $\mathscr{E} = \Sigma_X^\infty Y$ is a fiberwise suspension spectrum, then $\{Z, \mathscr{E}\}_{\mathscr{R}(X)} = \{Z, Y\}_{\mathscr{R}(X)}$.

2.7 Homology and cohomology

Let $\mathscr{E}$ be a fiberwise spectrum over X (which we take to be fibrant). Then an object $Z \in \mathscr{T}(X)$ (which we take to be cofibrant) with structure map $p\colon Z \to X$ gives rise to a fiberwise spectrum over Z,

$$p^* \mathscr{E},$$

whose k^{th} space is the pullback of $\mathscr{E}_k \to X$ along p. Let $(p^*\mathscr{E})^\flat$ denote the effect of making $p^*\mathscr{E}$ cofibrant. Then for each $n \geq 0$ we have a cofibration $Z \to (p^*\mathscr{E})_n^\flat$ and as n varies the quotient spaces $(p^*\mathscr{E})_n^\flat / Z$ form a spectrum, denoted by $H_\bullet(Z; \mathscr{E})$. The *homology groups* of Z with coefficients in $\mathscr{E}$ are the homotopy groups of this spectrum.

To define cohomology we take, for each n, the space of sections of $\mathscr{E}_n \to X$ along the map $Z \to X$ (this is the same thing as the space of maps $Z \to \mathscr{E}_n$ which commute with the structure map to X). As n varies, these spaces form a spectrum $H^\bullet(Z;\mathscr{E})$. The *cohomology groups* of Z with coefficients in $\mathscr{E}$ are defined to be homotopy groups of this spectrum, ie

$$H^i(Z;\mathscr{E}) = \{Z^+, \Sigma^i_X\mathscr{E}\}_{\mathscr{R}(X)}.$$

2.8 Induction and restriction

Let $f\colon X \to Y$ be a map of spaces. Then a fiberwise spectrum $\mathscr{E}$ over Y gives rise to a fiberwise spectrum $f^*\mathscr{E}$ over X by taking base change. This operation defines a *restriction* functor from fiberwise spectra over Y to fiberwise spectra over X (the construction is homotopy-invariant when $\mathscr{E}$ is fibrant). Using f to regard X as an object of $\mathscr{T}(Y)$, we obtain a tautological identification $H^\bullet(X;\mathscr{E}) = H^\bullet(X, f^*\mathscr{E})$, where on the right side X is viewed as an object of $\mathscr{R}(X)$ using the identity.

Suppose $\mathscr{F}$ is a fiberwise spectrum over X. Then we obtain a fiberwise *pushforward* spectrum over Y, denoted $f_*\mathscr{F}$ in which $(f_*\mathscr{F})_k = (\mathscr{F}_k) \cup_f Y$ (the construction is homotopy-invariant when $\mathscr{F}$ is cofibrant). The operation $E \mapsto f_*E$ is also called *induction.* Note that $H_\bullet(X;\mathscr{F}) = H_\bullet(Y; f_*\mathscr{F})$ tautologically. Note also that (f_*, f^*) is an adjoint pair.

2.9 Poincaré duality

Let ξ be a finite-dimensional vector bundle over X. Let S^ξ denote the fiberwise one-point compactification of ξ. Then S^ξ is an object of $\mathscr{R}(X)$. More generally, if ξ is a virtual bundle, ie $\xi + \epsilon^j$ is identified with a finite-dimensional vector bundle η for some j, then we define S^ξ is this case to be a fiberwise spectrum over X given by the j–fold desuspension of S^η.

Given a fiberwise spectrum $\mathscr{E}$ over X, set

$${}^\xi\mathscr{E} := S^\xi \wedge_X \mathscr{E}.$$

When ξ is a vector bundle, the definition of the right side is given by the fiberwise smash products in each degree, ie $S^\xi \wedge_X \mathscr{E}_k$. In the virtual bundle case one merely fiberwise desuspends $S^\eta \wedge_X \mathscr{E}$ j times.

Theorem 2.7 (Poincaré duality [14; 15, Theorem 6.2; 20, Theorem 19.6.1]) *Suppose $f\colon P \to X$ is a map in which P is a closed smooth manifold of dimension d. Let*

$-\tau_P$ be the virtual stable normal bundle given by the negation of the tangent bundle of P. Then, for any fiberwise spectrum $\mathscr{E}$ over X, there is a preferred weak equivalence of spectra

$$H^\bullet(P;\mathscr{E}) \simeq H_\bullet(P;{}^{-\tau_P}f^*\mathscr{E}).$$

Remark 2.8 More generally, if P is an open manifold then there is a weak equivalence

$$H^\bullet_{\mathrm{cs}}(P;\mathscr{E}) \simeq H_\bullet(P;{}^{-\tau_P}f^*\mathscr{E}),$$

where the left side denotes cohomology with compact supports. The latter is defined by taking the spectrum of sections of $\mathscr{E}$ which coincide with the zero section near infinity.

3 Strongly cocartesian cubes

3.1 Cubical diagrams

For a finite set J, we let 2^J be the poset of consisting of the subsets of J partially ordered by inclusion. A *J–cube* in a category $\mathscr{C}$ is a contravariant functor

$$A_\bullet\colon 2^J \to \mathscr{C}, \quad S \mapsto A_S.$$

(If J has cardinality j, we also say that $A_\bullet$ is a *j–cube*.) Since $A_\bullet$ is contravariant, the initial vertex is A_J and the terminal vertex is $A_\varnothing$. When $J = \{i\}$ we usually write $A_S = A_i$.

In what follows we will only consider J–cubes in which the target category $\mathscr{C}$ is either $\mathscr{T}(X)$ or $\mathscr{R}(X)$ for some space X, and, often enough, we shall be interested in the case when X is a point.

A *weak equivalence* of T–cubes $A_\bullet \to B_\bullet$ is a natural transformation such that $A_S \to B_S$ is a weak equivalence for each S, ie an objectwise weak equivalence. Two J–cubes are said to be *weakly equivalent* if there is a finite zigzag of weak equivalences connecting them.

Definition 3.1 [4, Definition 1.3] A J–cube $A_\bullet$ is *r–cartesian* if the map

$$A_J \to \operatorname*{holim}_{S \subsetneq J} A_S \tag{6}$$

is r–connected. Similarly, $A_\bullet$ is *r–cocartesian* if the map

$$\operatorname*{hocolim}_{S \neq \varnothing} A_S \to A_\varnothing \tag{7}$$

is r–connected. In both cases r may be ∞.

We remark that when $A_\bullet$ is a cube in which the maps $A_S \to A_T$ are based for $|S| < j$, the target of (6) inherits a basepoint. In this case, we will say that $A_\bullet$ is *almost based*.

Definition 3.2 The *total homotopy cofiber* of $A_\bullet$ is the homotopy cofiber of the map (7). If $A_\bullet$ is an almost based J–cube, then its *total homotopy fiber* is the homotopy fiber of (7) taken at the preferred basepoint.

For fixed subsets $U \subset W \subset J$, one has a *(W, U)–face* of $A_\bullet$ given by restricting $A_\bullet$ to those A_S for which $U \subset S \subset W$. This is a $(W \setminus U)$–cube and every face of $A_\bullet$ arises in this fashion. When $|W \setminus U| = k$ we also call this a *k–face* of $A_\bullet$.

Definition 3.3 [4, Definition 2.1] A J–cube $A_\bullet$ is *strongly cocartesian* if each 2–face of $A_\bullet$ is ∞–cocartesian.

In Definition 3.3, it is enough to check the condition on each 2–face meeting the initial vertex A_J (ie those (W, U)–faces in which $|W \setminus U| = 2$ and $W = J$; see loc. cit.).

Henceforth, we set

$$J := \{1, 2, \dots, j\}.$$

Example 3.4 (wedge cubes) Let $X_1, \dots, X_j$ be cofibrant based spaces. For $T \subset J$, let A_T be the wedge $\bigvee_{i \in T} X_i$ (by convention $A_\varnothing$ is a point). This defines a strongly cocartesian j–cube $A_\bullet$ whose maps are given by projections onto summands.

More generally, let $X_1, \dots, X_j \in \mathscr{R}(X)$ be cofibrant. Let A_T be the fiberwise wedge of X_i as i varies in T. Then $A_\bullet$ is strongly cocartesian.

Example 3.5 (backwards wedge cubes) With $X_1, \dots, X_j \in \mathscr{R}(X)$ as above, let B_T be the fiberwise wedge of those X_i with $i \in J \setminus T$. The maps of this cube are inclusions of summands. Then $B_\bullet$ is strongly cocartesian.

Example 3.6 (suspension) Let $A_\bullet$ be a strongly cocartesian j–cube of $\mathscr{T}(X)$. Then the j–cube $S_X A_\bullet$ given by $T \mapsto S_X A_T$ is also strongly cocartesian. Similarly, if $A_\bullet$ is a strongly cocartesian j–cube of $\mathscr{R}(X)$, then the cube of reduced fiberwise suspensions $\Sigma_X A_\bullet$ is strongly cocartesian.

Lemma 3.7 *Let $A_\bullet$ be a strongly cocartesian j–cube of connected based spaces in which $A_\varnothing$ is a point. Then the suspended j–cube $\Sigma A_\bullet$ is weakly equivalent to a wedge cube $B_\bullet$ in which $B_i = \Sigma A_i$ for $i \in J$.*

Proof The following sketch was provided to us by Tom Goodwillie. Let B_T be the wedge of ΣA_i for all $i \in T$, but write this as the wedge, over all $i \in J$, of either

- ΣA_i if $i \in T$, or
- $*$ if $i \notin T$.

Define a map $\Sigma A_T \to B_T$ as follows. First do a pinch to go from ΣA_T to the wedge of j copies of ΣA_T indexed by $i \in J$. Now map that to B_T by sending the i^{th} copy of ΣA_T to ΣA_i using the original map $A_T \to A_i$ if $i \in T$, or the constant map to a point if $i \notin T$.

The above recipe defines a map of j–cubes $\Sigma A_\bullet \to B_\bullet$. By the Whitehead theorem, it suffices to show that the map $\Sigma A_T \to B_T$ is a homology isomorphism for all $T \subset J$. Let C_T be the homotopy cofiber of this map. Then $T \mapsto C_T$ is also a strongly cocartesian j–cube. It is enough to show that C_T has trivial reduced homology. If T is a singleton, this is clear since the maps $\Sigma A_i \to B_i$ are homotopic to the identity. By a straightforward induction argument, we can assume that C_T has trivial homology for $|T| \le j-1$. We are reduced to showing that C_J has trivial homology. But the homology of C_J coincides with the homology of the total homotopy cofiber of the cube $C_\bullet$ with a degree shift by j. Since $C_\bullet$ is strongly cocartesian, the total homotopy cofiber is contractible. Hence, C_J has trivial homology. □

Given a strongly cocartesian j–cube $A_\bullet$, let $C(A_\bullet)$ denote the homotopy colimit

$$\operatorname{hocolim}\big(A_\varnothing \leftarrow A_J \to \operatorname*{holim}_{S \neq J} A_S\big). \tag{8}$$

Then $C(A_\bullet)$ is a retractive space over $A_\varnothing$. In what follows we rename

$$X := A_\varnothing.$$

Then $C(A_\bullet) \in \mathscr{R}(X)$ and one has a homotopy cofiber sequence of $\mathscr{T}(X)$

$$A_J \to \operatorname*{holim}_{S \neq J} A_S \to C(A_\bullet). \tag{9}$$

Notation 3.8 For a sequence of integers $r_1, \dots, r_j$ we write

$$\Sigma = \sum_i r_i \quad \text{and} \quad \mu = \min_i r_i.$$

If $1 \le i \le j$ and $T \subset J$, set

$$T_i := T \setminus \{i\}.$$

Hypothesis 3.9 X is 0–connected. Furthermore, for $1 \le i \le j$, the map

$$A_J \to A_{J_i}$$

is (r_i+1)–connected, where $r_i \ge 0$.

Note that $A_T \to A_{T_i}$ is also (r_i+1)–connected for all $T \subset S$, since $A_\bullet$ is strongly cocartesian. We assume Hypothesis 3.9 holds throughout the rest of this section.

Proposition 3.10 *Let $Z \in \mathscr{T}(X)$ be an object of dimension $\le 1+\mu+\Sigma$. Then the sequence*

$$[Z, A_J]_{\mathscr{T}(X)} \to \big[Z, \operatorname*{holim}_{S\neq\varnothing} A_S\big]_{\mathscr{T}(X)} \to [Z, C(A_\bullet)]_{\mathscr{T}(X)}$$

is exact.

Remark 3.11 The set $[Z, C(A_\bullet)]_{\mathscr{T}(X)}$ is pointed. As in Corollary 2.3, exactness means that an element of $[Z, \operatorname{holim}_{S\neq\varnothing} A_S]_{\mathscr{T}(X)}$ pushes forward to the basepoint if and only if it lifts to an element of $[Z, A_J]_{\mathscr{T}(X)}$.

Proof The object $A_J \in \mathscr{T}(X)$ is μ–connected. The higher Blakers–Massey theorem for cubical diagrams [4, Theorem 2.5] (or see [5, Theorem 2.3]) says that $A_\bullet$ is $(1+\Sigma)$–cartesian, Consequently, $C(A_\bullet) \in \mathscr{T}(X)$ is a $(1+\Sigma)$–connected object. The conclusion now follows from Corollary 2.3. □

3.2 Identification of $C(A_\bullet)$

In the remainder of this section we identify $C(A_\bullet)$ up through dimension $1+\mu+\Sigma$.

Let

$$W_j := \bigvee\nolimits_{(j-1)!} S^{2-2j} \tag{10}$$

be the wedge of $(j-1)!$ copies of the $(2-2j)$–sphere spectrum.

Let

$$\mathscr{W}_j = X \times W_j$$

be the trivial fiberwise spectrum on W_j.

Theorem 3.12 *With respect to the above assumptions, there is a preferred map*

$$C(A_\bullet) \to \Omega^\infty\Big(\mathscr{W}_j \wedge_X \bigwedge\nolimits_{X\, i\in J} S_X A_i\Big), \tag{11}$$

which is $(2+\mu+\Sigma)$–connected.

The proof of Theorem 3.12 is deferred to Section 5. If we combine Theorem 3.12 with Proposition 3.10, we obtain:

Corollary 3.13 *Let* $Z \in \mathscr{T}(X)$ *be an object such that* $\dim Z \le 1+\mu+\Sigma$. *Then there is an exact sequence*

$$[Z, A_J]_{\mathscr{T}(X)} \to \Big[Z, \operatorname*{holim}_{S\neq\varnothing} A_S\Big]_{\mathscr{T}(X)} \to \Big\{Z^+, \mathscr{W}_j \wedge_X \bigwedge_{i\in J}{}_X\; S_X A_i\Big\}_{\mathscr{R}(X)}.$$

Remark 3.14 Corollary 3.13 is a robust generalization of a result of Barratt and Whitehead [1] and, independently, Toda [29].

3.3 The Euler class

Let $f\colon Z \to \operatorname{holim}_{S\neq J} A_S$ be a map of spaces. Then f is also a morphism of $\mathscr{T}(X)$. Using Theorem 3.12, we see that the composed map

$$Z^+ \xrightarrow{f} \operatorname*{holim}_{S\neq J} A_S \to C(A_\bullet)$$

gives rise to a fiberwise stable homotopy class

$$e(f) \in \Big\{Z^+, \mathscr{W}_j \wedge_X \bigwedge_{i\in J}{}_X\; S_X A_i\Big\}_{\mathscr{R}(X)},$$

which we call the *Euler class* of f. Equivalently, $e(f)$ resides in the cohomology group

$$H^0\Big(Z; \mathscr{W}_j \wedge_X \bigwedge_{i\in J}{}_X\; S_X A_i\Big).$$

Then, from Corollary 3.13, we deduce:

Corollary 3.15 *The Euler class* $e(f)$ *vanishes when* f *admits a homotopy factorization through* A_J. *Conversely, if* $\dim Z \le 1+\mu+\Sigma$ *and* $e(f)=0$, *then* f *admits a homotopy factorization through* A_J.

3.4 A special case

When $X = A_0$ is a point, the above results can be expanded upon as follows: There is a homotopy cofiber sequence of spaces

(12) $$A_J \to \operatorname*{holim}_{S\neq\varnothing} A_S \to C(A_\bullet)$$

and a $(2{+}\mu{+}\Sigma)$–connected map

(13) $$C(A_\bullet) \to \Omega^\infty\Big(W_j \wedge \bigwedge_{i\in J} SA_i\Big).$$

Furthermore, the space A_J is μ–connected. If we choose a basepoint in A_J then $A_\bullet$ becomes a cube of based spaces. Let $F(A_\bullet)$ be its total homotopy fiber. By the Blakers–Massey theorem applied to (12) and using the map (13), we infer:

Corollary 3.16 *There is a* $(1+\mu+\Sigma)$*–connected map*

$$F(A_\bullet) \to \Omega^\infty\Big(\Sigma^{j-1} W_j \wedge \bigwedge_{i\in J} A_i\Big) \simeq \prod_i^{(j-1)!} Q(\Sigma^{1-j} A_1 \wedge \cdots \wedge A_j).$$

Remark 3.17 The proof we give of Theorem 3.12 implies that the map of Corollary 3.16 is natural with respect to morphisms of based cubes $A_\bullet \to B_\bullet$.

4 Proof of Theorems A and C

In this section we give the proof of Theorems A and C modulo the proof of Theorem 3.12. The proof of the latter result will appear in Section 5.

Returning to the situation of Section 1, we are given pairwise disjoint, connected, closed submanifolds $Q_1, \dots, Q_j \subset N$. Let $N \setminus Q_\bullet$ denote the j–cubical diagram of $\mathscr{R}(N)$ defined by

$$S \mapsto N \setminus Q_S, \quad S \subset J.$$

Note that $N \setminus Q_\bullet$ satisfies Hypothesis 3.9 since $n - q_i \ge 2$.

Proof of Theorem A Recall that we are given a map

$$f\colon P \to \operatorname*{holim}_{S\subsetneq J}(N \setminus Q_S)$$

and we wish to identify the obstructions to deforming it into $N \setminus Q_J$. By transversality, the map $N \setminus Q_J \to N \setminus Q_{J-\{i\}}$ is $(n-q_i-1)$–connected for $1 \le i \le j$. By Corollary 3.15, we infer:

Proposition 4.1 *If* $P \to \operatorname{holim}_{S\subsetneq J} N \setminus Q_S$ *admits a homotopy factorization through* $N \setminus Q_J$*, then* $e(f) = 0$*. The converse is true provided* $p \le 1 + \mu + \Sigma$*, where* $\Sigma = \sum_i (n - q_i - 2)$ *and* $\mu_i = \min_i (n - q_i - 2)$*.*

Proof This follows from Corollary 3.15 since a closed manifold P of dimension p admits the structure of a cell complex of dimension p. □

Let ν_i be the normal bundle of Q_i in N. The tubular neighborhood theorem gives a weak equivalence of $\mathscr{R}(N)$,

$$S_N(N \setminus Q_1) \simeq D(\nu_i) \cup_{S(\nu_i)} N =: T_N(\nu_i),$$

where the right side is the fiberwise Thom space of ν_i over N.

Stably, we can identify ν_i with the virtual bundle $\xi_i := f^*\tau_N - \tau_{Q_i}$, given by the difference of tangent bundles. We write $T_N(\xi_i)$ for the associated fiberwise Thom spectrum. With these notational changes, $e(f)$ can be regarded as residing in the cohomology group

$$H^0\Big(P; \mathscr{W}_j \wedge_N \bigwedge_{i \in J}{}_N \, T_N(\xi_i)\Big). \tag{14}$$

The remainder of the proof of Theorem A will involve application of Poincaré duality (Theorem 2.7) to the cohomology group (14).

4.1 The Euler characteristic

By Poincaré duality (Theorem 2.7), $e(f)$ corresponds to a homology class

$$\chi(f) \in H_0\Big(P; {}^{-\tau_P} f^*\Big(\mathscr{W}_j \wedge_N \bigwedge_{i \in J}{}_N \, T_N(\xi_i)\Big)\Big).$$

Using the induction isomorphism (Section 2.8), the group where $\chi(f)$ resides can alternatively be written as

$$H_0\Big(N; f_*{}^{-\tau_P} f^*\Big(\mathscr{W}_j \wedge_N \bigwedge_{i \in J}{}_N \, T_N(\xi_i)\Big)\Big).$$

By definition, the latter is the stable homotopy group in degree zero of the spectrum

$$\Big(\mathscr{W}_j \wedge_N T_N(-\tau_P) \wedge_N \bigwedge_{i \in J}{}_N \, T_N(\xi_i)\Big)/N.$$

Using Lemma 2.4 in virtual form, we deduce that the fiberwise spectrum

$$\mathscr{W}_j \wedge_N T_N(-\tau_P) \wedge_N \bigwedge_{i \in J}{}_N \, T_N(\xi_i)$$

can be rewritten up to homotopy as

$$\mathscr{W}_j \wedge_N T_N(\xi),$$

where ξ is the virtual bundle over $E(P, Q_\bullet)$ that was defined in Section 1.

Recall that $\mathscr{W}_j$ is just the fiberwise wedge of $(j-1)!$ copies of the fiberwise spectrum $N \times S^{2-2j}$. From this we infer

$$(\mathscr{W}_j \wedge_N T_N(\xi))/N \simeq \textstyle\bigvee_{(j-1)!} \Sigma^{2-2j} E(P, Q_\bullet)^\xi.$$

Since

$$\pi_0(\Sigma^{2-2j} E(P, Q_\bullet)^\xi) \cong \Omega_{2j-2}(E(P, Q_\bullet); \xi),$$

we have deduced that the obstruction $\chi(f)$ resides in the abelian group

$$\bigoplus_{(j-1)!} \Omega_{2j-2}(E(P, Q_\bullet); \xi).$$

By Proposition 4.1, $\chi(f)$ vanishes whenever $f\colon P \to \operatorname{holim}_{S \subsetneq J}(N \setminus Q_S)$ admits a homotopy factorization through $N \setminus Q_J$. Conversely, if $p \le 1+\mu+\Sigma$, then $\chi(f) = 0$; we have shown there is such a factorization of f. □

Proof of Theorem C Given a multirelative intersection problem f, recall that the solution space $\mathscr{L}(f)$ is the space of homotopy factorizations of f of the form

$$P \to N \setminus Q_J \to \operatorname*{holim}_{S \subsetneq J} N \setminus Q_S,$$

where we have suppressed the lifting homotopy. Consider the ∞–cocartesian square of spaces

(15)
$$\begin{tikzcd} N \setminus Q_J \arrow[r] \arrow[d] & \operatorname{holim}_{S \subsetneq J} N \setminus Q_S \arrow[d] \\ N \arrow[r] & C(N \setminus Q_\bullet) \end{tikzcd}$$

whose horizontal maps are $(1+\Sigma)$–connected (by the higher Blakers–Massey theorem applied to the j–cube $N \setminus Q_\bullet$ [4, Theorem 2.5]) and whose vertical maps are $(1+\mu)$–connected. By the Blakers–Massey theorem, the square is $(1+\mu+\Sigma)$–cartesian. Hence, if $\mathscr{F}$ is defined as the homotopy pullback of the diagram given by deleting $N \setminus Q_J$ from the square, then the map $N \setminus Q_J \to \mathscr{F}$ is $(1+\mu+\Sigma)$–connected.

Suppose that the given multirelative intersection problem comes equipped with a preferred solution $\hat{f}\colon P \to N \setminus Q_J$ (where again the lifting homotopy is suppressed). The solution gives a preferred null-homotopy of the composite

(16)
$$P \xrightarrow{f} \operatorname*{holim}_{S \subsetneq T} N \setminus Q_S \to C(N \setminus Q_\bullet)$$

as a morphism of $\mathscr{T}(N)$.

In other words, we have a map

$$\mathscr{L}(f) \to \mathscr{N}(f), \tag{17}$$

where $\mathscr{L}(f)$ is the solution space and $\mathscr{N}(f)$ is the space of null-homotopies of the composite (16). With respect to the preferred basepoint of $\mathscr{L}(f)$, this is a map of based spaces.

Furthermore, $\mathscr{N}(f)$ can be interpreted as the moduli space of homotopy factorizations of f of the form

$$P \to \mathscr{F} \to \operatorname*{holim}_{S \subsetneq T} N \setminus Q_S.$$

Since the map $N \setminus Q_J \to \mathscr{F}$ is $(1{+}\mu{+}\Sigma)$–connected, we infer by elementary obstruction theory that the map $\mathscr{L}(f) \to \mathscr{N}(f)$ is $(1{-}p{+}\mu{+}\Sigma)$–connected. The rest of the proof involves identifying $\mathscr{N}(f)$.

On the one hand, rather than considering null-homotopies in $\mathscr{T}(N)$, we can equivalently add a disjoint copy of N to P to get a null-homotopy in $\mathscr{R}(N)$ of the associated morphism

$$P^+ \to C(N \setminus Q_\bullet). \tag{18}$$

Then $\mathscr{N}(f)$ can be equivalently defined as the mapping space of null-homotopies of (18) in $\mathscr{R}(N)$.

On the other hand, the (derived) mapping space

$$\hom_{\mathscr{R}(N)}(\Sigma_N P^+, C(N \setminus Q_\bullet))$$

acts on the space $\mathscr{N}(f)$ (this is the fiberwise analogue of the classical fact that for a null-homotopic map of spaces $X \to Y$, the moduli space of null-homotopies, ie the space of extensions of the map to the cone on X, is a torsor over the space of maps $\Sigma X \to Y$). The orbit of the basepoint of $\mathscr{N}(f)$ with respect to this action gives a preferred weak equivalence

$$\hom_{\mathscr{R}(N)}(\Sigma_N P^+, C(N \setminus Q_\bullet)) \simeq \mathscr{N}(f).$$

Using the adjunction between Σ_N and Ω_N, we infer that for $k := 1 - p + \mu + \Sigma$ there is a preferred k–connected (weak) map

$$\mathscr{L}(f) \to \hom_{\mathscr{R}(N)}(P^+, \Omega_N C(N \setminus Q_\bullet)). \tag{19}$$

By Theorem 3.12 we also have a $(2+\mu+\Sigma)$–connected map

$$C(N\setminus Q_\bullet)\to \Omega^\infty_N(\mathscr{W}_j\wedge_N T_N(\xi)).$$

Applying to the latter the fiberwise loop functor Ω_N, then applying $\hom_{\mathscr{R}(N)}(P^+,-)$, and composing with (19) we get a $(1-p+\mu+\Sigma)$–connected (weak) map

$$\mathscr{L}(f)\to \hom_{\mathscr{R}(N)}\bigl(P^+,\Omega^{\infty+1}_N(\mathscr{W}_j\wedge_N T_N(\xi))\bigr). \tag{20}$$

By definition, the target of the map (20) is identified with the infinite loop space associated with the cohomology spectrum

$$H^\bullet(P;\Sigma^{-1}_N\mathscr{W}_j\wedge_N T_N(\xi)).$$

By the Poincaré duality argument appearing in the proof of Theorem A above, this spectrum is weakly equivalent to

$$\bigvee_{(j-1)!} E(P,Q_\bullet)^{\xi+(1-2j)\epsilon}.$$

Assembling, we have produced a $(1-p+\mu+\Sigma)$–connected (weak) map

$$\mathscr{L}(f)\to \prod_{(j-1)!}\Omega^\infty(E(P,Q_\bullet)^{\xi+(1-2j)\epsilon}).$$

This completes the proof of Theorem C. □

4.2 The euclidean case

When $N=\mathbb{R}^n$, we have a corollary to Corollary 3.16. Consider an embedding $Q_J\subset\mathbb{R}^n$, where now each Q_i is a manifold admitting a handle decomposition with handles having index at most q_i, where $n-q_i\ge 3$.

Consider the j–cubical diagram $\mathbb{R}^n\setminus Q_\bullet$. Choose a basepoint in $\mathbb{R}^n\setminus Q_J$. Then the j–cube is based and we consider its total homotopy fiber,

$$\Phi(\mathbb{R}^n\setminus Q_\bullet).$$

For $A\subset\mathbb{R}^n$ let $A^*=\mathbb{R}^n\setminus A$ denote its complement.

Corollary 4.2 *There is a $(1+\mu+\Sigma)$–connected map*

$$\Phi(\mathbb{R}^n\setminus Q_\bullet)\to\prod_{i=1}^{(j-1)!}\Omega^\infty(\Sigma^{1-j}Q_1^*\wedge\cdots\wedge Q_j^*), \tag{21}$$

where $\mu=\min_i(n-q_i-2)$ and $\Sigma=\sum_i(n-q_i-2)$.

Remark 4.3 The target of the map (21) may also be identified with the infinite loop space associated with the wedge of $(j-1)!$ copies of the spectrum

$$\Sigma^{1-jn} D_+(Q_1 \times \cdots \times Q_j),$$

where $D_+(X) = F(X_+, S^0)$ is the Spanier–Whitehead dual of X_+.

5 Proof of Theorem 3.12

The proof of Theorem 3.12 relies on basic results arising in the calculus of the identity functor which we now summarize. Let

$$\mathbb{I}\colon \mathscr{T} \to \mathscr{T}$$

be the identity functor. By [5] one has a tower of natural transformations

$$\cdots \to P_2\mathbb{I} \to P_1\mathbb{I} \to P_0\mathbb{I} = *$$

and compatible natural transformations $\mathbb{I} \to P_j\mathbb{I}$. Furthermore, the functor $P_j\mathbb{I}$ is j–excisive in the sense that it transforms strongly cocartesian $(j+1)$–cubes into ∞–cartesian ones. In what follows, we abbreviate notation by setting $P_j := P_j\mathbb{I}$.

If Y is r–connected, then the map $Y \to P_jY$ is $(jr+1)$–connected. In particular, when $r > 0$, the map

$$Y \to \lim_{j\to\infty} P_jY$$

is a weak homotopy equivalence.

If Y is a based space, then the j^{th} layer of the tower, that is, the homotopy fiber of $P_jY \to P_{j-1}Y$, is isomorphic in the homotopy category of functors to the infinite loop space valued functor

$$Y \mapsto \Omega^\infty \mathbb{D}_jY,$$

where $\mathbb{D}_j$ takes values in spectra.

The functor $\mathbb{D}_j$ is classified by a certain spectrum with Σ_j–action, denoted by L_j, whose underlying homotopy type is that of a wedge of $(j-1)!$ copies of the $(1-j)$–sphere spectrum [13; 5, page 706]. Then

$$\mathbb{D}_jY \simeq L_j \wedge_{h\Sigma_j} Y^{[j]}, \tag{22}$$

where $Y^{[j]}$ denotes the j–fold fiberwise smash product Y. This description of $\mathbb{D}_j$ enables one to extend its domain of definition to the category of spectra, ie if A is a spectrum then $\mathbb{D}_jA$ is the spectrum $L_j \wedge_{h\Sigma_j} A^{[j]}$.

Remark 5.1 The maps of the tower $P_j Y \to P_{j-1} Y$ are principal fibrations in the sense that there is a homotopy fiber sequence

$$P_j Y \to P_{j-1} Y \to BD_j Y,$$

where $BD_j Y$ is the delooping of $D_j Y$ given by $\Omega^\infty(\Sigma \mathbb{D}_j Y)$ (see [5, page 653]).

We now consider the strongly cocartesian j–cube $A_\bullet$ of $\mathscr{R}(X)$. Assume for now that X is contractible. Without loss in generality we can replace X by the one-point space. The assignment $S \mapsto P_k A_S$ defines a j–cube, denoted by $P_k A_\bullet$. A choice of basepoint in A_J equips $A_\bullet$ with the structure of a based j–cube. Then $D_k A_\bullet$ is a j–cube of infinite loop spaces. Let

$$\operatorname{fib}(D_k(A_\bullet))$$

denote its total homotopy fiber.

Proposition 5.2 *The total homotopy fiber of $D_k A_\bullet$ is $(\mu+\Sigma)$–connected if $k \geq j+1$. Furthermore, when $k = j$ there is a $(1+\mu+\Sigma)$–connected map*

$$\operatorname{fib}(D_j(A_\bullet)) \to \Omega^\infty(L_j \wedge A_1 \wedge \cdots \wedge A_j).$$

Proof Suppose first that $A_\bullet$ is a wedge cube on the based spaces $X_1, \dots, X_j$. Then X_i is r_i–connected. Using (22), the total homotopy fiber of $D_k(A_\bullet)$ may be identified with the infinite loop space associated with the total homotopy fiber of the j–cube of spectra

$$S \mapsto L_k \wedge_{h\Sigma_k} X_S^{[k]}, \tag{23}$$

where X_S is the wedge of the spaces X_i for $i \in S$. Applying the binomial theorem to expand $X_S^{[k]}$, direct calculation shows that the total homotopy fiber of (23) decomposes into a wedge of terms of the form

$$L_k \wedge_{h\Sigma_{s_\bullet}} (X_1^{[s_1]} \wedge \cdots \wedge X_j^{[s_j]}), \tag{24}$$

where

- $\sum_i s_1 = k$ with $s_i \geq 1$ for all i,
- $\Sigma_{s_\bullet} := \Sigma_{s_1} \times \cdots \times \Sigma_{s_j} \subset \Sigma_k$.

If $k \geq j+1$ then there is always at least one term $s_i \geq 2$. It follows that the displayed spectrum is at least $(\mu+\Sigma)$–connected. Hence, the total homotopy fiber $\operatorname{fib}(D_k(A_\bullet))$ is also $(\mu+\Sigma)$–connected when $k \geq j+1$.

When $k = j$, we can ignore those terms in which $s_i \geq 2$ since they are highly connected: the projection away from those terms produces the $(1+\mu+\Sigma)$–connected map

$$\mathrm{fib}(D_j(A_\bullet)) \to \Omega^\infty(L_j \wedge A_1 \wedge \cdots \wedge A_j).$$

This completes the proof in the case of wedge cubes.

Turning to the general case, we use the fact that $\mathbb{D}_k$ is defined on the category of spectra. By Lemma 3.7, the j–cube of spectra $\Sigma^\infty A_\bullet$ is weakly equivalent to a wedge cube on the spectra $\Sigma^\infty A_1, \dots, \Sigma^\infty A_j$. Replacing the spaces X_i of the previous case by the spectra $\Sigma^\infty A_i$ and making the same kind of calculation, the conclusion follows. □

Corollary 5.3 *Assume that X is contractible and $k \geq j+1$. Then the $(j+1)$–cube*

$$P_k A_\bullet \to P_{k-1} A_\bullet$$

is $(1+\mu+\Sigma)$–cartesian.

Proposition 5.4 *Assume that X is contractible. Then the $(j+1)$–cube*

$$A_\bullet \to P_j A_\bullet$$

is $(1+\mu+\Sigma)$–cartesian.

Proof If $r_i \geq 1$ for all i, the result follows easily from induction, Corollary 5.3 and the convergence of the tower for the identity functor for 1–connected spaces. In the general case one must proceed differently, using the higher Blakers–Massey theorem. We are indebted to the referee for communicating the following argument.

We first recall how $Y \mapsto P_j Y$ is defined in terms of an auxiliary functor $Y \mapsto T_j Y$ as in [5, Section 1]. The latter is given by taking the homotopy limit of the functor

$$U \mapsto Y * U,$$

where $*$ means topological join and U ranges over the poset of nonempty subsets of $\{1, \dots, j+1\}$. There is an evident natural transformation $Y \to T_j Y$ and $P_j Y$ is defined to be the homotopy colimit of the diagram

$$Y \to T_j Y \to T_j^2 Y \to \cdots.$$

For the rest of the proof we set $\underline{k} = \{1, 2, \dots, k\}$ to avoid notational clutter.

We first determine how cartesian the $(j+1)$–cube $A_\bullet \to T_j A_\bullet$ is. This is the same as asking the degree to which the $(2j+1)$–cube

$$(T, U) \mapsto A_T * U$$

is cartesian, where $T \subset \underline{j}$ and $U \subset \underline{j+1}$ (note: by our conventions this functor is contravariant in the first variable and covariant in the second).

For fixed T, the $(j+1)$–cube $U \mapsto A_T * U$ is strongly cocartesian. Similarly, for fixed U, the j–cube $T \mapsto A_T * U$ is strongly cocartesian. Any pair (T, U) corresponds to a subcube whose initial term is $A_T * U$. It follows that this subcube will be ∞–cocartesian whenever $|T| \geq 2$ or $|U| \geq 2$. Consequently, there are three remaining types of pairs (T, U) to consider:

(1) $|T| = 1$ and $|U| = 0$.

(2) $|T| = 0$ and $|U| = 1$.

(3) $|T| = |U| = 1$.

By inspection, one finds for a type (1) pair that the subcube is (r_i+1)–cocartesian. Similarly, for a type (2) pair the subcube is $(\mu+1)$–cocartesian and for a type (3) pair the subcube is (r_i+2)–cocartesian.

Given a partition of $\underline{j} \sqcup \underline{j+1}$ consisting of sets of these types only, the sum of these numbers indexed over the sets of the partition is given by

$$\Sigma + j + D + (j + 1 - D)(\mu + 1), \tag{25}$$

where D is the number of times a set of type (3) occurs in the partition. To see this, note that any such partition is determined by a choice of injections $a\colon \underline{D} \to \underline{j}$ and $b\colon \underline{D} \to \underline{j}$, in which the complement of the image of a defines the type (1) singletons of the partition and the complement of the image of b defines the singletons of type (2). Hence, the sum of the numbers for such a partition is given by

$$\sum_{i \notin a(\underline{D})} (r_i + 1) + \sum_{i \notin b(\underline{D})} (\mu_i + 1) + \sum_{i \in a(\underline{D})} (r_i + 2),$$

which clearly coincides with the expression (25).

Observe that (25) achieves a minimum when D is at its maximal value j. It follows that the minimal value is $1 + \mu + \Sigma + 2j$. Since we are dealing with a $(2j+1)$–cube, we subtract $2j$ to get $1+\mu+\Sigma$, which is how cartesian the cube is by [4, Theorem 2.5]. Hence, the $(j+1)$–cube $A_\bullet \to T_j A_\bullet$ is $(1+\mu+\Sigma)$–cartesian.

The next step is to consider $T_j A_\bullet \to T^2_{j+1} A_\bullet$. For each fixed nonempty $U \subset \underline{j+1}$, the map of j–cubes

$$A_\bullet * U \to T_j(A_\bullet * U)$$

is of the kind we considered above with the number r_i increased by 1 (so Σ is increased by j) and μ increased by 1. Hence, the corresponding $(j+1)$–cube is $(1+(1+\mu)+(\Sigma+j))$–cartesian. Moreover, taking the homotopy limit over U yields the map of j–cubes $T_j A_\bullet \to T^2_j A_\bullet$. In taking this homotopy limit the degree to which the latter is cartesian is decreased by j. We infer that $T_j A_\bullet \to T^2_j A_\bullet$ is $(2+\mu+\Sigma)$–cartesian, which is one better than the estimate we obtained for $A_\bullet \to T_j A_\bullet$. Repeating this argument, we infer that $T^k_j A_\bullet \to T^{k+1}_j A_\bullet$ is $(k+1+\mu+\Sigma)$–cartesian for any $k \ge 0$. It follows that $A_\bullet \to P_j A_\bullet$ is $(1+\mu+\Sigma)$–cartesian. □

Proof of Theorem 3.12 The proof is a verification in two cases.

Case 1 (X is contractible) There is no loss in generality in assuming that X is a point. Equip A_J with a basepoint. Then $A_\bullet$ is a j–cube of 1–connected based spaces. Consider the commutative diagram

$$\begin{tikzcd}
P_j A_J \arrow[r, "a_1"] \arrow[d, "b_1"'] & P_{j-1} A_J \arrow[r, "a_2"] \arrow[d, "b_2"', "\simeq"] & BD_j A_J \arrow[d, "b_3"] \\
\operatorname*{holim}_{S \subsetneq J} P_j A_S \arrow[r, "a_3"'] & \operatorname*{holim}_{S \subsetneq J} P_{j-1} A_S \arrow[r, "a_4"'] & \operatorname*{holim}_{S \subsetneq J} BD_j A_S
\end{tikzcd}$$

in which the top and bottom rows form fibration sequences. The map b_2 is a homotopy equivalence since P_{j-1} is $(j-1)$–excisive. The map b_3 is equivalent to a principal fibration in the following sense: it may be identified with the map of infinite loop spaces arising from the map of spectra

$$\Sigma\mathbb{D}_j(A_\bullet) \to \operatorname*{holim}_{S \subsetneq J} \Sigma\mathbb{D}_j(A_S)$$

associated with the j–cube $\Sigma\mathbb{D}_j(A_\bullet)$.

Set $W_j := \Sigma^{1-j} L_j$. By Proposition 5.2 there is a $(2+\mu+\Sigma)$–connected map of spectra

$$\Sigma\mathrm{fib}(\mathbb{D}_j(\Sigma^\infty A_\bullet)) \to W_j \wedge SA_1 \wedge \cdots \wedge SA_j, \tag{26}$$

where we have implicitly identified $\Sigma(L_j \wedge A_1 \wedge \cdots \wedge A_j) \simeq W_j \wedge SA_1 \wedge \cdots \wedge SA_j$ to avoid displaying the choice of basepoint. The infinite loop space associated with the source of (26) is identified with the homotopy fiber of the map b_3.

Consequently,

$$BD_j A_J \xrightarrow{b_3} \operatorname*{holim}_{S\subsetneq J} BD_j A_S \to \Omega^\infty(\Sigma W_j \wedge SA_1 \wedge \cdots \wedge SA_j)$$

is a homotopy fiber sequence in degrees $\le 2+\mu+\Sigma$.

Hence, by Lemma 5.5 below there is a homotopy fiber sequence in degrees $\le 1+\mu+\Sigma$ of the form

$$P_j A_J \xrightarrow{b_1} \operatorname*{holim}_{S\subsetneq J} P_j A_S \to \Omega^\infty(W_j \wedge SA_1 \wedge \cdots \wedge SA_j). \tag{27}$$

According to Proposition 5.4, the square

$$\begin{tikzcd}
A_J \arrow[r] \arrow[d] & P_j A_J \arrow[d, "b_1"] \\
\operatorname*{holim}_{S\subsetneq J} A_S \arrow[r] & \operatorname*{holim}_{S\subsetneq J} P_j A_S
\end{tikzcd}$$

is $(1+\mu+\Sigma)$–cartesian. Let $\operatorname{holim}_{S\subsetneq J} A_S \to \Omega^\infty(W_j \wedge SA_1 \wedge \cdots \wedge SA_j)$ be the composition of the bottom map of the square with the second map of (27). Then

$$A_J \to \operatorname*{holim}_{S\subsetneq J} A_S \to \Omega^\infty(W_j \wedge SA_1 \wedge \cdots \wedge SA_j) \tag{28}$$

is also a homotopy fiber sequence in degrees $\le 1+\mu+\Sigma$. By the dual Blakers–Massey theorem, we conclude that (28) is also a homotopy cofiber sequence in degrees $\le 2+\mu+\Sigma$.

Consequently, the induced map

$$C(A_\bullet) \to \Omega^\infty(W_j \wedge SA_1 \wedge \cdots \wedge SA_j)$$

is $(2+\mu+\Sigma)$–connected.

Case 2 (X is general) Let $\tilde{X} \to X$ be a universal principal bundle for X with topological structure group G. Then $\tilde{X}$ is contractible. Let $\tilde{A}_\bullet$ be the strongly cocartesian j–cube of G–spaces given by the fiber product

$$\tilde{A}_S := \tilde{X} \times_X A_S.$$

The terminal vertex of this cube is then contractible, and one checks that the argument in Case 1 preserves equivariance. It follows that there is a $(2+\mu+\Sigma)$–connected map of based G–spaces

$$C(\tilde{A}_\bullet) \to \Omega^\infty(W_j \wedge S\tilde{A}_1 \wedge \cdots \wedge S\tilde{A}_j). \tag{29}$$

The result follows by applying the Borel construction $-\times_G \widetilde{X}$ to (29) to obtain a $(2+\mu+\Sigma)$–connected map of $\mathscr{R}(X)$,

$$C(A_\bullet) \to \Omega^\infty_X(\mathscr{W}_j \wedge_X S_X A_1 \wedge \cdots \wedge_X S_X A_j). \qquad \square$$

The section ends with an elementary result about fibrations that was used in the proof of Theorem 3.12. Let

$$\begin{tikzcd}
F_1 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d, "\simeq"] & B_1 \arrow[d] \\
F_2 \arrow[r] & E_2 \arrow[r] & B_2
\end{tikzcd}$$

be a commutative diagram of connected spaces in which the rows are fibration sequences and the map $E_1 \to E_2$ is a homotopy equivalence. Here $B_1 \to B_2$ is a map of based spaces and the fiber over the basepoint of B_i is F_i.

Lemma 5.5 *Assume in addition that the map $B_1 \to B_2$ sits in a homotopy fiber sequence $B_1 \to B_2 \to B_3$ in degrees $\le s$. Then the map $F_1 \to F_2$ sits in a homotopy fiber sequence $F_1 \to F_2 \to \Omega B_3$ in degrees $\le s-1$.*

Proof Equip B_3 with the basepoint from B_2. The composition $E_1 \to E_2 \to B_2 \to B_3$ is null-homotopic. Hence, $E_2 \to B_2 \to B_3$ is also null-homotopic. Let $E_2 \to PB_3$ be adjoint to a null-homotopy, where PB_3 is the based path space. Then the diagram

$$\begin{tikzcd}
E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & PB_3 \arrow[d] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3
\end{tikzcd}$$

commutes. The result follows by taking fibers vertically. $\square$

6 Multiple disjunction

Let P, $Q_1, \dots, Q_j$ and N be as in Section 1. Let

$$E(P, N)$$

denote the space of smooth embeddings from P to N. Then $S \mapsto E(P, N \setminus Q_S)$ forms a j–cube of spaces, denoted by $E(P, N \setminus Q_\bullet)$. The natural transformation from embeddings to functions

$$E(P, N \setminus Q_\bullet) \to F(P, N \setminus Q_\bullet) \tag{30}$$

is a map of j–cubes. One of the main results of [6] is:

Theorem 6.1 [6, Theorem E] *Assume $p, q_i \le n-3$. Then the $(j+1)$–cube (30) is $(n-2p-1+\Sigma)$–cartesian.*

Proof of Theorem D Let

$$f \in \operatorname*{holim}_{S \neq J} \operatorname{emb}(P, N \setminus Q_S)$$

be any point. Then f is represented by a map of $(j+1)$–ads

$$\Delta^{j-1} \to E(P, N),$$

where the i^{th} face of Δ^{j-1} is constrained to map into the subspace $E(P, N \setminus Q_{i+1})$ for $i = 0, 1, \dots, j-1$. Note that by forgetting information, we may also regard f as a map $P \to \operatorname{holim}_{S \neq J} N \setminus Q_S$, and therefore we have an associated multirelative intersection problem. Consequently, Theorem D follows by combining Theorem 6.1 with Theorem A. □

Remark 6.2 Theorem D is a multirelative version of [12, Theorem 2.2].

7 The embedding tower

In [16, Section 13; 15], we described an invariant $\mu(f)$ which was shown to be a complete obstruction to regularly homotoping an immersion $f \colon P \to N$ to an embedding in the metastable range. The goal of this section is to generalize this result beyond the metastable range when N is highly connected.

7.1 Construction of the embedding tower

Let P be a smooth manifold of dimension p without boundary and let N be a smooth manifold of dimension n. We let $E(P, N)$ denote the space of embeddings of P in N, defined as the geometric realization of the simplicial set whose k–simplices are the smooth families of embeddings from P to N that are parametrized by the standard k–simplex.

Assume P is compact. Let $\mathcal{O}_j := \mathcal{O}_j(P)$ be the partially ordered set whose elements are open subsets $U \subset P$ such that U is diffeomorphic to $\mathbb{R}^p \times T$, where T is a set of cardinality at most j. A morphism $U \to V$ is given by an inclusion of subsets. The j^{th} stage of the Goodwillie–Weiss embedding tower is defined by

$$E_j(P, N) := \operatorname*{holim}_{U \in \mathcal{O}_j} E(U, N).$$

The inclusion $\mathscr{O}_{j-1} \to \mathscr{O}_j$ induces a map $E_j(P,N) \to E_{j-1}(P,N)$. The map $E(P,N) \to E_j(P,N)$ is given by restricting embeddings to elements of $\mathscr{O}_j$.

If $p \le n-1$ then $E_1(P,N)$ is homotopy equivalent to $I(P,N)$, the space of immersions from P to N (by a reformulation of Smale–Hirsch theory). Hence, a basepoint of $E_1(P,N)$ amounts to selecting an immersion $P \to N$ up to contractible choice. In what follows, we fix such a basepoint and define

$$\bar{E}_j(P,N) := \mathrm{fiber}(E_j(P,N) \to E_1(P,N)).$$

It follows that the square

$$\begin{tikzcd}
\bar{E}_j(P,N) \arrow[r] \arrow[d] & E_j(P,N) \arrow[d] \\
\bar{E}_{j-1}(P,N) \arrow[r] & E_{j-1}(P,N)
\end{tikzcd} \tag{31}$$

is homotopy cartesian. Furthermore, the tower $\{\bar{E}_j(P,N)\}$ is the manifold calculus tower associated with the functor $U \mapsto \bar{E}(U,N)$, where U varies throughout the open subsets of P. Call this the *reduced embedding tower*. Note that $\bar{E}_1(P,N)$ is the one-point space.

7.2 Configuration spaces

For a set J of cardinality j, set

$$E_J(N) := E(J,N).$$

If we equip J with a total ordering, then $E_J(N)$ is the configuration space of finite ordered subsets of N of cardinality j. A choice of embedding $J \to N$ equips $E_J(N)$ with a basepoint. To each $T \subset U \subset J$ there is a projection map $E_U(N) \to E_T(N)$. These assemble into a j–cube of based spaces $E_\bullet(N)$.

Lemma 7.1 *The j–cube $E_\bullet(N)$ is $((j-1)(n-2)+1)$–cartesian.*

Proof The j–cube $E_\bullet(N)$ can be written as a map of $(j-1)$–cubes

$$E_{S\cup 1}(N) \to E_S(N),$$

where $S \subset J_1 := \{2,\dots,j\}$. The displayed map is a fibration whose fiber at the basepoint is the based space $N \setminus S$. These form a strongly cocartesian $(j-1)$–cube $N_\bullet$, all of whose maps are $(n-1)$–connected. Then $N_\bullet$ is $((j-1)(n-2)+1)$–cartesian by the higher Blakers–Massey theorem. □

7.3 The unstable obstruction

For $j \geq 2$ let $\binom{P}{j}$ denote the configuration space of subsets $S \subset P$ of cardinality j. Over this space we consider two fibrations. The first fibration

$$E \to \binom{P}{j}$$

has fiber over $S \in \binom{P}{j}$ given by the configuration space $E_S(N)$.

The second fibration

$$D \to \binom{P}{j}$$

has fiber over S given by $\operatorname{holim}_{T \subsetneq S} E_T(N)$.

Then one has an evident map of fibrations

$$E \to D. \tag{32}$$

A point $x \in E_{j-1}(P, N)$ determines a section $t = t(x)$ of $D \to \binom{P}{j}$. It also determines a partial section $s = s(x)$ of $E \to \binom{P}{j}$ along an open collar of the boundary of a compactification of $\binom{P}{j}$. The sections agree with respect to the map (32).

The following is essentially just a reformulation of Weiss's description of the layers of the embedding tower.

Lemma 7.2 *Assume $j \geq 2$. The homotopy fiber of $\bar{E}_j(P, N) \to \bar{E}_{j-2}(P, N)$ taken at x is homotopy equivalent to the space of sections of $E \to \binom{P}{j}$ which are compatible with t and which coincide with s near infinity. In particular, x lifts to a point of $E_j(P, N)$ if and only if this section space is nonempty.*

Remark 7.3 Another formulation of the lemma is that the square

$$\begin{tikzcd}
\bar{E}_j(P, N) \arrow[r] \arrow[d] & \Gamma(E) \arrow[d] \\
\bar{E}_{j-1}(P, N) \arrow[r] & \Gamma_\infty(E) \times_{\Gamma_\infty(D)} \Gamma(D)
\end{tikzcd}$$

is ∞–cartesian, where Γ denotes the space of sections and Γ_∞ denotes the space of germs of sections near infinity.

Proof of Lemma 7.2 Given x, define a third fibration

$$F \to \binom{P}{j}$$

whose fiber at S is the total homotopy fiber of the cube $T \mapsto E_T(N)$ for $T \subset S$. Denote this fiber by $\Phi_S(N; x)$. It is an unbased space. Note that $\Phi_S(N; x)$ is well defined since when $T \subsetneq S$, each of the spaces $E(T, N)$ is based using x.

Moreover, x gives a partial section of this fibration at infinity. Weiss shows that the space of compactly supported sections of this fibration (ie the space of sections agreeing with the partial section near infinity) coincides with the homotopy fiber of $\bar{E}_j(P, N) \to \bar{E}_{j-1}(P, N)$ at x. The latter space is homotopy equivalent to the space in the statement of the lemma. □

7.4 A cohomological obstruction

If we suspend the fibers of $D \to \binom{P}{j}$, then the obstruction to finding a compactly supported section lies in a spectrum cohomology group. If certain dimensional restrictions are present, then nothing is lost in suspending.

When X is an unbased space, we define its suspension spectrum be the homotopy fiber of the map of spectra $\Sigma^\infty X_+ \to S^0$ that is induced by the map from X to the one-point space. By slight abuse in notation, denote the homotopy fiber by $\Sigma^\infty X$.

Definition 7.4 Let

$$\mathscr{D} \to \binom{P}{j}$$

be the fiberwise spectrum whose fiber at S given by $\Sigma^\infty \Phi_S(N; x)$. This comes equipped with a section near infinity. Note that $\mathscr{D}$ depends on the choice of x.

The total obstruction $e(x)$ to finding a compactly supported section of $\mathscr{D}$ lies in π_{-1} in the spectrum of compactly supported sections, that is,

$$e(x) \in H^{-1}_{\mathrm{cs}}\left(\binom{P}{j}; \mathscr{D}\right).$$

Lemma 7.5 *If $x \in E_{j-1}(P, N)$ lifts to $E_j(P, N)$, then $e(x)$ vanishes. The converse is true provided that $2(j-1)(n-2) - jp + 1 \geq 0$.*

Proof The "if" part is clear. For the converse, one observes that the map $\Phi_S(N;x) \to \Omega^\infty\Sigma^\infty\Phi_S(N;x)$ is $(2(j-1)(n-2)+1)$–connected using the Freudenthal suspension theorem and fact that $\Phi_S(N;x)$ is $((j-1)(n-2))$–connected by Lemma 7.1. It follows that the map of compactly supported section spaces is $(2(j-1)(n-2)-jp+1)$–connected. □

7.5 Highly connected manifolds

When N is highly connected, the obstruction to lifting simplifies considerably.

Definition 7.6 For $S \subset \binom{P}{j}$ let

$$C_S(N)$$

denote the mapping cone of the map

$$E_S(N) \to \operatorname*{holim}_{T \subsetneq S} E_T(N).$$

Remark 7.7 In contrast with $\Phi_S(N;x)$, the space $C_S(N)$ doesn't depend on x and it has a preferred basepoint.

Lemma 7.8 *Assume $j \geq 2$ and N is r–connected, where $r \leq n-2$. Then the square*

$$\begin{tikzcd}
E_S(N) \arrow[r] \arrow[d] & \operatorname{holim}_{T \subsetneq S} E_T(N) \arrow[d] \\
C \arrow[r] & C_S(N)
\end{tikzcd}$$

is $((j-1)(n-2)+r+1)$–cartesian, where C is the cone on $E_S(N)$.

Proof By definition, the square is ∞–cocartesian. Furthermore, the map $E_S(N) \to \operatorname{holim}_{T \subsetneq S} E_T(N)$ is $((j-1)(n-2)+1)$–connected by Lemma 7.1.

Since N is r–connected and $r \leq n-2$, it follows that $E_S(N)$ is r–connected. Hence, the left vertical map is $(r+1)$–connected. The conclusion now follows from the Blakers–Massey theorem. □

Let

$$\mathscr{C} \to \binom{P}{j} \tag{33}$$

be the fiberwise spectrum whose fiber at S is $\Sigma^\infty C_S(N)$. This fiberwise spectrum doesn't depend on x.

The section t induces another section of (33); call it t'. The latter section is homotopic to the zero section near infinity. Then an obstruction to lifting $x \in E_{j-1}(P,N)$ to $E_j(P,N)$ is given by the associated compactly supported spectrum cohomology class of t':

$$e'(x) \in H^0_{\mathrm{cs}}\Bigl(\binom{P}{j};\mathscr{C}\Bigr).$$

Lemma 7.9 *Assume $j \geq 2$ and N is r–connected with $r \leq n-2$. If x lifts to an element of $E_j(P,N)$, then $e'(x)$ vanishes. Furthermore, the converse holds if $r \geq p-1-(j-1)(n-p-2)$.*

Proof The proof uses the commutative square

$$\begin{tikzcd}
\Sigma\Phi_S(N;x) \arrow[r] \arrow[d] & C_S(N) \arrow[d] \\
\Omega^\infty\Sigma^\infty\Phi_S(N;x) \arrow[r] & \Omega^\infty\Sigma^\infty C_S(N)
\end{tikzcd}$$

Since $C_S(N)$ and $\Sigma\Phi_S(N;x)$ are $((j-1)(n-2)+1)$–connected (by Lemma 7.1), the vertical maps are $(2(j-1)(n-2)+3)$–connected by the Freudenthal suspension theorem.

By Lemma 7.8, the horizontal maps are $((j-1)(n-2)+r+2)$–connected. Hence, the composite

$$\Sigma\Phi_S(N;x) \to C_S(N) \to \Omega^\infty\Sigma^\infty C_S(N)$$

is $((j-1)(n-2)+r+2)$–connected. By elementary obstruction theory the obstructions $e'(x)$ and $e(x)$ contain the same information when $jp < (j-1)(n-2)+r+2$, that is, when $r \geq p-1-(j-1)(n-p-2)$. □

Corollary 7.10 *Assume $j \geq 2$. If N is contractible, then x lifts to an element of $E_j(P,N)$ if and only if $e'(x) = 0$.*

Proof In this case we can take $r = n-2$. Then the inequality of Lemma 7.9 becomes $n-2 \geq p-1-(j-1)(n-p-2)$, which is automatically satisfied because $p \leq n-3$. □

7.5.1 Equivariant reformulation Set $J := \{1,\dots,j\}$. Then the map $E_J(P) \to \binom{P}{j}$ which assigns to an embedding its image is a regular covering space with structure group Σ_j, where the latter acts on $E_J(P)$ via the automorphisms of J.

The pullback of $\mathscr{C} \to \binom{P}{j}$ along $E_J(P) \to \binom{P}{j}$ coincides with the fiberwise spectrum with Σ_j–action

$$E_J(P) \times \mathscr{C}_J \to E_J(P), \tag{34}$$

where $\mathscr{C}_J := \Sigma^\infty C_J(N)$ is a spectrum with Σ_j–action (recall that $C_J(N)$ is the total homotopy cofiber of the j–cube $E_\bullet(N)$; the action of Σ_j arises from the evident action of Σ_j on the cube). Note that Σ_j acts diagonally on $E_J(P) \times \mathscr{C}_J$. When considered unequivariantly, (34) is a trivial fiberwise spectrum.

Then the obstruction $e'(x)$ may be interpreted as an element of the equivariant cohomology group

$$H^0_{\mathrm{cs},\Sigma_j}(E_J(P); \mathscr{C}_J),$$

or, alternatively, as an element of the function space of compactly supported Σ_j–equivariant stable maps from $E_J(P)$ to $\mathscr{C}_J$.

7.5.2 The homological invariant

By Poincaré duality, there is an equivalence of spectra

$$H^0_{\mathrm{cs},\Sigma_j}(E_J(P); \mathscr{C}_J) \cong H_0^{\Sigma_j}(E_J(P); {}^{-\tau}\mathscr{C}_J),$$

where ${}^{-\tau}\mathscr{C}_J$ is the twist of $\mathscr{C}_J$ by the inverse of the tangent bundle of $E_J(P)$ (the latter is just the restriction of the product of j copies of the tangent bundle of P).

Definition 7.11 Let

$$\mu(x) \in H_0^{\Sigma_j}(E_J(P); {}^{-\tau}\mathscr{C}_J)$$

be the class that corresponds to $e'(x)$ via the Poincaré duality isomorphism.

Proof of Theorem E The procedure described above defines a function

$$\mu\colon \pi_0(\bar{E}_{j-1}(P,N)) \to H_0^{\Sigma_j}(E_J(P); {}^{-\tau}\mathscr{C}_J)$$

such that $\mu(x) = 0$ when x lifts to $\pi_0(\bar{E}_j(P,N))$ By Lemma 7.9 the converse is true provided $r \geq p-1-(j-1)(n-p-2)$. □

8 Spaces of link maps

Given manifolds $P_1, \dots, P_j$ of dimension $\dim P_i = p_i$ and a connected n–manifold N without boundary, a *link map* is a continuous map

$$f\colon P_1 \sqcup \cdots \sqcup P_j \to N$$

such that $f(P_i) \cap f(P_k) = \varnothing$ for $i \neq k$. We will typically assume that P_i is connected and boundaryless. Set $\boldsymbol{P} := \langle P_1, \dots, P_j \rangle$ and write

$$\mathscr{L}(\boldsymbol{P}, N)$$

for the space of link maps in the compact–open topology.

Recall that $J = \{1, 2, \dots, j\}$. For a subset $S \subset J$, set

$$P_S := \coprod_{i \in S} P_i \quad \text{and} \quad P^{(S)} := \prod_{i \in S} P_i .$$

Then, to each $S \subset J$, we have a space

$$\mathscr{L}^S(\boldsymbol{P}, N)$$

whose points are the maps

$$f \colon P_J \to N$$

such that $f(P_i) \cap f(P_k) = \varnothing$ for each pair of distinct elements $i, k \in S$. Note that $\mathscr{L}^J(\boldsymbol{P}, N) = \mathscr{L}(\boldsymbol{P}, N)$ is the space of link maps and if $|S| \leq 1$ then $\mathscr{L}^S(\boldsymbol{P}, N) = F(P_J, N)$ is the function space of maps with no constraint. The assignment

$$S \mapsto \mathscr{L}^S(\boldsymbol{P}, N)$$

is contravariant and defines a j–cube of spaces, which we denote by

$$\mathscr{L}^\bullet(\boldsymbol{P}, N).$$

Remark 8.1 There is a related j–cube

$$\mathscr{L}(\boldsymbol{P}_\bullet, N)$$

whose value at $S \subset J$ is the space of link maps $f \colon P_S \to N$. Then the evident map of j–cubes

$$\mathscr{L}^\bullet(\boldsymbol{P}, N) \to \mathscr{L}(\boldsymbol{P}_\bullet, N)$$

is ∞–cartesian because for each S we have a homotopy fiber sequence

$$F(P_{(J \backslash S)}, N) \to \mathscr{L}^S(\boldsymbol{P}, N) \to \mathscr{L}(\boldsymbol{P}_S, N),$$

and the j–cube $S \mapsto F(P_{(J \backslash S)}, N)$ is ∞–cartesian if $j > 1$.

8.1 Homotopy coherent Brunnian links

Henceforth, we fix an embedding

$$J \subset N$$

and identify J with its image. Let $c\colon \bigsqcup_i P_i \to N$ be the link map which sends P_i to i. Call c the *trivial link map*. Then c equips $\mathscr{L}^\bullet(\boldsymbol{P}, N)$ with the structure of a j–cube of based spaces. If $n \geq 2$, then the component of the basepoint is independent of the choice of embedding $J \subset N$.

Remark 8.2 Milnor [21] considers the case of link maps $f\colon \bigsqcup_{i=1}^{j} P_i \to N$ in euclidean space $N = \mathbb{R}^3$ in which each P_i is a circle S^1. Milnor defines f to be "trivial" if there is an extension of f to a link map $\bigsqcup_i D^2 \to \mathbb{R}^3$. Note that f is trivial in Milnor's sense if and only if f is homotopic through link maps to the trivial link map c.

Definition 8.3 The space of *homotopy coherent Brunnian link maps*

$$\mathscr{B}(\boldsymbol{P}, N)$$

is the total homotopy fiber of the j–cube of based spaces $\mathscr{L}^\bullet(\boldsymbol{P}, N)$.

Remarks 8.4 By Remark 8.1, an equivalent definition up to homotopy of $\mathscr{B}(\boldsymbol{P}, N)$ is given by taking the total homotopy fiber of the j–cube $\mathscr{L}(\boldsymbol{P}_\bullet, N)$.

A point of $\mathscr{B}(\boldsymbol{P}, N)$ is given by data consisting of a link map $f\colon P_J \to N$ together with a homotopy coherent set of rules which to each $S \subsetneq J$ associates a path from the associated point of $\mathscr{L}^S(\boldsymbol{P}, N)$ to the basepoint.

By contrast, Milnor [21, Section 5] defines a link map $f\colon \bigsqcup_{i=1}^{j} S^1 \to \mathbb{R}^3$ to be *almost trivial* if every proper sublink map of f is trivial.[3] If $j \geq 4$ then this notion of Brunnian fails to be homotopy coherent. Thus, a homotopy coherent Brunnian link map gives an almost trivial link map but not conversely.

Note that there is an evident map

$$\mathscr{B}(\boldsymbol{P}, N) \to \operatorname{fiber}\Bigl(\mathscr{L}^J(\boldsymbol{P}, N) \to \prod_{i=1}^{j} \mathscr{L}^{J_i}(\boldsymbol{P}, N)\Bigr),$$

---

[3] Subsequent authors call Milnor's notion of almost trivial link map a Brunnian link map. The earliest reference employing this language seems to be [3].

where $J_i = J \setminus \{i\}$, $\boldsymbol{P} = \langle S^1, \dots, S^1 \rangle$ and $N = \mathbb{R}^3$. However, if $j \geq 4$, this map is not a weak equivalence. Milnor's almost trivial link maps are those link maps whose components are in the image of the displayed homotopy fiber.

Terminology 8.5 As we only consider homotopy coherent Brunnian link maps in this paper, we henceforth refer to $\mathscr{B}(\boldsymbol{P}, N)$ simply as the space of *Brunnian link maps*, despite the different usage of this term in the literature.

8.2 The invariants

For each $S \subset J$, one has a map

$$\mathscr{L}^S(\boldsymbol{P}, N) \to F(P^{(J)}, E_S(N)), \tag{35}$$

where the target is the function space of maps $P^{(J)} \to E_S(N)$. One defines (35) by mapping a link map f to the map

$$(x_1, \dots, x_j) \mapsto \prod_{i \in S} f(x_i).$$

Remark 8.6 When $S = J$, the map (35) is Koschorke's *κ–invariant* $\mathscr{L}(\boldsymbol{P}, N) \to F(P^{(J)}, E_J(N))$.

If we let S vary, (35) defines a map of j–cubes of based spaces

$$\mathscr{L}^\bullet(\boldsymbol{P}, N) \to F(P^{(J)}, E_\bullet(N)). \tag{36}$$

Remark 8.7 For $S \subset J$, let $N^J(S)$ be the space of j–tuples $x \in N^J$ such that the image of x under the projection $N^J \to N^S$ lies in the subspace $E_S(N) \subset N^S$ (here $N^S := F(S, N)$). In other words, there is a pullback diagram

$$\begin{tikzcd}
N^J(S) \arrow[r] \arrow[d] & N^J \arrow[d] \\
E_S(N) \arrow[r] & N^S
\end{tikzcd}$$

The collection $\{N^J(S)\}_{S \subset J}$ forms both a stratification of N^J and a j–cube of based spaces.

The operation $S \mapsto F(P^{(J)}, N^J(S))$ is a j–cube of based spaces, which we denote by $F(P^{(J)}, N^J(\bullet))$. Then we have a commutative diagram of j–cubes

(37)
$$\begin{tikzcd}
\mathscr{L}^{\bullet}(\boldsymbol{P}, N) \arrow[r] \arrow[d] & F(P^{(J)}, N^J(\bullet)) \arrow[d] \\
\mathscr{L}(\boldsymbol{P}_{\bullet}, N) \arrow[r] & F(P^{(J)}, E_{\bullet}(N))
\end{tikzcd}$$

in which the vertical maps form ∞–cartesian $(j+1)$–cubes (even more is true if N happens to be contractible: in this case the vertical maps are objectwise weak equivalences of j–cubes). The map (36) is just the composition of the maps in diagram (37).

The top horizontal map of diagram (37) can be viewed as a kind of *coassembly map* which records the passage from global to local linking data. More precisely, set $\boldsymbol{J} := \langle 1, 2, \dots, j \rangle$, where we think of $i \in J$ as a manifold of dimension zero. Then, by definition,

$$N^J(S) = \mathscr{L}^S(\boldsymbol{J}, N),$$

and the top horizontal map of (37) associates to $f \colon \bigsqcup_i P_i \to N$ the map which sends a j–tuple $(x_1, \dots, x_j) \in P^{(J)}$ to the composed map $\bigsqcup_i x_i \subset \bigsqcup_i P_i \to N$.

One has a similar description of the bottom horizontal map by reinterpreting the configuration space $E_S(N)$ as the space of link maps $\mathscr{L}(\boldsymbol{S}, N)$.

Definition 8.8 Let

$$\Phi E_{\bullet}(N)$$

be the total homotopy fiber of the j–cube $E_{\bullet}(N)$ taken with respect to the given embedding $J \to N$. (Alternatively, $\Phi E_{\bullet}(N)$ can be defined as the total homotopy fiber of the $(j-1)$–cube $N_{\bullet}$ appearing in the proof of Lemma 7.1.)

Then the map of j–cubes (36) induces a map of total homotopy fibers

$$\ell \colon \mathscr{B}(\boldsymbol{P}, N) \to F(P^{(J)}, \Phi E_{\bullet}(N)), \tag{38}$$

called the *higher unstable linking number map*.

Remark 8.9 Let $\mathscr{O}_{\boldsymbol{P}}$ be the partially ordered set given by $\boldsymbol{U} = \langle U_1, \dots, U_j \rangle$, in which U_i is an open set in P_i, and $\boldsymbol{U} \le \boldsymbol{U}'$ if and only if $U_i \subset U_i'$ for all i. Then

$$\boldsymbol{U} \mapsto \mathscr{B}(\boldsymbol{U}, N)$$

defines a contravariant functor $\mathscr{O}_{\boldsymbol{P}} \to \mathscr{T}_*$. Its multilinearization in the sense of Weiss's manifold calculus coincides up to homotopy with the higher unstable linking number map ℓ (see [23; 22]).

Conjecture 8.10 *The map* ℓ (see (38)) *is* $(1+\Sigma')$*–connected, where*

$$\Sigma' = \sum_i (n - 2p_i - 2).$$

Remark 8.11 The $j = 2$ case of Conjecture 8.10 is known in the affirmative: it is the main result of [7].

8.2.1 The euclidean case, stabilization Assume $N = \mathbb{R}^n$. Then $\Phi(E_\bullet(\mathbb{R}^n))$ coincides with the total homotopy fiber of the based $(j-1)$–cube

$$S \mapsto \mathbb{R}^n \setminus S$$

for $S \subset J_1$ (see the proof of Lemma 7.1). By this identification and Corollary 4.2 applied to $Q_i := \{i\} \subset \mathbb{R}^n$, we infer there is a $(j(n-2)+1)$–connected map

$$\Phi E_\bullet(\mathbb{R}^n) \to \prod_{i=1}^{(j-2)!} \Omega^\infty S^{(j-1)(n-2)+1}. \tag{39}$$

Applying the functor $F(P^{(J)}, -)$ to (39), one obtains a map of function spaces

$$F(P^{(J)}, \Phi E_\bullet(\mathbb{R}^n)) \to \prod_{i=1}^{(j-2)!} F^{\mathrm{st}}(P^{(J)}, S^{(j-1)(n-2)+1}) \tag{40}$$

which is $(1+\Sigma)$–connected, where $\Sigma = \sum_{i=1}^{j}(n - p_i - 2)$. The composition of (38) with (40) defines the *higher stable linking number map*

$$\lambda\colon \mathscr{B}(\boldsymbol{P}, \mathbb{R}^n) \to \prod_{i=1}^{(j-2)!} F^{\mathrm{st}}(P^{(J)}, S^{(j-1)(n-2)+1}). \tag{41}$$

A version of (41) also appears in the work of Munson [22]. Note that [22, Corollary 1.2] gives a connectivity estimate one less than ours (see [22, Remark 3.6]).

Example 8.12 Let $n = j = 3$ and $P_i = S^1$ for $i = 1, 2, 3$. Then the higher stable linking number map λ is of the form

$$\mathscr{B}(\boldsymbol{P}, \mathbb{R}^3) \to F^{\mathrm{st}}((S^1)^{\times 3}, S^3).$$

Taking path components gives a function $\pi_0(\mathscr{B}(S^1_\bullet, \mathbb{R}^3)) \to \mathbb{Z}$. This can be described as the rule which assigns to a three-component Brunnian link in $\mathbb{R}^3$ a certain Massey product in the link complement [24].

Since $1 + \Sigma \geq 1 + \Sigma'$, we infer that Conjecture 8.10 with $N = \mathbb{R}^n$ is equivalent to the following:

Conjecture 8.13 *The higher stable linking number map* λ (see (41)) *is* $(1+\Sigma')$*–connected.*

8.3 Evidence for Conjecture 8.13

In this subsection we prove Theorem H, which we submit as evidence for Conjecture 8.13.

As above, $P_1, \dots, P_j$ are closed manifolds, but now we suppose that each P_i embeds in $\mathbb{R}^n$. In what follows, we don't require the P_i to be pairwise disjoint and we will not need to assume that P_1 is a submanifold of $\mathbb{R}^n$.

Recall the fixed embedding $J \subset \mathbb{R}^n$. Choose n–balls $B(i)$ containing $i \in J \setminus 1$ and assume that the collection $\{B(i)\}$ is pairwise disjoint. Choose an embedding $P_i \subset B(i)$ for $i \neq 1$. Using the inclusions $B(i) \subset \mathbb{R}^n$, we obtain an embedding

$$P_2 \sqcup \cdots \sqcup P_j \subset \mathbb{R}^n.$$

Consider the $(j-1)$–cube of function spaces

$$S \mapsto F(P_1, \mathbb{R}^n \setminus P_S), \quad S \subset J_1.$$

This is a based cube, where the basepoint of $F(P_1, \mathbb{R}^n \setminus P_S)$ is the constant map having value $1 \in \mathbb{R}^n \setminus P_S$. Consequently, the total homotopy fiber of this cube is given by

$$F(P_1, \Phi(\mathbb{R}^n \setminus P_\bullet)), \tag{42}$$

where now the convention is that $\mathbb{R}^n \setminus P_\bullet$ is the $(j-1)$–cube given by $\mathbb{R}^n \setminus P_S$ in which S ranges through subsets of J_1.

For $S \subset J_1$, consider the commutative diagram

$$\begin{tikzcd}
F(P_1, \mathbb{R}^n \setminus P_S) \arrow[r] \arrow[d, "a_S"'] & \mathscr{L}^{S \sqcup 1}(\boldsymbol{P}, \mathbb{R}^n) \arrow[r] \arrow[d, "b_S"] & F(P^{(J)}, E_{S \sqcup 1}(\mathbb{R}^n)) \arrow[d, "c_S"] \\
F(P_1, \mathbb{R}^n) \arrow[r] & \mathscr{L}^{S}(\boldsymbol{P}, \mathbb{R}^n) \arrow[r] & F(P^{(J)}, E_S(\mathbb{R}^n))
\end{tikzcd}$$

As S varies, each of the vertical maps assembles to a morphism of $(j-1)$–cubes, ie each gives a j–cube $a_\bullet$, $b_\bullet$ and $c_\bullet$, respectively. The j–cube $b_\bullet$ is just $\mathscr{L}^\bullet(\boldsymbol{P},\mathbb{R}^n)$. Similarly, $c_\bullet$ is the j–cube $F(P^{(J)},E_\bullet(\mathbb{R}^n))$. If we consider $a_\bullet$ as a map of $(j-1)$–cubes, then its target is the constant $(j-1)$–cube on the contractible space $F(P_1,\mathbb{R}^n)$; in particular, the target of $a_\bullet$ is ∞–cartesian. Hence, the total homotopy fiber $\Phi(a_\bullet)$ is identified with the total homotopy fiber of the source of $a_\bullet$, and the latter coincides with $F(P_1,\Phi(\mathbb{R}^n\setminus P_\bullet))$, ie the source of the map (42). Consequently, taking the total homotopy fibers of $a_\bullet$, $b_\bullet$ and $c_\bullet$ and composing with the map (40) results in a commutative diagram

$$\begin{tikzcd}
F(P_1,\Phi(\mathbb{R}^n\setminus P_\bullet)) \arrow[r] & \mathscr{B}(\boldsymbol{P},\mathbb{R}^n) \arrow[r, "\ell"] \arrow[dr, "\lambda"'] & F(P^{(J)},\Phi E_\bullet(\mathbb{R}^n)) \arrow[d] \\
 & & F^{\mathrm{st}}(P^{(J)},S^{(j-1)(n-2)+1})
\end{tikzcd} \tag{43}$$

such that the right vertical map is $(1+\Sigma)$–connected (see (40)).

Remark 8.14 In the above, we've neglected to mention that the map of cubes $a_\bullet\to b_\bullet$ isn't basepoint-preserving. This means that the map doesn't define a map of total homotopy fibers in an obvious way.

However, the map is easily seen to be basepoint-preserving up to a preferred path (the path is defined by the radial deformation retraction of each ball $B(i)$ onto its center i). It is this preferred path that enables us to define the map from the total homotopy fiber of $a_\bullet$ to the total homotopy fiber of $b_\bullet$, which is the leftmost map in (43).

Claim 8.15 *The horizontal composite*

$$F(P_1,\Phi(\mathbb{R}^n\setminus P_\bullet))\to F(P^{(J)},\Phi E_\bullet(\mathbb{R}^n)) \tag{44}$$

of diagram (43) is $(1-\hat{p}+\Sigma)$–connected.

The claim, proved below, gives evidence for the validity of Conjecture 8.13: it implies that ℓ is a retraction on homotopy in degrees $\le 1-\hat{p}+\Sigma$ (the same is true for λ since the vertical map of (43) is $(1+\Sigma)$–connected). Furthermore, we have $1-\hat{p}+\Sigma\ge 1+\Sigma'$, so λ will be a retraction in degrees $\le 1+\Sigma'$. Consequently, the proof of Theorem H has been reduced to verification of the claim.

Proof of Claim 8.15 For $S \subset J_1$, consider the pullback diagram

$$\begin{tikzcd}
\mathscr{E}_S \arrow[r] \arrow[d] & E_{S \sqcup 1}(\mathbb{R}^n) \arrow[d] \\
P^{(J_1)} \arrow[r] & E_S(\mathbb{R}^n)
\end{tikzcd}$$

where the right vertical map is given by projection and the bottom horizontal map is the projection $P^{(J_1)} \to P^{(S)}$ followed by the inclusion $P^{(S)} \subset E_S(\mathbb{R}^n)$. Observe that the fiber of $\mathscr{E}_S \to P^{(J_1)}$ at a point $(x_2, \dots, x_j)$ is given by $\mathbb{R}^n \setminus \{x_i\}_{i \in S}$.

The map $P^{(J_1)} \to E_S(\mathbb{R}^n)$ factors through the contractible space $B^{(J_1)} := \prod_i B(i)$, so the fibration $\mathscr{E}_S \to P^{(J_1)}$ is trivializable. Let $\Gamma(\mathscr{E}_S)$ be the space of sections of $\mathscr{E}_S \to P^{(J_1)}$. Define a map

$$\mathbb{R}^n \setminus P_S \to \Gamma(\mathscr{E}_S)$$

by sending a point $z \in \mathbb{R}^n \setminus P_S$ to the section given by $(x_2, \dots, x_j) \mapsto z$. This makes sense since z also lies in $\mathbb{R}^n \setminus \{x_i\}_{i \in S}$.

As S varies we obtain a map of J_1–cubes

$$\mathbb{R}^n \setminus P_\bullet \to \Gamma(\mathscr{E}_\bullet), \tag{45}$$

and applying the functor $F(P_1, -)$ to the induced map of total homotopy fibers of (45) yields the map of the claim.

Hence, it suffices to prove that (45) is $(1+\mu_2+\Sigma_2)$–cartesian, where

$$\mu_2 := \min_{2 \le i \le j} (n - p_i - 2), \quad \Sigma_2 := \sum_{i=2}^{j} (n - p_i - 2), \tag{46}$$

since $F(P_1, -)$ reduces connectivity by p_1 and

$$1 + \mu_2 + \Sigma_2 - p_1 = 1 - \hat{p} + \Sigma.$$

We will explain the proof when $2 \le j \le 3$. The remaining cases are analogous to the case $j = 3$ and we will leave them for the reader to verify.

When $j = 2$, it is readily checked that the statement to be proved amounts to the assertion that the map

$$\mathbb{R}^n \setminus P_2 \to F(P_2, S^{n-1})$$

given by

$$z \mapsto \left(x \mapsto \frac{x - z}{|x - z|} \right)$$

is $(1+2(n-p_2-2))$–connected. This follows from the commutative diagram

$$\begin{tikzcd}
\mathbb{R}^n \setminus P_2 \arrow[r] \arrow[d] & F(P_2, S^{n-1}) \arrow[d] \\
\Omega^\infty\Sigma^\infty(\mathbb{R}^n \setminus P_2) \arrow[r] & F^{\mathrm{st}}(P_2, S^{n-1})
\end{tikzcd}$$

where the left vertical map is $(1+2(n-p_2-2))$–connected by the Freudenthal suspension theorem, the right vertical map is $(1-p+2(n-2))$–connected, also by the Freudenthal suspension theorem, and the lower horizontal map is a homotopy equivalence by Spanier–Whitehead duality.

When $j=3$ one proceeds as follows: We think of the square $\mathbb{R}^n \setminus P_S$ for $S \subset \{2,3\}$ as defining an isotopy functor $\phi\colon \mathscr{O}_{P_2} \times \mathscr{O}_{P_3} \to T_*$ which assigns to an open set $U \subset P_2$ and an open set $V \subset P_3$ the total homotopy fiber of the square

$$\begin{tikzcd}
U^* \cap V^* \arrow[r] \arrow[d] & V^* \arrow[d] \\
U^* \arrow[r] & \mathbb{R}^n
\end{tikzcd}$$

where A^* denotes the complement of $A \subset \mathbb{R}^n$. Similarly, one has an isotopy functor $\phi^\sharp\colon \mathscr{O}_{P_2} \times \mathscr{O}_{P_3} \to T_*$ associated with the total homotopy fiber of the square $S \mapsto \Gamma(\mathscr{E}_S)$. In fact, the latter is easy to identify: it is given by

$$(U,V) \mapsto F(U \times V, S^{n-1} \flat S^{n-1}),$$

where $S^{n-1} \flat S^{n-1}$ is the total homotopy fiber of the wedge square on S^{n-1}. The natural map

$$\phi(U,V) \to \phi^\sharp(U,V) \tag{47}$$

is a kind of bilinearization (or coassembly) in the sense that

- its value when U and V are open balls is a homotopy equivalence;
- $\phi^\sharp(U,V)$ is linear in each variable in the sense of isotopy calculus.

Furthermore, (47) is initial with respect to these properties. On the other hand, Corollary 3.16 (see Corollary 4.2 and Remark 3.17) defines a natural transformation

$$\phi(U,V) \to \Omega^\infty\Sigma^\infty(S^{-1} \wedge U^* \wedge V^*) \tag{48}$$

whose connectivity can be described as follows: If U is a tubular neighborhood of a closed manifold of dimension k_1 and V is a tubular neighborhood of a closed manifold of dimension k_2, then (48) is $\bigl(1+\min(n-k_1-2, n-k_2-2)+\sum(n-k_i-2)\bigr)$–connected. In particular, it is $(3n-5)$–connected when U and V are balls.

The functor $(U, V) \mapsto \Omega^\infty(S^{-1} \wedge U^* \wedge V^*)$ is also bilinear. In fact, by Spanier–Whitehead duality it is naturally equivalent to the functor ψ given by

$$(U, V) \mapsto F(U \times V, \Omega^\infty\Sigma^\infty(S^{2n-3})).$$

As $\phi \to \phi^\sharp$ is initial in the homotopy category of functors, there is a natural transformation

$$\phi^\sharp \to \psi \tag{49}$$

that yields a factorization $\phi \to \phi^\sharp \to \psi$. Clearly, (49) is induced by a map of spaces $S^{n-1} \flat\, S^{n-1} \to \Omega^\infty\Sigma^\infty(S^{2n-3})$. Furthermore, it is automatic that the map $\phi^\sharp(U, V) \to \psi(U, V)$ is $(3n-5)$–connected when U and V are balls.

It follows that the map $\phi^\sharp(P_2, P_3) \to \psi(P_2, P_3)$ is $(3n-5-p_2-p_3)$–connected. As $3n-5-p_2-p_3$ is strictly larger than $1+\mu_2+\Sigma_2$, it follows that the map $\phi(P_2, P_3) \to \phi^\sharp(P_2, P_3)$ is $(1+\mu_2+\Sigma_2)$–connected, as was to be shown. □

Example 8.16 Let $\boldsymbol{P} = \langle S^1, \dots, S^1\rangle$ be an ordered j–tuple of circles and let $n = 3$. By Theorem H,

$$\pi_0(\lambda)\colon \pi_0(\mathscr{B}(\boldsymbol{P}, \mathbb{R}^3)) \to \prod_{i=1}^{(j-2)!} \mathbb{Z}$$

is surjective. We conjecture that $\pi_0(\lambda)$ coincides with Milnor's μ–invariants [21, Section 5] on the set of (classical) Brunnian link maps.

8.4 Postscript: the two-component case

When $j = 2$ there is some additional evidence for Conjecture 8.13 with the numerical improvements suggested by Theorem H. Let $\boldsymbol{P} = \langle P, Q\rangle$, with $p := \dim P$ and $q := \dim Q$. In this situation, λ is the classical stable linking pairing

$$\mathscr{L}(\langle P, Q\rangle, \mathbb{R}^n) \to F^{\mathrm{st}}(P \times Q, S^{n-1}), \tag{50}$$

which associates to a link map $f \sqcup g\colon P \sqcup Q \to \mathbb{R}^n$ the map

$$(x, y) \mapsto \frac{f(x) - g(y)}{|f(x) - g(y)|}.$$

On path components the above gives a function of pointed sets

$$\alpha\colon \pi_0(\mathscr{L}(\langle P,Q\rangle,\mathbb{R}^n)) \to \{P_+\wedge Q_+, S^{n-1}\}, \tag{51}$$

where we have identified the set of path components of $F^{\mathrm{st}}(P\times Q, S^{n-1})$ with the abelian group of stable homotopy classes of based maps $P_+\wedge Q_+\to S^{n-1}$.

Suppose A and B are pointed sets. We denote the basepoint in each case by $*$. A basepoint-preserving map $h\colon A\to B$ is said to be *weakly injective* if there are no nontrivial solutions to the equation $h(x)=*$. If h is a homomorphism of groups, then weak injectivity implies injectivity (compare [9, Lemma 1.1]).

Proposition 8.17 *Assume that $Q\subset\mathbb{R}^n$ is a submanifold of codimension ≥ 3. Then the function α is a surjection on path components if $2n-2q-p-3\geq 0$. Furthermore, if $2n-2q-p-3>0$ then α is weakly injective.*

Remarks 8.18 (1) Proposition 8.17 gives a better estimate than [7], but at the expense of an additional hypothesis on Q.

(2) The number $2n-2q-p-3$ may be rewritten in the form $1-q+\Sigma$, where $\Sigma=(n-p-2)+(n-q-2)$. This is the number of Theorem H when $j=2$. Hence, only weak injectivity needs to be verified.

(3) Proposition 8.17 suggests that the connectivity estimate of Conjecture 8.13 might be improved to $1-\hat{p}+\Sigma$ under the additional assumption that $P_2,\ldots,P_j\subset\mathbb{R}^n$ are submanifolds of codimension ≥ 3.

(4) Proposition 8.17 delivers more information in the spherical case $P=S^p$ and $Q=S^q$ with $q\leq n-3$. Then $\pi_0(\mathscr{L}(\langle S^p,S^q\rangle,\mathbb{R}^n))$ possesses a group structure (see [28; 17, page 765]) and the function α becomes a homomorphism. Consequently, weak injectivity implies injectivity and we recover [28, page 190]. We infer that Proposition 8.17 implies that α is an isomorphism when $2n-2q-p-3>0$. According to [9, Theorem 1.1], in the spherical case α is actually an isomorphism if $3n-2q-2p-4>0$ and $p,q\geq 1$.

Proof of Proposition 8.17 As pointed out above, we only need to verify the last part of the statement. Let

$$x := f\sqcup g\in\mathscr{L}(\langle P,Q\rangle,\mathbb{R}^n)$$

be any point. We can assume without loss in generality that $f\colon P\to\mathbb{R}^n$ is a smooth map. We first show how to find a path in $\mathscr{L}(\langle P,Q\rangle,\mathbb{R}^n)$ from x to $x'=(f,h)$ in which h is a smooth embedding. It then suffices to prove that if the stable linking number of x' is trivial then the map $f\colon P \to \mathbb{R}^n \setminus h(Q)$ is null-homotopic.

Consider the commutative square

$$\begin{tikzcd}
E(Q, \mathbb{R}^n \setminus f(P)) \arrow[r] \arrow[d] & F(Q, \mathbb{R}^n \setminus f(P)) \arrow[d] \\
E(Q, \mathbb{R}^n) \arrow[r] & F(Q, \mathbb{R}^n)
\end{tikzcd}$$

in which $E(-,-)$ denotes the space of embeddings. By Lemma 8.19 below, the square is $(2n{-}2q{-}p{-}3)$–cartesian. If particular, if we use the preferred basepoint of $E(Q, \mathbb{R}^n)$, it follows that, when $2n - 2q - p - 3 \geq 0$, we can find an isotopy of the submanifold $Q \subset \mathbb{R}^n$ to an embedding $h\colon Q \to \mathbb{R}^n \setminus f(P)$ such that the underlying map of this embedding is homotopic to the map $g\colon Q \to \mathbb{R}^n \setminus f(P)$. Then $x' = (f, h) \in \mathscr{L}(\langle P, Q\rangle, \mathbb{R}^n)$ is in the same path component as x.

But, as we've seen above, the composition

$$F(P, \mathbb{R}^n \setminus h(Q)) \to \mathscr{L}(\langle P, Q\rangle, \mathbb{R}^n) \to F^{\mathrm{st}}(P \times Q, S^{n-1})$$

is $(2n{-}2q{-}p{-}3)$–connected. In particular, if $2n - 2q - p - 3 > 0$ then the triviality of the stable linking number of x' implies that the map $P \to \mathbb{R}^n \setminus h(Q)$ is null-homotopic.

□

The following result was used in the proof of Proposition 8.17:

Lemma 8.19 *Assume N is a connected smooth n–manifold, and let P and Q be closed smooth manifolds of dimensions p and q. Assume $q \leq n-3$. Let $f\colon P \to N$ be a smooth map. Then the square*

$$\begin{tikzcd}
E(Q, N \setminus f(P)) \arrow[r] \arrow[d] & F(Q, N \setminus f(P)) \arrow[d] \\
E(Q, N) \arrow[r] & F(Q, N)
\end{tikzcd}$$

is $(2n{-}2q{-}p{-}3)$–cartesian.

Remark 8.20 When f is an embedding, this amounts to the $j = 2$ case of [6, Theorem E].

Proof sketch The argument was communicated to us by Tom Goodwillie. If we replace embeddings with immersions, then the analogous diagram is ∞–cartesian by Smale–Hirsch theory (in this instance we only need to assume $q \le n-1$). Hence, it suffices to show that the square

$$\begin{tikzcd}
E(Q, N \setminus f(P)) \arrow[r] \arrow[d] & I(Q, N \setminus f(P)) \arrow[d] \\
E(Q, N) \arrow[r] & I(Q, N)
\end{tikzcd}$$

is $(2n-2q-p-3)$–cartesian, where $I(-,-)$ denotes the space of immersions.

The proof then proceeds by comparing the homotopy fibers of the horizontal maps of the square. The map $N \setminus f(P) \to N$ is $(n-p-1)$–connected by transversality. If $q \le n-3$, then the Goodwillie–Weiss embedding calculus applied to the embedding spaces $E(Q, N \setminus f(P))$ and $E(Q, N)$ gives towers for these homotopy fibers, where the first nontrivial layer is in degree $j \ge 2$. The homotopy-theoretic model for these layers provided by [30] implies that the map of the j^{th} layers is $(2n-2q-p-3)$–connected for all j. The conclusion then follows from the five lemma. □

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Rips filtrations for quasimetric spaces and asymmetric functions with stability results

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The Rips filtration over a finite metric space and its corresponding persistent homology are prominent methods in topological data analysis to summarise the "shape" of data. Crucial to their use is the stability result that says if X and Y are finite metric spaces then the (bottleneck) distance between the persistence diagrams constructed via the Rips filtration is bounded by $2d_{\mathrm{GH}}(X,Y)$ (where d_{GH} is the Gromov–Hausdorff distance). A generalisation of the Rips filtration to any *symmetric* function $f\colon X \times X \to \mathbb{R}$ was defined by Chazal, de Silva and Oudot (Geom. Dedicata 173 (2014) 193–214), where they showed it was stable with respect to the correspondence distortion distance. Allowing asymmetry, we consider four different persistence modules, definable for pairs (X,f) where $f\colon X \times X \to \mathbb{R}$ is any real valued function. These generalise the persistent homology of the symmetric Rips filtration in different ways. The first method is through symmetrisation. For each $a \in [0,1]$ we can construct a symmetric function $\mathrm{sym}_a(f)(x,y) = a\min\{d(x,y),d(y,x)\} + (1-a)\max\{d(x,y),d(y,x)\}$. We can then apply the standard theory for symmetric functions and get stability as a corollary. The second method is to construct a filtration $\{\mathcal{R}^{\mathrm{dir}}(X)_t\}$ of ordered tuple complexes where $(x_0,x_2,\dots,x_p) \in \mathcal{R}^{\mathrm{dir}}(X)_t$ if $d(x_i,x_j) \le t$ for all $i \le j$. Both our first two methods have the same persistent homology as the standard Rips filtration when applied to a metric space, or more generally to a symmetric function. We then consider two constructions using an associated filtration of directed graphs or preorders. For each t we can define a directed graph $\{D(X)_t\}$ where directed edges $x \to y$ are included in $D(X)_t$ whenever $\max\{f(x,y), f(x,x), f(y,y)\} \le t$ (note this is when $d(x,y) \le t$ for $f=d$ a quasimetric). From this we construct a preorder where $x \le y$ if there is a path from x to y in $D(X)_t$. We build persistence modules using the strongly connected components of the graphs $D(X)_t$, which are also the equivalence classes of the associated preorders. We also consider persistence modules using a generalisation of poset topology to preorders.

The Gromov–Hausdorff distance, when expressed via correspondence distortions, can be naturally extended as a correspondence distortion distance to set–function pairs (X,f). We prove that all these new constructions enjoy the same stability as persistence modules built via the original persistent homology for symmetric functions.

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1 Introduction

The Rips filtration over a finite metric space (X,d) is a filtration of simplicial complexes $\{\mathcal{R}(X,d)_t\}_{t\in[0,\infty)}$, where $\mathcal{R}(X,d)_t$ is the clique complex over the graph whose vertex set is X and edge set $\{[x,y]\colon d(x,y)\le t\}$. It adds topological structure to an otherwise disconnected set of points. The persistent homology of the Rips filtration is widely used in topological data analysis because it encodes useful information about the geometry and topology of the underlying metric space; see Chazal, Cohen-Steiner. Guibas, Mémoli and Oudot [4], Ghrist [10], Lee, Chung, Kang, Kim and Lee [14] and Xia and Wei [20]. There are many potential applications for studying data whose structure is a quasimetric space. Examples include the web hyperlink quasimetric space, road networks, and quasimetrics induced from weighted directed graphs found throughout science (for example biological interaction graphs — see Klamt and von Kamp [12] — or the connections in neural systems; see Kaiser [11] and Reimann, Nolte, Scolamiero, Turner, Perin, Chindemi, Dłotko, Levi, Hess and Markram [18]). More generally we wish to define and show stability of Rips filtrations for sublevel sets of any (not necessarily symmetric) function $f\colon X\times X\to\mathbb{R}$.

Historically the Rips filtration was defined as a special increasing family of simplicial complexes built from a finite metric space. A metric space is a set X equipped with a distance function $d\colon X\times X\to\mathbb{R}$ that satisfies the following properties:

(i) **Nonnegativity** $d(x,y)\ge 0$ for all $x\in X$.

(ii) **Symmetry** $d(x,y)=d(y,x)$ for all $x,y\in X$.

(iii) **Triangle inequality** $d(x,z)\le d(x,y)+d(y,z)$.

(iv) **Identity of indiscernibles** $d(x,y)=0=d(y,x)$ if and only if $x=y$.

For any $r\ge 0$ we define the Rips complex of X at length scale r, denoted by $\mathcal{R}(X,d)_r$, as the abstract simplicial complex where $[x_0,x_1,\dots,x_k]\in\mathcal{R}(X,d)_r$ whenever $d(x_i,x_j)\le r$ for all i and j. We can think of $\mathcal{R}(X,d)_r$ as adding a topological structure of length scale r. It is easy to check that if $r\le s$ then $\mathcal{R}(X,d)_r\subset\mathcal{R}(X,d)_s$. We thus can define the Rips filtration of X as the increasing family of simplicial complexes $\{\mathcal{R}(X,d)_r\}_{r\in[0,\infty)}$.

Two classic types of examples of Rips filtrations are examples that come from finite point clouds sitting inside some larger space (such as Euclidean space) and examples built from graphs. If $X\subset\mathbb{R}^d$ is a set of points then it inherits a finite metric space structure from that of $\mathbb{R}^d$; the distance function is just the restriction of the Euclidean distance function to the set X. Given a graph G (with or without lengths on the edges) we can let the vertices of the graph be the finite set X and then construct a distance function on X by defining $d(x,y)$ as the shortest path length of all the paths from x to y in G.

From the Rips filtration we can produce a persistence module which describes its persistent homology. A persistence module is a family of vector spaces $\{V_t : t \in \mathbb{R}\}$ equipped with linear maps $\phi_s^t\colon V_s \to V_t$ for each pair $s \leq t$ such that $\phi_t^t = \mathrm{id}$ and $\phi_s^t = \phi_s^r \circ \phi_r^t$ whenever $s \leq r \leq t$. The persistence module we construct from the persistent homology of a Rips filtration over (X,d) has vector spaces $\{H_*(\mathcal{R}(X,d)_t)\}_{t\in[0,\infty)}$ along with maps on homology induced by inclusions, $\phi_s^t = \iota_*\colon H_*(\mathcal{R}(X,d)_s) \to H_*(\mathcal{R}(X,d)_t)$ when $s \leq t$.

Arguably the most important theoretical results in topological data analysis are the stability theorems. These stability results come in a variety of forms but generally say that if two sets of input data are close then various persistence modules computed from them are also close. To be specific we need to quantify what is meant by "close" for these different kinds of objects.

We can measure how close persistence modules are via whether there exist suitable families of interleaving maps. This distance is closely related to the bottleneck distance between the corresponding persistence diagrams or barcodes. Two persistence modules, $(\{V_t\},\{\phi_s^t\})$ and $(\{U_t\},\{\psi_s^t\})$, are called ϵ–interleaved when there exist families of linear maps $\{\alpha_t\colon V_t \to U_{t+\epsilon}\}$ and $\{\beta_t\colon U_t \to V_{t+\epsilon}\}$ satisfying natural commuting conditions. There is a pseudometric on the space of persistence modules called the interleaving distance, d_{int}, which is the infimum of the set of $\epsilon > 0$ such that there exists an ϵ–interleaving. More details about the interleaving distance are provided in Section 3. In this paper we will be considering a variety of different persistence modules, but we will always use the interleaving distance to quantify "closeness".

Gromov–Hausdorff distance is a classical distance between metric spaces. There are many equivalent formulations of Gromov–Hausdorff distance but for the purposes of this paper we will focus on that using correspondences. The set $\mathcal{M} \subset X \times Y$ is a *correspondence* between X and Y if for all $x \in X$ there exists some $y \in Y$ with $(x,y) \in \mathcal{M}$ and for all $y \in Y$ there is some $x \in X$ with $(x,y) \in \mathcal{M}$. Using correspondences we can define the Gromov–Hausdorff distance between X and Y as

$$(1\text{-}1)\quad d_{\mathrm{GH}}(X,Y) = \frac{1}{2} \inf_{\{\text{correspondences } \mathcal{M}\}} \sup_{(x_1,y_1),(x_2,y_2)\in\mathcal{M}} |d_X(x_1,x_2) - d_Y(y_1,y_2)|.$$

Here $\sup_{(x_1,y_1),(x_2,y_2)\in\mathcal{M}} |d_X(x_1,x_2)-d_Y(y_1,y_2)|$ is the distortion of the correspondence $\mathcal{M}$. We can define the *correspondence distortion distance* between set–function pairs $(X,\ f\colon X\times X\to\mathbb{R})$ and $(Y,\ g\colon Y\times Y\to\mathbb{R})$ by

$$d_{\mathrm{CD}}((X,f),(Y,g)) \\ = \frac{1}{2} \inf_{\mathcal{M}\text{ correspondence between } X \text{ and } Y} \ \sup_{(x_1,y_1),(x_2,y_2)\in\mathcal{M}} |f(x_1,x_2)-g(y_1,y_2)|.$$

This agrees with the standard definition for the Gromov–Hausdorff distance when (X,d_X) and (Y,d_Y) are metric spaces. More background and details about the correspondence distortion distance are presented in Section 2.

Useful as the Rips filtration for finite metric spaces is, there are scenarios where the input is not a finite metric space. For example, it is common in data analysis to considers data sets X equipped with a dissimilarity measure. A dissimilarity measure is a map $d_X\colon X\times X\to\mathbb{R}$ that satisfies $d_X(x,x)=0$ and $d_X(x,y)=d_X(y,x)$ for all $x,y\in X$, but is not required to satisfy any of the other metric space axioms. In [5], Chazal, de Silva and Oudot generalised the notion of a Rips filtration to cover dissimilarity measures and more generally for any symmetric function $f\colon X\times X\to\mathbb{R}$. Just as in the finite metric space case, the Rips complex of X with parameter r, denoted by $\mathcal{R}(X,f)_r$, is defined as the abstract simplicial complex where $[x_0,x_1,\dots,x_k]\in\mathcal{R}(X,f)_r$ whenever $f(x_i,x_j)\le r$ for all i and j (including $i=j$).

Persistent homology can be applied to any increasing family of topological spaces, so it is then natural to define persistence modules from the persistent homology of Rips filtrations built from any symmetric function. This was shown to be stable in [5].

Theorem *Let $f\colon X\times X\to\mathbb{R}$ and $g\colon Y\times Y\to\mathbb{R}$ be symmetric functions and $\mathcal{R}(X,f)$ and $\mathcal{R}(Y,g)$ their corresponding Rips filtrations. If $d_{\mathrm{CD}}((X,f),(Y,g))$ is finite then for all $\epsilon > d_{\mathrm{CD}}((X,f),(Y,g))$, the k^{th} homology persistence modules of $\mathcal{R}(X,f)$ and $\mathcal{R}(Y,g)$ are ϵ–interleaved. In particular, when (X,d_X) and (Y,d_Y) are compact metric spaces, $\mathcal{R}(X,d_X)$ and $\mathcal{R}(Y,d_Y)$ are ϵ–interleaved for all $\epsilon > 2d_{\mathrm{GH}}(X,Y)$.*

The proofs of the interleaving results in [5] didn't have any requirement on the function $f\colon X\times X\to\mathbb{R}$ except that it had to be symmetric. The purpose of this paper is to complete this generalisation procedure to lose that symmetry requirement. However, there are multiple ways to use asymmetry information, and so we have explored a variety of different constructions.

One method is to study related symmetric functions. We can take our original function f and construct a parametric family of related symmetric functions $\mathrm{sym}_a(f)$ where

$a \in [0,1]$ and

$$\mathrm{sym}_a(f)(x,y) = a\min\{f(x,y), f(y,x)\} + (1-a)\max\{f(x,y), f(y,x)\}.$$

We can then construct the Rips filtration as in [5] for the set–function pair $(X, \mathrm{sym}_a(f))$. Notably, if f is a symmetric function to begin with then $\mathrm{sym}_a(f) = f$ for all $a \in [0,1]$ and hence this symmetrisation process does give a generalisation of Rips filtrations to any set–function pair. We can show that the correspondence distortion distance between $(X, \mathrm{sym}_a(f))$ and $(Y, \mathrm{sym}_a(g))$ is bounded by that between (X, f) and (Y, g). We gain stability for these persistence modules constructed through this symmetrisation process as a corollary.

A limitation with using a filtration of simplicial complexes is that a simplex is an inherently symmetric object. An alternative is to use ordered tuple complexes (shortened to OT complexes). An OT complex K is a sets of ordered tuples $(v_0, v_1, \dots, v_p)$ such that if $(v_0, v_1, \dots, v_p) \in K$ then $(v_0, v_1, \dots, \hat{v}_i, \dots, v_p) \in K$ for all i. Note that repetitions of the v_j are allowed. Chain complexes, boundary maps, homology and persistent homology can analogously be defined for OT complexes. We will define the *directed Rips filtration* of OT complexes for $f\colon X \times X \to \mathbb{R}$, as the filtration $\{\mathcal{R}^{\mathrm{dir}}(X, f)_t\}$ of ordered tuple complexes where $(x_0, x_2, \dots, x_p) \in \mathcal{R}^{\mathrm{dir}}(X, f)_t$ if $f(x_i, x_j) \le t$ for all $i \le j$. We call the persistence module produced using the OT homology of the directed Rips filtration the *directed Rips persistence module*.

For each simplicial complex there is a canonical OT complex with isomorphic homology group. Furthermore, since these homology isomorphisms commute with the maps induced by inclusion, the persistence modules of these corresponding complexes are also isomorphic. This implies that these directed Rips filtrations are truly generalisations of the Rips filtration built from a symmetric function. We will prove that the persistence modules constructed from these Rips filtrations are stable with respect to the correspondence distortion distance.

The third generalisation considers connected components. The standard dimension 0 homology can be viewed as the vector space whose elements are linear combinations of connected components in the 1–skeleton (ie the graph containing the 0– and 1–cells). When working with directed graphs there are two notions of connected components: weakly and strongly connected. Completely analogous to the traditional connected components story, we can consider vector space whose elements are formal linear combinations the equivalence classes of *strongly* connected components in the directed graph which is the 1–skeleton of the directed Rips filtration.

Given a function $f\colon X \times X \to \mathbb{R}$, for each real number t we can create a directed graph D_t related to the sublevel set $f^{-1}(-\infty, t]$. The graph D_t should have vertex set $\{x \in X \mid f(x,x) \leq t\}$ and directed edge set $\{x \to y \mid \max\{f(x,x), f(x,y), f(y,y)\} \leq t\}$. We can not include a directed edge $x \to y$ just when $f(x,y) \leq t$ because of the closure conditions a directed graph has to satisfy. For each $t \in \mathbb{R}$ we have a vector space V_t of the formal linear combinations the equivalence classes of strongly connected components (SCCs) of D_t. Whenever $s \leq t$ we have an inclusion map $D_s \subset D_t$ which induces a linear map from V_s to V_t. This process directly constructs a persistence module, which we call the *strongly connected components persistence module*. We prove that these persistence modules are stable with respect to the correspondence distortion distance. We also provide some pseudocode on how to compute the barcode decomposition of the strongly connected components persistence module using a modification of the union-find algorithm.

We also note that the persistence modules generated from formal linear combinations of the weakly connected components have already been covered as the dimension 0 persistent homology of the filtration by sublevel sets of $\mathrm{sym}_1(f)$.

Our fourth method uses the directed graphs described above to create a filtration of preorders. Given a directed graph D over vertices X we say $x \leq y$ if there is a path from x to y. From a filtration of directed graphs we obtain a filtration of preorders. We then can construct persistence modules using poset topology (which can be generalised for all preorders, not just posets, discussed in the appendix). We will call these *preorder persistence modules*. We prove that these preorder persistence modules are stable with respect to the correspondence distortion distance. If $f\colon X \times X \to \mathbb{R}$ is a symmetric function, then the dimension 0 preorder persistence module is the same as that of the persistent homology of the standard Rips filtration $\mathcal{R}(X, f)$ and its higher-dimensional preorder persistence modules are always trivial. This implies that preorder persistence modules are describing asymmetry information.

1.1 Related other works

Other related work involves approaches in topological data analysis for incorporating asymmetry information. Ordered set homology is used in [18] in order to study the topology of brain networks. There has been a series of papers by Chowdhury and Mémoli [6; 8; 7] about other constructions of persistence modules which incorporate asymmetry information.

2 Directed graphs, quasi- and pseudometric spaces and the correspondence distortion distance

The original stability result in topological data analysis for Rips filtrations was for filtrations of simplicial complexes built from metric spaces and the bound between persistence modules in terms of the Gromov–Hausdorff distance. This was generalised in [5] to consider symmetric functions and the bound between the functions was the correspondence distance. However, there are many applications where asymmetry naturally arises, of which important examples are quasimetric spaces, such as those constructed as the path metric of some directed graph (with or without weights on the directed edges).

Definition 1 A *directed graph* is an ordered pair $D=(V,A)$ where V is a set whose elements are called *vertices* and A is a set of ordered pairs of vertices called *directed edges* or arrows. A *weighted directed graph* is a directed graph where each arrow is given a nonnegative weight.

Note that a graph can be thought of as a directed graph such that whenever a directed edge $v\to w$ is in A its opposite direction $w\to v$ must also be in A.

Definition 2 Let X be a nonempty set and $d\colon X\times X\to\mathbb{R}$. Consider the following potential properties of d:

(1) $d(x,x')\geq 0$ for all $x,x'\in X$.

(2) $d(x,x')=d(x',x)$ for all $x,x'\in X$.

(3) For all $x,x'\in X$, $x=x'$ if and only if $d(x,x')=0$ and $d(x',x)=0$.

(4) $d(x,x'')\leq d(x,x')+d(x',x'')$ for all $x,x',x''\in X$.

If (X,d) satisfies (1), (2), (3) and (4), it is called a metric space. If (X,d) satisfies (1), (3) and (4), it is called a *quasimetric space* and we can call d a *quasimetric*. If (X,d) satisfies (1), (2) and (4), it is called a *pseudometric space* and we can call d a *pseudometric*. If (X,d) satisfies (1) and (4), it is called a *pseudoquasimetric space* and we can call d a *pseudoquasimetric*.

We can build examples of these different types of spaces using weighted directed graphs. Given a weighted directed graph $D=(V,A)$ and two vertices $x,y\in V$, we call $x=v_0,v_1,v_2,\dots,v_m=y$ a path from x to y if all of the arrows $v_i\to v_{i+1}$ are in A. The length of that path $(x=v_0,v_1,v_2,\dots,v_m=y)$ is the sum of the weights

$\sum_{i=0}^{m-1} w(v_i \to v_{i+1})$. Construct $d\colon V \times V \to \mathbb{R}$ by setting $d(x,y)$ to be the length of the shortest path from x to y (and ∞ if no path exists). Since each arrow has nonnegative weight, the function d automatically satisfies (1) in Definition 2. By considering the concatenation of paths, we can easily see that d also automatically satisfies (4) in Definition 2. Thus, (V,d) must always be a quasipseudometric space.

More generally, we can consider any function $f\colon X \times X \to \mathbb{R}$ not necessarily satisfying any of the properties (1)–(4). It is in this most general setting that we will prove stability theorems.

The *Gromov–Hausdorff distance* between metric spaces (X, d_X) and (Y, d_Y) is often defined by

$$d_{\mathrm{GH}}(X,Y) = \inf_{Z,\, f\colon X\to Z,\, g\colon Y\to Z} d_{H,Z}(f(X), g(Y)),$$

where the infimum is taken over all metric spaces Z and isometric embeddings f and g to Z from X and Y, respectively, and $d_{H,Z}$ is the Hausdorff distance between subsets of Z. It is a standard result that the Gromov–Hausdorff distance is a metric on the space of compact metric spaces.

A useful alternative, but equivalent, formula for the Gromov–Hausdorff distance can be given through correspondences. The set $\mathcal{M} \subset X \times Y$ is a *correspondence* between X and Y if for all $x \in X$ there exists some $y \in Y$ with $(x,y) \in \mathcal{M}$ and for all $y \in Y$ there is some $x \in X$ with $(x,y) \in \mathcal{M}$. Using correspondences we can write

$$\begin{aligned} &d_{\mathrm{GH}}(X,Y) \\ &\quad = \frac{1}{2} \inf_{\mathcal{M} \text{ correspondence between } X \text{ and } Y} \ \sup_{(x_1,y_1),(x_2,y_2)\in\mathcal{M}} |d_X(x_1,x_2) - d_Y(y_1,y_2)|. \end{aligned}$$

More generally, given functions $f\colon X \times X \to \mathbb{R}$ and $g\colon Y \times Y \to \mathbb{R}$ we can call $\mathrm{dis}_{(X,f),(Y,g)}(\mathcal{M}) = \sup_{(x_1,y_1),(x_2,y_2)\in\mathcal{M}} |f(x_1,x_2) - g(y_1,y_2)|$ the *distortion* of the correspondence $\mathcal{M}$. We can then define the correspondence distortion distance by minimising this correspondence distortion.

Definition 3 For set–function pairs $(X,\ f\colon X \times X \to \mathbb{R})$ and $(Y,\ g\colon Y \times Y \to \mathbb{R})$ the *correspondence distance* between them can be defined as

$$\begin{aligned} &d_{\mathrm{CD}}((X,f),(Y,g)) \\ &\qquad = \frac{1}{2} \inf_{\mathcal{M} \text{ correspondence between } X \text{ and } Y} \mathrm{dis}_{(X,f),(Y,g)}(\mathcal{M}) \\ &\qquad = \frac{1}{2} \inf_{\mathcal{M} \text{ correspondence between } X \text{ and } Y} \ \sup_{(x_1,y_1),(x_2,y_2)\in\mathcal{M}} |f(x_1,x_2) - g(y_1,y_2)|. \end{aligned}$$

This agrees with the standard definition for the Gromov–Hausdorff distance when (X, d_X) and (Y, d_Y) are metric spaces. It is straightforward to verify that d_{CD} is a pseudometric on the space of all set–function pairs and a metric on the space of finite quasimetric spaces. The proofs are analogous to that for metric spaces discussed in [1].

3 Background: persistence modules

In this section we will cover some background theory on persistence modules and the interleaving distance between persistence modules. This is important because the interleaving distance between persistence modules bounds the bottleneck distance between their corresponding persistence diagrams. To introduce and motivate the concepts we will provide a brief summary of the theory of persistent homology. We will omit most of the details as we will be phrasing all results in later sections in terms of persistence modules. For more details about the history and applications of persistent homology we refer the reader to [19; 10; 9; 2].

Persistent homology describes how the homology groups evolve over an increasing family of topological spaces. Throughout this section let $K = \{K_t\}$ denote a family of reasonable topological spaces such that $K_s \subset K_t$ whenever $s \leq t$. Given $s \leq t$ the k^{th}*–dimensional persistent homology group* for K from s to t consists of the k^{th}–dimensional homology classes in K_s that "persist" until K_t, that is, $Z_k(K_s)/(Z_k(K_t) \cup B_k(K_s))$. This is isomorphic to the image of the induced map on homology $\iota_*\colon H_k(K_s) \to H_k(K_t)$ from the inclusion $K_s \subset K_t$.

Barcodes and persistence diagrams were introduced as discrete summaries of persistent homology information. Each barcode consists of a multiset of real intervals called bars. The barcode corresponding to the k^{th}–dimensional persistent homology of K is $\{I_1, I_2, \dots, I_n\}$ if, for all $s \leq t$, the dimension of $\mathrm{im}(\iota_*\colon H_k(K_s) \to H_k(K_t))$ equals the number of bars in $\{I_1, I_2, \dots, I_n\}$ that contain $[s, t)$. The corresponding persistence diagram is the multiset $\{(a_i, b_i)\}$ of points in $\mathbb{R}^2$, where a_i and b_i are the endpoints of the bar I_i, alongside infinitely many copies of every point along the diagonal (these diagonal points are acting the role of empty intervals).

Barcodes and persistence diagrams have played a prominent role in applied topology as topological summaries of data. In particular, they can provide insight into the "shape" of point cloud data through the persistent homology of the Rips filtration over that point cloud. Much of the power behind the use of barcodes and persistence diagrams comes from stability theorems, such as the stability theorem for the persistent homology of the Rips filtration over a finite metric space.

Persistence, such as persistent homology of a filtration of simplicial complexes, can be defined directly at an algebraic level. In [21], Zomorodian and Carlsson introduced the concept of a persistence module and proved that barcodes (and equivalently persistence diagrams) can be defined for persistence modules satisfying reasonable finiteness conditions. It was shown in [3] that we can define a distance between persistence modules (called the interleaving distance) and that the interleaving distance between persistence modules is a bound on the bottleneck distance of their corresponding persistence diagrams. Throughout this paper we will work directly with persistence modules.

Definition 4 Let R be a commutative ring with unity. A *persistence module* over $A \subset \mathbb{R}$ is a family $\{P_t\}_{t\in A}$ of R–modules indexed by real numbers, together with a family of homomorphism $\{\iota_t^s\colon P_t \to P_s\}$ such that $\iota_t^r = \iota_s^r \circ \iota_t^s$ for all $t \le s \le r$ and $\iota_t^t = \operatorname{id} P_t$.

If R is a field then the P_t are all vector fields and the ι_t^s are linear maps. As is standard in topological data analysis, we will assume throughout that R is the fixed field F (usually taken to be $\mathbb{F}_2$ for computational reasons). In the theory of persistence modules there are technical requirements about tameness. We say $\mathcal{P}$ is *tame* if $\operatorname{rank} \iota_t^s$ is always finite for any $s < t$. A sufficient condition for tameness is that X is finite, which is almost always true in any application. It is less straightforward in the constructions involving asymmetry to provide other nice sufficient conditions which would ensure the resulting persistence modules are tame (see the future directions). When the persistence modules are tame, the interleaving results will immediately imply a stability theorem for the persistence diagrams/barcodes.

The space of persistence modules is a pseudometric space under the interleaving distance function. Here we will define the interleaving distance between two persistence modules as the infimum of $\epsilon > 0$ such that they are ϵ–interleaved. In this we slightly differ from [3], where they define both strongly and weakly ϵ–interleaved, both of which are weaker than our notion of interleaving. More details about the pseudometric space structure of persistence modules and how the interleaving distance between persistence modules relates to the distances between corresponding persistence diagrams can be found in [3; 10; 21].

Definition 5 Two persistence modules $\mathcal{P}^X$ and $\mathcal{P}^Y$ are *ϵ–interleaved* if there exist families of homomorphisms $\{\alpha_t\colon P_t^X \to P_{t+\epsilon}^Y\}_{t\in\mathbb{R}}$ and $\{\beta_t\colon P_t^Y \to P_{t+\epsilon}^X\}_{t\in\mathbb{R}}$ such that the following diagrams commute:

(3-1)
$$\begin{tikzcd}
P_t^X \arrow[r, "\iota"] \arrow[dr, "\alpha_t"] & P_{t'}^X \arrow[dr, "\alpha'_t"] & \\
& P_{t+\epsilon}^Y \arrow[r, "\iota"'] & P_{t'+\epsilon}^Y
\end{tikzcd}
\qquad
\begin{tikzcd}
& P_{t+\epsilon}^X \arrow[r, "\iota"] & P_{t'+\epsilon}^X \\
P_t^Y \arrow[ur, "\beta_t"] \arrow[r, "\iota"'] & P_{t'+\epsilon}^Y \arrow[ur, "\beta'_t"] &
\end{tikzcd}$$

(3-2)
$$\begin{tikzcd}
P_t^X \arrow[rr, "\iota"] \arrow[dr, "\alpha_t"] & & P_{t+2\epsilon}^X \\
& P_{t+\epsilon}^Y \arrow[ur, "\beta_{t+\epsilon}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& P_{t+\epsilon}^X \arrow[dr, "\alpha_{t+\epsilon}"] & \\
P_t^Y \arrow[ur, "\beta_t"] \arrow[rr, "\iota"'] & & P_{t+2\epsilon}^Y
\end{tikzcd}$$

Definition 6 Two persistence modules $\mathcal{P}^X$ and $\mathcal{P}^Y$ are *isomorphic* if they are 0–interleaved.

The diagrams in (3-1) and (3-2) are slightly different from those given in [3] but the diagrams here commuting will imply that theirs also commute.

If $\mathcal{P}^X$ and $\mathcal{P}^Y$ are ϵ_1–interleaved and $\mathcal{P}^Y$ and $\mathcal{P}^Z$ are ϵ_2–interleaved then composing homomorphisms shows that $\mathcal{P}^X$ and $\mathcal{P}^Z$ are $(\epsilon_1+\epsilon_2)$–interleaved. We can define a pseudodistance on the space of persistence modules, called the *interleaving distance*, where the interleaving distance between $\mathcal{P}^X$ and $\mathcal{P}^Y$ is the infimum of the set of $\epsilon > 0$ such that $\mathcal{P}^X$ and $\mathcal{P}^Y$ are ϵ–interleaved. It is worth noting that two persistence modules might have interleaving distance 0 and yet not be 0–interleaved (and thus not isomorphic).

4 Existing stability results and Rips filtrations constructed from related symmetric functions

In this section we will recall the definition for the Rips filtration of a metric space and more generally for sublevel sets of a symmetric function $f\colon X\times X\to\mathbb{R}$. We will also recall the existing stability results for their persistent homology. Given a function $f\colon X\times X\to\mathbb{R}$ we construct a family of related symmetric functions $\mathrm{sym}_a(f)$ (for $a\in[0,1]$). We show that the persistent homology constructed from the $\mathrm{sym}_a(f)$ is stable as a corollary of the stability results for symmetric functions under the correspondence distortion distance.

Definition 7 Given a set X and a symmetric function $f\colon X\times X\to\mathbb{R}$, the *Rips filtration* of (X,f) is a family of finite simplicial complexes $\mathcal{R}(X,f)=\{\mathcal{R}(X,f)_t\}_{t\geq 0}$ with $\mathcal{R}(X,f)_t$ the clique complex on the graph with vertices $X_t=\{x\in X: f(x,x)\leq t\}$ and edges $\{[x_1,x_2]\in X_t\times X_t: f(x_1,x_2)\leq t\}$.[1]

Theorem 8 *Let $f\colon X\times X\to\mathbb{R}$ and $g\colon Y\times Y\to\mathbb{R}$ be symmetric functions and $\mathcal{R}(X,f)$ and $\mathcal{R}(Y,g)$ their corresponding Rips filtrations. If $d_{\mathrm{CD}}((X,f),(Y,g))$ is finite then for all $\epsilon>d_{\mathrm{CD}}((X,f),(Y,g))$, the k^{th} homology persistence modules of $\mathcal{R}(X,f)$ and $\mathcal{R}(Y,g)$ are ϵ–interleaved. In particular, when (X,d_X) and (Y,d_Y) are compact metric spaces, $\mathcal{R}(X,d_X)$ and $\mathcal{R}(Y,d_Y)$ are ϵ–interleaved for all $\epsilon>2d_{\mathrm{GH}}(X,Y)$.*

Since the only condition required is symmetry of the filtration function, one approach for analysing general functions $f\colon X\times X\to\mathbb{R}$ is to construct related symmetric functions. We will consider a one-parameter family of possible symmetric filtrations. We then prove stability for the Rips filtrations of these symmetric constructions in terms of the correspondence distortion distance between the original set–function pairs.

Definition 9 Let (X,f) be a finite set $X=\{x_1,\dots,x_N\}$ equipped with function $f\colon X\times X\to\mathbb{R}$. For any $a\in[0,1]$ we can define a symmetric function

$$\begin{aligned}&\mathrm{sym}_a(f)\colon X\times X\to\mathbb{R},\\&\qquad\qquad(x,y)\mapsto a\min\{f(x,y),f(y,x)\}+(1-a)\max\{f(x,y),f(y,x)\}.\end{aligned}$$

Since $\mathrm{sym}_a(f)$ is symmetric we can construct its Rips filtration $\{\mathcal{R}(X,\mathrm{sym}_a(f))_t\}$, where $\mathcal{R}(X,\mathrm{sym}_a(f))_t$ is the simplicial complex containing $[x_0,x_1,\dots,x_p]$ whenever $\mathrm{sym}_a(f)(x_i,x_j)\leq t$ for all i and j. We call this the *Rips filtration under* sym_a. If f is a symmetric function then $\mathrm{sym}_a(f)=f$ for all a, which implies that the Rips filtration under sym_a generalises the symmetric Rips filtration.

As a corollary of the stability for symmetric functions we have stability for the symmetrised functions.

Corollary 10 *Fix $a\in[0,1]$ and a homology dimension k. Let (X,f) and (Y,g) be set–function pairs such that $d_{\mathrm{CD}}((X,f),(Y,g))$ is finite. Let P^X and P^Y be*

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[1] Readers need to be warned that sometimes the Rips filtration is defined by adding the edge $[x_1,x_2]$ when $d_X(x_1,x_2)\leq\frac{1}{2}t$ instead of $d_X(x_1,x_2)\leq t$, so sometimes results may differ from here by a corresponding factor of 2.

the corresponding k^{th}–dimensional homology persistence modules constructed from the corresponding Rips filtrations under sym_a *(*$\mathcal{R}(X, \mathrm{sym}_a(f))$ *and* $\mathcal{R}(Y, \mathrm{sym}_a(g))$*, respectively). Then* $d_{\mathrm{int}}(P^X, P^Y) \leq 2d_{\mathrm{CD}}\big((X, \mathrm{sym}_a(f)), (Y, \mathrm{sym}_a(g))\big)$*.*

Unfortunately this method of constructing Rips filtrations is somewhat crude. We can show that in the process of symmetrising we dampen dissimilarities. This is not surprising as the space of symmetric functions is much smaller than that of functions generally. In particular, we will show in Theorem 12 that $d_{\mathrm{CD}}\big((X, \mathrm{sym}_a(f)), (Y, \mathrm{sym}_a(g))\big) \leq 2d_{\mathrm{CD}}((X, f), (Y, g))$ for all $a \in [0, 1]$. There are many examples where this inequality is strict. For asymmetric functions, $d_{\mathrm{CD}}\big((X, \mathrm{sym}_a(f)), (Y, \mathrm{sym}_a(g))\big)$ is often significantly smaller than $2d_{\mathrm{CD}}((X, f), (Y, g))$. Suppose $X = Y$, $f\colon X \times X \to \mathbb{R}$ is an antisymmetric function and $g = -f$. Then, by construction, $\mathrm{sym}_a(f) = \mathrm{sym}_a(g)$ for all a but for nonzero f, we generally have $d_{\mathrm{CD}}((X, f), (X, -f)) > 0$.

The dampening process through symmetrisation is encapsulated in the following lemma:

Lemma 11 *Let* $w, \widehat{w}, z, \widehat{z} \in \mathbb{R}$. *Then*

(i) $|\max\{w, \widehat{w}\} - \max\{z, \widehat{z}\}| \leq \max\{|w - z|, |\widehat{w} - \widehat{z}|\}$,

(ii) $|\min\{w, \widehat{w}\} - \min\{z, \widehat{z}\}| \leq \max\{|w - z|, |\widehat{w} - \widehat{z}|\}$.

Proof We can prove (i) through a series of cases. If $w \leq \widehat{w}$ and $z \leq \widehat{z}$ then $|\max\{w, \widehat{w}\} - \max\{z, \widehat{z}\}| = |\widehat{w} - \widehat{z}|$. If $w \geq \widehat{w}$ and $z \geq \widehat{z}$ then $|\max\{w, \widehat{w}\} - \max\{z, \widehat{z}\}| = |w - z|$.

If $w \leq \widehat{w}$ and $z \geq \widehat{z}$, then

$$\begin{aligned} |\max\{w, \widehat{w}\} - \max\{z, \widehat{z}\}| = |\widehat{w} - z| &\leq \begin{cases} |\widehat{w} - \widehat{z}| & \text{if } \widehat{z} \leq w, \\ |w - z| & \text{if } \widehat{z} \geq w \end{cases} \\ &\leq \max\{|w - z|, |\widehat{w} - \widehat{z}|\}. \end{aligned}$$

Reversing the roles of the letters, we also see that

$$|\max\{w, \widehat{w}\} - \max\{z, \widehat{z}\}| \leq \max\{|w - z|, |\widehat{w} - \widehat{z}|\}$$

whenever $w \geq \widehat{w}$ and $z \leq \widehat{z}$

We can infer (ii) from (i) by replacing each of w, $\widehat{w}$, z and $\widehat{z}$ by their negatives. □

Theorem 12 *Fix* $a \in [0, 1]$ *and a homology dimension* k. *Let* (X, f) *and* (Y, g) *be set–function pairs. Then* $d_{\mathrm{CD}}\big((X, \mathrm{sym}_a(f)), (Y, \mathrm{sym}_a(g))\big) \leq 2d_{\mathrm{CD}}((X, f), (Y, g))$.

Proof It is sufficient to show that $\mathrm{dis}_{(X,\mathrm{sym}_a(f)),(Y,\mathrm{sym}_a(g))}(\mathcal{M}) \leq \mathrm{dis}_{(X,f),(Y,g)}(\mathcal{M})$ for every correspondence $\mathcal{M}$.

Fix some correspondence $\mathcal{M} \subset X \times Y$ and let $(x_1, y_1), (x_2, y_2) \in \mathcal{M}$. From Lemma 11 (using $w = f(x_1, x_2)$, $\widehat{w} = f(x_2, x_1)$, $z = g(y_1, y_2)$ and $\widehat{z} = g(y_2, y_1)$) we know that both

$$|\min\{f(x_1, x_2), f(x_2, x_1)\} - \min\{g(y_1, y_2), g(y_2, y_1)\}| \\ \le \max\{|f(x_1, x_2) - g(y_1, y_2)|, |f(x_2, x_1) - g(y_2, y_1)|\}$$

and

$$|\max\{f(x_1, x_2), f(x_2, x_1)\} - \max\{g(y_1, y_2), g(y_2, y_1)\}| \\ \le \max\{|f(x_1, x_2) - g(y_1, y_2)|, |f(x_2, x_1) - g(y_2, y_1)|\}.$$

Taking a convex combination of these equations tells us that

$$|\mathrm{sym}_a(f)(x, \hat{x}) - \mathrm{sym}_a(g)(y, \hat{y})| \\ \le \max\{|f(x, \hat{x}) - g(y, \hat{y})|, |f(\hat{x}, x) - g(\hat{y}, y)|\}. \tag{4-1}$$

By taking the supremum on both sides over all pairs $\{(x, y), (\hat{x}, \hat{y})\} \in \mathcal{M}$ we see that

$$\mathrm{dis}_{(X, \mathrm{sym}_a(f)), (Y, \mathrm{sym}_a(g))}(\mathcal{M}) \le \mathrm{dis}_{(X, f), (Y, g)}(\mathcal{M}). \qquad \square$$

5 Persistent homology of OT complexes

Ordered tuple complexes are an alternative to simplicial complexes. We will find them useful as they have more flexibility with regard to order; we can have asymmetric roles within the same tuple.

Definition 13 An *ordered tuple* is a sequence of $(v_0, v_1, v_2, \dots, v_n)$, potentially including repeats. A *ordered tuple complex* (shortened to *OT complex*) is a collection K of ordered tuples such that if $(v_0, v_1, v_2, \dots, v_n) \in K$ then $(v_0, \dots, \widehat{v}_i, \dots, v_n) \in K$ for all i (where $(v_0, \dots, \widehat{v}_i, \dots, v_n)$ is the ordered tuple with v_i removed).

It is worth emphasising that each ordered tuple is determined by the ordered sequence and not just the underlying vertices; (v_1, v_2, v_3) and (v_3, v_1, v_2) are distinct and not even linearly related.

The ideas of homology and persistent homology naturally extend to OT complexes. Throughout $\mathbb{F}$ will be a fixed field.

Definition 14 Given an OT complex K we can build a chain complex $C_*(K)$ where $C_p(K)$ is the set of all the $\mathbb{F}$–linear combinations of the ordered tuples in K with length $p+1$. This is an $\mathbb{F}$–vector space whose basis vectors are the ordered tuples in K of length $p+1$. We define a boundary map $\partial_p\colon C_p(K) \to C_{p-1}(K)$ by

$$\partial_p((v_0, v_1, v_2, \dots, v_p)) = \sum_{i=0}^{k} (-1)^i (v_0, \dots, \widehat{v}_i, \dots, v_p)$$

and extending linearly. We define the k^{th} homology group of the OT complex K as $H_k(K) = \ker(\partial_{k-1})/\operatorname{im}(\partial_k)$.

When $K_1 \subset K_2$ are both OT complexes, the inclusion of chains induces a map on their homology groups, $\iota_*\colon H_*(K_1) \to H_*(K_2)$.

Definition 15 We say $\mathcal{K} = \{K_t\}$ is filtration of OT complexes if $K_t \subset K_r$ whenever $t \leq r$. We define the *k^{th}–dimensional ordered tuple persistence module* corresponding to $\mathcal{K}$ as follows:

- For each t set the vector space $V_t = H_k(K)$ computed over $\mathbb{F}$.
- For each pair $s \leq t$ we have a linear map induced from inclusion,

$$\iota_{t \to s}\colon H_*(K_s) \to H_*(K_t).$$

It is easy to check that this does satisfy the requirements of a persistence module.

We can define the directed Rips filtration as a filtration of OT complexes where the condition for when an ordered tuple is included is dependent on the order in which the points in the tuple appear. From this filtration of OT complexes we can construct directed Rips persistence modules.

Definition 16 Let (X, f) be a set–function pair. Set $\{\mathcal{R}^{\text{dir}}(X, f)_t\}$ to be the filtration of OT complexes where $(v_0, v_1, \dots, v_p) \in \mathcal{R}^{\text{dir}}(X, f)_t$ when $f(v_i, v_j) \leq t$ for all $i \leq j$. We call $\{\mathcal{R}^{\text{dir}}(X, f)_t\}$ the *directed Rips filtration* of (X, f). For each dimension k, we will define the *k^{th}–dimensional directed Rips persistence module* as the k^{th}–dimensional ordered tuple persistence module of $\{\mathcal{R}^{\text{dir}}(X, f)_t\}$.

We claim that these directed Rips persistence modules are a generalisation of the Rips persistence modules constructed from symmetric functions. To do this we need to recall some classical relationships between the homology of OT complexes and simplicial complexes. Indeed, a common first example of an OT complex is via a simplicial complex. For a simplicial complex K there is an OT complex K^{OT} where

$(v_0, v_1, \dots, v_p) \in K^{\mathrm{OT}}$ whenever $[v_0, v_1 \dots, v_p]$, after removing any repeats, is a simplex in K. In [17], Munkres calls the chain complex $C_*(K^{\mathrm{OT}})$ the *ordered chain complex of* K, and shows that the simplicial homology of K and the OT complex homology of K^{OT} are isomorphic. This isomorphism result holds also for persistence modules of filtrations of simplicial complexes as the isomorphisms on homology groups commute with the induced maps on homology by inclusions. This implies that if $f\colon X \times X \to \mathbb{R}$ is a symmetric function then the k^{th}–dimensional ordered tuple persistence module of $\{\mathcal{R}^{\mathrm{dir}}(X, f)_t\}$ is isomorphic to the k^{th}–dimensional persistence module of $\{\mathcal{R}(X, f)_t\}$.

5.1 Stability of the directed Rips persistence modules

We will want to prove that the directed Rips persistence modules enjoy stability with respect to the correspondence distortion distance. To do this we will compare set–function pairs over different sets via their induced set–function pairs over a common set constructed via a fixed correspondence.

Given functions $f\colon X \times X \to \mathbb{R}$ and $g\colon Y \times Y \to \mathbb{R}$ along with a correspondence $\mathcal{M} \subset X \times Y$, we can pull back the functions f and g to corresponding functions on $\mathcal{M} \times \mathcal{M}$ via the projection maps

$$f^{\mathcal{M}}\colon \mathcal{M} \times \mathcal{M} \to \mathbb{R}, \quad (x_1, y_1) \times (x_2, y_2) \mapsto f(x_1, x_2),$$

$$g^{\mathcal{M}}\colon \mathcal{M} \times \mathcal{M} \to \mathbb{R}, \quad (x_1, y_1) \times (x_2, y_2) \mapsto g(y_1, y_2).$$

The proof of the following lemma follows directly from the definitions of $f^{\mathcal{M}}$ and $g^{\mathcal{M}}$.

Lemma 17 *Let (X, f) and (Y, g) be set–function pairs and $\mathcal{M} \subset X \times Y$ a correspondence. Then*

$$\|f^{\mathcal{M}} - g^{\mathcal{M}}\|_\infty = 2\,\mathrm{dis}_{(X,f),(Y,g)}(\mathcal{M}).$$

We will also need to prove that the directed Rips persistence modules over (X, f) and $(\mathcal{M}, f^{\mathcal{M}})$ are isomorphic. To do this we will introduce the notion of the expansion of an OT complex.

Definition 18 Let K be an OT complex. We say that K is *closed under adjacent repeats* if whenever $(v_0, v_1, \dots, v_p) \in C_p(K)$ then $(v_0, \dots, v_i, v_i, \dots, v_p) \in C_{p+1}(K)$ for all $i = 0, 1, \dots, p$.

It is worth observing that, by construction, $\{\mathcal{R}^{\mathrm{dir}}(X, f)_t\}$ is closed under adjacent repeats for any set–function pair (X, f).

Definition 19 Let K and $\widetilde{K}$ be OT complexes, both closed under adjacent repeats, over vertex sets V and $\widetilde{V}$, respectively. We say that $\widetilde{K}$ is an *expansion* of K if there exists a surjective map $\pi\colon \widetilde{V} \to V$ and an injective map $\iota\colon V \to \widetilde{V}$ such that $\pi \circ \iota = \mathrm{id}_V$ and $(v_0, v_1, \dots, v_p) \in \widetilde{K}$ if and only if $(\pi(v_0), \pi(v_1), \dots, \pi(v_p)) \in K$.

Let $\mathcal{K} = \{K_t\}$ and $\widetilde{\mathcal{K}} = \{\widetilde{K}_t\}$ be filtrations of OT complexes over vertex sets V and $\widetilde{V}$, respectively. We say that $\widetilde{\mathcal{K}}$ is an *expansion* of $\mathcal{K}$ if there exists a surjective map $\pi\colon \widetilde{V} \to V$ and an injective map $\iota\colon V \to \widetilde{V}$ such that $\pi \circ \iota = \mathrm{id}_V$ and, for all t, $(v_0, v_1, \dots, v_p) \in \widetilde{K}_t$ if and only if $(\pi(v_0), \pi(v_1), \dots, \pi(v_p)) \in K_t$.

Proposition 20 *If $\mathcal{K} = \{K_t\}$ and $\widetilde{\mathcal{K}} = \{\widetilde{K}_t\}$ are filtrations of OT complexes such that $\widetilde{\mathcal{K}}$ is an expansion of $\mathcal{K}$ then the OT persistence modules of $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ are isomorphic.*

Proof Without loss of generality we can relabel the points in V to consider it as a subset of $\widetilde{V}$ (relabelling $v \in V$ as $\iota(v) \in \widetilde{V}$). In this case ι is the inclusion map and π is a projection map.

Both $\pi\colon \widetilde{K}_t \to K_t$ and $\iota\colon K_t \to \widetilde{K}_t$ induce chain maps, $\pi_\#\colon C_*(\widetilde{K}_t) \to C_*(K_t)$ and $\iota_\#\colon C_*(K_t) \to C_*(\widetilde{K}_t)$. Observe that $\pi_\# \circ \iota_\# = \mathrm{id}\colon C_*(K_t) \to C_*(K_t)$, so $\pi_* \circ \iota_* = \mathrm{id}\colon H_*(K_t) \to H_*(K_t)$ for all t.

Suppose $(v_0, v_1, \dots, v_i, \dots, v_p) \in C_p(\widetilde{K}_t)$. To construct a prism operator later we want to show that

$$(v_0, v_1, \dots, v_i, \pi(v_i), \dots, \pi(v_p)) \in C_{p+1}(\widetilde{K}_t).$$

To do this we use that $\widetilde{K}_t$ is closed under adjacent repeats, the definition of expansions (twice) and the property that π is a projection (so $\pi(\pi(v_j)) = \pi(v_j)$):

$$\begin{aligned}
&(v_0, v_1, \dots, v_i, \dots, v_p) \in C_p(\widetilde{K}_t) \\
&\qquad\qquad \Longrightarrow\ (v_0, v_1, \dots, v_i, v_i, \dots, v_p) \in C_{p+1}(\widetilde{K}) \\
&\qquad\qquad \Longrightarrow\ (\pi(v_0), \pi(v_1), \dots, \pi(v_i), \pi(v_i), \dots, \pi(v_p)) \in C_{p+1}(K_t) \\
&\qquad\qquad \Longrightarrow\ \big(\pi(v_0), \pi(v_1), \dots, \pi(v_i), \pi(\pi(v_i)), \dots, \pi(\pi(v_p))\big) \in C_{p+1}(K_t) \\
&\qquad\qquad \Longrightarrow\ (v_0, v_1, \dots, v_i, \pi(v_i), \dots, \pi(v_p)) \in C_{p+1}(\widetilde{K}_t).
\end{aligned}$$

Consider the prism operator

$$P((v_0, v_1, \dots, v_p)) = \sum_{i=0}^{p} (-1)^i ((v_0, v_1, \dots, v_i, \pi(v_i), \pi(v_{i+1}), \dots, \pi(v_p)).$$

Routine algebra shows that $\partial P + P\partial = i_\# \circ \pi_\# - \mathrm{id}$ and thus $i_\# \circ \pi_\#$ is chain homotopic to the identity. This implies $i_* \circ \pi_* \colon H_*(\widetilde{K}_t) \to H_*(\widetilde{K}_t)$ is the identity function.

The chain maps $\pi_\#$ and $i_\#$ commute with the inclusion maps for the filtrations of OT complexes and hence the following diagrams commute:

$$\begin{tikzcd} H_*(\widetilde{K}_s) \arrow[r, "\iota_*"] \arrow[d, "\pi_*"] & H_*(\widetilde{K}_t) \arrow[d, "\pi_*"] \\ H_*(K_s) \arrow[r, "\iota_*"] & H_*(K_t) \end{tikzcd} \qquad \begin{tikzcd} H_*(K_s) \arrow[r, "\iota_*"] \arrow[d, "i_*"] & H_*(K_t) \arrow[d, "i_*"] \\ H_*(\widetilde{K}_s) \arrow[r, "\iota_*"] & H_*(\widetilde{K}_t) \end{tikzcd}$$

Since $i_* \circ \pi_* = \mathrm{id} \colon H_*(\widetilde{K}_t) \to H_*(\widetilde{K}_t)$ and $\pi_* \circ i_* = \mathrm{id} \colon H_*(K_t) \to H_*(K_t)$ for all t we see that $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ are isomorphic. □

Theorem 21 *Let (X, f) and (Y, g) be set–function pairs such that $d_{\mathrm{CD}}((X, f), (Y, g))$ is finite. Let P^X and P^Y be the corresponding k^{th}–dimensional homology persistence modules constructed from the corresponding directed Rips filtrations $\{\mathcal{R}^{\mathrm{dir}}(X, f)_t\}$ and $\{\mathcal{R}^{\mathrm{dir}}(Y, g)_t\}$. Then $d_{\mathrm{int}}(P^X, P^Y) \le 2d_{\mathrm{CD}}((X, f), (Y, g))$.*

Proof Since $d_{\mathrm{CD}}((X, f), (Y, g))$ is finite, there exists some correspondence $\mathcal{M} \subset X \times Y$ with $\mathrm{dis}_{(X,f),(Y,g)}(\mathcal{M})$ finite. Fix a correspondence $\mathcal{M} \subset X \times Y$ with $\mathrm{dis}_{(X,f),(Y,g)}(\mathcal{M})$ finite. From this correspondence construct directed Rips filtrations $\{\mathcal{R}^{\mathrm{dir}}(\mathcal{M}, f^{\mathcal{M}})_t\}$ and $\{\mathcal{R}^{\mathrm{dir}}(\mathcal{M}, g^{\mathcal{M}})_t\}$ with corresponding k^{th}–dimensional persistence modules $P^{(X,\mathcal{M})}$ and $P^{(Y,\mathcal{M})}$.

By construction $\{\mathcal{R}^{\mathrm{dir}}(\mathcal{M}, f^{\mathcal{M}})_t\}$ is an expansion of $\{\mathcal{R}^{\mathrm{dir}}(X, f)_t\}$ and thus by Proposition 20 we know that the persistence modules P^X and $P^{(X,\mathcal{M})}$ are isomorphic. Similarly, we can also show that P^Y and $P^{(Y,\mathcal{M})}$ are isomorphic.

By Lemma 17 we know $\|f^{\mathcal{M}} - g^{\mathcal{M}}\|_\infty \le 2\,\mathrm{dis}_{(X,f),(Y,g)}(\mathcal{M})$. There is an inclusion

$$\mathcal{R}^{\mathrm{dir}}(\mathcal{M}, f^{\mathcal{M}})_t \subset \mathcal{R}^{\mathrm{dir}}(\mathcal{M}, g^{\mathcal{M}})_{t+2\,\mathrm{dis}_{(X,f),(Y,g)}(\mathcal{M})}$$

for all t as

$$\begin{aligned} &(v_0, v_1, \dots, v_n) \in \mathcal{R}^{\mathrm{dir}}(\mathcal{M}, f^{\mathcal{M}})_t \\ &\qquad \implies f^{\mathcal{M}}(v_i, v_j) \le t && \text{for all } i \le j \\ &\qquad \implies g^{\mathcal{M}}(v_i, v_j) \le t + \mathrm{dis}_{(X,f),(Y,g)}(\mathcal{M}) && \text{for all } i \le j \\ &\qquad \implies (v_0, v_1, \dots, v_n) \in \mathcal{R}^{\mathrm{dir}}(\mathcal{M}, g^{\mathcal{M}})_{t+2\,\mathrm{dis}_{(X,f),(Y,g)}(\mathcal{M})}. \end{aligned}$$

Symmetrically, there are also inclusions

$$\mathcal{R}^{\mathrm{dir}}(\mathcal{M}, g^{\mathcal{M}})_t \subset \mathcal{R}^{\mathrm{dir}}(\mathcal{M}, f^{\mathcal{M}})_{t+2\,\mathrm{dis}_{(X,f),(Y,g)}(\mathcal{M})}$$

for all t. These inclusion maps induce a $2\,\mathrm{dis}_{(X,f),(Y,g)}(\mathcal{M})$ interleaving between $P^{(X,\mathcal{M})}$ and $P^{(Y,\mathcal{M})}$. This implies that P^X and P^Y are $2\,\mathrm{dis}_{(X,f),(Y,g)}(\mathcal{M})$–interleaved.

By considering the infimum of the interleavings constructed by correspondences we see that $d_{\mathrm{int}}(P^X, P^Y)$ is at most $2d_{\mathrm{CD}}((X,f),(Y,g))$. □

5.2 Comparison to ordered-set persistent homology

It is possible to construct homology groups and persistence modules using ordered sets instead of ordered tuples. As a preemptive attempt to reduce confusion, this section will compare this ordered-tuple persistent homology to ordered-set persistent homology. In ordered-set homology we effectively restrict our chains to only contain ordered tuples where there are no repeats. We can still define homology, persistent homology and persistence modules. Furthermore, in some applications this may better reflect the connectivity structure (such as in the analysis of the blue brain project in [18]) but there are two important reasons why we are not considering ordered-set persistence modules as a generalisation of the Rips persistence modules. The first reason is that when we restrict to symmetric functions we do not get persistence modules isomorphic to the standard Rips persistence modules. The second reason is that these persistence modules are not stable with respect to the correspondence distortion distance.

For example, consider the set $X = \{x, y\}$ with the f the zero function. For $t < 0$, then, the corresponding ordered sets complexes are empty with trivial homology. The ordered tuple complexes and Rips simplicial complexes are also empty and have trivial homology. For $t \geq 0$, the corresponding ordered set complex consists of the ordered sets (x), (y), (x, y) and (y, x). It has nontrivial 1–dimensional homology. To see this first observe that $(x, y) + (y, x)$ is a cycle but the space of 2–chains is trivial, so there are no nontrivial 1–chain boundaries. In comparison, the Rips simplicial complex is $[x, y]$, which has no nontrivial 1–cycles. The ordered-tuple complex is more complicated but everything ends up cancelling each other. For example, this cycle of concern in the ordered-set homology, $(x, y) + (y, x)$, is a boundary in the setting of OT homology: $(x, y) + (y, x) = \partial((x, y, x) + (x, x, x))$.

To see that the ordered-set persistence modules are not stable with respect to the correspondence distortion distance, compare (X, f) in the example in the paragraph above to the single-point space $Z=\{z\}$ with function $g(z)=0$. The first-dimensional ordered set homology for Z is also trivial and so its first-dimensional persistence module is also trivial. The correspondence $\{(x,z),(y,z)\}\subset X\times Z$ has zero distortion but the ordered-set persistence modules are not ϵ–interleaved for any ϵ.

6 Persistence modules via strongly connected components and preorder homology

In this section we will consider constructions using an associated filtration of directed graphs or preorders. For each t we can define a directed graph $\{D(X)_t\}$ where $x\to y$ is included in $D(X)_t$ when $\max\{f(x,y),f(x,x),f(y,y)\}\leq t$. From a directed graph we can induce a natural preorder via the existence of paths. That is a preorder where $x\leq y$ if there is a path from x to y in $D(X)_t$. We will construct persistence modules using the strongly connected components of the graphs $D(X)_t$, which are also the equivalence classes of the associated preorders. We also consider persistence modules using ordered-tuple complexes constructed over preorders.

Let us first introduce the construction of directed graphs and preorders from set–function pairs.

Definition 22 Let X be a set with a binary relationship $\leq$. Consider the following potential properties of $(X,\leq)$:

(i) **Reflexive** $x\leq x$ for all $x\in X$.

(ii) **Antisymmetric** For all $x,y\in X$, if $x\leq y$ and $y\leq x$ then $x=y$.

(iii) **Transitive** For all $x,y,z\in X$, if $x\leq y$ and $y\leq z$ then $x\leq z$.

We say that $(X,\leq)$ is a *poset* is it satisfies (i), (ii) and (iii). We say $(X,\leq)$ is a *preorder* if it satisfies (i) and (iii).

There is a natural equivalence relation on X where $x\sim y$ when $x\leq y$ and $y\leq x$. If we quotient a preorder by this equivalence relation we are left with a poset.

One way to construct preorders is via directed graphs. Given a directed graph $G=(V,E)$ and vertices $x,y\in V$, we say there is a *path* from x to y when there is a finite sequence of vertices $x=x_0,x_1,\dots,x_n=y$ such that (x_i,x_{i+1}) is a directed edge. To create a preorder on V we declare that $x\leq y$ whenever there is a path from x to y. The strongly connected components of a directed graph are the equivalence classes of points where $v \sim w$ when there exists both a path from v to w and a path from w to v. Thus, we see that the equivalence classes of this poset are precisely the strongly connected components of the directed graph it was built from. Suppose we start with a directed graph and we consider the preorder defined by the existence of paths. If we quotient by the equivalence relation to get a poset, then on the directed graph level we are collecting the vertices into the strongly connected components and then we have directed edges between these strongly connected components if there is a path between them. This will create an acyclic directed graph.

We will first need to construct directed graphs from the sublevel sets of a set–function pair. From this we can consider filtrations of directed graphs and of preorders.

Definition 23 Given a set–function pair (X, f) there is a natural filtration of directed graphs $\{\mathcal{D}(X)_t : t \in [0, \infty)\}$ associated to X by setting $\mathcal{D}(X, f)_t$ to the directed graph with vertices $\{x \in X : f(x,x) \leq t\}$ and including the directed edge $x \to y$ whenever $\max\{f(x,x), f(y,y), f(x,y)\} \leq t$. We will call this the *associated filtration of directed graphs* of (X, f).

It is necessary for the inclusion rule for the directed edges to occur at the maximum of $\{f(x,x), f(y,y), f(x,y)\}$ (rather than at $f(x,y)$, which may occur earlier) to ensure that $D(X, f)_t$ will satisfy the closure conditions for a directed graph. In the case where $f = d$ is a quasimetric, $d(x,x) = 0 = d(y,y)$ and $d(x,y) \geq 0$ and so the edge from x to y is included at $t = d(x,y)$.

We define a filtration of preorders to be a parametrised family of preorders

$$\{(X_t, \leq_t) : t \in \mathbb{R}\}$$

such that for all $s \leq t$ we have $X_s \subset X_t$ and if $x, y \in X_s$ with $x \leq_s y$ then $x \leq_t y$. From a filtration of associated graphs for a set–function pair we can construct a natural filtration of preordered spaces as follows:

Definition 24 Let (X, f) be a set–function pair and let $\{D(X, f)_t\}$ be its associated filtration of directed graphs. For each $t \geq 0$ construct a preordered space $(X_t, \leq_t)$ with X_t the set of points in $D(X, f)_t$ and $x \leq_t y$ when there exists a path in $D(X, f)_t$ from x to y. We call this the *associated filtration of preorders*.

The following is a useful lemma for proving the interleaving results for the persistence modules constructed with strongly connected components or with preorder homology:

Lemma 25 *Let X and Y be sets and (X,f) and (Y,g) be set–function pairs with $d_{\mathrm{CD}}((X,f),(Y,g))$ finite. Let $\mathcal{D}(X,f)=\{D(X,f)_t\}$ and $\mathcal{D}(Y,g)=\{D(Y,g)_t\}$ be the associated filtrations of directed graphs. Let $\mathcal{M}\subset X\times Y$ be a correspondence with $\mathrm{dis}_{(X,f),(Y,g)}(\mathcal{M})$ finite.*

(i) *If $(x,y)\in\mathcal{M}$ and $x\in D(X,f)_t$ then $y\in D(Y,g)_{t+\mathrm{dis}(\mathcal{M})}$.*

(ii) *If $(x_1,x_2),(y_1,y_2)\in\mathcal{M}$ and there exists a directed path from x_1 to x_2 in $D(X,f)_t$ then there exists a directed path from y_1 to y_2 in $D(Y,g)_{t+\mathrm{dis}(\mathcal{M})}$.*

Proof (i) If $x\in D(X,f)_t$ then $f(x,x)\le t$. Since $(x,y)\in\mathcal{M}$, we know $g(y,y)\le t+\mathrm{dis}(\mathcal{M})$ and hence $y\in D(Y,g)_{t+\mathrm{dis}(\mathcal{M})}$.

(ii) Suppose that there is a path from x_1 to x_2 in $D(X,f)_t$. This means that there exists a sequence of points $(x_1=a_1,a_2,\dots,a_k=x_2)$ in X such that $f(a_i,a_{i+1})\le t$. There exists a sequence of points in Y, $y_1=b_1,b_2,\dots,b_k=y_2$, where $(a_i,b_i)\in\mathcal{M}$. By (i) we know each of the b_i lie in $D(Y,g)_{t+\mathrm{dis}_{(X,f),(Y,g)}(\mathcal{M})}$. Since each $(a_i,b_i)\in\mathcal{M}$, we have

$$|f(a_i,a_{i+1})-g(b_i,b_{i+1})|\le\mathrm{dis}_{(X,f),(Y,g)}(\mathcal{M})$$

for each i and hence $(y_1=b_1,b_2,\dots,b_k=y_2)$ is a path in $D(Y,g)_{t+\mathrm{dis}_{(X,f),(Y,g)}(\mathcal{M})}$. □

The lemma can be rewritten in terms of preorders; for $(x_1,y_1),(x_2,y_2)\in\mathcal{M}$, if $x_1\le_t^f x_2$ then $y_1\le^g_{t+\mathrm{dis}_{(X,f),(Y,g)}(\mathcal{M})} y_2$.

6.1 Strongly connected components persistence

Dimension 0 persistent homology is all about tracking the evolution of connected components. For directed graphs, unlike graphs, there is choice in how to interpret what a connected component is, with each interpretation providing their own corresponding persistence module. Here we will consider the persistence of weakly and strongly connected components.

Weakly connected components are the components of the graph when the directions are forgotten. Given a filtration of a directed graph by edge weights, the weakly connected persistence would be the same as the dimension 0 persistent homology of the Rips filtration under sym_1 in Section 4, and to the dimension 0 directed Rips persistence module in Section 5.

Studying strongly connected components will provide new information. Recall the strongly connected components of a directed graph are the equivalence classes of points where $v \sim w$ when there exists both a path from v to w and a path from w to v. Given a filtration of directed graphs we can construct a persistence module based on linear combinations of strongly connected components (analogous to dimension 0 homology being interpreted as the space of formal linear combinations of connected components).

Definition 26 We call $\mathcal{D} = \{D_t : t \in \mathbb{R}\}$ a filtration of directed graphs if D_t is directed graph for all t such that if $s \leq t$ then D_s is a directed subgraph of D_t. Given a filtration of directed graphs $\mathcal{D} = \{D_t\}$, let $[v]_t$ denote the strongly connected component of D_t containing v. We define the *strongly connected persistence module* corresponding to $\mathcal{D}$ as follows:

- For each $t \in \mathbb{R}$ set the vector space V_t to be the vector space of finite linear combinations of strongly connected components $\big($that is, elements are of the form $\sum_{i=1}^{k} \lambda_i [v_i]_t$ with $\lambda_i \in \mathbb{F}\big)$.
- For each pair $t \leq s$ we have a linear map induced from inclusion,
$$\iota_{t\to s}\left(\sum_{i=1}^{k} \lambda_i [v_i]_t\right) = \sum_{i=1}^{k} \lambda_i [v_i]_s.$$

We will now check that the strongly connected component persistence module does satisfy the requirements of a persistence module. Whenever we have an inclusion of directed graphs $D_t \subset D_s$, whenever there is a path from v to w in D_t, there is also a path from v to w in D_s. This implies that the maps $\iota_{t\to s}$ are well defined. Furthermore, for $u \leq t \leq s$ we have $\iota_{t\to s}\big(\iota_{u\to t}\big(\sum_{i=1}^{k} \lambda_i [v_i]_u\big)\big) = \sum_{i=1}^{k} \lambda_i [v_i]_s = \iota_{u\to s}\big(\sum_{i=1}^{k} \lambda_i [v_i]_u\big)$. Whenever the directed graphs D_t are all finite (which is true in almost any application) we automatically know that the V_t are all finite-dimensional and hence the strongly connected persistence module is tame.

We can create strongly connected persistence modules from set–function pairs via its associated filtration of directed graphs.

Theorem 27 *Let X and Y be sets and (X, f) and (Y, g) be set–function pairs with $d_{\mathrm{CD}}((X, f), (Y, g))$ finite. Let $\mathcal{D}(X, f) = \{D(X, f)_t\}$ and $\mathcal{D}(Y, g) = \{D(Y, g)_t\}$ be the associated filtrations of directed graphs. Let $\mathcal{P}^X$ and $\mathcal{P}^Y$ be the strongly connected component persistence modules for $\mathcal{D}(X, f)$ and $\mathcal{D}(Y, g)$. Then $d_{\mathrm{int}}(\mathcal{P}^X, \mathcal{P}^Y) \leq d_{\mathrm{CD}}((X, f), (Y, g))$.*

Proof Fix a correspondence $\mathcal{M} \subset X \times Y$ with $\operatorname{dis}_{(X,f),(Y,g)}(\mathcal{M})$ finite.

Construct a map $\alpha\colon X \to Y$ where for each x we arbitrarily fix a representative from $\{y \in Y : (x,y) \in \mathcal{M}\}$, and construct a map $\beta\colon Y \to X$ where for each y we arbitrarily fix a representative from $\{x \in X : (x,y) \in \mathcal{M}\}$.

If $[x_1]_t = [x_2]_t$ then there exist paths in $D(X,f)_t$ from x_1 to x_2 and from x_2 to x_1. By Lemma 25 there exist paths in $D(Y,g)_{t+\operatorname{dis}_{(X,f),(Y,g)}(\mathcal{M})}$ from $\alpha(x_1)$ to $\alpha(x_2)$ and from $\alpha(x_2)$ to $\alpha(x_1)$. This means that α induces a well-defined linear map

$$\alpha_*\colon P^X_t \to P^Y_{t+\operatorname{dis}_{(X,f),(Y,g)}(\mathcal{M})}, \quad [x]_t \mapsto [\alpha(x)]_{t+2\operatorname{dis}_{(X,f),(Y,g)}(\mathcal{M})}.$$

Similarly, β induces a linear map $\beta_*\colon P^Y_t \to P^X_{t+2\operatorname{dis}_{(X,f),(Y,g)}(\mathcal{M})}$ where $[y]_t \mapsto [\beta(y)]_{t+2\operatorname{dis}_{(X,f),(Y,g)}(\mathcal{M})}$.

It only remains to show that α_* and β_* satisfy an $2\operatorname{dis}_{(X,f),(Y,g)}(\mathcal{M})$ interleaving. That (3-1) commutes follows directly from the construction of α and β.

Let $f(x,x) = t$, whence $x \in D(X,f)_t$. From our construction of α and β we know that $(x,\alpha(x))$ and $(\beta(\alpha(x)),\alpha(x))$ are both in $\mathcal{M}$. By Lemma 25 this implies that there are directed paths in both directions between $\beta(\alpha(x))$ and x in the directed graph $D(X,f)_{t+2\operatorname{dis}_{(X,f),(Y,g)}(\mathcal{M})}$, and hence they lie in the same strongly connected component.

Similarly, for every $y \in Y$ with $g(y,y) = t$, we know that $\alpha(\beta(y))$ and y lie in the same strongly connected component in $D(Y,g)_{t+2\operatorname{dis}_{(X,f),(Y,g)}(\mathcal{M})}$. This ensures that we satisfy (3-2).

By taking the infimum over all correspondences we see that the interleaving distance between P^X and P^Y is bounded above by $2d_{\mathrm{CD}}((X,f),(Y,g))$. □

We provide some pseudocode (the algorithm in the appendix) for an algorithm that computes the interval decomposition of the strongly connected component persistence module from a filtration of directed graphs. It is a modification of the union-find algorithm used to compute the standard dimension 0 persistent homology. In the union-find algorithm each connected component is represented by a root vertex with an additional data of its birth time. The main difference for strongly connected components is that we have to also keep track of when directed paths exist between the various strongly connected components. These are stored as a list of the root vertices of "in" and "out" connected components. Here "in" means a connected component that has a path pointing into the current component and "out" means there is a path pointing out of the current component. Note that for any root vertex these in and out sets are disjoint, as being in both would imply they are the same strongly connected component. The main challenge in this modification is to ensure that at each stage the list of in and out strongly connected components listed by the root vertices are referred to by their root vertex.

6.2 OT complexes constructed using the preorder structure

In the theory of partially ordered sets ("posets"), the order complex of a poset is the set of all finite chains. Its homology contains important information about the poset. Preorders are a generalisation of posets where we drop the antisymmetry condition. Poset homology naturally extends to preorders, where we will call it preorder homology. It is easier and more flexible to construct filtrations of preorders than of posets.

From the associated filtration of directed graphs of a set–function pair we can create a filtration of preorders which we will call the preorder Rips filtration. From the filtration of preorders we can construct persistence modules using preorder homology to generate preorder Rips persistence modules. These persistence modules enjoys stability with respect to the correspondence distortion distance. The homology dimension 0 preorder Rips persistence module is isomorphic to that of its weakly connected components, its directed Rips persistence module and the standard Rips persistence module under sym_1. If the input is a symmetric function then its higher-dimensional preorder Rips persistence modules are all trivial, showing that preorder Rips persistence module describes asymmetry information.

In this paper we will generalise to preorders some constructions normally defined for posets. The homology of a poset has been defined and studied via its corresponding Alexandrov topology. Preorders are in bijective correspondence with Alexandrov topologies, with the antisymmetry condition (which is the axiom that makes a preorder a poset) translating to those topologies that are T_0. For each preorder there is a canonical poset over its equivalence classes, and this poset corresponds to the Kolmogorov quotient of the Alexandrov topology of that original preorder. Because these quotient spaces are weakly homotopy equivalent, standard references for Alexandrov topology often state they will restrict their analysis to T_0 spaces/posets (eg [16; 15]). It is for this reason that definitions are usually only stated for posets and not more generally for preorders. In the appendix we will go into more detail into this background material and justify why the definitions given in this section are the natural generalisation of those traditionally given for posets.

Let us now construct an OT complex from a preorder.

Definition 28 Given a preorder $(X, \leq)$, let $\mathcal{O}(X, \leq)$ be the OT complex containing $(x_0, x_1, \ldots, x_p)$ when $x_0 \leq x_1 \leq \cdots \leq x_p$. We call $\mathcal{O}(X, \leq)$ the *preorder OT complex* of $(X, \leq)$.

Definition 29 Given a preorder $(X, \leq)$, its associated order complex $\Delta(X, \leq)$ is an abstract simplicial complex whose vertices are the elements of X and whose faces are the chains (subsets where each pair is comparable) of $(X, \leq)$.

From a filtration of preorders we can construct a filtration of OT complexes. From this, persistence modules can be constructed as standard with OT homology classes as the vector spaces and induced maps from inclusions as the transition maps.

Definition 30 Let $\mathcal{O}(X, f) = \{\mathcal{O}(X, f)_t\}$ be the filtration of OT complexes corresponding to the filtration of posets $\{(X_t, \leq_t)\}$. We call $\mathcal{O}(X, f)$ the *preorder filtration* of (X, f).

In the appendix we see that the simplicial homology of the order complex $\Delta(X, \leq)$ is naturally isomorphic to the homology of the preorder OT complex $\mathcal{O}(X, \leq)$. Moreover, isomorphisms between the simplicial homology of the order complexes and the homology of the preorder OT complexes will extend to persistent homology as they commute with the maps on homology induced by inclusions.

Definition 31 We define the k^{th}–dimensional *preorder persistence module* corresponding to the filtration of preorders $\mathcal{X} = \{(X_t, \leq_t)\}$ as the dimension k OT homology persistence module for the filtration of OT complexes $\{\mathcal{O}(X, \leq_t)\}_{t\in\mathbb{R}}$.

Just as in the previous constructions in this paper we can prove that the corresponding persistence modules built from functions $f\colon X \times X \to \mathbb{R}$ and $g\colon Y \times Y \to \mathbb{R}$ are stable with respect to the correspondence distortion distance.

Theorem 32 *Let (X, f) and (Y, g) be set–function pairs with preorder Rips filtrations $\mathcal{O}(X, f)$ and $\mathcal{O}(Y, g)$. Let $\mathcal{P}^X$ and $\mathcal{P}^Y$ be the k^{th}–dimensional persistence modules for $\mathcal{O}(X, f)$ and $\mathcal{O}(Y, g)$, respectively. Then $d_{\text{int}}(\mathcal{P}^X, \mathcal{P}^Y) \leq 2d_{\text{CD}}((X, f), (Y, g))$.*

Proof Since $d_{\mathrm{CD}}((X,f),(Y,g))$ is finite, there exists some correspondence $\mathcal{M} \subset X \times Y$ with $\mathrm{dis}_{(X,f),(Y,g)}(\mathcal{M})$ finite. Fix a correspondence $\mathcal{M} \subset X \times Y$ with $\mathrm{dis}_{(X,f),(Y,g)}(\mathcal{M})$ finite. From this correspondence construct preorder Rips filtrations $\{\mathcal{O}(\mathcal{M}, f^{\mathcal{M}})_t\}$ and $\{\mathcal{O}(\mathcal{M}, g^{\mathcal{M}})_t\}$ with corresponding k^{th}–dimensional persistence modules $P^{(X,\mathcal{M})}$ and $P^{(Y,\mathcal{M})}$.

By construction, $\{\mathcal{O}(\mathcal{M}, f^{\mathcal{M}})_t\}$ is an expansion of $\{\mathcal{O}(X,f)_t\}$ and thus by Proposition 20 we know that the persistence modules P^X and $P^{(X,\mathcal{M})}$ are isomorphic. Similarly we can also show that P^Y and $P^{(Y,\mathcal{M})}$ are isomorphic.

If $((x_0,y_0),(x_1,y_1),\dots,(x_n,y_n)) \in \mathcal{O}(\mathcal{M}, f^{\mathcal{M}})_t$ then $x_0,\dots,x_n \in D(X)_t$ and there exist directed paths from x_i to x_j in $D(X,f)_t$ for all $i \le j$. By Lemma 25 there must exist a directed path from y_i to y_j in $D(Y,g)_{t+\mathrm{dis}_{(X,f),(Y,g)}(\mathcal{M})}$ for all $i \le j$. This implies that $\mathcal{O}(\mathcal{M}, f^{\mathcal{M}})_t \subset \mathcal{O}(\mathcal{M}, g^{\mathcal{M}})_{t+\mathrm{dis}_{(X,f),(Y,g)}(\mathcal{M})}$ for all t. Similarly, $\mathcal{O}(\mathcal{M}, g^{\mathcal{M}})_t \subset \mathcal{O}(\mathcal{M}, f^{\mathcal{M}})_{t+\mathrm{dis}_{(X,f),(Y,g)}(\mathcal{M})}$.

These inclusion maps induce a $\mathrm{dis}_{(X,f),(Y,g)}(\mathcal{M})$ interleaving between $P^{(X,\mathcal{M})}$ and $P^{(Y,\mathcal{M})}$. This implies that P^X and P^Y are $\mathrm{dis}_{(X,f),(Y,g)}(\mathcal{M})$–interleaved.

By considering the infimum of the interleavings constructed by correspondences we see that the interleaving distance between P^X and P^Y is bounded above by $2d_{\mathrm{CD}}((X,f),(Y,g))$. □

As shown in Theorem 36 (in the appendix), the simplicial homology of the order complex is naturally isomorphic to the OT homology of $\mathcal{O}(X,\le)$. Furthermore, this isomorphism result holds also for persistence modules of filtrations of simplicial complexes as the isomorphisms on homology groups commute with the induced maps on homology by inclusions. This implies that interval decomposition of the k^{th} preorder persistence modules can be computed via the simplicial persistent homology over the filtration of simplicial complexes $\{\Delta(X_t, \le_t^f)\}$.

7 Future directions

There are many future directions related to the research in this paper. Examples include:

- Applying the constructions in this paper to quasimetric spaces to see what they reveal about their quasimetric structure, or to use as a method of getting a lower bound on the correspondence distortion distance between different quasimetric spaces.

- Adapting these methods to construct persistence modules for sublevel set filtrations of special functions on quasimetric spaces and proving related stability results. For example, we conjecture that all four constructions built from a suitably defined sublevel set of the extremity function of a quasimetric space (analogous to constructions in [4]) could have correspondence distortion distance stability with respect to the original quasimetric distance functions. This would provide another way of capturing the "shape" of a quasimetric space.
- Finding nice sufficient conditions on functions $f\colon X \times X \to \mathbb{R}$, with $|X|$ infinite, as to when these various Rips constructions create tame persistence modules. Even when restricting to the case of quasimetric spaces it is not even clear how we should define an ϵ–sampling or compactness. In the symmetric case, definitions have been used to describe sufficient conditions for metric spaces that result in tame persistence modules (such as in [5]).
- Algorithmic techniques for computing OT persistent homology efficiently. In particular, is there a related filtration of simplicial complexes that have isomorphic OT persistent homology, at least in low homology dimensions?

Appendix

A.1 Algorithm to compute interval decomposition of the strongly connected persistence module

INPUT: List L of vertices $V = \{v_1, v_2, \dots, v_n\}$ and directed edges

$$\{(v_{i_1} \to v_{j_1}), \dots, (v_{i_m} \to v_{j_m})\},$$

each with a real-valued height such that $h(v_i \to v_j) \geq \max\{h(v_i), h(v_j)\}$. These vertices and directed edges are ordered in a combined list L by increasing height values. All the vertices at a height value occur before the edges at that same height.

OUTPUT: Interval decomposition of the strongly connected component persistence module from filtration of sublevel sets of the height function

```
function FIND($x$)
    while root($x$) != $x$ do
        $x$ = root($x$)
    end while
    return $x$
end function
```

```
procedure UNION($v_{\text{tail}}$, $v_{\text{head}}$, height)
    $W = v_{\text{tail}}.\text{in} \cap v_{\text{head}}.\text{out}$
    $\widehat{w} :=$ earliest $w \in W$ to appear in list $L$
    for $w \in W$, $w \neq \widehat{w}$ do
        root$(w) = \widehat{w}$
        if $h(w) <$ height then
            append $[h(w), \text{height})$ to BARCODE
        end if
        $\widehat{w}$.in = {FIND($x$) for $x \in v_{\text{tail}}$.in}  ▷ An SCC has a path to $\widehat{w}$ if and only if it has a path to $v_{\text{tail}}$.
        $\widehat{w}$.out = {FIND($x$) for $x \in v_{\text{head}}$.out}  ▷ An SCC has a path from $\widehat{w}$ if and only if it has a path from $v_{\text{head}}$.
        for $x \in \widehat{w}$.in do
            $x$.out = {FIND(y) for $y \in x.\text{out} \cup \widehat{w}.\text{out}$}
        end for
        for $x \in \widehat{w}$.out do
            $x$.in = {FIND(y) for $y \in x.\text{in} \cup \widehat{w}.\text{in}$}
        end for
    end for
end procedure

procedure UPDATEINOUT($v_{\text{tail}}$, $v_{\text{head}}$, height)
    for $x \in v_{\text{tail}}$.in do
        $x$.out = {FIND(y) for $y \in v_{\text{head}}.\text{out} \cup x.\text{out}$}
    end for
    for $y \in v_{\text{head}}$.out do
        $y$.in = {FIND(x) for $x \in v_{\text{tail}}.\text{in} \cup y.\text{in}$}
    end for
end procedure

for $i = 1$ to length$(L)$ do
    if $L(i)$ is a vertex $v_k$ then
        Add a vertex to $A$. Label it with (height $= h(L(i))$, root $= v_k$, in $= \{v_k\}$, out $= \{v_k\}$).
```

```
        if $L(i)$ is a directed edge $v_j \to v_k$ then
            $v_{\text{tail}} :=$ FIND$(v_j)$, $v_{\text{head}} :=$ FIND$(v_k)$
            if $v_{\text{head}} \notin v_{\text{tail}}$.out then
                if $v_{\text{head}} \notin v_{\text{tail}}$.in then    ▷ We need to update the paths between SCCs.
                    UPDATEINOUT$(v_{\text{tail}}, v_{\text{head}})$
                end if
                if $v_{\text{head}} \in v_{\text{tail}}$.in then    ▷ This is when various SCCs merge.
                    UNION$(v_{\text{tail}}, v_{\text{head}}, h(L(i)))$
                end if
            end if
        end if
    end if
end for
RemainingComponents := {FIND$(x)$ for $x \in V$}    ▷ Final set of strongly connected components.
for $x \in$ RemainingComponents do
    Append $[h(x), \infty)$ to BARCODE
end for
```

A.2 Homology of a poset

There are multiple ways to compute the homology of a poset, including via Alexandrov topological spaces and order simplicial complexes. For each preorder there is a canonical poset we call its equivalence class poset. In this subsection we show that the definitions of homology of a poset can naturally be extended to preorders. Furthermore, the resulting homology of a preorder is naturally isomorphic to the homology of its equivalence class poset. This justifies the constructions in Section 6.2.

An *Alexandrov topology* is a topology in which the intersection of any family of open sets is open. It is an axiom of topology that the intersection of any finite family of open sets is open; in Alexandrov topologies the finiteness restriction is dropped. Given an Alexandrov topology we can construct a special preorder, called its specialisation preorder.

Definition 33 Let $X = (X, \tau)$ be an Alexandrov space. The *specialisation preorder* on X is the preorder where $x \leq y$ if and only if x is in the closure of $\{y\}$.

In the other direction, given a preorder $(X, \leq)$ there is a unique Alexandrov topology whose specialisation preorder is $(X, \leq)$. To construct this, let the open sets τ on X be the upper sets,

$$\tau = \{U \subset X : \forall x, y \in X \text{ if } x \leq y \text{ and } x \in U \text{ then } y \in U\}.$$

The corresponding closed sets for τ are the lower sets,

$$\{S \subseteq X : \forall x, y \in X \text{ if } x \in S \text{ and } y \leq x \text{ then } y \in S\}.$$

The topology τ is generated by the sets $U_x = \{y : x \leq y\}$.

A topological space X is a T_0 *space* if for any pair of points in X there exists an open set containing one and only one of them. It is an exercise to see how the antisymmetry condition of posets directly corresponds to the Alexandrov topologies that are T_0.

We can construct T_0 spaces by taking Kolmogorov quotients. The *Kolmogorov quotient* of a topological space is defined as its quotient by the equivalence relation of topological indistinguishability, equipped with the quotient topology.

There is a natural way of constructing a poset from a preorder by using quotients. For $(X, \leq)$ a preorder, define an equivalence relation $x \sim y$ when $x \leq y$ and $y \leq x$. Let $\widetilde{X} = X/{\sim}$ be the quotient space on these equivalence classes. It is easy to check that the binary relation $\leq$ is now well defined on $\widetilde{X}$ and that $(\widetilde{X}, \leq)$ is a poset. We will call $(\widetilde{X}, \leq)$ the *equivalence class poset* of $(X, \leq)$. The following lemma states the relationship between a preorder and its equivalence class poset is analogous to taking the Kolmogorov quotient of its Alexandrov topology. The proof for finite spaces is Lemmas 8 and 9 in [16], and the extension to infinite spaces can be proved similarly (see [13]).

Lemma 34 *Let $(X, \leq)$ be a finite preorder with equivalence class poset $(\widetilde{X}, \leq)$. The Alexandrov topology of $(\widetilde{X}, \leq)$ is the Kolmogorov quotient of the Alexandrov topology of $(X, \leq)$. Furthermore, the Alexandrov topologies of $(X, \leq)$ and $(\widetilde{X}, \leq)$ are homotopy equivalent.*

Since homology is defined up to weak homotopy equivalence, often in analysis researchers restrict their analysis from general topological spaces to T_0 spaces as they do not lose any homological information by taking the Kolmogorov quotient. Thus, many definitions are stated as for posets even though they could be defined for all preorders.

One definition of the homology of a poset is the singular homology of the Alexandrov topology which has that poset as its specialisation order. Since the specialisation orders of Alexandrov topologies provide a one-to-one correspondence between Alexandrov topologies and preorders, we can generalise this to define the homology of a preorder as the singular homology of the Alexandrov topology which has that preorder as its specialisation order.

A *chain* in a poset is defined as a subset of elements which are all pairwise comparable. Note that there is no order of the elements given as part of the information of the chain but that the transitivity of a preorder will ensure that there exists a total ordering of any chain. In a poset the antisymmetry condition ensures that this order is unique. In a general preorder multiple possible orders might be possible.

In a poset, the unique ordering of elements in a chain means we can define chain complexes and homology groups for a poset directly via chains. We thus say that an *m–chain* of a poset P is a totally ordered subset $c = (x_0 < x_1 < \cdots < x_m)$ of P written in order. We can construct a chain complex by setting $C_j(P, R)$ to be the R–module freely generated by j–chains, and defining boundary maps $\partial_j \colon C_j(P) \to C_{j-1}(P)$ by $\partial_j(x_0 < x_1 < \cdots < x_m) = \sum_{i=0}^{m} (-1)^i (x_0 < x_1 < \cdots \hat{x}_i \cdots < x_m)$ and extending linearly.

We can observe that this chain complex is exactly that for ordered sets (see Section 5.2). If we specify the order of each chain, we can extend this definition to preorders as the OS homology. Generally the OS homology and the OT homology are not isomorphic (see Section 5.2). However, in the special case of posets they do define isomorphic homology groups, as proved below in Theorem 35.

An alternative definition for the homology of a poset is via the construction of its associated order simplex. The associated order complex $\Delta(X, \leq)$ for the poset $(X, \leq)$ is the abstract simplicial complex whose vertices are the elements of X and whose faces are the chains (subsets where each pair is comparable) of $(X, \leq)$. The definition of the associated order complex of a preorder given in Section 6.2 restricts to the standard definition for posets.

The following theorem presents some relationships between these different homology constructions:

Theorem 35 *Let $(\widetilde{X}, \leq)$ be a poset. The following homology groups are isomorphic:*

(i) *OS homology of the finite chains of $(\widetilde{X}, \leq)$.*

(ii) *OT homology of the preorder OT complex* $\mathcal{O}(\widetilde{X}, \leq)$*.*

(iii) *Simplicial homology of the order complex* $\Delta(\widetilde{X}, \leq)$*.*

(iv) *Singular homology of the Alexandrov topology with specialisation order* $(\widetilde{X}, \leq)$*.*

Proof The proof that (ii) and (iv) are isomorphic is in [16, Theorem 2]. The isomorphism between (i) and (iii) is via the unique total orderings of each simplex in the order complex. It is the induced map on homology of $(x_0 < x_1 < \cdots < x_k) \mapsto [x_0, x_1, \dots, x_k]$.

We will now prove that (i) is isomorphic to (ii). The set of ordered tuples forms a basis B for $\mathrm{OT}(\widetilde{X}, \leq)$. Define $\Phi\colon B \to \{\text{subcomplexes of } \mathrm{OT}(\widetilde{X}, \leq)\}$ by setting $\Phi(\tau)$ to be the subcomplex of $\mathrm{OT}(\widetilde{X}, \leq)$ containing only ordered tuples with elements within τ. Since τ is an ordered tuple, it has a smallest element x and for any $\alpha \in \Phi(\tau)$ the ordered tuple concatenating x in front of α (which we will denote by $(x\alpha)$) is also an element in $\Phi(\tau)$. Given a boundary α, we can see that $\partial(x\alpha) = \alpha - (x\partial(\alpha)) = \alpha$. This implies that Φ is an acyclic carrier.

Set $f\colon \mathrm{OT}(\widetilde{X}, \leq) \to \mathrm{OT}(\widetilde{X}, \leq)$ by $f(\tau)$ the identity when τ does not contain repeats (ie lives in $OS(\widetilde{X}, \leq)$ and $f(\tau) = 0$ if τ contains a repeat). Then f commutes with the boundary map because all repeats of a particular element in a tuple must be consecutive when working with posets. It is this claim that does not hold more generally between OT complexes and OS complexes. Since both f and the identity map are both carried by Φ, the acyclic carrier theorem (see [17]) ensures that f and the identity map are chain homotopic and hence the OS homology of the finite chains of $(\widetilde{X}, \leq)$ and the OT homology of preorder OT complex $\mathcal{O}(\widetilde{X}, \leq)$ are isomorphic. □

Each of these four different constructions of homology groups for posets can be generalised to preorders. Three of these generalise in a way that the homology groups are invariant under taking equivalence class posets (or equivalently under taking Kolmogorov quotients). The OS homology of chains is the odd one out in this respect. A counterexample is the preorder $X = \{x, y\}$ with $x \leq y$ and $y \leq x$. It has nontrivial OS homology in dimension one but its equivalence class poset $\widetilde{X} = \{[x]\}$ has trivial OS homology in dimension one.

Theorem 36 *Let* $(X, \leq)$ *be a preorder with equivalence class poset* $(\widetilde{X}, \leq)$*. Then:*

(a) *The preorder OT complex* $\mathcal{O}(X, \leq)$ *is an expansion of* $\mathcal{O}(\widetilde{X}, \leq)$ *and hence has the same OT homology.*

(b) *There is a natural projection map from* $\Delta(X,\leq)$ *to* $\Delta(\widetilde{X},\leq)$*. This projection map induces an isomorphism on their simplicial homology groups.*

(c) *The singular homology of the Alexandrov topology with specialisation order* $(X,\leq)$ *is isomorphic to the singular homology of the Alexandrov topology with specialisation order* $(\widetilde{X},\leq)$*.*

Proof (a) The OT complexes $\mathcal{O}(X,\leq)$ and $\mathcal{O}(\widetilde{X},\leq)$ are closed under adjacent repeats by construction. The quotient map sending X to its equivalence class poset $\widetilde{X}$ shows that $\mathcal{O}(X,\leq)$ is an expansion of $\mathcal{O}(\widetilde{X},\leq)$. We conclude that they are isomorphic by applying Proposition 20.

(b) Construct a map $f\colon \widetilde{X}\to X$ by fixing a representative $x\in X$ for each equivalence class $[x]\in\widetilde{X}$. We can embed $\Delta(\widetilde{X},\leq)$ into $\Delta(X,\leq)$ via the induced map of f. A straight line homotopy provides a deformation from $\Delta(X,\leq)$ to $f(\Delta(\widetilde{X},\leq))$. The result then follows because deformation retractions induce isomorphisms on homology classes.

(c) The proof follows from Lemma 34 as homotopic topological spaces have isomorphic singular homology groups. □

Combining these theorems we conclude that the OT homology of preorder OT complexes, simplicial homology of the associated order complex of a preorder and the singular homology of the Alexandrov topology of a preorder are all isomorphic. These isomorphisms extend to persistent homology as they commute with the maps on homology induced by inclusions.

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C^*–algebraic drawings of dendroidal sets

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In recent years the theory of dendroidal sets has emerged as an important framework for higher algebra. We introduce the concept of a C^**–algebraic drawing* of a dendroidal set. It depicts a dendroidal set as an object in the category of presheaves on C^*–algebras. We show that the construction is functorial and, in fact, it is the left adjoint of a Quillen adjunction between combinatorial model categories. We use this construction to produce a bridge between the two prominent paradigms of noncommutative geometry via adjunctions of presentable ∞–categories, which is the primary motivation behind this article. As a consequence we obtain a single mechanism to construct bivariant homology theories in both paradigms. We propose a (conjectural) roadmap to harmonize algebraic and analytic (or topological) bivariant K–theory. Finally, a method to analyze graph algebras in terms of trees is sketched.

46L85, 55P48; 18D50, 46L87, 55U10

0. Introduction 1171
1. Dendroidal sets 1176
2. C^*–algebras associated with trees: noncommutative dendrices 1182
3. Draw–dendraw adjunction and the bridge 1189
4. Prospects: commutative spaces and graph algebras 1195
Appendix. The model structure on $\mathcal{P}(\mathtt{SC}^{*\,\mathrm{op}}_{\mathrm{un}})$ 1198
References 1204

0 Introduction

Dendroidal sets provide a convenient model for ∞–operads (see Heuts, Hinich and Moerdijk [21] for a comparison with Lurie's model [31] for ∞–operads without constants). The category of dendroidal sets `dSet` was introduced by Moerdijk and Weiss [44; 45] so that (inter alia) it can serve as a receptacle for the nerve functor on the category of operads `Operad`. The following commutative diagram is explanatory:

$$\begin{tikzcd}
\mathtt{Cat} \arrow[r] \arrow[d, "\mathrm{N}"'] & \mathtt{Operad} \arrow[d, "\mathrm{N_d}"] \\
\mathtt{sSet} \arrow[r] & \mathtt{dSet}
\end{tikzcd}$$

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where the vertical arrow N (resp. $\mathrm{N_d}$) denotes the nerve (resp. dendroidal nerve) functor. Cisinski and Moerdijk [10] constructed a cofibrantly generated model structure on `dSet` such that the fibrant objects are precisely the ∞–operads [31]. Over the last decade the theory of dendroidal sets has reached an advanced stage, subsuming several aspects of the theory of operads and that of simplicial sets; see Cisinski and Moerdijk [11; 12].

For a small category $\mathcal{C}$ let $\mathcal{P}(\mathcal{C})$ denote the category of `Set`–valued presheaves on $\mathcal{C}$. Let $\mathtt{SC}^*_{\mathrm{un}}$ denote the category of *nonzero* separable unital C^*–algebras equipped with unit-preserving $*$–homomorphisms. The Gelfand–Naĭmark duality implies that ${\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}}$ can be regarded as the category of nonempty compact second countable noncommutative spaces with continuous maps. Let Ω denote the small category of trees, so that $\mathtt{dSet} := \mathcal{P}(\Omega)$ is the category of *dendroidal sets*. In this article we prove the following results:

(1) We construct a *noncommutative dendrices* functor $D\colon \Omega \to {\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}}$.

(2) We construct an *operadic model structure* on $\mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}})$, an instance of Cisinksi's model structure on presheaves.

(3) We observe that the canonical adjoint pair induced by the noncommutative dendrices functor via left Kan extension

$$\mathtt{dr}\colon \mathtt{dSet} \rightleftarrows \mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}}) :\!\mathtt{dd}$$

is a Quillen pair between combinatorial model categories.

We call the image of a dendroidal set under the left adjoint functor $\mathtt{dr}\colon \mathtt{dSet} \to \mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}})$ the *C^*–algebraic drawing* of the dendroidal set.

These results constitute the first steps towards a bigger objective, which we briefly explain below. There are two prevalent perspectives on noncommutative geometry: analytic and algebraic. The analytic approach was pioneered by Connes [13] — see also Connes and Marcolli [14] — whereas the algebraic approach builds upon work of Drinfeld, Keller, Kontsevich, Lurie, Manin, Mahanta, Tabuada, Toën and several others [39; 25; 27; 31; 50; 34]. Table 1 compares the two approaches as of now.

The space X above in each case must satisfy some reasonable hypotheses. The ∞–category $\mathtt{Perf}_\infty(X)$ is stable and in some contexts stability is included in the definition. This article is primarily motivated by the author's desire to reconcile the two viewpoints. In view of the disparate nature of the basic ingredients of the two paradigms, a *bridge* between the basic objects of the two worlds in the form (a zigzag of) ∞–categorical

	Analytic	Algebraic
objects	C^*–algebras	∞–categories
morphisms	$*$–homomorphisms	∞–functors
how to subsume traditional spaces	$X \mapsto C(X)$	$X \mapsto \mathtt{Perf}_{\infty}(X)$

Table 1: Comparison between the analytic and algebraic approaches

adjunctions subject to a reasonable requirement (explained below) seems to be a sensible target. While constructing the bridge we have resorted to ∞–categories, which reflects the state of the art.

Let $\mathtt{N}\mathcal{S}$ denote the compactly generated ∞–category of (unpointed) noncommutative spaces, whose construction is presented in Section 3.1. The following diagram of adjunctions between presentable ∞–categories summarizes our list of results and puts them in the broader context (see also Remark 3.6):

(1)

$$\begin{tikzcd}
 & \mathrm{N}(\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})^{\circ}) \arrow[ld, bend left, "\mathbf{R}\mathtt{dd}"] \arrow[rd, dashed, bend left] & \\
\mathrm{N}(\mathtt{dSet}^{\circ}) \arrow[ru, bend left, "\mathbf{L}\mathtt{dr}"] & & \mathtt{N}\mathcal{S} \arrow[lu, dashed, bend left]
\end{tikzcd}$$

Here $\mathrm{N}(\mathcal{M}^{\circ})$ denotes the underlying ∞–category of a model category $\mathcal{M}$. The ∞–categorical adjunction $\mathbf{L}\mathtt{dr}\colon \mathrm{N}(\mathtt{dSet}^{\circ}) \rightleftarrows \mathrm{N}(\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})^{\circ}) :\!\mathbf{R}\mathtt{dd}$ is induced by the Quillen adjunction $\mathtt{dr}\colon \mathtt{dSet} \rightleftarrows \mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}}) :\!\mathtt{dd}$ between combinatorial model categories mentioned earlier (see also Remark 3.4). However, the dashed pair between $\mathtt{N}\mathcal{S}$ and $\mathrm{N}(\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})^{\circ})$ is merely a zigzag of adjunctions that is constructed at the level of ∞–categories. This construction actually passes through a *mixed model structure*, denoted by $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})_{\mathrm{mix}}$, on $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$ which is a left Bousfield localization of the operadic model structure (see Definition 3.12). Diagram (1) is our proposed *bridge* between the two paradigms of noncommutative geometry.

0.1 Bivariant homology theories

Given any stable presentable ∞–category $\mathcal{C}$, a colimit-preserving functor

$$\mathrm{B}_{\mathcal{C}}\colon \mathrm{N}(\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})^{\circ}) \to \mathcal{C}$$

can be viewed as a $\mathcal{C}$–valued bivariant homology theory on $\mathrm{N}(\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})^{\circ})$. For a presentable ∞–category $\mathcal{D}$, let $\mathtt{Sp}(\mathcal{D})$ denote its stabilization. The functor $\mathrm{B}_{\mathcal{C}}$ factors as $\mathrm{N}(\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})^{\circ}) \to \mathtt{Sp}\big(\mathrm{N}(\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})^{\circ})\big) \to \mathcal{C}$.

There must be a unified framework for bivariant homology theories in the two paradigms of noncommutative geometry. In order to realize this objective one must construct a functor $\mathrm{B}_{\mathcal{C}}$ that passes the following two acid tests:

(i) the composite functor $\mathrm{N}(\mathtt{dSet}^{\circ}) \to \mathrm{N}(\mathcal{P}(\mathtt{SC}^{*}_{\mathtt{un}}{}^{\mathrm{op}})^{\circ}) \xrightarrow{\mathrm{B}_{\mathcal{C}}} \mathcal{C}$ should lead to the (nonconnective version of) algebraic K–theory of ∞–operads as in Nikolaus [46], and

(ii) the composite functor $\mathtt{N}\mathcal{S} \to \mathrm{N}(\mathcal{P}(\mathtt{SC}^{*}_{\mathtt{un}}{}^{\mathrm{op}})^{\circ}) \xrightarrow{\mathrm{B}_{\mathcal{C}}} \mathcal{C}$ should recover the opposite of the bivariant K–theory of (pointed) noncommutative spaces as in Mahanta [36] after stabilization.

Let us provide a pictorial description of our vision:

$$\begin{tikzcd}
\mathrm{N}(\mathtt{dSet}^{\circ}) \arrow[dr, "\mathrm{F}_1"'] \arrow[drr, bend left, "\text{algebraic K–theory}"] & & \\
& \mathrm{N}(\mathcal{P}(\mathtt{SC}^{*}_{\mathtt{un}}{}^{\mathrm{op}})^{\circ}) \arrow[r, dashed, "\mathrm{B}_{\mathcal{C}}"] & \mathcal{C} \\
\mathtt{N}\mathcal{S} \arrow[ur, "\mathrm{F}_2"] \arrow[urr, bend right, "\mathtt{KK}_{\infty}{}^{\mathrm{op}}\text{–theory}"'] & &
\end{tikzcd} \tag{2}$$

Here the functors F_1 and F_2 are furnished by those of diagram (1), so that $\mathrm{F}_1 = \mathbf{L}\mathtt{dr}$. For any $X \in \mathrm{N}(\mathtt{dSet}^{\circ})$ we require $\mathcal{C}(\mathrm{B}_{\mathcal{C}} \circ \mathrm{F}_1(\mathbf{1}), \mathrm{B}_{\mathcal{C}} \circ \mathrm{F}_1(X))$ to be the (nonconnective version of) algebraic K–theory of X, where $\mathbf{1}$ is a unit object. Moreover, for any pair $A, B \in \mathtt{N}\mathcal{S}$ we require the equivalence of spectra

$$\mathcal{C}\big(\mathtt{Sp}(\mathrm{B}_{\mathcal{C}}) \circ \mathtt{Sp}(\mathrm{F}_2)(\Sigma^{\infty}_{+}(A)), \mathtt{Sp}(\mathrm{B}_{\mathcal{C}}) \circ \mathtt{Sp}(\mathrm{F}_2)(\Sigma^{\infty}_{+}(B))\big) \simeq \mathtt{KK}_{\infty}{}^{\mathrm{op}}(\mathrm{k}_{+}{}^{\mathrm{op}}(A), \mathrm{k}_{+}{}^{\mathrm{op}}(B)),$$

where $\mathrm{k}_{+}{}^{\mathrm{op}}$ is the composite functor $\mathtt{N}\mathcal{S} \to \mathtt{N}\mathcal{S}_{*} \xrightarrow{\mathrm{k}^{\mathrm{op}}} \mathtt{KK}_{\infty}{}^{\mathrm{op}}$ [36]. Varying $\mathrm{B}_{\mathcal{C}}$, one can construct new bivariant homology theories using the above mechanism in both paradigms. For more generalities on bivariant homology theories of noncommutative spaces in the setting of ∞–categories and model categories, the reader may refer to Mahanta [38] or Barnea, Joachim and Mahanta [2]. One possible application of this vision is outlined in Remark 4.9.

Remark A knowledgeable reader might contend that *spectral triples* constitute the notion of a space in noncommutative geometry à la Connes. Let us clarify that by a *space* we really mean a *topological space*. A spectral triple (A, H, D) should be regarded as a noncommutative manifold, whose underlying topological space is determined by the C^{*}–algebra A. Therefore, our proposed bridge (1) exists in the realm of noncommutative topology.

Remark There is also a Quillen adjunction $i_!\colon \mathtt{sSet} \rightleftarrows \mathtt{dSet} : i^*$ that connects the theory of ∞–categories with that of ∞–operads. In this case the relevant model structure on $\mathtt{sSet}$ is the Joyal model structure, whose fibrant objects are ∞–categories. Via the Yoneda embedding ${\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}} \hookrightarrow \mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}})$ the category ${\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}}$ acquires a new class of weak equivalences from the operadic model structure on $\mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}})$ as in Theorem A.11. We call these weak equivalences on ${\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}}$ the *weak operadic equivalences*. The associated homotopy theory is different from (the opposite of) the standard homotopy theory of C^*–algebras endowed with the C^*–homotopy equivalences. The exact difference between the two homotopy theories is not clear to the author (see Remark 3.5).

Remark The technology developed in this article works for all dendroidal sets. But from the viewpoint of topology it is preferable to restrict one's attention to *open dendroidal sets*, which model ∞–operads without constants (see Remark 3.6).

Notation and conventions Unless otherwise stated, a graph means a finite directed graph and a presheaf is considered to be $\mathtt{Set}$–valued. For the sake of definiteness we adopt the quasicategorical model for ∞–categories. An operad always means a coloured operad. We are mostly going to deal with the category of *nonzero* unital separable C^*–algebras $\mathtt{SC}^*_{\mathrm{un}}$ with unit-preserving $*$–homomorphisms (except for in Section 3.1). Including the zero C^*–algebra from the viewpoint of trees and operads does not seem appropriate.

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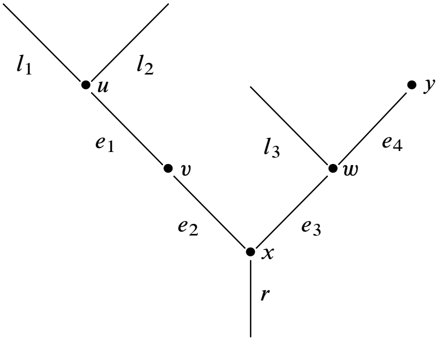
1 Dendroidal sets

We are going to assume familiarity with the theory of (coloured) operads and simplicial sets. For the uninitiated we recommend the following good sources of knowledge: [41; 8; 40; 29; 19; 6], a list that is obviously nonexhaustive. Since the article is written for topologists as well as operator algebraists, we review the theory of dendroidal sets from [51; 44; 45; 10], which is a simultaneous generalization of both operads and simplicial sets. The exposition is quite brief and necessarily not entirely self-contained.

Trees have played an important role in the theory of operads ever since its inception. We provide an informal and very concise introduction to trees. We follow the nomenclature and presentation in [44; 43]. A tree is a finite directed graph whose underlying undirected graph is connected and acyclic. The vertices will be marked by • as shown below:

(3)

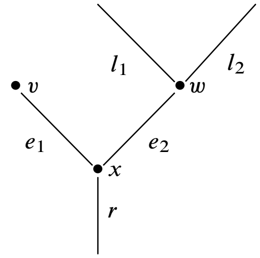


An edge that is connected to two vertices is called an *inner edge*; the rest are called *outer edges*. Amongst the outer edges, ie those that are attached to only one vertex, there is a distinguished one called the *root*; the other outer edges are called *leaves*. A *nonplanar rooted tree* is a nonempty tree with both inner and outer edges with the choice of one distinguished outer edge as the root. Henceforth, unless otherwise stated, by a tree we shall mean a nonplanar rooted tree. Such a tree will be drawn with the root at the bottom and all arrows directed from top to bottom (with arrowheads deleted) as shown above. For instance, in the above tree there are three leaves l_1, l_2 and l_3, four inner edges e_1, e_2, e_3 and e_4, and the root is r. Note that the number of inner edges as well as leaves in a tree could be zero. The simplest possible tree is

which is called the *unit tree*.

The category of simplicial sets, denoted by `sSet`, is the category of `Set`–valued presheaves on the category of simplices Δ, ie $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathtt{Set})$. The notion of a morphism between trees is described in Section 1.1, and this allows us to define a category Ω of trees. Then, in analogy with simplicial sets, we define dendroidal sets to be $\mathtt{dSet} = \mathrm{Fun}(\Omega^{\mathrm{op}}, \mathtt{Set})$, the category of `Set`–valued presheaves on Ω. It will be clear from the definition of the objects and the morphisms of Ω that it can be viewed as a full subcategory of the category of symmetric coloured operads. There is a fully faithful functor $i\colon \Delta \hookrightarrow \Omega$, leading to an adjunction $i_!\colon \mathtt{sSet} \rightleftarrows \mathtt{dSet} :i^*$. The functor $i_!$ is fully faithful and hence the category of dendroidal sets is a generalization of that of simplicial sets. Since $\mathtt{dSet} = \mathrm{Fun}(\Omega^{\mathrm{op}}, \mathtt{Set})$, it suffices to describe the category Ω. The objects of Ω are nonplanar rooted trees as described above. Note that in a *planar* rooted tree the incoming edges at each vertex have a prescribed linear ordering, which does not exist in a nonplanar rooted tree. Hence, each such planar (resp. nonplanar) rooted tree generates a nonsymmetric (resp. symmetric) coloured operad $\Omega[T]$. The set of morphisms $\Omega(S,T)$ between two nonplanar rooted trees S and T is by definition the set of coloured operad maps between $\Omega[S]$ to $\Omega[T]$. Thus, by construction, Ω is the full subcategory of the category of symmetric coloured operads spanned by the objects of the form $\Omega[T]$. The colours of the operad $\Omega[T]$ correspond to the edges of T and a morphism between such operads is completely determined by its effect on colours. Each vertex v of a tree T with outgoing edge e and a labelling of the incoming edges $e_1, \dots, e_n$ defines an operation $v \in \Omega[T](e_1, \dots, e_n; e)$. Consider the nonplanar rooted tree T:

(4)

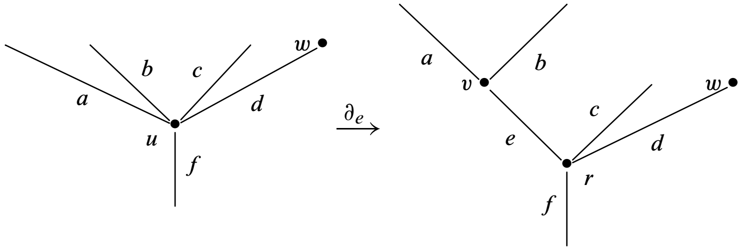


The operad $\Omega[T]$ that it generates has five colours, l_1, l_2, e_1, e_2 and r. The generating operations are $v \in \Omega[T](;e_1)$, $w \in \Omega[T](l_1, l_2; e_2)$ and $x \in \Omega[T](e_1, e_2; r)$. There are also operations that arise from the action of the symmetric group in the nonplanar case. For instance, if $\sigma \in \Sigma_2$, then $w \circ \sigma \in \Omega[T](l_2, l_1; e_2)$ is another operation. There are also the unit operations 1_{l_1}, 1_{l_2}, 1_{e_1}, 1_{e_2} and 1_r and compositions like $x \circ_2 w \in \Omega[T](e_1, l_1, l_2; r)$. We refrain from documenting a complete list of all operations and the relations they satisfy, which the reader can herself/himself reproduce from the above diagram. Instead, we turn towards a more concrete (and pictorial) description of the morphisms in Ω that will be needed later.

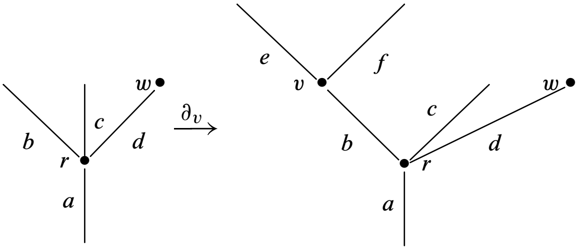
1.1 Face and degeneracy maps

We illustrate the face and degeneracy maps in Ω by examples that are taken directly from [44], where one can find a more elaborate discussion. These maps provide an explicit description of all morphisms in the category Ω, as we shall see at the end of this subsection.

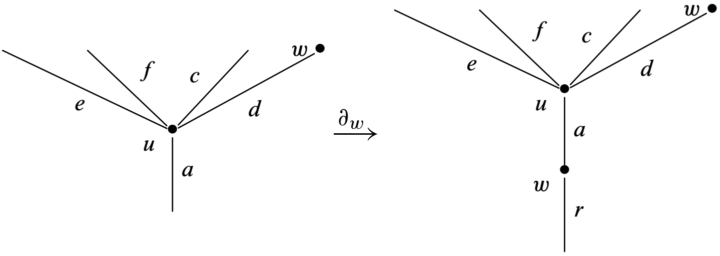
(1) If e is an inner edge in T, then one obtains an *inner face* map $\partial_e\colon T/e \to T$, where T/e is constructed by contracting the edge e as shown below:



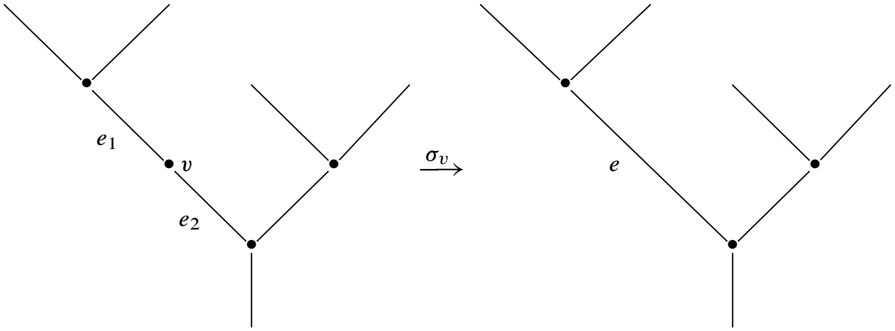
(2) If a vertex v in T has exactly one inner edge attached to it, one obtains the *outer face* map $\partial_v\colon T/v \to T$, where T/v is constructed by deleting v and all the outer edges attached to it as shown below:



It is also possible to remove the root and the vertex that it is attached to by this process, as shown below:



(3) If a vertex $v \in T$ has exactly one incoming edge, there is a tree $T\backslash v$, obtained from T by deleting the vertex v and merging the two edges e_1 and e_2 on either side of v into one new edge e. This defines the *degeneracy map* $\sigma_v\colon T \to T\backslash v$ as shown below:



The following lemma explains the importance of these maps:

Lemma 1.1 [44, Lemma 3.1] *Any arrow $f\colon S \to T$ in Ω decomposes as*

$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[d, "\sigma"'] & T \\
S' \arrow[r, "\varphi"] & T' \arrow[u, "\delta"']
\end{tikzcd}$$

where $\sigma\colon S \to S'$ is a composition of degeneracy maps, $\varphi\colon S' \to T'$ is an isomorphism and $\delta\colon T' \to T$ is a composition of face maps.

Remark 1.2 We have quoted the statement of Lemma 1.1 from the original source. If one carefully inspects its proof (see Lemma 2.3.2 of [43]) one notices immediately that the factorization $f = \delta \circ \varphi \circ \sigma$ is unique. Hence, the degeneracy maps and the face maps of Ω actually constitute a factorization system.

1.2 Face and degeneracy identities

These face and degeneracy maps satisfy numerous identities. We illustrate them in terms of various commuting diagrams in Ω (with the existence of certain nonobvious arrows as assertions). The interested reader is referred to [44; 43] for further details and also the discussion of a couple of special cases that we have left out (see Remark 1.3).

(I) If e and f are distinct inner edges, then $(T/e)/f = (T/f)/e$ and the following diagram commutes:

$$\begin{tikzcd}
(T/e)/f \arrow[r, "\partial_f"] \arrow[d, "\partial_e"'] & T/e \arrow[d, "\partial_e"] \\
T/f \arrow[r, "\partial_f"] & T
\end{tikzcd}$$

(II) Assume T has at least three vertices and let ∂_v and ∂_w be distinct outer face maps. Then $(T/v)/w = (T/w)/v$ and the following diagram commutes:

$$\begin{tikzcd}
(T/v)/w \arrow[r, "\partial_w"] \arrow[d, "\partial_v"'] & T/v \arrow[d, "\partial_v"] \\
T/w \arrow[r, "\partial_w"] & T
\end{tikzcd}$$

(III) If e is an inner edge that is not adjacent to a vertex v, then $(T/e)/v = (T/v)/e$ and the following diagram commutes:

$$\begin{tikzcd}
(T/v)/e \arrow[r, "\partial_e"] \arrow[d, "\partial_v"'] & T/v \arrow[d, "\partial_v"] \\
T/e \arrow[r, "\partial_e"] & T
\end{tikzcd}$$

(IV) Let e be an inner edge that is adjacent to a vertex v and let w be the other adjacent vertex. In T/e the two vertices combine to contribute a vertex z (expressing the composition of v and w in some order). Then the outer face $\partial_z\colon (T/e)/z \to T/e$ exists if and only if the outer face $\partial_w\colon (T/v)/w \to T/v$ exists, and in this case $(T/e)/z = (T/v)/w$. Summarizing the setup, the following diagram commutes:

$$\begin{tikzcd}
(T/v)/w \arrow[r, equal] \arrow[d, "\partial_w"'] & (T/e)/z \arrow[r, "\partial_z"] & T/e \arrow[d, "\partial_e"] \\
T/v \arrow[rr, "\partial_v"] & & T
\end{tikzcd}$$

(V) If σ_v and σ_w are two degeneracies of T, then $(T\backslash v)\backslash w = (T\backslash w)\backslash v$ and the following diagram commutes:

$$\begin{tikzcd}
T \arrow[r, "\sigma_v"] \arrow[d, "\sigma_w"'] & T\backslash v \arrow[d, "\sigma_w"] \\
T\backslash w \arrow[r, "\sigma_v"] & (T\backslash v)\backslash w
\end{tikzcd}$$

(VI) Let $\sigma_v\colon T \to T\backslash v$ be a degeneracy and $\partial\colon T' \to T$ be any face map such that T' still contains v and its two adjacent edges as a subtree. Then the following diagram commutes:

$$\begin{tikzcd}
T \arrow[r, "\sigma_v"] & T\backslash v \\
T' \arrow[u, "\partial"] \arrow[r, "\sigma_v"] & T'\backslash v \arrow[u, "\partial"']
\end{tikzcd}$$

(VII) Let $\sigma_v\colon T \to T\backslash v$ be a degeneracy map and $\partial\colon T' \to T$ be a face map induced by one of the adjacent edges to v or the removal of v (if that is possible). Then $T' = T\backslash v$ and the following diagram commutes:

$$\begin{tikzcd}
T\backslash v \arrow[rr, "\mathrm{id}_{T\backslash v}"] \arrow[dr, "\partial"'] & & T\backslash v \\
& T \arrow[ur, "\sigma_v"'] &
\end{tikzcd}$$

Remark 1.3 We have left out the following special cases of dendroidal identities:

- Outer face identities when T has fewer than three vertices.
- Predictable identities expressing the compatibility of the face and degeneracy maps with isomorphisms (see for instance Section 2.3.1 of [43]).

1.3 The model structure on `dSet`

The formalism of model categories was introduced by Quillen [48] as an abstract framework for homotopy theory. For a modern treatment the reader may refer to [24; 23]. We review the model structure on `dSet` constructed by Cisinski and Moerdijk [10] that generalizes the Joyal model structure on `sSet`.

The construction of the model structure on `dSet` exploits the Cisinski model structure on any category of presheaves [9] (see the appendix) and also a transfer principle. Typically one begins with certain desired features on the model structure based on intended applications. Keeping in mind the Joyal model structure on `sSet`, it is natural to expect that in the would-be model structure on `dSet` (certain) monomorphisms should be cofibrations, some class of objects (generalizing ∞–categories) should be fibrant, and certain morphisms (generalizing categorical equivalences) should be weak equivalences.

A monomorphism of dendroidal sets $X \to Y$ is *normal* if for any $T \in \Omega$, the action of $\mathrm{Aut}(T)$ on $Y(T) \setminus X(T)$ is free. If e is an inner edge of a tree T, then one obtains an *inner horn inclusion* $\Lambda^e[T] \to \Omega[T]$, where $\Lambda^e[T]$ is obtained as the union of the images of all the elementary face maps apart from $\partial_e\colon T/e \to T$. A map of dendroidal sets is called an *inner anodyne extension* if it belongs to the smallest class of maps which is stable under pushouts, transfinite compositions and retracts, and which contains the inner horn inclusions. There is an adjunction $\tau_{\mathrm{d}}\colon \mathtt{dSet} \rightleftarrows \mathtt{Operad} :\mathrm{N}_{\mathrm{d}}$, where τ_{d} is called the *operadic realization* functor. The model structure on $\mathtt{dSet}$ can be described as (see Theorem 2.4 of [10]):

- the cofibrations are the *normal monomorphisms*;
- the fibrant objects are the ∞–operads;
- the fibrations between fibrant objects are the inner Kan fibrations (see [45; 10, Section 2.1]), whose image under τ_{d} is an operadic fibration, ie a fibration in the canonical model structure on operads;
- the class of weak equivalences is the smallest class $\mathtt{W}$ of maps in $\mathtt{dSet}$ satisfying
  (a) the 2-out-of-3 property;
  (b) that inner anodyne extensions are in $\mathtt{W}$;
  (c) that trivial fibrations between ∞–operads are in $\mathtt{W}$.

We omit further details but explain an additional property of this model category that is relevant for our purposes. Let κ be regular cardinal. A category $\mathcal{A}$ is said to be κ–*accessible* if there is a small category $\mathcal{C}$ such that $\mathcal{A} \cong \mathrm{Ind}_\kappa(\mathcal{C})$. A *locally* κ*–presentable* category is a κ–accessible category that, in addition, possesses all small colimits. A category is *locally presentable* if it is locally κ–presentable for some regular cardinal κ. If $\mathcal{C}$ is a small category, the category of presheaves on $\mathcal{C}$ (eg $\mathtt{dSet} = \mathrm{Fun}(\Omega^{\mathrm{op}}, \mathtt{Set})$) is locally ω–presentable (see for instance [1]). Recall that a model category is said to be *combinatorial* if it is cofibrantly generated and its underlying category is locally presentable. It is also shown in Proposition 2.6 of [10] that the model category $\mathtt{dSet}$ is combinatorial. The set of generating cofibrations I consists of the boundary inclusions of trees, ie $I = \{\partial\Omega[T] \to \Omega[T] \mid T \in \Omega\}$.

2 C^*–algebras associated with trees: noncommutative dendrices

The description of a tree presented in the previous section differs slightly from the one that one might encounter in graph theory. For instance, in the graph algebra literature a *directed graph* $G=(E^0,E^1,r,s)$ consists of two (countable) sets E^0 and E^1 and functions $r,s\colon E^1\to E^0$. The elements of E^0 are called the *vertices* and those of E^1 are called the *edges* of G. For an edge e, the vertex $s(e)$ is its *source* and the vertex $r(e)$ is its *range*. Thus, in a directed graph one does not have edges attached only to one vertex like the leaves or the root that we considered in the previous section. In a graph a *path of length* n is a sequence $\mu=e_1e_2\cdots e_n$ of edges such that $s(e_i)=r(e_{i+1})$ for all $i\leqslant i\leqslant n-1$. For such a path $\mu=e_1e_2\cdots e_n$ we denote by $\operatorname{edge}(\mu)=\{e_1,e_2,\ldots,e_n\}$ the set of all edges traversed by it.

The C^*–algebra associated with a tree that we are going to describe shortly is to some extent inspired by the construction of noncommutative simplicial complexes in [16]. However, we design the C^*–algebra from the edges of the tree, since from the categorical (or operadic) viewpoint the edges are more fundamental than the vertices.

Definition 2.1 Given a set G of generators and a set R of relations, the *universal C^*–algebra*, denoted by $C^*(G,R)$, is a C^*–algebra equipped with a set map $\iota\colon G\to C^*(G,R)$ that satisfies the following universal property: for every C^*–algebra A and set map $\iota_A\colon G\to A$ such that the relations R are fulfilled inside A, there is a unique $*$–homomorphism $\theta\colon C^*(G,R)\to A$ satisfying $\theta\circ\iota=\iota_A$.

This is a subtle concept; for instance, if $G=\{x\}$ and $R=\varnothing$, then the universal C^*–algebra $C^*(G,R)$ does not exist. In other words, free (or relation-free) objects do not exist in the category of C^*–algebras. It follows from two simple facts:

(1) Every element in a C^*–algebra has a finite norm $\|\cdot\|$, ie a real number.

(2) Every $*$–homomorphism is norm-decreasing, ie $\phi\colon A\to B$ implies $\|\phi(a)\|\leqslant\|a\|$.

If $C^*(G=\{x\},R=\varnothing)$ were to exist, then the generator x would have a finite norm $\|x\|$. Now choose any C^*–algebra A and an element $a\in A$ with $\|a\|>\|x\|$, which can evidently be done. Then it is manifestly clear that one cannot find the desired $*$–homomorphism $\iota\colon C^*(G=\{x\},R=\varnothing)\to A$ with $\iota(x)=a$ that satisfies requirement (2) above. If the relations R put a nonstrict bound on the norm of each generator, then typically one obtains an interesting nontrivial universal C^*–algebra (although it can be trivial in certain cases).

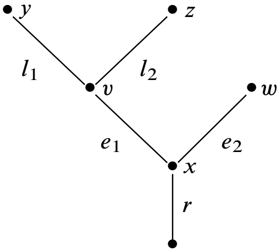
Definition 2.2 Given any tree $T=(E^0,E^1)$ (viewed as a graph as described above) we define its *associated C^*–algebra* as the universal unital C^*–algebra generated by $\{q_e\mid e\in E^1\}$ satisfying

1. $q_e \geqslant 0$ for all $e \in E^1$,
2. $\sum_{e\in E^1} q_e = 1$, and
3. $q_{e_1}q_{e_2}\cdots q_{e_n} = 0$ unless there is a path μ with $\{e_1, e_2, \dots, e_n\} \subseteq \mathrm{edge}(\mu)$ (inclusion of sets disregarding order).

Remark 2.3 Let us briefly clarify the motivation behind the relations.

- The relations (1) and (2) clearly put a bound on the norm of each generator and hence the existence of the universal C^*–algebra is clear.
- Relation (3) encodes the compositional nature of trees. It retains those terms that lie in a path (and hence bound a simplex). However, it also retains reorderings and repetitions of edges within the path because we want the canonical abelianization map to be surjective (see Remark 2.5 and Example 2.7).

Example 2.4 Note that repetitions are allowed amongst the e_i in relation (3) above. For instance, if T is



then $q_{l_2}q_{e_1}q_{e_2} = q_{l_1}q_{l_2} = q_{e_2}q_{e_1}q_{l_2} = 0$, whereas $q_r q_{e_1} q_{l_1} \neq 0$ and $q_{e_1}q_{l_2}q_{e_1} \neq 0$.

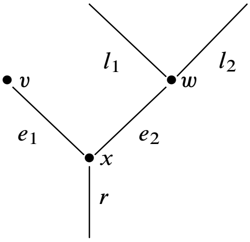
Given any nonplanar rooted tree T we construct its associated C^*–algebra $D(T)$ as follows:

(a) insert a vertex at each of the top tips of the leaves (if any) and the bottom tip of the root;

(b) construct the universal C^*–algebra of the modified tree as explained above.

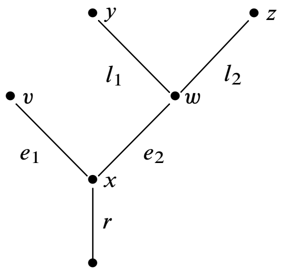
For instance, given the tree

(5)



then according to procedure (a) we modify the tree as

(6)



and then construct its universal C^*–algebra.

Remark 2.5 In the above construction we can add the relation that the generators commute, ie $q_e q_f = q_f q_e$ for all $e, f \in E^1$ to obtain a commutative C^*–algebra $D^{\mathrm{ab}}(T)$.

Definition 2.6 The C^*–algebra $D(T)$ associated with a nonplanar rooted tree T is called a *noncommutative dendrex*. Note that if $X \in \mathtt{dSet}$ and $T \in \Omega$, then $X(T)$ is viewed as the set of T–shaped dendrices in X.

Example 2.7 An object $[n] \in \Delta$ can be viewed as a linear tree L_n as

$$\leftarrow \bullet_1 \leftarrow \cdots \leftarrow \bullet_n \leftarrow$$

(drawn horizontally instead of vertically with arrowheads inserted to indicated the direction). This association $[n] \mapsto L_n$ defines a fully faithful functor $\Delta \hookrightarrow \Omega$ that produces the adjunction $\mathtt{sSet} \rightleftarrows \mathtt{dSet}$. After modification L_n produces the tree

$$\bullet_0 \leftarrow \bullet_1 \leftarrow \cdots \leftarrow \bullet_{n+1},$$

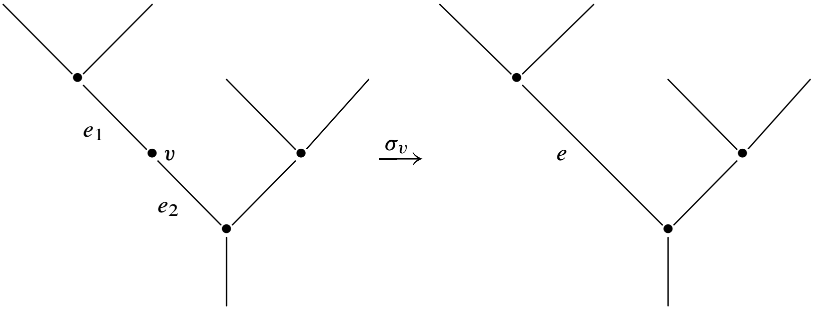
whose associated C^*–algebra is the universal unital C^*–algebra generated by $n+1$ positive generators $\{q_1, \dots, q_{n+1}\}$ such that $\sum_{i=1}^{n} q_i = 1$. Its associated commutative C^*–algebra (see Remark 2.5) is isomorphic to $\mathrm{C}(\Delta^n)$, where Δ^n is the standard n–simplex (see Proposition 2.1 of [16]). Our choice for the noncommutative dendrex construction was guided by this consideration. Observe that $D(L_0) = \mathbb{C}$, since [0] corresponds to the unit tree

whose modified tree is simply

with only one edge. This phenomenon reflects the fact that the edges of a tree correspond to the colours of its associated operad.

2.1 Functoriality

The aim of this subsection is to establish the (contravariant) functoriality of the above construction $T \mapsto D(T)$ with respect to morphisms of Ω. To this end we begin by defining the $*$–homomorphisms that the faces and degeneracies induce. If $\sigma_v\colon T \to T\backslash v$ is a degeneracy map (see Lemma 1.1) like



then define $\sigma_v^*\colon D(T\backslash v) \to D(T)$ as

$$q_f \mapsto \begin{cases} q_f & \text{if } f \neq e, \\ q_{e_1} + q_{e_2} & \text{otherwise.} \end{cases}$$

Remark 2.8 The notation employed in the definition of σ_v^* is potentially ambiguous. In the domain q_f is a generator of $D(T\backslash v)$ and in the codomain it is a generator of $D(T)$. One should ideally differentiate them by writing $q_f^{T\backslash v}$ and q_f^T (or something similar) to indicate the dependence on the tree. For notational simplicity we avoid doing this.

Lemma 2.9 *The map $\sigma_v^*\colon D(T\backslash v) \to D(T)$ is a $*$–homomorphism.*

Proof We need to verify that the set $\{\sigma_v^*(q_f) \mid f \text{ an edge in } T\backslash v\}$ satisfies the relations (1), (2) and (3) in $D(T)$ that define the universal C^*–algebra $D(T\backslash v)$.

For (1) note that q_{e_1} and q_{e_2} are both positive in $D(T)$, whence so is $q_{e_1}+q_{e_2}$. Clearly each q_f is also positive in $D(T)$. Let $E^1(T)$ be the set of edges in T. We verify (2) by computing

$$\sum_{f\in E^1(T\backslash v)} \sigma_v^*(q_f) = \sum_{f\neq e} q_f + (q_{e_1}+q_{e_2}) = \sum_{f\in E^1(T)} q_f = 1.$$

For (3) one can check by inspection that if f_1 and f_2 are two edges in $T\backslash v$ that do not lie in a path, then they cannot lie in a path in T. □

Note that every face map can be viewed as an injective map on edges (or colours of the associated operad). Thus, if $\partial_e\colon T/e\to T$ is an inner face map then define a $*$–homomorphism $\partial_e^*\colon D(T)\to D(T/e)$ as

$$q_f \mapsto \begin{cases} q_f & \text{if } f\neq e,\\ 0 & \text{otherwise.}\end{cases}$$

Similarly, if $\partial_v\colon T/v\to T$ is an outer face map then define $\partial_v^*\colon D(T)\to D(T/v)$ as

$$q_f \mapsto \begin{cases} q_f & \text{if } f \text{ has not been removed},\\ 0 & \text{otherwise.}\end{cases}$$

Lemma 2.10 *The maps*

$$\partial_e^*\colon D(T)\to D(T/e) \quad \textit{and} \quad \partial_v^*\colon D(T)\to D(T/v)$$

are $*$–homomorphisms.

Proof One needs to again verify that the set $\{\partial_e^*(q_f)\mid f \text{ an edge in } T\}$ satisfies the relations (1), (2) and (3) in $D(T/e)$ that define the universal C^*–algebra $D(T)$. Relations (1) and (2) are clearly satisfied; for relation (3) one needs to observe that if two edges e and f in T do not lie in a path, then this property continues to hold in T/e or T/v. A similar argument is applicable to ∂_v^*. □

Remark 2.11 If $\theta\colon S\to T$ is an isomorphism in Ω then $\theta^*\colon D(T)\to D(S)$ acts on the generators as $q_e\mapsto q_{\theta^{-1}(e)}$. One can readily verify that θ^* is a unital $*$–homomorphism.

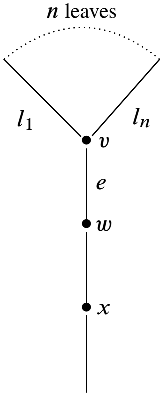
Let $\mathtt{SC}^*_{\mathrm{un}}$ denote the category of separable unital C^*–algebras with unit-preserving $*$–homomorphisms. Extending the Gelfand–Naĭmark duality, ${\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}}$ is regarded as the category of compact Hausdorff noncommutative spaces with continuous maps.

Proposition 2.12 *The association of a noncommutative dendrex with a tree $T \mapsto D(T)$ defines a functor $D\colon \Omega \to {\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}}$.*

Proof In view of Lemma 1.1 it suffices to show that the $*$–homomorphisms ∂_e^*, ∂_v^*, σ_v^* and θ^* satisfy the face and degeneracy identities (see Section 1.2). Note that thanks to the universal property of universal C^*–algebras we simply need to verify that various combinations of these $*$–homomorphisms governed by the identities agree on generators.

It is easy to verify that identities (I), (II), (III) and (V) are satisfied. The point is to observe that the order in which a certain number of generators are sent to 0 or sums of two other generators does not affect the final outcome.

For (IV) let us suppose that the tree around e looks like



Now $\partial_z^*\partial_e^*$ will first send q_e to 0 and then $q_{l_1},\dots,q_{l_n}$ to 0. On the other hand, $\partial_w^*\partial_v^*$ will first send $q_{l_1},\dots,q_{l_n}$ to 0 and then q_e to 0. The end result is evidently the same.

For (VI) we begin with the commutative diagram

$$\begin{tikzcd}
T \arrow[r, "\sigma_v"] & T\backslash v \\
T' \arrow[u, "\partial"] \arrow[r, "\sigma_v"] & T'\backslash v \arrow[u, "\partial"']
\end{tikzcd}$$

Let us suppose that the face map ∂ removes edges $f_1,\dots,f_n$. Since T' still contains v and its two adjacent edges (say e_1 and e_2), one can merge them to a new edge e. Thus, ∂^* is defined by $q_{f_i} \mapsto 0$ for $i = 1,\dots,n$ and σ_v^* by $q_e \mapsto q_{e_1} + q_{e_2}$. Hence, it is clear that $\partial^* \sigma_v^* = \sigma_v^* \partial^*$. The verifications of (VII) and the special cases (see Remark 1.3) are similar and omitted.

Let us observe that $D(T)$ is unital for every $T \in \Omega$ and the $*$–homomorphisms ∂_e^*, ∂_v^*, σ_v^* and θ^* are all unit-preserving, whence the essential image of the functor D is indeed ${\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}}$.

Note that for a map $\tau\colon S \to T$ in Ω the induced map is $\tau^*\colon D(T) \to D(S)$. It remains to check that the association $\tau \mapsto \tau^*$ respects composition of morphisms. It is clear that this association preserves composition of face maps as well as composition of degeneracy maps. To complete the proof we now simply invoke Remark 1.2. □

3 Draw–dendraw adjunction and the bridge

For a small category $\mathcal{C}$ let $\mathcal{P}(\mathcal{C})$ denote the category of $\mathtt{Set}$–valued presheaves on $\mathcal{C}$, ie $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathtt{Set})$. Thus, setting $\mathcal{C} = \Omega$ we find $\mathcal{P}(\Omega) = \mathtt{dSet}$. Since $\mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}})$ is cocomplete, using the covariant functoriality of the category of presheaves (via left Kan extension) one obtains the dashed functor below:

(7)
$$\begin{tikzcd}
\Omega \arrow[r, "D"] \arrow[d] & {\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}} \arrow[d] \\
\mathtt{dSet} \arrow[r, dashed] & \mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}})
\end{tikzcd}$$

where the vertical functors are the canonical Yoneda embeddings and the top horizontal functor $D\colon \Omega \to {\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}}$ is the one constructed in the previous section (see Proposition 2.12). Let $\mathtt{dr}$ denote the dashed functor in the above diagram (7). There is an adjunction

$$\mathtt{dr}\colon \mathtt{dSet} \rightleftarrows \mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}}) :\!\mathtt{dd},$$

where the right adjoint $\mathtt{dd}$ is defined as $[\mathtt{dd}(Y)](T) = Y(D(T))$ for any $Y \in \mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}})$.

Definition 3.1 For any $X \in \mathtt{dSet}$ the object $\mathtt{dr}(X)$ is its *C^*–algebraic drawing*. We call the functor $\mathtt{dr}$ (resp. $\mathtt{dd}$) the *draw* (resp. *dendraw*) functor.

Remark 3.2 In sheaf-theoretic notation, $\mathtt{dr} = D_!$ and $\mathtt{dd} = D^*$. The dendraw functor $\mathtt{dd}$ also admits a right adjoint $D_*\colon \mathtt{dSet} \to \mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}})$, whence it preserves colimits.

Recall from Section 1.3 that the category $\mathtt{dSet}$ admits a combinatorial model structure.

Theorem 3.3 *There is a combinatorial model structure on* $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$ *such that the draw–dendraw adjunction*

$$\mathtt{dr}\colon \mathtt{dSet} \rightleftarrows \mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}}) :\!\mathtt{dd}$$

becomes a Quillen adjunction.

Proof The model structure on $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$ that we are referring to is constructed in Theorem A.11 (see the appendix). The left adjoint $\mathtt{dr}$ sends generating cofibrations in $\mathtt{dSet}$ to cofibrations in $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$ (see Proposition A.6 below) and generating trivial cofibrations to trivial cofibrations in $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$ (see Remark A.13 below). Now, using Lemma 2.1.20 of [24], one concludes that the draw–dendraw adjunction is actually a Quillen adjunction. □

Remark 3.4 Associated with any (combinatorial) model category $\mathcal{M}$ there is an underlying (presentable) ∞–category $\mathrm{N}(\mathcal{M}^{\circ})$ (see Definition 1.3.1 of [22]). Moreover, a Quillen adjunction between (combinatorial) model categories (like $\mathtt{dr}\colon \mathtt{dSet} \rightleftarrows \mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}}) :\!\mathtt{dd}$) induces an ∞–categorical adjunction between the underlying (presentable) ∞–categories (like $\mathbf{L}\mathtt{dr}\colon \mathrm{N}(\mathtt{dSet}^{\circ}) \rightleftarrows \mathrm{N}(\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})^{\circ}) :\!\mathbf{R}\mathtt{dd}$)—see [22, Proposition 1.5.1; 42, Theorem 2.1]. Although we are mainly interested in the ∞–categorical adjunction pair $(\mathbf{L}\mathtt{dr}, \mathbf{R}\mathtt{dd})$, it is often convenient to have at our disposal an explicit Quillen adjunction modelling it.

Remark 3.5 Viewing $\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}}$ inside the category of presheaves $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$ via the Yoneda functor, we obtain a new homotopy theory for (the opposite category of) separable unital C^*–algebras, whose weak equivalences are called *weak operadic equivalences*. This new class of weak operadic equivalences is potentially interesting in its own right. The weak operadic equivalences on $\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}}$ are different from those inherited from the model structure on $\mathrm{Ind}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$ (see [2]) via the embedding $\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}} \hookrightarrow \mathrm{Ind}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$. These two classes of weak equivalences on $\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}}$ give rise to different homotopy theories. The class of weak operadic equivalences is not contained in the class of standard homotopy equivalences on $\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}}$ (see Remark A.12); it is not clear to the author whether the other containment holds. Those readers who prefer to stick to the category of C^*–algebras (and not venture into the category of presheaves) may try to classify the objects in it up to weak operadic equivalences.

Remark 3.6 A vertex that has no incoming edges is called a *stump*, eg in the 0–corolla

the top vertex is a stump. A tree devoid of stumps is called an *open tree*. Let Ω_o denote the full subcategory of Ω spanned by the open trees. The canonical inclusion $\Omega_o \hookrightarrow \Omega$ induces an adjunction $\mathtt{dSet}_o := \mathcal{P}(\Omega_o) \rightleftarrows \mathcal{P}(\Omega) = \mathtt{dSet}$ such that the left adjoint $\mathtt{dSet}_o \hookrightarrow \mathtt{dSet}$ is fully faithful. The objects of $\mathtt{dSet}_o$ are called *open dendroidal sets*. The category $\mathtt{dSet}_o$ inherits a combinatorial model structure via the adjunction $\mathtt{dSet}_o \rightleftarrows \mathtt{dSet}$, making it a Quillen pair (see Section 2.3 of [21]). The fully faithful functor $\mathtt{sSet} \to \mathtt{dSet}$ factors through $\mathtt{dSet}_o$. The fibrant objects of $\mathtt{dSet}_o$ are ∞–operads without constants. It was noticed by Moerdijk that our construction of the noncommutative dendrices functor does not distinguish between a leaf and an edge whose top vertex is a stump; in particular, the C^*–algebra associated with the unit tree and the 0–corolla are both $\mathbb{C}$. Thus, our draw–dendraw adjunction should be restricted to open dendroidal sets via the composite adjunction

$$\mathtt{dSet}_o \rightleftarrows \mathtt{dSet} \rightleftarrows \mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}}).$$

So far we have constructed the solid adjunctions in the following diagram of ∞–categories:

$$\begin{tikzcd}
\mathrm{N}(\mathtt{dSet}^\circ_o) \arrow[dr, bend left] & & \mathrm{N}(\mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}})^\circ) \arrow[dl, bend left, "\mathbf{Rdd}"] \arrow[dr, dashed, bend left] & \\
& \mathrm{N}(\mathtt{dSet}^\circ) \arrow[ul, bend left] \arrow[ur, bend left, "\mathbf{Ldr}"] & & \mathtt{N}\mathcal{S} \arrow[ul, dashed, bend left]
\end{tikzcd}$$

Now we define the ∞–category of noncommutative spaces $\mathtt{N}\mathcal{S}$. Then we complete the connection between ∞–operads and noncommutative spaces via a sequence of ∞–categorical adjunctions. The dashed pair above actually represents a zigzag of adjunctions.

3.1 The rest of the bridge between $\mathtt{N}\mathcal{S}$ and $\mathbf{N}(\mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}})^\circ)$

Earlier we constructed the compactly generated ∞–category of pointed noncommutative spaces generalizing the category of pointed compact noncommutative spaces (see Definition 2.13 of [37]). Let ${\mathtt{SC}^*}^{\mathrm{op}}$ denote the opposite topological category of separable C^*–algebras with all (not necessarily unit-preserving) $*$–homomorphisms.

We view it as a topological category by endowing the morphism sets with the point-norm topology. Let ${\mathtt{SC}^*_\infty}^{\mathrm{op}}$ denote the topological nerve of $\mathtt{SC}^{*\mathrm{op}}$. It is shown in Proposition 2.7 of [37] that ${\mathtt{SC}^*_\infty}^{\mathrm{op}}$ admits finite colimits.

Definition 3.7 We set $\mathtt{N}\mathcal{S}_* = \mathrm{Ind}_\omega({\mathtt{SC}^*_\infty}^{\mathrm{op}})$ and call it the compactly generated ∞–category of pointed noncommutative spaces.

Similarly, there exists a compactly generated ∞–category $\mathtt{N}\mathcal{S}$ of noncommutative (unpointed) spaces whose construction is outlined below.

Definition 3.8 Let $\mathcal{C}$ denote the opposite of the topological category of separable unital C^*–algebras with unit-preserving $*$–homomorphisms. We again view it as a topological category by endowing the morphism sets with the point-norm topology.

Here we have included the zero C^*–algebra in the topological category $\mathcal{C}$. The zero C^*–algebra should be viewed as the (unital) C^*–algebra of continuous functions on the empty space. Therefore, for every separable unital C^*–algebra A there is a unique unital $*$–homomorphism $A \to 0$, ie the opposite category $\mathcal{C}$ has an initial object. But the zero $*$–homomorphism $0 \to A$ is *not unital* unless $A = 0$.

Definition 3.9 Let $\mathtt{N}\mathcal{S}^{\mathtt{fin}}$ denote the topological nerve of the topological category $\mathcal{C}$. Here it is vitally important to consider the point-norm topology on the morphism spaces while constructing the topological nerve.

One can show as in Proposition 2.7 of [37] that $\mathtt{N}\mathcal{S}^{\mathtt{fin}}$ admits finite colimits. For the rest of this section we set $\mathrm{Ind} = \mathrm{Ind}_\omega$, which denotes the ∞–categorical ind–completion.

Definition 3.10 We set $\mathtt{N}\mathcal{S} := \mathrm{Ind}(\mathtt{N}\mathcal{S}^{\mathtt{fin}})$ and call it the compactly generated ∞–category of (unpointed) noncommutative spaces.

Remark 3.11 This ∞–categorical construction of noncommutative spaces $\mathtt{N}\mathcal{S}$ is simple and practical. It incorporates homotopy theory and analysis in a systematic manner; the analytical aspects are contained within the world of C^*–algebras. More complicated topological algebras like pro-C^*–algebras can be viewed within this setup via the homotopy theory of diagrams of C^*–algebras. The mechanism is explained in our earlier work [37; 36].

There is a canonical fully faithful embedding of (topological) categories $\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}} \hookrightarrow \mathcal{C}$. This functor induces an adjunction of the corresponding categories of presheaves $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}}) \rightleftarrows \mathcal{P}(\mathcal{C})$. A map $f\colon C \to D$ in $\mathcal{C}$ is a *C^*–homotopy equivalence* if there is another map $g\colon D \to C$ and homotopies $fg \simeq \mathrm{id}_D$ and $gf \simeq \mathrm{id}_C$. The set of C^*–homotopy equivalences gives rise to a set of maps in $\mathcal{P}(\mathcal{C})$ that, finally, gives rise to another set of maps in $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$ via the adjunction $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}}) \rightleftarrows \mathcal{P}(\mathcal{C})$.

Definition 3.12 (mixed model structure on $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$) The left Bousfield localization of the combinatorial model category $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$ equipped with the operadic model structure (see Theorem A.11) along the set of maps induced by the C^*–homotopy equivalences is the *mixed model structure* on $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$. We denote the mixed model category by $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})_{\mathrm{mix}}$, which again turns out to be combinatorial.

The Bousfield localization $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}}) \to \mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})_{\mathrm{mix}}$ of combinatorial model categories induces an adjunction of underlying presentable ∞–categories $\mathrm{N}(\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})^{\circ}) \rightleftarrows \mathrm{N}(\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})^{\circ}_{\mathrm{mix}})$ that exhibits $\mathrm{N}(\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})^{\circ}_{\mathrm{mix}})$ as a localization of $\mathrm{N}(\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})^{\circ})$. Let θ denote the composition of the functors

$$\mathcal{C} \xhookrightarrow{j} \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}}) \xrightarrow{(-)^{\mathrm{f}}} \mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})^{\mathrm{f}}_{\mathrm{mix}},$$

where j is the Yoneda embedding, $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})^{\mathrm{f}}_{\mathrm{mix}}$ is the full subcategory of (bi)fibrant objects of $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})_{\mathrm{mix}}$ and $(-)^{\mathrm{f}}$ denotes a fibrant replacement functor in the mixed model category $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})_{\mathrm{mix}}$. Let us view $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})^{\mathrm{f}}_{\mathrm{mix}}$ as a *relative category* in the sense of [4] via the weak equivalences inherited from the model category $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})_{\mathrm{mix}}$. We can also view $\mathcal{C}$ as a relative category with the C^*–homotopy equivalences as the weak equivalences.

Lemma 3.13 *The functor $\theta\colon \mathcal{C} \to \mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})^{\mathrm{f}}_{\mathrm{mix}}$ is a morphism of relative categories.*

Proof We need to verify that θ preserves weak equivalences. Our construction of the mixed model category $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})_{\mathrm{mix}}$ ensures this property (see Definition 3.12). □

For any relative category $\mathcal{A}$ we denote the underlying ∞–category by $\mathcal{A}_{\infty}$ (see Section 1.2 of [42]). The morphism of relative categories $\theta\colon \mathcal{C} \to \mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})^{\mathrm{f}}_{\mathrm{mix}}$ induces a morphism of underlying ∞–categories $\theta\colon \mathcal{C}_{\infty} \to (\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})^{\mathrm{f}}_{\mathrm{mix}})_{\infty}$. For any ∞–category $\mathcal{A}$ there is an ∞–category of ∞–presheaves $\mathcal{P}_{\infty}(\mathcal{A})$ (see [30]). Note the subtle difference in notation — for an ordinary category $\mathcal{A}$ we denote by $\mathcal{P}(\mathcal{A})$ the category of $\mathtt{Set}$–valued presheaves on $\mathcal{A}$, whereas for an ∞–category $\mathcal{A}$ we denote by $\mathcal{P}_{\infty}(\mathcal{A})$ the ∞–category of ∞–presheaves on $\mathcal{A}$.

Proposition 3.14 *The morphism of ∞–categories $\theta\colon \mathcal{C}_\infty \to (\mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}})^{\mathrm{f}}_{\mathrm{mix}})_\infty$ induces a colimit-preserving functor $\widetilde{\theta}\colon \mathcal{P}_\infty(\mathcal{C}_\infty) \to \mathrm{N}(\mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}})^{\circ}_{\mathrm{mix}})$.*

Proof The canonical inclusion $\mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}})^{\mathrm{f}}_{\mathrm{mix}} \hookrightarrow \mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}})_{\mathrm{mix}}$ induces an equivalence of underlying ∞–categories [18] (see also Lemma 2.8 of [42]). Thanks to the universal property of the category of presheaves $\mathcal{P}_\infty(-)$ in the setting of ∞–categories (see Theorem 5.1.5.6 of [30]), it suffices to show that $(\mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}})^{\mathrm{f}}_{\mathrm{mix}})_\infty \simeq \mathrm{N}(\mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}})^{\circ}_{\mathrm{mix}})$ admits small colimits. Since the model category $\mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}})_{\mathrm{mix}}$ is combinatorial, its underlying ∞–category is presentable (see Corollary 1.5.2 of [22]), ie it is cocomplete. □

The following result is proven in Proposition 3.18 of [2] using the formalism of weak (co)fibration categories [3].

Lemma 3.15 *There is an equivalence of ∞–categories $\mathrm{Ind}(\mathcal{C}_\infty) \simeq \mathtt{N}\mathcal{S}$.*

Remark 3.16 Actually Proposition 3.18 of [2] proves a pointed version of the above lemma. The desired result can be shown using similar methods and hence its proof is omitted.

Theorem 3.17 *There is a diagram of adjunctions of presentable ∞–categories*

$$\begin{tikzcd}
 & \mathrm{N}(\mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}})^{\circ}_{\mathrm{mix}}) \arrow[r, bend right] \arrow[dl, bend left] & \mathcal{P}_\infty(\mathcal{C}_\infty) \arrow[l, bend right, "\widetilde{\theta}"'] \arrow[dr, bend left] & \\
\mathrm{N}(\mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}})^{\circ}) \arrow[ur, bend left] & & & \mathrm{Ind}(\mathcal{C}_\infty) \simeq \mathtt{N}\mathcal{S} \arrow[ul, bend left]
\end{tikzcd}$$

Proof The presentability of each ∞–category in the above diagram is clear. Observe that $\widetilde{\theta}\colon \mathcal{P}_\infty(\mathcal{C}_\infty) \to \mathrm{N}(\mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}})^{\circ}_{\mathrm{mix}})$ is a colimit-preserving functor between presentable ∞–categories (see Proposition 3.14). Hence, using the adjoint functor theorem (see Corollary 5.5.2.9 of [30]) we deduce that it admits a right adjoint. The existence of the adjunction pair $\mathcal{P}_\infty(\mathcal{C}_\infty) \rightleftarrows \mathrm{Ind}(\mathcal{C}_\infty) \simeq \mathtt{N}\mathcal{S}$ is standard (see for instance Theorem 5.5.1.1 of [30]). The adjunction $\mathrm{N}(\mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}})^{\circ}) \rightleftarrows \mathrm{N}(\mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}})^{\circ}_{\mathrm{mix}})$ has already been explained above. □

Remark 3.18 For the benefit of the reader we explain briefly the meaning and significance of this result. It is the author's perception that several results in the two paradigms of noncommutative geometry use very similar techniques, albeit in different contexts. For example, the constructions of the bivariant K–theory category and the category of noncommutative motives are philosophically almost identical (only applied to different notions of spaces). That led to the vision of abstracting away the commonalities and providing a framework whereby results can be transferred back and forth, creating synergies (see Section 0.1). In what follows we substantiate this assertion with a few potential directions for development.

4 Prospects: commutative spaces and graph algebras

It is known how to view commutative spaces (or motives) inside their noncommutative counterparts in the algebrogeometric setting [27; 50; 7]. We briefly explain how the ∞–category of spaces (not necessarily compact) sits inside that of noncommutative spaces via a colocalization in the setting of Connes. We also highlight how noncommutative dendrices naturally interpolate between the two canonical notions of *building blocks*.

4.1 Commutative spaces via colocalization

Let $\mathcal{S}$ (resp. $\mathcal{S}_*$) denote the ∞–category of spaces (resp. pointed spaces). It is shown in Theorem 1.9(1) of [36] that there is a fully faithful ω–continuous functor $\mathcal{S}_* \hookrightarrow \mathtt{N}\mathcal{S}_*$. In the same vein one can show that there is a fully faithful ω–continuous functor $\mathcal{S} \hookrightarrow \mathtt{N}\mathcal{S}$.

Proposition 4.1 *The fully faithful ω–continuous functor $\mathcal{S}_* \hookrightarrow \mathtt{N}\mathcal{S}_*$ (as well as $\mathcal{S} \hookrightarrow \mathtt{N}\mathcal{S}$) admits a right adjoint, ie it is colimit-preserving.*

Proof Due to the Gelfand–Naĭmark correspondence there is a fully faithful functor $f\colon \mathcal{S}_*^{\mathrm{fin}} \hookrightarrow \mathtt{SC}_\infty^{*\ \mathrm{op}}$ that induces the fully faithful ω–continuous functor $\mathrm{Ind}_\omega(f)\colon \mathcal{S}_* \hookrightarrow \mathtt{N}\mathcal{S}_*$ of Theorem 1.9(1) of [36]. The functor f preserves finite colimits, whence it is right exact. Therefore, by Proposition 5.3.5.13 of [31], the functor $\mathrm{Ind}_\omega(f)$ admits a right adjoint. The proof of the corresponding assertion for $\mathcal{S} \hookrightarrow \mathtt{N}\mathcal{S}$ is similar. □

Definition 4.2 We denote the right adjoint of $\mathcal{S}_* \hookrightarrow \mathtt{N}\mathcal{S}_*$ (resp. $\mathcal{S} \hookrightarrow \mathtt{N}\mathcal{S}$) in the above Proposition 4.1 by $\mathrm{US}_*\colon \mathtt{N}\mathcal{S}_* \to \mathcal{S}_*$ (resp. $\mathrm{US}\colon \mathtt{N}\mathcal{S} \to \mathcal{S}$) and call it the *underlying pointed space* (resp. *underlying space*) functor. Since US_* and US admit fully faithful left adjoints they are colocalizations, ie they constitute the commutative (pointed) space approximation of a noncommutative (pointed) space.

Now we are going to demonstrate how noncommutative dendrices interconnect simplices and matrices. Let T_n denote the linear graph

$$\bullet_0 \xleftarrow{e_1} \bullet_1 \xleftarrow{e_2} \cdots \xleftarrow{e_n} \bullet_n,$$

whose graph algebra $C^*(T_n)$ is isomorphic to $M_{n+1}(\mathbb{C})$ (the construction of the graph algebra is explained below in Section 4.2). Let $D^{\mathrm{ab}}(T_n)$ denote the commutative unital C^*–algebra generated by requiring the generators $\{q_{e_1},\dots,q_{e_n}\}$ of $D(T_n)$ to commute (see Remark 2.5). There is a canonical surjective $*$–homomorphism $\pi_n\colon D(T_n) \to D^{\mathrm{ab}}(T_n)$ that is identity on the generators. It follows from Proposition 2.1 of [16] that $D^{\mathrm{ab}}(T_n)$ is isomorphic to the commutative C^*–algebra $\mathrm{C}(\Delta^n)$. There is also a canonical $*$–homomorphism $s_n\colon D(T_n) \to C^*(T_{n-1}) \cong M_n(\mathbb{C})$, sending $q_{e_i} \mapsto e_{ii}$. Note that $\sum_{i=1}^n e_{ii}$ is the identity matrix that is the unit in the graph algebra $C^*(T_{n-1}) \cong M_n(\mathbb{C})$. Thus, we have a zigzag of arrows

$$\begin{tikzcd}
 & D(T_n) \arrow[dl, "\pi_n"'] \arrow[dr, "s_n"] & \\
D^{\mathrm{ab}}(T_n) \cong \mathrm{C}(\Delta^n) & & C^*(T_{n-1}) \cong M_n(\mathbb{C})
\end{tikzcd} \tag{8}$$

The set of $*$–homomorphisms $\{s_n \mid n \in \mathbb{N}\}$ defines a set of maps M in the ∞–category noncommutative spaces $\mathtt{N}\mathcal{S}$ via the functor $j\colon \mathtt{N}\mathcal{S}^{\mathtt{fin}} \to \mathtt{N}\mathcal{S}$. Thus, we are going to invert the maps in M to construct the simplex–matrix-identified version of $\mathtt{N}\mathcal{S}$. It is quite natural to consider matrix algebras as noncommutative simplices.

Definition 4.3 The accessible localization $L_M\colon \mathtt{N}\mathcal{S} \to M^{-1}\mathtt{N}\mathcal{S} =: \mathtt{N}\mathcal{S}^{\mathtt{SM}}$, which admits a fully faithful right adjoint, is defined to be the ∞–category of *simplex–matrix-identified noncommutative spaces*.

Remark 4.4 Since $\mathtt{N}\mathcal{S}$ is a presentable ∞–category, so is $\mathtt{N}\mathcal{S}^{\mathtt{SM}}$.

Remark 4.5 The composite functor $\mathtt{N}\mathcal{S}^{\mathtt{SM}} \hookrightarrow \mathtt{N}\mathcal{S} \xrightarrow{\mathrm{US}} \mathcal{S}$ defines the underlying space functor on $\mathtt{N}\mathcal{S}^{\mathtt{SM}}$. The subcategory of simplex–matrix-identified noncommutative spaces $\mathtt{N}\mathcal{S}^{\mathtt{SM}}$ is a tractable part of the entire ∞–category of noncommutative spaces $\mathtt{N}\mathcal{S}$ and it would be nice to explore it further.

Remark 4.6 Let $\mathtt{CW}^{\mathtt{fin}}$ denote the category of finite CW complexes. The geometric realization functor $|\cdot|\colon \mathtt{sSet} \to \mathrm{Ind}(\mathtt{CW}^{\mathtt{fin}})$ preserves (tensor) products and detects weak equivalences, whose counterpart in the world of dendroidal sets has been treated in [20; 5]. It is plausible (and desirable) that one could modify the functor

$\mathtt{dr}\colon \mathtt{dSet} \to \mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})$ to produce yet another *C^*–algebraic* or *noncommutative* geometric realization of dendroidal sets that fits into the commutative diagram

$$\begin{tikzcd}
\mathtt{sSet} \arrow[r, "|\cdot|"] \arrow[d] & \mathrm{Ind}(\mathtt{CW}^{\mathtt{fin}}) \arrow[d] \\
\mathtt{dSet} \arrow[r, "?"] & \mathrm{Ind}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}}) \subset \mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})
\end{tikzcd}$$

We leave it as an open problem.

4.2 Graph algebras

There is a vast literature on graph algebras (or graph C^*–algebras) with several interesting results relating structural aspects of the graph algebra (like simplicity) to purely graph-theoretic properties. We encourage the interested reader to consult for instance [49].

Let E be a finite *directed* graph and let $\mathcal{H}$ be a fixed separable Hilbert space. A *Cuntz–Krieger E–family $\{S, P\}$ on $\mathcal{H}$* (abbreviated as CK E–family) consists of a set $P = \{P_v \mid v \in E^0\}$ of mutually orthogonal projections on $\mathcal{H}$ and a set $S = \{S_e \mid e \in E^1\}$ of partial isometries on $\mathcal{H}$ such that

(CK1) $S_e^* S_e = P_{s(e)}$ for all $e \in E^1$; and

(CK2) $P_v = \sum_{\{e \in E^1 : r(e) = v\}} S_e S_e^*$ provided $\{e \in E^1 : r(e) = v\} \neq \varnothing$.

The graph algebra of E, denoted by $C^*(E)$, is by definition the universal C^*–algebra generated by $\{S, P\}$ subject to relations (CK1) and (CK2). It is known that $C^*(E)$ is unital if and only if the set of vertices E^0 is finite (see Proposition 1.4 of [28]).

Remark 4.7 Some authors prefer to write the relations (CK1) and (CK2) differently, viz the roles of r and s are interchanged. We have adopted the convention from [49]. The advantage of this viewpoint is that juxtaposition of edges in a path corresponds to composition of partial isometries on the Hilbert space $\mathcal{H}$.

Example 4.8 The graph algebra corresponding to the graph  is the Cuntz algebra $\mathcal{O}_2$.

The left Quillen functor $\mathtt{dr}\colon \mathtt{dSet} \to \mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})$ is obtained by the left Kan extension of $\Omega \xrightarrow{D} \mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}} \to \mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})$ along $\Omega \to \mathtt{dSet}$. Explicitly it is given by the formula

$$[\mathtt{dr}(X)](A) = \operatorname*{colim}_{f\colon D(T) \to A} X(T),$$

where the colimit is taken over the comma category $(D \downarrow A)$. The Quillen adjunction descends to an adjunction of homotopy categories

$$\mathbf{L}\mathtt{dr}\colon \mathrm{Ho}(\mathtt{dSet}) \rightleftarrows \mathrm{Ho}(\mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}})) :\mathbf{R}\mathtt{dd},$$

after taking the total derived functors of $\mathtt{dr}$ and $\mathtt{dd}$ ($\mathbf{L}\mathtt{dr}$ and $\mathbf{R}\mathtt{dd}$, respectively).

The composite $\mathbf{L}\mathtt{dr}\circ\mathbf{R}\mathtt{dd}$ defines a comonad on $\mathrm{Ho}(\mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}}))$. Viewing any separable unital C^*–algebra A inside $\mathrm{Ho}(\mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}}))$ via the Yoneda functor, we may consider the map given by the counit of the adjunction $\mathbf{L}\mathtt{dr}\circ\mathbf{R}\mathtt{dd}(A) \to \mathrm{Id}(A)$. It is presumably not an isomorphism; nevertheless, one should consider its comonadic resolution. If A is a graph algebra, this resolution can be viewed as a *resolution of the underlying graph by trees*. It would be nice to classify C^*–algebras up to this dendroidal invariant.

Remark 4.9 In the world of C^*–algebras a celebrated result of Kirchberg asserts that topological K–theory acts as a complete invariant on the subcategory of so-called *stable Kirchberg algebras* that satisfy UCT [26]. It was shown in [35; 15] that for such C^*–algebras (in fact for a larger subcategory of C^*–algebras) algebraic K–theory is naturally isomorphic to topological K–theory (see Theorem 2.4 and Remark 1 of [35]). If the vision outlined in the introduction can be realized, viz if one can show that algebraic K–theory and KK–theory can be recovered from diagram (2), then the above-mentioned construction would provide a *higher invariant* that has the potential to act as a complete invariant on a bigger subcategory than that of stable Kirchberg algebras satisfying UCT. Observe that topological K–theory is also the primary classification tool for graph algebras. It would be actually more prudent to analyze this construction for a graph algebra at the level of underlying ∞–categories (and not at the level of homotopy categories), possibly after passing to the stabilization.

Appendix The model structure on $\mathcal{P}({\mathtt{SC}^*_{\mathrm{un}}}^{\mathrm{op}})$

For any small category $\mathcal{C}$ there is a *Cisinski model structure* on $\mathcal{P}(\mathcal{C})$ [9], whose construction is described below. A *functorial cylinder object* is an endofunctor

$$I \otimes (-)\colon \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C})$$

such that for every $X \in \mathcal{P}(\mathcal{C})$ there are natural morphisms ∂^0_X, ∂^1_X and σ_X that satisfy:

(1) The following diagram commutes:

$$\begin{tikzcd}
X \arrow[dr, "\partial^0_X"'] \arrow[drr, bend left, "\mathrm{id}_X"] & & \\
& I \otimes X \arrow[r, "\sigma_X"] & X \\
X \arrow[ur, "\partial^1_X"] \arrow[urr, bend right, "\mathrm{id}_X"'] & &
\end{tikzcd}$$

(2) The canonical morphism $X \amalg X \to I \otimes X$ induced by ∂^0_X and ∂^1_X is a monomorphism.

The choice of a functorial cylinder object $\mathcal{J} = (I \otimes (-), \partial^0_{(-)}, \partial^1_{(-)}, \sigma_{(-)})$ constitutes an *elementary homotopical datum* if $\mathcal{J}$ satisfies the following two additional conditions:

(i) the functor $I \otimes (-)$ commutes with small colimits, and

(ii) for every monomorphism $j \colon K \to L$ in $\mathcal{P}(\mathcal{C})$ for $e = 0, 1$, the diagram

$$\begin{tikzcd}
K \arrow[r, "j"] \arrow[d, "\partial^e_K"'] & L \arrow[d, "\partial^e_L"] \\
I \otimes K \arrow[r, "I\otimes j"] & I \otimes L
\end{tikzcd}$$

is a pullback square.

Using the functorial cylinder object $\mathcal{J}$, one can define an *elementary $\mathcal{J}$–homotopy* between two maps in $\mathcal{P}(\mathcal{C})$, viz two maps $f, g \colon X \to Y$ are elementary $\mathcal{J}$–homotopic if there is a map $\eta \colon I \otimes X \to Y$ making the following diagram commute:

$$\begin{tikzcd}
X \arrow[d, "\partial^0_X"'] \arrow[dr, "f"] & \\
I \otimes X \arrow[r, "\eta"] & Y \\
X \arrow[u, "\partial^1_X"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Let $\mathrm{Ho}_{\mathcal{J}}\mathcal{P}(\mathcal{C})$ denote the category whose objects are those of $\mathcal{P}(\mathcal{C})$ and whose morphisms are the elementary $\mathcal{J}$–homotopy classes of morphisms of $\mathcal{P}(\mathcal{C})$.

Definition A.1 There is a canonical functor $\mathcal{P}(\mathcal{C}) \to \mathrm{Ho}_{\mathcal{J}}\mathcal{P}(\mathcal{C})$ and the morphisms that descend to isomorphisms under this functor are called *$\mathcal{J}$–homotopy equivalences*. This notion obviously depends on the choice of $\mathcal{J}$.

The model structure on $\mathcal{P}(\mathcal{C})$ depends on another choice, viz a class $\mathtt{An}$ of *anodyne extensions*. For a class M of maps of $\mathcal{P}(\mathcal{C})$ we denote by $\mathrm{llp}(M)$ (resp. $\mathrm{rlp}(M)$) the class of maps that satisfy the left (resp. right) lifting property with respect to M. For any cartesian square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

in $\mathcal{P}(\mathcal{C})$ with $Y \to W$ and $Z \to W$ monomorphisms, the canonical map $Y \amalg_X Z \to W$ is also a monomorphism. For brevity this monomorphism is suggestively written as $Y \cup Z \to W$.

Definition A.2 Let $\mathcal{J}$ be an elementary homotopy datum on $\mathcal{P}(\mathcal{C})$. Then a *class of anodyne extensions* $\mathtt{An}$ *relative to* $\mathcal{J}$ is a class of morphisms in $\mathcal{P}(\mathcal{C})$ such that

(a) $\mathtt{An} = \mathrm{llp}(\mathrm{rlp}(M))$ for a small set of maps M,

(b) for any monomorphism $K \to L$ and $e = 0, 1$, the induced map $I \otimes K \cup \{e\} \otimes L \to I \otimes L$ belongs to $\mathtt{An}$, and

(c) if $K \to L$ belongs to $\mathtt{An}$, then so does $I \otimes K \cup \partial I \otimes L \to I \otimes L$, where $\partial I \otimes L = L \amalg L$.

Remark A.3 It is shown in Proposition 1.3.13 of [9] that for any small set S of monomorphisms of $\mathcal{P}(\mathcal{C})$ there is a smallest class of anodyne extensions relative to $\mathcal{J}$ that is generated by S. This class of morphisms is denoted by $\mathtt{An}_{\mathcal{J}}(S)$.

Theorem A.4 [9, Théorème 1.3.22] *Let $\mathcal{J}$ be an elementary homotopy datum on $\mathcal{P}(\mathcal{C})$ and $\mathtt{An}_{\mathcal{J}}(S)$ be a class of anodyne extensions relative to $\mathcal{J}$ that is generated by a small set S of monomorphisms. Then there is a combinatorial model structure on $\mathcal{P}(\mathcal{C})$ satisfying*

(1) *the cofibrations are the monomorphisms,*

(2) *$X \in \mathcal{P}(\mathcal{C})$ is fibrant if the map $X \to \star$, where $\star$ is the terminal object, satisfies the right lifting property with respect to all anodyne extensions $\mathtt{An}_{\mathcal{J}}(S)$, and*

(3) *a map $f\colon X \to Y$ is a weak equivalence if for all fibrant objects Z the induced map $f^*\colon \mathrm{Ho}_{\mathcal{J}}\mathcal{P}(\mathcal{C})(Y, Z) \to \mathrm{Ho}_{\mathcal{J}}\mathcal{P}(\mathcal{C})(X, Z)$ is bijective.*

Remark A.5 The Cisinksi model structure on $\mathcal{P}(\mathcal{C})$ admits a functorial fibrant replacement. A set of generating cofibrations can be chosen to be those monomorphisms whose codomains are quotients of representable presheaves (see Proposition 1.2.27 of [9]). Every object of $\mathcal{P}(\mathcal{C})$ is cofibrant and its homotopy category is equivalent to the full subcategory of $\mathrm{Ho}_{\mathcal{J}}\mathcal{P}(\mathcal{C})$ spanned by the fibrant objects (see 1.3.23 of [9]). Moreover, a morphism between two fibrant objects is a weak equivalence if and only if it is a $\mathcal{J}$–homotopy equivalence.

Proposition A.6 *The functor* $\mathtt{dr}\colon \mathtt{dSet} \to \mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})$ *preserves cofibrations.*

Proof The set of generating cofibrations in $\mathtt{dSet}$ is $\{\partial\Omega[T] \to \Omega[T] \mid T \in \Omega\}$. Each face map $\partial\colon T' \to T$ of trees induces a monomorphism of representable presheaves, whose image is specified by the datum of this monomorphism of representable presheaves (see Chapter IV of [32]). For any tree T the boundary inclusion $\partial\Omega[T] \to \Omega[T]$ is obtained as a union of the images of such face maps. We know that $\mathtt{dr}$ sends the representable presheaf of T to that of $D(T)$. Each face map $\partial\colon T' \to T$ in Ω induces a surjective $*$–homomorphism $\partial^*\colon D(T) \to D(T')$ in $\mathtt{SC}^*_{\mathrm{un}}$ (see Section 2.1). It induces a monomorphism in $\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}}$ and the Yoneda embedding preserves monomorphisms, whence $\mathtt{dr}(\partial)\colon \mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}}(-, D(T')) \to \mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}}(-, D(T))$ is a monomorphism in $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})$. It follows from the universal property of the noncommutative dendrices construction that $\mathtt{dr}$ sends the generating cofibrations of $\mathtt{dSet}$ to monomorphisms of $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})$. Note that the cofibrations of $\mathcal{P}(\mathtt{SC}^{*\ \mathrm{op}}_{\mathrm{un}})$ are precisely the monomorphisms, whence Lemma 2.1.20 of [24] shows that $\mathtt{dr}$ preserves cofibrations. □

Remark A.7 It is clear that the above proposition does not depend on the choice of $\mathcal{J}$.

For the choice of the elementary homotopy datum we have a few possibilities at our disposal.

Example A.8 [9, Example 1.3.9] Let $\mathcal{C}$ be any small category. For an object $C \in \mathcal{C}$ let us denote the representable presheaf of C in $\mathcal{P}(\mathcal{C})$ by h_C. Let $\mathcal{L}$ denote the presheaf that associates with every $C \in \mathcal{C}$ the set $\mathcal{L}(C) = \{\text{subobjects of } h_C\}$. For every map $u\colon C \to D$ in $\mathcal{C}$ the map $\mathcal{L}(D) \to \mathcal{L}(C)$ is induced by pullback along u. The presheaf $\mathcal{L}$ turns out to be a subobject classifier, ie $\mathcal{P}(\mathcal{C})(X, \mathcal{L}) \simeq \{\text{subobjects of the presheaf } X\}$. If $\star$ is the final object of $\mathcal{P}(\mathcal{C})$, then it has exactly two subobjects $\star \hookrightarrow \star$ and $\varnothing \hookrightarrow \star$, where $\varnothing$ denotes the initial object of $\mathcal{P}(\mathcal{C})$. These define two morphisms $\lambda_0, \lambda_1\colon \star \to \mathcal{L}$. The tuple $(\mathcal{L}, \lambda_0, \lambda_1)$ gives rise to an elementary homotopy datum by setting $I \otimes X = \mathcal{L} \times X$, $\partial^e_X = \lambda_e \times \mathrm{id}_X$ for $e = 0, 1$ and $\sigma_X = \mathrm{pr}_2\colon \mathcal{L} \times X \to X$. This elementary homotopy datum is called the *Lawvere cylinder*, and exists in any category of presheaves like $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$.

Example A.9 For any *nonzero* separable unital C^*–algebra A there is a sequence of two $*$–homomorphisms $A \xrightarrow{\iota} A[0,1] := \mathrm{C}([0,1], A) \xrightarrow{\mathrm{ev}_t} A$ for any $t \in [0,1]$ (natural in A), whose composition is the identity $*$–homomorphism on A. Here $\iota(a)$ is the constant a–valued function on $[0,1]$ for every $a \in A$ and ev_t is the evaluation at $t \in [0,1]$. For $A = \mathbb{C}$, after reversing the arrows and passing to the representable presheaves in $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$ we get the square

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & h_{\mathbb{C}} \arrow[d, "\partial^1 = \mathrm{ev}_1^*"] \\ h_{\mathbb{C}} \arrow[r, "\partial^0 = \mathrm{ev}_0^*"] & h_{\mathrm{C}([0,1])} \end{tikzcd} \tag{9}$$

where $\varnothing$ is the initial object (empty presheaf) of $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$. Note that $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$ are $\mathtt{Set}$–valued covariant functors on $\mathtt{SC}^{*}_{\mathrm{un}}$ and we do not notationally distinguish between objects in a category and in its opposite. For every $A \in \mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}}$ we find that the diagram

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & h_{\mathbb{C}}(A) \arrow[d, "\mathrm{ev}_1^*"] \\ h_{\mathbb{C}}(A) \arrow[r, "\mathrm{ev}_0^*"] & h_{\mathrm{C}([0,1])}(A) \end{tikzcd}$$

is a pullback square in $\mathtt{Set}$. Indeed, $h_{\mathbb{C}}(A) = \mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}}(A, \mathbb{C}) = \{\mathbf{1}_A\}$, where $\mathbf{1}_A$ is the unique unital $*$–homomorphism $\mathbb{C} \to A$ and $(\mathbf{1}_A \circ \mathrm{ev}_t^*)(f) = f(t)\mathbf{1}_A$ for $t = 0, 1$ and for every $f \in \mathbb{C}[0,1] = \mathrm{C}([0,1], \mathbb{C})$. In this argument it is crucial that A is a *nonzero* separable unital C^*–algebra. Since limits are computed objectwise in $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$ we conclude that diagram (9) is a pullback square. It follows from Example 1.3.8 of [9] that

$$\mathfrak{J} = (I \times X,\ \partial^0 \times \mathrm{id}_X,\ \partial^1 \times \mathrm{id}_X,\ \mathrm{pr}_X\colon I \times X \to X)$$

defines an elementary homotopy datum.

Example A.10 (continuous cylinder) Consider again the sequence of $*$–homomorphisms $A \xrightarrow{\iota} A[0,1] \xrightarrow{\mathrm{ev}_t} A$ (natural in A), whose composition is the identity $*$–homomorphism on A. Given any representable object h_A we set $I \otimes h_A = h_{A[0,1]}$ and extend the cylinder construction to all objects of $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$ by commuting with colimits, ie if $X \cong \mathrm{colim}_i\, h_{A_i}$, then we set $I \otimes X \cong \mathrm{colim}_i\, h_{A_i[0,1]}$.

We choose the elementary homotopy datum of Example A.8 since it is the most canonical choice for the Cisinski model structure on any presheaf category. Subsequently we are going to localize our model structure based on our requirements. Let X be a set of generating trivial cofibrations of `dSet` and set $S = \mathtt{dr}(X)$. By Proposition A.6, S is a set of monomorphisms of $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$ that generates a class of anodyne extensions $\mathtt{An}_{\mathcal{J}}(S)$ relative to $\mathcal{J}$ (see Remark A.3). As a consequence of Theorem A.4 we obtain:

Theorem A.11 (operadic model structure) *With the choice of the elementary homotopy datum $\mathcal{J}$ of Example A.8 and the class of anodyne extensions $\mathtt{An}_{\mathcal{J}}(S)$ relative to $\mathcal{J}$ described above, $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$ acquires the structure of a combinatorial model category.*

Remark A.12 The Lawvere cylinder is different from the continuous cylinder of Example A.10. Hence, the evaluation map $A[0,1] \xrightarrow{\mathrm{ev}_t} A$ is not a weak equivalence in the operadic model structure; it roughly mirrors the Joyal model structure on the category of simplicial sets, in which $\Delta^1 \to \Delta^0$ is not a weak equivalence.

Remark A.13 It is shown in Lemma 1.3.31 of [9] that every anodyne extension is a weak equivalence. Since $\mathtt{dr}(X) = S \subset \mathtt{An}_{\mathcal{J}}(S)$, where X is the set of generating trivial cofibrations of `dSet`, we observe that, by construction, the functor `dr` sends generating trivial cofibrations of `dSet` to trivial cofibrations of $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$.

Remark A.14 The construction of the Cisinski model structure can be profitably used in other contexts. For instance, one can start with a small category $\mathcal{A}$ of topological algebras (Banach, Fréchet or locally convex) with some mild hypotheses. Then one can simply start with the minimal model structure on $\mathcal{P}(\mathcal{A}^{\mathrm{op}})$ by choosing the Lawvere cylinder (see Example A.8) for the elementary homotopy datum $\mathcal{J}$ and $\mathtt{An}_{\mathcal{J}}(\varnothing)$ for the class of anodyne extensions. Now one can localize this combinatorial model category by inverting a small set of morphisms like differentiable homotopy equivalences between the representable objects in $\mathcal{P}(\mathcal{A}^{\mathrm{op}})$. This would produce an unstable model category to start with that can be (∞–categorically) stabilized and localized further according to one's requirements; for instance, one can aim for a stable ∞–category whose morphism groups model the Cuntz kk–groups for locally convex algebras [17]. Østvær developed his homotopy theory of C^*–algebras adopting a similar strategy in the setting of cubical set-valued presheaves on the category of separable C^*–algebras [47] but we do not expect a Quillen equivalence between his unstable model category for *cubical C^*–spaces* and $\mathcal{P}(\mathtt{SC}^{*}_{\mathrm{un}}{}^{\mathrm{op}})$ equipped with the operadic model structure as in Theorem A.11. This is because the evaluation map $A[0,1] \xrightarrow{\mathrm{ev}_t} A$ of the continuous cylinder construction (see Example A.10) is not a weak equivalence in the operadic model structure. One final observation — all the ingredients needed to develop a Waldhausen K–theory of noncommutative spaces are now at our disposal.

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Examples of nontrivial contact mapping classes for overtwisted contact manifolds in all dimensions

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We construct (infinitely many) examples in all dimensions of contactomorphisms of closed overtwisted contact manifolds that are smoothly isotopic but not contact-isotopic to the identity.

53D10, 57R17

1 Introduction

One of the problems in the field of contact topology is to understand the topology of the space of contactomorphisms $\mathcal{D}(V,\xi)$ of a given contact manifold (V,ξ) in comparison with that of the space of diffeomorphisms $\mathcal{D}(V)$ of the underlying smooth manifold V or, more specifically, the problem of understanding the map $j_*\colon \pi_k(\mathcal{D}(V,\xi)) \to \pi_k(\mathcal{D}(V))$ induced by the natural inclusion $j\colon \mathcal{D}(V,\xi) \to \mathcal{D}(V)$.

If $\Xi(V)$ denotes the space of all the contact structures on V, in the case of closed manifolds the natural map $\mathcal{D}(V) \to \Xi(V)$ given by $\phi \mapsto \phi_*\xi$ helps to understand the properties of the j_*, and shows that the relation between the topology of $\mathcal{D}(V,\xi)$ and that of $\mathcal{D}(V)$ is mediated by the topology of $\Xi(V)$. Indeed, (the proof of) Gray's theorem implies, modulo a general fibration criterion, that this map is a locally trivial fibration with fiber $\mathcal{D}(V,\xi)$; see for instance Giroux and Massot [21] for an explanation of this result or Massot [26] for a more detailed proof (the reader can also consult Geiges and Gonzalo Perez [14] for a proof of the fact that the map is a Serre fibration). Then the exact long sequence of homotopy groups

$$\cdots \to \pi_{k+1}(\Xi(V)) \to \pi_k(\mathcal{D}(V,\xi)) \xrightarrow{j_*} \pi_k(\mathcal{D}(V)) \to \pi_k(\Xi(V)) \to \cdots$$

associated to the fibration gives a relationship between the topologies of the three spaces $\mathcal{D}(V)$, $\mathcal{D}(V,\xi)$ and $\Xi(V)$.

As far as the 3–dimensional case is concerned, the availability of classification results for the isotopy classes of tight contact structures on particular 3–manifolds V gives

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some explicit results about the lower homotopy groups in the long exact sequence above for these specific manifolds. The reader can consult Geiges and Gonzalo Perez [14], Bourgeois [5], Ding and Geiges [9], Geiges and Klukas [15] and Giroux and Massot [21] for results on $\pi_1(\Xi(V),\xi)$ as well as Giroux [19], Giroux and Massot [21] for results on $\pi_0(\mathcal{D}(V,\xi))$.

The situation in higher dimensions is more complicated, due to the lack of classification results. The only results known so far are contained in Bourgeois [5], Massot and Niederkrüger [27] and Lanzat and Zapolsky [23]. In the first paper, Bourgeois gives results on some homotopy groups $\pi_k(\Xi(V),\xi)$, for particular contact manifolds (V,ξ), using tools from contact homology. In [27], the authors give examples of contact manifolds (V,ξ) for which $\ker\bigl(\pi_0(\mathcal{D}(V,\xi))\to\pi_0(\mathcal{D}(V))\bigr)$ is nontrivial; these examples rely on constructions in Massot, Niederkrüger and Wendl [28], which we will also use in the following. The last paper, dealing with the noncompact case, contains examples of embeddings of braid groups in the contactomorphism group of contactizations of certain noncompact symplectic manifolds.

All the examples recalled so far are given on *tight* contact manifolds. For the 3–dimensional case, the dichotomy tight–overtwisted has been well known since Eliashberg [10] and plays an important role in the classification results on which the cited examples are based. In the higher-dimensional case, a clear definition of overtwistedness is given in Borman, Eliashberg and Murphy [3] and according to it the three examples above are also tight.

As far as the class of overtwisted manifolds is concerned, the only result known at the moment is the classification result of the path components of the space of contactomorphisms for all overtwisted contact structures on the 3–sphere. This result, without proof until recently, is attributed to Chekanov according to Eliashberg and Fraser [11, Remark 4.16]. Vogel published a complete proof of this classification in [31], where it is also proven, using 3–dimensional techniques, that the space of embeddings of overtwisted disks in one of the overtwisted contact structures on $\mathbb{S}^3$ is not path-connected. This gives in particular the first known examples of contactomorphisms of overtwisted 3–manifolds that are smoothly isotopic but not contact-isotopic to the identity (we recall that, according to Cerf [8], each orientation-preserving diffeomorphism of the 3–sphere is smoothly isotopic to the identity).

In this article we give other explicit examples of overtwisted (V,ξ) such that the kernel of $\pi_0(\mathcal{D}(V,\xi))\to\pi_0(\mathcal{D}(V))$ is nontrivial. Though, we bypass here the problem of understanding the π_0 of the space of embeddings of overtwisted disks, about which nothing is known so far in high dimensions; the advantage of our approach is then that it gives (infinitely many) examples in each odd dimension.

More precisely, we start by proving the following result:

Theorem 1.1 *Consider a closed manifold W of dimension $2n \geq 2$ and let ξ be a coorientable contact structure on the manifold $V := \mathbb{S}^1 \times W$. Suppose that the first Chern class $c_1(\xi) \in H^2(V;\mathbb{Z})$ is toroidal and that, for each natural $k \geq 2$, the pullback $\pi_k^*\xi$ of ξ via the k–fold cover $\pi_k\colon \mathbb{S}^1 \times W \to \mathbb{S}^1 \times W$ given by $\pi_k(s,p) = (ks,p)$ satisfies $c_1(\pi_k^*\xi) = k \cdot c_1(\xi)$ modulo the submodule $H^2_{\mathrm{ator}}(V;\mathbb{Z})$ of atoroidal classes.*

Then the contact transformation $f\colon (\mathbb{S}^1 \times W, \pi_k^*\xi) \to (\mathbb{S}^1 \times W, \pi_k^*\xi)$ defined by $f(s,p) = \bigl(s + \frac{2\pi}{k}, p\bigr)$ is smoothly isotopic but not contact-isotopic to the identity.

Recall that a class $c \in H^2(V;\mathbb{Z})$ is called *toroidal* if there is $f\colon \mathbb{T}^2 \to V$ such that $f^*c \neq 0 \in H^2(\mathbb{T}^2;\mathbb{Z})$, and *atoroidal* otherwise.

Remark Theorem 1.1 also holds (with similar proof) if one exchanges

($*$) $c_1(\xi)$ is toroidal and $c_1(\pi_k^*\xi) = k \cdot c_1(\xi) \bmod H^2_{\mathrm{ator}}(V;\mathbb{Z})$ for each natural $k \geq 2$,

with the condition

($*'$) $c_1(\xi)$ is not torsion and $c_1(\pi_k^*\xi) = k \cdot c_1(\xi)$ for each natural $k \geq 2$.

Notice that $a \in H^2(V;\mathbb{Z})$ is toroidal if and only if $[a] \in H^2(V;\mathbb{Z})/H^2_{\mathrm{ator}}(V;\mathbb{Z})$ is not torsion, because $H^2(\mathbb{T}^2;\mathbb{Z}) \simeq \mathbb{Z}$. In particular, ($*$) is equivalent to

$c_1(\xi)$ is not torsion modulo $H^2_{\mathrm{ator}}(V;\mathbb{Z})$ and $c_1(\pi_k^*\xi) = k \cdot c_1(\xi) \bmod H^2_{\mathrm{ator}}(V;\mathbb{Z})$,

hence it is just a variation modulo $H^2_{\mathrm{ator}}(V;\mathbb{Z})$ of ($*'$) (and it is neither stronger nor weaker than ($*'$)). Slightly anticipating what follows, we also point out that the contact structures given in Theorem 1.2, Proposition 1.4 and Theorem 1.3(i) below actually satisfy both ($*$) and ($*'$); on the other hand, working modulo $H^2_{\mathrm{ator}}(V;\mathbb{Z})$, ie with ($*$), is necessary for Theorem 1.3(ii) We hence decided to formulate everything in terms of ($*$), even though ($*'$) would give (everywhere but in Theorem 1.3(ii)) slightly more direct proofs.

We then give, for each natural $n \geq 1$, an infinite number of *explicit* overtwisted contact manifolds $(\mathbb{S}^1 \times W^{2n}, \xi)$ satisfying the hypothesis of Theorem 1.1:

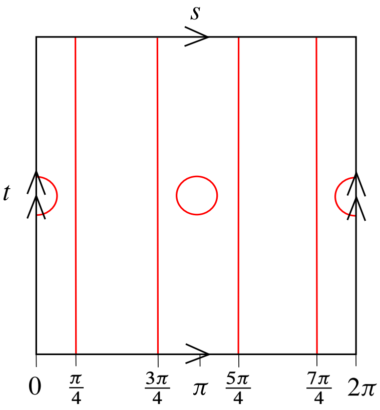

Figure 1: Dividing set on the torus $\mathbb{T}^2_{(s,t)} \times \{\theta_0\}$

Theorem 1.2 *Let $(M^{2n-1}, \alpha_+, \alpha_-)$ be one of the infinitely many Liouville pairs constructed in Massot, Niederkrüger and Wendl [28]. Consider the (coorientable) contact structure $\eta = \ker\bigl(\frac{1}{2}(1+\cos s)\alpha_+ + \frac{1}{2}(1-\cos s)\alpha_- + \sin s\, dt\bigr)$ on the manifold $V := \mathbb{T}^2_{(s,t)} \times M$ (here, the notation $\mathbb{T}^2_{(s,t)}$ denotes the choice of coordinates (s,t) on $\mathbb{T}^2$) and denote by ξ the overtwisted contact structure obtained from η via a half Lutz–Mori twist along $\{(0,0)\} \times M$, as defined in Massot, Niederkrüger and Wendl [28].*

Then $c_1(\xi) \in H^2(V;\mathbb{Z})$ is toroidal and, for each natural $k \geq 2$, we have $c_1(\pi_k^*\xi) = k \cdot c_1(\xi) \bmod H^2_{\mathrm{ator}}(V;\mathbb{Z})$, where $\pi_k\colon \mathbb{T}^2_{(s,t)} \times M \to \mathbb{T}^2_{(s,t)} \times M$ is given by $\pi_k(s,t,q) = (ks,t,q)$.

Example If $n = 3$, $(M, \alpha_\pm) = (\mathbb{S}^1, \pm d\theta)$. Moreover, if $k = 2$, the contact structure $\pi_2^*\xi$ on $V := \mathbb{T}^2 \times M$ is the unique (up to isotopy) contact structure which is invariant by the left action by multiplication of $M = \mathbb{S}^1$ on V, invariant by the $f(s,t,\theta) = (s+\pi, t, \theta)$ defined in the statement and such that each torus $\mathbb{T}^2_{(s,t)} \times \{\theta_0\}$ is convex with dividing set as in Figure 1. Theorems 1.2 and 1.1 then say that f is not contact-isotopic to the identity; to our knowledge, even in this simple and very explicit setting, there is no trace of this result in the literature.

If one is just interested in giving examples, in each odd dimension, of nontrivial elements in the kernel of the map $\pi_0(\mathcal{D}(V,\xi)) \to \pi_0(\mathcal{D}(V))$, without wanting the underlying overtwisted contact manifolds (V,ξ) to be as explicit as those from Theorem 1.2, the following result can also be proven using the existence of adapted open-book decompositions proven by Giroux [19]:

Theorem 1.3 *Consider W a closed $2n$–dimensional manifold and η a coorientable overtwisted contact structure on $V := \mathbb{S}^1 \times W$. Suppose that $c_1(\eta)$ is toroidal and that, for each $k \geq 2$, the pullback of η via the k–fold covering $\pi_k \colon V \to V$, given by $\pi_k(s, p) = (ks, p)$, satisfies $c_1(\pi_k^* \eta) = k \cdot c_1(\eta) \mod H^2_{\mathrm{ator}}(V; \mathbb{Z})$. Then:*

(i) *Each contact structure ξ on $V \times \mathbb{T}^2$ obtained via the Bourgeois construction [4] from (V, η) (is coorientable and) has first Chern class also satisfying the above conditions, with respect to the covering $\mu_k := (\pi_k, \mathrm{Id}) \colon V \times \mathbb{T}^2 \to V \times \mathbb{T}^2$.*

(ii) *Let $\nu \colon V \times \Sigma_g \to V \times \mathbb{T}^2$ be induced by a covering $\Sigma_g \to \mathbb{T}^2$ branched over two points* (here, Σ_g denotes the closed surface of genus $g \geq 2$). *Then every contact branched covering ξ_g of ξ on $V \times \Sigma_g$ (is coorientable and) has first Chern class satisfying the above conditions, with respect to the covering $\mu_k^g := (\pi_k, \mathrm{Id}) \colon V \times \Sigma_g \to V \times \Sigma_g$. Moreover, if η is overtwisted and g is large enough, ξ_g is also overtwisted.*

By an induction on the dimension, Theorem 1.3 gives, for any integer $n \geq 2$, examples of $(\mathbb{S}^1 \times W^{2n}, \xi)$ whose first Chern class satisfies the desired conditions. As far as point (ii) is concerned, the reader can consult Geiges [12] for a construction and Gironella [17] for a definition of *contact branched coverings*. We also point out that the optimal integer g to guarantee overtwistedness of η_g is actually 2, according to an observation due to Massot and Niederkrüger (see Gironella [17, Observation 5.10]).

Using the h–principle of Borman, Eliashberg and Murphy [3], an even bigger class of (nonexplicit) examples can be obtained:

Proposition 1.4 *Consider a closed connected manifold W^{2n} which is almost complex, spin and satisfies $H^1(W; \mathbb{Z}) \neq \{0\}$. Then there is a coorientable overtwisted contact structure ξ on $V := \mathbb{S}^1 \times W$ such that $c_1(\xi)$ is toroidal and $c_1(\pi_k^* \xi) = k \cdot c_1(\xi) \mod H^2_{\mathrm{ator}}(V; \mathbb{Z})$, where $\pi_k \colon \mathbb{S}^1_s \times W \to \mathbb{S}^1_s \times W$ is given by $\pi_k(s, p) = (ks, p)$.*

Outline

Section 2 contains a proof by contradiction of Theorem 1.1. Assuming that the contactomorphism f is contact-isotopic to the identity, we construct a contactomorphism between two contact structures ξ_1 and ξ_2; on the other hand, the hypothesis on the first Chern class of ξ implies that ξ_1 and ξ_2 are not even isomorphic as almost contact structures.

Section 3 shows how to obtain examples of contact manifolds $(\mathbb{S}^1 \times W^{2n}, \xi)$ satisfying the hypothesis of Theorem 1.1 starting from Massot, Niederkrüger and Wendl [28].

More precisely, Sections 3.1 and 3.2 recall, respectively, the definition of half Lutz–Mori twist and the explicit constructions of Liouville pairs, both from Massot, Niederkrüger and Wendl [28]. Then Section 3.3 describes the effects of a half Lutz–Mori twist on Chern classes in this context and Section 3.4 contains a proof of Theorem 1.2.

Finally, in Section 4 we show how to get examples of contactomorphisms smoothly isotopic but not contact-isotopic to the identity using the existence of adapted open-book decompositions proven by Giroux [20] and the h–principle of Borman, Eliashberg and Murphy [3]. More precisely, Theorem 1.3 and Proposition 1.4 are proven in Sections 4.2 and 4.1, respectively.

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2 Proof of Theorem 1.1

As each contactomorphism gives in particular an isomorphism of the underlying almost contact structures, Theorem 1.1 directly follows from the two following lemmas:

Lemma 2.1 *Let* $(\mathbb{S}^1 \times W^{2n}, \xi)$ *be a contact manifold, with* ξ *coorientable. For each natural* $k \geq 2$*, denote by* $\pi_k\colon \mathbb{S}^1 \times W \to \mathbb{S}^1 \times W$ *the* k*–fold cover* $\pi_k(s,p) = (ks, p)$ *and by* $f\colon (\mathbb{S}^1 \times W, \pi_k^*\xi) \to (\mathbb{S}^1 \times W, \pi_k^*\xi)$ *the contactomorphism* $f(s,p) = \left(s + \frac{2\pi}{k}, p\right)$*.*

If f *is contact-isotopic to the identity, then there is a contactomorphism*

$$\phi\colon (\mathbb{S}^1 \times W, \pi_{kN}^*\xi) \xrightarrow{\sim} (\mathbb{S}^1 \times W, \pi_{kN+1}^*\xi).$$

Lemma 2.2 *Let* $(V := \mathbb{S}^1 \times W, \xi)$*,* π_k *and* f *be as in Lemma 2.1. If moreover* $c_1(\xi)$ *is toroidal and* $c_1(\pi_m^*\xi) = m \cdot c_1(\xi) \bmod H^2_{\mathrm{ator}}(V;\mathbb{Z})$ *for every natural* $m \geq 2$*, then* $\pi_m^*\xi$ *and* $\pi_{m+1}^*\xi$ *are not isomorphic as almost contact structures.*

Proof of Lemma 2.1 In order to find the desired contactomorphism ϕ, we use an idea that already appeared in Geiges and Gonzalo Perez [14] and in Marinković and Pabiniak [25], and which consists in cutting off contact hamiltonians on a particular cover of the manifold we are working with.

By hypothesis, the contactomorphism $f\colon (\mathbb{S}^1 \times W, \pi_k^*\xi) \to (\mathbb{S}^1 \times W, \pi_k^*\xi)$ defined by $f(s,p) = \left(s + \frac{2\pi}{k}, p\right)$ is contact-isotopic to the identity. Call $(F_r)_{r\in[0,1]}$ the isotopy, so that $F_0 = \mathrm{Id}$, $F_1 = f$ and F_r is a contactomorphism for all $r \in [0,1]$.

Take now the universal cover $\mathbb{R}_s$ of the factor $\mathbb{S}^1_s$ of the manifold $\mathbb{S}^1_s \times W$. Then pull back $\pi_k^*\xi$ to a contact structure η_k on the covering $\mathbb{R}_s \times W$ of $\mathbb{S}^1_s \times W$ and lift the contact isotopy F_r to a contact isotopy Φ_r of $(\mathbb{R}_s \times W, \eta_k)$ starting at the identity. Fix a certain contact form β_k for η_k and denote by $H_r\colon \mathbb{R}_s \times W \to \mathbb{R}$ the path of contact hamiltonians $\beta_k(Y_r)$ associated to the contact vector field Y_r generating the isotopy Φ_r (see for instance Geiges [13, Section 2.3] for more details on contact hamiltonians).

Now, by compactness of W and $[0,1]$, there is an $N > 0$ such that, for each $r \in [0,1]$, $\Phi_r(\{0\}_s \times W)$ is contained in $(-2(N-1)\pi, +\infty)_s \times W$.

Consider then an $\epsilon > 0$ very small and a smooth function $\rho\colon \mathbb{R} \to \mathbb{R}$ such that $\rho(x) = 0$ for $x < -2N\pi + \epsilon$ and $\rho(x) = 1$ for $x > -2(N-1)\pi - \epsilon$. We can then construct a new contact hamiltonian, $K_r(s,p) := \rho(s) \cdot H_r(s,p)$ for all $(s,p) \in \mathbb{R}_s \times W$.

We claim that the contact vector field Z_r associated to this new hamiltonian K_r (ie the unique contact vector field Z_r such that $\beta_k(Z_r) = K_r$; see for instance [13, Section 2.3]) can be integrated to a contact isotopy $(\Psi_r)_{r\in[0,1]}$ of $(\mathbb{R}_s \times W, \eta_k)$

starting at the identity. Indeed, Z_r is zero for $s < -2N\pi + \epsilon$ and equal to the contact field Y_r for $s > -2(N-1)\pi - \epsilon$, which means in particular that it is integrable outside of a compact set of $\mathbb{R}_s \times W$ (note that Y_r is trivially integrable, because it comes from a contact isotopy); this implies integrability on all $\mathbb{R} \times W$. Moreover, $\Psi_r|_{\{0\}\times W} = \Phi_r|_{\{0\}\times W}$ and $\Psi_r|_{\{-2N\pi\}\times W} = \mathrm{Id}|_{\{-2N\pi\}\times W}$ for all $r \in [0,1]$.

In particular, Ψ_1 maps $[-2N\pi, 0] \times W$ contactomorphically to $\left[-2N\pi, \frac{2\pi}{k}\right] \times W$, where we consider on the domain and on the codomain the contact structure η_k.

Now, by the periodicity of η_k, we can identify the two boundary components of $[-2N\pi, 0] \times W$, so that the restriction of η_k induces a well-defined contact structure on the quotient. More precisely, the quotient contact manifold obtained is $(\mathbb{S}^1_s \times W, \pi^*_{kN}\xi)$. The analogous procedure for the codomain $\left[-2N\pi, \frac{2\pi}{k}\right] \times W$ of Ψ_1 gives as quotient the contact manifold $(\mathbb{S}^1_s \times W, \pi^*_{kN+1}\xi)$.

Lastly, because $\Psi_1\colon [-2N\pi, 0] \times W \to \left[-2N\pi, \frac{2\pi}{k}\right] \times W$ is the identity on a neighborhood of $\{-2N\pi\} \times W$ and a lift of the translation f on a neighborhood of $\{0\} \times W$, it induces on the quotient contact manifolds a well-defined contactomorphism

$$\phi\colon (\mathbb{S}^1_s \times W, \pi^*_{kN}\xi) \xrightarrow{\sim} (\mathbb{S}^1_s \times W, \pi^*_{kN+1}\xi). \qquad \square$$

Proof of Lemma 2.2 Suppose by contradiction that there is an isomorphism of almost contact structures $\psi\colon (V, \pi^*_m\xi) \xrightarrow{\sim} (V, \pi^*_{m+1}\xi)$; in particular,

$$\psi_* c_1(\pi^*_m\xi) = c_1(\pi^*_{m+1}\xi). \tag{1}$$

Because the submodule $H^2_{\mathrm{ator}}(V;\mathbb{Z})$ of atoroidal classes is natural (ie it is preserved by pullbacks induced by continuous maps $V \to V$), the map ψ_* induces a well-defined endomorphism, which is moreover an isomorphism, of the $\mathbb{Z}$–module $N := H^2(V;\mathbb{Z})/H^2_{\mathrm{ator}}(V;\mathbb{Z})$. We then have $\psi_*(\pi^*_n\xi) = n\psi_* c_1(\xi) \bmod H^2_{\mathrm{ator}}(V;\mathbb{Z})$ for each natural $n \geq 2$, so that (1) becomes

$$m\psi_* c_1(\xi) = (m+1)c_1(\xi) \mod H^2_{\mathrm{ator}}(V;\mathbb{Z}). \tag{2}$$

Notice also that N is a finitely generated $\mathbb{Z}$–module without torsion. In particular, there is a well-defined *divisibility* map

$$d\colon N \setminus \{0\} \to \mathbb{N} \setminus \{0\}, \quad a \mapsto \max\{k \in \mathbb{N} \mid a = kb \text{ for some } b \in N\},$$

which also satisfies $d(ha) = hd(a)$ and $d(\psi_* a) = d(a)$ for each $a \in N \setminus \{0\}$ and $h \in \mathbb{N} \setminus \{0\}$. Because $c_1(\xi)$ is toroidal, we can then apply d to both the left- and right-hand sides of (2), thus obtaining the desired contradiction. $\square$

3 Examples from Liouville pairs and half Lutz–Mori twists

The idea of the proof of Theorem 1.2 is the following. The contact structure η on the manifold $V = \mathbb{S}^1 \times W$ in the statement has trivial Chern classes (better, it is trivializable as a complex bundle). We then apply a semilocal modification to η and obtain another contact structure ξ; the explicit nature of this modification (as well as the explicit nature of the original contact manifold (V, η)) allows us to compute the first Chern class of ξ, and to show that it satisfies the desired conditions.

This section is structured in the following way. We recall in Sections 3.1 and 3.2, respectively, the notion of half Lutz–Mori twist and the construction of Liouville pairs, both from Massot, Niederkrüger and Wendl [28]. We then describe in Section 3.3 how half Lutz–Mori twists (along contact submanifolds belonging to one of the Liouville pairs constructed in [28]) affect the Chern classes of the underlying almost contact structure. Finally, Section 3.4 contains the proof of Theorem 1.2.

3.1 The half Lutz–Mori twist

Developing some ideas introduced by Mori in [29] in the 5–dimensional case, Massot, Niederkrüger and Wendl introduce in [28] the notion of *Lutz–Mori twist* along a submanifold belonging to a *Liouville pair* as a generalization of the known 3–dimensional Lutz twists. In this section, we briefly recall how to perform the *half* version of the Lutz–Mori twist, which we will use in the following.

We start by recalling the notion of Liouville pair:

Definition 3.1 [28] Let M^{2n-1} be an oriented manifold. A *Liouville pair on M* is a pair of contact forms (α_+, α_-) such that $\pm\alpha_\pm \wedge (d\alpha_\pm)^{n-1} > 0$ and such that the form $e^r \alpha_+ + e^{-r}\alpha_-$ is a Liouville form (ie its differential is symplectic) on $\mathbb{R}_r \times M$.

We point out that the existence of Liouville pairs on closed manifolds is not trivial; at the moment, the only known examples in high dimensions are given by the construction in [28, Section 8], which is nonetheless a source of infinitely many nonhomeomorphic manifolds with Liouville pairs in each (odd) dimension. In Section 3.2 we will recall the properties of this construction which are needed in order to prove Theorem 1.1.

Let now (V, η) be a contact manifold having as a codimension-2 contact submanifold (M, ξ_+) such that α_+ defining ξ_+ belongs to a Liouville pair (α_+, α_-). We want to describe how to perform a half Lutz–Mori twist on (V, η) along (M, ξ_+).

Consider then the 1–form

$$\alpha = \tfrac{1}{2}(1+\cos s)\alpha_+ + \tfrac{1}{2}(1-\cos s)\alpha_- + \sin s\, dt$$

on $[\pi, 2\pi]_s \times \mathbb{S}^1_t \times M$; notice that this is a contact form because (α_+, α_-) is a Liouville pair on M. Let then (U, ξ_U) be the *blow-down* of $([\pi, 2\pi]_s \times \mathbb{S}^1_t \times M, \ker\alpha)$ along $\{\pi\} \times \mathbb{S}^1_t \times M$, as defined in [28, Section 5.1].

More explicitly, (U, ξ_U) is obtained as follows. The hypersurface — or, better, *round hypersurface*, as defined in [28, Section 5.1] — $\{\pi\} \times \mathbb{S}^1_t \times M$ admits a neighborhood of the form $([0,\epsilon)_x \times \mathbb{S}^1_t \times M, \ker(\alpha_- + x\, dt))$ inside $([\pi, 2\pi]_s \times \mathbb{S}^1_t \times M, \ker\alpha)$ in such a way that $\{\pi\}_s \times \mathbb{S}^1_t \times M$ corresponds to $\{0\}_x \times \mathbb{S}^1_t \times M$; this follows from the fact that the restriction of the two contact structures to the two hypersurfaces coincide (see [28, Lemma 5.1]). We can then remove the hypersurface $\{\pi\}_s \times \mathbb{S}^1_t \times M$ inside $([\pi, 2\pi]_s \times \mathbb{S}^1_t \times M, \ker\alpha)$ and glue $(D^2_{\sqrt{\epsilon}} \times M, \ker(\alpha_- + r^2\, d\varphi))$ (here (r,φ) are polar coordinates on the 2–disc $D^2_{\sqrt{\epsilon}}$ centered at the origin and of radius $\sqrt{\epsilon}$) thanks to the contactomorphism from $((D^2_{\sqrt{\epsilon}} \setminus \{0\}) \times M, \ker(\alpha_- + r^2\, d\varphi))$ to $((0,\epsilon)_x \times \mathbb{S}^1_t \times M, \ker(\alpha_- + x\, dt))$ (seen as a subset of $((\pi, 2\pi]_s \times \mathbb{S}^1_t \times M, \ker\alpha)$) given by $(r, \varphi, p) \mapsto (r^2, \varphi, p)$. The resulting contact manifold (with one boundary component) is the desired (U, ξ_U).

At this point, performing a half Lutz–Mori twist along (M, ξ_+) means replacing a neighborhood of (M, ξ_+) in (V, η) with (U, ξ_U).

More precisely, one can see that the boundary component $\{2\pi\} \times \mathbb{S}^1_t \times M$ of (U, ξ_U) also admits a neighborhood $((-\epsilon, 0]_x \times \mathbb{S}^1_t \times M, \ker(\alpha_+ + x\, dt))$ inside (U, ξ_U), in such a way that $\{2\pi\}_s \times \mathbb{S}^1_t \times M$ corresponds to $\{0\}_x \times \mathbb{S}^1_t \times M$. Now, (M, ξ_+) is a codimension-2 contact submanifold with trivial normal bundle in (V, η); hence, by the contact neighborhood theorem (see Geiges [13, Theorem 2.5.15]), there is $\delta > 0$ such that (M, ξ_+) admits a neighborhood $(D^2_\delta \times M, \eta_0 := \ker(\alpha_+ + r^2\, d\varphi))$ inside (V, η) (here, (r, φ) are polar coordinates on D^2_δ) in such a way that (M, ξ_+) corresponds to $(\{0\} \times M, \eta_0|_{\{0\}\times M})$. Because $((D^2_\delta \setminus \{0\}) \times M, \ker(\alpha_+ + r^2\, d\varphi))$ is contactomorphic to $((0, \delta^2)_x \times \mathbb{S}^1_t \times M, \ker(\alpha_+ + x\, dt))$ via $(r, \varphi, p) \mapsto (r^2, \varphi, p)$, we can then glue (U, ξ_U) to $(V \setminus M, \eta)$ and obtain a well-defined contact manifold (V, ξ) (notice that the underlying smooth manifold is still V).

The above construction does not depend, up to isotopy, on any choice made.

Definition 3.2 [28, Remark 9.6] (V, ξ) is said to be obtained from (V, η) by a *half Lutz–Mori twist* along the contact submanifold $(M, \xi_+ = \ker(\alpha_+))$ belonging to the Liouville pair (α_+, α_-).

We point out that performing a half Lutz–Mori twist makes the contact manifold overtwisted. Indeed, it is explained in Massot, Niederkrüger and Wendl [28, Remark 9.6] that this twist always gives a PS-overtwisted manifold, which then is also overtwisted according to Casals, Murphy and Presas [7] and Huang [22].

3.2 Construction of Liouville pairs

We recall here the construction in Massot, Niederkrüger and Wendl [28, Section 8], leaving the details that are not important for our purposes.

Consider the product manifold $\mathbb{R}^m \times \mathbb{R}^{m+1}$ with the pair of contact structures ξ_+ and ξ_- induced by the pair of contact forms

$$\alpha_\pm := \pm e^{t_1+\cdots+t_m}\,d\theta_0 + e^{-t_1}\,d\theta_1 + \cdots + e^{-t_m}\,d\theta_m,$$

where we use coordinates $(t_1,\dots,t_m)$ on $\mathbb{R}^m$ and $(\theta_0,\dots,\theta_m)$ on $\mathbb{R}^{m+1}$. A direct computation shows that (α_+,α_-) is a Liouville pair on $\mathbb{R}^m \times \mathbb{R}^{m+1}$.

We now remark that there are two Lie groups acting explicitly on $\mathbb{R}^m \times \mathbb{R}^{m+1}$ by strict contact transformations for both α_+ and α_-.

Indeed, the left action of the group $\mathbb{R}^{m+1}$ on $\mathbb{R}^m \times \mathbb{R}^{m+1}$ given by the translations

$$(\varphi_0,\dots,\varphi_m)\cdot(t_1,\dots,t_m,\theta_0,\dots,\theta_m) := (t_1,\dots,t_m,\theta_0+\varphi_0,\dots,\theta_m+\varphi_m)$$

and the left action of $\mathbb{R}^m$ given by the law

$$\begin{aligned}&(\tau_1,\dots,\tau_m)\cdot(t_1,\dots,t_m,\theta_0,\dots,\theta_m)\\ &\qquad := (t_1+\tau_1,\dots,t_m+\tau_m,e^{-\tau_1+\cdots-\tau_m}\theta_0,e^{\tau_1}\theta_1,\dots,e^{\tau_m}\theta_m)\end{aligned}$$

are Lie group left actions on $\mathbb{R}^m \times \mathbb{R}^{m+1}$ and they both preserve the contact forms α_+ and α_-.

Moreover, these two actions allow us to produce a *compact* contact manifold from $\mathbb{R}^m \times \mathbb{R}^{m+1}$. Indeed, there are lattices Λ and Λ' of $\mathbb{R}^m$ and $\mathbb{R}^{m+1}$, respectively, such that the Λ–action on $\mathbb{R}^m \times \mathbb{R}^{m+1}$ induced by the action of $\mathbb{R}^m$ preserves $\mathbb{R}^m \times \Lambda'$. This implies that, by first taking the quotient of $\mathbb{R}^m \times \mathbb{R}^{m+1}$ by Λ' and then quotienting it by the (well defined by the above property) induced action of Λ, we obtain a compact manifold M.

Finally, this manifold M naturally inherits a Liouville pair, still denoted by (α_+,α_-), from the Liouville pair on the covering $\mathbb{R}^m \times \mathbb{R}^{m+1}$, because $\mathbb{R}^m$ and $\mathbb{R}^{m+1}$ act on $\mathbb{R}^m \times \mathbb{R}^{m+1}$ by strict contactomorphisms for both α_+ and α_-.

We point out that this construction actually gives an infinite number of nonhomeomorphic manifolds M, hence an infinite number of nonisomorphic Liouville pairs, in each odd dimension greater than or equal to 3.

Indeed, the existence of the lattices Λ and Λ' follows from number theory arguments and the manifold M obtained depends on the choice of a totally real field of real numbers $\mathbb{k}$ with finite dimension over $\mathbb{Q}$. Now, for each dimension ≥ 2 over $\mathbb{Q}$, there are infinitely such fields $\mathbb{k}$ and the corresponding manifolds are nonhomeomorphic. See [28, Lemma 8.3] for the details.

As far as Theorem 1.2 is concerned, this means that we have, in each odd dimension $2n+1 \geq 5$, a contact structure satisfying the hypothesis of Theorem 1.1 on infinitely many different smooth manifolds $\mathbb{T}^2 \times M^{2n-1}$; in dimension 3, we obtain one contact structure on $\mathbb{T}^2 \times M^1 = \mathbb{T}^3$. In both cases, Theorem 1.1 then gives examples of contactomorphisms smoothly isotopic but not contact-isotopic to the identity for the countably many contact structures $(\pi_k^* \xi)_{k \geq 2}$ on each $\mathbb{T}^2 \times M$.

3.3 Effects of half Lutz–Mori twists on Chern classes

Chern classes are global invariants of complex vector bundles E over a manifold V. In our setting, we then have to find a way to study how local modifications (ie over an open set $\mathcal{U}$ of V) of the complex vector bundle E affect its Chern classes. The solution is either to use a relative version of Chern classes or to shift to another point of view more local in nature.

Aguilar, Cisneros-Molina and Frías-Armenta [1] adopt in particular this second strategy and this allows them to prove a generalization of the classical fact that the top Chern class of E is the Poincaré dual of the zero locus of a section of E which is transverse to the zero section. In order to achieve such generalization, they deal with the following technical issue: when $1 < k \leq r = \mathrm{rk}_{\mathbb{C}}(E)$, the locus S_k of points where k–sections $s_1, \dots, s_k$ are $\mathbb{C}$–linearly dependent may not be a smooth submanifold of V, even for a "generic" choice of $s_1, \dots, s_k$, hence it has a priori no well-defined homology class. In [1] it is hence proved that S_k can be desingularized to a smooth submanifold Z_k of $V \times \mathbb{CP}^{k-1}$ in such a way that the $(r-k+1)^{\mathrm{st}}$ Chern class of E can be interpreted as the Poincaré dual of the pushforward in V of the class of $Z_k \subset V \times \mathbb{CP}^{k-1}$ via the map induced in homology by the projection $V \times \mathbb{CP}^{k-1} \to V$.

In our context of half Lutz–Mori twists along particular contact submanifolds, the results proven by Aguilar, Cisneros-Molina and Frías-Armenta [1] give the following:

Proposition 3.3 *Let* (V^{2m+3},ξ) *be a contact manifold containing the* (M^{2m+1},ξ_+) *of Section 3.2 as a codimension-2 contact submanifold with trivial normal bundle. Then, if we denote by* ξ' *the contact structure on* V *obtained by performing a half Lutz–Mori twist along the submanifold* (M,ξ_+) (where we consider M with the orientation given by ξ_+), *we have the following:*

1. *For all* $i=2,\dots,m+1$, $c_i(\xi')-c_i(\xi)=0$ *in* $H^{2i}(V;\mathbb{Z})$.
2. $c_1(\xi')-c_1(\xi)=-2\,\mathrm{PD}(j_*[M])$ *in* $H^2(V;\mathbb{Z})$*, where* $j\colon M\to V$ *is the inclusion,* $j_*\colon H_{2m+1}(M;\mathbb{Z})\to H_{2m+1}(V;\mathbb{Z})$ *is the induced map and* $\mathrm{PD}(\alpha)$ *denotes the Poincaré dual of the homology class* $\alpha\in H_*(V;\mathbb{Z})$.

Remark This result is not in contradiction with Massot, Niederkrüger and Wendl [28, Theorem 9.5], where the authors prove that the contact structures before and after a full Lutz–Mori twist (as defined in [28, Section 9.1]) are homotopic through almost contact structures, hence have the same Chern classes. Indeed, the result ξ'' of a full Lutz–Mori twist can be interpreted as a pair of successive half twists. More precisely, we first perform a half twist along a submanifold (M,ξ_+) to obtain ξ'; this changes the core of the tube where we perform the twist from (M,ξ_+) to (M,ξ_-). We then perform another half twist, this time along the new core (M,ξ_-), to obtain ξ''. Hence, applying Proposition 3.3 twice and using the fact that ξ_- induces an orientation that is opposite to that induced by ξ_+, we get that $c_i(\xi'')=c_i(\xi')=c_i(\xi)$ for all $i=2,\dots,m+1$ and that

$$c_1(\xi'')=c_1(\xi')-2\,\mathrm{PD}(j_*[-M])=c_1(\xi)-2\,\mathrm{PD}(j_*[M])-2\,\mathrm{PD}(j_*[-M])=c_1(\xi),$$

as we expected from [28, Theorem 9.5].

The proof of Proposition 3.3 relies on the explicit results in [1]; we hence made the choice to omit it in this paper, in order to avoid lengthy technical digressions and keep the focus on the motivating contact geometric problem, ie the research of examples of contactomorphisms smoothly isotopic but not contact-isotopic to the identity on overtwisted contact manifolds of high dimensions. A detailed proof of Proposition 3.3 (together with the necessary background from [1]) can be found in Gironella [18, Section 4.2.3 and Appendix A].

3.4 Proof of Theorem 1.2

We use in this section the notation introduced in the statement of Theorem 1.2.

The contact structure η on the manifold $\mathbb{T}^2_{(s,t)} \times M$ can be explicitly written as the kernel of $\alpha := \sum_{i=1}^{m} e^{-t_i}\, d\theta_i + \cos(s) e^{\sum_{i=1}^{m} t_i}\, d\theta_0 + \sin s\, dt$, where we use locally on M the coordinates $(t_1, \dots, t_m, \theta_0, \dots, \theta_m)$ induced by the covering $\mathbb{R}^m \times \mathbb{R}^{m+1} \to M$, as described in Section 3.2. Then η admits a trivialization as a complex vector bundle given by the following sections and choice of $d\alpha|_\eta$–compatible complex structure J:

(1) $S_i := \partial_{t_i}$ for $i = 1, \dots, m$, and $S_{m+1} := \partial_s$

(2) $J(S_i) := e^{-\sum_{j=1}^{m} t_j} \cos(s) \partial_{\theta_0} - e^{t_i} \partial_{\theta_i} + \sin(s) \partial_t$ for $i = 1, \dots, m$, and $J(S_{m+1}) := -e^{-\sum_{j=1}^{m} t_j} \sin(s) \partial_{\theta_0} + \cos(s) \partial_t$.

(An explicit computation shows that these sections are indeed well defined on $\mathbb{T}^2_{(s,t)} \times M$ and not only on $\mathbb{T}^2_{(s,t)} \times \mathbb{R}^m \times \mathbb{R}^{m+1}$.)

In particular, all the Chern classes of η are zero. Hence, applying Proposition 3.3 to the pair (ξ, η) we get the following: if we denote by $j\colon M \to \mathbb{T}^2_{(s,t)} \times M$ the inclusion $j(p) = (0, 0, p)$ and by $j_*\colon H_{2m+1}(M; \mathbb{Z}) \to H_{2m+1}(\mathbb{T}^2 \times M; \mathbb{Z})$ the induced map in homology, then $c_1(\xi) = -2\,\mathrm{PD}(j_*[M])$ in $H^2(\mathbb{T}^2 \times M; \mathbb{Z})$.

We now prove that $c_1(\xi)$ is toroidal. Fix a $p \in M$ and consider $f\colon \mathbb{T}^2 \to \mathbb{T}^2 \times M$ given by $f(\theta, \varphi) = (\theta, \varphi, p)$ for every $(\theta, \varphi) \in \mathbb{T}^2$. Because f is transverse to $j(M)$, we have $f^*\,\mathrm{PD}_{\mathbb{T}^2 \times M}(j_*[M]) = \mathrm{PD}_{\mathbb{T}^2}([f^{-1}(j(M))])$; here, the notation PD_X means that we are considering the Poincaré duality on the compact manifold X. Now, $\mathrm{PD}_{\mathbb{T}^2}\big([f^{-1}(j(M))]\big) = \mathrm{PD}_{\mathbb{T}^2}\big([\{(0,0)\}]\big)$ generates $H^2(\mathbb{T}^2; \mathbb{Z}) \simeq \mathbb{Z}$; in other words, $\mathrm{PD}(j_*[M])$ is toroidal. As $H^2(V; \mathbb{Z})/H^2_{\mathrm{ator}}(V; \mathbb{Z})$ is torsion-free, $c_1(\xi)$ is also toroidal.

The only thing left to show is that $c_1(\pi_k^* \xi) = k c_1(\xi) \bmod H^2_{\mathrm{ator}}(V; \mathbb{Z})$ for each $k \geq 2$.

Because η is a trivial complex vector bundle over $\mathbb{T}^2 \times M$, the same is true for each $\pi_k^* \eta$; in particular, each $\pi_k^* \eta$ has trivial Chern classes. Notice that $\pi_k^* \xi$ can also be seen as obtained from $\pi_k^* \eta$ by performing a half Lutz–Mori twist along each of the k submanifolds $\big\{\big(\frac{2l\pi}{k}, 0\big)\big\} \times M$ with $l = 0, \dots, k-1$. Then Proposition 3.3 tells that $c_1(\pi_k^* \xi) = -2k\,\mathrm{PD}(j_*[M]) = k c_1(\xi)$, so that $c_1(\pi_k^* \xi) = k c_1(\xi) \bmod H^2_{\mathrm{ator}}(V; \mathbb{Z})$ too.

4 Examples from adapted open books and the h–principle

In this section, we show how to obtain examples of $(\mathbb{S}^1 \times W, \xi)$ as in the hypothesis of Theorem 1.1 using the existence of adapted open-book decompositions due to Giroux [20] and the h–principle of Borman, Eliashberg and Murphy [3].

In the following, we are going to adopt two (homotopically equivalent) points of view on (coorientable) almost contact structures on V^{2n+1}. More precisely, in Sections 4.1 and 4.2 we look at them as, respectively, pairs (ξ,ω_ξ) and (ξ,J_ξ), where ξ is a coorientable hyperplane field on V, ω_ξ is a symplectic structure on ξ and J_ξ is a complex structure on it.

4.1 Proof of Theorem 1.3

In order to prove Theorem 1.3, we need the following lemma, which describes the effects of the Bourgeois construction [4] and of its branched coverings at the level of almost contact structures as well as a sufficient condition for overtwistedness in the case of branched covers:

Lemma 4.1 *Let (V^{2n-1},η) be a contact manifold, where η is coorientable, (B,φ) an open-book decomposition supporting η and α a contact form defining η and adapted to the open book. Then we have the following:*

1. *The Bourgeois construction [4] on (V,η) and (B,φ,α) gives a contact structure ξ on $V\times\mathbb{T}^2$ which is homotopic, as an almost contact structure, to $(\eta\oplus T\mathbb{T}^2, d\alpha\oplus\omega_T)$, where ω_T is a volume form on $\mathbb{T}^2$.*
2. *Any contact branched covering ξ_g of ξ via a branched covering $\nu\colon V\times\Sigma_g\to V\times\mathbb{T}^2$, induced by a covering $\Sigma_g\to\mathbb{T}^2$ branched over two points, is homotopic, as an almost contact structure, to $(\eta\oplus T\Sigma_g, d\alpha\oplus\omega_g)$, where ω_g is a volume form on Σ_g.*
3. *Suppose η is overtwisted. Then, if g is large enough, ξ_g is overtwisted too.*

Notice that point (1) above has already been pointed out by Lisi, Marinković and Niederkrüger [24, Remark 2.1].

We now prove, in this order, Theorem 1.3 and Lemma 4.1:

Proof of Theorem 1.3 We use the notation of Theorem 1.3. Denote also the natural projections by

$$p\colon V\times\mathbb{T}^2\to V,\quad p_g\colon V\times\Sigma_g\to V\quad\text{and}\quad p_g'\colon V\times\Sigma_g\to\Sigma_g.$$

Lemma 4.1(1)–(2) imply that $c_1(\xi)=p^*c_1(\eta)$ and $c_1(\xi_g)=p_g^*c_1(\eta)+(p_g')^*c_1(T\Sigma_g)$. Recall now that every continuous map from $\mathbb{T}^2$ to Σ_g has degree 0 (here, we use $g\geq 2$); in particular, for each $f\colon\mathbb{T}^2\to V\times\Sigma_g$, we have

$$f^*(p_g')^*c_1(T\Sigma_g)=(p_g'\circ f)^*c_1(T\Sigma_g)=0\in H^2(\mathbb{T}^2;\mathbb{Z}),$$

ie $(p'_g)^* c_1(T\Sigma_g)$ is atoroidal. We hence have that

$$\begin{aligned} c_1(\xi) &= p^* c_1(\eta) \quad \text{mod } H^2_{\text{ator}}(V \times \mathbb{T}^2; \mathbb{Z}), \\ c_1(\xi_g) &= p_g^* c_1(\eta) \quad \text{mod } H^2_{\text{ator}}(V \times \Sigma_g; \mathbb{Z}). \end{aligned} \tag{3}$$

We now claim that both p and p_g pull back toroidal classes on V to toroidal classes on, respectively, $V \times \mathbb{T}^2$ and $V \times \Sigma_g$. By equation (3) and the fact that $c_1(\eta)$ is toroidal by hypothesis, this would then directly imply that $c_1(\xi)$ and $c_1(\xi_g)$ are toroidal too.

Let $a \in H^2(V;\mathbb{Z})$ be toroidal, ie there is $t\colon \mathbb{T}^2 \to V$ with $t^* a \neq 0$; we then want to prove that $p^* a \in H^2(V \times \mathbb{T}^2; \mathbb{Z})$ is toroidal too. Consider any $h\colon \mathbb{T}^2 \to V \times \mathbb{T}^2$ such that $p \circ h = t$; for instance, let $q_0 \in \mathbb{T}^2$ and take $h(\cdot) := (t(\cdot), q_0)$. Then

$$h^*(p^* a) = (p \circ h)^* a = t^* a \neq 0 \in H^2(\mathbb{T}^2; \mathbb{Z}),$$

ie $p^* a$ is toroidal, as desired. An analogous argument shows that $p_g^* a$ is toroidal too.

The fact that ξ and ξ_g satisfy

$$\begin{aligned} c_1(\mu_k^* \xi) &= k c_1(\xi) \qquad \text{mod } H^2_{\text{ator}}(V \times \mathbb{T}^2; \mathbb{Z}), \\ c_1((\mu_k^g)^* \xi_g) &= k c_1(\xi_g) \quad \text{mod } H^2_{\text{ator}}(V \times \Sigma_g; \mathbb{Z}) \end{aligned}$$

follows, by a direct computation, from equation (3), from the equalities $\pi_k \circ p = p \circ \mu_k$ and $\pi_k \circ p_g = p_g \circ \mu_k^g$ and from the fact that $c_1(\pi_k^* \eta) = k c_1(\eta)$ mod $H^2_{\text{ator}}(V;\mathbb{Z})$.

Lastly, if η is overtwisted, Lemma 4.1(3) gives the overtwistedness of ξ_g for g large enough, thus concluding the proof. □

Proof of Lemma 4.1 We start by proving (1). The Bourgeois construction [4] on (V,η) and (B,φ,α) gives a function $\Phi = (f,g)\colon V \to \mathbb{R}^2$ defining the open book (B,φ) and such that ξ on $V \times \mathbb{T}^2_{(x,y)}$ is defined by $\beta := \alpha + f\,dx - g\,dy$. Then an explicit homotopy of almost contact structures from $(\xi, d\beta|_\xi)$ to $(\eta \oplus T\mathbb{T}^2, d\alpha|_\eta + dx \wedge dy)$ is given by the $[0,1]_t$–family of hyperplane fields ξ_t given by the kernel of

$$\alpha + (1-t)(f\,dx - g\,dy),$$

together with the symplectic structures given by the restriction of

$$d\alpha + (1-t)[df \wedge dx - dg \wedge dy] + t\,dx \wedge dy$$

to ξ_t.

As far as point (2) is concerned, as explained in Geiges [12], an explicit contact branched covering ξ_g on $V \times \Sigma_g$ is given by the kernel of a differential 1–form $\nu^*\beta + \epsilon h(r) r^2\,d\theta$;

here, (r, θ) are radial coordinates on the D^2–factor of a neighborhood $D^2 \times \{p, q\}$ of the upstairs branching locus $\{p, q\}$ of the branched covering $\Sigma_g \to \mathbb{T}^2$, the constant $\epsilon > 0$ is very small and $h = h(r)$ is a smooth function with support in $D^2 \times \{p, q\}$, equal to 1 on the branching locus and strictly decreasing in r. As contact branched coverings are unique up to isotopy (see Gironella [17, Section 2.2]), it's enough to prove that this specific η_g is homotopic to the desired almost contact structure.

Now, an explicit computation (analogous to the one in [17, Section 6.5]) shows that the desired homotopy of almost contact structures is given by the $[0,1]_t$–family of hyperplane fields ξ_g^t defined as the kernel of $\nu^*\alpha + (1-t)[\nu^*(f\,dx - g\,dy) + \epsilon h r^2\,d\theta]$, together with the symplectic structures given by the restriction of

$$\nu^*\,d\alpha + (1-t)[\nu^*(df \wedge dx - dg \wedge dy) + \epsilon\,d(hr^2) \wedge d\theta] + t\omega_g$$

to ξ_g^t.

Point (3) has already been discussed in [17, Section 7.2]; more precisely, it essentially follows from the following three facts. Firstly, the contact branched covering ξ_g can be chosen (up to isotopy) in such a way that it induces on each fiber of $V \times \Sigma_g \to \Sigma_g$ the original overtwisted contact structure η. Secondly, Niederkrüger and Presas [30, page 724] describe how the "size" of a contact neighborhood of each connected component (V, ξ) of the branching set of $V \times \Sigma_g \to V \times \mathbb{T}^2$ is diverging to $+\infty$ as the index g of the branched covering is going to $+\infty$; see also [17, Lemma 7.10]. Then, according to Casals, Murphy and Presas [7, Theorem 3.1], topologically trivial contact neighborhoods of overtwisted manifolds in codimension 2 are themselves overtwisted provided they are sufficiently "large". This concludes the proof of Lemma 4.1. □

4.2 Proof of Proposition 1.4

The proof is structured as follows. We start from a natural almost contact structure η_0 on $V := \mathbb{S}^1 \times W$ and we modify it to an almost contact structure η with first Chern class $c_1(\eta)$ satisfying the desired conditions. Then the h–principle from Borman, Eliashberg and Murphy [3] says that η can be deformed to an overtwisted contact structure ξ on V; the first Chern class of such a ξ will then satisfy the desired properties too.

Before entering into the details of the proof of Proposition 1.4, we state a lemma from algebraic topology, whose proof is postponed:

Lemma 4.2 *Let η_0 be a (coorientable) almost contact structure on V^{2n+1}. For each $u \in H^2(V; \mathbb{Z})$, there is an almost contact structure η_u on V with $c_1(\eta_u) = c_1(\eta_0) + 2u$.*

Proof of Proposition 1.4 The hyperplane field $\eta_0 = \{0\} \oplus TW$ on $V = \mathbb{S}^1 \times W$ is a (coorientable) almost contact structure thanks to the almost complex structure J_W on W. Moreover, its first Chern class $c_1(\eta_0)$ is equal to $\pi_W^* c_1(W)$, where $\pi_W\colon \mathbb{S}^1 \times W \to W$ is the projection on the second factor.

The hypothesis that W is spin means that the 2nd Stiefel Whitney class $w_2(W) \in H^2(W;\mathbb{Z}_2)$ of W is trivial. Because $w_2(W)$ is the reduction modulo 2 of $c_1(W)$, there is $\lambda \in H^2(W;\mathbb{Z})$ such that $c_1(W) = 2\lambda$. Hence, $c_1(\eta_0) = \pi_W^* c_1(W) = 2\pi_W^*\lambda$.

Consider then a nontrivial $c \in H^1(W;\mathbb{Z}) \neq \{0\}$, and let v be a generator of $H^1(\mathbb{S}^1;\mathbb{Z})$. Using Künneth's decomposition theorem, we can see $H^1(\mathbb{S}^1;\mathbb{Z}) \otimes H^1(W;\mathbb{Z})$ as a submodule of $H^2(\mathbb{S}^1 \times W;\mathbb{Z})$. An application of Lemma 4.2 with $u = v \otimes c - \pi_W^*\lambda$ then gives an almost contact structure η with $c_1(\eta) = 2v \otimes c$.

Notice that the map π_k^*, induced on $H^2(\mathbb{S}^1 \times W;\mathbb{Z})$ by π_k, acts as multiplication by k on the submodule $H^1(\mathbb{S}^1;\mathbb{Z}) \otimes H^1(W;\mathbb{Z})$ of $H^2(\mathbb{S}^1 \times W;\mathbb{Z})$. In particular, the fact that $c_1(\eta) = 2v \otimes c$ implies that $c_1(\pi_k^*\eta) = kc_1(\eta) \bmod H^2_{\mathrm{ator}}(V;\mathbb{Z})$.

We also claim that $c_1(\eta)$ is toroidal. Indeed, according to the universal coefficient theorem and the Hurewicz theorem,

$$H^1(W;\mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(H_1(W;\mathbb{Z});\mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(\pi_1(W);\mathbb{Z});$$

in particular, as $c \neq 0 \in H^1(W;\mathbb{Z})$, there is $\gamma\colon \mathbb{S}^1 \to W$ such that $\gamma^* c \neq 0$ in $H^1(\mathbb{S}^1;\mathbb{Z})$. If we define $f := (\mathrm{Id}, \gamma)\colon \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times W$, we then have $f^* c_1(\eta) = 2v \otimes \gamma^* c \neq 0$ in $H^1(\mathbb{S}^1;\mathbb{Z}) \otimes H^1(\mathbb{S}^1;\mathbb{Z}) \subset H^2(\mathbb{T}^2;\mathbb{Z})$, ie $c_1(\eta)$ is toroidal, as desired.

The h–principle from Borman, Eliashberg and Murphy [3] then gives the desired contact structure ξ as a deformation of η. □

We now give a proof of the lemma used above:

Proof of Lemma 4.2 Bowden, Crowley and Stipsicz [6, Lemma 2.17(1)] states that if V is a closed connected manifold of dimension $2n+1$ and ζ is a stable almost complex structure on it, then there is an almost contact structure η on V whose stabilization gives ζ. Recall that a *stable almost complex structure* on V is the stable isomorphism class of a complex structure on $TV \oplus \varepsilon_V^k$, where ε_V is the trivial real vector bundle of dimension 1 over V, and the *stabilization of* η is the stable isomorphism class of the complex structure induced by η on $TV \oplus \varepsilon_V$. In particular, in order to prove Lemma 4.2, it's enough to find a stable almost complex structure ζ_u such that $c_1(\zeta_u) = c_1(\eta_0) + 2u$.

The existence of such a ζ_u follows, for instance, from Geiges [13, Remark 8.1.4], of which we recall here the idea.

There is a bijective correspondence, given by the first Chern class, between isomorphism classes of complex line bundles over V and cohomology classes in $H^2(V;\mathbb{Z})$. Let then L_u be the complex line bundle over V satisfying $c_1(L_u) = u$. Consider then a complex vector bundle E_u over V such that there are $m \in \mathbb{N}_{>0}$ and an isomorphism $\nu\colon L_u^* \oplus_{\mathbb{C}} E_u \simeq (\varepsilon_V^{\mathbb{C}})^m$ of complex vector bundles over V, where $\varepsilon_V^{\mathbb{C}}$ denotes the complexification of ε_V; for a proof of the existence of such a complement E_u, see for instance Atiyah [2, Corollary 1.4.14]. We then claim that the complex vector bundle $F_u := \eta_0 \oplus L_u \oplus E_u$ can be used to define the desired stable complex structure.

The fact that $L_u^* \oplus_{\mathbb{C}} E_u$ is a trivial complex vector bundle implies in particular that $c_1(E_u) = -c_1(L_u^*) = u$; hence, $c_1(F_u) = c_1(\eta) + u + u = c_1(\eta) + 2u$.

Now, because L_u^* and L_u are isomorphic as real vector bundles, ν induces an isomorphism of real vector bundles $\nu'\colon L_u \oplus_{\mathbb{R}} E_u \simeq \varepsilon_V^{2m}$. Moreover, the choice of a vector field X on V transverse to η_0 gives an isomorphism of real vector bundles $\Psi\colon \eta_0 \oplus \varepsilon_V \simeq TV$. We then have an isomorphism θ of real vector bundles over V given by the composition

$$F_u = \eta_0 \oplus L_u \oplus E_u \xrightarrow[\simeq]{\mathrm{Id}\oplus\nu'} \eta_0 \oplus \varepsilon_V^{2m} =_{\mathbb{R}} (\eta_0 \oplus \varepsilon_V) \oplus \varepsilon_V^{2m-1} \xrightarrow[\simeq]{\Psi\oplus\mathrm{Id}} TV \oplus \varepsilon_V^{2m-1}.$$

In particular, the pushforward $\theta_* J$ of the complex structure J on F_u via θ gives the desired stable almost complex structure ζ_u on V. □

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Uniform exponential growth for CAT(0) square complexes

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We start the inquiry into proving uniform exponential growth in the context of groups acting on CAT(0) cube complexes. We address free group actions on CAT(0) square complexes and prove a more general statement. This says that if F is a finite collection of hyperbolic automorphisms of a CAT(0) square complex X, then either there exists a pair of words of length at most 10 in F which freely generate a free semigroup, or all elements of F stabilize a flat (of dimension 1 or 2 in X). As a corollary, we obtain a lower bound for the growth constant, $\sqrt[10]{2}$, which is uniform not just for a given group acting freely on a given CAT(0) cube complex, but for all groups which are not virtually abelian and have a free action on a CAT(0) square complex.

20F65

1 Introduction

Given a group G and a finite generating set S, we let $\mathcal{C}(G,S)$ denote the Cayley graph of G relative to S. The length of an element $g \in G$ with respect to the word metric relative to S is denoted by $|g|_S$ and we let $B(S,n)$ denote the ball of radius n in $\mathcal{C}(G,S)$. The *exponential growth rate of G relative to S* is defined to be the following limit (which always exists):

$$\omega(G,S) = \lim_{n\to\infty} |B(S,n)|^{1/n}.$$

The *exponential growth rate of G* is then given by

$$\omega(G) = \inf\{\omega(G,S) \mid \text{finite generating sets } S\}.$$

The group G is said to have exponential growth if $\omega(G,S) > 1$ for some and therefore for all finite generating sets S. Moreover, G is said to have uniform exponential growth if $\omega(G) > 1$. See de la Harpe [9] for details.

Gromov asked if every group of exponential growth is also of uniform exponential growth. The first example of a group with exponential growth which is not of uniform

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exponential growth was constructed by Wilson [15]. Wilson's group and future counterexamples were finitely generated. Whether Gromov's question has an affirmative answer for finitely presented groups remains open.

Uniform exponential growth is known to hold for groups with virtually free quotients, hyperbolic groups, soluble groups, linear groups in characteristic zero and groups acting on trees in the sense of Bass–Serre theory (see [9]). Uniform exponential growth is typically established by constructing free semigroups; see Alperin and Noskov [1].

Lemma *Let G be a group. Suppose there exists a constant $C > 0$ such that for any finite generating set S of G, one can find two elements $u, v \in G$ with $\max\{|u|_S, |v|_S\} < C$ that freely generate a free semigroup. Then $\omega(G) \geq \sqrt[C]{2}$.*

This method and variations of it often allow one to establish "uniform uniform exponential growth". Bucher and de la Harpe considered actions on trees and showed in [10] that the constant in the above lemma is $\sqrt[4]{2}$ for nondegenerate amalgams and HNN extensions. Mangahas [12] proved that finitely generated subgroups of the mapping class group $\mathrm{Mod}(S)$ of a surface S which are not virtually abelian have uniform exponential growth with minimal growth rate bounded below by a constant depending exclusively on the surface S. Breuillard [2, Main Theorem] established a different sort of uniformity for linear groups: for every $d \in \mathbb{N}$ there is $N(d) \in \mathbb{N}$ such that if K is any field and F a finite symmetric subset of $\mathrm{GL}_d(K)$ containing 1, either $F^{N(d)}$ contains two elements which freely generate a nonabelian free group, or the group generated by F is virtually solvable. We refer the reader to Button [5] for further examples.

In this paper we start the inquiry into proving uniform exponential growth in the context of groups acting on CAT(0) cube complexes. We address free group actions on CAT(0) square complexes. We do this by proving a more general statement about groups generated by hyperbolic elements.

Theorem 1 *Let F be a finite collection of hyperbolic automorphisms of a* CAT(0) *square complex. Then either*

1. *there exists a pair of words of length at most 10 in F which freely generate a free semigroup, or*
2. *there exists a flat (of dimension 1 or 2) in X stabilized by all elements of F.*

As a corollary, we obtain a "uniform uniform" type result, which says that there is a uniform lower bound for growth, not just for a given group, but for all groups acting freely on any CAT(0) square complex.

Corollary 2 *Let G be a finitely generated group acting freely on a* CAT(0) *square complex. Then either $w(G) \geq \sqrt[10]{2}$ or G is virtually abelian.*

We expect that a similar result will hold for all dimensions, in that for a finitely generated group G acting freely on a CAT(0) cube complex of dimension n, G will be virtually abelian or $w(G) \geq w_0 > 1$ where, w_0 will depend only on the dimension n, and not on the group or the complex.

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2 Hyperplanes and group elements

We review some relevant basic facts regarding hyperplanes and halfspaces. See for example [6] or [13] for more details. We let X be a CAT(0) square complex. We use $\mathfrak{h}$, $\mathfrak{k}$ to denote halfspaces, $\hat{\mathfrak{h}}$, $\hat{\mathfrak{k}}$ to denote the corresponding hyperplanes and $\mathfrak{h}^*$, $\mathfrak{k}^*$ to denote the complementary halfspaces.

We let $\mathrm{Aut}(X)$ denote the collection of cubical, inversion-free automorphisms of X. (An inversion is an isometry of X that preserves a hyperplane and inverts the corresponding halfspaces.) If G is an action on X which contains inversions, then we may subdivide X so that there are no inversions.

In a CAT(0) cube complex of dimension n, any collection of $n+1$ hyperplanes contains a disjoint pair. In particular, in the case of our 2–dimensional complex, if $g \in \mathrm{Aut}(X)$ and $\hat{\mathfrak{h}}$ is a hyperplane, then the triple $\{\hat{\mathfrak{h}}, g\hat{\mathfrak{h}}, g^2\hat{\mathfrak{h}}\}$ contains a pair that is either disjoint or equal. Thus, either $g^2\hat{\mathfrak{h}} = \hat{\mathfrak{h}}$, or one of the pairs $\{\hat{\mathfrak{h}}, g\hat{\mathfrak{h}}\}$ or $\{\hat{\mathfrak{h}}, g^2\hat{\mathfrak{h}}\}$ is a disjoint pair.

Given a hyperplane $\hat{\mathfrak{h}}$ in X and $g \in \mathrm{Aut}(X)$ a hyperbolic isometry of X, we say that g *skewers* $\hat{\mathfrak{h}}$ if for some choice of halfspace $\mathfrak{h}$ associated to $\hat{\mathfrak{h}}$, we have $g^2\mathfrak{h} \subset \mathfrak{h}$ (note that this includes the case $g\mathfrak{h} \subset \mathfrak{h}$). This property is equivalent to saying that any axis for g intersects $\hat{\mathfrak{h}}$ in a single point.

We say that a hyperbolic isometry $g \in \mathrm{Aut}(X)$ is *parallel* to $\widehat{\mathfrak{h}}$ if any axis for g is a bounded distance from $\widehat{\mathfrak{h}}$, and a hyperbolic isometry is *peripheral* to $\widehat{\mathfrak{h}}$ if it neither skewers $\widehat{\mathfrak{h}}$ nor is parallel to $\widehat{\mathfrak{h}}$. In this case, any axis lies in a halfspace $\mathfrak{h}$ bounded by the hyperplane $\widehat{\mathfrak{h}}$ and is not contained in any neighborhood of $\widehat{\mathfrak{h}}$. It follows that either $g\mathfrak{h}^* \subset \mathfrak{h}$ or $g^2\mathfrak{h}^* \subset \mathfrak{h}$.

Definition 3 Given a hyperbolic isometry $g \in \mathrm{Aut}(X)$, we define the *skewer set of* g, denoted by $\mathrm{sk}(g)$, as the collection of all hyperplanes skewered by g. We define a *disjoint skewer set* for g as a collection of disjoint hyperplanes in $\mathrm{sk}(g)$ which is invariant under g^2.

If g is parallel to a hyperplane $\widehat{\mathfrak{h}}$, then any hyperplane in $\mathrm{sk}(g)$ intersects $\widehat{\mathfrak{h}}$. Since there are no intersecting triples of hyperplanes in X, this means that no two hyperplanes in $\mathrm{sk}(g)$ intersect. Furthermore, any two translates of $\widehat{\mathfrak{h}}$ under $\langle g\rangle$ are parallel to g and hence cross every hyperplane in $\mathrm{sk}(g)$. Again, by the 2–dimensionality of X, this means that the two translates of $\widehat{\mathfrak{h}}$ under $\langle g\rangle$ are disjoint. We record this observation, since we will make use of it.

Observation 4 *If g is parallel to $\widehat{\mathfrak{h}}$, then all the hyperplanes in* $\mathrm{sk}(g)$ *are disjoint and two distinct hyperplanes in the orbit of $\widehat{\mathfrak{h}}$ under $\langle g\rangle$ are disjoint.*

Lemma 5 *Let g be a hyperbolic automorphism of X; then* $\mathrm{sk}(g)$ *is a union of finitely many disjoint skewer sets.*

Proof Consider $\widehat{\mathfrak{h}} \in \mathrm{sk}(g)$. If $g\widehat{\mathfrak{h}} \cap \widehat{\mathfrak{h}} = \varnothing$, we let $P_1 = \{g^n(\widehat{\mathfrak{h}}) \mid n \in \mathbf{Z}\}$. Otherwise, since X is 2–dimensional, we have $g^2\widehat{\mathfrak{h}} \cap \widehat{\mathfrak{h}} = \varnothing$. We then set $P_1 = \{g^{2n}(\widehat{\mathfrak{h}}) \mid n \in \mathbf{Z}\}$ and $P_2 = \{g^{2n+1}\widehat{\mathfrak{h}} \mid n \in \mathbf{Z}\}$. Thus, P_1 and P_2 break up the orbit of $\widehat{\mathfrak{h}}$ under $\langle g\rangle$ into two disjoint skewer sets. Since there are finitely many orbits of hyperplanes in $\mathrm{sk}(g)$ under the action of $\langle g\rangle$, this breaks up $\mathrm{sk}(g)$ into finitely many disjoint skewer sets. □

Example 6 Let X denote the Euclidean plane, squared in the usual way by unit squares. Let g be an integer translation in the vertical direction. Then the skewer set of g is the collection of horizontal hyperplanes and the number of disjoint skewer sets depends on the translation length of g.

Example 7 Again, let X denote the Euclidean plane. Let g be a glide reflection along the diagonal axis, $g(x, y) = (y + 1, x + 1)$. Then the skewer set of g is a union of four disjoint skewer sets, each invariant under g^2.

3 The parallel subset of an element

Given a hyperbolic $g \in \mathrm{Aut}(X)$, we describe combinatorially a certain invariant subcomplex associated to g which consists of all the lines parallel to axes in G. (This subcomplex is discussed also in [11] and is slightly different than the minimal set of G, as described in [3] or [7].)

We consider the following partition of hyperplanes $\widehat{\mathcal{H}}$ of X. Let

$$\widehat{\mathcal{H}}_{\|}(g) = \{\hat{\mathfrak{h}} \mid \hat{\mathfrak{h}} \text{ intersects every hyperplane in } \mathrm{sk}(g)\}, \quad \widehat{\mathcal{H}}_{\boldsymbol{P}}(g) = \widehat{\mathcal{H}} - (\mathrm{sk}(g) \cup \widehat{\mathcal{H}}_{\|}(g)).$$

Since the elements of $\widehat{\mathcal{H}}_{\boldsymbol{P}}(g)$ are peripheral to g, it follows that for each hyperplane $\hat{\mathfrak{h}} \in \widehat{\mathcal{H}}_{\boldsymbol{P}}(g)$, there exists a well-defined halfspace $\mathfrak{h}$ containing all the axes of g. Recall that the collection of cubes intersecting a hyperplane $\hat{\mathfrak{h}}$ has a product structure $\hat{\mathfrak{h}} \times [0,1]$. We let $N(\hat{\mathfrak{h}}) = \hat{\mathfrak{h}} \times (0,1)$. For a halfspace $\mathfrak{h}$ we let $R(\mathfrak{h}) = \mathfrak{h} - N(\hat{\mathfrak{h}})$.

We define

$$Y_g = \bigcap_{\ell_g \in \mathfrak{h} \text{ and } \hat{\mathfrak{h}} \in \widehat{\mathcal{H}}_{\boldsymbol{P}}(g)} R(\mathfrak{h}).$$

The subspace Y_g is a $\langle g \rangle$–invariant convex subcomplex of X, and as Y_g contains the axes of g, it is nonempty.

The hyperplanes intersecting Y_g are the hyperplanes of $\mathrm{sk}(g)$ and $\widehat{\mathcal{H}}_{\|}(g)$. Since $\mathrm{sk}(g)$ and $\widehat{\mathcal{H}}_{\|}(g)$ are transverse collections of hyperplanes, we obtain (by [6]) that Y_g admits a product structure $Y_g \cong E_g \times T_g$, where E_g is defined by the hyperplanes $\mathrm{sk}(g)$ and T_g is defined by the hyperplanes in $\widehat{\mathcal{H}}_{\|}(g)$. Note that $\mathrm{sk}(g)$ does not contain any disjoint facing triples of hyperplanes. As g does not skewer any hyperplane in $\widehat{\mathcal{H}}_{\|}(g)$, g fixes a vertex in T_g. Since Y_g is 2–dimensional, there are two possibilities:

(1) $E_g = \mathbf{R}$ and T_g is isomorphic to a tree.

(2) E_g is 2–dimensional and T_g is a point.

We call Y_g the *parallel set* of g and E_g its *Euclidean factor.*

We need a further understanding of E_g in order to conclude that groups that stabilize it have nice properties.

Lemma 8 *Let E_g be the Euclidean factor of Y_g. Then either E_g is a Euclidean plane or E_g contains an* $\mathrm{Aut}(E_g)$*–invariant line.*

Proof See [4] or [6] for a discussion of ultrafilters, intervals and medians, which are used in the following argument. We claim first that E_g is an *interval complex*. That is, there exist two ultrafilters α and β on $\mathcal{H}$ such that $\bar{E}_g = [\alpha, \beta]$ (where $\bar{E}_g$ denotes the ultrafilter closure of E_g). To see this, choose a point on an axis ℓ_g for g and let R^+ and R^- be the two subrays of ℓ_g defined by p. Define two ultrafilters

$$\alpha_+ = \{\mathfrak{h} \in \mathcal{H} \mid R^+ \cap \mathfrak{h} \text{ is unbounded}\},$$
$$\alpha_- = \{\mathfrak{h} \in \mathcal{H} \mid R^- \cap \mathfrak{h} \text{ is unbounded}\}.$$

Note that since ℓ_g intersects every hyperplane of E_g, α_+ and α_- are ultrafilters. Moreover, α_+ and α_- make the opposite choices for each hyperplane, which is to say $\alpha_+ \cap \alpha_- = \varnothing$. It follows that for every other ultrafilter β, we have that

$$\operatorname{med}(\alpha_+, \alpha_-, \beta) = (\alpha_+ \cap \alpha_-) \cup (\alpha_+ \cap \beta) \cup (\alpha_- \cap \beta) = \beta.$$

This means that $\bar{E}_g = [\alpha_+, \alpha_-]$, as claimed.

It follows, by [4, Theorem 1.16], that E_g embeds isometrically in the standard squaring of the Euclidean plane. We can thus assume that E_g is an isometrically embedded subset of the standard squaring of the Euclidean plane. It follows that the hyperplanes in E_g are either lines, rays or closed intervals. Since $g \in \operatorname{Aut}(E_g)$ is a hyperbolic element, we also have that there are finitely many orbits of hyperplanes under the action of $\operatorname{Aut}(E_g)$ on E_g.

If all the hyperplanes are lines, then we obtain that E_g is itself a Euclidean plane and we are done. If some hyperplane, say a horizontal one, is a ray, then we claim that all the other horizontal hyperplanes are rays. For if some horizontal hyperplane were a line, then by the fact that g is acting cofinitely on the hyperplanes, we would obtain two horizontal line hyperplanes, separated by a horizontal ray hyperplane. This would contradict the fact that E_g is isometrically embedded in the Euclidean plane. By the same reasoning, there can be no closed interval horizontal hyperplanes, for we would obtain two ray intervals a bounded Hausdorff distance apart in E_g separated by a closed interval hyperplane. From this it follows that all the vertical hyperplanes are rays as well and we have that E_g is a "staircase", as in Figure 1.

In this "stairstep" case, the space of lines which coarsely contains the endpoints of the hyperplanes is itself a ray R which is $\operatorname{Aut}(E_g)$–invariant; hence, there is an $\operatorname{Aut}(E_g)$ fixed point in R and hence an $\operatorname{Aut}(E_g)$–invariant line in E_g.

If there exists a hyperplane in E_g which is a closed interval, then by similar considerations as above, we may conclude that all hyperplanes are closed intervals. Since $\langle g\rangle$

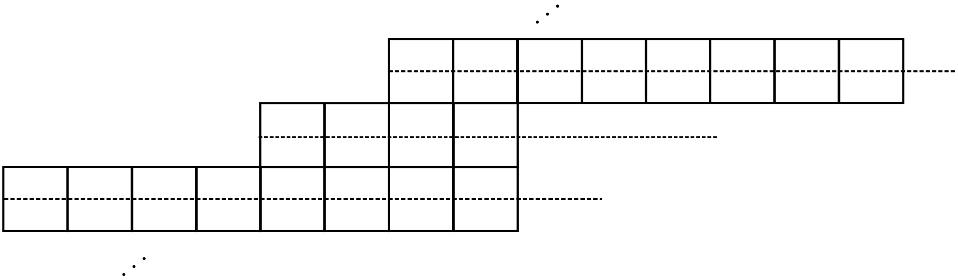

Figure 1: The case in which all hyperplanes in E_g are rays. The endpoints of the rays are invariant, and hence any line in E_g a bounded distance from all endpoints is $\mathrm{Aut}(E_g)$–invariant.

acts cocompactly on E_g, it follows that all lines in E_g are parallel and the space of such lines is a compact interval I. Since the action of $\mathrm{Aut}(E_g)$ on I has a fixed point, it then follows that there is an $\mathrm{Aut}(E_g)$–invariant line. □

4 The ping pong lemma and hyperplane patterns that yield free semigroups

We will use the following version of the ping pong lemma (see for example [8]):

Lemma 9 (semigroup ping pong) *Suppose that a group G is acting on a set X and U and V are* disjoint *subsets of X. If the elements $a, b \in G \backslash \{1\}$ satisfy*

- $a(U \cup V) \subset U$,
- $b(U \cup V) \subset V$,

then a and b freely generate a free subsemigroup in G.

Proof Let Σ be the semigroup generated by a and b in G. Observe that for any $g, h \in \Sigma \subset G$, $ag = ah$ or $bg = bh$ in Σ if and only if $g = h$ in Σ. Therefore, it is enough to check that two words of the form ag and bh cannot be equal in Σ. But, $ag(U \cup V) \subset U$ and $bh(U \cup V) \subset V$. Since $U \cap V = \varnothing$, $ag \neq bh$. □

4.1 On groups acting on trees

To warm up, and to record a few observations we use later on, we first explore what happens for a pair of hyperbolic isometries acting on a tree. We include the proofs here because we will need these types of arguments. However, this is not new. See for example [1]. Let T be a simplicial tree. Recall if an element g of $\mathrm{Aut}(T)$ is hyperbolic then there is a unique geodesic ℓ_g (called the axis of g) which is invariant under g, on which g induces a translation.

Proposition 10 *If a and b are two hyperbolic automorphisms of a tree T, then one of the following occurs:*

- *a and b share the same axis.*
- *$a^{\pm 1}$ and $b^{\pm 1}$ freely generate a free semigroup.*

Proof Suppose that $\ell_a \neq \ell_b$. First assume that $\ell_a \cap \ell_b$ is nonempty and contains an edge $e = [p,q]$. (See Figure 2.) Choose e so that q is a point of bifurcation of ℓ_a and ℓ_b. Let T_q be the component of $T - \mathrm{interior}(e)$ containing q. After possibly replacing a by a^{-1} and/or b by b^{-1}, we see that $ae \subset T_q$ and $be \subset T_q$. Set $U = aT_q$ and $V = bT_q$. Then U and V satisfy the hypothesis of Lemma 9. We will generalize this argument in our context.

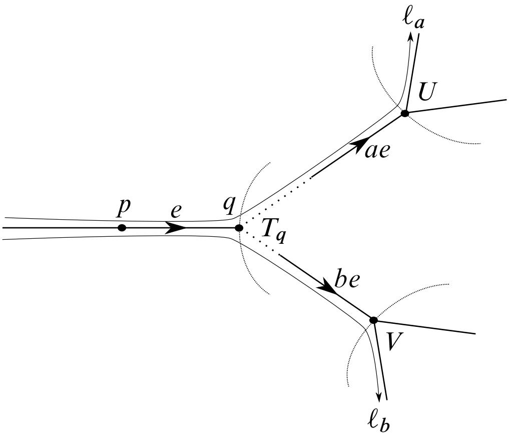

Figure 2: The hyperbolic isometries a and b have unequal but overlapping axes.

The case when $\ell_a \cap \ell_b = \varnothing$ calls for a different argument (see Figure 3). Consider an edge $e = [p,q]$ situated along the geodesic arc joining ℓ_a and ℓ_b. Let T_p be the component of $T - \mathrm{interior}(e)$ containing p and T_q be the component of $T - \mathrm{interior}(e)$ containing q. Suppose (without loss of generality) that $\ell_a \subset T_p$ and $\ell_b \subset T_q$. Then, letting $U = \bigcup_{n>0} a^n T_q$ and $V = \bigcup_{n>0} b^n T_p$, we see that $a(U \cup V) \subset U$ and $b(U \cup V) \subset V$, as required. In fact, in this case, we can argue that a and b generate a

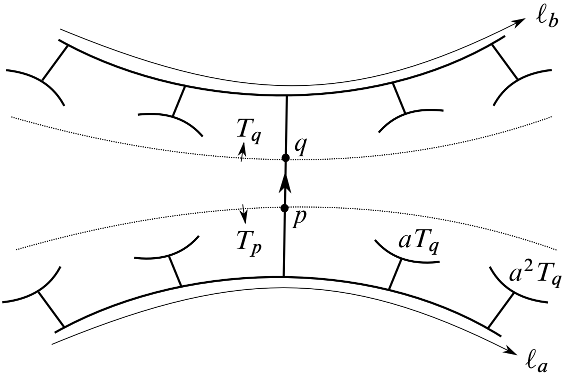

Figure 3: The hyperbolic isometries a and b have disjoint axes.

free group by adjusting U and V to include all nonzero powers of a and b, but we will not need this fact. Note that there is a singular case in which ℓ_a and ℓ_b intersect in a single point. In this case, we simply use the intersecting vertex to separate T into two subtrees, each containing a different axis, and proceed in the same manner. □

4.2 Back to CAT(0) cube complexes

The following lemma works in any dimension and so, just for the paragraph below, we let X be an n–dimensional CAT(0) cube complex.

Lemma 11 *Let $g_1, g_2 \in \mathrm{Aut}(X)$ and suppose that there exists a halfspace $\mathfrak{h}$ of X such that $g_i\mathfrak{h} \subset \mathfrak{h}$ and $g_1\mathfrak{h} \subset g_2\mathfrak{h}^*$. Then g_1 and g_2 generate a free semigroup.*

Proof This argument resembles the first case in the proof of Proposition 10. Set $U = g_1\mathfrak{h}$ and set $V = g_2\mathfrak{h}$ and apply Lemma 9. □

We call the triple $\{\mathfrak{h}, g_1\mathfrak{h}, g_2\mathfrak{h}\}$ a *ping pong triple* for g_1 and g_2.

5 Main argument

Now, let X be a CAT(0) square complex.

Lemma 12 (all or nothing) *Let a and b be hyperbolic isometries of X and let P be a disjoint skewer set for a. Suppose that no pair of words of length at most 6 in a and b generate a free semigroup; then either b skewers every hyperplane in P or b does not skewer any hyperplane in P.*

Proof Recall that for any $\hat{\mathfrak{h}}$ in $\mathrm{sk}(a)$, there exists an associated halfspace $\mathfrak{h}$ such that $a^2\mathfrak{h} \subset \mathfrak{h}$. If b skewers some element in P, but not all, we may also choose $\mathfrak{h}$ such that $\mathfrak{h}$ is skewered by b but $a^2\mathfrak{h}$ is not skewered by b. After replacing b possibly by b^{-1}, we may assume that $b^2\mathfrak{h} \subset h$. Note that b, and hence b^2, is peripheral to $a^2\hat{\mathfrak{h}}$.

Now, by the 2–dimensionality of X, either $b^2a^2\hat{\mathfrak{h}} \cap a^2\hat{\mathfrak{h}} = \varnothing$ or $b^4a^2\hat{\mathfrak{h}} \cap a^2\hat{\mathfrak{h}} = \varnothing$. We further have that $b^2a^2\mathfrak{h} \subset b^2\mathfrak{h} \subset \mathfrak{h}$ and $b^4a^2\mathfrak{h} \subset b^4\mathfrak{h} \subset \mathfrak{h}$.

We thus have that either $\{\mathfrak{h}, a^2\mathfrak{h}, b^2a^2\mathfrak{h}\}$ or $\{\mathfrak{h}, a^2\mathfrak{h}, b^4a^2\mathfrak{h}\}$ is a ping pong triple of halfspaces for the pairs $\{a^2, b^2a^2\}$ or $\{a^2, b^4a^2\}$. See Figure 4. In either case, we obtain words of length at most 6 freely generating a free semigroup, a contradiction. □

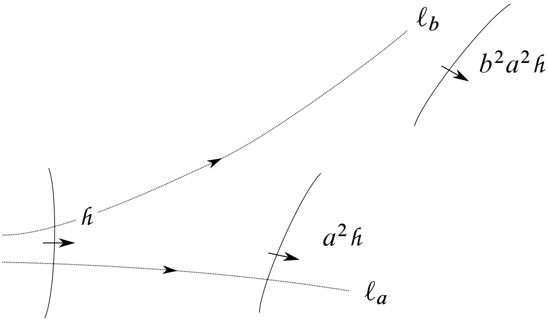

Figure 4: The element b skewering $\mathfrak{h}$ but not $a\mathfrak{h}$

Proposition 13 (not skewering means parallel) *Let a and b be hyperbolic isometries of X and let P be a disjoint skewer set for a. Let ℓ_b be an axis for b. Suppose that b does not skewer any element of P and that no pair of words of length no more than 10 freely generate a free semigroup. Then:*

(1) *The axis ℓ_b is parallel to every hyperplane $\hat{\mathfrak{h}} \in P$.*

(2) *$bP \in \mathrm{sk}(a)$.*

(3) *b^2 stabilizes every hyperplane in P.*

Proof The disjoint skewer set P decomposes as a finite union of $\langle a^2\rangle$–orbits. So, the assumption that b does not skewer any hyperplane in P holds for each orbit. If the conclusion of the proposition holds for each $\langle a^2\rangle$–orbit, then it holds for all of P. Therefore, it suffices to prove the proposition for when P is a single $\langle a^2\rangle$–orbit: there exists $\mathfrak{h} \in P$ such that $a^2\mathfrak{h} \subset \mathfrak{h}$ and $P = \{a^{2k}\hat{\mathfrak{h}} \mid k \in \mathbf{Z}\}$. We set $c = a^2$. Since b

does not skewer any hyperplane in P, we may assume that $\ell_b \subset \mathfrak{h} \cap c\mathfrak{h}^*$. (We are using here that the action is without inversions, so that if $\ell_b \subset \hat{\mathfrak{h}}$ for some hyperplane, there is a parallel axis for b on either side of $\hat{\mathfrak{h}}$.) We will now use our assumptions to remove the possibility that b is peripheral to $\hat{\mathfrak{h}}$ or $c\hat{\mathfrak{h}}$.

First, suppose b is peripheral to both $\hat{\mathfrak{h}}$ and $c\hat{\mathfrak{h}}$. We claim that we can find a facing triple of hyperplanes of the form $\{\hat{\mathfrak{h}}, b^s\hat{\mathfrak{h}}, b^t\hat{\mathfrak{h}}\}$ with $|s|, |t| \leq 4$.

To see this, consider the six translates $\{b^{-2}\hat{\mathfrak{h}}, b^{-1}\hat{\mathfrak{h}}, \hat{\mathfrak{h}}, b\hat{\mathfrak{h}}, b^2\hat{\mathfrak{h}}, b^3\hat{\mathfrak{h}}\}$. Construct the intersection graph Γ for these six hyperplanes: the vertices of Γ are the elements of $\{b^{-2}\hat{\mathfrak{h}}, b^{-1}\hat{\mathfrak{h}}, \hat{\mathfrak{h}}, b\hat{\mathfrak{h}}, b^2\hat{\mathfrak{h}}, b^3\hat{\mathfrak{h}}\}$, and two vertices are joined by an edge if and only if the respective hyperplanes cross. Since $R(3,3) = 6$, the graph Γ possesses a clique or an anticlique on three vertices. However, as in a CAT(0) square complex, three distinct hyperplanes cannot pairwise intersect; the intersection graph Γ must have an anticlique T consisting of three hyperplanes. If T contains $\hat{\mathfrak{h}}$, then we are done; else, we take a suitable translate of T. The highest exponents appear when $T = \{b^{-2}\hat{\mathfrak{h}}, b^2\hat{\mathfrak{h}}, b^3\hat{\mathfrak{h}}\}$, and, in this case, we take $b^{-2}T$ as our chosen set of facing triples.

We now have s and t of absolute value at most 4, such that $\hat{\mathfrak{h}}$, $b^s\hat{\mathfrak{h}}$ and $b^t\hat{\mathfrak{h}}$ are disjoint and form a facing triple. Translating by c, we get that $c\hat{\mathfrak{h}}$, $cb^s\hat{\mathfrak{h}}$ and $cb^t\hat{\mathfrak{h}}$ form a facing triple of hyperplanes. As b is also peripheral to $c\hat{\mathfrak{h}}$, there exists $\eta \leq 2$ such that $b^\eta c\hat{\mathfrak{h}} \cap c\hat{\mathfrak{h}} = \varnothing$. Now, $cb^s\mathfrak{h}^*$ and $cb^t\mathfrak{h}^*$ are both disjoint halfspaces that lie inside the halfspace $b^\eta c\mathfrak{h}^*$. This implies that the two elements $cb^sc^{-1}b^{-\eta}$ and $cb^tc^{-1}b^{-\eta}$ (each of length ≤ 10) freely generate a free semigroup, a contradiction.

Let us now assume that b is parallel to $\hat{\mathfrak{h}}$ but peripheral to $c\hat{\mathfrak{h}}$. It follows from Observation 4 that for any $i \in \mathbf{Z}$, $b^i\hat{\mathfrak{h}} = \hat{\mathfrak{h}}$ or $b^i\hat{\mathfrak{h}} \cap \hat{\mathfrak{h}} = \varnothing$. First let us consider the case that $b^2\hat{\mathfrak{h}} = \hat{\mathfrak{h}}$. Note that since we are assuming that $\mathrm{Aut}(X)$ acts with no inversions, we have that $b^2\mathfrak{h} = \mathfrak{h}$. Now, since b is peripheral to $c\hat{\mathfrak{h}}$, for $k = 1$ or 2 we have that $b^{2k}c\hat{\mathfrak{h}} \cap c\hat{\mathfrak{h}} = \varnothing$. We thus obtain a ping pong triple of halfspaces $\{\mathfrak{h}, c\mathfrak{h}, b^{2k}c\mathfrak{h}\}$ for the elements c and $b^{2k}c$. From Lemma 11 we see that c and $b^{2k}c$ freely generate a free semigroup, a contradiction since these are words of length at most 6 in a and b. (See Figure 5.)

We may thus assume that $b\hat{\mathfrak{h}} \cap \hat{\mathfrak{h}} = \varnothing$ and $b^2\hat{\mathfrak{h}} \cap \hat{\mathfrak{h}} = \varnothing$. Only one of $b\hat{\mathfrak{h}}$ or $b^2\hat{\mathfrak{h}}$ can separate $\hat{\mathfrak{h}}$ and $c\hat{\mathfrak{h}}$, for otherwise we would have $b\mathfrak{h} \subset b^2\mathfrak{h}$ or $b^2\mathfrak{h} \subset \mathfrak{h}$. So for some $\epsilon = 1$ or 2, we can assume that $b^\epsilon\hat{\mathfrak{h}}$ does not separate $\hat{\mathfrak{h}}$ and $c\hat{\mathfrak{h}}$. Note also that since $c\hat{\mathfrak{h}}$ is peripheral to b, one cannot have $b^\epsilon\hat{\mathfrak{h}} \subset c\mathfrak{h}$.

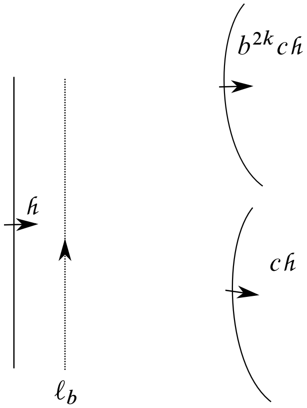

Figure 5: If b stabilizes h, we obtain a ping pong triple of hyperplanes.

If $c\hat{\mathfrak{h}} \cap b^{\epsilon}\hat{\mathfrak{h}} = \varnothing$, then we obtain a ping pong triple of halfspaces $\{c\mathfrak{h}^*, \mathfrak{h}^*, b^{\epsilon}\mathfrak{h}^*\}$ for the words c^{-1} and $b^{\epsilon}c^{-1}$. Since these are words of length at most 4 in a and b, we have a contradiction. (See Figure 6.)

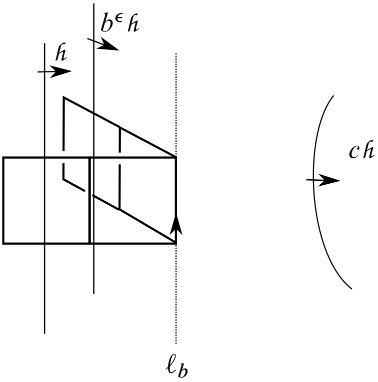

Figure 6: If $c\hat{\mathfrak{h}} \cap b^{\epsilon}\hat{\mathfrak{h}} = \varnothing$ and $b^{\epsilon}\hat{\mathfrak{h}}$ does not separate $\hat{\mathfrak{h}}$ and $c\hat{\mathfrak{h}}$, we obtain a ping pong triple.

Thus, we assume that $b^{\epsilon}\hat{\mathfrak{h}} \cap c\hat{\mathfrak{h}} \neq \varnothing$ and refer to Figure 7. Since, by Observation 4, any hyperplane in $\mathrm{sk}(b)$ intersects $b^{\epsilon}\hat{\mathfrak{h}}$, and we are assuming that $b^{\epsilon}\hat{\mathfrak{h}} \cap c\hat{\mathfrak{h}} \neq \varnothing$, the 2–dimensionality of X implies that any hyperplane in $\mathrm{sk}(b)$ is disjoint from $c\hat{\mathfrak{h}}$. Moreover, by Observation 4, we have that for any hyperplane $\hat{\mathfrak{k}}$ in $\mathrm{sk}(b)$, $b\mathfrak{k} \subset k$ for some choice of halfspace $\mathfrak{k}$ associated to $\hat{\mathfrak{k}}$. We may further choose $\mathfrak{k}$ such that $c\mathfrak{h} \subset \mathfrak{k} \cap b\mathfrak{k}^*$.

Applying c^{-1}, we see that $\mathfrak{h} \subset c^{-1}\mathfrak{k} \cap c^{-1}b\mathfrak{k}^*$. Applying b^ϵ, we see that $b^\epsilon c^{-1}\hat{\mathfrak{k}} \subset b^\epsilon \mathfrak{h}^* \subset \mathfrak{h}$. Thus, we have a ping pong triple of half spaces $\{c^{-1}b\mathfrak{k}^*, c^{-1}\mathfrak{k}^*, b^\epsilon c^{-1}\mathfrak{k}^*\}$ for the elements $c^{-1}b^{-1}c$ and $b^\epsilon c^{-1}b^{-1}c$. So, by Lemma 9 we have that $c^{-1}b^{-1}c$ and $b^\epsilon c^{-1}b^{-1}c$ generate a free semigroup and these are words of length at most 7.

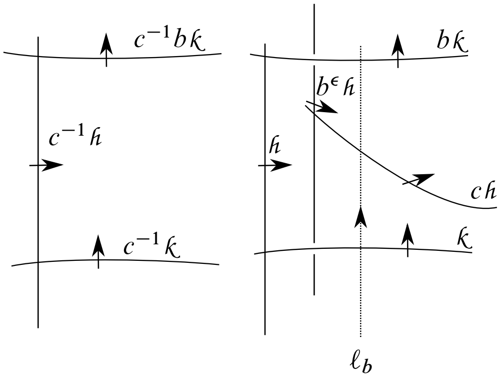

Figure 7: If $c\hat{\mathfrak{h}} \cap b^\epsilon \hat{\mathfrak{h}} \neq \varnothing$, we obtain a ping pong triple.

We may thus assume that b is parallel to both $\hat{\mathfrak{h}}$ and $c\hat{\mathfrak{h}}$. Assume that $d(\ell_b, \hat{\mathfrak{h}}) \leq d(\ell_b, c\hat{\mathfrak{h}})$. (There is no loss of generality here, for if $d(\ell_b, c\hat{\mathfrak{h}}) \leq d(\ell_b, \hat{\mathfrak{h}})$, we will reverse the roles of $\hat{\mathfrak{h}}$ and $c\hat{\mathfrak{h}}$ in the following argument.)

As before, we first consider what happens if $\hat{\mathfrak{h}}$ is not stabilized by b^2. Here we obtain that $\hat{\mathfrak{h}}$, $b\hat{\mathfrak{h}}$ and $b^2\hat{\mathfrak{h}}$ are disjoint. We cannot have that $b\hat{\mathfrak{h}} = c\hat{\mathfrak{h}}$ or $b^2\hat{\mathfrak{h}} = c\hat{\mathfrak{h}}$, for then we would obtain that $c^{-1}b\hat{\mathfrak{h}}$ or $c^{-1}b^2\hat{\mathfrak{h}}$ is an inversion of $\hat{\mathfrak{h}}$. Thus, we have that $b\hat{\mathfrak{h}} \subset c\mathfrak{h}^*$ and $b^2\hat{\mathfrak{h}} \subset c\mathfrak{h}^*$. We now proceed as in the case in which $c\hat{\mathfrak{h}}$ is peripheral to b to produce a ping pong triple of halfspaces $\{c\mathfrak{h}^*, \mathfrak{h}^*, b^\epsilon \mathfrak{h}^*\}$ for the words c^{-1} and $b^\epsilon c^{-1}$. (The configuration is the same as in Figure 6 except that here $c\hat{\mathfrak{h}}$ is parallel to ℓ_b.)

So assume $b^2\hat{\mathfrak{h}} = \hat{\mathfrak{h}}$. Again, as above, if b^2 did not also stabilize $c\hat{\mathfrak{h}}$, we would obtain a small ping pong triple. Thus, b^2 stabilizes $c\hat{\mathfrak{h}}$ as well. Since b^2 stabilizes $c\hat{\mathfrak{h}}$ (and the action is inversion-free), we have an axis for b^2 in $c\mathfrak{h} \cap c^2\mathfrak{h}^*$. We can now carry out all the above arguments, replacing $\hat{\mathfrak{h}}$ and $c\hat{\mathfrak{h}}$ with $c\hat{\mathfrak{h}}$ and $c^2\hat{\mathfrak{h}}$, to conclude that b^2 stabilizes $c^2\hat{\mathfrak{h}}$. Proceeding in this way we see that b is parallel to every hyperplane of P and that $b^2P = P$.

We are left to show that $bP \subset \mathrm{sk}(a)$. We now argue as in the proof of Lemma 12 using the pair bab^{-1} and a. The pairs $\{ba^2b^{-1}, a^2ba^2b^{-1}\}$ and $\{ba^2b^{-1}, a^4ba^2b^{-1}\}$ made of words of length at most 8 in a and b may freely generate free semigroups. But we have assumed that there are no such free semigroups. Hence, in our current case, Lemma 12 implies that a skewers every hyperplane in bP or none of the hyperplanes in bP. In the former case, we get $bP \subset \mathrm{sk}(a)$, as required. So suppose that a does not skewer any hyperplane in bP. Note that $b\hat{\mathfrak{h}}$ must be disjoint from $\hat{\mathfrak{h}}$ and $c\hat{\mathfrak{h}}$ because ℓ_b is parallel to all three. Similarly, $bc\hat{\mathfrak{h}}$ is disjoint from $\hat{\mathfrak{h}}$ and $c\hat{\mathfrak{h}}$. Since $\ell_b \subset \mathfrak{h} \cap c\mathfrak{h}^*$, we have either $b\mathfrak{h}^* \subset \mathfrak{h} \cap c\mathfrak{h}^*$ or $bc\mathfrak{h} \subset \mathfrak{h} \cap c\mathfrak{h}^*$, depending on which of $\hat{\mathfrak{h}}$ or $c\hat{\mathfrak{h}}$ is closer to ℓ_b. In either case, we then get a small ping pong triple, a contradiction. □

If a and b are elements such that there exists a disjoint skewer set P for a as in Proposition 13, then we say that b *is subparallel to* a.

Corollary 14 *Given hyperbolic isometries a and b such that no words of length at most 10 generate a free semigroup of rank 2, b is subparallel to a if and only if* $\mathrm{sk}(a) - \mathrm{sk}(b) \neq \varnothing$.

Proof If b is subparallel to a, then, by definition, there exists a disjoint skewer set for a such that b is parallel to all the hyperlanes in P. Thus, $P \subset \mathrm{sk}(a) - \mathrm{sk}(b)$. Conversely, if there exists $\hat{\mathfrak{h}} \in \mathrm{sk}(a) - \mathrm{sk}(b)$, then by Lemma 12, the entire disjoint parallel set P for a containing $\hat{\mathfrak{h}}$ is not skewered by b. Then, by Proposition 13, b is subparallel to a. □

From this corollary, we see that there are three possibilities for two hyperbolic elements a and b such that words of length at most 10 do not freely generate a free semigroup:

(I) $\mathrm{sk}(a) = \mathrm{sk}(b)$.

(II) b is subparallel to a and a is subparallel to b.

(III) b is subparallel to a and a is not subparallel to b (or the same with the roles of a and b reversed).

We claim that in each of these cases, we can find an invariant line or flat for $\langle a, b\rangle$.

Proposition 15 *Let a and b be hyperbolic isometries such that no words in a and b of length at most 10 freely generate a free semigroup; then there exists a Euclidean subcomplex of X invariant under $\langle a, b\rangle$.*

Proof We analyze the three cases above. Suppose we are in case I, so that $\mathrm{sk}(a) = \mathrm{sk}(b)$. Then we consider $Y = Y_a = Y_b = E \times T$. If T is trivial (ie a single point), then we have that both a and b leave E invariant, as required. Otherwise we have that $Y = \mathbf{R} \times T$, where a and b both act by vertical translation. We consider the action of a and b on T. Both a and b have nonempty fixed point sets, which we denote by

F_a and F_b. If $F_a \cap F_b \neq \varnothing$, then, choosing $p \in F_a \cap F_b$ we have that both a and b stabilize the line $\mathbf{R} \times \{p\} \subset \mathbf{R} \times T$.

So suppose that $F_a \cap F_b = \varnothing$. As in [14], we have that ab is hyperbolic in its action on T, stabilizing a line ℓ which intersects both F_a and F_b. We claim that a stabilizes ℓ. For, otherwise, consider the line $a\ell$. This is stabilized by the element $u = a(ab)a^{-1}$. If $a\ell \neq \ell$, then we obtain that $(ab)^{\pm 1}$ and $u^{\pm 1}$ freely generate a free semigroup by Proposition 10, contradicting our assumption. Similarly, we see that b stabilizes ℓ as well. Thus, $\langle a, b\rangle$ stabilizes the flat $\mathbf{R} \times \ell \subset \mathbf{R} \times T$, as required.

We now consider case II, where a and b are subparallel to one another. Note that since an axis for a is parallel to a hyperplane (in $\operatorname{sk}(b)$), then all the hyperplanes in $\operatorname{sk}(a)$ are disjoint. Similarly all the hyperplanes in $\operatorname{sk}(b)$ are disjoint. Note also every hyperplane in $\operatorname{sk}(a)$ crosses every hyperplane in $\operatorname{sk}(b)$, so that they determine a flat $E = Y_a \cap Y_b$. Moreover, since b is parallel to one of the hyperplanes in $\operatorname{sk}(a)$, it is parallel or peripheral to all hyperplanes in $\operatorname{sk}(a)$. But then Proposition 13 implies that for all disjoint skewer sets $P \subset \operatorname{sk}(a)$, we have $bP \subset \operatorname{sk}(a)$. Thus, $b\operatorname{sk}(a) \subset \operatorname{sk}(a)$. By the same argument, we obtain $b^{-1}\operatorname{sk}(a) \subset \operatorname{sk}(a)$, so that $b\operatorname{sk}(a) = \operatorname{sk}(a)$.

Similarly, we have that $a\operatorname{sk}(b) = \operatorname{sk}(b)$. We thus have that $\langle a, b\rangle$ stabilizes the flat E.

Finally, we consider case III. In this case there exists a disjoint skewer set P for a, so that b is parallel to P. However, since a is not subparallel to b, a also skewers every element in $\operatorname{sk}(b)$. Since the hyperplanes in $\operatorname{sk}(b)$ all intersect the hyperplanes in P, we have that $\operatorname{sk}(a)$ has crossing hyperplanes. It follows that the parallel set Y_a for a is of the form $Y_a = E \times \{\text{point}\}$. It is also easy to see that b stabilizes E, so that $\langle a, b\rangle$ stabilizes E. □

We are now ready to prove Theorem 1, which we restate here for convenience:

Theorem 1 *Let F be a finite collection of hyperbolic automorphisms of a* CAT(0) *square complex. Then either*

(1) *there exists a pair of words of length at most* 10 *in F which freely generate a free semigroup, or*

(2) *there exists a flat (of dimension* 1 *or* 2*) in X stabilized by all elements of F.*

Proof Consider $F = \{s_1, s_2, \dots, s_n\}$. Each of the pairs $\{s_i, s_j\}$ satisfy one of the cases I, II or III, above.

If there exists a pair of type III, without loss of generality, assume that is the pair $\{s_1, s_2\}$, with s_2 subparallel to s_1 and s_1 not subparallel to s_2. In this case, the parallel set $Y_{s_1} = E \times \{\text{point}\}$. In this case, for every other s_i, we have that the pair $\{s_1, s_i\}$ is either of type I or III. In either case, we obtain that s_i stabilizes E and we are done.

So we suppose that no pair $\{s_i, s_j\}$ is of type III. Suppose, that there exists a pair, say $\{s_1, s_2\}$, which is of type II. Let E be the flat in X on which $\langle s_1, s_2 \rangle$ acts. For any other s_i, we have that the pairs $\{s_1, s_i\}$ and $\{s_2, s_i\}$ are of type I or II. It cannot be that both pairs are of type I since $\operatorname{sk}(s_1) \cap \operatorname{sk}(s_2) = \varnothing$. Also, it cannot be that s_i is subparallel to both s_1 and s_2, for otherwise ℓ_{s_i} would be parallel to hyperplanes in $\operatorname{sk}(s_1)$ and in $\operatorname{sk}(s_2)$, but every hyperplane in $\operatorname{sk}(s_1)$ crosses every hyperplane in $\operatorname{sk}(s_2)$ in a single point. Thus, a line cannot be parallel to a hyperplane in $\operatorname{sk}(s_1)$ and a hyperplane in $\operatorname{sk}(s_2)$. It follows that, without loss of generality, s_i is subparallel to s_1 and $\operatorname{sk}(s_i) = \operatorname{sk}(s_2)$. It then follows that s_i stabilizes E.

Finally, suppose that all the pairs $\{s_i, s_j\}$ are of type I. Thus, $\operatorname{sk}(s_i) = \operatorname{sk}(s_j)$ for all i and j. Thus, G stabilizes $Y = E \times T = E_{s_i} \times T_{s_i}$. If E contains squares, then T is trivial and s_i stabilizes E, as required. So suppose that $Y = \mathbf{R} \times T$, and each s_i acts "vertically". That is, s_i acts by translation along $\mathbf{R}$ and has a fixed point in T.

We now examine the action of G on T. Let F_i denote the fixed set of s_i. If for each pair i and j, $F_i \cap F_j \neq \varnothing$, then by a standard result, $X_n = \bigcap_{i=1}^{n} F_i \neq \varnothing$. Choose a vertex $p_n \in X_n$. Then $H_n = \langle s_1, \dots, s_n \rangle$ acts on $\ell_n = \mathbf{R} \times p_n$ by translations. Thus, H_n stabilizes a flat in X.

So suppose that there exists a pair, say F_1 and F_2, such that $F_1 \cap F_2 = \varnothing$. In this case, as in the proof of Proposition 15, there exist a line $\ell \subset T$ on which $\langle s_1, s_2 \rangle$ acts as a dihedral group. As in the proof of Proposition 15, we also obtain that for every i, s_i stabilizes ℓ. Thus, G stabilizes ℓ, and therefore the flat $\mathbf{R} \times \ell$, as required. □

Remark 16 The proof of the theorem shows that in case (1), there is a subset F_0 of F made of two or three elements and a pair of words of length ≤ 10 in F_0 which generate the free semigroup of rank 2.

Corollary 2 now follows from the main theorem since when the action of a group is free, stabilizing a flat implies the group is virtually abelian, by the Bieberbach theorem.

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Commensurability invariance for abelian splittings of right-angled Artin groups, braid groups and loop braid groups

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We prove that if a right-angled Artin group A_Γ is abstractly commensurable to a group splitting nontrivially as an amalgam or HNN extension over $\mathbb{Z}^n$, then A_Γ must itself split nontrivially over $\mathbb{Z}^k$ for some $k \le n$. Consequently, if two right-angled Artin groups A_Γ and A_Δ are commensurable and Γ has no separating k–cliques for any $k \le n$, then neither does Δ, so "smallest size of separating clique" is a commensurability invariant. We also discuss some implications for issues of quasi-isometry. Using similar methods we also prove that for $n \ge 4$ the braid group B_n is not abstractly commensurable to any group that splits nontrivially over a "free group–free" subgroup, and the same holds for $n \ge 3$ for the loop braid group LB_n. Our approach makes heavy use of the Bieri–Neumann–Strebel invariant.

20F65; 20F36, 57M07

Introduction

We say two groups are *abstractly commensurable* or for brevity *commensurable* if they contain isomorphic finite-index subgroups. It has been an ongoing problem to understand commensurability for right-angled Artin groups, or RAAGs for short. This can mean either to understand when a group is commensurable to a given RAAG, or to understand when two RAAGs are commensurable to each other. For instance, a RAAG is commensurable to a nonabelian free group if and only if it itself is a nonabelian free group, and on the other hand $\mathbb{Z}^n$ is not commensurable to any RAAG except itself. Related questions include all of the above replacing "commensurable" with "quasi-isometric" everywhere, and the "rigidity" question asking for which RAAGs does quasi-isometry imply commensurability.

Recall that for a finite simplicial graph Γ, the RAAG A_Γ is defined by the presentation with a generator for each vertex of Γ and the relations that two generators commute if and only if their corresponding vertices span an edge in Γ. A great deal of work has been done toward understanding the above questions for RAAGs A_Γ assuming

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various restrictions on Γ. For example, in [14] Huang proved that if A_Γ has finite outer automorphism group, which is equivalent to saying that Γ has no separating closed stars and no instances of $\mathrm{lk}\, v \subseteq \mathrm{st}\, w$ for vertices $v \neq w$, then a RAAG A_Δ is commensurable to A_Γ if and only if it is quasi-isometric. Moreover, if A_Γ and A_Δ both have finite outer automorphism group then they are quasi-isometric if and only if $\Gamma \cong \Delta$. Other examples of past work include Huang [13; 15], Casals-Ruiz, Kazachkov and Zakharov [6], Casals-Ruiz [5], Behrstock, Januszkiewicz and Neumann [2] and Kim and Koberda [16; 17]. In all of these examples, results are shown assuming the RAAG or RAAGs in question have defining graphs falling into certain classes. For example, there are results if the graph is a tree, or a join, or an atomic graph, or a cyclic graph, or has some other such global structure.

In this paper we do not focus on any particular graph or class of graphs, but rather inspect the commensurability problem in terms of some more local features of the graph, with an eye on separating cliques. These correspond to nontrivial splittings over free abelian groups. Recall that a *nontrivial splitting* of a group G over a subgroup C is a decomposition $G = A *_C B$ with $G \neq A, B$ or $G = A *_C$ with $G \neq A$. Our main results are:

Theorem 3.5 *Let Γ be a finite simplicial nonclique graph with no separating k–cliques for any $k \leq n$. Then A_Γ is not commensurable to any group splitting nontrivially over $\mathbb{Z}^n$.*

Corollary 3.6 *If A_Γ and A_Δ are commensurable and Γ has no separating k–cliques for any $k \leq n$, then neither does Δ.*

An equivalent way to phrase Theorem 3.5 is to say that such an A_Γ does not virtually split nontrivially over $\mathbb{Z}^n$. Another equivalent formulation is: if a RAAG virtually splits nontrivially over $\mathbb{Z}^n$ then it must actually split nontrivially over $\mathbb{Z}^k$ for some $k \leq n$. Corollary 3.6 can be phrased informally as "'smallest size of separating clique' is a commensurability invariant for RAAGs". We suspect that the conclusion of Theorem 3.5 is true even if we only assume Γ has no separating n–cliques, though proving this would require new ideas (for instance, even in the proof of Proposition 2.3, concerning when A_Γ itself splits, we cannot precisely control the size of the cliques that arise).

Say that a group is *NF* if it contains no nonabelian free subgroups (so, colloquially, it is a "free group–free group"). It is a fact that RAAGs satisfy a strong Tits alternative, namely every NF subgroup of a RAAG is abelian; even more strongly, every pair of elements in a RAAG either commute or generate a copy of F_2; see Baudisch [1], Carr [4] and Kim and Koberda [18]. This leads to the following corollary in the case when Γ has no separating cliques at all:

Corollary 3.7 *Let Γ be a finite simplicial nonclique graph with no separating cliques. Then A_Γ is not commensurable to any group splitting nontrivially over an NF subgroup.*

The key to proving Theorem 3.5 is understanding the Bieri–Neumann–Strebel (BNS) invariant well enough to produce nontrivial characters of the groups of interest that contain certain prescribed subgroups in their kernels while still lying in the BNS invariant. The BNS invariant of an arbitrary RAAG is known from work of Meier and VanWyk [22]. There has been some other recent interest in using the BNS invariants of RAAGs to distinguish groups; for instance, Koban and Piggott [20] determined precisely when the pure symmetric automorphism group of a RAAG is itself a RAAG, and Day and Wade [11] used a new homology theory to produce similar results for the "outer" version.

Using the BNS invariant to approach questions of commensurability is a natural endeavor, but to the best of our knowledge it has not been exploited in the literature. We expect that our techniques could be used in the future to get similar commensurability results for other groups whose BNS invariants are known. In the interest of providing other explicit examples, we inspect braid groups and loop braid groups, and use similar methods to those used for RAAGs to get the following results:

Theorem 5.1 *For $n \geq 4$ the braid group B_n is not commensurable to any group that splits nontrivially over an NF subgroup.*

Theorem 5.2 *For $n \geq 3$ the loop braid group LB_n is not commensurable to any group that splits nontrivially over an NF subgroup.*

The BNS invariant of the (loop) braid group is known but turns out not to be useful here, since it is too small (characters tend to become trivial as soon as they kill interesting subgroups). Instead we use the BNS invariants of the pure braid group PB_n and pure loop braid group PLB_n, which are known from work of Koban, McCammond and Meier [19] and Orlandi-Korner [23], and are robust enough to use for these purposes. Another relevant comment here is that Clay, Leininger and Margalit proved that for $n \geq 4$ the group B_n is not commensurable to any RAAG [9].

This paper is organized as follows. In Section 1 we recall the BNS invariant and establish some results about kernels of characters. In Section 2 we discuss RAAGs and their BNS invariants, and refine a result of Groves and Hull [12] about which RAAGs split over which abelian subgroups. In Section 3 we prove our main commensurability results, Theorem 3.5 and Corollaries 3.6 and 3.7, about RAAGs, and in Section 4 we discuss the consequences our results have for questions of quasi-isometry. Finally, in Section 5 we prove related commensurability results, Theorems 5.1 and 5.2, about braid groups and loop braid groups.

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1 Characters of a group

A *character* of a group G is a homomorphism $G \to \mathbb{R}$. In this section we recall the definition of the BNS invariant and establish some facts about the behavior of kernels of characters.

1.1 The BNS invariant

The BNS invariant $\Sigma^1(G)$ of a finitely generated group G is a certain subset of the *character sphere*

$$S(G) := \{[\chi] \mid 0 \neq \chi \in \operatorname{Hom}(G,\mathbb{R})\}$$

of G. Here $[\chi]$ is the equivalence class of the character $\chi \in \operatorname{Hom}(G,\mathbb{R})$ under the equivalence relation given by $\chi \sim \chi'$ whenever $\chi = a\chi'$ for some $a \in \mathbb{R}_{>0}$. The character sphere is thus the "sphere at infinity" for the euclidean vector space $\operatorname{Hom}(G,\mathbb{R})$. The invariant $\Sigma^1(G)$ is the subset of $S(G)$ defined as follows:

Definition 1.1 (BNS invariant) Let G be a finitely generated group and let $\operatorname{Cay}(G)$ be its Cayley graph with respect to some finite generating set. For $0 \neq \chi \in \operatorname{Hom}(G,\mathbb{R})$ let $\operatorname{Cay}(G)^{\chi \geq 0}$ be the induced subgraph of $\operatorname{Cay}(G)$ supported on those vertices g with $\chi(g) \geq 0$. The *BNS invariant* $\Sigma^1(G)$ is defined to be

$$\Sigma^1(G) := \{[\chi] \in S(G) \mid \operatorname{Cay}(G)^{\chi \geq 0} \text{ is connected}\}.$$

Denote by $\Sigma^1(G)^c$ the complement $S(G) \setminus \Sigma^1(G)$. For various reasons it will be convenient to adopt the convention that the trivial character 0 lies in $\Sigma^1(G)^c$ (but note that this runs counter to the definition).

In general the BNS invariant can be very difficult to compute. It contains a huge amount of information; for example, it reveals exactly which (normal) subgroups $N \le G$ containing $[G,G]$ are finitely generated or not, namely N is finitely generated if and only if $[\chi] \in \Sigma^1(G)$ for all $0 \ne \chi$ such that $\chi(N) = 0$.

Even if $\Sigma^1(G)$ is completely known, it can still be very difficult to compute $\Sigma^1(H)$ for H a finite-index subgroup of G. There is a region of $S(H)$ that can be understood based just on knowing $\Sigma^1(G)$, namely the region given by characters of H that are restrictions of characters of G:

Citation 1.2 [24, Proposition B1.11] *Let G be a finitely generated group and H a finite-index subgroup. Let $\chi \in \mathrm{Hom}(G,\mathbb{R})$ and consider the restriction $\chi|_H \in \mathrm{Hom}(H,\mathbb{R})$ of χ to H. We have that $[\chi|_H] \in \Sigma^1(H)$ if and only if $[\chi] \in \Sigma^1(G)$.*

1.2 Kernels of characters

In this subsection we find a way to control which generators of a group must lie in the kernel of a character, given the knowledge that some prescribed subgroup lies in the kernel. The main result is Proposition 1.4.

Fix a finitely generated group G. Let V denote the $\mathbb{R}$–vector space $(G/[G,G]) \otimes \mathbb{R}$. Let $\phi\colon G \to V$ be the "euclideanization" map obtained by composing the abelianization map $G \to G/[G,G]$ with the map $G/[G,G] \to (G/[G,G]) \otimes \mathbb{R}$.

Definition 1.3 Given a subset A of G, define the *radical* $\sqrt{A}$ of A to be the set $\{g \in G \mid g^q \in A \text{ for some } q \in \mathbb{Z} \setminus \{0\}\}$. Note that $A \subseteq \sqrt{A} \subseteq G$, and if A is a subgroup of G containing $[G,G]$ then $\sqrt{A}$ is a subgroup of G.

For $J \le G$, if a character $\chi \in \mathrm{Hom}(G,\mathbb{R})$ contains J in its kernel then it necessarily contains $\sqrt{J[G,G]}$. This next proposition says, first, that χ does not necessarily contain more than this, and, second, that under an addition restriction on G (which will be satisfied by our future groups of interest), the number of generators of J controls the number of generators of G that can lie in $\ker(\chi)$.

Proposition 1.4 (kill J and little else) *Let G be a finitely generated group, and let $J \le G$. Then there exists $\chi \in \mathrm{Hom}(G,\mathbb{R})$ with $\ker(\chi) = \sqrt{J[G,G]}$. Moreover, if G admits a finite generating set S such that $\dim_{\mathbb{R}}(V) = |S|$, and J is generated by n elements, then at most n elements of S lie in $\ker(\chi)$.*

Proof The quotient $G/\sqrt{J[G,G]}$ is a finitely generated torsion-free abelian group (ie a free abelian group), and hence can be embedded in $\mathbb{R}$. Composing this embedding with $G \to G/\sqrt{J[G,G]}$ yields a character $\chi \in \operatorname{Hom}(G,\mathbb{R})$ with $\ker(\chi) = \sqrt{J[G,G]}$. Now suppose G admits a finite generating set S such that $\dim_{\mathbb{R}}(V) = |S|$, and J is generated by n elements $j_1,\dots,j_n$. We claim that the image of $\sqrt{J[G,G]}$ in V spans a subspace W of dimension at most n. It suffices to prove that every element of $\sqrt{J[G,G]}$ maps under ϕ to a vector of V in the span of the $\phi(j_i)$. Let $g \in \sqrt{J[G,G]}$, say $g^q = jc$ for $q \neq 0$, $j \in J$ and $c \in [G,G]$. Then $\phi(g) = \frac{1}{q}\phi(g^q) = \frac{1}{q}\phi(jc) = \frac{1}{q}\phi(j)$, which indeed lies in the span of the $\phi(j_i)$. Now, since $\dim_{\mathbb{R}}(V) = |S|$ and $\phi(S)$ spans V, we must have that ϕ is injective on S and $\phi(S)$ is also linearly independent. Hence, at most n elements of S can map into W, and hence at most n elements of S can lie in $\sqrt{J[G,G]} = \ker(\chi)$. □

2 Right-angled Artin groups

A *right-angled Artin group* or *RAAG* is a group admitting a finite presentation in which each relator is a commutator of two generators. Given a finite simplicial graph Γ, with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, we get a RAAG, denoted by A_Γ, by taking a generator for each vertex and declaring that two vertices commute if and only if they share an edge. For example, if $E(\Gamma) = \varnothing$ then $A_\Gamma \cong F_{|V(\Gamma)|}$, the free group on $|V(\Gamma)|$ generators, and if Γ is a *clique*, ie a graph where every pair of vertices spans an edge, then $A_\Gamma \cong \mathbb{Z}^{|V(\Gamma)|}$.

The BNS invariants of RAAGs were fully computed by Meier and VanWyk [22]. We recall the computation here.

Definition 2.1 (living/dead subgraph) Given a character $\chi \in \operatorname{Hom}(A_\Gamma,\mathbb{R})$, define the *$\chi$–living subgraph* Γ_χ^* to be the induced subgraph of Γ supported on those vertices v with $\chi(v) \neq 0$, and the *χ–dead subgraph* $\Gamma_\chi^\dagger$ to be the induced subgraph of Γ supported on those vertices v with $\chi(v) = 0$.

Citation 2.2 (BNS of RAAG [22]) $[\chi] \in \Sigma^1(A_\Gamma)$ *if and only if the χ–living subgraph Γ_χ^* is connected and dominating in Γ.*

Here a subgraph Δ of Γ is called *dominating* (*in* Γ) if every vertex of $\Gamma \setminus \Delta$ is adjacent to a vertex of Δ.

In [12], Groves and Hull proved that the only way a nonabelian RAAG can split nontrivially over an abelian subgroup is if its defining graph admits a (possibly empty) separating clique. Recall that a subgraph Δ of Γ is called *separating* (*for* Γ) if $\Gamma \setminus \Delta$ is disconnected.

We now inspect the details of Groves and Hull's proof of their Theorem A to get the following refined result:

Proposition 2.3 (splittings and cliques) *Let Γ be a finite simplicial graph that is not a clique. The minimal $n \geq 0$ such that A_Γ splits nontrivially over $\mathbb{Z}^n$ equals the minimal $n \geq 0$ such that Γ admits a separating n–clique, with n taken to be ∞ whenever such splittings or cliques do not exist.*

To clarify, by *n–clique* we mean a clique with n vertices, ie the 1–skeleton of an $(n-1)$–simplex.

Proof of Proposition 2.3 The $n = \infty$ case is immediate from [12, Theorem A], so assume $n < \infty$. Note that if Γ has a separating n–clique then A_Γ splits nontrivially over $\mathbb{Z}^n$, so the thing to prove is that if A_Γ splits nontrivially over $\mathbb{Z}^n$ then Γ admits a separating k–clique for some $k \leq n$. The splitting gives us an action of A_Γ on a tree T with edge stabilizers isomorphic to $\mathbb{Z}^n$, no global fixed points and no edge inversions, and we will inspect this action using the proof of Theorem A in [12] as an outline.

First suppose some $v \in V(\Gamma)$ acts hyperbolically on T. Let e be any edge of the axis of v in T, so $\mathrm{Stab}_{A_\Gamma}(e) \cong \mathbb{Z}^n$. Let u be a vertex in $\mathrm{lk}_\Gamma v$, so u stabilizes the axis of v in T. Hence, there exist $n_u, m_u \in \mathbb{Z}$ with $n_u \neq 0$ such that $u^{n_u} v^{m_u}$ fixes this axis pointwise, and in particular $u^{n_u} v^{m_u} \in \mathrm{Stab}_{A_\Gamma}(e)$. Since this holds for every $u \in \mathrm{lk}_\Gamma v$, and since $\mathrm{Stab}_{A_\Gamma}(e)$ is abelian, we conclude that $[u^{n_u}, w^{n_w}] = 1$ for any $u, w \in \mathrm{lk}_\Gamma v$, which implies that $\mathrm{lk}_\Gamma v$ is a clique (this conclusion is also in [12]), and, even more precisely, since $\mathrm{Stab}_{A_\Gamma}(e) \cong \mathbb{Z}^n$ we conclude that $\mathrm{lk}_\Gamma v$ is a k–clique for some $k \leq n$. Since $\mathrm{lk}_\Gamma v$ separates v from $\Gamma \setminus \mathrm{st}_\Gamma v$ (and the latter is nonempty since Γ is not a clique but $\mathrm{st}_\Gamma v$ is), we have our separating k–clique.

Now assume that every $v \in V(\Gamma)$ acts elliptically on T. Groves and Hull define a map $F\colon \Gamma \to T$ that in particular takes each $v \in V(\Gamma)$ to some point of T that it fixes. There is a special point p, at the midpoint of an edge, that is the image under F of every v fixing it. Since the action does not invert edges, all these v even fix the edge containing p. As Groves and Hull show, $F^{-1}(p)$ is a separating clique in Γ, but even more precisely it is a separating k–clique for some $k \leq n$, since the edge stabilizer is isomorphic to $\mathbb{Z}^n$. □

As a remark, the reason to exclude the case when Γ is a clique is that while cliques have no separating cliques, technically $\mathbb{Z}^n$ does split nontrivially over $\mathbb{Z}^{n-1}$, as the HNN extension $\mathbb{Z}^n = \mathbb{Z}^{n-1} *_t$ where the stable element t conjugates $\mathbb{Z}^{n-1}$ to itself via the identity map. Another remark is that the $n = 1$ case was previously proved by Clay [8], and Groves and Hull remarked in [12, Remark 0.1] that their approach could recover Clay's result. Finally, to reiterate a point made in the introduction, the reason that the statement of the proposition is only about the minimal n is that in the proof we cannot control the size of the cliques produced, only an upper bound. However, we suspect that the stronger statement "A_Γ splits nontrivially over $\mathbb{Z}^n$ if and only if Γ admits a separating n–clique" is true.

3 Commensurability results for RAAGs

In this section we prove our main results about RAAGs, Theorem 3.5 and Corollaries 3.6 and 3.7. We first prove a proposition about general finitely generated groups that shows, outside a trivial case, that if a group G is commensurable to a group G' that splits over a subgroup L, then G contains a copy of a finite-index subgroup of L that cannot be killed by any pair of opposite characters $\pm\chi$ in the BNS invariant of G.

Proposition 3.1 *Let L be a group and let G be a finitely generated group that is not virtually of the form $K \rtimes \mathbb{Z}$ for any finite-index subgroup K of L. Suppose G is commensurable to a group G' that splits nontrivially over L. Then there exists $K \le G$, with K isomorphic to a finite-index subgroup of L, such that for any $\chi \in \mathrm{Hom}(G, \mathbb{R})$, if $\chi(K) = 0$ then at least one of $[\pm\chi]$ lies in $\Sigma^1(G)^c$.*

Proof Let H be a finite-index subgroup of G that embeds with finite index into G'. We will abuse notation and write H also for the finite-index image of H in G'. Since G' splits nontrivially over L, we know H decomposes as the fundamental group of a finite reduced graph of groups $\mathcal{G}$ whose edge groups are H intersected with conjugates of L in G'. Since H has finite index in G', these edge groups are all isomorphic to finite-index subgroups of L. Let $K \le H$ be one of these edge groups; for example, just take $K := H \cap L$. First suppose $\mathcal{G}$ is a strictly ascending HNN extension, say $H = K *_t$. Then, for any $\psi \in \mathrm{Hom}(H, \mathbb{R})$ such that $\psi(K) = 0$, if moreover $\psi(t) = 0$ then $\psi = 0$ and $[\pm\psi] \in \Sigma^1(H)^c$ by our convention. If $\psi(K) = 0$ and $\psi(t) \neq 0$ then either $[\psi]$ or $[-\psi]$ lies in $\Sigma^1(G)^c$ (see for instance [3, Theorem 2.1]). Next suppose $\mathcal{G}$ is an ascending HNN extension that is not strict, ie $H \cong K \rtimes \mathbb{Z}$. Then G is virtually of the form $K \rtimes \mathbb{Z}$, which we ruled out. Finally suppose $\mathcal{G}$ is not an ascending HNN extension. Then [7, Proposition 2.5] says that for any $\psi \in \mathrm{Hom}(H, \mathbb{R})$, if $\psi(K) = 0$ then $[\psi] \in \Sigma^1(H)^c$. In any case, for any $\chi \in \mathrm{Hom}(G, \mathbb{R})$ with $\chi(K) = 0$, at least one of $[\pm\chi|_H] \in \Sigma^1(H)^c$, so by Citation 1.2 also at least one of $[\pm\chi] \in \Sigma^1(G)^c$. □

Now we specialize to RAAGs.

Lemma 3.2 *Let Γ be a finite simplicial graph and let K be an abelian subgroup of A_Γ. Let $\Delta_K \subseteq \Gamma$ be the induced subgraph supported on those vertices v such that $v \in \sqrt{K[A_\Gamma, A_\Gamma]}$. Then Δ_K is a clique.*

Proof Suppose v and w are distinct vertices in Δ_K, say with $v^q c, w^r d \in K$ for $q, r \in \mathbb{Z} \setminus \{0\}$ and $c, d \in [A_\Gamma, A_\Gamma]$. Since K is abelian, $v^q c$ and $w^r d$ commute. Now suppose v and w are not adjacent, so there is a retract $\pi\colon A_\Gamma \to F_2 = \langle v, w\rangle$. We have that $\pi(v^q c) = v^q \pi(c)$ and $\pi(w^r d) = w^r \pi(d)$ commute in F_2. Since neither is trivial, this means that $(v^q \pi(c))^a = (w^r \pi(d))^b$ for some $a, b \in \mathbb{Z} \setminus \{0\}$. Abelianizing F_2 to $\mathbb{Z}^2 = \langle \overline{v}, \overline{w}\rangle$, this produces $qa\overline{v} = rb\overline{w}$, with $qa, rb \neq 0$, which is absurd. □

Corollary 3.3 *Let Γ be a finite simplicial graph and let $K \leq A_\Gamma$ with $K \cong \mathbb{Z}^n$. Then there exists $\chi \in \mathrm{Hom}(A_\Gamma, \mathbb{R})$ such that $\chi(K) = 0$ and the χ–dead subgraph $\Gamma_\chi^\dagger$ is a k–clique for some $0 \leq k \leq n$.*

Proof Choose χ as in Proposition 1.4 with $G = A_\Gamma$, $S = V(\Gamma)$ and $J = K$. Then $\chi(K) = 0$ and $v \in V(\Gamma)$ satisfies $v \in \Gamma_\chi^\dagger$ if and only if $v \in \sqrt{K[A_\Gamma, A_\Gamma]}$. Since the abelianization of A_Γ is $\mathbb{Z}^{|V(\Gamma)|}$, Proposition 1.4 also says that at most n vertices satisfy this, and Lemma 3.2 says they must span a clique. □

Proposition 3.1 applied to the $L = \mathbb{Z}^n$ case said that a RAAG commensurable to a group splitting over $\mathbb{Z}^n$ contains a copy of $\mathbb{Z}^n$ that cannot be killed by a pair of opposite characters $\pm\chi$ in the BNS invariant. This next proposition says that for a RAAG that does not obviously split over $\mathbb{Z}^n$, any copy of $\mathbb{Z}^n$ can be killed by a pair of opposite characters $\pm\chi$ in the BNS invariant.

Proposition 3.4 *Let Γ be a finite simplicial graph with no separating k–cliques for any $k \leq n$. Then, for any proper subgroup $K \cong \mathbb{Z}^n$ of A_Γ, there exists a character $\chi \in \mathrm{Hom}(A_\Gamma, \mathbb{R})$ such that $\chi(K) = 0$ but $[\pm\chi] \in \Sigma^1(A_\Gamma)$.*

Proof Choose χ as in Corollary 3.3, so $\chi(K) = 0$ and $\Gamma_\chi^\dagger$ is a k–clique for some $0 \leq k \leq n$. If Γ is a clique, then since K is a proper subgroup of A_Γ we know $\Gamma_\chi^\dagger$ is not all of Γ, so in this case Γ_χ^* is connected and dominating. Now assume Γ is not a clique. Since Γ has no separating k–cliques, Γ_χ^* is connected. Also, it must be dominating since if $\mathrm{st}_\Gamma(v)$ lies in $\Gamma_\chi^\dagger$ then $\mathrm{st}_\Gamma(v)$ is an ℓ–clique for some $\ell \leq k$, and since Γ is not a clique this means $\mathrm{lk}_\Gamma(v)$ is a separating $(\ell-1)$–clique, which we have ruled out. In either case Citation 2.2 says $[\chi] \in \Sigma^1(A_\Gamma)$. Since $\Gamma_\chi^* = \Gamma_{-\chi}^*$, we also have $[-\chi] \in \Sigma^1(A_\Gamma)$. □

Now we can prove our main results.

Theorem 3.5 *Let Γ be a finite simplicial nonclique graph with no separating k–cliques for any $k \le n$. Then A_Γ is not commensurable to any group splitting nontrivially over $\mathbb{Z}^n$.*

Proof Suppose A_Γ is commensurable to a group splitting nontrivially over $\mathbb{Z}^n$. By Proposition 3.1 using $L = \mathbb{Z}^n$ (which applies since A_Γ contains F_2 and hence cannot be virtually of the form $\mathbb{Z}^n \rtimes \mathbb{Z}$) there exists a subgroup $K \cong \mathbb{Z}^n$ of A_Γ such that for any $\chi \in \mathrm{Hom}(A_\Gamma, \mathbb{R})$ if $\chi(K) = 0$ then at least one of $[\pm\chi]$ lies in $\Sigma^1(A_\Gamma)^c$ (in fact both do since $\Sigma^1(A_\Gamma)$ happens to be closed under inverting characters). But by Proposition 3.4 we know that there exists a character $\chi \in \mathrm{Hom}(A_\Gamma, \mathbb{R})$ such that $\chi(K) = 0$ but $[\pm\chi] \in \Sigma^1(A_\Gamma)$, a contradiction. □

We immediately get the following commensurability invariant for RAAGs:

Corollary 3.6 *If A_Γ and A_Δ are commensurable and Γ has no separating k–cliques for any $k \le n$, then neither does Δ.*

Proof If Γ is itself a clique then we must have $\Gamma = \Delta$. If Γ is not a clique then the result is immediate from Proposition 2.3 and Theorem 3.5. □

We also get the following corollary in the special case where Γ has no separating cliques at all. Recall from the introduction that any NF subgroup (that is, one containing no nonabelian free subgroups) of a RAAG is abelian.

Corollary 3.7 *Let Γ be a finite simplicial nonclique graph with no separating cliques. Then A_Γ is not commensurable to any group splitting nontrivially over an NF subgroup.*

Proof Suppose A_Γ is commensurable to a group that splits nontrivially over an NF subgroup. By Proposition 3.1, which applies since A_Γ is not (virtually) NF, there exists an NF subgroup $K \le A_\Gamma$ such that for any $\chi \in \mathrm{Hom}(A_\Gamma, \mathbb{R})$ if $\chi(K) = 0$ then at least one of $[\pm\chi]$ lies in $\Sigma^1(A_\Gamma)^c$. Since NF subgroups of RAAGs are abelian, in fact K is abelian, so by Proposition 1.4 and Lemma 3.2 we can choose $\chi \in \mathrm{Hom}(A_\Gamma, \mathbb{R})$ such that $\chi(K) = 0$ and $\Gamma_\chi^\dagger$ is a clique. Since Γ has no separating cliques, this implies $[\pm\chi] \in \Sigma^1(A_\Gamma)$, as explained in the proof of Proposition 3.4, a contradiction. □

As a remark, if A_Γ is commensurable to a group splitting nontrivially over an NF subgroup generated by n elements, then in general we cannot control the number of generators of the subgroup K described in the proof, and hence cannot control the size of the clique $\Gamma_\chi^\dagger$. Of course, if the NF subgroup is $\mathbb{Z}^n$ then K is also $\mathbb{Z}^n$, since finite-index subgroups of $\mathbb{Z}^n$ are isomorphic to $\mathbb{Z}^n$ (which is why Theorem 3.5 worked), but in general we do not get a statement like Corollary 3.7 if we merely rule out separating cliques up to some size; we really need to rule out all separating cliques.

4 Quasi-isometry results for RAAGs

This brief section amounts to a collection of examples of results about quasi-isometry, which follow immediately from our results about commensurability together with results by Huang [14; 15; 13] tying commensurability to quasi-isometry.

First we need one technical lemma, the proof of which is essentially due to Jingyin Huang.

Lemma 4.1 *Let* Γ *be a finite simplicial graph. Suppose* $\mathrm{Out}(A_\Gamma)$ *is finite. Then* Γ *has no separating cliques.*

Proof (Jingyin Huang) Since $\mathrm{Out}(A_\Gamma)$ is finite we know Γ has no separating closed stars, and no instances of $\mathrm{lk}\, v \subseteq \mathrm{st}\, w$ for vertices $v \neq w$. Now suppose Γ has a separating clique K, say the connected components of its complement are $C_1, \dots, C_k$, so $k \geq 2$. If $K = \varnothing$ (ie it is a 0–clique) then Γ is disconnected and has infinite outer automorphism group, so we know $K \neq \varnothing$. Pick a vertex $v \in K$, so $K \subseteq \mathrm{st}\, v$. Since $\mathrm{st}\, v$ is not separating, at most one of the $C_i \setminus \mathrm{st}\, v$ can be nonempty. Since $k \geq 2$ this means at least one of the $C_i \setminus \mathrm{st}\, v$ must be empty, say without loss of generality $C_1 \setminus \mathrm{st}\, v = \varnothing$, ie $C_1 \subseteq \mathrm{st}\, v$. But now for any vertex w in C_1, we have $\mathrm{lk}\, w \subseteq C_1 \cup K \subseteq \mathrm{st}\, v$, a contradiction. □

Corollary 4.2 *Suppose* A_Γ *and* A_Δ *are quasi-isometric, and that* $\mathrm{Out}(A_\Gamma)$ *is finite, so by Lemma 4.1 we know* Γ *has no separating cliques. Then* Δ *also has no separating cliques.*

Proof This follows from [14, Theorem 1.2] and Corollary 3.6. □

Corollary 4.3 *Suppose A_Γ and A_Δ are quasi-isometric and Γ is of weak type I or type II as defined in [13]. Then, if Γ has no separating k–cliques for any $k \leq n$, neither does Δ.*

Proof This follows from [13, Theorems 1.3 and 1.6] and Corollary 3.6. □

Corollary 4.4 *Let G be a finitely generated group quasi-isometric to A_Γ. Suppose that every automorphism of Γ fixing a closed star of a vertex pointwise fixes all of Γ, that Γ contains no induced 4–cycles and that $\mathrm{Out}(A_\Gamma)$ is finite. Then G does not split nontrivially over $\mathbb{Z}^n$ for any n.*

Proof Since $\mathrm{Out}(A_\Gamma)$ is finite, Γ has no separating cliques by Lemma 4.1. The result now follows from [15, Theorem 1.2] and Theorem 3.5. □

In general, we would get similar sorts of results anytime there is a graph Γ for which quasi-isometry to A_Γ implies commensurability to A_Γ.

5 Commensurability results for (loop) braid groups

In this section we apply our approach to braid groups and loop braid groups.

5.1 Commensurability results for braid groups

The *n–strand braid group* is the group presented by

$$B_n = \left\langle s_1, \dots, s_{n-1} \;\middle|\; \begin{array}{c} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for all } 1 \leq i \leq n-2 \\ \text{and } s_i s_j = s_j s_i \text{ for all } |i-j| > 1 \end{array} \right\rangle.$$

There is a projection $B_n \to S_n$ given by adding the relations $s_i^2 = 1$ for all i, and the kernel of this map is the *n–strand pure braid group* PB_n.

We will work with a specific generating set of PB_n, namely $S = \{S_{i,j} \mid 1 \leq i < j \leq n\}$, where $S_{i,j} := s_i s_{i+1} \cdots s_{j-2} s_{j-1}^2 s_{j-2}^{-1} \cdots s_{i+1}^{-1} s_i^{-1}$. Visually, in $S_{i,j}$ the i^{th} strand crosses in front of all the strands between it and the j^{th} strand, spins around the j^{th} strand, and returns to where it came from, again crossing in front of the intermediate strands. An important fact we will use is that $\mathrm{PB}_3 \cong F_2 \times \mathbb{Z}$, with $S_{1,2}$ and $S_{1,3}$ serving as generators of the F_2 factor. We will also make use of the *standard projections* $\mathrm{PB}_n \to \mathrm{PB}_m$ for $m < n$, obtained by deleting some collection of $n-m$ strands.

The BNS invariant $\Sigma^1(\mathrm{PB}_n)$ was computed by Koban, McCammond and Meier [19]. We recall the computation here. If $n \geq 5$, then $[\chi] \in \Sigma^1(\mathrm{PB}_n)^c$ if and only if $\chi = \chi' \circ \pi$ for π one of the standard projections $\mathrm{PB}_n \to \mathrm{PB}_4$ or $\mathrm{PB}_n \to \mathrm{PB}_3$ given by deleting strands, and $[\chi'] \in \Sigma^1(\mathrm{PB}_3)^c$ or $\Sigma^1(\mathrm{PB}_4)^c$. In particular, to understand $\Sigma^1(\mathrm{PB}_n)^c$ we need only understand $\Sigma^1(\mathrm{PB}_3)^c$ and $\Sigma^1(\mathrm{PB}_4)^c$. For PB_3, we have $[\chi] \in \Sigma^1(\mathrm{PB}_3)^c$ if and only if $\chi(S_{1,2}) + \chi(S_{1,3}) + \chi(S_{2,3}) = 0$. For PB_4 we have $[\chi] \in \Sigma^1(\mathrm{PB}_4)^c$ if and only if either $\chi = \chi' \circ \pi$ for $[\chi] \in \Sigma^1(\mathrm{PB}_3)^c$ and $\pi\colon \mathrm{PB}_4 \to \mathrm{PB}_3$ one of the standard projections, or else χ satisfies the equations $\chi(S_{1,2}) = \chi(S_{3,4})$, $\chi(S_{1,3}) = \chi(S_{2,4})$, $\chi(S_{1,4}) = \chi(S_{2,3})$ and $\chi(S_{1,2}) + \chi(S_{1,3}) + \chi(S_{1,4}) = 0$. Note that these characterizations imply that, for any χ, $[\chi] \in \Sigma^1(\mathrm{PB}_n)$ if and only if $[-\chi] \in \Sigma^1(\mathrm{PB}_n)$.

We now use the ideas from the previous sections to prove the following:

Theorem 5.1 *For $n \geq 4$ the braid group B_n is not commensurable to any group that splits nontrivially over an NF subgroup.*

Note that $\mathrm{PB}_3 \cong F_2 \times \mathbb{Z} = \mathbb{Z}^2 *_{\mathbb{Z}} \mathbb{Z}^2$ and $\mathbb{Z}$ is NF, so the $n \geq 4$ restriction in the theorem is necessary. Also, the NF condition is obviously necessary, since for instance $B_n \cong [B_n, B_n] \rtimes \mathbb{Z}$ is a nontrivial HNN extension.

Proof of Theorem 5.1 We will work with the pure braid group PB_n, which is commensurable to B_n (being a finite-index subgroup). Suppose PB_n is commensurable to a group that splits nontrivially over an NF subgroup. Since PB_n is not NF, Proposition 3.1 implies that PB_n admits an NF subgroup K such that for any $\chi \in \mathrm{Hom}(\mathrm{PB}_n, \mathbb{R})$, if $\chi(K) = 0$ then at least one of $[\pm\chi]$ lies in $\Sigma^1(\mathrm{PB}_n)^c$, which means $[\chi] \in \Sigma^1(\mathrm{PB}_n)^c$. By Proposition 1.4, there exists $\chi \in \mathrm{Hom}(\mathrm{PB}_n, \mathbb{R})$ with $\ker(\chi) = \sqrt{K[\mathrm{PB}_n, \mathrm{PB}_n]}$. Since $\chi(K) = 0$ we know $[\chi] \in \Sigma^1(\mathrm{PB}_n)^c$, which implies that either $n = 4$ or else χ is induced from some standard projection onto PB_3 or PB_4.

In particular, if $n \geq 5$ then there exists j such that $\chi(S_{i,j}) = \chi(S_{j,k}) = 0$ for any $i < j$ or $j < k$ (just choose j to be the label of a strand getting deleted), which implies that $S_{i,j}, S_{j,k} \in \sqrt{K[\mathrm{PB}_n, \mathrm{PB}_n]}$ for any such i or k. Up to automorphisms (note that the BNS invariant is invariant under automorphisms) we can assume $j = 1$, so in particular $S_{1,2}, S_{1,3} \in \sqrt{K[\mathrm{PB}_n, \mathrm{PB}_n]}$. Choose $q, r \in \mathbb{Z} \setminus \{0\}$ and $c, d \in [\mathrm{PB}_n, \mathrm{PB}_n]$ such that $S_{1,2}^q c, S_{1,3}^r d \in K$, which, since K is NF, implies that $S_{1,2}^q c$ and $S_{1,3}^r d$ do not generate a copy of F_2. Now consider the standard projection $\pi\colon \mathrm{PB}_n \to \mathrm{PB}_3$ given by deleting all but the first three strands. Then $S_{1,2}^q \pi(c)$ and $S_{1,3}^r \pi(d)$ do not generate a copy of F_2 in PB_3, and so neither do their images in $\mathrm{PB}_3/Z(\mathrm{PB}_3) \cong F_2$.

Hence, these images commute,[1] and so modulo $Z(\mathrm{PB}_3)$, $S_{1,2}^q\pi(c)$ and $S_{1,3}^r\pi(d)$ have a common power, say $(S_{1,2}^q\pi(c))^a = (S_{1,3}^r\pi(d))^b z$ for $a, b \in \mathbb{Z}$ and $z \in Z(\mathrm{PB}_3)$. But modding out $Z(\mathrm{PB}_3)$ and abelianizing F_2 to $\mathbb{Z}^2 = \langle \bar{S}_{1,2}, \bar{S}_{1,3}\rangle$, this implies that $qa\bar{S}_{1,2} = rb\bar{S}_{1,3}$, which is absurd.

Now suppose $n = 4$. If χ is induced from a standard projection $\mathrm{PB}_4 \to \mathrm{PB}_3$ then we can use the above argument to get our contradiction, so suppose it is not. Hence, we have $\chi(S_{1,2}) = \chi(S_{3,4})$, $\chi(S_{1,3}) = \chi(S_{2,4})$, $\chi(S_{1,4}) = \chi(S_{2,3})$ and $\chi(S_{1,2}) + \chi(S_{1,3}) + \chi(S_{1,4}) = 0$. In particular,

$$S_{1,2}S_{3,4}^{-1}, S_{1,3}S_{2,4}^{-1} \in \ker(\chi) = \sqrt{K[\mathrm{PB}_4, \mathrm{PB}_4]},$$

so we can choose $q, r \in \mathbb{Z} \setminus \{0\}$ and $c, d \in [\mathrm{PB}_4, \mathrm{PB}_4]$ such that $(S_{1,2}S_{3,4}^{-1})^q c$ and $(S_{1,3}S_{2,4}^{-1})^r d$ lie in K, and hence do not generate a copy of F_2. Their images under the standard projection $\pi\colon \mathrm{PB}_4 \to \mathrm{PB}_3$ given by deleting all but the first three strands also do not generate a copy of F_2, so $S_{1,2}^q\pi(c)$ and $S_{1,3}^r\pi(d)$ do not generate a copy of F_2 in PB_3. We are now in the same situation as in the proof of the $n \geq 5$ case, and as in that proof we reach a contradiction. □

As a remark, it would not have worked to try and apply this technique to B_n itself, so working with PB_n really was necessary. Indeed, every $[\chi] \in \Sigma^1(B_n)$ satisfies $\ker(\chi) = [B_n, B_n]$, so it is impossible to find such a χ with an arbitrary NF subgroup lying in its kernel.

5.2 Commensurability results for loop braid groups

Much of this subsection proceeds very similarly to Section 5.1.

The *loop braid group* LB_n on n loops is the group of symmetric automorphisms of the free group F_n. Fixing a free generating set $\{x_1, \dots, x_n\}$ for F_n, an automorphism $\alpha \in \mathrm{Aut}(F_n)$ is called *symmetric* if it takes each x_i to a conjugate of some x_j or x_j^{-1}. Sometimes the word symmetric is reserved for those α taking each x_i to a conjugate of some x_j, not allowing x_j^{-1}; this produces a finite-index subgroup of what we are calling LB_n. This terminological ambiguity will not matter here, since we will actually work with the *pure loop braid group* PLB_n, the group of automorphisms $\alpha \in \mathrm{Aut}(F_n)$ taking each x_i to a conjugate of x_i, which is again a finite-index subgroup of LB_n.

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[1] Actually $S_{1,2}^q c$ and $S_{1,3}^r d$ already commute in PB_n by [21], but we have to pass to F_2 anyway, so it is not necessary to appeal to the result from [21].

The name loop braid group comes from viewing such automorphisms as pictures of n loops in 3–space moving around and through each other. See [10] for a great deal of background and more details.

The BNS invariant $\Sigma^1(\mathrm{PLB}_n)$ was computed by Orlandi-Korner [23]. We recall here some of her setup. First, PLB_n is generated by $\{\alpha_{i,j} \mid i \neq j\}$, where $\alpha_{i,j}$ is the automorphism of F_n taking x_i to $x_j^{-1} x_i x_j$ and x_k to itself for $k \neq i$. For $m < n$ a *standard projection* $\mathrm{PLB}_n \to \mathrm{PLB}_m$ is a map induced from some projection $F_n \to F_m$ given by sending some choice of $n-m$ generators to the identity and sending the remaining m generators to the generators of F_m. Now $\Sigma^1(\mathrm{PLB}_n)$ is described as follows: For $n \geq 4$, $[\chi] \in \Sigma^1(\mathrm{PLB}_n)^c$ if and only if $\chi = \chi' \circ \pi$ for π a standard projection $\mathrm{PLB}_n \to \mathrm{PLB}_2$ or $\mathrm{PLB}_n \to \mathrm{PLB}_3$ and $[\chi']$ in $\Sigma^1(\mathrm{PLB}_2)^c$ or $\Sigma^1(\mathrm{PLB}_3)^c$. For $n = 3$ we have that $[\chi] \in \Sigma^1(\mathrm{PLB}_3)^c$ if and only if it is induced from a standard projection to PLB_2 or else $\chi(\alpha_{2,1}) + \chi(\alpha_{3,1}) = 0$, $\chi(\alpha_{1,2}) + \chi(\alpha_{3,2}) = 0$ and $\chi(\alpha_{1,3}) + \chi(\alpha_{2,3}) = 0$. For $n = 2$ we have $\Sigma^1(\mathrm{PLB}_2) = \varnothing$ (in fact $\mathrm{PLB}_2 \cong F_2$). Note that a consequence of all this is that $[\chi] \in \Sigma^1(\mathrm{PLB}_n)$ if and only if $[-\chi] \in \Sigma^1(\mathrm{PLB}_n)$.

We now use the ideas from the previous sections to prove the following. The proof is very similar to the proof of Theorem 5.1.

Theorem 5.2 *For $n \geq 3$ the loop braid group LB_n is not commensurable to any group that splits nontrivially over an NF subgroup.*

The $n \geq 3$ restriction is necessary since $\mathrm{PLB}_2 \cong F_2$ splits nontrivially over $\{1\}$, and the NF condition is necessary for reasons similar to the braid group case.

Proof of Theorem 5.2 We will work with the pure loop braid group PLB_n, which is commensurable to LB_n (being a finite-index subgroup). Suppose PLB_n is commensurable to a group that splits nontrivially over an NF subgroup. Since PLB_n is not NF, Proposition 3.1 implies that PLB_n admits an NF subgroup K such that for any $\chi \in \mathrm{Hom}(\mathrm{PLB}_n, \mathbb{R})$, if $\chi(K) = 0$ then at least one of $[\pm\chi]$ lies in $\Sigma^1(\mathrm{PLB}_n)^c$, which means $[\chi] \in \Sigma^1(\mathrm{PLB}_n)^c$. By Proposition 1.4, there exists $\chi \in \mathrm{Hom}(\mathrm{PLB}_n, \mathbb{R})$ with $\ker(\chi) = \sqrt{K[\mathrm{PLB}_n, \mathrm{PLB}_n]}$. Since $\chi(K) = 0$ we know $[\chi] \in \Sigma^1(\mathrm{PLB}_n)^c$, which implies that either $n = 3$ or else χ is induced from some standard projection onto PLB_2 or PLB_3.

In particular, if $n \geq 4$ then there exists i such that $\chi(\alpha_{i,j}) = \chi(\alpha_{j,i}) = 0$ for all $i \neq j$ (just choose i such that x_i is sent to 1 in the projection of F_n inducing the standard projection of PLB_n), which implies that $\alpha_{i,j},\alpha_{j,i}\in\sqrt{K[\mathrm{PLB}_n,\mathrm{PLB}_n]}$ for all $i\neq j$. Up to automorphisms (note that the BNS invariant is invariant under automorphisms) we can assume $i=1$, so in particular $\alpha_{1,2},\alpha_{2,1}\in\sqrt{K[\mathrm{PLB}_n,\mathrm{PLB}_n]}$. Choose $q,r\in\mathbb{Z}\setminus\{0\}$ and $c,d\in[\mathrm{PLB}_n,\mathrm{PLB}_n]$ such that $\alpha_{1,2}^q c,\alpha_{2,1}^r d\in K$, which, since K is NF, implies that $\alpha_{1,2}^q c$ and $\alpha_{2,1}^r d$ do not generate a copy of F_2. Now consider the standard projection $\pi\colon \mathrm{PLB}_n\to\mathrm{PLB}_2$ given by sending all but the first two generators of F_n to 1 and the first two to the generators of F_2 (in order). Then $\alpha_{1,2}^q\pi(c)$ and $\alpha_{2,1}^r\pi(d)$ do not generate a copy of F_2 in PLB_2. Since $\mathrm{PLB}_2\cong F_2$, this means $\alpha_{1,2}^q\pi(c)$ and $\alpha_{2,1}^r\pi(d)$ have a common power, say $(\alpha_{1,2}^q\pi(c))^a=(\alpha_{2,1}^r\pi(d))^b$ for $a,b\in\mathbb{Z}$. But abelianizing F_2 to $\mathbb{Z}^2=\langle\bar\alpha_{1,2},\bar\alpha_{2,1}\rangle$, this implies that $qa\bar\alpha_{1,2}=rb\bar\alpha_{2,1}$, which is absurd.

Now suppose $n=3$. If χ is induced from a standard projection $\mathrm{PLB}_3\to\mathrm{PLB}_2$ then we can use the above argument to get our contradiction, so suppose it is not. Hence, we have $\chi(\alpha_{2,1})+\chi(\alpha_{3,1})=0$, $\chi(\alpha_{1,2})+\chi(\alpha_{3,2})=0$ and $\chi(\alpha_{1,3})+\chi(\alpha_{2,3})=0$. In particular, $\alpha_{1,2}\alpha_{3,2},\alpha_{2,1}\alpha_{3,1}\in\ker(\chi)=\sqrt{K[\mathrm{PLB}_3,\mathrm{PLB}_3]}$, so we can choose $q,r\in\mathbb{Z}\setminus\{0\}$ and $c,d\in[\mathrm{PLB}_3,\mathrm{PLB}_3]$ such that $(\alpha_{1,2}\alpha_{3,2})^q c$ and $(\alpha_{2,1}\alpha_{3,1})^r d$ lie in K, and hence do not generate a copy of F_2. Their images under the standard projection $\pi\colon\mathrm{PLB}_3\to\mathrm{PLB}_2$ induced by the projection $F_3\to F_2$ sending x_1 to x_1, x_2 to x_2 and x_3 to 1 also do not generate a copy of F_2, so $\alpha_{1,2}^q\pi(c)$ and $\alpha_{2,1}^r\pi(d)$ do not generate a copy of F_2 in PLB_2. We are now in the same situation as in the proof of the $n\geq 4$ case, and as in that proof we reach a contradiction. □

Much like in the braid group case, it would not have worked to try and apply this technique to LB_n itself, so working with PLB_n really was necessary. In fact LB_n has finite abelianization, so it is impossible to find nontrivial characters killing arbitrary NF subgroups simply because there no nontrivial characters at all.

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Occupants in simplicial complexes

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Let M be a smooth manifold and $K \subset M$ be a simplicial complex of codimension at least 3. Functor calculus methods lead to a homotopical formula of $M \setminus K$ in terms of spaces $M \setminus T$ where T is a finite subset of K. This is a generalization of the author's previous work with Michael Weiss (Contemp. Math. 682, Amer. Math. Soc., Providence, RI (2017) 237–259), where the subset K is assumed to be a smooth submanifold of M and uses his generalization of manifold calculus adapted for simplicial complexes.

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1 Introduction

Let K be a simplicial complex — that is, the geometric realization of an abstract simplicial complex. Let M be a smooth manifold with codimension $\dim M - \dim K \geq 3$. Throughout this paper we assume that K is a subset of M such that each (closed) simplex of K is smoothly embedded in M. We would like to recover the homotopy type of $M \setminus K$ from the homotopy types of the spaces $M \setminus T$ where T is a finite subset of K. The finite subset $T \subset K$ could be regarded as a finite set of occupants.

It turns out that it is possible to find such a homotopical formula, but only if we allow standard thickenings of the finite subsets $T \subset K$ and inclusions between them. We get an interesting poset regarded as a category — the configuration category $\mathrm{con}(K)$ of K. The objects of $\mathrm{con}(K)$ are pairs (T, ρ) where T is a finite subset of K and $\rho\colon T \to (0, \infty)$ is a function which assigns to each element $t \in T$ the radius $\rho(t)$ of the corresponding thickening using a standard metric on K. These pairs have to fulfill certain conditions, eg the thickenings of the elements $t \in T$ are pairwise disjoint (for a precise definition, see Section 3.1). For each object (T, ρ) in $\mathrm{con}(K)$, we get a corresponding open subset $V_K(T, \rho) \subset K$, which is the disjoint union of the open balls of radius $\rho(t)$ about the points $t \in T$. We note that for each element (T, ρ) of the configuration category, there is an inclusion

$$M \setminus K \to M \setminus V_K(T, \rho)$$

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and thus a map from $M \setminus K$ into the associated homotopy limit. The following theorem is our (technical) main result:

Theorem 1.1 *If the codimension* $\dim M - \dim K$ *is at least* 3, *the canonical map*

$$M \setminus K \to \operatorname*{holim}_{(T,\rho)\in \mathrm{con}(K)} M \setminus V_K(T,\rho)$$

is a weak equivalence.

The condition on the codimension is essential, that is, the result is not true for codimension ≤ 2. A nice counterexample is given in Tillmann and Weiss [11, 1.3.3].

Theorem 1.1 is an application of manifold calculus adapted for simplicial complexes, as developed in Tillmann [10]. In this paper the configuration category $\mathrm{con}(K)$ is a convenient replacement of the category of special open subsets $\bigcup_k \mathcal{O}k(K)$ there. Recall: the objects of $\bigcup_k \mathcal{O}k(K)$ are those open subsets V of K which have finitely many components and where each component of V is stratified isotopy equivalent to the open star of some simplex σ in K (intersection of the open stars of the vertices of σ). Roughly speaking, a stratified isotopy equivalence is a simplexwise smooth isotopy equivalence.

As is to be expected from manifold calculus, there is a stronger version of our main result with restricted cardinalities (see Theorem 4.1). More precisely, the map from $M \setminus K$ into the homotopy limit over the full subcategory of $\mathrm{con}(K)$ of the set with restricted cardinality is highly connected, depending on that cardinality.

Now let M be a Riemannian manifold with boundary and let $L \subset M \setminus \partial M$ be a smooth submanifold without boundary. Using Theorem 1.1, we can prove an approximation theorem of $M \setminus L$ in some cases where no conditions on the codimension of M and L is needed. More precisely, we can recover the homotopy type of $M \setminus L$ from the homotopy types of the spaces $M \setminus T$ where T is a finite subset of L. Again, we need to allow thickenings of the finite subsets T in L and inclusions between them. Therefore, we consider the configuration category $\mathrm{con}(L)$ of L (see Section 5.3 for a precise definition). For each object (T,ρ) in $\mathrm{con}(L)$, we have again a corresponding open subset $V_L(T,\rho)$ (using the Riemannian metric), which is the union of the open balls of radius $\rho(t)$ about the points $t \in T$. The inclusions

$$M \setminus L \to M \setminus V_L(T,\rho)$$

induce a map from $M \setminus L$ into the homotopy limit taken over the category $\mathrm{con}(L)$. Assume now that L is a *smooth thickening* of a compact simplicial complex $K \subset L$, as defined in Definition 5.2. In particular, this means that K is a retract of L weakly equivalent to it. This is our main application:

Theorem 1.2 *If the codimension* $\dim M - \dim K$ *is at least* 3*, the canonical map*

$$M \setminus L \to \operatorname*{holim}_{(T,\rho)\in\mathrm{con}(L)} M \setminus V_L(T,\rho)$$

is a weak equivalence.

In particular, we can prove an approximation theorem for the boundary of the manifold in some cases. Namely, if $M \setminus \partial M$ is a *smooth thickening* of a compact simplicial complex $K \subset M \setminus \partial M$, we get the following corollary:

Corollary 1.3 *If the codimension* $\dim M - \dim K$ *is at least* 3*, the canonical map*

$$\partial M \to \operatorname*{holim}_{(T,\rho)\in\mathrm{con}(M\setminus\partial M)} M \setminus V_{M\setminus\partial M}(T,\rho)$$

is a weak equivalence.

In this case we also have a stronger version with restricted cardinalities (see Corollary 5.11) and it generalizes one of the main results in [11]. In the absence of the calculus for simplicial complexes as developed in [10], there we had to assume the existence of a smooth disk fiber bundle $M \to L$ with fiber dimension $c \geq 3$ where L is a closed smooth submanifold of M. This condition is a special case of our *smooth thickening* condition here (see Examples 5.4).

The ideas and strategies of [11] and of the generalization here thus intersect, so we feel compelled to indicate the substantial technical issues needed to establish the generalization. The main issue is to reformulate the key definitions. We give two basic examples: First, the definition of the configuration category $\mathrm{con}(K)$ of a simplicial complex K is quite different from its analogue, the configuration category of a smooth manifold (see Remark 4.4 for a comparison). Since we will apply manifold calculus for simplicial complexes, the technical conditions introduced in [10] go into the definition of $\mathrm{con}(K)$. Using these technical conditions, it becomes clear that in order to prove the main theorem, we also have to solve new technical challenges. Second, the definition of a smooth thickening of a simplicial complex involves various technical conditions. Again we have to verify that this definition is a convenient replacement of its analogue, the smooth fiber bundle condition, in [11].

In an application we will study the following question: Let M be a smooth manifold with boundary. It is well known that the boundary ∂M can be recovered as the homotopy link of the basepoint in $M/\partial M \cong (M \setminus \partial M) \cup \infty$. Therefore, it is possible to say that there is an action of the homeomorphism group $\mathrm{homeo}(M \setminus \partial M)$ on the pair $(M, \partial M)$ by homotopy automorphisms, ie each homeomorphism of $M \setminus \partial M$ determines a homotopy automorphism of the pair $(M, \partial M)$. But it is also well known that there is a canonical map of topological grouplike monoids (if an explanation is needed, see Section 6)

$$\mathrm{homeo}(M \setminus \partial M) \hookrightarrow \mathrm{haut}_{N\mathrm{Fin}}(\mathrm{con}(M \setminus \partial M)),$$

where $N\mathrm{Fin}$ is the nerve of the category of finite sets and maps between finite sets and $\mathrm{haut}_{N\mathrm{Fin}}(\mathrm{con}(M \setminus \partial M))$ is the space of the homotopy automorphisms of $\mathrm{con}(M \setminus \partial M)$ over $N\mathrm{Fin}$. In [14] Weiss studies the question in what cases the action of $\mathrm{homeo}(M \setminus \partial M)$ on the pair $(M, \partial M)$ by homotopy automorphisms extends to an action of $\mathrm{haut}_{N\mathrm{Fin}}(\mathrm{con}(M \setminus \partial M))$ on the pair $(M, \partial M)$ by homotopy automorphisms. This has also applications in Weiss [15]. We can generalize his result (see Theorem 6.5): the action can be extended if the condition in Theorem 1.2 is satisfied.

Our paper with Weiss [11] attracted attention in applied topology because of possible relevance in the study of sensor network problems (for an introduction from the topological point of view see Adams and Carlsson [1] and de Silva and Ghrist [9]). At the moment there is no application of the theory developed in this paper outside the smooth setting, but we give a short explanation why there are potential ones in the context of sensor networks: In [1] movable sensor networks and evasion paths are studied. More concretely, let X be a subspace of a euclidean space. Assume we have a collection of points in X, each point equipped with a sensor. Each sensor covers a neighborhood of its location, for simplicity a ball of fixed radius. Then an evasion path is a specific embedding of a one-dimensional space into X minus the sensor region, which is the space covered by the union of all sensors. The spaces involved are usually not equipped with a smooth manifold structure, so the authors explicitly ask for an extension of the Goodwillie–Weiss manifold calculus to the setting of nonmanifold spaces [1, Section 7]. In particular, the theory developed in this paper could be a relevant application of manifold calculus for simplicial complexes because complements in manifolds are studied and the sensor region can be represented as a simplicial complex.

Outline

In Section 2 we recall the basic results of manifold calculus adapted for simplicial complexes. Using Goodwillie's homotopy functor calculus, we give general criteria for when a functor is analytic or polynomial and manifold calculus can be applied.

In Section 3 we will introduce the configuration categories of a simplicial complex and a smooth manifold. The configuration category carries a continuous structure. We will take this into account when we define homotopy limits. This leads to the notion of the *continuous homotopy limit*. We prove that in cases important to us it is weakly equivalent to the ordinary (or discrete) homotopy limit.

In Section 4 we will formulate Theorem 1.1 more precisely as well as the stronger version with restricted cardinalities and compare it with the situation in [11], where K is replaced by a smooth submanifold. Then we use manifold calculus (adapted for simplicial complexes) to prove it.

In Section 5 we will define a *smooth thickening* of a simplicial complex embedded in a smooth manifold and explain how this is a generalization of a smooth disk bundle over a smooth manifold. We will prove Theorem 1.2 and its stronger version with restricted cardinalities. In Section 6 these results will be applied in our study of homotopy automorphisms of the pair $(M, \partial M)$.

Notation The category (Top) is the category of topological spaces. By a simplex S of a simplicial complex, we mean a nondegenerate closed simplex. For such a simplex S, we denote by $\mathrm{op}(S)$ the open simplex. For a positive integer k, we set $[k] := \{0, 1, \dots, k\}$ and $\underline{k} := \{1, \dots, k\}$.

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2 Manifold calculus adapted for simplicial complexes

In [10] we develop a generalization of manifold calculus where the smooth manifold is replaced by a simplicial complex. The main results of this paper are applications of this theory. Therefore, we introduce the constructions and main results of [10] and compare them with the homotopy functor calculus. The comparison leads to criteria which help us to apply manifold calculus (adapted to simplicial complexes).

2.1 Definitions and main results

All the constructions and results can be found in [10]. We define the category $\mathcal{O} = \mathcal{O}(K)$ as follows: the objects are the open subsets of K and the morphisms are inclusions, ie for $U, V \in \mathcal{O}$ there is exactly one morphism $U \to V$ if $U \subset V$ and there are no morphisms otherwise.

Definition 2.1 Let $U, V \in \mathcal{O}$ be open subsets and let $f_0, f_1 \colon U \to V$ be two maps such that $f_i|_{U\cap S}$ is a smooth embedding from $U \cap S$ into $V \cap S$ for all simplices S of K and $i = 0, 1$. We call f_0 and f_1 *stratified isotopic* if there is a continuous map $H \colon U \times [0,1] \to V$ such that

$$H|_{(U\cap S)\times[0,1]} \colon (U \cap S) \times [0,1] \to (V \cap S)$$

is a smooth isotopy from $f_0|_{U\cap S}$ to $f_1|_{U\cap S}$ for all simplices S of K. In this case we call H a *stratified isotopy* (from f_0 to f_1).

Note: for an n–dimensional simplex S, we can regard $U \cap S$ as a subspace in the euclidean space $\mathbb{R}^{n+1}$.

Definition 2.2 Let $U, V \in \mathcal{O}$ be two open subsets with $U \subset V$. The inclusion $i \colon U \to V$ is a *stratified isotopy equivalence* if there is a map $e \colon V \to U$ such that $e|_{V\cap S}$ is an embedding from $V \cap S$ into $U \cap S$ for all simplices S of K and $i \circ e$ (resp. $e \circ i$) is stratified isotopic to id_V (resp. id_U).

In the manifold calculus of Goodwillie and Weiss we consider functors which take smooth isotopy equivalences between open subsets of a fixed manifold to weak equivalences. In the version for simplicial complexes, stratified isotopy equivalences are replacing these smooth isotopy equivalences.

Definition 2.3 A contravariant functor $F \colon \mathcal{O} \to (\mathrm{Top})$ is *good* if:

1. F takes stratified isotopy equivalences to weak homotopy equivalences.
2. For every family $\{V_i\}_{i\in\mathbb{N}}$ of objects in $\mathcal{O}$ with $V_i \subset V_{i+1}$ for all $i \in \mathbb{N}$, the following canonical map is a weak homotopy equivalence:

$$F\Big(\bigcup_i V_i\Big) \to \operatorname*{holim}_i F(V_i).$$

Recall: For a positive integer k, let $\mathcal{P}([k])$ be the power set of $[k]$. Then a functor from $\mathcal{P}([k])$ to topological spaces is a $(k+1)$–cube of spaces.

Definition 2.4 Let χ be a cube of spaces. The *total homotopy fiber* of χ is the homotopy fiber of the canonical map

$$\chi(\varnothing) \to \operatorname*{holim}_{\varnothing \neq T \subset [k]} \chi(T).$$

If this map is a weak homotopy equivalence, we call the cube χ *(weak homotopy) cartesian.*

Now we define polynomial functors. To this end, let F be a good functor, let $V \in \mathcal{O}$ be an open subset of K and let $A_0, A_1, \dots, A_k$ be pairwise disjoint closed subsets of V (for a positive integer k). Define a k–cube by

$$T \mapsto F\Bigl(V \setminus \bigcup_{i \in T} A_i\Bigr). \tag{2-1}$$

Definition 2.5 The functor F is *polynomial of degree* $\leq k$ if the k–cube defined in (2-1) is cartesian for all $V \in \mathcal{O}$ and pairwise disjoint closed subsets $A_0, A_1, \dots, A_k$ of V.

Notation Let $x \in K$ be given and let $\mathcal{S}_x$ be the open star of the open simplex containing x, ie $\mathcal{S}_x := \bigcup_S \operatorname{op}(S)$, where the union ranges over all closed simplices S of K such that x is an element of S.

Definition 2.6 For a positive integer k, we define a full subcategory $\mathcal{O}k(K) = \mathcal{O}k$ of $\mathcal{O}$. Its objects are the open subsets $V \subset K$ with the following properties: V has at most k connected components and, for each component V_0 of V, there is an $x \in K$ such that $V_0 \subset \mathcal{S}_x$ and the inclusion $V_0 \to \mathcal{S}_x$ is a stratified isotopy equivalence. An element of $\mathcal{O}k$ (for some k) is called a *special open set*.

Theorem 2.7 *Let $F_1 \to F_2$ be a natural transformation between two k–polynomial functors. If $F_1(V) \to F_2(V)$ is a weak equivalence for all $V \in \mathcal{O}k$, it is a weak equivalence for all $V \in \mathcal{O}$.*

Let $F\colon \mathcal{O} \to (\mathrm{Top})$ be a good functor. There is a concept of (relative) handle index in a simplicial complex [10, Section 3.1]. We can use it to define analyticity for F. To this end, let P be a compact codimension-zero subobject of K and let ρ be a fixed integer. Suppose $A_0, A_1, \dots, A_r$ are pairwise disjoint compact codimension-zero subobjects of $K \setminus \operatorname{int}(P)$ with relative handle index $q_{A_i} \leq \rho$ (relative to P). For $T \subset [r]$, we set $A_T := \bigcup_{i \in T} A_i$ and assume $r \geq 1$.

Definition 2.8 The functor F is called *ρ–analytic with excess c* if, in these circumstances, the cube

$$T \mapsto F(\mathrm{int}(P \cup A_T)), \quad T \subset [r],$$

is $\left(c + \sum_{i=0}^{r} (\rho - q_{A_i})\right)$–cartesian for some integer c.

Theorem 2.9 [10, Theorem 3.6] *Let F be a ρ–analytic functor with excess c and let $V \in \mathcal{O}$ be an open subset. Then the map*

$$\eta_{k-1}(V)\colon F(V) \to T_{k-1}F(V)$$

is $(c+k(\rho - \dim K))$–connected for every $k > 1$.

Remark 2.10 Theorem 2.9 is weaker than [10, Theorem 3.6], which uses the homotopy dimension of V [10, Definition 3.4] in order to increase the connectivity. For our purposes we do not need this stronger version.

Corollary 2.11 *Let F be a ρ–analytic functor with $\rho > \dim K$. For all open sets $V \in \mathcal{O}(K)$, the canonical map*

$$F(V) \to T_\infty F(V) = \operatorname*{holim}_k T_k F(V)$$

is a weak equivalence.

2.2 Comparison with homotopy functor calculus

In the last section we introduced a version of manifold calculus for simplicial complexes. We saw that in order to apply the approximation theorem, Theorem 2.9, we need to assume analyticity of the functor. Therefore, we should look for criteria which imply that a functor is analytic. Surprisingly, the homotopy functor calculus introduced by Goodwillie [6] helps to find such criteria.

Functor calculus investigates (covariant) *homotopy functors* from topological spaces to themselves. A functor $G\colon (\mathrm{Top}) \to (\mathrm{Top})$ is called homotopy functor if it takes weak equivalences to weak equivalences. If G is such a functor, we can compose it with a contravariant functor F from $\mathcal{O}(K)$ to (Top). The composition $G \circ F$ is a contravariant functor from $\mathcal{O}(K)$ to (Top). We will examine this composition.

Definition 2.12 A cube of spaces is called *strongly cocartesian* if each sub-2–face is a homotopy pushout.

Definition 2.13 A homotopy functor G from (Top) to itself is called *polynomial of degree* $\leq k$ if it takes any strongly cocartesian $(k+1)$–cube to a weakly cartesian $(k+1)$–cube.

Let $V \in \mathcal{O}(K)$ be an open subset of K, let $A_0, A_1, \dots, A_k$ be pairwise disjoint closed subsets of V (for a positive integer k) and let $A_T := \bigcup_{i \in T} A_i$, where T is a subset of $[k]$. The following proposition is an easy observation:

Proposition 2.14 *Let* $F\colon \mathcal{O}(K) \to \text{(Top)}$ *be a good (contravariant) functor* (see Definition 2.3) *such that*

$$\begin{tikzcd} F(V \setminus A_{T \cap T'}) \arrow[r] \arrow[d] & F(V \setminus A_T) \arrow[d] \\ F(V \setminus A_{T'}) \arrow[r] & F(V \setminus A_{T \cup T'}) \end{tikzcd}$$

is a homotopy pushout for all $T, T^{'} \subset [k]$ *and all choices of* $V, A_0, \dots, A_k$ *as above and let* $G\colon \text{(Top)} \to \text{(Top)}$ *be a (covariant) homotopy functor. We suppose that* G *is* k*–polynomial in the sense of homotopy functor calculus* (see Definition 2.13). *Then the composition* $G \circ F$ *is* k*–polynomial in the sense of manifold calculus (adapted for simplicial complexes).*

We would like to have a similar statement for analyticity.

Definition 2.15 Let ρ be an integer and let χ be a cocartesian k–cube of spaces such that the maps $\chi(\varnothing) \to \chi(\{i\})$ are k_i–connected with $k_i > \rho$ for all $i \in [k]$. A homotopy functor G is called ρ*–analytic* with excess c if the cube $G \circ \chi$ is $\bigl(c + \sum_{i \in [k]} (k_i - \rho)\bigr)$–cartesian (for all choices of χ).

Example 2.16 According to the Blakers–Massey theorem [5], for any strongly cocartesian cube χ where the map $\chi(\varnothing) \to \chi(\{i\})$ is κ_i–connected for each $i \in [k]$, the cube χ is κ–cartesian with $\kappa = 1 + \sum_{i \in [k]} (\kappa_i - 1)$. Therefore, by definition, the identity functor id: (Top) → (Top) is 1–analytic with excess 1.

Let $F\colon \mathcal{O}(K) \to \text{(Top)}$ be a good functor (see Definition 2.3). Recall that there is a concept of relative handle index in a simplicial complex [10, Section 3.1]. Let P be a compact codimension-zero subobject of K and let ρ be a fixed integer. Suppose $A_0, A_1, \dots, A_r$ are pairwise disjoint compact codimension-zero subobjects of $K \setminus \operatorname{int}(P)$ with relative handle index $q_{A_i} \leq \rho$ (relative to P). For $T \subset [k]$, we set $A_T := \bigcup_{i \in T} A_i$ and assume $k \geq 1$.

Proposition 2.17 *Suppose that the cube*

$$T \mapsto F(\mathrm{int}(P \cup A_T)), \quad T \subset [k],$$

is strongly cocartesian and suppose that there is a positive integer δ such that the maps

$$F(\mathrm{int}(P \cup A_{[k]})) \to F(\mathrm{int}(P \cup A_{[k]\setminus\{i\}}))$$

are $(\delta - q_{A_i})$–connected. Then F is $(\delta - 1)$–analytic with excess 1 (in the sense of Definition 2.8).

Proof The idea is to apply the Blakers–Massey theorem. By assumption, the cube $T \mapsto F(\mathrm{int}(P \cup A_T))$ is strongly cocartesian. We consider the cube

$$T \mapsto \mathrm{id} \circ F(\mathrm{int}(P \cup A_T)), \quad T \subset [k].$$

By applying Example 2.16, we deduce that the cube is $\bigl(1 + \sum_{i \in [k]} (\delta - q_{A_i} - 1)\bigr)$–cartesian. □

Remark 2.18 In the last proposition we use the analyticity of the identity map in topological spaces to find a criteria for analyticity of F, where F is a good functor. More generally, the following statement holds: for a ρ–analytic functor $G\colon (\mathrm{Top}) \to (\mathrm{Top})$ with excess c and F as above, the composition $G \circ F$ is a $(\delta - \rho)$–analytic functor with excess c and where δ is as above.

For an additional short note on the relationship of manifold calculus (for smooth manifolds) and homotopy functor calculus, see [11, Remark 1.3.2].

3 Background

In this section we provide some background which we will need for the discussions in the next sections. We introduce the configuration category of a simplicial complex and the continuous homotopy limit.

3.1 Configuration category of a simplicial complex

We will need the configuration category of a manifold as well as the configuration category of a simplicial complex. First, we recall the Riemannian model of the configuration category of a smooth manifold. Note that there are several equivalent definitions of the configuration category of a manifold [2].

Let M be a smooth manifold without boundary of dimension m and suppose that we have fixed a Riemannian metric on M. Then the configuration category $\mathrm{con}(M)$ of M is a topological poset. The objects are pairs (T,ρ) where T is a finite subset of M and $\rho\colon T\to(0,\infty)$ is a function such that:

(1) For each $t\in T$, the exponential map $\exp_t$ is defined and regular on the compact disk of radius $\rho(t)$ about the origin in the tangent space T_tM.

(2) The images in M of these disks under the exponential maps $\exp_t$ are pairwise disjoint.

For such a pair (T,ρ), let $V_M(T,\rho)\subset M$ be the union of the open balls of radius $\rho(t)$ about $t\in T$. Then $V_M(T,\rho)$ is an open subset of M which is diffeomorphic to $T\times\mathbb{R}^m$. All these pairs form a topological poset $\mathrm{con}(M)$ by

$$(T,\rho)\le(T',\rho')\iff V_M(T,\rho)\subset V_M(T',\rho').$$

This poset can also be regarded as a category. We would like to adapt this definition and introduce the configuration category $\mathrm{con}(K)$ of the simplicial complex K. Therefore, we should start with the following observation:

Remark 3.1 Let x be an element of K and let $\mathcal{S}_x$ be the open star neighborhood of x in K. The closure $K_x:=\mathrm{cl}(\mathcal{S}_x)$ of $\mathcal{S}_x$ in K carries a canonical metric $d=d_x$ induced by the euclidean structure of each simplex. The precise definition is technical and can be done by distinguishing the following two cases: If two elements $y,y'\in K_x$ are in the same simplex, we can use the euclidean structure of the simplex to define $d(y,y')\in[0,\infty)$ as the distance between y and y' in the euclidean space. If they are not in the same simplex, we set

$$d(y,y'):=\min_{z\in S_y\cap S_{y'}} d(y,z)+d(z,y'),$$

where S_y (resp. $S_{y'}$) is the simplex of maximal dimension which includes y (resp. y'). By definition, we can use again the euclidean structure.

We wrote d instead of d_x to avoid the index x. In fact, $d(y,y')$ is independent of the element x in K: if x and x' are two elements of K with $y,y'\in\mathcal{S}_x\cap\mathcal{S}_{x'}$, then $d_x(y,y')=d_{x'}(y,y')$.

Now we introduce the configuration category $\mathrm{con}(K)$. The objects are again pairs (T,ρ) where T is a finite subset of K and $\rho\colon T\to(0,\infty)$ is a function fulfilling the following two conditions:

(1) For each $t \in T$, there is an element $x \in K$ such that $t \in \mathcal{S}_x$ and the open ball $B^d_{\rho(t)}(t) \subset K_x = \mathrm{cl}(\mathcal{S}_x)$ of radius $\rho(t)$ about t determined by the metric $d = d_x$ is a subset of the open star neighborhood $\mathcal{S}_x$ and the inclusion $B^d_{\rho(t)}(t) \hookrightarrow \mathcal{S}_x$ is a stratified isotopy equivalence (see Definition 2.2). In particular, $B^d_{\rho(t)}(t) \in \mathcal{O}1$ is a special open set (see Definition 2.6).

(2) The open balls $B^d_{\rho(t)}(t) \subset K$ with origin t and radius $\rho(t)$ are pairwise disjoint.

For such a pair (T, ρ), let $V_K(T, \rho) \subset M$ be the union of the open balls $B^d_{\rho(t)}(t) \subset K$ of radius $\rho(t)$ about $t \in T$. Then $V_K(T, \rho)$ is a special open subset of K (see Definition 2.6). By analogy with the manifold case, we form the topological poset $\mathrm{con}(K)$ by

$$(T, \rho) \le (T', \rho') \iff V_K(T, \rho) \subset V_K(T', \rho').$$

This poset can also be regarded as a category.

Remark 3.2 Since this is a very technical notion, we feel compelled to give a short explanation why this category $\mathrm{con}(K)$ is nonempty. Let T be a configuration in K. If we choose ϵ small enough, then the function $\rho\colon T \to (0, \infty)$ mapping all elements of T to ϵ fulfills all conditions in the definition of $\mathrm{con}(K)$. More precisely, the inclusion of the open ball $B^d_\epsilon(t)$ about an element $t \in T$ of radius ϵ into the open star $\mathcal{S}_t$ of t is a stratified isotopy equivalence. If ϵ is small enough, the open balls for different elements of T are also pairwise disjoint.

Now we want to take a closer look at the configuration category $\mathrm{con}(K)$. But note that the following results are also true for $\mathrm{con}(M)$, the configuration category of a smooth manifold M (without boundary).

Remark 3.3 The configuration category $\mathrm{con}(K)$ is a topological poset, ie the objects as well as the morphisms form a topological space. More generally, if $N(\mathrm{con}(K))$ is the nerve of the category $\mathrm{con}(K)$, then $N_r(\mathrm{con}(K))$ is a topological space for all $r \ge 0$. This is obvious since $N_r(\mathrm{con}(K))$ is the space of all strings

$$(T_0, \rho_0) \le (T_1, \rho_1) \le \cdots \le (T_r, \rho_r),$$

where (T_i, ρ_i) for $0 \le i \le r$ is an element of $\mathrm{con}(K)$.

Now we want to investigate the homotopy type of the configuration category $\mathrm{con}(K)$ as a topological space. It is very reminiscent of the configuration spaces.

Definition 3.4 We define $C_r(K)$ to be the space of unordered configurations of r points in K: Let $F_r(K)$ be the space of ordered r–configurations of K given by

$$F_r(K) := \{(x_1, \dots, x_r) \in K^r \mid x_i \neq x_j \text{ for all } i \neq j\}.$$

The symmetric group Σ_r acts freely on $F_r(K)$. Then

$$C_r(K) := F_r(K)/\Sigma_r$$

is the space of unordered r–configurations.

Remark 3.5 What is the relation between the configuration category and the configuration spaces? Let $r \geq 0$ be a fixed integer. We define the space $C_r^{\mathrm{fat}}(K)$ to be the space of all pairs $(T, \rho) \in \mathrm{con}(K)$ with $|T| = r$. Then we have a forgetful projection map

$$C_r^{\mathrm{fat}}(K) \to C_r(K),$$

which is a fiber bundle with contractible fibers. Therefore, this map is a weak equivalence of spaces.

3.2 Continuous homotopy limit

Let $\mathrm{con}(K)$ be the configuration category of K and let $N(\mathrm{con}(K))$ be its nerve. We saw that $N_r(\mathrm{con}(K))$ is a topological space for all $r \geq 0$. We are studying the functor Φ from $\mathrm{con}(K)$ to topological spaces defined by

$$\Phi((T, \rho)) := M \setminus V_K(T, \rho)$$

and its homotopy limit

$$\underset{\mathrm{con}(K)}{\mathrm{holim}}\, \Phi = \underset{(T,\rho)\in\mathrm{con}(K)}{\mathrm{holim}}\, M \setminus V_K(T, \rho).$$

During our study of this homotopy limit, we would like to integrate the continuous structure of the nerve of $\mathrm{con}(K)$. To this end, we will introduce the continuous homotopy limit of Φ using the topological structure of the configuration category.

We recall that the ordinary (or discrete) homotopy limit $\mathrm{holim}_{\mathrm{con}(K)}\, \Phi$ of the contravariant functor Φ is defined to be the totalization of the cosimplicial space

$$[r] \mapsto \prod_{(T_0,\rho_0)\leq\cdots\leq(T_r,\rho_r)\in N_r(\mathrm{con}(K))} \Phi((T_r, \rho_r)).$$

By definition, the right-hand side is equal to the space of all sections from $N_r(\mathrm{con}(K))$ equipped with the discrete topology to

$$\coprod_{(T_0,\rho_0)\le\cdots\le(T_r,\rho_r)\in N_r(\mathrm{con}(K))} \Phi((T_r,\rho_r)).$$

Equivalently, it is equal to the space of all maps $f\colon N_r(\mathrm{con}(K)) \to M$ such that

$$f((T_0,\rho_0)\le\cdots\le(T_r,\rho_r)) \in M \setminus V_K(T_r,\rho_r),$$

where $N_r(\mathrm{con}(K))$ is again given the discrete topology. Using the continuous structure of $\mathrm{con}(K)$, we introduce the following notation:

Definition 3.6 We define $\Gamma_r(\Phi)$ as the space of all continuous maps $f\colon N_r(\mathrm{con}(K)) \to M$ such that $f((T_0,\rho_0)\le\cdots\le(T_r,\rho_r)) \in M \setminus V_K(T_r,\rho_r)$.

If we define $E_r^!(\Phi)$ to be the space

$$\coprod_{(T_0,\rho_0)\le\cdots\le(T_r,\rho_r)\in N_r(\mathrm{con}(K))} \Phi((T_r,\rho_r))$$

equipped with the subspace topology of $N_r(\mathrm{con}(K)) \times M$, then the projection map $E_r^!(\Phi) \to N_r(\mathrm{con}(K))$ is a fiber bundle and $\Gamma_r(\Phi)$ is the space of all continuous sections of this fiber bundle.

Definition 3.7 The *continuous homotopy limit* $\mathrm{cts\,holim}_{\mathrm{con}(K)}\Phi$ of Φ is defined to be the totalization of the cosimplicial space $[r] \mapsto \Gamma_r(\Phi)$.

Lemma 3.8 *The canonical inclusion map*

$$\operatorname*{ctsholim}_{\mathrm{con}(K)} \Phi \to \operatorname*{holim}_{\mathrm{con}(K)} \Phi$$

is a weak equivalence.

We skip the proof because it is equal to the proof of [11, Lemma 1.2.1]. (If we replace the manifold L appearing in [11, 1.2.1] by the simplicial complex K, then we get a proof for Lemma 3.8.)

Using this result, we can work in the following with either of these homotopy limits — the discrete homotopy limit or the continuous homotopy limit.

Remark 3.9 For an open subset U of K, let $\mathrm{con}(K)|_U$ be the full subcategory of $\mathrm{con}(K)$ such that the objects are all elements (T,ρ) in $\mathrm{con}(K)$ with $V_K(T,\rho) \subset U$.

For $r \geq 0$, let $\Gamma_r(\Phi)|_U$ be the space of all continuous maps $f\colon N_r(\mathrm{con}(K)|_U) \to M$ such that

$$f((T_0, \rho_0) \leq \cdots \leq (T_r, \rho_r)) \in M \setminus V_K(T_r, \rho_r).$$

Now we define $\mathrm{ctsholim}_{\mathrm{con}(K)|_U} \Phi$ to be the totalization of the cosimplicial space $r \mapsto \Gamma_r(\Phi)|_U$. There is a canonical inclusion map

$$\underset{\mathrm{con}(K)|_U}{\mathrm{ctsholim}}\, \Phi \to \underset{\mathrm{con}(K)|_U}{\mathrm{holim}}\, \Phi,$$

which is a weak equivalence. The proof is equal to that of Lemma 3.8.

Remark 3.10 The cosimplicial space $r \mapsto \Gamma_r(\Phi)|_U$ is Reedy fibrant for every open subset U of K. The verification is the same as that in [11, 1.1.3]. Recall that for a map $X \to Y$ between cosimplicial spaces which is a degreewise weak equivalence, the map of their totalizations $\mathrm{Tot}(X) \to \mathrm{Tot}(Y)$ is a weak equivalence.

4 The main theorem

We formulate the main theorem and apply manifold calculus (adapted to simplicial complexes) in order to prove it.

4.1 The formulation of the problem

We remind the reader that M is a smooth manifold and $K \subset M$ is a simplicial complex such that each (closed) simplex of K is smoothly embedded in M. For each element (T, ρ) of the configuration category $\mathrm{con}(K)$, there is an inclusion map

$$M \setminus K \to M \setminus V_K(T, \rho),$$

where $V_K(T, \rho)$ is the open subset of K corresponding to the pair (T, ρ). If we define a contravariant functor Φ from $\mathrm{con}(K)$ to topological spaces by $\Phi((T, \rho)) := M \setminus V_K(T, \rho)$, then the inclusion maps induce a canonical map

$$M \setminus K \to \underset{\mathrm{con}(K)}{\mathrm{holim}}\, \Phi. \tag{4-1}$$

We can ask if the canonical map is a weak equivalence. There is a variant with restricted cardinalities. Let $n \geq 0$ be an integer. Then we define $\mathrm{con}_{\leq n}(K)$ to be the full subcategory of $\mathrm{con}(K)$ where the objects are all elements (T, ρ) of $\mathrm{con}(K)$ with $|T| \leq n$. Again, we get a canonical map

$$M \setminus K \to \underset{\mathrm{con}_{\leq n}(K)}{\mathrm{holim}}\, \Phi \tag{4-2}$$

induced by inclusions. In this case we do not expect that this map is a weak equivalence. But, we can ask if it is highly connected. In the following theorem we use the notation $m := \dim M$ and $\kappa := \dim K$.

Theorem 4.1 *If $\kappa + 3 \leq m$, then the canonical map (4-1) is a weak equivalence and (4-2) is $(1+(n+1)(m-\kappa-2))$–connected.*

Remark 4.2 The homotopy limit appearing in (4-1) is the ordinary (or discrete) homotopy limit. By Lemma 3.8, we could also use the continuous homotopy limit and the theorem would still hold. Using similar arguments, we could also use the continuous homotopy limit in (4-2).

Remark 4.3 We assumed that the codimension of K in M is at least three. In fact, the theorem would be false without this assumption. There is a nice counterexample in codimension two [11, Remark 1.3.3].

Remark 4.4 The theorem is a generalization of [11, Theorem 1.1.1]. Let L be a compact, smooth submanifold (without boundary) of M where the codimension of L in M is at least three. We can choose a triangulation of L and get a simplicial complex K, ie $K = L$ as a topological space but the configuration categories $\mathrm{con}(L)$ and $\mathrm{con}(K)$ are quite distinct because the structure of K as a simplicial complex goes into the definition of $\mathrm{con}(K)$.

Let $\bigcup_k \mathcal{O}k(L)$ be the category of all special open subsets of L [12]. These are all the open subsets of L which are diffeomorphic to a disjoint union of open disks. Then we have the inclusions of categories

$$\mathrm{con}(L) \hookrightarrow \bigcup_k \mathcal{O}k(L) \hookleftarrow \mathrm{con}(K)$$

and we get a zigzag

$$\operatorname*{holim}_{(T,\rho)\in\mathrm{con}(L)} \Phi((T,\rho)) \leftarrow \operatorname*{holim}_{U\in\bigcup_k \mathcal{O}k(L)} M \setminus U \rightarrow \operatorname*{holim}_{(T,\rho)\in\mathrm{con}(K)} \Phi((T,\rho)).$$

These projection maps of homotopy limits given by inclusion of categories are both weak equivalences.

4.2 A good functor

In order to prove Theorem 4.1, we would like to apply manifold calculus (adapted to simplicial complexes). Naively, one could suggest to apply the approximation theorem

(Theorem 2.9) to the contravariant functor which maps an open subset $V \subset K$ to the topological space $M \setminus V$. Unfortunately, this functor is not good because in general it does not take stratified isotopy equivalences to weak equivalences (for a counterexample, see [11, 1.3]). Therefore, we need a modification.

Definition 4.5 We define the functor F from the category $\mathcal{O}(K)$ of open subsets of K to topological spaces by

$$F(V) := \operatorname*{holim}_{C \subset V} M \setminus C,$$

where C runs over all compact subsets of V.

We will see that F is an appropriate replacement of the functor $V \mapsto M \setminus V$. The proof in the following lemma is similar to that of [11, 1.3.1]. For the sake of completeness, we will give all required arguments.

Lemma 4.6 *The functor F is good* (in the sense of Definition 2.3).

Proof First, we notice that the (co)limit axiom is fulfilled. This is obvious. In order to show that the functor takes stratified isotopy equivalences to weak homotopy equivalences, we will use the reformulation of stratified isotopy equivalences as given in Remark 4.7. To this end, let V_0 and V_1 be two open subsets of K with $V_0 \subset V_1$ and let $e_t\colon V_0 \to V_1$ for $t \in [0,1]$ be a stratified isotopy such that e_0 is the inclusion and, for each simplex S of K, e_1 is a homeomorphism such that $e_1|_S\colon S \cap V_0 \to S \cap V_1$ can be extended to a diffeomorphism (see Remark 4.7).

Let $\{C_i\}_{i \geq 0}$ be a sequence of compact subsets of V_1 such that $C_i \subset C_{i+1}$ for all $i \geq 0$ and such that, for every compact subset C of V_1, there is an element C_i of this sequence with $C \subset C_i$. By definition, the inclusion

$$\{C_i\}_{i \geq 0} \to \{C \subset V_1 \mid C \text{ compact}\}$$

is homotopy terminal. (Note that the morphisms are the inclusions of compact subsets.) Therefore, the canonical map

$$F(V_1) \to \operatorname*{holim}_{i} M \setminus C_i$$

is a weak equivalence. Now we define the compact sets $C_{t,i} := e_t(e_1^{-1}(C_i))$. Note that $C_{1,i} = C_i$. By definition, the inclusion

$$\{C_{0,i}\}_{i \geq 0} \to \{C \subset V_0 \mid C \text{ compact}\}$$

is homotopy terminal and induces a weak equivalence

$$F(V_0) \to \operatorname*{holim}_i M \setminus C_{0,i}.$$

We fix the notation

$$Y_i := \{w\colon [0,1] \to M \mid w(t) \notin M \setminus C_{t,i}\}.$$

There are evaluation maps $Y_i \to M \setminus C_{0,i}$ and $Y_i \to M \setminus C_{1,i}$. Using the isotopy extension theorem [8, 6.5], it is straightforward to find homotopy inverses. For a comment on the isotopy extension theorem for stratified spaces, see Remark 4.8. We get homotopy equivalences

$$M \setminus C_{0,i} \longleftrightarrow Y_i \longleftrightarrow M \setminus C_{1,i}.$$

Since the evaluation maps are natural, we get weak equivalences

$$\operatorname*{holim}_i M \setminus C_{0,i} \leftarrow \operatorname*{holim}_i Y_i \to \operatorname*{holim}_i M \setminus C_{1,i}$$

To summarize, we have shown that the spaces $F(V_1)$ and $F(V_0)$ are weakly equivalent. Now we have to argue that the canonical map $F(V_1) \to F(V_0)$ induced by inclusion is a weak equivalence.

Let $g\colon \mathbb{N} \to \mathbb{N}$ be a monotone injective function such that for every $i \in \mathbb{N}$ and $t \in [0,1]$, the compact set $C_{t,i}$ is a subset of $C_{1,g(i)}$. We consider the composition

$$\Psi\colon \operatorname*{holim}_i M \setminus C_{1,i} \to \operatorname*{holim}_i M \setminus C_{1,g(i)} \to \operatorname*{holim}_i M \setminus C_{0,i},$$

where the first map is induced by the inclusion $\{C_{1,g(i)}\}_i \to \{C_{1,i}\}_i$ of categories and the second map is induced by the inclusions $C_{0,i} \hookrightarrow C_{1,g(i)}$ of spaces for $i \in \mathbb{N}$. In order to verify that the composition Ψ is a weak equivalence, we consider the homotopy commutative triangle

$$\begin{tikzcd}
\operatorname{holim}_i Y_i \arrow[rr, "\cong"] \arrow[dr, "\cong"'] & & \operatorname{holim}_i M \setminus C_{1,i} \arrow[dl, "\Psi"] \\
& \operatorname{holim}_i M \setminus C_{0,i} &
\end{tikzcd}$$

It does not seem to be trivial that the triangle is homotopy commutative. But, by careful inspection, the definition of the homotopy limit provides a homotopy whereby the triangle is homotopy commutative. Using the same argument, we get a homotopy commutative square

$$\begin{tikzcd}
F(V_1) \arrow[r] \arrow[d, "\cong"] & F(V_0) \arrow[d, "\cong"] \\
\operatorname{holim}_i M \setminus C_{1,i} \arrow[r, "\Psi"] & \operatorname{holim}_i M \setminus C_{0,i}
\end{tikzcd}$$

Since Ψ is a weak equivalence, the canonical map $F(V_1) \to F(V_0)$ is also a weak equivalence. □

Remark 4.7 We need a slight reformulation of a stratified isotopy equivalence. According to Definition 2.2, an inclusion $i\colon V_0 \to V_1$ of open subsets of K is a stratified isotopy equivalence if there is a continuous map $e\colon V_1 \to V_0$ such that $e|_{V_1 \cap S}$ is a smooth embedding from $V_1 \cap S$ into $V_0 \cap S$ for all simplices S of K and if there are a stratified isotopy from $i \circ e$ to id_{V_1} and a stratified isotopy from $e \circ i$ to id_{V_0}. The following definition would also be appropriate: we could call an inclusion $i\colon V_0 \to V_1$ of open subsets of K a stratified isotopy equivalence if i is stratified isotopic to a homeomorphism $e\colon V_0 \to V_1$ such that $e|_{V_0 \cap S}$ is a diffeomorphism from $V_0 \cap S$ to $V_1 \cap S$ for all simplices S of K. (Note that S is not a manifold, so more precisely we should say: the map $e|_{V_0 \cap S}$ from $V_0 \cap S$ to $V_1 \cap S$ can be extended to a diffeomorphism using that S is canonically embedded in an euclidean space.)

Why is the second definition of stratified isotopy equivalences also appropriate? We do not know if these definitions are equivalent, but it is straightforward to verify the following claim: Let $G\colon \mathcal{O}(K) \to (\mathrm{Top})$ be a contravariant functor. Then G takes stratified isotopy equivalences as in Definition 2.2 to weak equivalences if and only if G takes stratified isotopy equivalences as in the second definition to weak equivalences.

Remark 4.8 In the proof of the last lemma we can use a continuous version of the isotopy extension theorem for stratified spaces as provided in [8, 6.5]: Let $C \subset V_0$ be a compact subset, where $V_0 \subset K$ is an open subset as above. We consider a continuous family of open topological embeddings $f_t\colon C \to K$ for $0 \le t \le 1$, with $f_0 = \mathrm{id}_C$. Then there is a continuous family of homeomorphisms $H_t\colon K \to K$ such that $H_t|_C = f_t$ and $H_0 = \mathrm{id}_K$.

We can use this theorem in the proof above as follows: Let $e_t\colon V_0 \to V_1$ for $t \in [0,1]$ be a stratified isotopy as above. In particular, e_0 is the inclusion of V_0 into V_1 and e_1 is a homeomorphism. For a positive integer i, we define $C := e_1^{-1}(C_i)$ and $f_t := e_t|_C\colon C \to V_1 \subset K$. Using the isotopy extension theorem, we get a continuous family of homeomorphisms $H_t\colon K \to K$ such that $H_t|_C = f_t$ and $H_0 = \mathrm{id}_K$. Then a homotopy inverse of the evaluation map $Y_i \to K \setminus C_{0,i}$ given by $w \mapsto w(0)$ can be defined by $x \mapsto (t \mapsto H_t(x))$.

4.3 Proof of the main theorem

Now we prove Theorem 4.1, ie we show that the top horizontal arrow in the commutative diagram

$$\begin{tikzcd}
M \setminus K \arrow[r] \arrow[d] & \operatorname*{holim}_{(T,\rho)\in\mathrm{con}(K)} M \setminus V_K(T,\rho) \arrow[d] \\
F(K) \arrow[r] & \operatorname*{holim}_{(T,\rho)\in\mathrm{con}(K)} F(V_K(T,\rho))
\end{tikzcd}$$

is a weak equivalence. The left vertical arrow is a weak equivalence because K is a maximal element in the category (poset) of all compact subsets of K. The right vertical arrow is a weak equivalence because for every $(T,\rho) \in \mathrm{con}(K)$, the category of all compact subsets of $V_K(T,\rho)$ has a directed subcategory which is homotopy terminal. Therefore, we have to show that the bottom horizontal arrow is a weak equivalence. To this end, we will use the good properties of the functor F and manifold calculus (adapted to simplicial complexes). The bottom arrow equals the composition

$$F(K) \to \operatorname*{holim}_{U\in\bigcup_k \mathcal{O}k(K)} F(U) \to \operatorname*{holim}_{(T,\rho)\in\mathrm{con}(K)} F(V_K(T,\rho)),$$

where the first map is the canonical map and the second map is induced by the inclusion of posets

$$\mathrm{con}(K) \to \bigcup_k \mathcal{O}k(K)$$

given by $(T,\rho) \mapsto V_K(T,\rho)$. Therefore, the following two lemmas complete the proof. (The proof of the case with restricted cardinalities follows similar lines.)

Lemma 4.9 *The canonical projection map*

$$\operatorname*{holim}_{U\in\bigcup_k \mathcal{O}k(K)} F(U) \to \operatorname*{holim}_{(T,\rho)\in\mathrm{con}(K)} F(V_K(T,\rho))$$

induced by the inclusion $\mathrm{con}(K) \to \bigcup_k \mathcal{O}k(K)$ *is a weak equivalence.*

Proof By [3, Theorem 6.14], it remains to show that the canonical map

$$F(U) \to \operatorname*{holim}_{(T,\rho)\in\mathrm{con}(K)|_U} F(V_K(T,\rho))$$

is a weak equivalence for all $U \in \bigcup_k \mathcal{O}k(K)$. Recall that $\mathrm{con}(K)|_U$ is the full subcategory of $\mathrm{con}(K)$ where the objects are all elements (T,ρ) in $\mathrm{con}(K)$ with $V_K(T,\rho) \subset U$. For a fixed $U \in \bigcup_k \mathcal{O}k(K)$, we choose an element $(T',\sigma) \in \mathrm{con}(K)|_U$ such that the map $F(U) \to F(V_k(T,\sigma))$ is a weak equivalence. We set $W := V_K(T',\sigma)$ and consider the commutative diagram

$$\begin{tikzcd}
F(U) \arrow[r] \arrow[d] & \operatorname*{holim}_{(T,\rho)\in\mathrm{con}(K)|_U} F(V_K(T,\rho)) \arrow[d] \\
F(W) \arrow[r] & \operatorname*{holim}_{(T,\rho)\in\mathrm{con}(K)|_W} F(V_K(T,\rho))
\end{tikzcd}$$

The bottom arrow is a weak equivalence because W is a maximal element in $\mathrm{con}(K)|_W$. In order to show that the right vertical arrow is a weak equivalence, we will consider the two homotopy limits as continuous homotopy limits. This is allowed by Remark 3.9. Then we compare the two spaces $\mathrm{con}(K)|_W$ and $\mathrm{con}(K)|_U$. By definition of their topologies, the inclusion $\mathrm{con}(K)|_W \to \mathrm{con}(K)|_U$ is a weak equivalence. Similarly, the maps of section spaces $\Gamma_r(\Phi)|_U \to \Gamma_r(\Phi)|_W$ are weak equivalences for all $r \geq 0$. So they induce a weak equivalence of continuous homotopy limits. □

Lemma 4.10 *If* $\dim K + 3 \leq \dim M$*, the canonical map*

$$F(K) \to \operatorname*{holim}_{U\in\bigcup_k \mathcal{O}k(K)} F(U)$$

is a weak equivalence.

Proof Note that we have already shown that F is good (Lemma 4.6). Let P be a smooth compact codimension-zero subobject of K and let $A_0, A_1, \dots, A_r$ be compact codimension-zero subobjects of $K \setminus \mathrm{int}(P)$ with relative handle index q_{A_i} (relative to P). For $T \subset [r]$, we define

$$W_T := \mathrm{int}\biggl(P \cup \bigcup_{i\in T} A_i\biggr),$$

where $\mathrm{int}(-)$ is the interior in K. We have to show that the cube

$$T \mapsto F(W_T), \quad T \subset [r],$$

is strongly cocartesian and that, for every $0 \leq i \leq r$, the maps

$$F(W_{[r]}) \to F(W_{[r]\setminus\{i\}})$$

are $((m-1)-q_{A_i})$–connected, where m is the dimension of M. Note that W_S is the interior of a compact codimension-zero subobject of K. Therefore, instead of using the functor F, we can work with the cube

$$T \mapsto G(W_T) := M \setminus W_T.$$

Why can we use this cube? Because of the special assumption, there is a directed homotopy terminal subcategory in the category of all compact subsets of W_T. Thus, the canonical map $G(W_T) \to F(W_T)$ is a weak equivalence.

Let $i, j \in [r]$ be two distinct elements. In order to show that the cube induced by G is strongly cocartesian, we need to investigate if the canonical map from the homotopy pushout of

$$G(W_{[r]\setminus\{i\}}) \leftarrow G(W_{[r]}) \rightarrow G(W_{[r]\setminus\{j\}})$$

to $G(W_{\underline{r}\setminus\{i,j\}})$ is a weak equivalence. But this can easily be seen. In fact, using the assumptions that all A_i are pairwise disjoint, we can find a copy of $G(W_{\underline{r}\setminus\{i,j\}})$ in the homotopy pushout which is a retract of the homotopy pushout. Likewise, it is not difficult to check that for a fixed $i \in [r]$, the map

$$G(W_{[r]}) \rightarrow G(W_{[r]\setminus\{i\}})$$

is $(m-q_{A_i}-1)$–connected since the target is homotopy equivalent to the source with attached cells of dimension $\geq m - q_{A_i}$. □

5 Occupants in the interior of a manifold

In this section, let M be a manifold with boundary and let L be a smooth submanifold without boundary. We discuss Theorem 1.2, where the homotopy type of $M \setminus L$ is recovered from the homotopy types of the spaces $M \setminus T$ with $T \subset L$ finite. To this end, we give the definition of a *smooth thickening* of a simplicial complex (in M) and discuss first observations and examples. Then we prove the tube lemma, Lemma 5.6, which we will need in order to prove Theorem 1.2.

5.1 Smooth thickenings of a simplicial complex

We consider the following situation: Let M be a manifold with boundary. Let $L \subset M \setminus \partial M$ be a smooth submanifold without boundary of dimension l.

Definition 5.1 Let $K \subset L$ be a simplicial complex. We say that $p\colon L \to K$ is a *nice projection map* if the following conditions hold:

(1) $p|_K = \mathrm{id}_K$.

(2) The open set $p^{-1}(V_K(T,\rho)) \subset L$ is diffeomorphic to $T \times \mathbb{R}^l$ for every element (T,ρ) of the configuration category $\mathrm{con}(K)$ of K.

Definition 5.2 We say that L is a *smooth thickening of K in M* if each (closed) simplex of K is smoothly embedded in L and if there exists a nice projection map $p\colon L \to K$ such that the inclusion $M \setminus p^{-1}(V) \to M \setminus V$ is a weak equivalence for all open sets $V \in \mathcal{O}(K)$.

Definition 5.3 If $M \setminus \partial M$ is a smooth thickening of K in M, then we just say that M (which is a manifold with boundary) is a *smooth thickening of K*.

Examples 5.4 (1) The definition of smooth thickening weakens the strong condition in [11, 2.1.1] in the following sense: Let L be a smooth closed manifold and let $p\colon M \to L$ be a smooth disk bundle, ie a smooth fiber bundle where each fiber is diffeomorphic to a (closed) disk D^r of fixed dimension $r \geq 0$. Then L can be considered as a subset of M by using the zero section of p. We can choose a triangulation of L and then L is a smooth thickening of its triangulation in M.

(2) We consider the 1–dimensional simplicial complex K with four vertices $\{a,b,c,d\}$ and 1–simplices $\{\{a,b\},\{a,c\},\{b,c\},\{b,d\},\{c,d\}\}$, ie we have two triangles which coincide in exactly one simplex, namely $\{b,c\}$. Now it is an easy exercise to build up a compact manifold M of dimension $m = 2$ such that the interior $M \setminus \partial M$ is a smooth thickening of K in M, ie M is a smooth thickening of K. We ought to consider M as a manifold with four 0–handles and five 1–handles. This example can easily be generalized to all dimensions $m \geq 2$ and/or to an one-dimensional simplicial complex which consists of more than two triangles.

Lemma 5.5 *We assume that* $\dim K + 3 \leq m$ *and that* L *is a smooth thickening of* K *in* M. *Let* $p\colon L \to K$ *be a nice projection map. Then the canonical map*

$$M \setminus L \to \operatorname*{holim}_{(T,\rho)\in \mathrm{con}(K)} M \setminus p^{-1}(V_K(T,\rho))$$

is a weak equivalence.

Proof We consider the five homotopy equivalences

$$\begin{aligned} M \setminus L &\simeq M \setminus K \\ &\simeq M \setminus (K \cup \partial M) \\ &\simeq \operatorname*{holim}_{(T,\rho)} M \setminus (V_K(T,\rho) \cup \partial M) \\ &\simeq \operatorname*{holim}_{(T,\rho)} M \setminus V_K(T,\rho) \\ &\simeq \operatorname*{holim}_{(T,\rho)} M \setminus p^{-1}(V_K(T,\rho)), \end{aligned}$$

where the three homotopy limits are taken over all (T,ρ) in $\mathrm{con}(K)$. By definition of smooth thickenings in M, the first equivalence can be verified, as well as the fifth equivalence. By Theorem 4.1, the third map is a weak equivalence. The second and the fourth map are weak equivalences since $M \cong M \setminus \partial M$. □

5.2 Tube lemma

Now we adapt the results of [11, 2.2] for a nice projection map. Note that for the following lemma we do not have to require that the codimension be at least three. It could also be zero.

Lemma 5.6 *Let L be a smooth manifold without boundary and let K be a compact simplicial complex K. Let $p\colon L \to K$ be a nice projection map* (see Definition 5.1). *Then the canonical map*

$$\operatorname*{hocolim}_{(T,\rho)\in\mathrm{con}(K)} C_n\big(p^{-1}(V_K(T,\rho))\big) \to C_n(L) \tag{5-1}$$

is a weak equivalence.

Proof We are going to show that the map is a microfibration with contractible fibers. Then the lemma will follow [13, Lemma 2.2]. Let T be an element of the configuration space $C_n(L)$. The fiber of the map (5-1) over the configuration T is identified with the classifying space of the poset of all $(T,\rho) \in \mathrm{con}(K)$ with $T \in p^{-1}(V_K(T,\rho))$, ie $p(T) \in V_K(T,\rho)$. The inclusion of the directed poset

$$\Big\{(T,\rho) \in \mathrm{con}(K) \mid \exists n \in \mathbb{N}\ \forall t \in T\ \rho(t) = \frac{1}{n}\Big\}$$

into the above described poset is a homotopy initial functor. (We consider the posets as categories.) Therefore, the fiber is contractible.

Now we verify the lifting condition. We start with an observation: The projection map and the map (5-1) determine an injective, continuous map

$$\operatorname*{hocolim}_{(T,\rho)\in\operatorname{con}(K)} C_n\bigl(p^{-1}(V_K(T,\rho))\bigr) \to |N\operatorname{con}(K)| \times C_n(L).$$

(This map is not an embedding, ie a homeomorphism onto its image. See also Remark 5.7.) We call this map $g = (g_1, g_2)$.

Let Z be a compact CW–space. We consider the diagram

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & \operatorname*{hocolim}_{(T,\rho)\in\operatorname{con}(K)} C_n\bigl(p^{-1}(V_K(T,\rho))\bigr) \arrow[d] \\
Z \times I \arrow[r] & C_n(L)
\end{tikzcd}$$

We call the upper horizontal map f and we can consider it as a pair of maps $f = (f_1, f_2)$ if we define $f_i := g_i \circ f$ for $i = 1, 2$. We call the bottom horizontal map h. The right vertical arrow is equal to g_2. We can define a small lift

$$H\colon Z \times [0,\epsilon] \to \operatorname*{hocolim}_{(T,\rho)\in\operatorname{con}(K)} C_k\bigl(p^{-1}(V_K(T,\rho))\bigr)$$

by $H := (f_1, h)$.

How can we describe the map H? Let $z \in Z$ be given. By the formula $H := (f_1, h)$, the map

$$\{z\} \times [0,\epsilon] \xrightarrow{H} \operatorname*{hocolim}_{(T,\rho)\in\operatorname{con}(K)} C_n\bigl(p^{-1}(V_K(T,\rho))\bigr) \xrightarrow{g_1} N\operatorname{con}(K)$$

is constant; more precisely, $g_1 \circ H(\{z\} \times [0,\epsilon]) = \{f_1(z)\}$.

How can we find an $\epsilon > 0$ such that H is well defined? Let S be an r–simplex of $|N\operatorname{con}(K)|$, let E be the corresponding open simplex and let $(T_0, \rho_0) \le \cdots \le (T_r, \rho_r)$ be the corresponding element in $N_r\operatorname{con}(K)$. We define

$$Z_S := f_1^{-1}(S) = f^{-1}(g_1^{-1}(S)) \subset Z,$$
$$Z_E := f_1^{-1}(E) = f^{-1}(g_1^{-1}(E)) \subset Z.$$

We take a close look at the map

$$f_2|_{Z_S}\colon Z_S \xrightarrow{f} \operatorname*{hocolim}_{(T,\rho)\in\operatorname{con}(K)} C_n\bigl(p^{-1}(V_K(T,\rho))\bigr) \xrightarrow{g_2} C_n(L).$$

First, we note that $f_2(Z_E) \subset C_n\bigl(p^{-1}(V_K(T_0,\rho_0))\bigr)$ by definition. By definition (of smooth thickening), $p^{-1}(V_K(T_j,\rho_j))$ is a special open set for every $0 \le j \le r$. In the spirit of Remark 5.7, we conclude that $f_2(Z_S)$ is also a subset of $C_n\big(p^{-1}(V_K(T_0,\rho_0))\big)$. (For an easier example of this argument, see [11, 2.2.1].) Since $f_2(Z_S)=h(Z_S\times\{0\})$ is compact, there is an $\epsilon_S>0$ with

$$h(Z_S\times[0,\epsilon_S])\subset C_n\big(p^{-1}(V_K(T_0,\rho_0))\big).$$

The image of Z is contained in a finite union of open cells of $|N\mathrm{con}(K)|$. Therefore, there is a finite number of simplices S such that Z_S is nonempty. We can define ϵ to be the minimum of all ϵ_S, where the minimum ranges over all simplices S such that Z_S is nonempty. □

Remark 5.7 Let $U\in\mathbb{R}^n$ be a bounded open subset. Then the mapping cylinder of the inclusion $U\to\mathbb{R}^n$ is not homeomorphic to a subspace of $\mathbb{R}^{n+1}$. The quotient topology equips the mapping cylinder with a different structure. In fact, it is not metrizable [11, 2.2.2].

Corollary 5.8 *The canonical map*

$$\operatorname*{hocolim}_{(T,\rho)\in\mathrm{con}(K)} N_0\mathrm{con}\big(p^{-1}(V_K(T,\rho))\big)\to N_0\mathrm{con}(L)$$

determined by the inclusions is a weak equivalence.

Proof We remind the reader that for an open set $U\subset K$, we defined $\mathrm{con}(U)$ to be the full subcategory of $\mathrm{con}(K)$ with all objects (T,ρ) such that $V_K(T,\rho)$ is a subset of U. There is a commutative square

$$\begin{tikzcd}
\operatorname*{hocolim}_{(T,\rho)\in\mathrm{con}(K)} N_0\mathrm{con}\big(p^{-1}(V_K(T,\rho))\big) \arrow[r] \arrow[d] & N_0\mathrm{con}(L) \arrow[d] \\
\operatorname*{hocolim}_{(T,\rho)\in\mathrm{con}(K)} \coprod_n C_n\big(p^{-1}(V_K(T,\rho))\big) \arrow[r] & \coprod_n C_n(L)
\end{tikzcd}$$

where the vertical arrows are weak equivalences (the left one is induced by a natural transformation). Therefore, we only have to verify that the bottom map is a weak equivalence. But this follows from the fact that the homotopy colimit commutes with disjoint union. □

Corollary 5.9 *For every $r\geq 0$, the canonical map*

$$\operatorname*{hocolim}_{(T,\rho)\in\mathrm{con}(K)} N_r\mathrm{con}\big(p^{-1}(V_K(T,\rho))\big)\to N_r\mathrm{con}(L)$$

induced by the inclusions is a weak equivalence.

Proof We consider the commutative square

$$\begin{tikzcd}
\operatorname*{hocolim}_{(T,\rho)\in\mathrm{con}(K)} N_r\mathrm{con}\big(p^{-1}(V_K(T,\rho))\big) \arrow[r] \arrow[d] & N_r\mathrm{con}(L) \arrow[d] \\
\operatorname*{hocolim}_{(T,\rho)\in\mathrm{con}(K)} N_0\mathrm{con}\big(p^{-1}(V_K(T,\rho))\big) \arrow[r] & N_0\mathrm{con}(L)
\end{tikzcd}$$

Here the vertical arrows are given by the ultimate target operator and the horizontal arrows are the canonical maps induced by the inclusions. We can check that this is a (strict) pullback square and that the right vertical arrow is a fibration. Since (Top) is a proper model category [7, 13.1.11] and the bottom arrow is a weak equivalence, we conclude that the upper arrow is also a weak equivalence. □

5.3 Boundary recovered

Let M be a manifold with boundary ∂M and let L be a smooth submanifold without boundary. We recover the homotopy type of $M \setminus L$ from the homotopy types of the spaces $M \setminus T$ where T is a finite subset of L. Again, we need to allow thickenings of the finite subsets T and inclusions between them. We recall that for each object (T,ρ) in the configuration category $\mathrm{con}(L)$ of L, there is a corresponding open subset $V_L(T,\rho)$ in L. We can define a contravariant functor ψ from $\mathrm{con}(L)$ to the category of topological spaces by $\psi((T,\rho)) := M \setminus V_L(T,\rho)$. We get a canonical map

$$M \setminus L \to \operatorname*{holim}_{(T,\rho)\in\mathrm{con}(L)} M \setminus V_L(T,\rho), \tag{5-2}$$

induced by the inclusions $M \setminus L \to M \setminus V_L(T,\rho)$. We can ask if this map is a weak equivalence. There is also a variant with restricted cardinalities. Let $\mathrm{con}_{\leq n}(L)$ be the full subcategory of $\mathrm{con}(L)$ where the objects are all pairs $(T,\rho)\in\mathrm{con}(L)$ with $|T|\leq n$. Again, we get a canonical map

$$M \setminus L \to \operatorname*{holim}_{(T,\rho)\in\mathrm{con}_{\leq n}(L)} M \setminus V_L(T,\rho), \tag{5-3}$$

induced by inclusions. We can ask whether this map is highly connected and whether there is a lower bound for the connectivity. The following theorem, where we use again the notation $\kappa := \dim K$ and $m := \dim M$, answers these questions.

Theorem 5.10 *The canonical map (5-2) is a weak equivalence if the following condition holds: there is a compact simplicial complex $K \subset M$ of dimension κ with $\kappa + 3 \leq m$ such that L is a smooth thickening of K in M* (see Definition 5.2). *In this case, the canonical map (5-3) is $(1+(n+1)(m-\kappa-2))$–connected.*

Corollary 5.11 *The canonical map*

$$\partial M \to \operatorname*{holim}_{(T,\rho)\in \mathrm{con}_{\le n}(M\setminus\partial M)} M \setminus V_{M\setminus\partial M}(T,\rho)$$

is a weak equivalence if the following condition holds: there is a compact simplicial complex $K \subset M$ of dimension κ with $\kappa + 3 \le m$ such that M is a smooth thickening of K (see Definition 5.3). *In this case, the canonical map*

$$\partial M \to \operatorname*{holim}_{(T,\rho)\in \mathrm{con}_{\le n}(M\setminus\partial M)} M \setminus V_{M\setminus\partial M}(T,\rho)$$

is $(1+(n+1)(m-\kappa-2))$–connected.

Remark 5.12 In (5-2) and (5-3), the discrete (or ordinary) homotopy limit can be replaced by the continuous homotopy limit without changing the (weak) homotopy type. This can be justified with arguments which are provided in [11, 1.2] (and in Section 2.2).

Remark 5.13 This corollary is a generalization of [11, Theorem 2.1.1]; compare Example 5.4(2). It can be applied in the proof of [14, Theorems 5.2.1 and 5.3.1], whereby we get a weaker condition in these theorems (this will extensively be studied in Section 6).

In order to prove that (5-2) is a weak equivalence, we consider the following diagram, where all arrows are the canonical maps and $p\colon L \to K$ is a nice projection map:

$$\begin{tikzcd}
M \setminus L \arrow[r] \arrow[d] & \operatorname*{holim}\limits_{(T,\rho)\in \mathrm{con}(K)} M \setminus p^{-1}(V_K(T,\rho)) \arrow[d] \\
\operatorname*{holim}\limits_{(T',\sigma)\in \mathrm{con}(L)} \psi(T',\sigma) \arrow[r] & \operatorname*{holim}\limits_{(T,\rho)\in \mathrm{con}(K)} \operatorname*{holim}\limits_{\substack{(T',\sigma)\in \mathrm{con}(L) \\ p(V_L(T',\sigma))\subset V_K(T,\rho)}} \psi(T',\sigma)
\end{tikzcd}$$

It commutes because both compositions factorize through the ordinary limit and the two maps through the ordinary limit are clearly the same. In Lemma 5.5 we have already shown that the upper horizontal arrow is a weak equivalence. Therefore, the first part of the theorem follows from the next two lemmas.

Lemma 5.14 *The right vertical arrow is a weak equivalence.*

Proof Let $(T,\rho) \in \mathrm{con}(K)$ be fixed. Since the map under investigation is induced by a natural transformation, it suffices to show that the map

$$M \setminus p^{-1}(V_K(T,\rho)) = M \setminus U \to \operatorname*{holim}_{(T',\sigma)\in \mathrm{con}(U)} \psi(T',\sigma)$$

is a weak equivalence, where, for simplicity, U is defined to be the open set

$$U := p^{-1}(V_K(T,\rho)) \subset L.$$

Note that by definition, the open set U is diffeomorphic to $T \times \mathbb{R}^l$. We consider the composition of maps

$$M \setminus U \to \operatorname*{holim}_{(T',\sigma)\in \mathrm{con}(U)} \psi(T',\sigma) \to \operatorname*{holim}_{(T',\sigma)\in \mathrm{con}(U)} F(V_K(T',\sigma)),$$

where F is the functor from the category $\mathcal{O}(U)$ of open subsets of U to topological spaces given by $F(W) := \operatorname{holim}_{C\subset W} M \setminus C$, where C runs through the compact subsets of W. Note that the category of all compact subsets of $V_K(T',\sigma)$ has a directed subcategory which is homotopy terminal. Therefore, the canonical map $\psi(T',\sigma) \to F(V_K(T',\sigma))$ is a weak equivalence for every $(T',\sigma) \in \mathrm{con}(U)$. Using the homotopy invariance of the homotopy limit, the second map is a weak equivalence. So, in order to prove that the first map is a weak equivalence, we have to show that the composition is a weak equivalence. To this end, we consider another composition

$$M \setminus U \to F(U) \to \operatorname*{holim}_{W\in \bigcup_k \mathcal{O}k(U)} F(W) \to \operatorname*{holim}_{(T',\sigma)\in \mathrm{con}(U)} F(V_K(T',\sigma)).$$

First of all, we note that the two compositions give the same map since both compositions factorize through the ordinary limit and the two maps through the ordinary limit are clearly the same. The first map in this composition is a weak equivalence because the category of all compact subsets of U has a directed subcategory which is homotopy terminal. The third map is a weak equivalence by an argument which we have seen in Lemma 4.9. The second map is a weak equivalence because the open set U is a maximal element in $\bigcup_k \mathcal{O}k(U)$. □

Lemma 5.15 *The bottom horizontal arrow is a weak equivalence.*

Proof If replace the homotopy limit by the continuous homotopy limit, the source is the totalization of the cosimplicial space $[r] \mapsto \Gamma_r(\Psi)$, where $\Gamma_r(\Psi)$ is the space of all sections from $N_r\mathcal{P}(L)$ to $E^!(\Psi)$. (All notation is introduced in Section 3.2.) If replace the second homotopy limit in the target by the continuous homotopy limit (compare Remark 3.9), the target is isomorphic to the totalization of the cosimplicial space $[r] \mapsto \widetilde{\Gamma}_r(\Psi)$, where $\widetilde{\Gamma}_r(\Psi)$ is the space of all sections from

$$\operatorname*{hocolim}_{(T,\rho)\in \mathrm{con}(K)} N_r \mathrm{con}\big(p^{-1}(V_K(T,\rho))\big)$$

to $E^!(\Psi)$. The bottom horizontal arrow in the above diagram is induced by composition with the map in Corollary 5.9,

$$\operatorname*{hocolim}_{(T,\rho)\in\mathrm{con}(K)} N_r\mathrm{con}\big(p^{-1}(V_K(T,\rho))\big) \to N_r\mathrm{con}(L) \to E^!(\Psi).$$

Using Corollary 5.9, this map is a weak equivalence. □

Now we investigate the case with restricted cardinalities. To this end, we fix $n \geq 0$. Let j be an integer with $0 \leq j \leq n$ be given. There is the following modification of the tube lemma, Lemma 5.6. The canonical map

$$\operatorname*{hocolim}_{(T,\rho)\in\mathrm{con}_{\leq n}(K)} C_j\big(p^{-1}(V_K(T,\rho))\big) \to C_j(L)$$

is a weak equivalence. The proof is the same: The projection map is a microfibration with contractible fibers. Why do we need that $j \leq n$? In the proof of Lemma 5.6 we introduced a homotopy initial subposet, in order to show that the fibers are contractible. In the restricted case, this poset is defined if and only if $j \leq n$.

Using this observation, the proof of the restricted case follows similar lines. In particular, there is a commutative diagram

$$\begin{tikzcd}
M \setminus L \arrow[r] \arrow[d] & \operatorname*{holim}\limits_{(T,\rho)\in\mathrm{con}_{\leq n}(K)} M \setminus p^{-1}(V_K(T,\rho)) \arrow[d] \\
\operatorname*{holim}\limits_{(T',\sigma)\in\mathrm{con}_{\leq n}(L)} \psi(T',\sigma) \arrow[r] & \operatorname*{holim}\limits_{(T,\rho)\in\mathrm{con}_{\leq n}(K)} \operatorname*{holim}\limits_{\substack{(T',\sigma)\in\mathrm{con}_{\leq n}(L)\\ p(V_L(T',\sigma))\subset V_K(T,\rho)}} \psi(T',\sigma)
\end{tikzcd}$$

By Theorem 4.1 (and Lemma 5.5), the top horizontal map is $(1+(n+1)(m-k-2))$–connected. Using a modification of Corollary 5.9, the bottom horizontal arrow is a weak equivalence. In order to justify that the right vertical arrow is a weak equivalence, we can use arguments which we have seen in Lemma 5.14.

6 Homotopy automorphisms

Let M be a smooth, compact manifold with boundary.

Definition 6.1 We define the *homotopy link* $\mathrm{holink}(M/\partial M, *)$ of the basepoint in $M/\partial M$ to be the space of paths $\gamma\colon [0,1] \to M/\partial M$ which satisfy the condition

$\gamma^{-1}(\{*\}) = \{0\}$. The topology is the compact–open topology. We define the map

$$q_M\colon \operatorname{holink}(M/\partial M, *) \to M \setminus \partial M$$

by $\gamma \mapsto \gamma(1)$.

Remark 6.2 It is well known that the map q_M is a good homotopical substitute for the inclusion map $\partial M \hookrightarrow M$: if we define Z_M to be the space of paths $\gamma\colon [0,1] \to M$ which satisfy the condition $\gamma^{-1}(\partial M) = \{0\}$ (with the compact–open topology), we get a homotopy commutative diagram

$$\begin{tikzcd}
\operatorname{holink}(M/\partial M, \star) \arrow[rr, "q_M"] & & M \setminus \partial M \arrow[d, "\simeq"] \\
Z_M \arrow[u, "\simeq"] \arrow[r, "\simeq"] & \partial M \arrow[r, hook] & M
\end{tikzcd}$$

Let $\operatorname{homeo}(M)$ be the homeomorphism group of M. Evidently, there is a canonical action of $\operatorname{homeo}(M)$ on the complete diagram. This action extends to an action of the homeomorphism group $\operatorname{homeo}(M \setminus \partial M)$ on q_M. But, unfortunately, the action does not extend to an action of the homeomorphism group $\operatorname{homeo}(M \setminus \partial M)$ on the inclusion map $\partial M \hookrightarrow M$. We are interested in this extension. That is why we introduced the homotopical substitute q_M.

Definition 6.3 Let c be an object in a model category $\mathcal{C}$. We define $\operatorname{haut}(c)$ to be the *space of derived homotopy automorphisms* of c in $\mathcal{C}$, ie $\operatorname{haut}(c)$ is the union of the homotopy invertible path components of the derived mapping space $\mathbb{R}\mathrm{map}(c,c)$. With composition, $\operatorname{haut}(c)$ is a grouplike topological or simplicial monoid. (For a suitable definition of simplicial mapping spaces, we follow [4].)

Note that the map q_M can be regarded as a functor from the totally ordered set $\{0,1\}$ to the category of topological spaces. The category of such functors has well-known standard model category structures. If we choose one of them, we can study the space of derived homotopy automorphisms $\operatorname{haut}(q_M)$ of q_M. In particular, since $\operatorname{homeo}(M \setminus \partial M)$ acts on q_M, each homeomorphism of $M \setminus \partial M$ determines a (derived) homotopy automorphism of q_M. Therefore, we get a map

$$B\operatorname{homeo}(M \setminus \partial M) \to B\operatorname{haut}(q_M) \tag{6-1}$$

of classifying spaces.

Let Fin be the category of finite sets and maps between them. The nerve $N\mathrm{Fin}$ is a simplicial set. We introduced the Riemannian model of the configuration category $\mathrm{con}(M \setminus \partial M)$. The nerve of this category is a simplicial space over $N\mathrm{Fin}$.

Definition 6.4 Let X be a simplicial space over $N\mathrm{Fin}$. We define $\mathrm{haut}_{N\mathrm{Fin}}(X)$ to be the *space of derived homotopy automorphisms* of X over $N\mathrm{Fin}$, ie $\mathrm{haut}(X)$ is the union of the homotopy invertible path components of the derived mapping space $\mathbb{R}\mathrm{map}_{N\mathrm{Fin}}(X,X)$ of X over $N\mathrm{Fin}$. (If an introduction to derived mapping spaces of simplicial spaces is needed, we refer the reader to [14, Section 3].) With composition, $\mathrm{haut}_{N\mathrm{Fin}}(X)$ is a grouplike topological or simplicial monoid.

If we use the particle model [2, Section 3.1; 14, Section 1] of the configuration category $\mathrm{con}(M \setminus \partial M)$, it is easy to see that each homeomorphism of $M \setminus \partial M$ determines a (derived) homotopy automorphism of the nerve of $\mathrm{con}(M \setminus \partial M)$ over $N\mathrm{Fin}$.

Particle model In this model, the space of objects of the configuration category $\mathrm{con}(M \setminus \partial M)$ is

$$\coprod_{k\geq 0} \mathrm{emb}(\underline{k}, M \setminus \partial M).$$

A morphism from $f \in \mathrm{emb}(\underline{k}, M \setminus \partial M)$ to $g \in \mathrm{emb}(\underline{l}, M \setminus \partial M)$ is a map $v\colon \underline{k} \to \underline{l}$ and a homotopy

$$(\gamma_t)_{t\in[0,a]}\colon \underline{k} \to M \setminus \partial M$$

from f to gv which satisfies the *stickiness condition*: if $\gamma_s(b_1) = \gamma_s(b_2)$ for $s \in [0,a]$ and $b_1, b_2 \in \underline{k}$, then $\gamma_t(b_1) = \gamma_t(b_2)$ for all $t \in [s,a]$. Therefore, the space of morphisms of the configuration category $\mathrm{con}(M \setminus \partial M)$ in the particle model is

$$\coprod_{k,l\geq 0,\, v\colon \underline{k}\to\underline{l}} \Lambda(v).$$

Here $\Lambda(v)$ is the space of all triples (f,g,γ) where $f \in \mathrm{emb}(\underline{k}, M \setminus \partial M)$, $g \in \mathrm{emb}(\underline{l}, M \setminus \partial M)$ and γ is a homotopy from f to gv which satisfies the stickiness condition. The Riemannian model of the configuration category and the particle model are equivalent [2, Section 3.2].

Using the particle model of the configuration category $\mathrm{con}(M \setminus \partial M)$, there is an inclusion of topological grouplike monoids from $\mathrm{homeo}(M \setminus \partial M)$ to $\mathrm{haut}_{N\mathrm{Fin}}(\mathrm{con}(M \setminus \partial M))$. We get a map of classifying spaces

$$B\mathrm{homeo}(M \setminus \partial M) \to B\mathrm{haut}_{N\mathrm{Fin}}(\mathrm{con}(M \setminus \partial M)). \tag{6-2}$$

Now we can ask whether the map (6-1) has a factorization through the map (6-2).

Theorem 6.5 *We assume that the following condition holds: there is a compact simplicial complex $K \subset M$ of dimension κ with $\kappa + 3 \le m$ such that M is a smooth thickening* (see Definition 5.3) *of K. Then the broken arrow in the homotopy commutative diagram*

$$\begin{tikzcd}
B\mathrm{homeo}(M \setminus \partial M) \arrow[r, "\text{(6-1)}"] \arrow[d, equal] & B\mathrm{haut}(q_M) \\
B\mathrm{homeo}(M \setminus \partial M) \arrow[r, "\text{(6-2)}"] & B\mathrm{haut}_{N\mathrm{Fin}}(\mathrm{con}(M \setminus \partial M)) \arrow[u, dotted]
\end{tikzcd}$$

can be supplied.

Using Corollary 5.11, the proof is equal to that of [14, Theorem 5.2.1]. There is also a variant with restricted cardinalities. Following [14, 5.3], we need a Postnikov decomposition of the map q_M. It is well known that for each integer $a \ge 0$, there is a decomposition

$$\partial M \to \wp_a \partial M \to M$$

of the inclusion map $\partial M \hookrightarrow M$ such that the homotopy groups of $\wp_a \partial M$ are zero in dimension $\ge a+2$ and equal to the homotopy groups of ∂M in dimension $\le a+1$. ($\wp_a \partial M$ is obtained from ∂M, as a space over M, by killing the relative homotopy groups of $\partial M \to M$ in dimensions $\ge a+2$.) By analogy with this construction, there is a decomposition

$$\mathrm{holink}(M/\partial M, *) \to \wp_a(q_M) \to M \setminus \partial M$$

of the map q_M, where $\wp_a(q_M)$ has the same properties as $\wp_a \partial M$.

Theorem 6.6 *We assume that the following condition holds: there is a compact simplicial complex $K \subset M$ of dimension κ with $\kappa + 3 \le m$ such that M is a smooth thickening* (see Definition 5.3) *of K. Then the broken arrow in the homotopy commutative diagram*

$$\begin{tikzcd}
B\mathrm{homeo}(M \setminus \partial M) \arrow[r, "\text{action}"] \arrow[d, equal] & B\mathrm{haut}(\wp_{(j+1)(m-\kappa-2)}(q_M)) \\
B\mathrm{homeo}(M \setminus \partial M) \arrow[r, "\text{action}"] & B\mathrm{haut}_{N\mathrm{Fin}}(\mathrm{con}_{\le j}(M \setminus \partial M)) \arrow[u, dotted]
\end{tikzcd}$$

can be supplied. Here the two action maps are the maps (6-1) *and* (6-2) *applied to the restricted case.*

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On uniqueness of end sums and 1–handles at infinity

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For oriented manifolds of dimension at least 4 that are simply connected at infinity, it is known that end summing is a uniquely defined operation. Calcut and Haggerty showed that more complicated fundamental group behavior at infinity can lead to nonuniqueness. We examine how and when uniqueness fails. Examples are given, in the categories TOP, PL and DIFF, of nonuniqueness that cannot be detected in a weaker category (including the homotopy category). In contrast, uniqueness is proved for Mittag-Leffler ends, and generalized to allow slides and cancellation of (possibly infinite) collections of 0– and 1–handles at infinity. Various applications are presented, including an analysis of how the monoid of smooth manifolds homeomorphic to $\mathbb{R}^4$ acts on the smoothings of any noncompact 4–manifold.

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1 Introduction

Since the early days of topology, it has been useful to combine spaces by simple gluing operations. The connected sum operation for closed manifolds has roots in nineteenth century surface theory, and its cousin, the boundary sum of compact manifolds with boundary, is also classical. These two operations are well understood. In the oriented setting, for example, the connected sum of two connected manifolds is unique, as is the boundary sum of two manifolds with connected boundary. The boundary sum has an analogue for open manifolds, the *end sum*, which has been used in various dimensions since the 1980s, but is less well known and understood. The first author and Haggerty showed in 2014 [7] that, in contrast with boundary sums, end sums of one-ended oriented manifolds need not be uniquely determined, even up to proper homotopy. The present paper explores uniqueness and its failure in more detail. To illustrate the subtlety of the issue, we present examples in various categories (homotopy, TOP, PL and DIFF) where uniqueness fails, but the failure cannot be detected in weaker categories. In counterpoint, we find general hypotheses under which the operation *is* unique in all categories and apply this result to exotic smoothings of open 4–manifolds.

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Our results naturally belong in the broader context of *attaching handles at infinity*. We obtain general uniqueness results for attaching collections of 0– and 1–handles at infinity, generalizing handle sliding and cancellation. We conclude that end sums, and, more generally, collections of handles at infinity with index at most one, can be controlled in broad circumstances, although deep questions remain.

End sums are the natural analogue of boundary sums. To construct the latter, we choose codimension-zero embeddings of a disk into the boundaries of the two summands, then use these to attach a 1–handle. For an end sum of open manifolds, we attach a 1–handle at infinity, guided by a properly embedded ray in each summand. Informally, we can think of the 1–handle at infinity as a piece of tape joining the two manifolds; see Definition 2.1 for details. Boundary summing two compact manifolds then has the effect of end summing their interiors. While this notion of end summing seems obvious, the authors have been unable to find explicit appearances of it before the second author's 1983 paper [18] and sequel [19] on exotic smoothings of $\mathbb{R}^4$. However, the germ of the idea may be perceived in Mazur's 1959 paper [33] and Stallings' 1965 paper [38]. End summing was used in [19] to construct infinitely many exotic smoothings of $\mathbb{R}^4$. The appendix of that paper showed that the operation is well defined in that context, so is independent of choice of rays and their order (even for infinite sums). Since then, the second author and others have continued to use end summing with an exotic $\mathbb{R}^4$ for constructing many exotic smoothings on various open 4–manifolds, eg Taylor [39, Theorem 6.4] in 1997 and Gompf [23, Section 7] in 2017. The operation has also been subsequently used in other dimensions, for example by Ancel (unpublished) in the 1980s to study high-dimensional Davis manifolds, and by Tinsley and Wright [40] in 1997 and Myers [35] in 1999 to study 3–manifolds. In 2012, the first author, with King and Siebenmann, gave a somewhat general treatment [8] of end sum (called *CSI*, for *connected sum at infinity*, therein) in all dimensions and categories (TOP, PL and DIFF). One corollary gave a classification of multiple hyperplanes in $\mathbb{R}^n$ for all $n \neq 3$, which was used by Belegradek [2] in 2014 to study certain interesting open aspherical manifolds. Most recently, Sparks [37] in 2018 used infinite end sums to construct uncountably many contractible topological 4–manifolds obtained by gluing two copies of $\mathbb{R}^4$ along a subset homeomorphic to $\mathbb{R}^4$.

While [19] showed that end sums are uniquely determined for oriented manifolds homeomorphic to $\mathbb{R}^4$, uniqueness fails in general for multiple reasons. The most obvious layer of difficulty already occurs for the simpler operation of boundary summing. In that case, when a summand has disconnected boundary, we must specify which boundary component to use. For example, nondiffeomorphic boundary components can lead to boundary sums with nondiffeomorphic boundaries. We must also be careful to specify orientations — a pair of disk bundles over S^2 with nonzero Euler numbers can be boundary summed in two different ways, distinguished by their signatures (0 or ± 2). In general, we should specify an orientation on each orientable boundary component receiving a 1–handle. Similarly, for end sums and 1–handles at infinity, we must specify which ends of the summands we are using and an orientation on each such end (if orientable).

Unlike boundary sums, however, end sums have a more subtle layer of nonuniqueness. One difficulty is specific to dimension 3: the rays in use can be knotted. Myers [35] showed that uncountably many homeomorphism types of contractible manifolds can be obtained by end summing two copies of $\mathbb{R}^3$ along knotted rays. For this reason, the present paper focuses on dimensions above 3. However, another difficulty persists in high dimensions: rays determining a given end need not be properly homotopic. The first author and Haggerty [7] constructed examples of pairs of one-ended oriented n–manifolds (for all $n \geq 3$) that can be summed in different ways, yielding manifolds that are not even properly homotopy-equivalent. We explore this phenomenon more deeply in Section 3. After sketching the key example of [7] in Example 3.2, we exhibit more subtle examples of nonuniqueness of end summing (and related constructions) on fixed oriented ends. Examples 3.3 include topological 5–manifolds with properly homotopy-equivalent but nonhomeomorphic end sums on the same pair of ends, and PL n–manifolds (for various $n \geq 9$) whose end sums are properly homotopy-equivalent but not PL homeomorphic. Unlike other examples in this section, those in Examples 3.3 have extra ends or boundary components; the one-ended case seems more elusive. Examples 3.4 provide end sums of smooth manifolds (for most $n \geq 8$) that are PL homeomorphic but not diffeomorphic. The analogous construction in dimension 4 gives smooth manifolds whose end sums are naturally identified in the topological category, but whose smoothings are not stably isotopic. Distinguishing their diffeomorphism types seems difficult.

These failures of uniqueness arise from complicated fundamental group behavior at the relevant ends, contrasting with uniqueness associated with the simply connected end of $\mathbb{R}^4$. Section 4 examines more generally when ends are simple enough to guarantee uniqueness of end sums and 1–handle attaching. In dimensions 4 and up, it suffices for the end to satisfy the *Mittag-Leffler* condition (also called *semistability*), whose definition we recall in Section 4. Ends that are simply connected or topologically collared are

Mittag-Leffler; in fact, the condition can only fail when the end requires infinitely many $(n-1)$–handles in any topological handle decomposition (Proposition 4.3). For example, Stein manifolds of complex dimension at least 2 have (unique) Mittag-Leffler ends. (See Corollaries 4.4 and 4.10, and Theorem 5.4 for an application to 4–manifold smoothing theory.) The Mittag-Leffler condition is necessary and sufficient to guarantee that any two rays approaching the end are properly homotopic. This fact traces back at least to Geoghegan in the 1980s, and appears to have been folklore since the preceding decade. (See also Edwards and Hastings [13], Mihalik [34, Theorem 2.1] and Geoghegan [17].) The first author and King worked out an algebraic classification of proper rays up to proper homotopy on an arbitrary end in 2002. This material was later excised from the 2012 published version of [8] due to length considerations and since a similar proof had appeared in Geoghegan's text [17] in the meantime. The present paper gives a much simplified version of the proof, dealing only with the Mittag-Leffler case, in order to highlight the topology underlying the algebraic argument (Lemma 4.11). This lemma leads to a general statement (Theorem 4.6) about attaching countable collections of 1–handles to an open manifold. The following theorem is a special case:

Theorem 1.1 *Let X be a (possibly disconnected) n–manifold, with $n \geq 4$. Then the result of attaching a (possibly infinite) collection of 1–handles at infinity to some oriented Mittag-Leffler ends of X depends only on the pairs of ends to which each 1–handle is attached, and whether their orientations agree.*

Note that uniqueness of end sums along Mittag-Leffler ends (preserving orientations) is a special case. Theorem 4.6 also deals with ends that are nonorientable or not Mittag-Leffler.

Theorem 4.6 has consequences for open 4–manifold smoothing theory, which we explore in Section 5. The theorem easily implies the result from [19] that the oriented diffeomorphism types of 4–manifolds homeomorphic to $\mathbb{R}^4$ form a monoid $\mathcal{R}$ under end sum, allowing infinite sums that are independent of order and grouping. This monoid acts on the set $\mathcal{S}(X)$ of smoothings (up to isotopy) of any given oriented 4–manifold X with a Mittag-Leffler end, and more generally a product of copies of $\mathcal{R}$ acts on $\mathcal{S}(X)$ through any countable collection of Mittag-Leffler ends (see Corollary 5.1). One can also deal with arbitrary ends by keeping track of a family of proper homotopy classes of rays. Similarly, one can act on $\mathcal{S}(X)$ by summing with exotic smoothings of $S^3 \times \mathbb{R}$ along properly embedded lines (Corollary 5.5), or modify smoothings along properly embedded star-shaped graphs. While summing with a fixed exotic $\mathbb{R}^4$ is unique for an oriented (or nonorientable) Mittag-Leffler end, Section 3 suggests that there should be examples of nonuniqueness when the end of X is not Mittag-Leffler. However, such examples seem elusive, prompting the following natural question:

Question 1.2 *Let X be a smooth, one-ended, oriented 4–manifold. Can summing X with a fixed exotic $\mathbb{R}^4$, preserving orientation, yield different diffeomorphism types depending on the choice of ray in X?*

We show (Proposition 5.3) that such examples would be quite difficult to detect.

Having studied the uniqueness problem for adding 1–handles at infinity, we progress in Section 6 to uniqueness of adding collections of 0– and 1–handles at infinity (Theorem 6.1). It turns out that, when adding countably many handles of index 0 and 1, the noncompact case is simpler than for compact handle addition. As an application of Theorem 6.1, we present (Theorem 6.2) a very natural and partly novel proof of the hyperplane unknotting theorem of Cantrell [9] and Stallings [38]: each proper embedding of $\mathbb{R}^{n-1}$ in $\mathbb{R}^n$ for $n \geq 4$ is unknotted (in each category DIFF, PL and TOP). An immediate corollary is the TOP Schoenflies theorem: the closures of the two complementary regions of a (locally flat) embedding of S^{n-1} in S^n for $n \geq 4$ are topological disks. Mazur's infinite swindle still lies at the heart of our proof of the hyperplane unknotting theorem. The novelty in our proof consists of the supporting framework of 0– and 1–handle additions, slides and cancellations at infinity.

Throughout the text, we take manifolds to be Hausdorff with countable basis, so with only countably many components. We allow boundary, and note that the theory is vacuous unless there is a noncompact component. *Open* manifolds are those with no boundary and no compact components. We work in a category CAT that can be DIFF, PL or TOP. For example, DIFF homeomorphisms are the same as diffeomorphisms. Embeddings (particularly with codimension zero) are not assumed to be proper. (Proper means the preimage of every compact set is compact.) In PL and TOP, embeddings are assumed to be locally flat (as is automatically true in DIFF). It follows that in each category, codimension-one two-sided embeddings in $\operatorname{Int} X$ are bicollared (Brown [6] in TOP; see Connelly [11] for a simpler proof in both TOP and PL). Furthermore, a CAT proper embedding $\gamma\colon Y \hookrightarrow X^n$ of a CAT 1–manifold Y with $b_1(Y) = 0$ and $\gamma^{-1}(\partial X) = \varnothing$ extends to a CAT proper embedding $\overline{\nu}\colon Y \times D^{n-1} \hookrightarrow X^n$ whose boundary (after rounding corners in DIFF) is bicollared. (This is easy in DIFF and PL, and follows in TOP by a classical argument: cover suitably by charts exhibiting Y as locally flat, then stretch one chart consecutively through the others.) If we radially identify $\mathbb{R}^{n-1}$ with $\operatorname{Int} D^{n-1}$, then $\overline{\nu}$ determines an embedding $\nu\colon Y \times \mathbb{R}^{n-1} \hookrightarrow X$. We call ν and $\overline{\nu}$ *tubular neighborhood maps*, and their images open (resp. closed) *tubular neighborhoods* of Y. Thus, an open tubular neighborhood extends to a closed tubular neighborhood by definition.

2 1–handles at infinity

We begin with our procedure for attaching 1–handles at infinity.

Definition 2.1 A *multiray* in a CAT n–manifold X is a CAT proper embedding $\gamma\colon S \times [0,\infty) \hookrightarrow X$, with $\gamma^{-1}(\partial X) = \varnothing$, for some discrete (so necessarily countable) set S, called the *index set* of γ. If the domain has a single component, γ will be called a *ray*. Given two multirays $\gamma^-, \gamma^+\colon S \times [0,\infty) \hookrightarrow X$ with disjoint images, choose tubular neighborhood maps $\nu^\pm\colon S \times [0,\infty) \times \mathbb{R}^{n-1} \hookrightarrow X$ with disjoint images, and let Z be the CAT manifold obtained by gluing $S \times [0,1] \times \mathbb{R}^{n-1}$ to X using identifications $\nu^\pm \circ (\mathrm{id}_S \times \varphi^\pm \times \rho^\pm)$, where $\varphi^-\colon \left[0,\frac{1}{2}\right) \to [0,\infty)$ and $\varphi^+\colon \left(\frac{1}{2},1\right] \to [0,\infty)$ and $\rho^\pm\colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ are diffeomorphisms, with $\rho^\pm$ chosen so that $\varphi^\pm \times \rho^\pm$ preserves orientation. Then Z is obtained by *attaching* 1*–handles at infinity* to X along γ^- and γ^+ (see Figure 1).

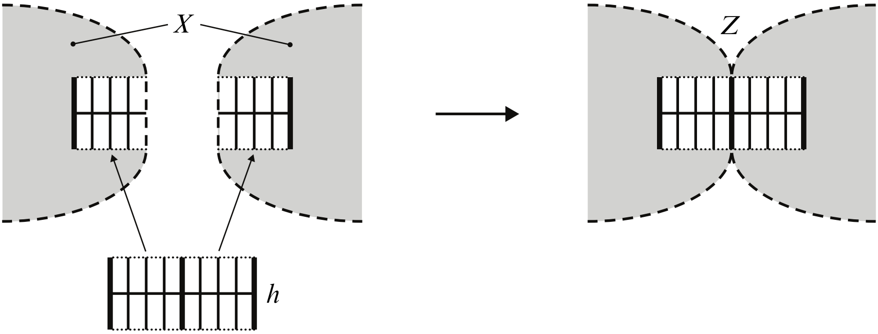

Figure 1: Data for attaching h, a 1–handle at infinity, to the n–manifold X (left) and resulting n–manifold Z (right)

The case of handle attaching where S is a single point and X has two components that are connected by the 1–handle at infinity is called the *end sum* or *connected sum at infinity* in the literature. In general, we will see that Z depends in a subtle way on the choice of images of $\gamma^\pm$ (Section 3), but not on the parametrizations of their rays. It depends on the orientations locally induced by $\nu^\pm$, but is otherwise independent of the choices of maps $\nu^\pm$, $\varphi^\pm$ and $\rho^\pm$. (Independence follows from the stronger Theorem 4.6 when $n \geq 4$, and by a similar method in lower dimensions.) By reparametrizing the maps $\varphi^\pm$, we can change their domains to smaller neighborhoods of the endpoints of $[0,1]$ without changing Z, making it more obvious that attaching compact 1–handles to the boundary of a compact manifold has the effect of attaching handles at infinity to the interior. Yet another description of handle attaching at infinity is to remove the interiors of the closed tubular neighborhoods from X and glue together the resulting $\mathbb{R}^{n-1}$ boundary components. Some articles (eg [8; 37]) use this perspective for defining end sums. It can be useful to start, more generally, with any countable collection of disjoint rays, allowing clustering (for example to preserve an infinite group action as in Gompf [25]). However, this gains no actual generality, since we can transform such a collection to a multiray by suitably truncating the domains of the rays to achieve properness of the combined embedding.

Remark The second author exploited higher-index handles at infinity in [24], but additional subtleties arise in that context. For example, a Casson handle CH can be attached to an unknot in the boundary of a 4–ball B so that the interior of the resulting smooth 4–manifold is not diffeomorphic to the interior of any compact manifold. However, Int CH is diffeomorphic to $\mathbb{R}^4$, so we can interchange the roles of Int CH and Int B, exhibiting the manifold as $\mathbb{R}^4$ with a 2–handle attached at infinity. The latter is attached along a properly embedded $S^1 \times [0,\infty)$ in $\mathbb{R}^4$ that is topologically unknotted but smoothly knotted, and cannot be smoothly compactified to an annulus in the closed 4–ball. This proper annulus seems analogous to a knotted ray in a 3–manifold, but is more subtle since it is unknotted in TOP.

Variations on the above 1–handle construction were recently applied to 4–dimensional smoothing theory by the second author [23]. Let X be a topological 4–manifold with a fixed smooth structure, and let R be an exotic $\mathbb{R}^4$ (a smooth, oriented manifold homeomorphic but not diffeomorphic to $\mathbb{R}^4$). Choose a smooth ray in X, and homeomorphically identify a smooth, closed tubular neighborhood N of it with the complement of a tubular neighborhood of a ray in R. Transporting the smooth structure from R to N, where it fits together with the original one on $X - \operatorname{Int} N$, we obtain a new smooth structure on X diffeomorphic to an end sum of X and R. The advantage of this description is that it fixes the underlying topological manifold, allowing us to assert, for example, that the two smooth structures are stably isotopic. Another variation from [23] is to sum a smooth structure with an exotic $\mathbb{R} \times S^3$ along a smooth, properly embedded line in each manifold, with one line topologically isotopic to $\mathbb{R} \times \{p\} \subset \mathbb{R} \times S^3$.

(We order the factors this way instead of the more commonly used $S^3 \times \mathbb{R}$ so that the obvious identification with $\mathbb{R}^4 - \{0\}$ preserves orientation.) One can similarly change a smooth structure on a high-dimensional PL manifold by summing along a line with $\mathbb{R} \times \Sigma$ for some exotic sphere Σ. We exhibit these operations in Section 5 as well-defined monoid actions on the set of isotopy classes of smoothings of a fixed topological manifold. One can also consider CAT sums along lines in general. We discuss nonuniqueness of this latter operation in Section 3 as a prelude to discussing subtle end sums.

There are several obvious sources of nonuniqueness for attaching 1–handles at infinity. For attaching 1–handles in the compact setting, the result can depend both on orientations and on choices of boundary components. We will consider orientations in Section 4, but now recall the noncompact analogue of the set of boundary components, the space of ends of a manifold. (See eg Hughes and Ranicki [29].) This only depends on the underlying TOP structure of a CAT manifold X (and generalizes to other spaces). A *neighborhood of infinity* in X is the complement of a compact set, and a *neighborhood system of infinity* is a nested sequence $\{U_i \mid i \in \mathbb{Z}^+\}$ of neighborhoods of infinity with empty intersection and with the closure of U_{i+1} contained in U_i for all $i \in \mathbb{Z}^+$.

Definition 2.2 For a fixed neighborhood system $\{U_i\}$ of infinity, the *space of ends* of X is given by $\mathcal{E} = \mathcal{E}(X) = \varprojlim \pi_0(U_i)$.

That is, an end $\epsilon \in \mathcal{E}(X)$ is given by a sequence $V_1 \supset V_2 \supset V_3 \supset \cdots$, where each V_i is a component of U_i. For two different neighborhood systems of infinity for X, the resulting spaces $\mathcal{E}(X)$ can be canonically identified: the set is preserved when we pass to a subsequence, but any two neighborhood systems of infinity have interleaved subsequences. A *neighborhood* of the end ϵ is an open subset of X containing one of the subsets V_i. This notion allows us to topologize the set $X \cup \mathcal{E}(X)$ so that X is homeomorphically embedded as a dense open subset and $\mathcal{E}(X)$ is totally disconnected. (The new basis elements are the components of each U_i, augmented by the ends of which they are neighborhoods.) The resulting space is Hausdorff with a countable basis. If X has only finitely many components, this space is compact and called the *Freudenthal* [16] or *end compactification* of X. In this case, $\mathcal{E}(X)$ is homeomorphic to a closed subset of a Cantor set.

Ends can also be described using rays, most naturally if we allow the rays to be singular. We call a continuous, proper map $\gamma\colon S \times [0,\infty) \to X$ (with S discrete and countable) a *singular multiray*, or a *singular ray* if S is a single point. Every singular ray γ in a manifold X determines an end $\epsilon_\gamma \in \mathcal{E}(X)$. This is because γ is proper, so every neighborhood U of infinity in X contains $\gamma([k,\infty))$ for sufficiently large k, and this image lies in a single component of U. In fact, an alternative definition of $\mathcal{E}(X)$ is as the set of equivalence classes of singular rays, where two such are considered equivalent if their restrictions to $\mathbb{Z}^+$ are properly homotopic. A singular multiray $\gamma\colon S \times [0,\infty) \hookrightarrow X$ then determines a function $\epsilon_\gamma\colon S \to \mathcal{E}(X)$ that is preserved under proper homotopy of γ. Attaching 1–handles at infinity depends on these functions for γ^- and γ^+, just as attaching compact 1–handles depends on choices of boundary components, with examples of the former easily obtained from the latter by removing boundary. We will find more subtle dependence on the defining multirays in the next section, but a weak condition preventing these subtleties in Section 4.

3 Nonuniqueness

We now investigate examples of nonuniqueness in the simplest setting. In each case, we begin with an open manifold X with finitely many ends, and attach a single 1–handle at infinity, at a specified pair of ends. We assume the 1–handle respects a preassigned orientation on X. For attaching 1–handles in the compact setting, this would be enough information to uniquely specify the result, but we demonstrate that uniqueness can still fail for a 1–handle at infinity. The first author and Haggerty showed in [7] that even the proper homotopy type need not be uniquely determined; Example 3.2 below sketches the simplest construction from that paper. Our subsequent examples are more subtle, having the same proper homotopy (or even CAT′ homeomorphism) type but distinguished by their CAT homeomorphism types.

All of our examples necessarily have complicated fundamental group behavior at infinity, since Section 4 proves uniqueness when the fundamental group is suitably controlled. We obtain the required complexity by the following construction, which generalizes examples of [7]:

Definition 3.1 For an oriented CAT manifold X, let $\gamma^-, \gamma^+\colon S \times [0,\infty) \hookrightarrow X$ be multirays with disjoint images. *Ladder surgery* on X along γ^- and γ^+ is orientation-preserving surgery on the infinite family of 0–spheres given by $\{\gamma^-(s,n), \gamma^+(s,n)\}$ for each $s \in S$ and $n \in \mathbb{Z}^+$. That is, we find disjoint CAT balls centered at the points $\gamma^\pm(s,n)$, remove the interiors of the balls and glue each resulting pair of boundary spheres together by a reflection (so that the orientation of X extends).

It is not hard to verify that the resulting oriented CAT homeomorphism type only depends on the end functions $\epsilon_{\gamma^\pm}$ of the multirays; see Corollary 4.13 for details and a generalization to unoriented manifolds. If X has two components X_1 and X_2, each with k ends, any bijection from $\mathcal{E}(X_1)$ to $\mathcal{E}(X_2)$ determines a connected manifold with k ends obtained by ladder surgery with $S=\mathcal{E}(X_1)$. Such a manifold will be called a *ladder sum* of X_1 and X_2. For closed, connected, oriented $(n-1)$–manifolds M and N, we let $\mathbb{L}(M,N)$ denote the ladder sum of the two-ended n–manifolds $\mathbb{R}\times M$ and $\mathbb{R}\times N$ for the bijection preserving the ends of $\mathbb{R}$. (This is a slight departure from [7], which used the one-ended manifold $[0,\infty)$ in place of $\mathbb{R}$.) Note that any ladder surgery transforms its multirays $\gamma^\pm$ into infinite unions of circles, and surgery on all these circles (with any framings) results in the manifold obtained from X by adding 1–handles at infinity along $\gamma^\pm$. (This is easily seen by interpreting the surgeries as attaching 1– and 2–handles to $I\times X$.)

The examples in [7] are naturally presented in terms of ladder sums and attaching 1–handles at infinity. They represent the simplest type of example, where a single 1–handle may be attached at infinity in essentially distinct ways, namely an orientation-preserving end sum of one-ended manifolds.

Example 3.2 Homotopy-inequivalent end sums (one-ended) [7] For a fixed prime $p>1$, let E denote the $\mathbb{R}^2$–bundle over S^2 with Euler number $-p$ (so E has a neighborhood of infinity diffeomorphic to $\mathbb{R}\times L(p,1)$). Let Y be the ladder sum of E and $\mathbb{R}^4$. We will attach a single 1–handle at infinity to the disjoint union $X=Y\sqcup E$ in two ways to produce distinct, one-ended, boundaryless manifolds Z_0 and Z_1. Let γ_0 and γ_1 be rays in Y, with γ_0 lying in the E summand and γ_1 lying in the $\mathbb{R}^4$ summand. Let γ be any ray in E, and let Z_i be obtained from X by attaching a 1–handle at infinity along γ_i and γ. The manifolds Z_0 and Z_1 are not properly homotopy-equivalent (in fact, their ends are not properly homotopy-equivalent) since they have nonisomorphic cohomology algebras at infinity [7]. The basic idea is that both manifolds Z_i have obvious splittings as ladder sums. For Z_0, one summand is $\mathbb{R}^4$, so all cup products from $H^1(Z_0;\mathbb{Z}/p)\otimes H^2(Z_0;\mathbb{Z}/p)$ are supported in the other summand in a 1–dimensional subspace of $H^3(Z_0;\mathbb{Z}/p)$. However, Z_1 has cup products on both sides, spanning a 2–dimensional subspace.

Our remaining examples are pairs with the same homotopy type, distinguished by more subtle means.

Examples 3.3 **(a)** **Homotopy-equivalent but nonhomeomorphic sums** It should not be surprising that the sum of two manifolds along a properly embedded line in each depends on more than just the ends and orientations involved. However, as a warm-up for end sums, we give an explicit example in TOP where moving one line changes the resulting homeomorphism type but not its proper homotopy type. Let P and Q, respectively, denote $\mathbb{C}P^2$ and Freedman's fake $\mathbb{C}P^2$ (eg [14]). Then there is a homotopy-equivalence between P and Q, restricting to a pairwise homotopy-equivalence between the complements of a ball interior in each. But P and Q cannot be homeomorphic since Q is unsmoothable. The ladder sum $\mathbb{L}(P,Q)$ is an unsmoothable topological 5–manifold with two ends. The lines $\mathbb{R}\times\{p\}\subset\mathbb{R}\times P$ and $\mathbb{R}\times\{q\}\subset\mathbb{R}\times Q$ can be chosen to lie in $\mathbb{L}(P,Q)$, with each spanning the two ends of $\mathbb{L}(P,Q)$, but they are dual to two different elements of $H^4(\mathbb{L}(P,Q);\mathbb{Z}/2)$ (see [7]), with $\mathbb{R}\times\{q\}$ dual to the Kirby–Siebenmann smoothing obstruction of $\mathbb{L}(P,Q)$. Clearly, there is a proper homotopy-equivalence of $\mathbb{L}(P,Q)$ interchanging the two lines. Thus, the two resulting ways to sum $\mathbb{L}(P,Q)$ along a line with $\mathbb{R}\times\overline{Q}$ (where the orientation on Q is reversed for later convenience) give properly homotopy-equivalent manifolds, namely $\mathbb{L}(\overline{Q}\,\#\,P,Q)$ and $L(P,Q\,\#\,\overline{Q})=L(P,P\,\#\,\overline{P})$. (The last equality follows from Freedman's classification of simply connected topological 4–manifolds [14].) These two manifolds cannot be homeomorphic, since the latter is a smooth manifold whereas the former is unsmoothable, with Kirby–Siebenmann obstruction dual to a pair of lines running along opposite sides of the ladder. (A discussion of the cohomology of such manifolds can be found in [7], but, more simply, there are subsets $(a,b)\times Q$ on which the Kirby–Siebenmann obstruction must evaluate nontrivially.)

(b) **Homotopy-equivalent but nonhomeomorphic end sums** We adapt the previous example to end sums. Instead of summing along a line, we end sum $\mathbb{L}(P,Q)$ with $\mathbb{R}\times\overline{Q}$ along their positive ends in two different ways (using rays obtained from the positive ends of the previous lines). We obtain a pair of properly homotopy-equivalent, unsmoothable, three-ended manifolds. In one case, the modified end has a neighborhood that is smoothable, and in the other case, all three ends fail to have smoothable neighborhoods since the Kirby–Siebenmann obstruction cannot be avoided. Thus, we have a pair of nonhomeomorphic, but properly homotopy-equivalent, manifolds, both obtained by an orientation-preserving end sum on the same pair of ends.

There are several other variations of the construction. We can replace the $\mathbb{R}$ factor by $[0,\infty)$ so that the ladder sum is one-ended, to get an example of nonuniqueness of summing one-ended topological manifolds with compact boundary. Unfortunately, we cannot cap off the boundaries to obtain one-ended open manifolds, since the Kirby–Siebenmann obstruction is a cobordism invariant of topological 4–manifolds. However, we can modify the original ladder sum so that we do ladder surgery on the positive end, but end sum on the negative end (which then has a neighborhood homeomorphic to $\mathbb{R} \times (\overline{P} \# \overline{Q})$). Now we have a connected, two-ended open manifold whose ends can be joined by an orientation-preserving 1–handle at infinity in two different ways, yielding properly homotopy-equivalent but nonhomeomorphic one-ended manifolds, only one of which has a smoothable neighborhood of infinity.

(c) Homotopy-equivalent but not PL homeomorphic end sums In higher dimensions, the Kirby–Siebenmann obstruction of a neighborhood V of an end cannot be killed by adding 1–handles at infinity (since $H^4(V;\mathbb{Z}/2)$ is not disturbed), but we can do the analogous construction using higher smoothing obstructions. This time, we obtain PL n–manifolds (for various $n \geq 9$) that are properly homotopy-equivalent but not PL homeomorphic. Let P and Q be homotopy-equivalent PL $(n-1)$–manifolds with P and $Q - \{q_0\}$ smooth but Q unsmoothable. (For an explicit 24–dimensional pair, see Anderson [1, Proposition 5.1].) The previous discussion applies almost verbatim with PL in place of TOP, with the smoothing obstruction in $H^{n-1}(X;\Theta_{n-2})$ for PL manifolds X in place of the Kirby–Siebenmann obstruction. The one change is that smoothability of $Q \# \overline{Q}$ follows since it is the double of the smooth manifold obtained from Q by removing the interior of a PL ball centered at q_0. (This time the orientation reversal is necessary since the smoothing obstruction need not have order 2.)

Examples 3.4 (a) PL homeomorphic but nondiffeomorphic end sums (one-ended) A similar construction shows that end summing along a fixed pair of ends can produce PL homeomorphic but nondiffeomorphic manifolds. Let Σ be an exotic $(n-1)$–sphere with $n > 5$. Then Σ is PL homeomorphic to S^{n-1}, so the ladder sum $\mathbb{L}(\Sigma, S^{n-1})$ is a two-ended smooth manifold with a PL self-homeomorphism that is not isotopic to a diffeomorphism. Since $\Sigma \# \overline{\Sigma} = S^{n-1}$, summing $\mathbb{L}(\Sigma, S^{n-1})$ along a line with $\mathbb{R} \times \overline{\Sigma}$ gives the two manifolds $\mathbb{L}(S^{n-1}, S^{n-1})$ and $\mathbb{L}(\Sigma, \overline{\Sigma})$. The first of these bounds an infinite handlebody made with 0– and 1–handles, as does its universal cover. Since a contractible 1–handlebody is a ball with some boundary points removed, it follows that the universal cover of $\mathbb{L}(S^{n-1}, S^{n-1})$ embeds in S^n. However, $\mathbb{L}(\Sigma, \overline{\Sigma})$ contains copies of Σ arbitrarily close to its ends. Since any homotopy $(n-1)$–sphere with $n > 5$ that embeds in S^n cuts out a ball, so is a standard sphere, it follows that no neighborhood of either end of $\mathbb{L}(\Sigma, \overline{\Sigma})$ has a cover embedding in S^n. Thus, the two manifolds have nondiffeomorphic ends, although they are PL homeomorphic. As before, we can modify this example to get a pair of end sums of two-ended manifolds, or a pair obtained from a two-ended connected manifold by joining its ends with a 1–handle in two different ways. This time, however, we can also interpret the example as end summing two one-ended open manifolds, by first obtaining one-ended manifolds with compact boundary, then capping off the boundary. (Note that Σ bounds a compact manifold. Unlike codimension-0 smoothing existence obstructions, the uniqueness obstructions are not cobordism invariants.) The resulting pair of one-ended DIFF manifolds are now easily seen to be PL homeomorphic (by Corollary 4.9, for example) but nondiffeomorphic.

(b) Nonisotopic DIFF = PL structures on a fixed TOP 4–manifold (one-ended) The previous construction has an analogue in dimension 4, where the categories DIFF and PL coincide. Replace $\mathbb{R}\times\Sigma$ by W, Freedman's exotic $\mathbb{R}\times S^3$. This is distinguished from the standard $\mathbb{R}\times S^3$ by the classical PL uniqueness obstruction in $H^3(\mathbb{R}\times S^3;\mathbb{Z}/2)\cong\mathbb{Z}/2$, dual to $\mathbb{R}\times\{p\}$. The ladder sum L of W with $\mathbb{R}\times S^3$ can be summed along a line with W in two obvious ways. These can be interpreted as smoothings on the underlying topological manifold $\mathbb{L}(S^3,S^3)$, and can be transformed to an example of end summing one-ended DIFF manifolds as before: To transform W into a one-ended DIFF manifold, cut it in half along a Poincaré homology sphere Σ, then cap it with an E_8–plumbing. The result E is a smoothing of a punctured Freedman E_8–manifold. (Alternatively, we can take E homeomorphic to a punctured fake $\mathbb{C}P^2$.) We ladder sum with $\mathbb{R}^4$. The two results of end summing with another copy of E are identified in TOP with a ladder sum of two copies of E (see Corollary 4.9). The smoothings are nonisotopic (even stably, ie after Cartesian product with $\mathbb{R}^k$), since the uniqueness obstruction by which they differ near infinity is dual to a pair of lines on opposite sides of the ladder. However, the authors have not been able to distinguish their diffeomorphism types. The problem with the previous argument is that the sum of two copies of W along a line is not diffeomorphic to $\mathbb{R}\times S^3$ (although the classical invariant vanishes). While W contains a copy of Σ separating its ends, so cannot embed in S^4, the sum of two copies of W contains $\Sigma\#\Sigma$, which also does not embed in S^4. The effect of summing with reversed orientation or switched ends, or replacing Σ by a different homology sphere, is less clear. This leads to the following question, which is discussed further in Section 5 (Question 5.6):

Question 3.5 *Are there two exotic smoothings on $\mathbb{R}\times S^3$ whose sum along a line is the standard $\mathbb{R}\times S^3$?*

If such smoothings exist, one of which has the additional property that every neighborhood of one end has a slice $(a,b)\times S^3$ (as seen in TOP) that cannot smoothly embed in S^4, then the method of Example 3.4(a) gives two one-ended open 4–manifolds that can be end summed in two homeomorphic but not diffeomorphic (or PL homeomorphic) ways.

4 Uniqueness for Mittag-Leffler ends

Having examined the failure of uniqueness in the last section, we now look for hypotheses that guarantee that 1–handle attaching at infinity *is* unique. There are several separate issues to deal with. In the compact setting, attaching a 1–handle to given boundary components can yield two different results if both boundary components are orientable, so uniqueness requires specified orientations in that case. The same issue arises for 1–handles at infinity. Beyond that, we must consider the dependence on the involved multirays. Since rays in $\mathbb{R}^3$ can be knotted, uncountably many homeomorphism types of contractible manifolds arise as end sums of two copies of $\mathbb{R}^3$ (Myers [35]; see also Calcut and Haggerty [7]). Thus, we assume more than 3 dimensions and conclude, not surprisingly, that the multirays affect the result only through their proper homotopy classes, and that the choices of (suitably oriented) tubular neighborhood maps cause no additional difficulties. We have already seen that different rays determining the same end can yield different results for end summing with another fixed manifold and ray, but we give a weak group-theoretic condition on an end that entirely eliminates dependence on the choice of rays limiting to it.

We begin with terminology for orientations. We will call an end ϵ of an n–manifold X *orientable* if it has an orientable neighborhood in X. An orientation on one connected, orientable neighborhood of ϵ determines an orientation on every other such neighborhood, through the component of their intersection that is a neighborhood of ϵ. Such a compatible choice of orientations will be called an *orientation of* ϵ, so every orientable end has two orientations. We let $\mathcal{E}_{\mathrm{O}}\subset\mathcal{E}(X)$ denote the open subset of orientable ends of X. (This need not be closed, as seen by deleting a sequence of points of X converging to a nonorientable end.) If γ is a singular multiray in a DIFF manifold X, the tangent bundle of X pulls back to a trivial bundle γ^*TX over $S\times[0,\infty)$. A fiber orientation on this bundle will be called a *local orientation of* X *along* γ, and if such an orientation is specified, γ will be called *locally orienting*. We apply the same terminology in PL and TOP, using the appropriate analogue of the tangent bundle, or, equivalently but more simply, using local homology groups $H_n(X, X - \{\gamma(s,t)\}) \cong \mathbb{Z}$. If γ is a (nonsingular) CAT multiray, a CAT tubular neighborhood map ν induces a local orientation of X along γ; if this agrees with a preassigned local orientation along γ, then ν will be called *orientation preserving*. A homotopy between two singular multirays determines a correspondence between their local orientations (eg by pulling back the tangent bundle to the domain of the homotopy). If a singular ray γ determines an orientable end $\epsilon_\gamma \in \mathcal{E}_{\mathrm{O}}$, then a local orientation along γ induces an orientation on the end, since $\gamma([k,\infty))$ lies in a connected, orientable neighborhood of ϵ_γ when k is sufficiently large.

We now turn to the group theory of ends. See Geoghegan [17] for a more detailed treatment. An *inverse sequence of groups* is a sequence $G_1 \leftarrow G_2 \leftarrow G_3 \leftarrow \cdots$ of groups and homomorphisms. We suppress the homomorphisms from the notation, since they will be induced by obvious inclusions in our applications. A *subsequence* of an inverse sequence is another inverse sequence obtained by passing to a subsequence of the groups and using the obvious composites of homomorphisms. Passing to a subsequence and its inverse procedure, along with isomorphisms commuting with the maps, generate the standard notion of equivalence of inverse sequences.

Definition 4.1 An inverse sequence $G_1 \leftarrow G_2 \leftarrow G_3 \leftarrow \cdots$ of groups is called *Mittag-Leffler* (or *semistable*) if for each $i \in \mathbb{Z}^+$ there is a $j \geq i$ such that all G_k with $k \geq j$ have the same image in G_i.

Clearly, a subsequence is Mittag-Leffler if and only if the original sequence is, so the notion is preserved by equivalences. After passing to a subsequence, we may assume $j = i + 1$ in the definition.

For a manifold X with a singular ray γ and a neighborhood system $\{U_i\}$ of infinity, we reparametrize γ so that $\gamma([i,\infty))$ lies in U_i for each $i \in \mathbb{Z}^+$.

Definition 4.2 The *fundamental progroup* of X based at γ is the inverse sequence of groups $\pi_1(U_i, \gamma(i))$, where the homomorphism $\pi_1(U_{i+1}, \gamma(i+1)) \to \pi_1(U_i, \gamma(i))$ is the inclusion-induced map to $\pi_1(U_i, \gamma(i+1))$ followed by the isomorphism moving the basepoint to $\gamma(i)$ along the path $\gamma|_{[i,i+1]}$.

This only depends on the TOP structure of X. Passing to a subsequence of $\{U_i\}$ replaces the fundamental progroup by a subsequence of it. Since any two neighborhood systems of infinity have interleaved subsequences, the fundamental progroup is independent, up to equivalence, of the choice of neighborhood system. It is routine to check that it is similarly preserved by any proper homotopy of γ, so it only depends on X and the proper homotopy class of γ. Furthermore, the inverse sequence is unchanged if we replace each U_i by its connected component containing $\gamma([i,\infty))$, so it is equivalent to use a neighborhood system of the end ϵ_γ. Beware, however, that even if there is only one end, the choice of proper homotopy class of γ can affect the fundamental progroup, and even whether its inverse limit vanishes. (See [17, Example 16.2.4]. The homomorphisms in the example are injective, but changing γ conjugates the resulting nested subgroups, changing their intersection.)

We call the pair (X,γ) *Mittag-Leffler* if its fundamental progroup is Mittag-Leffler. We will see in Lemma 4.11(a) below that this condition implies γ is determined up to proper homotopy by its induced end ϵ_γ, so the fundamental progroup of ϵ_γ is independent of γ in this case, and it makes sense to call ϵ_γ a *Mittag-Leffler end*. Note that this condition rules out ends made by ladder surgery, and hence the examples of Section 3. We will denote the set of Mittag-Leffler ends of X by $\mathcal{E}_{\mathrm{ML}} \subset \mathcal{E}(X)$, and its complement by $\mathcal{E}_{\mathrm{bad}}$.

Many important types of ends are Mittag-Leffler. *Simply connected* ends are (essentially by definition) the special case for which the given images all vanish. *Topologically collared* ends, with a neighborhood homeomorphic to $\mathbb{R}\times M$ for some compact $(n-1)$–manifold M, are *stable*, the special case for which the fundamental progroup is equivalent to an inverse sequence with all maps isomorphisms. Other important ends are neither simply connected nor collared, but still Mittag-Leffler if the maps are nontrivial surjections (Example 4.5). Any end admits a neighborhood system for which the maps are not even surjective, obtained from an arbitrary system by adding 1–handles to each U_i inside U_{i-1}; such ends may still be Mittag-Leffler. In the smooth category, we can analyze ends using a Morse function φ that is exhausting (ie proper and bounded below). For such a function, the preimages $\varphi^{-1}(i,\infty)$ for $i\in\mathbb{Z}^+$ form a neighborhood system of infinity.

Proposition 4.3 *Let X be a* DIFF *open n–manifold. If an end ϵ of X is not Mittag-Leffler, then for every exhausting Morse function φ on X and every $t\in\mathbb{R}$, there are infinitely many critical points of index $n-1$ in the component of $\varphi^{-1}(t,\infty)$ containing ϵ. In particular, if X admits an exhausting Morse function with only finitely many index-$(n-1)$ critical points, then all of its ends are Mittag-Leffler.*

Proof After perturbing φ and composing it with an orientation-preserving diffeomorphism of $\mathbb{R}$, we can assume each $\varphi^{-1}[i,i+1]$ is an elementary cobordism. Since ϵ is not Mittag-Leffler, its corresponding fundamental progroup must have infinitely many homomorphisms that are not surjective. Thus, there are infinitely many values of i for which $\varphi^{-1}[i,\infty)$ is made from $\varphi^{-1}[i+1,\infty)$ by attaching a 1–handle with at least one foot in the component of the latter containing ϵ. This handle corresponds to an index-1 critical point of $-\varphi$, or an index-$(n-1)$ critical point of φ. □

The Mittag-Leffler condition on an end of a CAT manifold is determined by its underlying TOP structure (in fact, by its proper homotopy type), so we are free to change the smooth structure on a manifold before looking for a suitable Morse function. This is especially useful in dimension 4. For example, an exhausting Morse function on an exotic $\mathbb{R}^4$ with nonzero Taylor invariant must have infinitely many index-3 critical points [39], but after passing to the standard structure, there is such a function with a unique critical point. (Furthermore, an exotic $\mathbb{R}^4$ is topologically collared and simply connected at infinity.) Proposition 4.3 is most generally stated in TOP, using topological Morse functions. (These are well behaved [31] and can be constructed from handle decompositions, which exist on all open TOP manifolds; see eg [14].)

Since every Stein manifold of complex dimension m (real dimension $2m$) has an exhausting Morse function with indices at most m, we conclude:

Corollary 4.4 *For every Stein manifold of complex dimension at least 2, the unique end of each component is Mittag-Leffler.* □

Example 4.5 For infinite-type Stein surfaces ($m=2$), the ends must be Mittag-Leffler, but they are typically neither simply connected nor stable (and hence not topologically collared). This is more generally typical for open 4–manifolds whose exhausting Morse functions require infinitely many critical points, but none of index above 2. As a simple example, let X be an infinite end sum of $\mathbb{R}^2$–bundles over S^2. (Its diffeomorphism type is independent of the choice of rays, by Theorems 4.6 and 6.1, but it is convenient to think of the bundles as indexed by $\mathbb{Z}^+$ and summed consecutively.) If each Euler number is less than -1, then X will be Stein. We get a neighborhood system of infinity with each U_i obtained from a collar of the end of the first i–fold sum by attaching the remaining (simply connected) summands. Then each group G_i is a free product of i cyclic groups, and each homomorphism is surjective, projecting out one factor. The inverse limit is not finitely generated, so the end is not stable. (Every neighborhood system of the end has a subsequence that can be interleaved by some of our neighborhoods U_i.)

We can now state our main theorem on uniqueness of attaching 1–handles. Its primary conclusion is that when we attach 1–handles at infinity, any locally orienting defining ray that determines a Mittag-Leffler end will affect the outcome only through the end and local orientation it determines. If the end is also nonorientable, then even the local orientation has no influence (as for a compact 1–handle attached to a nonorientable boundary component). To state this in full generality, we also allow rays determining ends that are not Mittag-Leffler, which are required to remain in a fixed proper homotopy class. That is, we allow an arbitrary multiray γ, but require its restriction to the subset $\epsilon_\gamma^{-1}(\mathcal{E}_{\mathrm{bad}})$ of the index set S (corresponding to rays determining ends that are not Mittag-Leffler) to lie in a fixed proper homotopy class. For each 1–handle with at least one defining ray determining a nonorientable Mittag-Leffler end, no further constraint is necessary, but otherwise we keep track of orientations. We do this through orientations of the end if they exist. In the remaining case, the end is not Mittag-Leffler, and we compare the local orientations of the rays through a proper homotopy. More precisely, we have:

Theorem 4.6 *For a* CAT *n–manifold X with $n \geq 4$, discrete S and $i = 0, 1$, let $\gamma_i^-, \gamma_i^+ \colon S \times [0, \infty) \hookrightarrow X$ be locally orienting* CAT *multirays whose images (for each fixed i) are disjoint, and whose end functions $\epsilon_{\gamma_i^\pm} \colon S \to \mathcal{E}(X)$ are independent of i. Suppose that*

- (a) *after γ_0^- and γ_1^- are restricted to the index subset $\epsilon_{\gamma_0^-}^{-1}(\mathcal{E}_{\mathrm{bad}})$, there is a proper homotopy between them;*
- (b) *for each $s \in \epsilon_{\gamma_0^-}^{-1}(\mathcal{E}_{\mathrm{bad}} \cup \mathcal{E}_{\mathrm{O}}) \cap \epsilon_{\gamma_0^+}^{-1}(\mathcal{E}_{\mathrm{bad}} \cup \mathcal{E}_{\mathrm{O}})$, the local orientations of the corresponding rays in γ_0^- and γ_1^- induce the same orientation of the end if there is one, and otherwise correspond under the proper homotopy of* (a)*;*
- (c) *the two analogous conditions apply to γ_i^+.*

Let Z_i be the result of attaching 1–handles to X along $\gamma_i^\pm$ (for any choice of orientation-preserving tubular neighborhood maps $\nu_i^\pm$). Then there is a CAT *homeomorphism from Z_0 to Z_1 sending the submanifold X onto itself by a* CAT *homeomorphism* CAT *ambiently isotopic in X to the identity map.*

It follows that 1–handle attaching is not affected by reparametrization of the rays (a proper homotopy), or changing the auxiliary diffeomorphisms $\varphi^\pm$ and $\rho^\pm$ occurring in Definition 2.1 (which only results in changing the parametrization and tubular neighborhood maps, respectively).

Corollary 4.7 *For an oriented* CAT *n–manifold X with $n \geq 4$, every countable multiset of* (*unordered*) *pairs of Mittag-Leffler ends canonically determines a* CAT *manifold obtained from X by attaching 1–handles at infinity to those pairs of ends, respecting the orientation.* □

Since the end of $\mathbb{R}^n$ is Mittag-Leffler, we immediately obtain cancellation of 0/1–handle pairs at infinity:

Corollary 4.8 *For $n \geq 4$, every end sum of a* CAT *n–manifold X with $\mathbb{R}^n$* (*or countably many copies of $\mathbb{R}^n$*) *is* CAT *homeomorphic to X.* □

See Section 6 for further discussion of 0–handles at infinity. This corollary shows that end summing with an exotic $\mathbb{R}^4$ doesn't change the homeomorphism type of a smooth 4–manifold (although it typically changes its diffeomorphism type); see Section 5. It also shows:

Corollary 4.9 *Suppose X_0 and X_1 are connected, oriented* CAT *n–manifolds with $n \geq 4$, and that X_0 has an end ϵ that is* CAT *collared by S^{n-1}. Then all manifolds obtained as the oriented end sum of X_0 with X_1 at the end ϵ are* CAT *homeomorphic.*

Proof Write X_0 as a connected sum $X \# \mathbb{R}^n$. Then any such end sum is $X \# X_1$. □

The following corollary shows that 1–handles at infinity respect Stein structures. This will be applied to 4–manifold smoothing theory in Theorem 5.4.

Corollary 4.10 *Every manifold Z obtained from a Stein manifold X by attaching 1–handles at infinity, respecting the complex orientation, admits a Stein structure. The resulting almost-complex structure on Z can be assumed to restrict to the given one on X, up to homotopy.*

Proof Since every open, oriented surface has a Stein structure and a contractible space of almost-complex structures, we assume X has real dimension $2m \geq 4$. Since X is Stein, it has an exhausting Morse function with indices at most m. It can then be described as the interior of a smooth (self-indexed) handlebody whose handles have index at most m. This is well known when there are only finitely many critical points. A proof of the infinite case is given in the appendix of [21], which also shows that when $m = 2$ one can preserve the extra framing condition that arises for 2–handles, encoding the given almost-complex structure. By Corollaries 4.4 and 4.7, we can realize the 1–handles at infinity by attaching compact handles to the handlebody before passing to the interior (after adding infinitely many canceling $0/1$ pairs if necessary to accommodate infinitely many new 1–handles, avoiding compactness issues). Now we can convert the handlebody interior back into a Stein manifold by Eliashberg's theorem; see [10]. The almost-complex structures then correspond by construction. □

The proof of Theorem 4.6 follows from two lemmas. The first guarantees that (a) Mittag-Leffler ends are well defined and (b) singular multirays with a given Mittag-Leffler end function are unique up to proper homotopy.

Lemma 4.11 (a) *If* (X,γ) *is a Mittag-Leffler pair, then every singular ray determining the same end as* γ *is properly homotopic to* γ*. In particular, the Mittag-Leffler condition for ends is independent of choice of singular ray, so the subset* $\mathcal{E}_{\mathrm{ML}} \subset \mathcal{E}$ *is well defined.*

(b) *Let* $\gamma_0, \gamma_1\colon S\times[0,\infty)\hookrightarrow X$ *be locally orienting singular multirays with the same end function. Suppose that this function* $\epsilon_{\gamma_0} = \epsilon_{\gamma_1}$ *has image in* $\mathcal{E}_{\mathrm{ML}}$*, and that for each* s *with* $\epsilon_{\gamma_0}(s) \in \mathcal{E}_{\mathrm{O}}$*, the corresponding locally orienting singular rays of* γ_0 *and* γ_1 *induce the same orientation (depending on* s*) of the end* $\epsilon_{\gamma_0}(s)$*. Then there is a proper homotopy from* γ_0 *to* γ_1*, respecting the given local orientations.*

The first sentence and its converse are essentially Proposition 16.1.2 of Geoghegan [17], which is presented as an immediate consequence of two earlier statements: Proposition 16.1.1 asserts that the set of proper homotopy classes of singular rays approaching an arbitrary end corresponds bijectively to the derived limit $\varprojlim^1 \pi_1(U_i,\gamma(i))$ of a neighborhood system U_i of infinity; Theorem 11.3.2 asserts that an inverse sequence of countable groups G_i is Mittag-Leffler if and only if $\varprojlim^1 G_i$ has only one element. We follow those proofs but considerably simplify the argument, eliminating use of derived limits, by focusing on the Mittag-Leffler case. This reveals the underlying geometric intuition: If an end ϵ is topologically collared by a neighborhood identified with $\mathbb{R}\times M$, and $\gamma=(\gamma_{\mathbb{R}},\gamma_M)\colon [0,\infty)\to\mathbb{R}\times M$ is a singular ray, we can assume after a standard proper homotopy of the first component that $\gamma_{\mathbb{R}}\colon [0,\infty)\to\mathbb{R}$ is inclusion. Then the proper homotopy $\gamma_s(t) = \bigl(t,\gamma_M((1-s)t)\bigr) = \frac{1}{1-s}\gamma((1-s)t)$ (where the last multiplication acts only on the first factor) stretches the image of γ, pushing any winding in M out toward infinity, so that when $s\to 1$ the ray becomes a standard radial ray. If, instead, ϵ only has a neighborhood system with π_1–surjective inclusions, we can compare two singular rays using an initial proper homotopy after which they agree on $\mathbb{Z}^+ \subset [0,\infty)$, and so only differ by a proper sequence of loops. Then π_1–surjectivity again allows us to push the differences out to infinity: inductively collapse loops by transferring their homotopy classes to more distant neighborhoods of infinity, so that the resulting homotopy sends one ray to the other. In the general Mittag-Leffler case, we still have enough surjectivity to push each loop to infinity after pulling it back a single level in the neighborhood system (with properness preserved because we only pull back one level). The following proof efficiently encodes this procedure with algebra:

Proof First we prove (a), showing that an arbitrary singular ray γ' determining the same Mittag-Leffler end as γ is properly homotopic to it. We also keep track of preassigned local orientations along the two singular rays. If ϵ_γ is orientable, we assume these local orientations induce the same orientation on ϵ_γ (as in (b)). Let $\{U_i\}$ be a neighborhood system of infinity, arranged (by passing to a subsequence if necessary) so that each j is $i+1$ in the definition of the Mittag-Leffler condition, and that the component of U_1 containing ϵ_γ is orientable if ϵ_γ is. Then reparametrize γ so that each $\gamma([i,\infty))$ lies in U_i. Reparametrize γ' similarly, then arrange it to agree with γ on $\mathbb{Z}^+$ by inductively moving γ' near each $i \in \mathbb{Z}^+$ separately, with compact support inside U_i. The limiting homotopy is then well defined and proper. If ϵ_γ is nonorientable, then so is the relevant component of each U_i, so we can assume (changing the homotopy via orientation-reversing loops as necessary) that the local orientations along the two singular rays agree at each i. (This is automatic when ϵ_γ is orientable.) The two singular rays now differ by a sequence of orientation-preserving loops, representing classes $x_i \in \pi_1(U_i, \gamma(i))$ for each $i \geq 1$. Inductively choose orientation-preserving classes $y_i \in \pi_1(U_i, \gamma(i))$ for all $i \geq 2$ starting from an arbitrary y_2, and for $i \geq 1$ choosing $y_{i+2} \in \pi_1(U_{i+2}, \gamma(i+2))$ to have the same image in $\pi_1(U_i, \gamma(i))$ as $x_{i+1}^{-1} y_{i+1} \in \pi_1(U_{i+1}, \gamma(i+1))$. (This is where the Mittag-Leffler condition is necessary.) For each $i \geq 1$, let $z_i = x_i y_{i+1} \in \pi_1(U_i, \gamma(i))$ (where we suppress the inclusion map). In that same group, we then have $z_i z_{i+1}^{-1} = x_i y_{i+1} y_{i+2}^{-1} x_{i+1}^{-1} = x_i$. After another proper homotopy, we can assume the two singular rays and their induced local orientations on X agree along $\frac{1}{2}\mathbb{Z}^+$ and give the sequence $z_1, z_2^{-1}, z_2, z_3^{-1}, \dots$ in $U_1, U_1, U_2, U_2, \dots$. Now a proper homotopy fixing $\mathbb{Z}^+ + \frac{1}{2}$ cancels all loops between these points and eliminates z_1 (moving $\gamma'(0)$), so that the two singular rays coincide. This completes the proof of (a), and also (since $\mathcal{E}_{\mathrm{ML}}$ is now well defined) the case of (b) with S a single point.

For the general case of (b), we wish to apply the previous case to each pair of singular rays separately. The only issue is properness of the resulting homotopy of singular multirays. Let $\{W_j\}$ be a neighborhood system of infinity with $W_1 = X$. For each $s \in S$, find the largest j such that W_j contains both rays indexed by s, and apply the previous case inside that W_j. Since the singular multirays are proper, each W_j contains all but finitely many pairs of singular rays, guaranteeing that the combined homotopy is proper. □

Remark To see the correspondence of this proof with the geometric description, first consider the case with all inclusion maps π_1–surjective. Then the argument simplifies: We can just define $z_1 = 1$, and inductively choose z_{i+1} to be any pullback of $x_i^{-1} z_i$. Then z_i is a pullback of $(x_1 \cdots x_{i-1})^{-1}$ to U_i, exhibiting the loops being transferred toward infinity.

To upgrade a proper homotopy of multirays to an ambient isotopy, we need the following lemma:

Lemma 4.12 *Suppose that X is a* CAT *n–manifold with $n \geq 4$ and Y is a* CAT *1–manifold with $b_1(Y) = 0$. Let $\Gamma\colon I \times Y \hookrightarrow \operatorname{Int} X$ be a topological proper homotopy, between* CAT *embeddings γ_i for $i = 0, 1$ that extend to* CAT *tubular neighborhood maps $\nu_i\colon Y \times \mathbb{R}^{n-1} \hookrightarrow X$ whose local orientations correspond under Γ. Then there is a* CAT *ambient isotopy $\Phi\colon I \times X \to X$, supported in a preassigned neighborhood of $\operatorname{Im}\Gamma$, such that $\Phi_0 = \mathrm{id}_X$ and $\Phi_1 \circ \nu_0$ agrees with ν_1 on a neighborhood of $Y \times \{0\}$ in $Y \times \mathbb{R}^{n-1}$.*

This lemma is well known when CAT $=$ DIFF or PL, but a careful proof seems justified by the subtlety of noncompactness: the corresponding statement in $\mathbb{R}^3$ is false even with Γ a proper (nonambient) isotopy of $Y = \mathbb{R}$. (Such an isotopy Γ can slide a knot out to infinity, changing the fundamental group of the complement, and this can even be done while fixing the integer points of $\mathbb{R}$.) The case CAT $=$ TOP is also known to specialists. We did not find a theorem in the literature from which it follows immediately. Instead, we derive it from much stronger results of Dancis [12] with antecedents dating back to pioneering work of Homma [28].

Proof First we solve the case CAT $=$ DIFF. By transversality, we may assume (after an ambient isotopy that we absorb into Φ) that γ_0 and γ_1 have disjoint images. Then we properly homotope Γ rel $\partial I \times Y$ to be smooth and generic, so it is an embedding if

$n \geq 5$ and an immersion with isolated double points if $n = 4$. After decomposing Y as a cell complex with 0–skeleton Y_0, we can assume Γ restricts to a smooth embedding on some neighborhood of $I \times Y_0$. Then there is a tubular neighborhood J of Y_0 in Y such that $\Gamma|_{I \times J}$ extends to an ambient isotopy. (Apply the isotopy extension theorem separately in disjoint compact neighborhoods of the components of $\Gamma(I \times Y_0)$.) After using this ambient isotopy to define Φ for parameter $t \leq \frac{1}{2}$, it suffices to assume Γ fixes J, and view Γ as a countable collection of path homotopies of the 1–cells of Y. We need the resulting immersed 2–disks to be disjoint. This is automatic when $n \geq 5$, but is the step that fails for knotted lines in $\mathbb{R}^3$. For $n = 4$, we push the disks off of each other by finger moves. This operation preserves properness of Γ since each compact subset of X initially intersects only finitely many disks, which have only finitely many intersections with other disks (and we do not allow finger moves over other fingers). Now we can extend to an ambient isotopy, working in disjoint compact neighborhoods of the disks. We arrange ν_0 to correspond with ν_1 by uniqueness of tubular neighborhoods and contractibility of the components of Y.

We reduce the PL and TOP cases to DIFF. As before, we can assume the images of γ_0 and γ_1 are disjoint. (We did not find a clean TOP statement of this. However, we can easily arrange $\gamma_0(Y_0)$ to be disjoint from $\gamma_1(Y)$, then apply [12, General Position Lemma 3]. While this lemma assumes the moved manifold is compact and without boundary, we can apply it to the remaining 1–cells of $\gamma_0(Y)$ by arbitrarily extending them to circles.) A tubular neighborhood N of $\gamma_0(Y) \sqcup \gamma_1(Y)$ now inherits a smoothing Σ from the maps ν_i. If $n = 4$, Σ extends over the entire manifold X except for one point in each compact component [14]. Homotoping Γ off of these points, we reduce to the case CAT $=$ DIFF. If $n \geq 5$, we again homotope Γ rel $\partial I \times Y$ to an embedding. (Again we found no clean TOP statement, but it follows by smoothing Γ on $\Gamma^{-1}(N)$, homotoping so that $\Gamma^{-1}(N)$ is a collar of $\partial I \times Y$, and applying [12, Corollary 6.1] in $X - N$.) Since $(I, \partial I) \times Y$ has no cohomology above dimension 1, there is no obstruction to extending Σ over a neighborhood of the image of Γ, again reducing to CAT $=$ DIFF. □

Proof of Theorem 4.6 For each $i = 0, 1$, the two multirays γ_i^- and γ_i^+ can be thought of as a single multiray γ_i with index set $S^* = S \times \{-1, 1\}$. For each index $(s, \sigma) \in \epsilon_{\gamma_0}^{-1}(\mathcal{E}_O) \subset S^*$, we arrange for the corresponding locally orienting rays in γ_0 and γ_1 to induce the same orientation of the end: If this is not already true, then hypothesis (b) of the theorem implies that the opposite end $\epsilon_{\gamma_0}(s, -\sigma)$ is Mittag-Leffler but nonorientable. In this case, reverse the local orientations along both rays in γ_1 parametrized by s. This corrects the orientations without changing Z_1, since the change extends as a reflection of the 1–handle $\{s\}\times[0,1]\times\mathbb{R}^{n-1}$. Now split γ_i into two multirays γ_i^{ML} and γ_i^{bad}, according to whether the rays determine Mittag-Leffler ends. By hypothesis (a), we have a proper homotopy from γ_0^{bad} to γ_1^{bad}, which respects the local orientations by hypothesis (b) after further possible flips as above when the opposite end is Mittag-Leffler but nonorientable. Lemma 4.11(b) then gives a proper homotopy from γ_0^{ML} to γ_1^{ML} respecting local orientations. Reassembling the multirays, we obtain a proper homotopy from γ_0 to γ_1 that respects local orientations. Now we apply Lemma 4.12 with $Y=S^*\times[0,\infty)$, and ν_i the given tubular neighborhood map for γ_i (after the above flips). We obtain a CAT ambient isotopy Φ of id_X such that $\Phi_1\circ\nu_0$ agrees with ν_1 on a neighborhood N of $S^*\times[0,\infty)\times\{0\}$ in $S^*\times[0,\infty)\times\mathbb{R}^{n-1}$. Note that the quotient space Z_i does not change if we cut back the 1–handles $S\times[0,1]\times\mathbb{R}^{n-1}$ to any neighborhood N' of $S\times\{\frac{1}{2}\}\times\mathbb{R}^{n-1}$ and use the restricted gluing map. Recall that the gluing map factors through an $\mathbb{R}^{n-1}$–bundle map $\mathrm{id}_S\times\varphi^{\pm}\times\rho^{\pm}$ to $S^*\times[0,\infty)\times\mathbb{R}^{n-1}$. We can assume that the resulting image of N' lies in some disk bundle (with radii increasing along the rays) inside $S^*\times[0,\infty)\times\mathbb{R}^{n-1}$. A smooth ambient isotopy supported inside a larger disk bundle moves this image into N. Conjugating with ν_i gives a CAT ambient isotopy $\Psi_{(i)}$ on X. Then $\Phi'=\Psi_{(1)}^{-1}\circ\Phi\circ\Psi_{(0)}$ is a CAT ambient isotopy for which $\Phi'_1\circ\nu_0$ agrees with ν_1 on N'. The CAT homeomorphism Φ'_1 extends to one sending Z_0 to Z_1 with the required properties. □

We can now address uniqueness of ladder surgeries. Note that their definition immediately extends to unoriented manifolds, provided that we use locally orienting multirays.

Corollary 4.13 *For a* CAT *manifold X, discrete S and $i=0,1$, let $\gamma_i^{\pm}\colon S\times[0,\infty)\hookrightarrow X$ be locally orienting* CAT *multirays with disjoint images (for each fixed i) such that the end functions $\epsilon_{\gamma_i^{\pm}}\colon S\to\mathcal{E}(X)$ are independent of i. Suppose that for each $s\in\epsilon_{\gamma_0^-}^{-1}(\mathcal{E}_{\mathrm{O}})\cap\epsilon_{\gamma_0^+}^{-1}(\mathcal{E}_{\mathrm{O}})$, the local orientations of the corresponding rays in $\gamma_i^{\pm}$ induce the same orientation of the end for $i=0,1$. Then the manifolds Z_i obtained by ladder surgery on X along $\gamma_i^{\pm}$ are* CAT *homeomorphic.*

Proof As in the previous proof, we assume that each ray of $\gamma_0^{\pm}$ determining an orientable end induces the same orientation of that end as the corresponding ray of $\gamma_1^{\pm}$, after reversing orientations on some mated pairs of rays (with the mate determining a nonorientable end). Since the end functions are independent of i, there is a proper homotopy of $\gamma_0^{\pm}$ for each choice of sign, after which $\gamma_i^{\pm}(s,n)$ is independent of i for each $s\in S$ and $n\in\mathbb{Z}^+$ (as in the proof of Lemma 4.11). We can assume the local orientations agree at each of these points, after possibly changing the homotopy on each ray determining a nonorientable end. The proper homotopy of $\gamma_0^{\pm}|_{S\times\mathbb{Z}^+}$ extends to an ambient isotopy as in the proof of Lemma 4.12, without dimensional restriction (since we only deal with the 0–skeleton Y_0). □

5 Smoothings of open 4–manifolds

Recall from Section 2 that end summing with an exotic $\mathbb{R}^4$ can be defined as an operation on the smooth structures of a fixed topological 4–manifold, and that one can similarly change smoothings of n–manifolds by summing with an exotic $\mathbb{R}\times S^{n-1}$ along a properly embedded line. (The latter is most interesting when $n=4$, but the comparison with higher dimensions is illuminating.) We now address uniqueness of both operations, expressing them as monoid actions on the set of isotopy classes of smoothings of a topological manifold. We define an *action* of a monoid $\mathcal{M}$ on a set $\mathcal{S}$ by analogy with group actions: each element of $\mathcal{M}$ is assigned a function $\mathcal{S}\to\mathcal{S}$, with the identity of $\mathcal{M}$ assigned $\mathrm{id}_{\mathcal{S}}$, and with monoid addition corresponding to composition of functions in the usual way.

We first consider end summing with an exotic $\mathbb{R}^4$. The second author showed in [19] that the set $\mathcal{R}$ of oriented diffeomorphism types of smooth manifolds homeomorphic to $\mathbb{R}^4$ admits the structure of a commutative monoid under end sum, with identity given by the standard $\mathbb{R}^4$, and such that countable sums are well defined and independent of order and grouping. (Infinite sums were defined as simultaneously end summing onto the standard $\mathbb{R}^4$ along a multiray in the latter. Thus, the statement follows from Theorem 4.6 with the two multirays γ_i^+ in $\mathbb{R}^4$ differing by a permutation of S, and with Corollary 4.8 addressing grouping; see also Section 6.) For any set S, the Cartesian product $\mathcal{R}^S$ inherits a monoid structure with the same properties, as does the submonoid $\mathcal{R}_c^S$ of S–tuples that are the identity except in countably many coordinates. Note that every action by such a monoid inherits a notion of infinite iteration, since we can sum infinitely many monoid elements together before applying them. In the case at hand, we obtain the following corollary of the lemmas of the previous section. We again split a multiray $\gamma\colon S\times[0,\infty)\to X$ into two multirays $\gamma_{\mathrm{ML}}\colon S_{\mathrm{ML}}\times[0,\infty)\to X$ and $\gamma_{\mathrm{bad}}\colon S_{\mathrm{bad}}\times[0,\infty)\to X$, according to which rays determine Mittag-Leffler ends.

Corollary 5.1 *Let X be a* TOP *4–manifold with a locally orienting* TOP *multiray $\gamma\colon S\times[0,\infty)\to X$. Then γ determines an action of $\mathcal{R}^S$ on the set $\mathcal{S}(X)$ of isotopy classes of smoothings of X. The action only depends on the proper homotopy class of* *the locally orienting multiray* γ_{bad}, *the function* $\epsilon_{\gamma_{\mathrm{ML}}}$ *and the subset of* S_{ML} *inducing a preassigned orientation on the orientable ends. In particular*, *if* X *is oriented* (*or orientations are specified on all orientable Mittag-Leffler ends*) *then the monoid* $\mathcal{R}_c^{\mathcal{E}_{\mathrm{ML}}(X)}$ *acts canonically on* $\mathcal{S}(X)$.

Note that orientation reversal induces an involution on the monoid $\mathcal{R}$, and changing the local orientations of γ changes the action by composing with this involution on the affected factors of $\mathcal{R}^S$.

Proof To define the action, fix a smoothing on X and an indexed set $\{R_s \mid s \in S\}$ of elements of $\mathcal{R}$. According to Quinn [36] — see also [14] — γ can be made smooth by a TOP ambient isotopy. For each $s \in S$, choose a smooth ray γ' in R_s, and use it to sum R_s with X along the corresponding ray in X. We do this by homeomorphically identifying the complement of a tubular neighborhood of γ' (with smooth $\mathbb{R}^3$ boundary) with a corresponding closed tubular neighborhood of the ray in X (preserving orientations), then transporting the smoothing of R_s to X. We assume the identification is smooth near each boundary $\mathbb{R}^3$, and then the smoothing fits together with the given one on the rest of X. This process can be performed simultaneously for all $s \in S$, provided that we work within a closed tubular neighborhood of γ. Each ray γ' is unique up to smooth ambient isotopy (Lemma 4.12), and the required identifications of neighborhoods (homeomorphic to the half-space $[0,\infty) \times \mathbb{R}^3$) are unique up to topological ambient isotopy that is smooth on the boundary (by the Alexander trick), so the resulting isotopy class of smoothings on X is independent of choices made in the R_s summands. Similarly, the resulting smoothing is changed by an isotopy if the original smoothing of X is isotoped or γ is changed by a proper homotopy (Lemma 4.12 again). In particular, the initial choice of smoothing of γ does not matter. Since the proper homotopy class of the locally orienting multiray γ_{ML} is determined by $\epsilon_{\gamma_{\mathrm{ML}}}$ and the orientation data (Lemma 4.11(b)), we have a well-defined function $\mathcal{S}(X) \to \mathcal{S}(X)$ determined by an element of $\mathcal{R}^S$ and the data given in the corollary.

The rest of the corollary is easily checked. To verify that we have a monoid action, consecutively apply two elements $\{R_s\}$ and $\{R'_s\}$ of $\mathcal{R}^S$. This uses the multiray Γ twice. After summing with each R_s, however, Γ lies in the new summands, so we are equivalently end summing X with the sum of the two elements of $\mathcal{R}^S$ as required. If we enlarge the index set S of $\{R_s\}$ while requiring all of the new summands R_s to be $\mathbb{R}^4$, the induced element of $\mathcal{S}(X)$ will be unchanged, so it is easy to deduce the last sentence of the corollary even when $\mathcal{E}_{\mathrm{ML}}$ is uncountable. □

In contrast with more general end sums, the action of $\mathcal{R}^S$ on $\mathcal{S}(X)$ is not known to vary with the choice of proper homotopy class of γ (for a fixed end function).

Question 5.2 *Suppose that two locally orienting multirays in X have the same end function, and that for each $s \in S$, the two corresponding rays induce the same orientation on the corresponding end, if it admits one. Can the two actions of $\mathcal{R}^S$ on $\mathcal{S}(X)$ be different?*

We can also ask about diffeomorphism types rather than isotopy classes as in Question 1.2. Clearly, any example of nonuniqueness must involve an end that fails to be Mittag-Leffler, such as one arising by ladder surgery. While such examples seem likely to exist, there are also reasons for caution, as we now discuss.

First, not every exotic $\mathbb{R}^4$ can give such examples. Freedman and Taylor [15] constructed a "universal" $\mathbb{R}^4$, $R_U \in \mathcal{R}$, which is characterized as being the unique fixed point of the $\mathcal{R}$–action on itself. They essentially showed that for any smoothing Σ of a 4–manifold X, the result of end summing with copies of R_U depends only on the subset of $\mathcal{E}(X)$ at which the sums are performed, regardless of whether those ends are Mittag-Leffler. Then $\mathcal{R}$ subsequently acts trivially on each of those ends. They also showed that the result of summing with R_U on a dense subset of ends creates a smoothing depending only on the stable isotopy class of Σ (classified by $H^3(X, \partial X; \mathbb{Z}/2)$). For such a smoothing, $\mathcal{R}^S$ acts trivially for any choice of multiray. The main point is that the universal property is obtained through a countable collection of disjoint compact subsets of R_U that allow h–cobordisms to be smoothly trivialized. If X is summed with R_U on one side of a ladder sum (for example), those compact subsets are also accessible on the other side by reaching through the rungs of the ladder.

A second issue is that examples of nonuniqueness would be subtle and hard to distinguish:

Proposition 5.3 *Let X be a* TOP *4–manifold with smoothing Σ. Let*

$$\gamma_0, \gamma_1 \colon S \times [0, \infty) \to X$$

be multirays as in the above question, inducing smoothings Σ_0 and Σ_1, respectively, via a fixed element of $\mathcal{R}^S$. Then, for every compact DIFF *4–manifold K, every Σ_0–smooth embedding $\iota\colon K \to X$ is* TOP *ambiently isotopic to a Σ_1–smooth embedding. After isotopy of Σ_1, every neighborhood of infinity in X contains another such neighborhood U such that whenever $\iota(K) \subset U$ and K is a 2–handlebody, the resulting isotopy can be assumed to keep $\iota(K)$ inside U.*

This shows that many of the standard 4–dimensional techniques for distinguishing smooth structures will fail in the above situation. One of the oldest techniques for distinguishing two smoothings on $\mathbb{R}^4$ is to find a compact DIFF manifold that smoothly embeds in one but not the other [19]. A newer incarnation of this idea is the Taylor invariant [39], distinguishing DIFF 4–manifolds via an exotic $\mathbb{R}^4$ embedded in one with compact closure. Clearly, such techniques must fail in the current situation. Most recently, the second author [23] constructed infinite families of smooth structures on many open 4–manifolds, distinguished by the minimal genera of smoothly embedded surfaces representing various homology classes. However, any such surface for the above smoothing Σ_0 will be homologous to one of the same genus for Σ_1 and vice versa. Minimal genera at infinity [23] will also fail: if we choose a system of neighborhoods U of infinity as in the proposition, any corresponding sequence of Σ_0–smooth surfaces in these will be homologous to a corresponding sequence for Σ_1 with the same genera. A possibility remains of distinguishing Σ_0 and Σ_1 by sequences of smoothly embedded 3–manifolds approaching infinity (such as by the engulfing index of Bižaca and Gompf [5]; see also [23, Remark 4.3(b)]) but there does not currently seem to be any good way to analyze such sequences. Note that the situation is not improved by passing to a cover, since the corresponding lifted smoothings will behave similarly. (The multirays γ_i will lift to multirays, and for each $s \in S$ the lifts of the corresponding rays of γ_0 and γ_1 will be multirays with end functions whose images have the same closure in $\mathcal{E}(\widetilde{X})$; see the last paragraph of the proof of [22, Theorem 8.1]. The proof below still applies to this situation.)

Proof For the first conclusion, let $\overline{\nu}_i\colon S \times [0,\infty) \times D^3 \to X$ be the closed tubular neighborhood maps of the multirays γ_i used for the end sums. By properness, both subsets $\overline{\nu}_i^{-1}\iota(K)$ are contained in a single subset of the form $T = S_0 \times [0,N] \times D^3$ for some finite $S_0 \subset S$ and $N \in \mathbb{Z}^+$. We need a Σ–smooth ambient isotopy Φ_t of id_X such that $\Phi_1 \circ \overline{\nu}_0 = \overline{\nu}_1$ on T, allowing no new intersections with $\iota(K)$, ie with $\overline{\nu}_1^{-1}\Phi_1\iota(K)$ still lying in T. This is easily arranged, since for each $s \in S_0$ the corresponding rays of γ_0 and γ_1 determine the same end and induce the same orientation on it if possible. This allows us to move $\gamma_0(s,N)$ to $\gamma_1(s,N)$ so that the local orientations agree, and then complete the isotopy following the initial segments of the rays. (The end hypothesis is needed when $X - \iota(K)$ is disconnected, for example.) After we perform the end sums, our isotopy will only be topological. However, $\Phi_1 \circ \iota$ will be Σ_1–smooth, as required, since the new smoothings correspond under Φ_1 on the images of T and the smoothing Σ is preserved elsewhere on $\iota(K)$.

For the second statement, assume (isotoping Σ_1) that the images of $\overline{\nu}_i$ for $i = 0, 1$ are disjoint. Given a neighborhood of infinity, pass to a smaller neighborhood U such that the two subsets $\overline{\nu}_i^{-1}(U)$ are equal, with complement of the form $S_1 \times [0, N'] \times D^3$ for some finite S_1 and $N' \in \mathbb{Z}^+$. For any K and ι with $\iota(K) \subset U$, we can repeat the previous argument. There is only one difficulty: if $K = M^3 \times I$, for example, some sheets of M may be caught between ∂U and the moving image of γ_0 during the final isotopy, and be pushed out of U. However, if K is a handlebody with all indices 2 or less, we can remove the image of K from the path of γ_0 (which will be following arcs of γ_1) by transversality. The statement now follows as before. □

Elements of $\mathcal{R}$ can be either *large* or *small*, depending on whether they contain a compact submanifold that cannot smoothly embed in the standard $\mathbb{R}^4$ (eg [26, Section 9.4]). Action on $\mathcal{S}(X)$ by small elements does not change the invariants discussed above (except for 3–manifolds at infinity), but still can yield uncountably many diffeomorphism types [23, Theorem 7.1]. However, large elements typically do change invariants. In particular, the minimal genus of a homology class can drop under end sum with, for example, the universal $\mathbb{R}^4$ [23, Theorem 8.1]. For Stein surfaces, the adjunction inequality gives a lower bound on minimal genera, which is frequently violated after such sums. Thus, the following application of Corollary 4.10 seems surprising:

Theorem 5.4 (Bennett [3, Corollary 4.1.3]) *There is a family $\{R_t \mid t \in \mathbb{R}\}$ of distinct large elements of $\mathcal{R}$ (with nonzero Taylor invariant) such that if Z is obtained from a Stein surface X by any orientation-preserving end sums with elements R_t then the adjunction inequality of X applies in Z.*

Nevertheless, we expect such sums to destroy the Stein structure, since every handle decomposition of each R_t requires infinitely many 3–handles. The idea of the proof is that [3] or [4] constructs such manifolds R_t embedded in Stein surfaces, in such a way that the sums can be performed pairwise. By Corollary 4.10, we obtain Z embedded in a Stein surface, so that the adjunction inequality is preserved.

Next we consider sums along properly embedded lines. For a fixed $n \geq 4$, let $\mathcal{Q}$ denote the set of oriented diffeomorphism types of manifolds homeomorphic to $\mathbb{R} \times S^{n-1}$, with a given ordering of their two ends. Each such manifold admits a DIFF proper embedding of a line, preserving the order of the ends, and this is unique up to DIFF ambient isotopy by Lemma 4.12. Thus, $\mathcal{Q}$ has a well-defined commutative monoid structure induced by summing along lines, preserving orientations on the lines and n–manifolds. (This time, properness prevents infinite sums.) The identity is $\mathbb{R} \times S^{n-1}$ with its standard smoothing. For $n = 5, 6, 7$, $\mathcal{Q}$ is trivial, and for $n > 5$, $\mathcal{Q}$ is canonically isomorphic to the finite group Θ_{n-1} of homotopy $(n-1)$–spheres [30] (by taking their product with $\mathbb{R}$). However, when $n = 4$, $\mathcal{Q}$ has much more structure: High-dimensional theory predicts that $\mathcal{Q}$ should be $\mathbb{Z}/2$, but in fact it is an uncountable monoid with an epimorphism to $\mathbb{Z}/2$ (analogous to the Rokhlin invariant of homology 3–spheres). Uncountability is already suggested by Corollary 5.1, but the structure of $\mathcal{Q}$ is richer than can be obtained just by acting by $\mathcal{R}$ at the two ends, as can be seen as follows. For $V, V' \in \mathcal{Q}$, call V a *slice* of V' if it embeds in V' separating the ends. (For this discussion, orientations and order of the ends do not matter.) Every known "large" exotic $\mathbb{R}^4$ has a neighborhood of infinity in $\mathcal{Q}$ with the property that disjoint slices are never diffeomorphic [19]. This neighborhood clearly has infinitely many disjoint slices, which form an infinite family in $\mathcal{Q}$ such that no two share a common slice. Thus, no two are obtained from a common element of $\mathcal{Q}$ by the action of $\mathcal{R} \times \mathcal{R}$. A similar family representing the other class in $\mathbb{Z}/2$ is obtained from the end of a smoothing of Freedman's punctured E_8–manifold.

To get an action on $\mathcal{S}(X)$ for $n \geq 4$, let $\gamma\colon S \times \mathbb{R} \to X$ (with S discrete) be a proper, locally orienting TOP embedding. Then $\mathcal{Q}^S$ has a well-defined action on $\mathcal{S}(X)$ (although without infinite iteration) by the same method as before, and this only depends on the proper homotopy class of γ. (We assume after proper homotopy that $\gamma^{-1}(\partial X) = \varnothing$. To see that a self-homeomorphism rel boundary of $\mathbb{R} \times D^{n-1}$ is isotopic to the identity, first use the topological Schoenflies theorem to reduce to the case where $\{0\} \times D^{n-1}$ is fixed.) Note that while $\mathcal{Q}$ admits only finite sums, the set S may be countably infinite. Examples 3.4 showed that the action of $\mathcal{Q}$ on $\mathcal{S}(X)$ for a two-ended 4–manifold X can depend on the choice of line spanning the ends, and in high dimensions, even the resulting diffeomorphism type can depend on the line. We next find fundamental group conditions eliminating such dependence.

To obtain such conditions, note that the fundamental progroup of X based at a ray γ has an inverse limit with well-defined image in $\pi_1(X, \gamma(0))$. In the Mittag-Leffler case, its image equals the image of $\pi_1(U_2, \gamma(2))$ for a suitably defined neighborhood system of infinity (ie with $j = i + 1$ in Definition 4.1). If γ is instead a line, it splits as a pair $\gamma_\pm$ of rays, obtained by restricting its parameter $\pm t$ to $[0, \infty)$, determining ends $\epsilon_\pm$ and images $G_\pm \subset \pi_1(X, \gamma(0))$ of the corresponding inverse limits. We will call the pair (ϵ_-, ϵ_+) a *Mittag-Leffler couple* if both ends are Mittag-Leffler and the double coset space $G_- \backslash \pi_1(X, \gamma(0)) / G_+$ is trivial. The proof below shows that γ is then uniquely determined up to proper homotopy by the pair of ends, so the condition is independent of choice of γ (as well as the direction of γ). A proper embedding $\gamma\colon S\times\mathbb{R}\to X$ now splits into γ_{ML} and γ_{bad} according to which lines connect Mittag-Leffler couples, and the restriction $\epsilon_{\gamma_{\mathrm{ML}}}$ of the end function $\epsilon_\gamma\colon S\times\{\pm1\}\to\mathcal{E}$ picks out the corresponding pairs of Mittag-Leffler ends. For simplicity, we now assume X is oriented.

Corollary 5.5 *Let X be an oriented topological n–manifold $($with $n\geq 4)$ with a proper embedding $\gamma\colon S\times\mathbb{R}\to X$. Then γ determines an action of $\mathcal{Q}^S$ on $\mathcal{S}(X)$, depending only on the proper homotopy classes of γ_{bad} and γ_{ML}. If the latter consists of finitely many lines, it only affects the action through its end function $\epsilon_{\gamma_{\mathrm{ML}}}$.*

If X is simply connected and $\mathcal{E}_{\mathrm{ML}}$ is finite, we obtain a canonical action of $\mathcal{Q}^{\mathcal{E}_{\mathrm{ML}}\times\mathcal{E}_{\mathrm{ML}}}$ on $\mathcal{S}(X)$.

Proof For a proper embedding γ of $\mathbb{R}$ determining a Mittag-Leffler couple $\epsilon_\pm$ as above, we show that any other embedding γ' determining the same ordered pair of ends is properly homotopic to γ. This verifies that Mittag-Leffler couples are well defined, and proves the corollary. (The finiteness hypothesis guarantees properness of the homotopy that we make using the proper homotopies of the individual lines.) Let $\{U_i\}$ be a neighborhood system of infinity as in the proof of Lemma 4.11, and reparametrize the four rays $\gamma_\pm$ and $\gamma'_\pm$ accordingly (fixing 0). As before, we can properly homotope γ' to agree with γ on $\mathbb{Z}\subset\mathbb{R}$, so that γ and γ' are related by a doubly infinite sequence of loops. The loop captured between ±2 (starting at $\gamma(0)$, then following γ_-, γ' and, backwards, γ_+) represents a class in $\pi_1(X,\gamma(0))$ that by hypothesis can be written in the form w_-w_+ with $w_\pm\in G_\pm$. After a homotopy of γ' supported in $[-2,2]$, we can assume that $\gamma'=\gamma$ on $[-1,1]$, and the innermost loops are given by $w_\pm$ pulled back to $\pi_1(U_1,\gamma(\pm1))$. Working with each sign separately, we now complete the proof of Lemma 4.11(a), denoting the pullback of $w_\pm$ by x_1 as before. By the definition of $G_\pm$, x_1 can be assumed to pull back further to $\pi_1(U_2,\gamma_\pm(2))$; let y_2 be the inverse of such a pullback. Completing the construction, we see that $z_1=1$, so that γ' is then properly homotoped to γ rel $[-1,1]$. □

Corollary 5.5 is most interesting when $n=4$, since classical smoothing theory reduces the higher-dimensional case to discussing the Poincaré duals of the relevant lines in $H^{n-1}(X,\partial X;\Theta_{n-1})$. When $n=4$, this same discussion applies to the classification of smoothings up to stable isotopy (isotopy after product with $\mathbb{R}$) by the obstruction group $H^3(X,\partial X;\mathbb{Z}/2)$, but one typically encounters uncountably many isotopy classes (and diffeomorphism types) within each stable isotopy class. Note that the above method can be used to study sums of more general CAT manifolds along collections of lines. In dimension 4, one can also consider actions on $\mathcal{S}(X)$ of the monoid $\mathcal{Q}_k$ of oriented smooth manifolds homeomorphic to a k–punctured 4–sphere Σ_k with an order on the ends, generalizing the cases $\mathcal{Q}_1 = \mathcal{R}$ and $\mathcal{Q}_2 = \mathcal{Q}$ considered above. (The monoid operation is summing along k–fold unions of rays with a common endpoint; see the end of Gompf [20] for a brief discussion.) However, little is known about this monoid beyond what can be deduced from Corollaries 5.1 and 5.5 and the structure of $\mathcal{R}$ and $\mathcal{Q}$. It follows formally from having infinite sums that $\mathcal{R}$ has no nontrivial invertible elements, and no nontrivial homomorphism to a group [19]; see also Theorem 6.2. However, the other monoids do not allow infinite sums. This leads to the following reformulation of Question 3.5:

Question 5.6 *Does $\mathcal{Q}$ (or more generally any $\mathcal{Q}_k$ with $k \geq 2$) have any nontrivial invertible elements? Is $H^3(\Sigma_k; \mathbb{Z}/2)$ the largest possible image of $\mathcal{Q}_k$ under a homomorphism to a group?*

6 1–handle slides and 0/1–handle cancellation at infinity

Our uniqueness result for adding 1–handles at infinity (Theorem 4.6) easily extends to adding both 0– and 1–handles at infinity, while allowing infinite slides and cancellation (Theorem 6.1). With *compact* handles of index 0 and 1, one may easily construct countable handlebodies that are contractible, but are distinguished by their numbers of ends. In this regard, adding 0– and 1–handles at infinity turns out to be simpler. For instance, in each dimension at least four, every (at most) countable, connected and oriented union of 0– and 1–handles at infinity is determined by its first Betti number. As an application of Theorem 6.1, we give a very natural and partly novel proof of the hyperplane unknotting theorem. The novelty here is that 0– and 1–handles at infinity provide the basic framework in which we employ Mazur's infinite swindle.

For simplicity, we assume throughout this section that all manifolds are oriented and all handle additions respect orientations.

Let X be a possibly disconnected CAT n–manifold with $n \geq 4$. Add to X a collection of 0–handles at infinity $W = \bigsqcup_{i \in J} w_i$ where each w_i is CAT homeomorphic to $\mathbb{R}^n$. The index set J and all others below are discrete and countable. Attach to $X \sqcup W$ a collection of 1–handles at infinity $H = \bigsqcup_{i \in S} h_i$ where each h_i is CAT homeomorphic

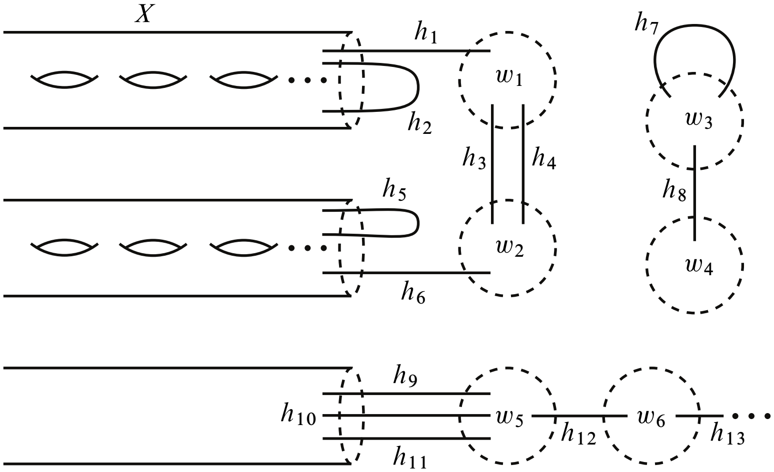

Figure 2: Manifold Z obtained from the manifold X by adding 0– and 1–handles at infinity, the latter denoted by arcs

to $[0,1]\times\mathbb{R}^{n-1}$ (see Figure 2). By Definition 2.1 and Theorem 4.6, H is determined by multiray data $\gamma^-,\gamma^+\colon S\times[0,\infty)\hookrightarrow X\sqcup W$ with disjoint images.

To this data, we associate a graph G defined as follows (see Figure 3). Let $\{v_i\mid i\in I\}$ be the set of proper homotopy classes of rays in the multiray data for H that lie in X. Each v_i has at least one representative of the form $\gamma^-(j_i)$ or $\gamma^+(j_i)$ for some $j_i\in S$. The vertex set V of G is

$$V:=\{v_i\mid i\in I\}\sqcup\{w_i\mid i\in J\}.$$

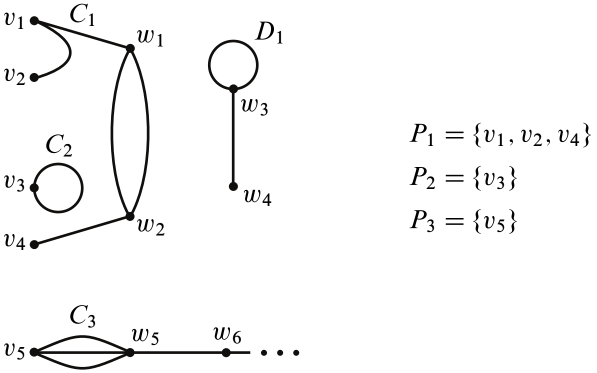

Figure 3: Graph G associated to the construction in Figure 2 and induced partition of the vertices v_i in X

The collection E of edges of G is bijective with the 1–handles at infinity H and thus is indexed by S. The edge e_i, with $i \in S$, corresponding to h_i is formally defined to be the multiset of the two vertices in V determined by the multiray data of h_i. In particular, E itself is a multiset, and the graph G is countable, but is not necessarily locally finite, connected or simple. Indeed, G may have multiple edges and loops. Let $C = \bigsqcup_{i \in I(C)} C_i$ be the connected components of G such that each component C_i contains a vertex $v_{j(i)}$ in X. Let $D = \bigsqcup_{i \in I(D)} D_i$ be the remaining components of G where each component D_i contains no vertex v_j in X. Notice that C induces a partition $\mathcal{P} = \{P_j \mid j \in I(C)\}$ of $\{v_i \mid i \in I\}$ where P_j is the subset of vertices in $\{v_i \mid i \in I\}$ that lie in C_j. Below, Betti numbers b_k are finite or countably infinite.

Theorem 6.1 *For a* CAT *n–manifold X with $n \geq 4$, the* CAT *oriented homeomorphism type of the manifold Z obtained by adding 0– and 1–handles at infinity to X as above is determined by:*

(a) *The set of pairs $(P_j, b_1(C_j))$ where $P_j \in \mathcal{P}$.*

(b) *The multiset with elements $b_1(D_i)$ where $i \in I(D)$.*

Thus, we only need to keep track of which proper homotopy classes of rays in X are used by at least one 1–handle (encoded as the vertices in each P_j), together with the most basic combinatorial data of the new handles. When the relevant ends are Mittag-Leffler, we can replace the ray data by the set of corresponding ends. The theorem implies that all 0–handles at infinity can be canceled except for one in each component of Z disjoint from X, and that we can slide 1–handles over each other whenever their attaching rays are properly homotopic (eg whenever they determine the same Mittag-Leffler end). Furthermore, any reasonable notion of infinitely iterated handle sliding is allowed.

Proof First, consider a component D_i of G. Let M denote the component of Z corresponding to D_i. By Corollary 4.7, we can and do assume that the rays used to attach 1–handles at infinity in M are radial (while still remaining proper and disjoint). Then, when D_i is a tree, we can easily describe M as a nested union of smooth n–disks, so it is a copy of $\mathbb{R}^n$. In general, a spanning tree T of D_i determines a copy of $\mathbb{R}^n$ in M (namely, one ignores a subset of the 1–handles at infinity). Thus, M is $\mathbb{R}^n$ with $b_1(D_i)$ 1–handles at infinity attached. By Corollary 4.7, such a manifold is determined by $b_1(D_i)$.

Second, consider a component C_j of G. Let N denote the component of Z corresponding to C_j. Let N' be the n–manifold obtained from N as follows. For each vertex v_k in C_j, introduce a $0/1$–handle pair at infinity where the new 1–handle at infinity attaches to a ray in the class v_k and to a ray in the new 0–handle at infinity. Also, the 1–handles at infinity in N attached to rays in the class of v_k attach in N' to rays in the new 0–handle at infinity. Theorem 4.6 implies that N and N' are CAT oriented homeomorphic. The graph C'_j corresponding to N' is obtained from C_j by adding a leaf to each v_k. Let T be a spanning tree of the connected graph obtained by removing the new leaves from C'_j. Then, T determines a copy of $\mathbb{R}^n$ in N'. This exhibits N' as the components of X containing the vertices in P_j, a single 0–handle at infinity w_0, $b_1(C_j)$ oriented 1–handles at infinity attached to w_0, and an oriented 1–handle at infinity from each $v_k \in P_j$ to w_0. □

As an application of 1–handle slides and $0/1$–handle cancellation at infinity, we prove the hyperplane unknotting theorem of Cantrell [9] and Stallings [38]. Recall that we assume CAT embeddings are locally flat.

Theorem 6.2 *Let $f\colon \mathbb{R}^{n-1} \to \mathbb{R}^n$ be a proper* CAT *embedding with $n \geq 4$, and let $H = f(\mathbb{R}^{n-1})$. Then there is a* CAT *homeomorphism of $\mathbb{R}^n$ that carries H to a linear hyperplane.*

A CAT ray in $\mathbb{R}^k$ is *unknotted* provided there is a CAT homeomorphism of $\mathbb{R}^k$ that carries the ray to a linear ray. Recall that each CAT ray in $\mathbb{R}^k$, $k \geq 4$, is unknotted. For CAT = PL and CAT = DIFF, this fact follows from general position, but for CAT = TOP it is nontrivial and requires Homma's method (see Lemma 4.12 above and [8, Section 7]). Thus, the following holds under the hypotheses of Theorem 6.2 by taking r to be the image under f of a linear ray in $\mathbb{R}^{n-1}$: *There is a* CAT *ray $r \subset H$ that is unknotted in both H and $\mathbb{R}^n$, where the former means $f^{-1}(r)$ is unknotted in $\mathbb{R}^{n-1}$.*

The hyperplane H separates $\mathbb{R}^n$ into two connected components by Alexander duality. Let A' and B' denote the closures in $\mathbb{R}^n$ of these two components as in Figure 4. So, $\partial A' = H = \partial B'$, and H has a bicollar neighborhood in $\mathbb{R}^n$. Using the bicollar, define

$$A := A' \cup (\text{open collar on } H \text{ in } B'),$$
$$B := B' \cup (\text{open collar on } H \text{ in } A'),$$

as in Figure 4. Figure 4 also depicts CAT rays $a \subset A$ and $b \subset B$ that are radial with respect to the collarings. Evidently, a and b are CAT ambient isotopic to r in A and B, respectively. (These simple isotopies have support in a neighborhood of the open collars).

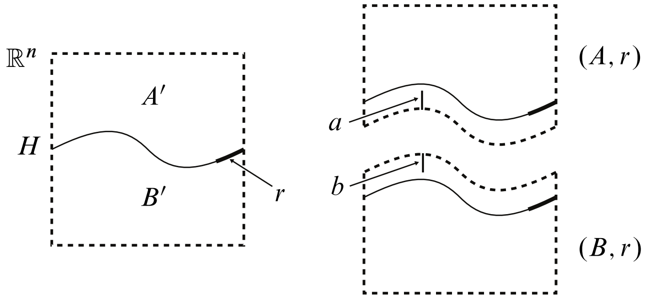

Figure 4: Closures A' and B' of the complement of H in $\mathbb{R}^n$ (left) and their unions A and B with open collars on H (right)

Lemma 6.3 *It suffices to show that A' and B' are* CAT *homeomorphic to the closed upper half-space $\mathbb{R}^n_+$.*

Proof We are given CAT homeomorphisms $g\colon A' \to \mathbb{R}^n_+$ and $h\colon B' \to \mathbb{R}^n_+$. Replace h by its composition with a reflection so that h maps $B' \to \mathbb{R}^n_-$. Note that g and h need not agree pointwise on H. Identify $\mathbb{R}^{n-1} \times \{0\}$ with $\mathbb{R}^{n-1}$. We have a CAT homeomorphism $j\colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ given by the restriction of $g \circ h^{-1}$ to $\mathbb{R}^{n-1}$. Define the CAT homeomorphism $k\colon B' \to \mathbb{R}^n_-$ by $k = (j \times \mathrm{id}) \circ h$ (that is, compose h with j at each height). Now, g and k agree pointwise on H. For CAT $=$ TOP and CAT $=$ PL, the proof of the lemma is complete. For CAT $=$ DIFF, one smooths along collars as in Hirsch [27, Theorem 1.9, page 182]. □

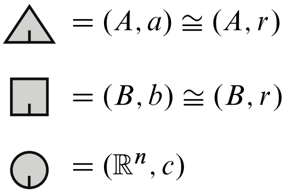

Figure 5: Notation for relevant manifold/ray pairs

We will use the symbols in Figure 5 to denote the indicated manifold/ray pairs. Here, c is a radial ray in $\mathbb{R}^n$. All rays in this proof, such as a and b, will be parallel (CAT ambient isotopic) to r or c. An added 1–handle at infinity will be denoted by an arc connecting such symbols as in Figure 6.

Lemma 6.4 *All three of the manifold/ray pairs in Figure 6 are* CAT *homeomorphic to one another.*

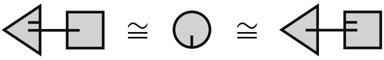

Figure 6: Isomorphic manifold/ray pairs

Proof First, we claim that adding a 1–handle at infinity to $(A,a) \sqcup (B,b)$ yields $\mathbb{R}^n$. Recalling the collars in Figure 4, the claim would be evident if we could choose the tubular neighborhood maps for the 1–handle at infinity to be the full collars in the $\mathbb{R}^{n-1}$ directions. However, an open tubular neighborhood must, by our definition, extend to a closed tubular neighborhood. So, instead we use smaller tubular neighborhoods inside the collars as follows. Identify the collar on H in A with $\mathbb{R}^{n-1} \times [0,1)$ so that H corresponds to $\mathbb{R}^{n-1} \times \{0\}$ and the ray a corresponds to $\{0\} \times \left[\frac{1}{2}, 1\right)$. For each $t \in \left[\frac{1}{2}, 1\right)$, there is an open horizontal $(n-1)$–disk in $\mathbb{R}^{n-1} \times [0,1)$ at height t, of radius $1/(1-t)$ and with center on a. The union of these disks is our desired open tubular neighborhood of a. Similarly, we obtain an open tubular neighborhood of b using the compatible collar in B. The claim follows by attaching the 1–handle at infinity using these tubular neighborhood maps and reparametrizing collars. Next, let a' and b' be the indicated rays in Figure 6 parallel to a and b, respectively. The lemma follows by shrinking the above tubular neighborhood maps in the $\mathbb{R}^{n-1}$ directions to be disjoint from a' and b', respectively. □

Lemma 6.5 *It suffices to prove that* (A,a) *and* (B,b) *are* CAT *homeomorphic as pairs to* $(\mathbb{R}^n, c)$.

Proof First, consider the cases CAT $=$ DIFF and CAT $=$ PL. The collar on H in A is a CAT *closed* regular neighborhood of a in A with boundary H. Using the hypothesis $(A,a) \cong (\mathbb{R}^n, c)$, apply uniqueness of such neighborhoods in $(\mathbb{R}^n, c)$ to see that A' is CAT homeomorphic to $\mathbb{R}^n_+$. Similarly, B' is CAT homeomorphic to $\mathbb{R}^n_+$. Now, apply Lemma 6.3.

For CAT $=$ TOP, we are given a homeomorphism $g\colon (A,a) \to (\mathbb{R}^n, c)$. Let $V \cong \mathbb{R}^n_+$ be the collar added to A' along H to obtain A as in Figure 4. Let $U \cong \mathbb{R}^n_+$ be a collar on H in A on the opposite side of H as in Figure 7. Recall that $\mathbb{R}^n$ itself is an open mapping cylinder neighborhood of c in $\mathbb{R}^n$ (see Kwun and Raymond [32] and Calcut, King and Siebenmann [8, pages 1816 and 1831]). Similarly, $U \cup V$ is an open mapping cylinder neighborhood of a in $U \cup V$. So, $g(U \cup V)$ is another open mapping cylinder neighborhood of c in $\mathbb{R}^n$. Uniqueness of such neighborhoods (see [32; 8]) implies there exists a homeomorphism $h\colon g(U \cup V) \to \mathbb{R}^n$ that fixes $g(V)$ pointwise.

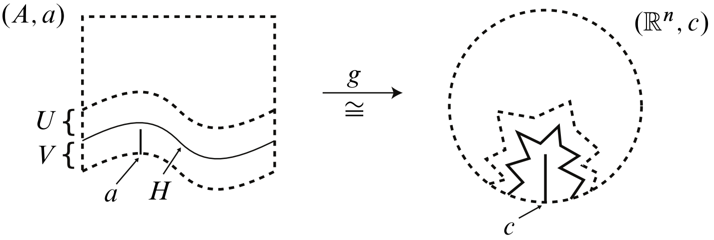

Figure 7: Homeomorphic manifold/ray pairs (A, a) and $(\mathbb{R}^n, c)$. Also depicted are the hyperplane H, the collar V added to A' to obtain A, a collar U on the other side of H, and their images in $\mathbb{R}^n$.

Therefore,

$$g(U) \cong \mathbb{R}^n - \operatorname{Int} g(V) = g(A').$$

Hence, $A' \cong U \cong \mathbb{R}^n_+$. Similarly, B' is homeomorphic to $\mathbb{R}^n_+$. Again, Lemma 6.3 completes the proof. □

Finally, we come to the heart of the proof of the hyperplane unknotting theorem. Mazur's infinite swindle [33] is realized as 1–handle slides and 0/1–handle cancellations at infinity. Figure 8 proves that (A, a) is CAT homeomorphic to $(\mathbb{R}^n, c)$. In Figure 8, the

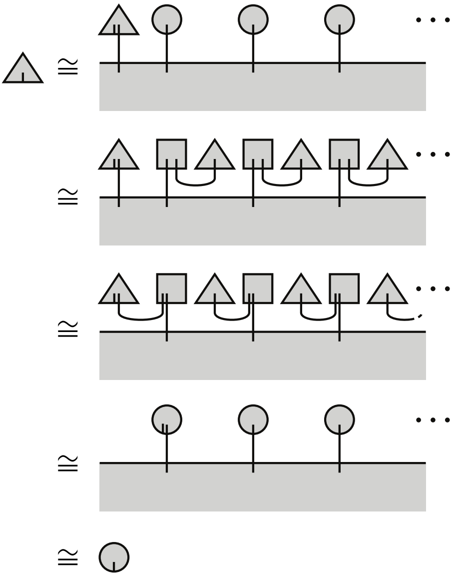

Figure 8: Mazur's infinite swindle as 1–handle slides and 0/1–handle cancellations at infinity

horizontal region is a copy of $\mathbb{R}^n$. The first, third and fifth isomorphisms in Figure 8 hold by Theorem 6.1. The second and fourth isomorphisms hold by Lemma 6.4. With $(A,a) \cong (\mathbb{R}^n, c)$, Figure 6 implies that $(B,b) \cong (\mathbb{R}^n, c)$. By Lemma 6.5, our proof of the hyperplane unknotting theorem is complete.

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The topology of arrangements of ideal type

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In 1962, Fadell and Neuwirth showed that the configuration space of the braid arrangement is aspherical. Having generalized this to many real reflection groups, Brieskorn conjectured this for all finite Coxeter groups. This in turn follows from Deligne's seminal work from 1972, where he showed that the complexification of every real simplicial arrangement is a $K(\pi,1)$–arrangement.

We study the $K(\pi,1)$–property for a certain class of subarrangements of Weyl arrangements, the so-called arrangements of ideal type $\mathscr{A}_{\mathcal{I}}$. These stem from ideals $\mathcal{I}$ in the set of positive roots of a reduced root system. We show that the $K(\pi,1)$–property holds for all arrangements $\mathscr{A}_{\mathcal{I}}$ if the underlying Weyl group is classical and that it extends to most of the $\mathscr{A}_{\mathcal{I}}$ if the underlying Weyl group is of exceptional type. Conjecturally this holds for all $\mathscr{A}_{\mathcal{I}}$. In general, the $\mathscr{A}_{\mathcal{I}}$ are neither simplicial nor is their complexification of fiber type.

14N20, 20F55, 52C35; 13N15

1 Introduction and results

By fundamental work of Fadell and Neuwirth [9], Brieskorn [4] and Deligne [8], all Coxeter arrangements are $K(\pi,1)$–arrangements, ie the complements of their complexifications are aspherical spaces.

While Coxeter arrangements are well studied, their subarrangements are considerably less well understood. In this paper we study the topology of the complements of certain arrangements which are associated with ideals in the set of positive roots of a reduced root system, so-called *arrangements of ideal type* $\mathscr{A}_{\mathcal{I}}$ (Definition 1.1); see Sommers and Tymoczko [19, Section 11]. We show that a combinatorial property introduced by Röhrle [18, Condition 1.10] combined with Terao's fibration theorem [21] gives an inductive method to show that a large class of (the complexifications of) the arrangements of ideal type $\mathscr{A}_{\mathcal{I}}$ are indeed $K(\pi,1)$–arrangements. This inductive technique was used in [18] to show that many of the arrangements $\mathscr{A}_{\mathcal{I}}$ are inductively

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free. In general a subarrangement of a Weyl arrangement need not be $K(\pi, 1)$; eg see Example 2.7.

Let Φ be an irreducible, reduced root system and let Φ^+ be the set of positive roots with respect to some set of simple roots Π. An *(upper) order ideal*, or simply *ideal* for short, of Φ^+, is a subset $\mathcal{I}$ of Φ^+ satisfying the following condition: if $\alpha \in \mathcal{I}$ and $\beta \in \Phi^+$ are such that $\alpha + \beta \in \Phi^+$, then $\alpha + \beta \in \mathcal{I}$. Recall the standard partial ordering $\preceq$ on Φ, $\alpha \preceq \beta$ provided $\beta - \alpha$ is a $\mathbb{Z}_{\geq 0}$–linear combination of positive roots or $\beta = \alpha$. Then $\mathcal{I}$ is an ideal in Φ^+ if and only if, whenever $\alpha \in \mathcal{I}$ and $\beta \in \Phi^+$ with $\alpha \preceq \beta$, also $\beta \in \mathcal{I}$.

Let β be in Φ^+. Then $\beta = \sum_{\alpha \in \Pi} c_\alpha \alpha$ for $c_\alpha \in \mathbb{Z}_{\geq 0}$. The *height* of β is defined to be $\operatorname{ht}(\beta) = \sum_{\alpha \in \Pi} c_\alpha$. Let $\mathcal{I} \subseteq \Phi^+$ be an ideal and let

$$\mathcal{I}^c := \Phi^+ \setminus \mathcal{I}$$

be its complement in Φ^+.

Following [19, Section 11], we associate with an ideal $\mathcal{I}$ in Φ^+ the arrangement consisting of all hyperplanes with respect to the roots in $\mathcal{I}^c$. Let $\mathscr{A}(\Phi)$ be the *Weyl arrangement* of Φ, ie $\mathscr{A}(\Phi) = \{H_\alpha \mid \alpha \in \Phi^+\}$, where H_α is the hyperplane in the Euclidean space $V = \mathbb{R} \otimes \mathbb{Z}\Phi$ orthogonal to the root α.

Definition 1.1 [19, Section 11] Let $\mathcal{I} \subseteq \Phi^+$ be an ideal. The *arrangement of ideal type* associated with $\mathcal{I}$ is the subarrangement $\mathscr{A}_\mathcal{I}$ of $\mathscr{A}(\Phi)$ defined by

$$\mathscr{A}_\mathcal{I} := \{H_\alpha \mid \alpha \in \mathcal{I}^c\}.$$

It was shown by Sommers and Tymoczko [19, Theorem 11.1] that each $\mathscr{A}_\mathcal{I}$ is free if the root system is classical or of type G_2. The general case was settled in a uniform manner for all types by Abe, Barakat, Cuntz, Hoge and Terao [1, Theorem 1.1]. The nonzero exponents are given by the dual of the height partition of the roots in $\mathcal{I}^c$.

Note that the complement $\mathcal{I}^c$ forms a lower ideal in Φ^+. Thus in particular, in type A_n the arrangements of ideal type $\mathscr{A}_\mathcal{I}$ are graphic arrangements corresponding to chordal graphs on $n+1$ vertices. The freeness of the latter is due to Stanley [20, Proposition 2.8].

In [2, Corollary 5.15], Barakat and Cuntz showed that every Weyl arrangement $\mathscr{A}(\Phi)$ is *inductively free*. It was shown in [18] that the free subarrangements $\mathscr{A}_\mathcal{I}$ of $\mathscr{A}(\Phi)$ are also inductively free with possible exceptions only in type E_8. The remaining instances in type E_8 were settled only recently by Cuntz, Röhrle and Schauenburg [7].

Note that if $\mathcal{I} = \varnothing$, then $\mathscr{A}_{\mathcal{I}} = \mathscr{A}(\Phi)$ is just the reflection arrangement of Φ and so $\mathscr{A}_{\varnothing}$ is $K(\pi, 1)$ by Deligne's result. So we may assume that $\mathcal{I} \neq \varnothing$.

Next we describe a combinatorial condition for an ideal $\mathcal{I} \subseteq \Phi^+$ from [18]. Using induction and Terao's fibration theorem [21], it allows us to show that a large class of arrangements of ideal type consists of $K(\pi, 1)$ arrangements. Let Φ_0 be a (standard) parabolic subsystem of Φ and let

$$\Phi_0^c := \Phi^+ \setminus \Phi_0^+,$$

the set of positive roots in the ambient root system which do not lie in the smaller one.

Condition 1.2 [18, Condition 1.10] Let $\mathcal{I} \neq \varnothing$ be an ideal in Φ^+ and let Φ_0 be a maximal parabolic subsystem of Φ such that $\Phi_0^c \cap \mathcal{I}^c \neq \varnothing$. Assume that, firstly, $\Phi_0^c \cap \mathcal{I}^c$ is linearly ordered with respect to $\preceq$, so that there is a unique root of every occurring height in $\Phi_0^c \cap \mathcal{I}^c$, and, secondly, for any $\alpha \neq \beta$ in $\Phi_0^c \cap \mathcal{I}^c$, there is a $\gamma \in \Phi_0^+$ such that α, β and γ are linearly dependent.

The instances when this condition is satisfied have been determined in [18].

Our first main result shows that Condition 1.2 entails the $K(\pi, 1)$–property for the associated arrangement of ideal type $\mathscr{A}_{\mathcal{I}}$.

Theorem 1.3 *Let $\mathcal{I} \neq \varnothing$ be an ideal in Φ^+ and let Φ_0 be a maximal parabolic subsystem of Φ such that Condition 1.2 is satisfied. Then $\mathscr{A}_{\mathcal{I}}$ is $K(\pi, 1)$.*

Specifically, this is the case if and only if one of the following holds:

(i) *Φ is of type A_n, B_n or C_n for $n \geq 2$ or G_2 and $\mathcal{I}$ is any ideal in Φ^+;*

(ii) *Φ is of type D_n for $n \geq 4$ and either $\mathcal{I}^c$ does not contain both $e_1 \pm e_n$ or $\mathcal{I}$ is generated by the root $e_{n-2} + e_{n-1}$;*

(iii) *Φ is of type F_4, E_6, E_7 or E_8 and $\mathcal{I}$ is as in* [18, Section 4].

In addition we use Thom's first isotopy lemma to construct explicit locally trivial fibrations in each of the remaining instances in type D_n not covered in Theorem 1.3(ii), ie when Φ is of type D_n and $\mathcal{I}^c$ does contain both $e_1 \pm e_n$. Combined with Theorem 1.3, this gives our second main result.

Theorem 1.4 *For Φ of classical type and $\mathcal{I}$ an ideal in Φ^+, we have that $\mathscr{A}_{\mathcal{I}}$ is $K(\pi, 1)$.*

In Table 1 we present the number of all arrangements of ideal type for each exceptional type in the first row. In the second row, we list the number of all $\mathscr{A}_{\mathcal{I}}$ when $\mathcal{I}$ satisfies Condition 1.2 with respect to a suitable parabolic subsystem; see [18, Table 1]. Thus, in these instances $\mathscr{A}_{\mathcal{I}}$ is $K(\pi, 1)$, by Theorem 1.3(iii).

Φ	E_6	E_7	E_8	F_4	G_2
all $\mathscr{A}_{\mathcal{I}}$	833	4160	25080	105	8
aspherical $\mathscr{A}_{\mathcal{I}}$	771	3433	18902	85	8

Table 1: Aspherical $\mathscr{A}_{\mathcal{I}}$ for exceptional Φ from Theorem 1.3

It is evident from Table 1 that with the possible exception of a relatively small number of cases in the exceptional types, all $\mathscr{A}_{\mathcal{I}}$ are $K(\pi, 1)$. The number of possible exceptions in types F_4, E_6, E_7 and E_8 are 20, 62, 727 and 6178, respectively. Thus, Theorems 1.3 and 1.4 give strong evidence for the following conjecture:

Conjecture 1.5 *Let Φ be a reduced root system with Weyl arrangement $\mathscr{A}(\Phi)$. Then any subarrangement of ideal type $\mathscr{A}_{\mathcal{I}}$ of $\mathscr{A}(\Phi)$ is a $K(\pi, 1)$–arrangement.*

Remarks 1.6 (i) Let Φ be of type F_4 and let $\mathcal{I}$ be the ideal generated by the root 0122 of height 5. Although $\mathcal{I}$ is not covered by Theorem 1.3, it turns out that $\mathscr{A}_{\mathcal{I}}$ is simplicial (see Cuntz and Heckenberger [6]), and so $\mathscr{A}_{\mathcal{I}}$ is $K(\pi, 1)$.

(ii) Since the $\mathscr{A}_{\mathcal{I}}$ in type E_6 and type E_7 are localizations of arrangements of ideal type in type E_8, thanks to Remark 2.2, the open cases in Conjecture 1.5 reduce to the ones in type F_4 and E_8.

Remark 1.7 It is worth emphasizing that Theorems 1.3 and 1.4 provide new examples for $K(\pi, 1)$–arrangements that are neither of fiber type nor simplicial. For instance, one can check that none of the nonsupersolvable arrangements $\mathscr{A}_{\mathcal{I}}$ in type E_6 that are shown to be $K(\pi, 1)$ by Theorem 1.3 are simplicial. See also Examples 3.6.

Note that in type D_n and type B_n, some of the arrangements $\mathscr{A}_{\mathcal{I}}$ that contain the full braid arrangement of A_{n-1} as a subarrangement are shown to be $K(\pi, 1)$ by Falk and Proudfoot [10, Section 5].

For general information about arrangements, Weyl groups and root systems, we refer the reader to Bourbaki [3] and Orlik and Terao [16].

2 Preliminaries

2.1 Hyperplane arrangements

Let $V = \mathbb{C}^n$ be an n–dimensional complex vector space. A *hyperplane arrangement* is a pair $(\mathscr{A}, V)$, where $\mathscr{A}$ is a finite collection of hyperplanes in V. Usually, we simply write $\mathscr{A}$ in place of $(\mathscr{A}, V)$.

The *lattice* $L(\mathscr{A})$ of $\mathscr{A}$ is the set of subspaces of V of the form $H_1 \cap \dots \cap H_i$, where $\{H_1, \dots, H_i\}$ is a subset of $\mathscr{A}$. For $X \in L(\mathscr{A})$, we have two associated arrangements: firstly $\mathscr{A}_X := \{H \in \mathscr{A} \mid X \subseteq H\} \subseteq \mathscr{A}$, the *localization of* $\mathscr{A}$ *at* X, and, secondly, the *restriction of* $\mathscr{A}$ *to* X, $(\mathscr{A}^X, X)$, where $\mathscr{A}^X := \{X \cap H \mid H \in \mathscr{A} \setminus \mathscr{A}_X\}$. The lattice $L(\mathscr{A})$ is a partially ordered set by reverse inclusion: $X \leq Y$ provided $Y \subseteq X$ for $X, Y \in L(\mathscr{A})$.

Throughout, we only consider arrangements $\mathscr{A}$ such that $0 \in H$ for each H in $\mathscr{A}$. These are called *central*. In that case the *center* $T(\mathscr{A}) := \bigcap_{H \in \mathscr{A}} H$ of $\mathscr{A}$ is the unique maximal element in $L(\mathscr{A})$ with respect to the partial order. A *rank* function on $L(\mathscr{A})$ is given by $r(X) := \operatorname{codim}_V(X)$. The *rank* of $\mathscr{A}$ is defined as $r(\mathscr{A}) := r(T(\mathscr{A}))$.

2.2 $K(\pi, 1)$–arrangements

A member X in $L(\mathscr{A})$ is said to be *modular* provided $X + Y \in L(\mathscr{A})$ for every $Y \in L(\mathscr{A})$ [16, Corollary 2.26]. The following is an immediate consequence of Terao's work [21] (see also [16, Section 5.5]). Indeed, $\mathscr{A}$ is strictly linearly fibered (see Definition 2.3) if and only if $L(\mathscr{A})$ admits a modular element of rank $r-1$; see [21, Corollary 2.14] (see also [16, Corollary 5.112]).

Lemma 2.1 *Let $\mathscr{A}$ be a complex arrangement of rank r. Suppose that $X \in L(\mathscr{A})$ is modular of rank $r-1$. If $\mathscr{A}_X$ is $K(\pi, 1)$, then so is $\mathscr{A}$.*

Remark 2.2 Thanks to an observation by Oka, if the complex arrangement $\mathscr{A}$ is $K(\pi, 1)$, then so is every localization $\mathscr{A}_X$ for X in $L(\mathscr{A})$; eg see [17, Lemma 1.1].

There is a standard construction for $K(\pi, 1)$–arrangements using locally trivial fibrations with $K(\pi, 1)$–spaces as bases and fibers. The long exact sequence in homotopy theory then gives that $\mathcal{M}(\mathscr{A})$ is a $K(\pi, 1)$–space; eg see [16, Theorem 5.9]. We recall two basic definitions due to Falk and Randell [11]; also see [16, Definitions 5.10 and 5.11].

Definition 2.3 An n–arrangement $\mathscr{A}$ is called *strictly linearly fibered* if, after a suitable linear change of coordinates, the restriction of the projection of $\mathcal{M}(\mathscr{A})$ to the first $n-1$ coordinates is a locally trivial fibration whose base space is the complement of an arrangement in $\mathbb{C}^{n-1}$ and whose fiber is the complex line $\mathbb{C}$ with finitely many points removed.

Definition 2.4 (i) The 1–arrangement $(\{0\}, \mathbb{C})$ is of *fiber type*.

(ii) For $n \geq 2$, the n–arrangement $\mathscr{A}$ is of *fiber type* if $\mathscr{A}$ is strictly linearly fibered with base $\mathcal{M}(\mathscr{B})$, where $\mathscr{B}$ is an $(n-1)$–arrangement of fiber type.

A repeated application of the homotopy exact sequence shows that a fiber-type arrangement $\mathscr{A}$ is $K(\pi, 1)$; eg see [16, Proposition 5.12].

The following important tool for proving that a given map is a locally trivial fibration is due to Thom [22]; see also [15].

Theorem 2.5 (Thom's first isotopy lemma) *Let M and P be smooth manifolds, $f\colon M \to P$ a smooth mapping and $S \subseteq M$ a closed subset which admits a Whitney stratification $\mathscr{S}$. Suppose $f|_S\colon S \to P$ is proper and $f|_X\colon X \to P$ is a submersion for each stratum $X \in \mathscr{S}$. Then $f|_S\colon S \to P$ is a locally trivial fibration and, in particular, $f|_X\colon X \to P$ is a locally trivial fibration for all $X \in \mathscr{S}$.*

Let $\mathscr{B}_n$ be the reflection arrangement of the hyperoctahedral group of type B_n. In the following example we consider a fiber-type subarrangement $\mathscr{J}_n$ of $\mathscr{B}_n$ which is used in Section 4 in the proof of Theorem 1.4.

Example 2.6 The subarrangement $\mathscr{J}_n$ of $\mathscr{B}_n$ is obtained by removing the antidiagonals from $\mathscr{B}_n$. So $\mathscr{J}_n$ is the union of the rank n Boolean arrangement and the braid arrangement $\mathscr{A}_{n-1}$, ie $\mathscr{J}_n$ has defining polynomial

$$Q(\mathscr{J}_n) := \prod_{i=1}^{n} x_i \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

One easily checks that $\mathscr{J}_n$ is of fiber type, eg by projecting onto the first $n-1$ coordinates and using induction on n.

We observe that the fiber-type arrangement $\mathscr{J}_n$ was already used by Brieskorn in his proof of the asphericity of the Coxeter arrangement in type D_n; see [4; 11, Section 5]. Also note that $\mathscr{J}_n$ is the irreducible version of the braid arrangement of type A_n. It is isomorphic to the restriction $\mathscr{A}(A_n)^X$, where $X = \ker(x_0)$; the hyperplane $\ker x_i$ in $\mathscr{J}_n$ then corresponds to the hyperplane $\ker(x_0 - x_i)$ in $\mathscr{A}(A_n)$.

The following related example shows that in general a subarrangement of a Coxeter arrangement need not be $K(\pi, 1)$ (nor free):

Example 2.7 Let $\mathscr{B}_n$ be as above and let $\mathscr{A}_{n-1}$ be its subarrangement consisting of the braid arrangement of type A_{n-1}. Let

$$\mathscr{K}_n := \mathscr{B}_n \setminus \mathscr{A}_{n-1}$$

be the complement of $\mathscr{A}_{n-1}$ in $\mathscr{B}_n$. As opposed to the subarrangement $\mathscr{J}_n$ of $\mathscr{B}_n$ from Example 2.6, rather than removing the antidiagonal hyperplanes from $\mathscr{B}_n$, for $\mathscr{K}_n$ we remove all the diagonals instead. Thus, $\mathscr{K}_n$ has defining polynomial

$$Q(\mathscr{K}_n) = \prod_{i=1}^{n} x_i \prod_{1 \le i < j \le n} (x_i + x_j).$$

We show by induction on n that $\mathscr{K}_n$ is not $K(\pi, 1)$ for $n \geq 3$. Owing to [12, (3.12)], $\mathscr{K}_3$ is not $K(\pi, 1)$. Now suppose that $n > 3$ and that the statement holds for $\mathscr{K}_{n-1}$. Let $X := \bigcap_{i=1}^{n-1} \ker x_i$. Then one readily checks that

$$(\mathscr{K}_n)_X \cong \mathscr{K}_{n-1}.$$

It follows from our induction hypothesis and Remark 2.2 that also $\mathscr{K}_n$ fails to be $K(\pi, 1)$.

In [12, (3.12)], Falk and Randell also observe that $\mathscr{K}_3$ is not free. Accordingly, by the argument above along with [16, Theorem 4.37], we see that $\mathscr{K}_n$ is not free for all $n \geq 3$.

So, while the construction of $\mathscr{K}_n$ is quite similar to that of $\mathscr{J}_n$, its combinatorial, algebraic and topological properties differ sharply from those of $\mathscr{J}_n$.

3 Proof of Theorem 1.3

Let Φ be a reduced root system of rank n with Weyl group W and *reflection arrangement* $\mathscr{A} = \mathscr{A}(\Phi) = \mathscr{A}(W)$. Let Φ^+ be the set of positive roots with respect to some set of simple roots Π of Φ. For Π_0 a proper subset of Π, the (*standard parabolic*) *subsystem* of Φ generated by Π_0 is $\Phi_0 := \mathbb{Z}\Pi_0 \cap \Phi$; see [3, Chapter VI, Section 1.7]. Define $\Phi_0^+ := \Phi_0 \cap \Phi^+$, the set of positive roots of Φ_0 with respect to Π_0. If the rank of Φ_0 is $n-1$, then Φ_0 is said to be *maximal*.

Set $X_0 := \bigcap_{\gamma \in \Phi_0^+} H_\gamma$. Then $\mathscr{A}(\Phi)_{X_0} = \mathscr{A}(\Phi_0)$. Therefore, the reflection arrangement $\mathscr{A}(W_{X_0})$ of the parabolic subgroup W_{X_0} is just $\mathscr{A}(\Phi_0)$, ie Φ_0 is the root system of W_{X_0} (see [16, Theorem 6.27, Corollary 6.28]).

Definition 3.1 Fix a standard parabolic subsystem Φ_0 of Φ. For $\mathcal{I}$ an ideal in Φ^+,

$$\mathcal{I}_0 := \mathcal{I} \cap \Phi_0^+$$

is an ideal in Φ_0^+. Thus,

$$\mathscr{A}_{\mathcal{I}_0} := \{H_\gamma \mid \gamma \in \mathcal{I}_0^c = \Phi_0^+ \setminus \mathcal{I}_0\}$$

is an arrangement of ideal type in $\mathscr{A}(\Phi_0)$, the Weyl arrangement of Φ_0.

Obviously, since $\mathcal{I}_0^c = \Phi_0^+ \setminus \mathcal{I}_0 = \mathcal{I}^c \cap \Phi_0^+ \subseteq \mathcal{I}^c$, we may view $\mathscr{A}_{\mathcal{I}_0}$ as a subarrangement of $\mathscr{A}_{\mathcal{I}}$ rather than as a subarrangement of $\mathscr{A}(\Phi_0)$. Note however, as such, $\mathscr{A}_{\mathcal{I}_0}$ is not of ideal type in $\mathscr{A}$ in general, since $\mathcal{I}_0$ need not be an ideal in Φ^+. We continue by recalling some basic facts from [18].

Lemma 3.2 [18, Lemma 3.1] *Viewing $\mathscr{A}_{\mathcal{I}_0}$ as a subarrangement of $\mathscr{A}_{\mathcal{I}}$, we have $\mathscr{A}_{\mathcal{I}_0} = (\mathscr{A}_{\mathcal{I}})_{X_0}$.*

The next observation shows that Condition 1.2 entails the presence of a modular element in $L(\mathscr{A}_{\mathcal{I}})$ of rank $r(\mathscr{A}_{\mathcal{I}}) - 1$.

Lemma 3.3 [18, Lemma 3.4] *If $\mathcal{I} \subseteq \Phi^+$ and Φ_0 satisfy Condition 1.2, then the center $Z := T((\mathscr{A}_{\mathcal{I}})_{X_0})$ of $(\mathscr{A}_{\mathcal{I}})_{X_0}$ is modular of rank $r(\mathscr{A}_{\mathcal{I}}) - 1$ in $L(\mathscr{A}_{\mathcal{I}})$.*

Observe that X_0 itself need not belong to $L(\mathscr{A}_{\mathcal{I}})$; eg see [18, Example 3.3].

Our next result shows that Condition 1.2 allows us to derive the $K(\pi,1)$–property for $\mathscr{A}_{\mathcal{I}}$ from that of $\mathscr{A}_{\mathcal{I}_0}$. It is just a consequence of Lemma 2.1.

Corollary 3.4 *Let $\mathcal{I}$ be an ideal in Φ^+ and let Φ_0 be a maximal parabolic subsystem of Φ such that either $\Phi_0^c \cap \mathcal{I}^c = \varnothing$ or Condition 1.2 is satisfied. Then $\mathscr{A}_{\mathcal{I}_0}$ is $K(\pi,1)$ if and only if $\mathscr{A}_{\mathcal{I}}$ is $K(\pi,1)$.*

Proof If $\Phi_0^c \cap \mathcal{I}^c = \varnothing$, then $\mathscr{A}_{\mathcal{I}}$ is the product of the empty 1–dimensional arrangement and $\mathscr{A}_{\mathcal{I}_0}$, and so the result is clear. Otherwise, $\mathscr{A}_{\mathcal{I}_0} = (\mathscr{A}_{\mathcal{I}})_{X_0} = (\mathscr{A}_{\mathcal{I}})_Z$, by Lemmas 3.2 and 3.3. Therefore, the forward implication follows from Lemmas 2.1 and 3.3, while the reverse implication is clear by Remark 2.2. □

We note that modular elements of corank 1 were constructed in [10, Lemma 5.4] for certain subarrangements of the reflection arrangement $\mathscr{B}_n$ of the hyperoctahedral group of type B_n that contain the full braid arrangement $\mathscr{A}_{n-1}$ of type A_{n-1}.

Remark 3.5 Let Φ be of type D_n for $n \geq 4$ and let Φ_0 be the standard subsystem of Φ of type D_{n-1}. Here and in Section 4 we use the notation for the positive roots from [3, Section 4.8, Planche IV]. Then $\Phi_0^c = \{e_1 \pm e_j \mid 2 \leq j \leq n\}$. Note that Φ_0^c is not linearly ordered by $\preceq$, as $e_1 \pm e_n$ both have height $n-1$.

Suppose that $\mathcal{I} \neq \varnothing$ fails to satisfy Condition 1.2 (with respect to our fixed Φ_0). This is precisely the case when both $e_1 \pm e_n$ belong to $\mathcal{I}^c$. Then $\mathcal{I}$ consists of roots from Φ^+ each of which admits the root $e_{n-2} + e_{n-1}$ of height 3 as a summand. Otherwise, at least one of $e_1 \pm e_n$ must belong to $\mathcal{I}$, as $\mathcal{I}$ is an ideal in Φ^+. This contradicts the assumption on $\mathcal{I}$. In turn this implies that if $\mathcal{I}_0 = \Phi_0 \cap \mathcal{I}$ is nonempty and fails to satisfy Condition 1.2 with respect to the maximal rank subsystem of Φ_0 of type D_{n-2}, then $\mathcal{I}$ fails to satisfy Condition 1.2 with respect to Φ_0. For, if each root in $\mathcal{I}_0$ admits the root $e_{n-3} + e_{n-2}$ as a summand, then necessarily each root in $\mathcal{I}$ has $e_{n-2} + e_{n-1}$ as a summand.

We conclude that if $\mathcal{I}$ satisfies Condition 1.2 with respect to Φ_0, then $\mathcal{I}_0$ satisfies Condition 1.2 with respect to the subsystem of Φ_0 of type D_{n-2}.

Proof of Theorem 1.3 (i) For Φ of type A_n, B_n or C_n for $n \geq 2$, it follows from [19, Section 7] that for Φ_0 the canonical maximal rank subsystem of type A_{n-1}, B_{n-1} or C_{n-1}, respectively, each $\mathcal{I}$ satisfies Condition 1.2, because irrespective of $\mathcal{I}$, in each case Φ_0^c is linearly ordered by $\preceq$. So the result follows in this instance from induction on the rank, Corollary 3.4 and the fact that central rank 2–arrangements are $K(\pi,1)$; see [16, Proposition 5.6]. The last result also implies that for Φ of type G_2 each arrangement of ideal type is $K(\pi,1)$. The very same inductive argument shows that in all these cases each $\mathscr{A}_{\mathcal{I}}$ is actually supersolvable; see [18, Theorem 1.5], and also [13, Theorems 6.6 and 7.1], where this is proved by different means.

(ii) Now let Φ be of type D_n for $n \geq 4$ and let Φ_0 be the standard subsystem of Φ of type D_{n-1}. We argue by induction on n. For $n = 4$, the result follows from [18, Lemma 6.1]. Indeed, each $\mathscr{A}_{\mathcal{I}}$ which satisfies the hypothesis of the theorem is already supersolvable.

Now suppose that $n \geq 5$ and that the result holds for root systems of type D of smaller rank. If $\mathcal{I}_0 = \Phi_0 \cap \mathcal{I} = \varnothing$, then $\mathscr{A}_{\mathcal{I}_0} = \mathscr{A}(D_{n-1})$. Being simplicial, the latter is $K(\pi,1)$. It follows from Corollary 3.4 that also $\mathscr{A}_{\mathcal{I}}$ is $K(\pi,1)$.

Now suppose that $\mathcal{I}_0 \neq \varnothing$. By Remark 3.5, $\mathcal{I}_0$ satisfies Condition 1.2 and so, by induction, $\mathscr{A}_{\mathcal{I}_0}$ is $K(\pi,1)$. Using Corollary 3.4 again, we conclude that $\mathscr{A}_{\mathcal{I}}$ is also $K(\pi,1)$, as desired.

Now let $\mathcal{I}$ be the ideal in Φ which is generated by $e_{n-2}+e_{n-1}$. Then one easily checks that $\mathcal{I}$ satisfies Condition 1.2 with respect to either one of the two subsystems of type A_{n-1}; see [18, Example 3.9]. So it follows from part (i) and Corollary 3.4 that $\mathscr{A}_{\mathcal{I}}$ is also $K(\pi,1)$ in this instance.

(iii) Now suppose that Φ is of type F_4, E_6, E_7 or E_8. All instances when $\mathcal{I}$ satisfies Condition 1.2 with respect to a suitably chosen maximal-rank subsystem Φ_0 are discussed in detail in [18, Section 4]. Perusing the arguments and in particular the data in Tables 6–9 in [18, Section 4], one checks that in each instance either $\mathcal{I}_0 = \varnothing$, or $\mathcal{I}_0 \neq \varnothing$ satisfies Condition 1.2 with respect to Φ_0^+. In the first instance we have $\mathscr{A}_{\mathcal{I}_0} = \mathscr{A}(\Phi_0)$, which is simplicial, and so it is $K(\pi,1)$. In the second instance, $\mathscr{A}_{\mathcal{I}_0}$ is $K(\pi,1)$ by induction. In both cases it follows from Corollary 3.4 that also $\mathscr{A}_{\mathcal{I}}$ is $K(\pi,1)$, as claimed. □

We illustrate the inductive arguments in the proof of Theorem 1.3(iii) in the following examples.

Examples 3.6 (a) Let Φ be of type E_6 and let $\mathcal{I}$ be the ideal generated by the root $\overset{00111}{\scriptstyle 0}$ of height 3. Then, according to the last entry for E_6 in [18, Table 6], $\mathcal{I}$ together with the subsystem Φ_0 of type D_5 satisfy Condition 1.2. Since $\mathcal{I}_0 = \varnothing$, $\mathscr{A}_{\mathcal{I}_0} = \mathscr{A}(\Phi_0)$ is the full reflection arrangement of type D_5, which is $K(\pi,1)$. Thus, so is $\mathscr{A}_{\mathcal{I}}$, by Corollary 3.4.

(b) Next consider Φ of type E_7 and let $\mathcal{I}$ be the ideal generated by the root $\overset{001110}{\scriptstyle 0}$ of height 3. Then according to the next to last entry for E_7 in [18, Table 6], $\mathcal{I}$ together with the subsystem Φ_0 of type E_6 satisfy Condition 1.2. Now $\mathcal{I}_0$ is just the ideal in E_6 considered in part (a). Consequently, $\mathscr{A}_{\mathcal{I}_0}$ is $K(\pi,1)$. But then so is $\mathscr{A}_{\mathcal{I}}$, again by Corollary 3.4.

(c) Finally, let Φ be of type E_8 and let $\mathcal{I}$ be the ideal generated by the root $\overset{0011100}{\scriptstyle 0}$ of height 3. Thanks to the data in the fifth row for E_8 in [18, Table 6], $\mathcal{I}$ together with the subsystem Φ_0 of type E_7 satisfy Condition 1.2. As $\mathcal{I}_0$ is the ideal in E_7 considered in part (b), we have that $\mathscr{A}_{\mathcal{I}_0}$ is $K(\pi,1)$ and so is $\mathscr{A}_{\mathcal{I}}$, thanks to Corollary 3.4.

Note that none of the three arrangements of ideal type $\mathscr{A}_{\mathcal{I}}$ considered in Examples 3.6 is supersolvable (see [13, Lemma 6.2]) and none of them is simplicial.

4 Proof of Theorem 1.4

Thanks to Theorem 1.3, Theorem 1.4 follows once the outstanding instances in type D_n not covered in Theorem 1.3(ii) are resolved. Accordingly, these are the instances when $\mathcal{I}$ consists of roots from Φ^+ each of which admits the root $e_{n-2}+e_{n-1}$ of height 3 as a summand, by Remark 3.5. In addition, by the proof of Theorem 1.3, we need not consider the case when $\mathcal{I}$ is the ideal in Φ which is generated by $e_{n-2}+e_{n-1}$. We list the different cases we need to consider below. We distinguish three different types of such ideals $\mathcal{I}$ according to their generators. In the first two instances, each $\mathcal{I}$ is generated by just a single root and by two in the third case:

(I) $0\dots01\dots1^1_1 = e_r + e_{n-1}$ for $1 \le r < n-2$. Here r is the first position with 1 as coefficient.

(II) $0\dots01\dots12\dots12^1_1 = e_s + e_t$, where $1 \le s < t < n-1$. Here s is the first position with a coefficient 1 and t is the first position labeled with 2.

(III) $0\dots01\dots1^1_1 = e_r + e_{n-1}$ for $1 \le r < n-2$ and $0\dots01\dots12\dots12^1_1 = e_s + e_t$, where $1 \le s < t < n-1$ and $r < s$. Note that the two roots are not comparable, since $r < s$.

In the following we give explicit locally trivial fibrations of the complements in each of the three cases above. First, we consider spaces that are going to serve as our bases for the locally trivial fibrations in these three instances. Recall the fiber-type subarrangement $\mathscr{J}_n$ of $\mathscr{B}_n$ from Example 2.6. In the following three lemmas, we exhibit three classes of subarrangements of $\mathscr{J}_n$ that are still of fiber type.

Lemma 4.1 *For $1 \le r < n-1$ fixed, the n–arrangement*

$$\mathscr{J}_n(r) := \mathscr{J}_n \setminus \{\ker(x_i - x_j) \mid 1 \le i \le r < j \le n\}$$

is of fiber type.

Proof We distinguish two cases: First, assume $r = 1$. Then the projection

$$\pi\colon \mathbb{C}^n \to \mathbb{C}^{n-1}, \quad (z_1,\dots,z_n) \mapsto (z_2,\dots,z_n),$$

induces a locally trivial fibration $\tilde{\pi}\colon \mathcal{M}(\mathscr{J}_n(r)) \to \mathcal{M}(\mathscr{J}_{n-1})$ with fiber the complex plane with one point removed.

Now assume that $r > 1$. Then we have $\mathscr{J}_n(r) = \mathscr{J}_r \times \mathscr{J}_{n-r}$.

Thus, in both cases, $\mathscr{J}_n(r)$ is of fiber type. □

Lemma 4.2 *For $1 \le s < t < n$ fixed, the n–arrangement*

$$\mathscr{I}_n(s,t) := \mathscr{I}_n \setminus \{\ker(x_i - x_j) \mid 1 \le i \le s < j \le t\}$$

is of fiber type.

Proof As in the proof of Lemma 4.1, let $\pi\colon \mathbb{C}^n \to \mathbb{C}^{n-1}$ be the projection

$$(z_1, \dots, z_n) \mapsto (z_2, \dots, z_n).$$

First, assume $s = 1$. Then π induces a locally trivial fibration

$$\tilde{\pi}\colon \mathcal{M}(\mathscr{I}_n(1,t)) \to \mathcal{M}(\mathscr{I}_{n-1})$$

with fiber the complex plane with $n-t+1$ points removed. So $\mathscr{I}_n(1,t)$ is of fiber type.

Now assume $s > 1$. Then π induces a locally trivial fibration

$$\tilde{\pi}\colon \mathcal{M}(\mathscr{I}_n(s,t)) \to \mathcal{M}(\mathscr{I}_{n-1}(s-1,t-1))$$

with fiber the complex plane with $n-t+s$ points removed. Thus, $\mathscr{I}_n(s,t)$ is of fiber type by induction on s. □

Lemma 4.3 *For $1 \le r < s < t < n$ fixed, the n–arrangement*

$$\mathscr{I}_n(r,s,t) := \mathscr{I}_n \setminus \{\ker(x_i - x_j) \mid 1 \le i \le r < j \le n \text{ or } r < i \le s < j \le t\}$$

is of fiber type.

Proof Take $\pi\colon \mathbb{C}^n \to \mathbb{C}^{n-1}$ to be the projection

$$(z_1, \dots, z_n) \mapsto (z_1, \dots, z_{s-1}, z_{s+1}, \dots, z_n).$$

If $s > r+1$, this projection induces a locally trivial fibration

$$\tilde{\pi}\colon \mathcal{M}(\mathscr{I}_n(r,s,t)) \to \mathcal{M}(\mathscr{I}_{n-1}(r,s-1,t-1)).$$

If $s = r+1$, it induces a locally trivial fibration

$$\tilde{\pi}\colon \mathcal{M}(\mathscr{I}_n(r,s,t)) \to \mathcal{M}(\mathscr{I}_{n-1}(r)).$$

In both cases the fiber is the complex plane with $n-r+s-t+1$ points removed. Now the result follows by induction on s and Lemma 4.1. □

We observe that the identification of $\mathscr{I}_n$ with a braid arrangement mentioned in Example 2.6 yields alternative proofs of Lemmas 4.1–4.3 via Stanley's theorem [20, Proposition 2.8]. For, the subarrangement $\mathscr{I}_n(r)$ corresponds to the graphic arrangement with underlying graph the union of the complete subgraphs on the vertices $\{0, 1, \dots, r\}$ and $\{0, r+1, \dots, n\}$. Further, $\mathscr{I}_n(s,t)$ corresponds to the union of the complete subgraphs on the vertices

$$\{0, \dots, s, t+1, \dots, n\} \quad \text{and} \quad \{0, s+1, \dots, t, t+1, \dots, n\}.$$

The arrangement $\mathscr{I}_n(r,s,t)$ then corresponds to the union of complete subgraphs on the vertices $\{0, 1, \dots, r\}$, $\{0, r+1, \dots, s, t+1, \dots, n\}$ and $\{0, s+1, \dots, t, t+1, \dots, n\}$. In all cases the graph is clearly chordal, so the arrangement is of fiber type, thanks to [20, Proposition 2.8].

Now let $\mathcal{I}$ be of type (I), (II) or (III) listed above, set $\mathscr{A} = \mathscr{A}_{\mathcal{I}}$ and, in types (I)–(III), let $\mathscr{B}$ be $\mathscr{I}_{n-1}(r)$, $\mathscr{I}_{n-1}(s,t)$ or $\mathscr{I}_{n-1}(r,s,t)$, respectively. Consider the map

$$\textbf{(4.4)} \qquad f\colon \mathcal{M}(\mathscr{A}) \to \mathcal{M}(\mathscr{B}), \quad (y_1, \dots, y_n) \mapsto (y_n^2 - y_1^2, \dots, y_n^2 - y_{n-1}^2).$$

Note that for $\mathcal{I} = \varnothing$, ie $\mathscr{A}_{\mathcal{I}} = \mathscr{A}(\Phi)$, and $\mathscr{B} = \mathscr{I}_{n-1}$, the map f was used in [4] to show asphericity in type D_n; see also [11, Section 5]. Our argument that the map f in (4.4) is a fibration over these larger bases is inspired by an argument due to Li Li (personal communication, 2006), who worked out the details of Brieskorn's approach [4].

Set $Y := \mathcal{M}(\mathscr{A})$ and $Z := \mathcal{M}(\mathscr{B})$. We can embed Y into $\mathbb{P}^n \times Z$ by the "graph" map $\iota\colon Y \to \mathbb{P}^n \times Z$ defined by

$$(y_1, \dots, y_n) \mapsto ((1 : y_1 : \dots : y_n), f(y_1, \dots, y_n))$$

and denote the image of Y by $C := \iota(Y)$. Then the map f is just $f = \pi|_C \circ \tilde{\iota}$, where $\tilde{\iota}\colon Y \to C$ is the homeomorphism induced by ι and $\pi|_C$ is the restriction of the projection $\pi\colon \mathbb{P}^n \times Z \to Z$ to C. Thus, f is a locally trivial fibration if and only if $\pi|_C$ is one.

Now let S_i be the hypersurface in $\mathbb{C}^n \times Z \subset \mathbb{P}^n \times Z$ defined by $z_i = y_n^2 - y_i^2$, so that $C = S_1 \cap \dots \cap S_{n-1}$. For $z = (z_1, \dots, z_{n-1}) \in Z$, let

$$(S_i)_z := S_i \cap (\mathbb{C}^n \times \{z\}) \subset \mathbb{P}^n \times \{z\} \quad \text{and} \quad C_z := (S_1)_z \cap \dots \cap (S_{n-1})_z,$$

ie C_z is the fiber of $\pi|_C$ over z. Moreover, let $\overline{C}$ and $\overline{C}_z$ denote the projective closures of C and C_z in $\mathbb{P}^n \times Z$, respectively. Then

$$\overline{C} = \overline{S}_1 \cap \dots \cap \overline{S}_{n-1} \quad \text{and} \quad \overline{C}_z = \overline{(S_1)}_z \cap \dots \cap \overline{(S_{n-1})}_z,$$

where $\bar{S}_i$ is the hypersurface in $\mathbb{P}^n \times Z$ given by $z_i y_0^2 = y_n^2 - y_i^2$ and for $z = (z_1, \dots, z_{n-1}) \in Z$,

$$\overline{(S_i)}_z := \bar{S}_i \cap (\mathbb{P}^n \times \{z\}).$$

Since $\bar{S}_i$ is defined by $y_n^2 - y_i^2 = z_i y_0$ for all $1 \le i \le n-1$ and the points at infinity are given by setting $y_0 = 0$, we get that $\bar{C}_z$ has the following 2^{n-1} points at infinity:

$$((0 : \pm 1 : \cdots : \pm 1 : 1), (z_1, \dots, z_{n-1})).$$

Lemma 4.5 *For each $z \in Z$, the projective closure $\bar{C}_z$ of C_z is a smooth curve.*

Proof The $\overline{(S_i)}_z$ intersect transversally, which can be seen by looking at the Jacobian $J = (\partial f_j / \partial t_i(y))$ of the polynomials given by

$$f_i \colon \bar{Y} \to \mathbb{C}, \quad (t_0 : t_1 : \cdots : t_n) \mapsto t_n^2 - t_i^2 - z_i t_0^2,$$

where $\bar{Y}$ is the projective closure of Y in $\mathbb{P}^n$. □

Moreover, we have the following:

Lemma 4.6 *For each $z \in Z$, $\bar{C}_z$ is connected.*

Proof Every point in $\bar{C}_z$ satisfies the equations

$$\frac{y_n^2 - y_1^2}{z_1} = \cdots = \frac{y_n^2 - y_{n-1}^2}{z_{n-1}} = y_0^2.$$

First take U_n to be the subset of $\bar{C}_z$ consisting of points $((y_0 : \cdots : y_n), (z_1, \dots, z_{n-1}))$ with $y_n \neq 0$. Thus, considering the change of coordinates $x_i := y_i / y_n$ and fixing some $1 \le j \le n-1$, we get that

$$x_i^2 = g_i^j(x_j) \quad \text{for all} \quad 1 \le i \le n-1 \qquad \text{and} \qquad x_0^2 = g_0^j(x_j),$$

where $g_i^j(x) = (z_i/z_j)x^2 + (z_j - z_i)/z_j$ and $g_0^j(x) = -(1/z_j)x^2 + 1/z_j$. Let α_0 and α_1 be the two branches of $y = x^2$. Then, for any point $p \in U_n$ there are indices $k_i \in \{0, 1\}$ such that

$$p = \Big(\big(\alpha_{k_0}(g_0^j(x_j)) : \cdots : \alpha_{k_{j-1}}(g_{j-1}^j(x_j)) : x_j : \alpha_{k_{j+1}}(g_{j+1}^j(x_j)) : \cdots : \alpha_{k_{n-1}}(g_{n-1}^j(x_j)) : 1 \big), \\ (z_1, \dots, z_{n-1}) \Big).$$

So, by choosing an appropriate path in $\mathbb{C}$, we may path-connect p to one of the points at infinity $((0 : \pm 1 : \cdots : \pm 1 : 1), (z_1, \dots, z_{n-1}))$. As $1 \le j \le n-1$ is arbitrary and $g_i^j(x) = g_i^j(-x)$, any point $p \in U_n$ is path-connected to the point $((0 : 1 : \cdots : 1), (z_1, \dots, z_{n-1}))$.

Now take U_1 to be the subset of $\overline{C}_z$ consisting of points $((y_0 : \cdots : y_n), (z_1, \ldots, z_{n-1}))$ with $y_1 \neq 0$ and observe that $U_1 \cup U_n = \overline{C}_z$. By a similar argument as the one above, for any point $q \in U_1$ there are indices $k_i \in \{0, 1\}$ such that

$$q = \big((\alpha_{k_0}(h_0(x_n)) : 1 : \alpha_{k_2}(h_2(x_n)) : \cdots : \alpha_{k_{n-1}}(h_{n-1}(x_n)) : x_n\big), (z_1, \ldots, z_{n-1})\big),$$

where $h_0(x) = (1/z_1)x_n^2 - 1/z_1$, $h_i(x) = ((z_1 - z_i)/z_1)x^2 + z_i/z_1$ and $x_i = y_i/y_1$. Now we can again choose a path in $\mathbb{C}$ that connects q to one of the points at infinity $((0 : \pm 1 : \cdots : \pm 1 : 1), (z_1, \ldots, z_{n-1}))$. Thus, $\overline{C}_z$ is connected. □

Note that this also proves that C_z is connected: as two points in C_z are connected by a path through finitely many points at infinity and $\overline{C}_z$ is locally homeomorphic to $\mathbb{C}$, we can alter the path around each of the points at infinity to get a path that completely lies inside C_z.

The above lemmas prove the following:

Corollary 4.7 *For each $z \in Z$, the curve $\overline{C}_z$ is a connected Riemann surface and C_z is a connected Riemann surface with 2^{n-1} puncture points.*

Theorem 4.8 *The map f defined in (4.4) is a locally trivial fibration.*

Proof Set $D = \overline{C} \setminus C$, the intersection of $\overline{C}$ with the infinity hyperplane. Then $\mathscr{S} = \{C, D\}$ is a Whitney stratification of $\overline{C}$: It is obviously locally finite and satisfies the condition of the frontier and, as C is open and D its boundary, $\mathscr{S}$ trivially satisfies Whitney condition B. The intersection of D with a fiber $\mathbb{P}^n \times \{z\}$ of the projection π is just the set of the 2^{n-1} points $((0 : \pm 1 : \cdots : \pm 1 : 1), (z_1, \ldots, z_{n-1}))$, which we can think of locally as 2^{n-1} sections of π. Thus, $\pi|_D$ is locally homeomorphic and therefore it is a submersion. The map $\pi|_C$ is a submersion as well, which can be seen by considering the Jacobian again. Moreover, $\pi|_{\overline{C}}$ is proper, as $\overline{C}$ is a closed subset of $\mathbb{P}^n \times Z$ and π is proper. Now, using Thom's first isotopy lemma, Theorem 2.5, $\pi|_{\overline{C}}$ is a locally trivial fibration and, in particular, $f = \pi|_C \circ \tilde{\imath}$ is a fibration as well. □

This proves the following:

Theorem 4.9 *If $\mathcal{I}$ is of type* (I), (II) *or* (III), *then $\mathscr{A}_{\mathcal{I}}$ is $K(\pi, 1)$.*

Proof Consider the map $f \colon Y \to Z$ from (4.4). Clearly, the fiber $f^{-1}(z)$ is homeomorphic to C_z, so by Corollary 4.7 it is a connected Riemann surface with 2^{n-1}

puncture points. Thus, by the uniformization theorem, it is a $K(\pi,1)$–space. By Lemmas 4.1, 4.2 and 4.3, Z is a $K(\pi,1)$–space as well. This proves the theorem. □

This concludes the proof of Theorem 1.4. Note that none of the arrangements of ideal type $\mathscr{A}_{\mathcal{I}}$ of types (I)–(III) considered here are supersolvable (see [13, Lemma 6.2]) and none of them are simplicial. So these families of $\mathscr{A}_{\mathcal{I}}$ also provide new classes of $K(\pi,1)$–arrangements.

Remarks 4.10 (i) If $\mathscr{A}$ is strictly linearly fibered over $\mathscr{B}$, then there always exists a section of the associated fibration of the complements $\mathcal{M}(\mathscr{A}) \to \mathcal{M}(\mathscr{B})$, eg see [5, Corollary 1.1.6]. As a consequence, by the splitting lemma, $\pi_1(\mathcal{M}(\mathscr{A}))$ is a semidirect product of $\pi_1(\mathcal{M}(\mathscr{B}))$ acting on the fundamental group of the fiber. In particular, this applies to each of the cases considered in Theorem 1.3.

(ii) One can also construct a cross-section to the fibration $f\colon Y \to Z$ used in the proof of Theorem 1.4 as follows: Let

$$y_n = y_n(z_1,\dots,z_{n-1}) = \sqrt{|z_1| + \cdots + |z_{n-1}|}.$$

Now, for all $(z_1,\dots,z_{n-1}) \in Z$, for all $1 \le i \le n-1$ the real part of $y_i^2 = y_n^2 - z_i$ is positive. Thus, choosing a branch α of the square root, we can define $y_i = \alpha(y_n^2 - z_i)$ continuously, yielding a cross-section $s\colon Z \to Y$. This section was initially constructed by Falk and Randell in [11, Section 5] in the case $\mathscr{A}$ is the full reflection arrangement of type D_n, which is strictly linearly fibered over $\mathscr{B} = \mathscr{J}_{n-1}$; see Example 2.6. See also [14, Section 1.1] for a locally trivial fibration in this case with a slightly different section.

As $f \circ s = \mathrm{id}_Z$, the short exact sequence of fundamental groups splits. Thus, by the splitting lemma we see that $\pi_1(Y)$ is a semidirect product of $\pi_1(Z)$ acting on $\pi_1(C_z)$, where C_z is the fiber over $z \in Z$ as above.

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Topological complexity of unordered configuration spaces of surfaces

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We determine the topological complexity of unordered configuration spaces on almost all punctured surfaces (both orientable and nonorientable). We also give improved bounds for the topological complexity of unordered configuration spaces on all aspherical closed surfaces, reducing it to three possible values. The main methods used in the proofs were developed in 2015 by Grant, Lupton and Oprea to give bounds for the topological complexity of aspherical spaces. As such this paper is also part of the current effort to study the topological complexity of aspherical spaces and it presents many further examples where these methods strongly improve upon the lower bounds given by zero-divisor cup-length.

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1 Introduction

In 2003 Farber introduced the topological complexity of a space to study the problem of robot motion planning from a topological perspective [5]. It is a numerical homotopy invariant which measures the minimal instability of every motion planner on this space. More explicitly, given a path-connected space X, the topological complexity $\mathsf{TC}(X)$ is the sectional category of the free path fibration $p_X\colon X^I \to X \times X$ (see Section 2).

Determining $\mathsf{TC}(X)$ is in general a hard problem. For over a decade the topological complexity of many spaces has been computed and diverse tools have been developed to that end.

In this context, configuration spaces have been extensively studied because they are of special interest from the point of view of robotics. Considering the problem of moving n objects on a space X avoiding collisions naturally leads to the definition of the *ordered configuration space* $F(X,n)$ of n distinct ordered points on X as

$$F(X,n) = \{(x_1,\dots,x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\}.$$

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These spaces model automated guided vehicles (AGVs) moving on a factory floor — see Ghrist [11] — or flying drones trying to avoid each other in the air.

Farber and Yuzvinsky determined the topological complexity of $F(\mathbb{R}^d, n)$ for $d = 2$ or d odd in [10]. Later, Farber and Grant extended the results to all dimensions d in [8]. The topological complexity of ordered configuration spaces of orientable surfaces has also been computed by Cohen and Farber in [2]. Many more related results can be found in the recent survey articles by Cohen [1] and Farber [7].

In the configuration spaces $F(X, n)$ considered above, the points of a configuration are labeled (or ordered) and the symmetric group $\mathfrak{S}_n$ acts on $F(X, n)$ by permuting the labels. However, in certain situations it greatly improves the efficiency to consider the points to be identical. For instance, consider a scenario in which all the AGVs perform the same tasks equally. In this case we are only interested in the positions of points in X up to permutation, in other words forgetting the labels assigned to the points. This leads to the *unordered configuration spaces* $C(X, n) = F(X, n)/\mathfrak{S}_n$, by definition the orbits of the symmetric group action.

As we saw above, there is a very complete picture of the topological complexity of *ordered* configuration spaces of 2–dimensional manifolds and beyond. In contrast to this, very little is known for *unordered* configuration spaces, as Cohen notes at the end of [1]. One of the main reasons for this discrepancy is that all the above results use a cohomological technique involving *zero-divisors*, which seems to be insufficient for unordered configuration spaces (at least with constant coefficients).

The results in this paper use a technique to bound the topological complexity of aspherical spaces developed in 2015 by Grant, Lupton and Oprea [12]. Being a homotopy invariant, the topological complexity of an aspherical space only depends on its fundamental group and the methods are algebraic in nature. An introduction to topological complexity of groups is given in Section 2.

The mentioned technique was already used in the recent paper [13], in which Grant and the second author computed the topological complexity of some mixed configuration spaces $F(\mathbb{R}^2, n)/(\mathfrak{S}_{n-k} \times \mathfrak{S}_k)$ on the plane, with $1 \le k \le n-1$. These spaces are in a sense intermediate between the ordered and the unordered case and they model the situation in which there are two different types of identical AGVs. It turns out that also in the mixed case the cohomological lower bounds used in previous results are insufficient.

It has to be mentioned that the topological complexity of unordered configuration spaces of trees was computed in many cases by Scheirer in [16]. To the best of the authors' knowledge that is the only previous computation of the topological complexity of an unordered configuration space with at least three points. It is worth noting that Scheirer uses the zero-divisor cup-length lower bound, which seems to be insufficient for unordered configuration spaces of surfaces.

In this paper we determine the topological complexity of the unordered configuration spaces of all punctured surfaces (orientable and nonorientable) except the disc and the Möbius band, and narrow it down to three values for all closed aspherical surfaces (orientable and nonorientable). For the Möbius band we narrow it down to two values and for the disc we give some improved bounds and a complete answer in the case of three points. Many of the proofs extend to ordered configuration spaces (this is discussed at the end of the paper).

All results except the ones for the disc are presented in the following theorem, which follows from Theorems 4.1, 4.2, 5.1 and 5.3. In the case of the annulus the upper bound is proven by finding an explicit motion planner.

Theorem 1.1

- *Let $\mathcal{S}$ be obtained from a closed surface by removing a positive number of points. If $\mathcal{S}$ is not the disc, the annulus or the Möbius band, then*

$$\mathsf{TC}(C(\mathcal{S},n)) = 2n.$$

- *Let $\mathcal{S}$ be a closed surface. If $\mathcal{S}$ is not the sphere or the projective plane, then*

$$2n \leq \mathsf{TC}(C(\mathcal{S},n)) \leq 2n+2.$$

- *If $\mathcal{A}$ denotes the annulus, then*

$$\mathsf{TC}(C(\mathcal{A},n)) = 2n-1.$$

- *If $\mathcal{M}$ denotes the Möbius band, then*

$$2n-1 \leq \mathsf{TC}(C(\mathcal{M},n)) \leq 2n.$$

Remark 1.2 Theorem 1.1 should be compared to the corresponding results for ordered configuration spaces of Cohen and Farber in [2]. They are consistent with the possibility that the values of the topological complexity of ordered and unordered configuration spaces of surfaces always agree. Note that in [2] the nonreduced version of topological complexity is used, which is 1 greater than the one used in this paper.

The only aspherical surface not covered by Theorem 1.1 is the disc. The best estimates we found for the disc are given in the following two theorems. Note that they greatly improve over the best previously known lower bounds

$$\mathsf{TC}(C(D,n)) \geq \mathsf{cat}(C(D,n)) = n-1$$

coming from the Lusternik–Schnirelmann category $\mathsf{cat}(C(D,n))$ (see [13]).

Theorem 1.3 *If D is the disc, then*

$$2n-2-\tfrac{1}{2}n \leq n-1+\mathsf{cd}([P_n, P_n]) \leq \mathsf{TC}(C(D,n)) \leq 2n-2.$$

Here cd is the cohomological dimension of a group and $[P_n, P_n]$ is the commutator subgroup of the pure braid group of the disc (see Section 3).

We expect that $\mathsf{cd}([P_n, P_n])$ is in fact the maximum possible, which would mean that Theorem 1.3 narrows $\mathsf{TC}(C(D,n))$ down to two possible values.

Conjecture 1.4 The cohomological dimension of $[P_n, P_n]$ is equal to $n-2$.

The following theorem gives a potentially better lower bound (depending on the actual value of $\mathsf{cd}([P_n, P_n])$, which is unknown to the authors). It also tells us that asymptotically $\mathsf{TC}(C(D,n))$ behaves like $2n$.

Theorem 1.5 *If D is the disc, then*

$$2n-2\left\lfloor \sqrt{n/2} \right\rfloor - 3 \leq \mathsf{TC}(C(D,n)) \leq 2n-2.$$

Finally, we compute the topological complexity of the unordered configuration space of three points on the disc by finding an explicit motion planner.

Theorem 1.6 *If D is the disc, then*

$$\mathsf{TC}(C(D,3)) = 3.$$

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2 Topological complexity of aspherical spaces

In this section we first define the topological complexity of a general topological space and then specialize it to aspherical spaces.

For a path-connected topological space X, let $p_X\colon X^I \to X \times X$ denote the free path fibration on X, with projection $p_X(\gamma) = (\gamma(0), \gamma(1))$.

Definition 2.1 The *topological complexity* of X, denoted by $\mathsf{TC}(X)$, is defined to be the minimal k such that $X \times X$ admits a cover by $k+1$ open sets $U_0, U_1, \dots, U_k$, on each of which there exists a local section of p_X (that is, a continuous map $s_i\colon U_i \to X^I$ such that $p_X \circ s_i = \mathrm{incl}_i\colon U_i \hookrightarrow X \times X$).

Note that here we use the reduced version of $\mathsf{TC}(X)$, which is 1 less than the number of open sets in the cover.

Let π be a discrete group. It is well known that there exists a connected CW–complex $K(\pi,1)$ with

$$\pi_i(K(\pi,1)) = \begin{cases} \pi & \text{if } i = 1, \\ 0 & \text{if } i \geq 2. \end{cases}$$

Such a space is called an *Eilenberg–Mac Lane space* for the group π. Furthermore, $K(\pi,1)$ is unique up to homotopy. Because the topological complexity $\mathsf{TC}(X)$ is a homotopy invariant of the space X (see [5]), the following definition is sensible:

Definition 2.2 The topological complexity of a discrete group π is given by

$$\mathsf{TC}(\pi) := \mathsf{TC}(K(\pi,1)).$$

In [6] Farber posed the problem of giving an algebraic description of $\mathsf{TC}(\pi)$. This problem is far from being solved but some progress has been made, including the following theorem:

Theorem 2.3 (Grant, Lupton and Oprea [12, Theorem 1.1]) *Let π be a discrete group and let A and B be subgroups of π. Suppose that $gAg^{-1} \cap B = \{1\}$ for every $g \in \pi$. Then*

$$\mathsf{TC}(\pi) \geq \mathsf{cd}(A \times B).$$

It is worth noting that this theorem has recently been generalized using different methods in [9, Corollary 3.5.4].

The corresponding problem for the Lusternik–Schnirelmann category of a group has been completely answered: $\mathsf{cat}(\pi) = \mathsf{cd}(\pi)$. This classical result is due to Eilenberg and Ganea [3] for $\mathsf{cd}(\pi) \neq 1$, while the remaining case follows from the later work by Stallings [17] and Swan [18].

We will also need the following standard result:

Lemma 2.4 $$\mathsf{TC}(\pi) \leq \mathsf{cd}(\pi \times \pi).$$

Proof This follows from the upper bound $\mathsf{TC}(X) \leq \mathsf{cat}(X \times X)$ given by Farber in [5]. □

3 The surface braid groups

In this section we introduce the surface braid groups and we recall their main properties.

Definition 3.1 A surface $\mathcal{S}$ is a connected closed 2–dimensional manifold possibly with a finite number of points removed, called *punctures*.

Recall from the introduction that the configuration space $F(\mathcal{S}, n)$ admits an action by the symmetric group $\mathfrak{S}_n$ which permutes the points in each configuration. The unordered configuration space

$$C(\mathcal{S}, n) = F(\mathcal{S}, n)/\mathfrak{S}_n$$

is by definition the orbit space of that action.

Definition 3.2 We call $P_n(\mathcal{S}) = \pi_1(F(\mathcal{S}, n))$ the *pure braid group* on n strands of the surface $\mathcal{S}$, and $B_n(\mathcal{S}) = \pi_1(C(\mathcal{S}, n))$ the *(full) braid group* on n strands of $\mathcal{S}$. When $\mathcal{S}$ is the disc D, we also abbreviate $P_n = P_n(D)$ and $B_n = B_n(D)$.

The covering $F(\mathcal{S}, n) \to C(\mathcal{S}, n)$ yields the short exact sequence

$$1 \to P_n(\mathcal{S}) \to B_n(\mathcal{S}) \to \mathfrak{S}_n \to 1.$$

The following theorem is due to Fadell and Neuwirth:

Theorem 3.3 (Fadell and Neuwirth [4]) *Denote by $\mathcal{S}_n$ the surface obtained from $\mathcal{S}$ by removing n points. There is a locally trivial fibration*

(1) $$\mathcal{S}_n \to F(\mathcal{S}, n+1) \to F(\mathcal{S}, n),$$

where the projection map forgets the last point of the ordered configuration.

It is well known that the only surfaces that are not aspherical are the sphere S^2 and the projective plane $\mathbb{R}P^2$. From now on all the surfaces that we will consider are assumed to be aspherical. The reason for this is that the methods in this paper only apply to aspherical spaces.

Corollary 3.4 *Let $\mathcal{S}$ be an aspherical surface. From the long exact sequence of the homotopy groups applied to the Fadell–Neuwirth fibrations (1) and induction it follows that the spaces $F(\mathcal{S},n)$ are also aspherical. Furthermore, we get the short exact sequence*

$$1 \to \pi_1(\mathcal{S}_n) \to P_{n+1}(\mathcal{S}) \to P_n(\mathcal{S}) \to 1. \tag{2}$$

We will need the following technical result, which we expect to be well known to the experts. However, we could not find a full proof in the literature and thus we will give a detailed proof here. The result appears as Proposition 2.2 in [15] but it relies on Lemma 3.6 below (Proposition 2.1 in [15]), which is stated there without a proof.

Theorem 3.5 *Let $\mathcal{S} \hookrightarrow \mathcal{T}$ be a smooth embedding of aspherical surfaces such that the induced homomorphism $\pi_1(\mathcal{S}) \rightarrowtail \pi_1(\mathcal{T})$ is injective.*

Then the corresponding inclusion $C(\mathcal{S},n) \hookrightarrow C(\mathcal{T},n)$ induces an injective homomorphism $B_n(\mathcal{S}) \to B_n(\mathcal{T})$.

In the proof of the theorem the following lemma will be essential. In that lemma a slightly different definition of nonclosed surface is needed, with open balls removed instead of points removed. This is the only place in which we make use of this definition. We stress that this is not an essential distinction because the configuration spaces of punctured surfaces and the configuration spaces of surfaces with boundary are homotopy equivalent.

Lemma 3.6 *Let $\mathcal{S} \hookrightarrow \mathcal{T}$ be a smooth embedding of aspherical surfaces, which we assume to be closed surfaces with (possibly) some open balls removed instead of points removed. Further assume that the image of $\mathcal{S}$ lies in the interior of $\mathcal{T}$. Then the induced homomorphism $\pi_1(\mathcal{S}) \to \pi_1(\mathcal{T})$ is injective if and only if no boundary component of $\mathcal{S}$ bounds a disc in $\mathcal{T} \setminus \mathcal{S}$.*

Proof Recall that we are assuming that surfaces are path-connected. Therefore, if $\mathcal{S}$ is closed, the embedding has to be surjective and the claim is trivial. Assume $\mathcal{S}$ is not closed. Because the boundary components of $\mathcal{S}$ are smooth simple closed curves inside $\mathcal{T}$, they separate $\mathcal{T}$ into $\mathcal{S}$ on one side and a disjoint union of surfaces on the other side.

We first assume that the homomorphism $\pi_1(\mathcal{S}) \to \pi_1(\mathcal{T})$ induced by the embedding is not injective and claim that there is a disc in $\mathcal{T} \setminus \mathcal{S}$ bounded by a boundary component of $\mathcal{S}$.

A nontrivial element in the kernel of $\pi_1(\mathcal{S}) \to \pi_1(\mathcal{T})$ can be represented by a smooth map $f\colon S^1 \to \mathcal{S}$ which extends to a smooth map on the disc $g\colon D \to \mathcal{T}$. We may assume that the image of f is in the interior of $\mathcal{S}$ and that g is transverse to $\partial\mathcal{S}$.

Observe that the image of g needs to have a nonempty intersection with the boundary of $\mathcal{S}$. Otherwise, g would yield a nullhomotopy of f inside $\mathcal{S}$, but by assumption f represents a nontrivial class in $\pi_1(\mathcal{S})$. Let B be a boundary component of $\mathcal{S}$ which intersects the image of g.

The preimage of B in D under g is now a nonempty, smooth 1–dimensional manifold. Since $f\colon S^1 \to \mathcal{S}$ doesn't intersect $\partial\mathcal{S}$, $g^{-1}(B)$ is a compact subset of the interior of D, hence it must be a closed $1-$manifold.

Therefore, given a path-component $C \subset D$ of $g^{-1}(B)$, we know that C is a smooth circle and, by the Jordan–Schoenflies curve theorem, C bounds a disc $\widetilde{D}$ in D on one side and an annulus A on the other side such that $\partial A = C \cup \partial D$. We can further assume, by choosing C to be *outermost* in D among the path-components of $g^{-1}(B)$, that there exists a collar neighborhood $U \supset C$ in D such that $g(U \cap A) \subseteq \mathcal{S}$. Indeed, by transversality we have, for a small collar neighborhood U, that $g(U \cap A)$ is contained either in $\mathcal{S}$ or in $\mathcal{T} \setminus \mathcal{S}$. If C is *outermost*, the former must be the case, as under this condition there is a path in A from C to ∂D only intersecting $g^{-1}(B)$ at the starting point, and $g(\partial D) \subset \mathcal{S}$.

The curve C gives an element in $\pi_1(B) \simeq \mathbb{Z}$. If this element is trivial then we can redefine g on $\widetilde{D}$ by a nullhomotopy living on B. After pushing the image of $\widetilde{D}$ along the collar neighborhood into the interior of $\mathcal{S}$, we get a replacement of g with (at least) one less connected component in $g^{-1}(\partial\mathcal{S})$ than for the original map.

Hence, there must exist a circle C such that $g|_C$ is a nontrivial element in $\pi_1(B)$, otherwise we would construct a nullhomotopy of f inside $\mathcal{S}$ after finitely many iterations of the above procedure. Therefore, there is a power of the generator $[B] \in \pi_1(B)$ that vanishes in $\pi_1(\mathcal{T})$. Because $\pi_1(\mathcal{T})$ is torsion-free (indeed $\mathcal{T}$ is a finite-dimensional classifying space for $\pi_1(\mathcal{T})$), $[B]$ is already trivial in $\pi_1(\mathcal{T})$.

Then B is a nullhomotopic simple closed curve and it must bound a disc in $\mathcal{T}$ by the classification of surfaces. There are two possibilities. Either this disc doesn't intersect the interior of $\mathcal{S}$ and it is glued to the boundary component B to obtain $\mathcal{T}$, or $\mathcal{S}$ is a punctured sphere and $\mathcal{T}$ is obtained from $\mathcal{S}$ by glueing discs onto all the path-components of $\partial\mathcal{S}$ different from B (there is at least one other boundary component because by assumption $\pi_1(\mathcal{S}) \to \pi_1(\mathcal{T})$ is not injective and therefore $\mathcal{S}$ is not a disc).

We showed that if $\pi_1(\mathcal{S}) \to \pi_1(\mathcal{T})$ is not injective, there must be a disc in $\mathcal{T} \setminus \mathcal{S}$ bounded by boundary component of $\mathcal{S}$.

Conversely, assume that $\mathcal{T} \setminus \mathcal{S}$ contains a disc D bounded by some boundary component B of $\partial\mathcal{S}$. Then the corresponding element $[B] \in \pi_1(\mathcal{S})$ vanishes in $\pi_1(\mathcal{T})$. Therefore, the homomorphism $\pi_1(\mathcal{S}) \to \pi_1(\mathcal{T})$ is not injective unless $[B]$ is already trivial in $\pi_1(\mathcal{S})$. Again by the classification of surfaces, this can only happen if $\mathcal{S}$ itself is a disc, but then $\mathcal{T}$ would be a sphere, contradicting the hypothesis that $\mathcal{T}$ is aspherical. □

Proof of Theorem 3.5 By the commutativity of the following diagram with exact rows, it suffices to show that $P_n(\mathcal{S}) \to P_n(\mathcal{T})$ is injective:

$$\begin{tikzcd}
1 \arrow[r] & P_n(\mathcal{S}) \arrow[r] \arrow[d] & B_n(\mathcal{S}) \arrow[r] \arrow[d] & \mathfrak{S}_n \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & P_n(\mathcal{T}) \arrow[r] & B_n(\mathcal{T}) \arrow[r] & \mathfrak{S}_n \arrow[r] & 1
\end{tikzcd}$$

We do this by induction using the Fadell–Neuwirth fibrations.

For $n = 1$, the homomorphism $\pi_1(\mathcal{S}) \to \pi_1(\mathcal{T})$ is injective by assumption.

Suppose now that $P_{n-1}(\mathcal{S}) \to P_{n-1}(\mathcal{T})$ is injective. The embedding $\mathcal{S} \hookrightarrow \mathcal{T}$ gives rise to an embedding $\mathcal{S}_n \hookrightarrow \mathcal{T}_n$, in which the n new punctures in $\mathcal{S}_n$ are sent to the n new punctures in $\mathcal{T}_n$. The short exact sequences (2) give rise to the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\mathcal{S}_{n-1}) \arrow[r] \arrow[d] & P_n(\mathcal{S}) \arrow[r] \arrow[d] & P_{n-1}(\mathcal{S}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi_1(\mathcal{T}_{n-1}) \arrow[r] & P_n(\mathcal{T}) \arrow[r] & P_{n-1}(\mathcal{T}) \arrow[r] & 1
\end{tikzcd}$$

The rows are exact and we assumed the vertical homomorphism on the right is injective. If the vertical homomorphism on the left were also injective, the vertical homomorphism in the middle would have to be injective, which would complete the induction argument.

It is not hard to see that the configuration spaces of punctured surfaces (points removed) and the configuration spaces of surfaces with boundary components (open balls removed) are homotopy equivalent. Because of this we might assume that $\mathcal{S}$ and $\mathcal{T}$ are surfaces with boundary and that $\mathcal{S}_{n-1}$ and $\mathcal{T}_{n-1}$ are the surfaces which result from removing $n-1$ open balls, in order to be able to use Lemma 3.6. Then the embedding $\mathcal{S} \hookrightarrow \mathcal{T}$ satisfies the assumptions of Lemma 3.6 if and only if $\mathcal{S}_{n-1} \hookrightarrow \mathcal{T}_{n-1}$ satisfies them. Therefore, the injectivity of the leftmost vertical homomorphism is equivalent to the injectivity of $\pi_1(\mathcal{S}) \rightarrowtail \pi_1(\mathcal{T})$, which is part of the assumptions. □

4 Lower bounds

Theorem 4.1 *Let $\mathcal{S}$ be an aspherical surface which is not the disc, the annulus or the Möbius band. Then*

$$\mathsf{TC}(C(\mathcal{S},n)) \geq 2n.$$

Proof Let $\mathcal{S}$ be a surface satisfying the assumptions in the theorem. Then, with the only exception of the Klein bottle, we have $\operatorname{rank}(H_1(\mathcal{S})) \geq 2$ and there are two smooth simple closed curves α and α' on $\mathcal{S}$ representing linearly independent classes of $H_1(\mathcal{S})$. We may assume that there exist tubular neighborhoods $\mathcal{A}$ of α and $\mathcal{A}'$ of α' that are annuli. If the tubular neighborhood of α were a Möbius band, then we could replace α by the boundary of this Möbius band.

The homomorphism $\pi_1(\mathcal{A}) \to \pi_1(\mathcal{S})$ is injective, as can be checked by further projecting to $H_1(\mathcal{S})$. Similarly, the homomorphism $\pi_1(\mathcal{A}') \to \pi_1(\mathcal{S})$ is injective.

For the Klein bottle $\mathcal{K}$, recall that the fundamental group $\pi_1(\mathcal{K})$ has a presentation

$$\pi_1(\mathcal{K}) = \langle a, b \mid aba^{-1}b \rangle,$$

where both a and b are represented by simple closed curves α and β in $\mathcal{K}$. Both subgroups $\langle a \rangle$ and $\langle b \rangle$ are infinite cyclic, and therefore the inclusions of collar neighborhoods $\mathcal{A}$ of α and $\mathcal{A}'$ of β in $\mathcal{K}$ are injective at the level of π_1; the collar neighborhood of α is a Möbius band so we replace α with its *double* as above.

Hence, by Theorem 3.5 the homomorphisms $P_n(\mathcal{A}) \to P_n(\mathcal{S})$ and $P_n(\mathcal{A}') \to P_n(\mathcal{S})$ are injective.

We now construct a subgroup $Z_n \subset P_n(\mathcal{A})$. Consider n parallel, disjoint copies $\alpha_1, \dots, \alpha_n$ of the curve α inside $\mathcal{A}$, and let $\mathfrak{T} \subset F(\mathcal{A}, n)$ be the subspace of ordered configurations $(x_1, \dots, x_n)$ with x_i lying on the curve α_i for all $1 \leq i \leq n$; then $\mathfrak{T}$ is an embedded n–fold torus in $F(\mathcal{A}, n)$, and at the level of fundamental groups we have a map $\mathbb{Z}^n \simeq \pi_1(\mathfrak{T}) \to P_n(\mathcal{A})$.

This map is injective: indeed the composition

$$\mathbb{Z}^n \simeq \pi_1(\mathfrak{T}) \to P_n(\mathcal{A}) = \pi_1(F(\mathcal{A}, n)) \to \pi_1(\mathcal{A}^n) \simeq \mathbb{Z}^n$$

is an isomorphism. We call $Z_n \simeq \mathbb{Z}^n \subset P_n(\mathcal{A})$ the image of this map.

In the same way we construct an $n-$fold torus $\mathfrak{T}' \subset F(\mathcal{A}', n)$ and get a subgroup $Z_n' \subset P_n(\mathcal{A}')$ as the image of the map between fundamental groups induced by the inclusion, with $Z_n' \simeq \mathbb{Z}^n$.

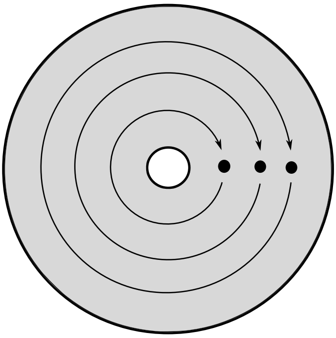

Figure 1: Braids from Z_n as seen from above

There is a homomorphism

$$P_n(\mathcal{S}) \to \prod_{k=1}^{n} \pi_1(\mathcal{S}) \to \bigoplus_{k=1}^{n} H_1(\mathcal{S}), \tag{3}$$

under which nontrivial elements in the image of Z_n and Z_n' inside $P_n(\mathcal{S})$ are mapped to elements which lie in different orbits under the action which permutes the summands in $\bigoplus_{k=1}^{n} H_1(\mathcal{S})$. This is because the image of each nontrivial element in Z_n will have at least one summand corresponding to a nontrivial multiple of the class in $H_1(\mathcal{S})$ represented by the curve α, whereas the image of each braid in Z_n' has only summands corresponding to multiples of the class represented by the curve α'. Notice that for the Klein bottle it suffices that the homology class represented by α is infinite cyclic, and the argument works even if the homology class represented by α' has order 2.

Now we observe that conjugating an element of $P_n(\mathcal{S})$ by an element of $B_n(\mathcal{S})$ has the effect of permuting the summands in $\bigoplus_{k=1}^n H_1(\mathcal{S})$ under the homomorphism (3). To see this first note that the homomorphism (3) consists of a sum of compositions of homomorphisms of the form

$$P_n(\mathcal{S}) \to \pi_1(\mathcal{S}) \to H_1(\mathcal{S})$$

given by forgetting all strands but one and then taking the abelianization. Given a braid $\gamma \in B_n(\mathcal{S})$, we can write $\gamma = \delta\epsilon$, where ϵ is supported on a disc and $\delta \in P_n(\mathcal{S})$. Therefore, conjugating by γ reduces to conjugating by ϵ and δ. Conjugating by ϵ permutes the order of the strands by the corresponding permutation under the canonical map $B_n \to \mathfrak{S}_n$. Conjugating by δ results in a conjugation inside $\pi_1(\mathcal{S})$ under the first homomorphism $P_n(\mathcal{S}) \to \pi_1(\mathcal{S})$, but this has no effect on the abelianization.

Therefore, no nontrivial element of Z_n is conjugate to an element of Z'_n in $B_n(\mathcal{S})$. By Theorem 2.3 this implies the lower bound $\mathsf{TC}(B_n(\mathcal{S})) \geq \mathsf{cd}(Z_n \times Z'_n) = 2n$. □

Theorem 4.2 *Let $\mathcal{S}$ be either the annulus or the Möbius band. Then*

$$\mathsf{TC}(C(\mathcal{S},n)) \geq 2n-1.$$

Proof In the same way as in the previous proof we can find an annulus $\mathcal{A}$ inside $\mathcal{S}$ and a subgroup Z_n in $P_n(\mathcal{A})$ isomorphic to $\mathbb{Z}^n$. Because $\pi_1(\mathcal{S}) \simeq H_1(\mathcal{S}) \simeq \mathbb{Z}$, this time we cannot find a second annulus inducing a linearly independent homology class, nor even a disjoint infinite cyclic subgroup of $\pi_1(\mathcal{S})$.

However, the inclusion of a disc D in $\mathcal{S}$ also induces a monomorphism $P_n(D) \to P_n(\mathcal{S})$ and no nontrivial element in $P_n(D)$ is conjugate to an element of Z_n inside $B_n(S)$.

Indeed, if we consider the map

$$P_n(\mathcal{S}) \to \prod_{k=1}^n \pi_1(\mathcal{S}) \to \bigoplus_{k=1}^n H_1(\mathcal{S}),$$

we see that no nontrivial element of Z_n is mapped to zero, whereas all elements of $P_n(D)$ are mapped to zero. As we saw in the proof of the previous theorem, conjugation inside $B_n(S)$ results only in a permutation of the coordinates of the target group $\bigoplus_{k=1}^n H_1(\mathcal{S})$, and the stated properties are therefore invariant under conjugation.

By Theorem 2.3 we get

$$\mathsf{TC}(C(\mathcal{S},n)) \geq \mathsf{cd}(\mathbb{Z}^n \times P_n(D)) = 2n-1.$$ □

5 Upper bounds

Theorem 5.1 *If $\mathcal{S}$ is a closed aspherical surface, then*

$$\mathsf{TC}(C(\mathcal{S},n)) \leq 2n+2.$$

If $\mathcal{S}$ is a punctured surface which is not the disc, then

$$\mathsf{TC}(C(\mathcal{S},n)) \leq 2n.$$

Proof It is well known that $\mathsf{cd}(\pi_1(\mathcal{S})) = 2$ for closed aspherical surfaces and that $\mathsf{cd}(\pi_1(\mathcal{S})) = 1$ for punctured surfaces (other than the disc). Using the short exact sequences (2) of Corollary 3.4, together with the fact that the cohomological dimension is subadditive under group extensions, and that $\mathsf{cd}(B_n(\mathcal{S})) = \mathsf{cd}(P_n(\mathcal{S}))$ because $B_n(\mathcal{S})$ is torsion-free and $P_n(\mathcal{S})$ is a finite-index subgroup, we see that $\mathsf{cd}(B_n(\mathcal{S})) \leq n+1$ if $\mathcal{S}$ is closed and $\mathsf{cd}(B_n(\mathcal{S})) \leq n$ if $\mathcal{S}$ has punctures and is not the disc (the two preceding inequalities are in fact equalities, but we don't need that stronger statement in this proof).

The upper bounds now follow from Lemma 2.4. □

Next we give an upper bound for the annulus which is 1 better than the one given in the previous theorem (it is in fact the optimal upper bound). For the proof we will need the following well-known technical lemma.

We defined the topological complexity in terms of the number of open sets in an open cover of $X \times X$, but for sufficiently nice spaces (CW–complexes for instance) there is an equivalent characterization in terms of decompositions into disjoint *Euclidean neighborhood retracts* (ENRs).

Lemma 5.2 [6] *Let X be an ENR* (for instance a finite-dimensional, locally finite CW–complex). *Then the topological complexity $\mathsf{TC}(X)$ equals the smallest integer k such that there exists a decomposition $X \times X = E_0 \sqcup E_1 \sqcup \cdots \sqcup E_k$ into $k+1$ disjoint ENRs, on each of which there is a local section $s_i\colon E_i \to X^I$.*

The existence of such a section $s_i\colon E_i \to X^I$ is equivalent to the existence of a deformation of E_i into the diagonal of $X \times X$, ie a homotopy between the inclusion $E_i \hookrightarrow X \times X$ and a map whose image lies entirely in the diagonal.

Theorem 5.3 *If $\mathcal{A}$ is the annulus, then*

$$\mathsf{TC}(C(\mathcal{A},n)) \leq 2n-1.$$

The proof of Theorem 5.3 occupies the rest of this section. By Lemma 5.2 we need to find a decomposition of $C_n(\mathcal{A}) \times C_n(\mathcal{A})$ into $2n$ disjoint ENRs which can be deformed into the diagonal. Note that such deformations can equivalently be viewed as an explicit motion planner with $2n$ different continuous rules and as such it is potentially relevant for applications.

5.1 Decomposition of $C_n(\mathcal{A}) \times C_n(\mathcal{A})$

The annulus can be identified with a product $\mathcal{A} = S^1 \times \mathbb{R}$ of a circle and the real line. The projection map $p\colon \mathcal{A} \to S^1$ induces a map

$$p_n\colon C_n(\mathcal{A}) \to \mathrm{Sym}_n(S^1),$$

where the latter space is the n–fold symmetric power of S^1, defined as the quotient of $(S^1)^{\times n}$ by the action of $\mathfrak{S}_n$ on the coordinates.

For a given pair of configurations $(x, y) \in C_n(\mathcal{A}) \times C_n(\mathcal{A})$ we interpret $p_n(x)$ and $p_n(y)$ as finite subsets of S^1, ie we forget the multiplicities of points in S^1. The cardinality $\deg(x, y) = |p_n(x) \cup p_n(y)|$ of the union of those subsets will be called the *degree* of the pair.

Notice that $\deg(x, y)$ is at least 1 and at most $2n$. This yields a decomposition of $C_n(\mathcal{A}) \times C_n(\mathcal{A})$ into $2n$ disjoint subspaces $L_k = \deg^{-1}(k)$, corresponding to the different values of deg; see Figure 2. Furthermore, L_k is a smooth embedded manifold and in particular an ENR.

5.2 Local motion planners

Given a pair $(x, y) \in L_k$, the union $p_n(x) \cup p_n(y)$ contains exactly k distinct points $q_1, \dots, q_k \in S^1$, ordered cyclically on S^1 in the clockwise direction. We need to

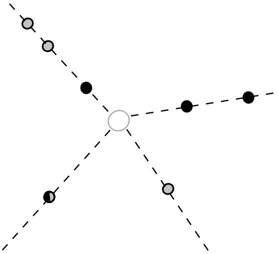

Figure 2: A pair of configurations in L_4, with one double point

introduce some notation. Let $n_{x,i}$ be the number of points in x mapped to q_i under p and let $n_{y,i}$ be the number of points in y mapped to q_i under p. Finally, let $\delta_i = n_{x,i} - n_{y,i}$ be the difference between those two numbers.

The following map is continuous and well defined:

$$\psi_k \colon L_k \to \Big\{(m_i)_i \in \mathbb{Z}^k \;\Big|\; \sum_{i=1}^{k} m_i = 0,\ \sum_{i=1}^{k} |m_i| \le 2n\Big\}\Big/(12\dots k), \qquad (x,y) \mapsto [(\delta_i)_i].$$

Here $(12\dots k) \in \mathfrak{S}_k$ is the *long cycle*, permuting the components m_i.

Because the preimages of different $[(\delta_i)_i]$ are topologically disjoint, we can define the local section of the free path fibration over L_k separately on each preimage.

Given a pair of configurations $(x,y) \in L_k$ lying in the preimage $\psi_k^{-1}([(\delta_i)_i])$, we need to construct a path between them, continuously over L_k.

If $\delta_i = 0$ for all i, we will simply move the points of x onto the points of y on each fiber of p by linear interpolation inside the fibers.

On the other hand, if there exists an i such that $\delta_i \neq 0$, first we need to construct a path from x to $\tilde{x}$ such that $\deg(\tilde{x}, y) = \tilde{k}$ for some $\tilde{k} \le k$, and such that $(\tilde{x}, y) \in \psi_{\tilde{k}}^{-1}((0)_i)$; then we concatenate this path with the fiberwise linear interpolation used above. The path from x to $\tilde{x}$ will consist in an iteration of one particular deformation which we describe in the following and which is illustrated in Figure 3.

5.3 First step

Let $(x,y) \in L_k$ as above and let x consist of the points $x_{i,l} \in \mathcal{A}$ for $1 \le i \le k$ and $1 \le l \le n_{x,i}$, where for each i the points $x_{i,l}$ are exactly those lying over $q_i \in S^1$ and the indices are chosen according to the order of the points on the fiber $p^{-1}(q_i) \simeq \mathbb{R}$.

We are going to deform x into another configuration, denoted by $x^{(1)}$.

Whenever $\delta_i > 0$, we move the δ_i top points of x in $p^{-1}(q_i)$ clockwise until they reach $p^{-1}(q_{i+1})$, on top of all points of x already in $p^{-1}(q_{i+1})$ (if any). More precisely, we move the points $x_{i,l}$ for $n_{y,i} + 1 \le l \le n_{x,i}$ to $p^{-1}(q_{i+1})$ so as to keep their order and their pairwise distances, and such that $x_{i,n_{y,i}+1}$ reaches the position $1 + \max\{0, x_{i+1,n_{x,i+1}}\}$ inside the fiber $p^{-1}(q_{i+1}) \simeq \mathbb{R}$. We move these points by linear interpolation along the interval $[q_i, q_{i+1}] \subset S^1$ and along $\mathbb{R}$. We do this simultaneously for all i for which $\delta_i > 0$. Note that the indices are considered modulo k. This is shown in Figure 3.

It is clear from the construction that this deformation is continuous within $\psi_k^{-1}([(\delta_i)_i])$.

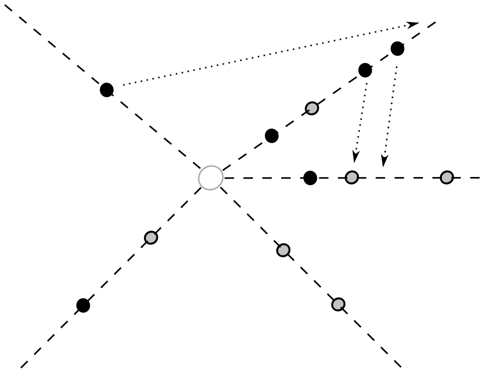

Figure 3: One iteration of the motion planner on the annulus. Notice that the positions of the gray points on a given fiber are disregarded when moving black points towards it because the points exist in two separate spaces.

5.4 Iterations of the first step

We started with a pair of configurations $(x, y) \in L_k$ and in the previous subsection we constructed a deformation of x into $x^{(1)}$. Clearly $k_1 = \deg(x^{(1)}, y) \leq k$. We can now repeat the process starting with the pair $(x^{(1)}, y)$ to get a new configuration $x^{(2)}$, again without changing y. Iterating this, we get a sequence of configurations $x^{(j)}$ and a sequence of degrees $k_j = \deg(x^{(j)}, y)$ which is weakly decreasing.

If this algorithm terminates after T steps, then it gives us a path from $(x, y) \in L_k$ to $(x^{(T)}, y) \in \psi_{\tilde{k}}^{-1}((0)_i)$. Furthermore, because each iteration is continuous it yields a continuous deformation of L_k into $\psi_{\tilde{k}}^{-1}((0)_i)$, which completes the proof.

To see that the algorithm does indeed terminate, note that there exists an $N \in \mathbb{N}$ such that $k_j = k_N$ for all $j \geq N$. After k_N further iterations we have that $\delta_i^{(N+k_N)} = 0$ for all $1 \leq i \leq k_N$ and we are done. This follows from the following three facts, which are easy to check:

(1) For all $j > N$, if $\delta_i^{(j)} > 0$, then $\delta_{i-1}^{(j-1)} > 0$.

(2) For all $j \geq N$, if $\delta_i^{(j)} \geq 0$, then $\delta_i^{(j+1)} \geq 0$.

(3) For all $j \geq N$ we have $\sum_i \delta_i^{(j)} = 0$.

Indeed, if $\delta_i^{(N+k_N)} \neq 0$ for some i, we may assume that $\delta_i^{(N+k_N)} > 0$ because of (3). By (1), this would imply that $\delta_{i-l}^{(N+k_N-l)} > 0$ for all $0 \leq l \leq k_N - 1$ and therefore, by (2), $\delta_{i-l}^{(N+k_N)} \geq 0$ for all $0 \leq l \leq k_N - 1$.

This would mean that $\sum_{l=1}^{k_N-1} \delta_{i-l}^{(N+k_N)} > 0$, which contradicts (3). This completes the proof of Theorem 5.3. □

6 Proof of Theorems 1.3 and 1.5

Proof of Theorem 1.3 Notice that the pure braid group on the disc P_n is isomorphic to the pure braid group $P_{n-1}(\mathcal{A})$ on the annulus with one less strand in the following way. Every braid in P_n can be chosen such that the last strand does not move and that strand is identified with the central hole in the annulus.

Consider the subgroup $Z_{n-1} \cong \mathbb{Z}^{n-1} \leq P_{n-1}(\mathcal{A}) \cong P_n$ given by the braids in $P_{n-1}(\mathcal{A})$ in which all strands move in concentric circles around the central hole.

Recall that, for the abelianization, $P_n^{\mathrm{ab}} \cong \mathbb{Z}^{\binom{n}{2}}$. The abelianization homomorphism is given by the collection over all unordered pairs $\{i, j\}$ of the maps $\psi_{i,j}\colon P_n \to P_2 \cong \mathbb{Z}$ forgetting all strands except the i^{th} and the j^{th} (measuring the linking number between the strands i and j).

Conjugating by an element $g \in B_n$ is compatible, under the abelianization, with the induced permutation of the components $\psi_{i,j}$ of P_n^{ab} coming from the canonical permutation in $\mathfrak{S}_n$ associated to g.

In light of the above, it is clear that the commutator subgroup $[P_n, P_n]$ (the kernel of the abelianization homomorphism) is not only normal in P_n but also in B_n. Furthermore, it also follows that Z_{n-1} is mapped injectively under the abelianization homomorphism and thus has a trivial intersection with $[P_n, P_n]$. Taken together this implies that the conjugates of a nontrivial element of $[P_n, P_n]$ cannot lie in Z_{n-1}.

The lower bound now follows from Theorem 2.3 together with Lemma 6.1 below.

Finally, the upper bound follows from Lemma 2.4 and $\mathrm{cd}(B_n) = n-1$, which can be shown using the Fadell–Neuwirth fibrations as for the other aspherical surfaces. □

Proof of Theorem 1.5 Let $(p_1, \dots, p_n) \in F(D, n)$ denote an ordered configuration of n points in the disc D and let $1 \leq k \leq n$, to be chosen suitably later. Recall that based loops in $F(D, n)$ represent braids in the pure braid group P_n and let $A \subset P_n$ consist of those pure braids represented by loops in which the points $p_1, p_2, \dots, p_k$ are fixed *in the middle* and $p_{k+1}, \dots, p_n$ independently rotate around this cluster in concentric orbits. Clearly we have $A \cong \mathbb{Z}^{n-k}$.

We now write $n = mk + r$ for appropriate $m \geq 0$ and $1 \leq r \leq k$. Notice that r is assumed to be positive.

Divide the points $p_1, \dots, p_n$ into m clusters of k points each plus an additional cluster of r points. Let B be the subgroup of P_n in which points of the same cluster interact freely and such that moreover the $m+1$ clusters are allowed to move around each other, so long as they don't mix and their trajectories describe an element in $[P_{m+1}, P_{m+1}]$.

More formally, let $E_2(m+1)$ be the space of ordered configurations of $m+1$ little discs $D_1, \dots, D_{m+1}$ inside the disc D. Each disc D_i is uniquely determined by its center and its (positive) radius and the little discs are required to have disjoint interiors (see [14] for an introduction to the operad of little cubes). There is a map

$$E_2(m+1) \times F(D,k) \times \cdots \times F(D,k) \times F(D,r) \to F(D,n)$$

given by embedding each configuration of k or r points into the corresponding disc D_i, using the only positive rescaling of D onto D_i. Because $E_2(m+1)$ is also a classifying space for P_{m+1}, there is a homomorphism on fundamental groups

$$\gamma\colon P_{m+1} \times P_k \times \cdots \times P_k \times P_r \to P_n.$$

To show that γ is injective, let ρ be the product of the following $m+2$ maps:

- One map $P_n \to P_{m+1}$ given by forgetting all strands but a chosen one in each cluster, so that exactly $m+1$ strands remain.
- The maps $P_n \to P_k$ and $P_n \to P_r$ given by forgetting all strands outside a given cluster.

It is easy to see that ρ is a retraction of γ and that therefore γ is injective. The subgroup $B \subset P_n$ is defined to be the image of the restriction of γ to $[P_{m+1}, P_{m+1}] \times (P_k)^m \times P_r$.

Next we need to check that A and B satisfy the assumptions of Theorem 2.3 as subgroups of B_n, ie $gAg^{-1} \cap B = \{1\}$ for all $g \in B_n$. For this we will use the abelianization of the pure braid group P_n. As we saw in the proof of Theorem 1.3, the abelianization detects the pairwise linking numbers between the braids and conjugation by $g \in B_n$ permutes those numbers by the induced permutation.

The following property of an element $\sigma \in P_n$ is invariant under conjugation by each $g \in B_n$:

There exists an index $1 \leq j \leq n$ and k other indices $i_1, \dots, i_k$ such that $\psi_{j,i_l}(\sigma) \neq 0$ for all $1 \leq l \leq k$.

Let $\alpha \in A$ be a nontrivial braid. In such a braid there is at least one point p_j, for $k+1 \le j \le n$, which rotates a nonzero number of times around the points $p_1, \dots, p_k$. Therefore, the numbers $\psi_{l,j}(\alpha)$ are all nonzero (and equal to each other) for $1 \le l \le k$.

However, no braid $\beta \in B$ has the property above. Indeed, $\psi_{i,j}(\beta)$ can be nonzero only if p_i and p_j are in the same cluster, and every cluster contains at most k points.

Hence, we get that, for each $1 \le k \le n$,

$$\begin{aligned}\mathsf{TC}(B_n) &\ge \mathsf{cd}(A \times B)\\ &= n-k+m(k-1)+r-1+\mathsf{cd}([P_{m+1}, P_{m+1}])\\ &\ge 2n-k-m-1+\tfrac{1}{2}m-1 && \text{(by Lemma 6.1)}\\ &= 2n-k-\tfrac{1}{2}m-\tfrac{3}{2}.\end{aligned}$$

Choosing $k = \lfloor \sqrt{n/2} \rfloor$, the inequality

$$n = mk + r \ge mk + 1$$

implies that

$$m \le \frac{n-1}{k}$$

and so

$$m \le \left\lfloor \frac{n-1}{k} \right\rfloor \le 2k+4$$

by the choice of k. Therefore,

$$\mathsf{TC}(B_n) \ge 2n - 2\lfloor \sqrt{n/2} \rfloor - 3 - \tfrac{1}{2}$$

and since $\mathsf{TC}(B_n)$ is an integer we can drop the term $\frac{1}{2}$. □

Lemma 6.1 *Let $[P_n, P_n]$ be the commutator subgroup of the pure braid group P_n. Then*

$$\mathsf{cd}([P_n, P_n]) \ge \tfrac{1}{2}(n-2).$$

Proof Like in the previous proof, let $E_2(3)$ denote the space of ordered configurations of three little discs D_1, D_2 and D_3 inside a disc D. There exists a map

$$E_2(3) \times F(D,3) \to F(D,5),$$

given by embedding the configurations in $F(D,3)$ into the first disc D_1 (after the appropriate rescaling) and by mapping the other two little discs to their center points. Iterating this construction $k-1$ times results in the map

$$\underbrace{E_2(3) \times E_2(3) \times \cdots \times E_2(3)}_{k-1} \times F(D,3) \to F(D, 2k+1).$$

On fundamental groups this yields a homomorphism

$$P_3^k \to P_{2k+1}. \tag{4}$$

Similarly to the previous proof, this homomorphism is injective. By construction the images of the different P_3 factors commute with each other.

Let $\mathbb{Z}$ be an infinite cyclic subgroup of $[P_3, P_3]$. The image of the homomorphism (4) restricted to $\mathbb{Z}^k \leq P_3^k$ is isomorphic to $\mathbb{Z}^k$ by the above observations and it is a subgroup of $[P_{2k+1}, P_{2k+1}]$. By the well-known properties of cohomological dimension,

$$\mathsf{cd}([P_{2k+1}, P_{2k+1}]) \geq \mathsf{cd}(\mathbb{Z}^k) = k.$$

This proves the claim for $n = 2k+1$ odd. For n even the claim immediately follows from $P_{n-1} \leq P_n$. □

7 Motion planner for the disc

Let D be the disc. In this section we are going to give an explicit motion planner which will imply that $\mathsf{TC}(C(D,3)) = 3$, as stated in Theorem 1.6. Observe that a motion planner on a subset of $X \times X$ is the same as a deformation into the diagonal.

Proof of Theorem 1.6 The lower bound follows from Theorem 1.3 because

$$\mathsf{cd}([P_3, P_3] \times Z_2) = 1 + 2 = 3.$$

We will work with the space $C_3 = C(\mathbb{C}, 3) \simeq C(D, 3)$ for the remainder of the proof.

To show $\mathsf{TC}(C_3) \leq 3$, it suffices to find a decomposition of $C_3 \times C_3$ into four disjoint ENRs such that each of them can be deformed to the diagonal, by Lemma 5.2.

In the next subsections we will first decompose $C_3 \times C_3$ into four disjoint ENRs E_0, E_1, E_2 and E_3 and discuss some geometric properties of these; then we will describe a motion planner on each E_i.

7.1 Decomposition of $C_3 \times C_3$

First we need a notion of *orientation* for configurations in C_3. For this we define a function $\Delta\colon C_3 \to \mathbb{C}^*$ by

$$\Delta(\{z_1, z_2, z_3\}) = (z_1 - z_2)^2 (z_2 - z_3)^2 (z_3 - z_1)^2,$$

and let $\delta = \Delta/|\Delta|\colon C_3 \to S^1$ be its normalization.

We say that two configurations $x, y \in C_3$ are *cooriented* if $\delta(x) = \delta(y)$. Let $P \subset C_3 \times C_3$ denote the closed subspace of pairs (x, y) for which x and y are cooriented and let $N = C_3 \times C_3 \setminus P$ denote its complement.

The Lie group S^1 on C_3 by rotations about the origin. Given a configuration $x \in C_3$ and an element $\theta \in S^1$,

$$\delta(\theta \cdot x) = \theta^6 \delta(x). \tag{5}$$

Let $L \subset C_3$ consist of those configurations for which all three points are on a line and let $T = C_3 \setminus L$ be its complement. The points in a configuration in T form a nondegenerate triangle; $L \subset C_3$ is closed and $T \subset C_3$ is open.

We define a deformation retraction of L onto the subspace $L_{\boldsymbol{R}}$ containing configurations of three aligned points, one at the origin and two on the unit circle and opposite to each other. Note that $L_{\boldsymbol{R}}$ is homeomorphic to a circle and is invariant under rotation. Given a configuration in L, we translate it so that the central point ends up at the origin and then slide the two outer points along the line which goes through all three points until they are both at distance 1 from the origin. This defines a deformation retraction $r_L\colon L \to L_{\boldsymbol{R}}$. The deformation preserves δ, because the direction determined by any two points in the configuration remains the same throughout the deformation.

Similarly we define a deformation retraction of T onto the subspace $T_{\boldsymbol{R}}$ containing configurations of three points on the unit circle that form an equilateral triangle. Note that $T_{\boldsymbol{R}}$ is also homeomorphic to a circle and invariant under rotation. Given a configuration in T, we translate it until the center of mass coincides with the origin. Then we slide all three points simultaneously along the lines going through the origin until the points land in the unit circle. Finally, we rotate the points until they are at equal distance from each other on the unit circle.

More precisely, let X, Y and Z be a configuration of three points on the unit circle, appearing in this order clockwise. Consider the lengths of the arcs XY, YZ and ZX. If the arcs are all of the same length, then we are done. If there is precisely one arc of minimal length, say XY, then we can slide X and Y at the same speed along the unit circle, gradually increasing the length of XY and decreasing both YZ and ZX, until the length of XY becomes equal to at least one of the other two arcs. Therefore, we may assume that there are exactly two arcs of minimal length. In this case there is one arc, say YZ, which is strictly longer than the other two arcs. Slide both Y and Z at the same speed along the unit circle, gradually decreasing the length of YZ and increasing the lengths of XY and ZX, until all three arcs are equal. See Figure 4.

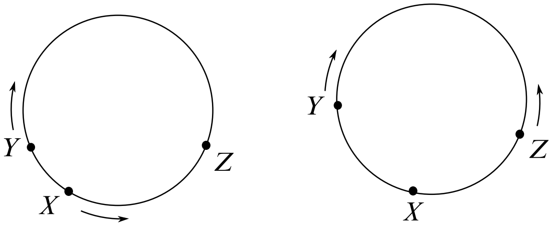

Figure 4: First step (left) and second step (right)

Additionally, we make sure that the above deformation preserves $\delta(x)$ by constantly rotating the configuration x about the origin during the whole process to compensate for the potential change of $\delta(x)$. More precisely, let $H\colon T\times[0,1]\to C_3$ be the homotopy described above, with $H(\,\cdot\,,1)\in T_R$, and consider the function $\overline{\delta}\colon T\times[0,1]\to S^1$ defined by

$$\overline{\delta}(x,t)=\delta(H(x,t))/\delta(x).$$

Then $\overline{\delta}(\,\cdot\,,0)\colon T\to S^1$ is the constant function 1 and it admits a lift to the universal covering $\mathbb{R}\to S^1$, namely the constant function 0. We can then extend this lift to all positive times, obtaining a map $\widetilde{\delta}\colon T\times[0,1]\to\mathbb{R}$. Now let $\widetilde{\rho}\colon T\times[0,1]\to\mathbb{R}$ be given by

$$\widetilde{\rho}(x,t)=\tfrac{1}{6}\widetilde{\delta}(x,t)$$

and denote by $\rho\colon T\times[0,1]\to S^1$ its projection onto S^1 along the universal covering map $\mathbb{R}\to S^1$.

Finally, consider the homotopy $\overline{H}\colon T\times[0,1]\to C_3$ given by

$$\overline{H}(x,t)=(\rho(x,t))^{-1}\cdot H(x,t).$$

Then $\overline{H}$ is a deformation retraction of T onto T_R preserving δ at all times; this follows easily from the construction and from formula (5).

Write $r_T=\overline{H}(\,\cdot\,,1)\colon T\to T_R$.

We are now ready to construct the decomposition into disjoint ENRs as follows:

- $E_0=P\cap(L\times L)$.
- $E_1=N\cap(L\times L)\sqcup P\cap(T\times L\sqcup L\times T)$.
- $E_2=N\cap(T\times L\sqcup L\times T)\sqcup P\cap(T\times T)$.
- $E_3=N\cap(T\times T)$.

Note that the subspaces E_i are semialgebraic sets and therefore ENRs.

Furthermore, the disjoint unions above are topological, ie they form disconnected components inside each E_i. This follows from the fact that the disjoint components are relatively open inside each E_i. For example, $N \cap (L \times L)$ and $P \cap (T \times L)$ are the intersections of E_1 with the open sets N and $T \times C_3$, respectively, and $N \cap (T \times L)$ is the intersection of E_2 with the open set $N \cap (T \times C_3)$.

7.2 Local motion planners

We show now that each E_i deformation retracts onto a disjoint union of circles. First we notice that for $A, B \in \{L, T\}$ the intersection $N \cap (A \times B)$ can be deformed to $P \cap (A \times B)$. Given a pair $(x, y) \in N \cap (A \times B)$, rotate x clockwise about the origin until x and y are cooriented. This can be done continuously thanks to formula (5).

The subspaces $P \cap (L \times L)$ and $P \cap (T \times T)$ deformation retract to $P \cap (L_R \times L_R)$ and $P \cap (T_R \times T_R)$, respectively, because the retractions r_L and r_T commute with δ.

The subspaces $P \cap (L_R \times L_R)$ and $P \cap (T_R \times T_R)$ in turn consist of a disjoint union of three circles and a disjoint union of two circles, respectively, where each circle is an orbit under the diagonal action of S^1 on $C_3 \times C_3$. Precisely one orbit in $P \cap (L_R \times L_R)$ and one orbit in $P \cap (T_R \times T_R)$ already lie in the diagonal of $C_3 \times C_3$. The remaining orbits consist of pairs of lines or pairs of triangles which are at a given angle from each other $\left(\frac{\pi}{3}\right.$ or $\frac{2\pi}{3}$ in the case of lines and $\frac{\pi}{3}$ in the case of triangles, to be precise$\left.\right)$. See Figures 5 and 6. They can be deformed into the diagonal by rotating the first configuration in every pair clockwise about the origin until it is equal to the second configuration in that pair.

Similarly, the space $P \cap (L \times T)$ can be deformed to $P \cap (L_R \times T_R)$, which consists of one single orbit under the diagonal S^1–action; see Figure 7. Specifically, it contains pairs of configurations (x, y), where the points in y form an equilateral triangle centered at the origin and the points in x lie on a line parallel to one of the sides of said triangle and are symmetrically distributed around the origin. We move the point in y opposite to the side parallel to x to the origin and the other two points in y to the

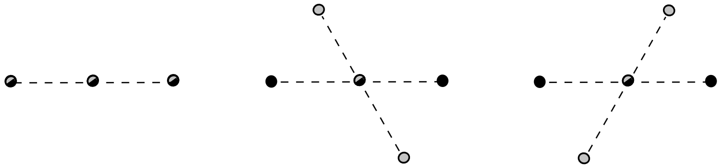

Figure 5: Path-components of $P \cap (L_R \times L_R)$ (up to rotation)

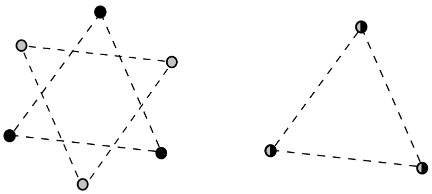

Figure 6: Path-components of $P \cap (T_R \times T_R)$ (up to rotation)

corresponding outer points in x. The pair (x,x) is obviously in the diagonal and so we are done.

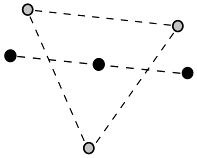

Figure 7: The subset $P \cap (T_R \times L_R)$ (up to rotation)

This completes the proof because the deformation can be defined separately on the different disconnected components of each E_i. □

8 Conclusions

The results in this paper can be viewed equivalently as finding the values for the topological complexity of either full braid groups of surfaces or unordered configuration spaces of surfaces, since, for aspherical surfaces $\mathcal{S}$,

$$\mathsf{TC}(C(\mathcal{S},n)) = \mathsf{TC}(B_n(\mathcal{S})).$$

All the results except the ones which rely on finding explicit motion planners (or equivalently deformations into the diagonal) extend to finite-index subgroups of $B_n(\mathcal{S})$ with the same proofs. To be precise, the results which generalize to finite-index subgroups are the ones given in Theorems 1.3, 1.5, 4.1, 4.2 and 5.1.

In particular, those results apply to the pure braid groups $P_n(\mathcal{S})$ and the mixed braid groups from [13]. Observe that for aspherical surfaces $\mathcal{S}$ the topological complexity $\mathsf{TC}(P_n(\mathcal{S}))$ of the pure braid groups of $\mathcal{S}$ is the same as the topological complexity $\mathsf{TC}(F(\mathcal{S},n))$ of the ordered configuration spaces of $\mathcal{S}$.

Thus, the methods in this paper yield an alternative proof for some of the results given by Cohen and Farber in [2], in particular the topological complexity of ordered configuration spaces for all nonclosed orientable surfaces (for the ordered configuration spaces of the disc one can use a slightly modified version of the proof of Theorem 5.3 to find explicit motion planners). Furthermore, it extends their results to all nonclosed nonorientable surfaces except the Möbius band.

It is worth noting that the results in this paper taken together with the results in [2] are consistent with the possibility that the topological complexities of the ordered and the unordered configuration spaces of a surface coincide for all surfaces.

The only remaining aspherical surface for which the gap between the lower bound and the upper bound for the topological complexity of its unordered configuration spaces is still arbitrarily large is, perhaps surprisingly, the disc.

If it is in fact true that $\mathsf{cd}([P_n, P_n]) = n-2$, then Theorem 1.3 would imply

$$\mathsf{TC}(C(D,n)) \geq 2n-3.$$

If additionally the upper bound for $n=3$ given in Theorem 1.6 generalized to higher n, this would completely determine $\mathsf{TC}(C(D,n))$. We make the following:

Conjecture 8.1 If D is the disc, then

$$\mathsf{TC}(C(D,n)) = \mathsf{TC}(B_n) = 2n-3.$$

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Hyperbolic extensions of free groups from atoroidal ping-pong

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We prove that all atoroidal automorphisms of $\mathrm{Out}(F_N)$ act on the space of projectivized geodesic currents with generalized north–south dynamics. As an application, we produce new examples of nonvirtually cyclic, free and purely atoroidal subgroups of $\mathrm{Out}(F_N)$ such that the corresponding free group extension is hyperbolic. Moreover, these subgroups are not necessarily convex cocompact.

20F28, 20F67; 37D40

1 Introduction

Let F_N be a free group of rank $N \geq 3$, and $\mathrm{Out}(F_N)$ be its outer automorphism group. Consider the short exact sequence

$$1 \to F_N \xrightarrow{\iota} \mathrm{Aut}(F_N) \xrightarrow{q} \mathrm{Out}(F_N) \to 1,$$

where ι sends an element of F_N to the corresponding inner automorphism, and q is the natural quotient map.

Given a subgroup $\Gamma < \mathrm{Out}(F_N)$, the preimage $E_\Gamma = q^{-1}(\Gamma)$ gives an extension of F_N. In fact, any extension of F_N produces an extension of this form; see Dowdall and Taylor [16, Section 2]. Motivated by a long history of investigating hyperbolic extensions of hyperbolic groups — see Bestvina and Feighn [1], Bestvina, Feighn and Handel [4], Farb and Mosher [17], Hamenstaedt [20], Kent and Leininger [26] and Mosher [31] — Dowdall and Taylor [16] initiated a systematic study of the following question:

What conditions on the group Γ guarantee that the extension group E_Γ is hyperbolic?

When the group Γ is infinite cyclic, generated by $\varphi \in \mathrm{Out}(F_N)$, combined work of Bestvina and Feighn [1] and Brinkmann [8] shows that E_Γ is hyperbolic if and only if φ is *atoroidal*, meaning that no power of φ fixes a nontrivial conjugacy class in F_N. Dowdall and Taylor [16] proved that if a finitely generated subgroup $\Gamma < \mathrm{Out}(F_N)$ is

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purely atoroidal (ie every infinite-order element is atoroidal) and the orbit map from Γ into the free factor complex is a quasi-isometric embedding, then the extension E_Γ is hyperbolic. The second condition also implies that every infinite-order element $\varphi \in \Gamma$ is *fully irreducible*, meaning that no power of φ fixes the conjugacy class of a proper free factor; see Section 2 for definitions.

So far the only known examples of hyperbolic extensions of free groups come from slight variations, or iterated applications of these two constructions, and Schottky-type subgroups generated by high powers of fully irreducible and atoroidal elements. The following subgroup alternative result allows us to produce more examples:

Theorem 1.1 *Let $\mathcal{H} < \mathrm{Out}(F_N)$ be a subgroup that contains an atoroidal element φ. Then one of the following occurs:*

(1) *There is some minimal $\mathcal{H}$–invariant free factor A of F_N such that the restriction of $\mathcal{H}$ to A is virtually cyclic in $\mathrm{Out}(A)$.*

(2) *There exists a subgroup $\Gamma \leq \mathcal{H}$ such that $\Gamma \cong \mathrm{F}_2$ and that every nontrivial element of Γ is atoroidal. Moreover, the corresponding extension group E_Γ is hyperbolic.*

Remark 1.2 Theorem 1.1 generalizes a well-known result of Bestvina, Feighn and Handel. Indeed, if the subgroup $\mathcal{H}$ is irreducible, namely no finite-index subgroup of $\mathcal{H}$ fixes a proper free factor, then $\mathcal{H}$ contains a fully irreducible element by a theorem of Handel and Mosher [21]; see also Horbez [22] for a concise and more general proof. Since $\mathcal{H}$ contains an atoroidal element, then Theorem 5.4 of Uyanik [36] implies that $\mathcal{H}$ contains an element which is both fully irreducible and atoroidal. In that case, Bestvina, Feighn and Handel [4] show that either $\mathcal{H}$ is virtually cyclic, or there is a nonabelian free subgroup Γ of $\mathcal{H}$ such that every nontrivial element of Γ is atoroidal and the corresponding free group extension is hyperbolic. A different proof of the aforementioned result of Bestvina, Feighn and Handel is given by Kapovich and Lustig [25], who additionally obtained that each nontrivial element is fully irreducible.

Remark 1.3 The subgroup $\mathcal{H}$ isn't necessarily irreducible or it doesn't have to preserve a free splitting of F_N. Theorem 1.1 gives new examples of hyperbolic extensions of free groups, which do not come from previously known constructions. In particular, they are not necessarily convex cocompact; see Dowdall, Taylor and Tiozzo [15; 16; 34].

The main ingredient in the proof of Theorem 1.1 is the following dynamical result. See Section 3.1 for definitions.

Theorem 1.4 *Let* $\varphi \in \mathrm{Out}(F_N)$ *be an atoroidal outer automorphism of a free group of rank* $N \geq 3$*. Then there exist a simplex of attraction* Δ_+ *and a simplex of repulsion* Δ_- *in* $\mathbb{P}\mathrm{Curr}(F_N)$ *such that* φ *acts on* $\mathbb{P}\mathrm{Curr}(F_N)$ *with generalized north–south dynamics from* Δ_- *to* $\Delta+$*.*

The space $\mathbb{P}\mathrm{Curr}(F_N)$ of *projectivized geodesic currents* contains positive multiples of conjugacy classes as a dense subset, and hence serves as a natural tool for detecting atoroidal outer automorphisms; see Section 2.5 for details. The proof of Theorem 1.4 builds on our earlier results with M Lustig about dynamics of reducible substitutions [29] and is modeled on the proof of the specific case where both φ and φ^{-1} admit absolute train track representatives as we treated in [28]. In this paper, we use completely split relative train track maps (CTs), which are particularly nice topological representatives introduced by Feighn and Handel [18]. The new key insight in the proof of Theorem 1.4 is to use the legal structure coming from the splitting units in the CT that represents $\varphi \in \mathrm{Out}(F_N)$ rather than using the classical legal structure coming from the edges.

As a byproduct of Theorem 1.4 we also obtain:

Corollary 1.5 *Let* $\varphi \in \mathrm{Out}(F_N)$ *be a fully irreducible and atoroidal outer automorphism. Then, for any atoroidal outer automorphism* $\psi \in \mathrm{Out}(F_N)$ *(not necessarily fully irreducible) which is not commensurable with* φ *(ie* $\varphi^t \neq \psi^s$ *for any* s *and* t*), there exists an exponent* $M > 0$ *such that, for all* $n, m > M$*, the subgroup* $\Gamma = \langle \varphi^n, \psi^m \rangle < \mathrm{Out}(F_N)$ *is purely atoroidal and the corresponding extension* E_Γ *is hyperbolic.*

Note that the subgroup Γ in Corollary 1.5 is irreducible, and since Γ is not purely fully irreducible the orbit map to the free factor graph is not a quasi-isometric embedding [16].

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2 Preliminaries

2.1 Graphs and graph maps

A *graph* G is a 1–dimensional cell complex, where 0–cells are called *vertices* and 1–cells are called topological edges. We denote the set of vertices by VG, and the set of topological edges by $E_{\mathrm{top}}G$. Identifying the interior of an edge with the open interval $(0,1)$ each edge admits exactly two orientations. We denote the set of *oriented* edges by EG. Choosing an orientation on each edge splits the set EG into two disjoint sets: the set $E^{+}G$ of positively oriented and the set $E^{-}G$ of negatively oriented edges. Given an oriented edge $e \in EG$, the initial vertex of e is denoted by $o(e)$ and the terminal vertex of e is denoted by $t(e)$, and the edge with the opposite orientation is denoted by e^{-1}.

An *edge path* γ in G is a concatenation $\gamma = e_1 e_2 \cdots e_n$ of edges in G such that $t(e_{i-1}) = o(e_i)$ for all $i = 2, \dots, n$. An edge path $\gamma = e_1 e_2 \cdots e_n$ is called *reduced (or tight)* if $e_i^{-1} \neq e_{i+1}$ for all $i = 1, \dots, n-1$. A reduced edge path $\gamma = e_1 e_2 \cdots e_n$ is called cyclically reduced if $o(\gamma) = t(\gamma)$ and in addition $e_n^{-1} \neq e_1$. We call cyclically reduced edge paths *circuits*.

Given an edge path γ, we denote the reduced edge path obtained by a homotopy relative to endpoints of γ by $[\gamma]$.

2.2 Markings and topological representatives

Let $\mathrm{R_N}$ denote the rose with N pedals, which is the finite graph with one vertex and N loop edges attached to that vertex. A *marking* is a homotopy equivalence $m\colon \mathrm{R_N} \to G$ where G is a finite graph all of whose vertices are at least valence 2.

A homotopy equivalence $f\colon G \to G$ is a *(topological) graph map* if it sends vertices to vertices, and its restriction to the interior of an edge is an immersion. Let $m'\colon G \to \mathrm{R_N}$ be a homotopy inverse to the marking $m\colon \mathrm{R_N} \to G$. We say that a topological graph map is a *topological representative* of an outer automorphism $\varphi \in \mathrm{Out}(F_N)$ if the induced map satisfies $(m' \circ f \circ m)_*\colon F_N \to F_N = \varphi$.

A *filtration* for a topological representative $f\colon G \to G$ is an ascending sequence of f–invariant subgraphs $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_k = G$. The closure of $G_r \setminus G_{r-1}$ is called the r^{th} *stratum*, and is denoted by H_r.

For each stratum H_r, there is an associated *transition matrix* M_r of H_r which is a nonnegative integer square matrix. The ij^{th} entry of H_r records the number of times $[f(e_i)]$ crosses e_j or e_j^{-1}. A nonnegative square matrix M is called *irreducible* if for each i and j, there exists $k = k(i,j)$ such that $M_{ij}^k > 0$, the matrix M is called *primitive* if k can be chosen independent of i and j. The stratum H_r is called *irreducible (resp. primitive)* if and only if M_r is irreducible (resp. primitive). If M_r is irreducible then it has a unique eigenvalue $\lambda \geq 1$, called the Perron–Frobenius (PF) eigenvalue, for which the associated eigenvector is positive. We say that H_r is an *exponentially growing stratum* or EG stratum if $\lambda > 1$ and *nonexponentially growing stratum* or NEG stratum if $\lambda = 1$. We say that H_r is a *zero stratum* if M_r is the zero matrix.

2.3 Train track maps

We first set up the relevant terminology to define relative train track maps, and their strengthened versions, CTs. The standard resources for this section are [6; 5; 18].

Let $f\colon G \to G$ be a topological graph map. A *direction* at a point $v \in G$ is the germ of an initial segment of an oriented edge. The map $f\colon G \to G$ induces a natural *derivative map* Df on the set of germs, and we say that a direction is *fixed* or *periodic* if it is fixed or periodic under the derivative map. A *turn* in G is an unordered pair of directions. We say that a turn is *degenerate* if the two directions are the same, and *nondegenerate* otherwise. A turn is called *illegal* if its image under some iterate of Df is degenerate, otherwise a turn is called *legal*. An edge path $\gamma = e_1 e_2 \cdots e_k$ is called legal if each turn (e_i^{-1}, e_{i+1}) is legal. We say that, a turn is contained in a stratum H_r if both directions are contained in H_r. An edge path γ is called *r–legal* if every turn in γ that is contained in H_r is legal. If H_r is an EG stratum, and γ is a path whose endpoints are in $H_r \cap G_{r-1}$, then γ is called a *connecting path*.

Definition 2.1 A homotopy equivalence $f\colon G \to G$ representing $\varphi \in \mathrm{Out}(F_N)$ is called a *relative train track* map if for every exponentially growing stratum H_r the following hold:

(RTT-i) Df maps the set of directions in H_r to itself.

(RTT-ii) For each connecting path γ for H_r, $[f(\gamma)]$ is a connecting path for H_r. In particular, $[f(\gamma)]$ is nontrivial.

(RTT-iii) If γ is r–legal, then $[f(\gamma)]$ is r–legal.

Definition 2.2 (Nielsen paths) A path ρ is a *periodic Nielsen path* if there is an exponent $k \geq 1$ such that $[f^k(\rho)] = \rho$. The minimal such k is called the *period*, and if $k = 1$ then ρ is called a *Nielsen path.* A periodic Nielsen path is called *indivisible* if it cannot be written as a concatenation of periodic Nielsen paths. We will denote the (periodic) indivisible Nielsen paths by (pINPs) INPs.

Definition 2.3 (taken and exceptional paths) A path $\gamma \in G$ is called *r–taken* by $f\colon G \to G$ if γ appears as a subpath of $f^k(e)$ for some $k \geq 1$ and for some edge $e \in H_r$ in an irreducible stratum. We will drop r and only say *taken* whenever r is irrelevant. Let e_i and e_j be linear edges as defined in Definition 2.10 below such that $f(e_i) = e_i w^{m_i}$ and $f(e_j) = e_j w^{m_j}$ for some root free Nielsen path w. Then a path of the form $e_i w^p e_j^{-1}$ for $p \in \mathbb{Z}$ is called an *exceptional path.*

Definition 2.4 (splittings and complete splittings) Let $f\colon G \to G$ be a relative train track map. A decomposition of a path γ in G into subpaths $\gamma = \gamma_1 \cdot \gamma_2 \cdots \gamma_m$ is called a *splitting* if $[f^k(\gamma)] = [f^k(\gamma_1)][f^k(\gamma_2)] \cdots [f^k(\gamma_m)]$. Namely, one can tighten the image $f^k(\gamma)$ by tightening the images of the subpaths γ_i. We use the "$\cdot$" notation for splittings.

A splitting $\gamma = \gamma_1 \cdot \gamma_2 \cdots \gamma_m$ is called a *complete splitting* if each term γ_i is one of the following:

(1) an edge in an irreducible stratum;

(2) an INP;

(3) an exceptional path;

(4) a connecting path in a zero stratum that is both maximal and taken.

The paths in the above list are called *splitting units*.

Lemma 2.5 [5; 18] *Every completely split path or circuit has a unique complete splitting.*

The properties of relative train track maps are not strong enough for our purposes. Hence, in order to study the dynamics of atoroidal outer automorphisms, we utilize *completely split train track maps* (CTs) introduced by Feighn and Handel. In what follows, rather than giving the defining properties of CTs we will list the relevant properties of CTs and cite the appropriate resources. We refer the reader to [18] for a detailed discussion of CTs. We begin with two definitions:

Definition 2.6 A subgroup $\mathrm{F} < F_N$ is called a *free factor* of F_N if there is another subgroup $\mathrm{F}' < F_N$ such that $\mathrm{F} * \mathrm{F}' = F_N$. We denote the conjugacy class of a free factor F with [F]. A *free factor system* is a collection $\mathcal{F} = \{[\mathrm{F}^1], \dots, [\mathrm{F}^k]\}$ of conjugacy classes of free factors of F_N such that there exists $\mathrm{F}' < F_N$ (possibly trivial) with the property that $F_N = \mathrm{F}^1 * \cdots * \mathrm{F}^k * \mathrm{F}'$. There is a partial order on the set of free factor systems as follows: given two free factor systems $\mathcal{F} = \{[\mathrm{F}^1], \dots, [\mathrm{F}^k]\}$ and $\mathcal{F}' = \{[\mathrm{F}'^1], \dots, [\mathrm{F}'^l]\}$, we say that $\mathcal{F} \sqsubset \mathcal{F}'$ if for each $[\mathrm{F}^i] \in \mathcal{F}$ there exists $[\mathrm{F}'^j] \in \mathcal{F}'$ such that $g\,\mathrm{F}^i\,g^{-1} < F'^j$ for some $g \in F_N$.

The *free factor graph* $\mathcal{FF}(F_N)$ is the (infinite) graph whose vertices correspond to conjugacy classes of proper free factors, and there is an edge between [F] and [F′] if either $\mathrm{F} < g\,\mathrm{F}'\,g^{-1}$ or $\mathrm{F}' < g\,\mathrm{F}\,g^{-1}$ for some $g \in F_N$. By declaring the length of each edge 1, $\mathcal{FF}(F_N)$ is equipped with a path metric d, and a result of Bestvina and Feighn says that $\mathcal{FF}(F_N)$ is hyperbolic [2]. The group $\mathrm{Out}(F_N)$ acts on $\mathcal{FF}(F_N)$ with simplicial isometries and fully irreducible elements are precisely the loxodromic isometries [2].

Definition 2.7 For any marked graph G and a subgraph K of G, the fundamental group of the noncontractible components of K determines a free factor system $[\pi_1(K)] = \mathcal{F}$ of F_N. We say that K *realizes* $\mathcal{F}$. Given a nested sequence $\mathcal{C}$ of free factor systems $\mathcal{F}^1 \sqsubset \mathcal{F}^2 \sqsubset \cdots \sqsubset \mathcal{F}^n$ we say that $\mathcal{C}$ is *realized* by a relative train track map $f\colon G \to G$ if there is an f–invariant filtration $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_k = G$ such that for all $1 \le i \le n$ we have $\mathcal{F}^i = [\pi_1(G_{k(i)})]$ for some $k(i)$.

The following theorem is the main existence result about CTs:

Theorem 2.8 [18, Theorem 4.28, Lemma 4.42] *There exists a uniform constant $M = M(N) \ge 1$ such that for any $\varphi^M \in \mathrm{Out}(F_N)$ and any nested sequence $\mathcal{C}$ of φ^M–invariant free factor systems, there exists a CT $f\colon G \to G$ that represents φ^M and realizes $\mathcal{C}$.*

We now state several results about structures of paths in CTs that will be relevant in the discussion follows.

Lemma 2.9 [18, Lemma 4.21] *If $f\colon G \to G$ is a CT, then every NEG stratum H_r consists of a single edge e_i. Moreover, either e_i is fixed, or $f(e_i) = e_i \cdot u_i$, where u_i is a nontrivial, completely split circuit in G_{i-1}.*

Definition 2.10 Let $e \in G$ be an NEG edge. The edge e is called a *fixed edge* if $f(e) = e$, a *linear edge* if $f(e) = e\eta$, where η is a nontrivial Nielsen path, and a *superlinear edge* otherwise.

Lemma 2.11 (properties of CTs [18, Definition 4.7, Lemma 4.13, Lemma 4.15, Corollary 4.19, Lemma 4.25]) (1) *For each edge e in an irreducible stratum, $f(e)$ is completely split. For each taken connecting path γ in a zero stratum, $[f(\gamma)]$ is completely split.*

(2) *For each filtration element G_r, H_r is a zero stratum if and only if H_r is a contractible component of G_r. In particular, there are only finitely many reduced connecting paths that are contained in some zero stratum.*

(3) *Every periodic indivisible Nielsen path (INP) has period one.*

(4) *The endpoints of all INPs are vertices. The terminal endpoint of each NEG edge is fixed.*

(5) *If γ is a circuit or an edge-path, then $[f^k(\gamma)]$ is completely split for all sufficiently large k.*

(6) *Each zero stratum H_i is enveloped by an EG stratum H_r, each edge in H_i is r–taken, and each vertex in H_i is contained in H_r and has link contained in $H_i \cup H_r$.*

(7) *If H_r is an EG stratum, then there is at most one indivisible Nielsen path ρ_r of height r that intersects H_r nontrivially. The initial edges of ρ_r and ρ_r^{-1} are distinct edges in H_r.*

(8) *If H_r is a zero stratum or an NEG superlinear stratum, then no Nielsen path crosses an edge of H_r.*

2.4 CTs representing atoroidal automorphisms

Given an atoroidal outer automorphism $\varphi \in \mathrm{Out}(F_N)$, let $f\colon G \to G$ be a CT with filtration $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_k = G$ that represents a suitable power of φ as given by Theorem 2.8. Observe that for such a CT, there are no exceptional paths in the complete splitting of $[f^n(e)]$ for any $e \in \Gamma$ as there are no linear edges in Γ (since it requires a closed Nielsen path). The following is an easy consequence of the definitions:

Fact 2.12 *Let $f\colon G \to G$ be a CT that represents an atoroidal outer automorphism. Then every Nielsen path is a legal concatenation of INPs and fixed edges.*

Definition 2.13 We call a splitting unit σ *expanding* if $\|[f^n(\sigma)]\| \to \infty$ as $n \to \infty$. If $f\colon G \to G$ is a CT that represents an atoroidal outer automorphism, then an expanding splitting unit is one of the following three types:

(1) an edge in an EG stratum;

(2) a superlinear edge in an NEG stratum;

(3) a maximal connecting path γ in a zero stratum such that the complete splitting of $[f^k(\gamma)]$ contains at least one of the above two types for some $k \geq 1$.

2.5 Geodesic currents

Let ∂F_N denote the Gromov boundary of F_N. Let $\partial^2 F_N$ be the double boundary, ie $\partial^2 F_N = \partial F_N \times \partial F_N \setminus \Delta$, where Δ denotes the diagonal. Let $\iota\colon \partial^2 F_N \to \partial^2 F_N$ be the *flip* map given by $\iota(\zeta_1, \zeta_2) = (\zeta_2, \zeta_1)$.

The group F_N acts on itself by left multiplication, which induces an action of F_N on ∂F_N and hence on $\partial^2 F_N$. A *geodesic current* on F_N is a locally finite (positive) Borel measure on $\partial^2 F_N$ which is both F_N–invariant and flip-invariant.

The space of geodesic currents on F_N is denoted by $\mathrm{Curr}(F_N)$, and endowed with the weak-$*$ topology it is a metrizable topological space [7]. The space of *projectivized geodesic currents* $\mathbb{P}\mathrm{Curr}(F_N)$ is the quotient of $\mathrm{Curr}(F_N)$, where two currents are equivalent if they are positive scalar multiples of each other. The space $\mathbb{P}\mathrm{Curr}(F_N)$ is compact; see [24].

Both $\mathrm{Aut}(F_N)$ and $\mathrm{Out}(F_N)$ act on the space of currents by homeomorphisms, and these actions descend to well-defined actions on $\mathbb{P}\mathrm{Curr}(F_N)$.

Let $g \in F_N$ be an element which is not a proper power. We define the *counting current* η_g corresponding to g as follows: for any Borel set $S \subset \partial^2 F_N$ the value $\eta_g(S)$ is the number of F_N–translates of $(g^{-\infty}, g^{\infty})$ or of $(g^{\infty}, g^{-\infty})$ that are contained in S. For any nontrivial element $h \in F_N$ we write $h = g^k$, where g is not a proper power, and set $\eta_h := k\eta_g$. A *rational current* is a nonnegative real multiple of a counting current. The set of rational currents forms a dense subset of $\mathrm{Curr}(F_N)$; see [23; 24; 30].

3 Dynamics of atoroidal automorphisms

3.1 North–south dynamics

Let X be a compact metric space, and G be a group acting on X by homeomorphisms. We say that $g \in G$ acts on X with *(uniform) north–south dynamics* if the action of g

on X has two distinct fixed points x_- and x_+ and, for any open neighborhood $U_\pm$ of $x_\pm$ and a compact set $K_\pm \subset X \setminus x_\mp$, there exists $M > 0$ such that

$$g^{\pm n} K \subset U_\pm$$

for all $n \geq M$.

North–south dynamics is a strong form of stability for the action of a group on a compact metric space, and allows one to deduce various structural results about the group itself. For example, a fully irreducible outer automorphism $\varphi \in \mathrm{Out}(F_N)$ acts on the closure $\overline{\mathrm{CV}}$ of the projectivized outer space with north–south dynamics [27]. Similarly, if φ is both fully irreducible and atoroidal, then φ acts on $\mathbb{P}\mathrm{Curr}(F_N)$ with north–south dynamics [30]; see also [35]. On the other hand, an atoroidal outer automorphism does not act on $\mathbb{P}\mathrm{Curr}(F_N)$ with classical north–south dynamics. Existence of invariant free factors makes them dynamically more complicated but, as we show below, they still exhibit a strong form of stability.

Definition 3.1 (generalized north–south dynamics) Let X be a compact metric space, and G be a group acting on X by homeomorphisms. We say that an element $g \in G$ acts on X with generalized north–south dynamics if the action of g on X has two invariant disjoint sets Δ_-, and Δ_+ (ie $g\Delta_- = \Delta_-$ and $g\Delta_+ = \Delta_+$) and, for any open neighborhood $U_\pm$ of $\Delta_\pm$ and a compact set $K_\pm \subset \mathbb{P}\mathrm{Curr}(F_N) \setminus \Delta_\mp$, there exists $M > 0$ such that

$$g^{\pm n} K_\pm \subset U_\pm$$

for all $n \geq M$.

We restate Theorem 1.4 from the introduction for the benefit of the reader, the proof of which is given at the end of this section.

Theorem 1.4 *Let $\varphi \in \mathrm{Out}(F_N)$ be an atoroidal outer automorphism of a free group of rank $N \geq 3$. Then there exist a simplex of attraction Δ_+ and a simplex of repulsion Δ_- in $\mathbb{P}\mathrm{Curr}(F_N)$ such that φ acts on $\mathbb{P}\mathrm{Curr}(F_N)$ with generalized north–south dynamics from Δ_- to $\Delta+$.*

The rest of this section is modeled on our earlier paper [28] with Lustig, and utilizes the dynamics of reducible substitutions as treated in [29]. In what follows we explain the subtleties that arise in this new setting carefully, while referring to [28] for arguments that follow by straightforward modifications from the old setting.

3.2 Symbolic dynamics and CTs

In this section we recall the relevant definitions in symbolic dynamics and results from our earlier paper [29], that allows us to describe the simplex of attraction and simplex of repulsion in Theorem 1.4 explicitly.

Let $A=\{a_1,\dots,a_n\}$ be a finite alphabet, and A^* denote the set of all finite words in A. A *substitution* $\xi\colon A\to A^*$ is a rule that assigns to each letter $a\in A$ a nonempty word w in A^*. A substitution induces a map, which we also denote by ξ, on the set of infinite words $A^{\mathbb{N}}$ by concatenation:

$$\xi\colon A^{\mathbb{N}}\to A^{\mathbb{N}},\quad x_1x_2\cdots\mapsto\xi(x_1)\xi(x_2)\cdots.$$

Given a substitution $\xi\colon A\to A^*$ there is an associated transition matrix M_ξ, where $\{M_\xi\}_{ij}$ is the number of occurrences of a_j in $\xi(a_i)$. A substitution ξ is called *irreducible* if for all $1\le i,j\le n$, there exists an exponent $k=k(i,j)\ge 1$ such that the letter a_i appears in the word $\xi^k(a_j)$. The substitution ξ is called *primitive* if k can be chosen independently. In what follows, up to passing to powers and rearranging the letters, we will assume that each transition matrix is a lower diagonal block matrix where each diagonal block is either primitive, or has bounded entries for all M^t for all $t\ge 1$ [29, Lemma 3.1]. We refer the reader to [32; 29] for a detailed account of substitutions.

Given a nonprimitive substitution we consider maximal invariant *subalphabets*

$$0=A_0\sqsubset A_1\sqsubset A_2\sqsubset\cdots\sqsubset A_n=A$$

and call $A_{i+1}\setminus A_i$ the i^{th} stratum in analogy with train tracks terminology [29, proof of Proposition 3.5].

Given two words w_1 and w_2 in A^*, let $|w_1|_{w_2}$ denote the number of occurrences of the word w_2 in w_1. The following is a slight variation of Theorem 1.2 and Corollary 1.3 of [29], a detailed proof of which is given in [29, Proposition 5.4, Case 1].

Proposition 3.2 [29] *Let ξ be a substitution on a finite alphabet A. Then there exists a positive power $\zeta=\xi^s$ such that for any nonempty word $w\in A^*$ and any letter $a_i\in A$, the limit frequency*

$$\lim_{t\to\infty}\frac{|\zeta^t(a_i)|_w}{|\zeta^t(a_i)|}$$

exists. Furthermore, if a_i is in a primitive stratum H_i, where the Perron–Frobenius eigenvalue of H_i is strictly bigger than those of the dependent strata, then the limit frequencies are independent of the chosen letter.

The next proposition shows how one can extract dynamical information from CTs by interpreting them as substitutions and invoking Proposition 3.2. Similar ideas were also used in our earlier work [35; 28] in the setting of train tracks and [19] in the CT setting for studying dynamics of *relative* outer automorphisms.

For any two reduced edge paths γ and γ' in a graph G, define

$$\langle \gamma, \gamma' \rangle := |\gamma'|_{\gamma} + |\gamma'|_{\gamma^{-1}}.$$

Proposition 3.3 *Let $f\colon G \to G$ be a CT that represents an atoroidal outer automorphism $\varphi \in \mathrm{Out}(F_N)$. For any splitting unit σ and any reduced edge path γ in G, the limit*

$$\sigma_{\gamma} := \lim_{n\to\infty} \frac{\langle \gamma, f^t(\sigma) \rangle}{|f^t(\sigma)|}$$

exists. Moreover, for any expanding splitting unit σ, the set of values

$$\{\sigma_{\gamma} \mid \gamma \text{ is a reduced edge path in } G\}$$

defines a geodesic current μ_{σ} on F_N.

Proof If the splitting unit is not expanding then there is a definite bound on the length $\|[f^n(\sigma)]\|$ for all $n \geq 1$. Hence, the image $[f^n(\sigma)]$ becomes periodic after sufficiently many iterations. Since every periodic Nielsen path has period one, the sequence of paths $[f^n(\sigma)]$ becomes eventually fixed, and the claim follows. For the remaining part of the proof we assume that σ is an expanding splitting unit and will prove the claim by induction on the height of the stratum. Let $r = 1$. Since φ is atoroidal, H_1 is necessarily an EG stratum, and the restriction of f to $G_1 = H_1$ is an absolute train track map. Hence, the result follows from [35, Proposition 2.4 and Lemma 3.7]. Now assume that the claim holds for $r \leq k-1$. There are three cases to consider.

First suppose that H_k is an EG stratum. A splitting unit of height k is either an edge $e \in H_k$, or an INP intersecting H_k. Since an INP is not expanding we just need to prove the claim for an edge $e \in H_k$. Let A be the alphabet whose letters consist of edges in irreducible strata that are in G_k, INPs contained in G_{k-1}, and maximal, taken connecting paths in a zero stratum that are in G_{k-1}. The fact that this alphabet is finite follows from the properties of the CT map that represents an atoroidal outer automorphism. Let $\zeta\colon A^* \to A^*$ be the substitution induced by the CT $f\colon G \to G$ on the alphabet A using the following rule: $\zeta(\sigma) = [f(\sigma)]$. For each "letter" in the above alphabet, the image is completely split and hence a reduced "word" in this alphabet.

Hence, the above formula is a substitution, and Proposition 3.2 gives the required convergence.

The latter claim that the set of values $\{\sigma_\gamma\}_{\gamma\in\mathcal{P}G}$ defines a unique geodesic current is easy to check. They satisfy Kirchhoff conditions, ie

(1) $0\le\sigma_\gamma\le 2<\infty$,

(2) $\sigma_\gamma=\sigma_{\gamma^{-1}}$,

(3) $\sigma_\gamma=\sum_{a\in A}\sigma_{a\gamma}=\sum_{a\in A}\sigma_{\gamma a}$,

as in [29, Proposition 3.13; 35, Lemma 3.7], and by the Kolmogorov measure extension theorem the result follows.

Now assume that H_k is an NEG stratum. Since σ is expanding it is necessarily a superlinear edge e. By properties of CTs, $f(e)=e\cdot u$, where u is a circuit in G_{k-1} such that u is completely split and the turn (u,u^{-1}) is legal. We can similarly define a substitution as in the EG case, where the alphabet consists of the edge e, and splitting units appearing in u, and all of its iterates. The frequency convergence for the corresponding substitution is now given by Proposition 3.2.

Finally, if H_k is a zero stratum, then σ is a maximal connecting taken path, whose image $[f(\sigma)]$ is completely split, and has height $\le k-1$. Hence, the claim follows by induction. □

Remark 3.4 Proposition 3.2 together with the arguments in the proof of Proposition 3.3 reveals that, for an EG stratum H_r where the PF–eigenvalue is strictly greater than those of the dependent strata, the currents μ_e are independent of the edge e chosen from H_r. Furthermore, combined with [29, Proposition 5.4], we have that for any other expanding splitting unit σ, the current μ_σ is a linear combination of currents coming from edges in EG strata.

Definition 3.5 Given a CT map $f\colon G\to G$ that represents an atoroidal outer automorphism $\varphi\in\mathrm{Out}(F_N)$, we define the *simplex of attraction* as the projective class of nonnegative linear combinations of currents obtained from Proposition 3.3. We define the *simplex of repulsion* similarly, using a CT map that represents φ^{-1}.

3.3 Goodness and legal structure

Lemma 3.6 (bounded cancellation lemma [11]) *Let $f\colon G\to G$ be a topological graph map. There exists a constant C_f such that for any reduced path $\rho=\rho_1\rho_2$ in G*

one has

$$\|[f(\rho)]\| \geq \|[f(\rho_1)]\| + \|[f(\rho_2)]\| - 2C_f.$$

That is, at most C_f *terminal edges of* $[f(\rho_1)]$ *are canceled with* C_f *initial edges of* $[f(\rho_2)]$ *when we concatenate them to obtain* $[f(\rho)]$*.*

Definition 3.7 (goodness) Let γ be a reduced edge path in G and $\gamma = \gamma_1 \cdot \gamma_2 \cdots \gamma_m$ be a splitting of γ into edge paths γ_i. Define $\mathfrak{g}_{\mathrm{CT}}(\gamma)$ to be the proportion of the sum of the lengths of the γ_i that have a complete splitting to the total length of γ. Define *goodness* of γ, denoted by $\mathfrak{g}(\gamma)$, as the supremum of $\mathfrak{g}_{\mathrm{CT}}(\gamma)$ over all splittings of γ into edge paths. Since there are only finitely many decompositions of an edge path into subedge paths, the value $\mathfrak{g}(\gamma)$ is realized for some splitting of γ. We will call the splitting for which $\mathfrak{g}(\gamma)$ is realized the *maximal edge splitting* of γ. The subpaths that are part of a complete splitting in the maximal edge splitting will be called *good*. The subpaths in the maximal edge splitting which do not admit complete splittings will be called *bad*.

Let $w \in F_N$ be a conjugacy class in F_N, and γ_w be the unique circuit in G that represents $w \in F_N$. We define the *goodness* of the conjugacy class w as $\mathfrak{g}(w) := \mathfrak{g}(\gamma_w)$.

Remark 3.8 The properties of CTs — see Lemma 2.11(1) and (5) — imply that forward images of good paths are always good, and forward images of bad paths are eventually good.

Proposition 3.9 *Let* $f\colon G \to G$ *be a CT representing an atoroidal outer automorphism* $\varphi \in \mathrm{Out}(F_N)$*. There exists* $s > 0$ *such that for any completely split edge path* σ *such that* $|\sigma|$ *is sufficiently big,*

$$\frac{\text{total length of expanding splitting units in } \sigma}{|\sigma|} \geq s.$$

Proof Let σ be a completely split edge path, and consider its complete splitting. By properties of CTs (Lemma 2.11(6)) each maximal connecting path in a zero stratum is necessarily followed by an edge in an EG stratum. Since zero strata are precisely the contractible components, there is an upper bound for the length of any maximal connecting path in a zero stratum, say Z_0. Since φ is atoroidal, there is also an upper bound for the length of any path that is a concatenation of INPs and fixed edges, say Z_1.

Let $Z = \max\{Z_0, Z_1\}$. From these two observations it follows that for any completely split edge path of length $\geq 2Z+1$, we have

$$\begin{aligned}\frac{\text{total length of expanding splitting units in } \sigma}{|\sigma|} & \\ &\geq \frac{\text{total length of EG or superlinear edges in } \sigma}{|\sigma|} \\ &\geq \frac{1}{2Z+1}\end{aligned}$$ □

Convention/Remark 3.10 The values Z_0, Z_1 and hence Z are valid for all powers of f. From now on, we will replace φ, and hence f, with a power (which we will still denote by f) so that each expanding splitting unit grows at least by a factor of $2(2Z+1)$.

Definition 3.11 (short and long good paths) In light of Proposition 3.9 we will call a good segment γ a *long good segment* if $|\gamma| \geq 2Z+1$ and *short good segment* if $|\gamma| \leq 2Z$.

Lemma 3.12 *Let C_f be the bounded cancellation constant and $C := \max\{C_f, 2Z+1\}$. Let $\gamma = \gamma_1\gamma_2$ be an edge path such that γ_1 and γ_2 are completely split. Then any edge that is C away from the turn $\{\gamma_1^{-1}, \gamma_2\}$ is good.*

Proof Since any completely split path of length $\geq 2Z+1$ grows at least by a factor of 2, the bounded cancellation lemma dictates that reducing $f(\gamma_1\gamma_2)$ will not result in any cancellation at edges C away from the concatenation point; hence, the claim follows. □

Lemma 3.13 *For any edge path γ the total length of bad subpaths in $[f^k(\gamma)]$ is uniformly bounded by $|\gamma|2C$.*

Proof This is an easy consequence of Lemma 3.12. □

We first show that, up to passing to further powers, the goodness is *monotone*.

Lemma 3.14 *Let $f\colon G \to G$ be a CT representing an atoroidal outer automorphism $\varphi \in \mathrm{Out}(F_N)$. There exists an exponent $t' \geq 1$ such that for any circuit γ with $1 > \mathfrak{g}(\gamma) > 0$ and for all $t \geq t'$, one has*

$$\mathfrak{g}([f^t(\gamma)]) > \mathfrak{g}(\gamma).$$

Proof Note, by definition, the total length of good subpaths in γ is $\mathfrak{g}(\gamma)|\gamma|$. Under iteration of f, each good segment remains good and the length of each good segment is nondecreasing. Therefore, the total length of good segments in $[f^k(\gamma)]$ is $\geq \mathfrak{g}(\gamma)|\gamma|$.

Let t' be an exponent such that for each edge path β of length $\leq 2C+1$, the edge path $[f^t(\gamma)]$ is completely split for all $t \geq t'$. Therefore, for any bad segment β such that $|\beta| \leq 2C+1$, the path $[f^t(\beta)]$ is completely split, and hence contains no bad edges. For any bad segment β of length $\geq 2C+1$, divide β into subsegments β_i of length $2C+1$, with the exception of the last segment being of length $\leq 2C+1$. By the choice of t', each $[f^t(\beta_i)]$ is completely split, where the turns at concatenation points are possibly illegal. The bounded cancellation lemma dictates that total length of bad segments decreases by at least the number of subsegments, and the conclusion of the lemma follows. □

Convention/Remark 3.15 In what follows, we pass to a further power of φ and f so that each expanding splitting unit grows at least by a factor of $2(2Z+1)$ and the goodness function is monotone. We furthermore consider the bounded cancellation constant for this new power, but we continue to use f and C_f.

The following is one of the key technical lemmas in this paper. It allows us to get convergence estimates while dealing with forward iterations of CTs.

Lemma 3.16 *Let $\delta > 0$ and $\epsilon > 0$ be given. There exists an exponent $m_+ = m_+(\delta, \epsilon)$ such that for all circuits γ with $\mathfrak{g}(\gamma) > \delta$, we have $\mathfrak{g}([f^m(\gamma)]) > 1-\epsilon$ for $m \geq m_+$.*

Proof Let γ be a cyclically reduced edge path such that $\mathfrak{g}(\gamma) = \delta > 0$. First consider the splitting of γ into maximal good segments a_i and maximal bad segments b_i. There are two cases to consider:

Case 1 First assume that

$$\frac{\text{total length of long good segments in } \gamma}{\text{total length of good segments in } \gamma} \geq \frac{1}{4Z+1}$$

This gives that

$$\frac{\text{total length of expanding splitting units in } \gamma}{\text{total length of good segments in } \gamma} \geq \frac{1}{(2Z+1)(4Z+1)}.$$

Note that by Lemma 3.13 the total length of bad segments in $[f^k(\gamma)]$ is uniformly bounded by $(1-\mathfrak{g}(\gamma))|\gamma|C$. On the other hand, the assumption above together with

Convention/Remark 3.15 implies that

$$\text{total length of good segments in } [f^k(\gamma)] \geq \mathfrak{g}(\gamma)|\gamma| \frac{1}{(2Z+1)(4Z+1)}(2Z+1)^k 2^k.$$

Therefore,

$$\begin{aligned}\mathfrak{g}([f^k(\gamma)]) &\geq \frac{\mathfrak{g}(\gamma)|\gamma| \frac{1}{(2Z+1)(4Z+1)}(2Z+1)^k 2^k}{(1-\mathfrak{g}(\gamma))|\gamma|C + \mathfrak{g}(\gamma)|\gamma| \frac{1}{(2Z+1)(4Z+1)}(2Z+1)^k 2^k} \\ &= \frac{\mathfrak{g}(\gamma) \frac{1}{4Z+1}(2Z+1)^{k-1} 2^k}{(1-\mathfrak{g}(\gamma))C + \mathfrak{g}(\gamma) \frac{1}{4Z+1}(2Z+1)^{k-1} 2^k},\end{aligned}$$

which converges to 1 as $k \to \infty$; hence, the conclusion of Lemma 3.14 follows for big enough k, say $k = m_+$.

Case 2 Otherwise, we have

$$\frac{\text{total length of long good segments in } \gamma}{\text{total length of good segments in } \gamma} < \frac{1}{4Z+1}.$$

Equivalently,

$$\frac{\text{total length of short good segments in } \gamma}{\text{total length of long good segments in } \gamma} \geq 4Z. \tag{3-1}$$

We now subdivide the path γ into subpaths as follows. Consider the maximal edge splitting of γ. First subpath starts at a good edge, and it stops after tracing a total length of $2Z+1$ good segments end at a vertex such that the next edge is good. The second subpath starts at where the first path stops, and traces a total length of $2Z+1$ good segments, and stops at a vertex such that the next edge is good. We inductively form subpaths $\gamma_1, \gamma_2, \ldots$ so that each of them contains good segments of length $2Z+1$, with the possible exception of the last subpath. Note that by construction, $\gamma_i \cdot \gamma_2 \cdots \gamma_s$ is a splitting of γ.

Observe that (3-1) implies that

$$\frac{\#\{\gamma_i \text{ containing bad segments}\}}{\#\{\gamma_i \text{ which are completely good}\}} \geq 4Z,$$

which, in turn, implies

$$\#\{\gamma_i \text{ containing bad segments}\} \geq \frac{s4Z}{4Z+1},$$

where s is the total number of subpaths in γ in the above subdivision.

Since

$$\text{total length of good segments in } \gamma \leq \frac{1-\mathfrak{g}(\gamma)}{\mathfrak{g}(\gamma)} s(2Z+1),$$

each γ_i above that contains a bad segment contains

$$\frac{1-\mathfrak{g}(\gamma)}{\mathfrak{g}(\gamma)} s(2Z+1) \frac{4Z+1}{s(4Z)} = \frac{(2Z+1)(4Z+1)}{4Z} \frac{(1-\mathfrak{g}(\gamma))}{\mathfrak{g}(\gamma)}$$

bad edges on *average*.

Therefore, for each γ with $\mathfrak{g}(\gamma) \geq \delta$, at least half of the subpaths contain bad segments of total length

$$\leq \frac{(2Z+1)(4Z+1)}{2Z} \frac{(1-\delta)}{\delta} =: C_b$$

Let $t_b > 0$ be an exponent such that for all edge paths γ with $|\gamma| \leq C_b$, the path $[f^k(\gamma)]$ is completely split for $k \geq t_b$. Therefore, at least half of the subsegments in the subdivision will be mapped to long good segments, and the result follows from Case 1. □

Lemma 3.17 *Let U a neighborhood of the simplex of attraction and a positive number $\delta_+ > 0$ be given. Then there exists an exponent $N = N(\delta, U)$ such that for any $w \in F_N$ with $\mathfrak{g}(w) > \delta$,*

$$(\varphi^N)^n(\eta_w) \in U$$

for all $n \geq 1$.

Proof We first apply a power of f so that for every conjugacy class w with $\mathfrak{g}(w) > \delta$, we have $\mathfrak{g}(\varphi(w)) > 1-\epsilon$ for small $\epsilon > 0$. The rest of the proof is nearly identical to the proof of Lemma 6.1 in [28], where edges are replaced by expanding splitting units. □

Lemma 3.18 *Let $f\colon G \to G$ be a CT that represents an atoroidal outer automorphism. Given $0 < \delta < 1$, there exists an exponent T such that, for any element $w \in F_N$, and for all $t \geq T$ either*

$$\mathfrak{g}(\varphi^t(w)) \geq \delta$$

or

$$\text{total length of bad segments in } f^t(\gamma_w) \leq \tfrac{1}{2} \text{ total length of bad segments in } \gamma_w,$$

where γ_w is the unique circuit in G representing w.

Proof Let γ_w be the unique circuit in G that represents $w \in F_N$. Consider the splitting of γ into maximal good segments a_i and maximal bad segments b_i. Recall that $C = \max\{C_f, 2Z + 1\}$. Let us call a bad segment b_i a *long bad segment* if $|b_i| > 10C$, and a *short bad segment* otherwise.

There are two cases to consider:

Case 1 First assume that

$$\frac{\text{total length of short bad segments in } \gamma_w}{\text{total length of bad segments in } \gamma_w} \geq \frac{1}{10}.$$

Since every maximal bad segment is followed by at least one good segment, we have

$$\text{total length of good segments in } \gamma_w \geq \frac{1}{10C} \text{ total length of short bad segments in } \gamma_w$$

and hence

$$\text{total length of good segments in } \gamma_w \geq \frac{1}{100C} \text{ total length of bad segments in } \gamma_w.$$

Therefore,

$$\mathfrak{g}(\gamma_w) \geq \frac{1}{100C+1}.$$

Now, invoking Lemma 3.16, there is an exponent T_1 such that

$$\mathfrak{g}(\varphi^t(w)) \geq \delta$$

for all $t \geq T_1$, which is clearly independent of the conjugacy class w.

Case 2 Now assume, on the other hand, that

$$\frac{\text{total length of long bad segments in } \gamma_w}{\text{total length of bad segments in } \gamma_w} \geq \frac{9}{10}. \tag{3-2}$$

Let T_2 be an exponent such that for all edge paths γ with $|\gamma| < 10C$, $[f^t(\gamma)]$ is completely split for all $t \geq T_2$. Then, for any long bad segment b, the bounded cancellation lemma implies that

$$\text{total length of bad segments in } [f^t(b)] \leq \tfrac{1}{5} \text{ total length of bad segments in } b.$$

Together with (3-2), we get

$$\text{total length of bad segments in } f^t(\gamma_w) \leq \tfrac{9}{50} \text{ total length of bad segments in } \gamma_w$$

for all $t \geq T_2$. Now set $T = \max\{T_1, T_2\}$, and the lemma follows. □

Lemma 3.19 *Let $h\colon G' \to G'$ be a CT that represents $\varphi^{-1} \in \mathrm{Out}(F_N)$. Define $\mathfrak{g}'(\gamma')$ for $\gamma' \in G'$, and $\mathfrak{g}'(w)$ for $w \in F_N$ analogously. Then, given $0 < \delta < 1$, there is an exponent $T > 0$ such that, up to replacing f and h with powers, for any element $w \in F_N$ either*

$$\mathfrak{g}(\varphi^t(w)) \geq \delta \quad \text{or} \quad \mathfrak{g}'(\varphi^{-t}(w)) \geq \delta$$

for all $t \geq T$.

Proof Let $h\colon G' \to G'$ be a CT that represents $\varphi^{-1} \in \mathrm{Out}(F_N)$ and $\mathfrak{g}'$ be the corresponding goodness function, and we pass to appropriate powers according to Convention/Remark 3.15. The proof is now nearly identical to that of Proposition 4.20 of [28], where the number of illegal turns is replaced by the total length of bad segments. □

Proposition 3.20 [28, Proposition 3.3] *Let $f\colon X \to X$ be a homeomorphism of a compact metrizable space X. Let $Y \subset X$ be a dense subset of X, and let Δ_+ and Δ_- be two f–invariant sets in X that are disjoint. Assume that the following criterion holds:*

For every neighborhood U of Δ_+ and every neighborhood V of Δ_- there exists an integer $m_0 \geq 1$ such that, for any $m \geq m_0$ and any $y \in Y$, one has either $f^m(y) \in U$ or $f^{-m}(y) \in V$.

Then f^2 has generalized uniform north–south dynamics from Δ_- to Δ_+.

Proposition 3.21 [28, Proposition 3.4] *Let $f\colon X \to X$ be a homeomorphism of a compact space X, and let Δ_+ and Δ_- be disjoint f–invariant sets. Assume that some power f^p with $p \geq 1$ has generalized uniform north–south dynamics from Δ_- to Δ_+.*

Then the map f, too, has generalized uniform north–south dynamics from Δ_- to Δ_+.

Proof of Theorem 1.4 The theorem now follows from a combination of Lemmas 3.17 and 3.19 and Propositions 3.20 and 3.21. □

4 Hyperbolic extensions of free groups

In this section we use the dynamics of atoroidal outer automorphisms to prove Theorem 1.1 from the introduction, which allows us to construct new examples of hyperbolic extensions of free groups.

In what follows we will utilize theory of laminations on free groups which appear as supports of currents on F_N. We refer the reader to [4; 5; 9; 12; 13; 14; 18; 21] for detailed discussions. A *lamination* is a closed subset of $\partial^2 F_N$ which is F_N–invariant, and flip-invariant. We say that a free factor F *carries* a lamination Λ if all lines in Λ are contained in ∂^2 F.

Convention 4.1 Throughout this section we assume that we pass to the finite-index characteristic subgroup $\mathrm{IA}_N(\mathbb{Z}_3)$ of $\mathrm{Out}(F_N)$, as in Handel–Mosher subgroup decomposition theory [21], so that for each outer automorphism every periodic conjugacy class is fixed, and every periodic free factor system is invariant.

Let $\mathcal{H}$ be a subgroup of $\mathrm{Out}(F_N)$ and $\mathcal{F}^1 \sqsubset \mathcal{F}^2 \sqsubset \cdots \sqsubset \mathcal{F}^n = F_N$ be a *maximal* $\mathcal{H}$–invariant filtration of F_N by free factor systems, meaning that if $\mathcal{H}(\mathcal{A}) = \mathcal{A}$ for some $\mathcal{F}^i \sqsubset \mathcal{A} \sqsubset \mathcal{F}^{i+1}$, then either $\mathcal{A} = \mathcal{F}^i$ or $\mathcal{A} = \mathcal{F}^{i+1}$. Let $\varphi \in \mathcal{H}$ be an atoroidal outer automorphism. Consider a (possibly trivial) refinement $\mathcal{A}_1 \sqsubset \mathcal{A}_2 \sqsubset \cdots \sqsubset \mathcal{A}_m = F_N$ of $\mathcal{F}^1 \sqsubset \mathcal{F}^2 \sqsubset \cdots \sqsubset \mathcal{F}^n = F_N$ which is a maximal invariant filtration for φ.

If $\mathcal{H}$ fixes the conjugacy class of a free factor F of F_N, we will call the image of $\mathcal{H}$ in Out(F) under the natural homomorphism Stab(F) → Out(F) *the restriction* of $\mathcal{H}$ to F and denote it by $\mathcal{H}|_{\mathrm{F}}$.

We say that an $\mathcal{H}$–invariant free factor F is *minimal* if $\mathcal{H}$ does not fix the conjugacy class of any proper free factor of F. Similar definition holds for φ by considering the cyclic subgroup $\langle\varphi\rangle$. Observe that for $\varphi \in \mathcal{H}$, each minimal φ–invariant free factor F^i_φ is contained in a unique minimal $\mathcal{H}$–invariant free factor $\mathrm{F}^i_{\mathcal{H}}$.

Definition 4.2 Let φ and ψ be two atoroidal outer automorphisms with attracting and repelling simplices $\Delta_\pm(\varphi)$ and $\Delta_\pm(\psi)$ given by Theorem 1.4. We say that φ and ψ are independent if $\Delta_\pm(\varphi) \cap \Delta_\pm(\psi) = \varnothing$.

Lemma 4.3 *Let $\varphi \in \mathcal{H}$ be an atoroidal outer automorphism. Suppose that the restriction of $\mathcal{H}$ to F_i is not virtually cyclic for each minimal $\mathcal{H}$–invariant free factor F_i of F_N. Then $\mathcal{H}$ contains two independent atoroidal outer automorphisms.*

Proof Let $\{\mathrm{F}^i\}_{i=1}^s$ be the set of all minimal $\mathcal{H}$–invariant free factors. For each $i = 1, \dots, s$ the restriction of $\mathcal{H}$ to F^i is irreducible; hence, by [21, Theorem A; 22, Theorem 0.1], $\mathcal{H}$ contains an element θ_i whose restriction to F^i is fully irreducible, and since $\mathcal{H}$ is not geometric (since it contains an atoroidal element), we can choose θ_i in a way that its restriction to F^i is both fully irreducible and atoroidal [36, Theorem 5.4].

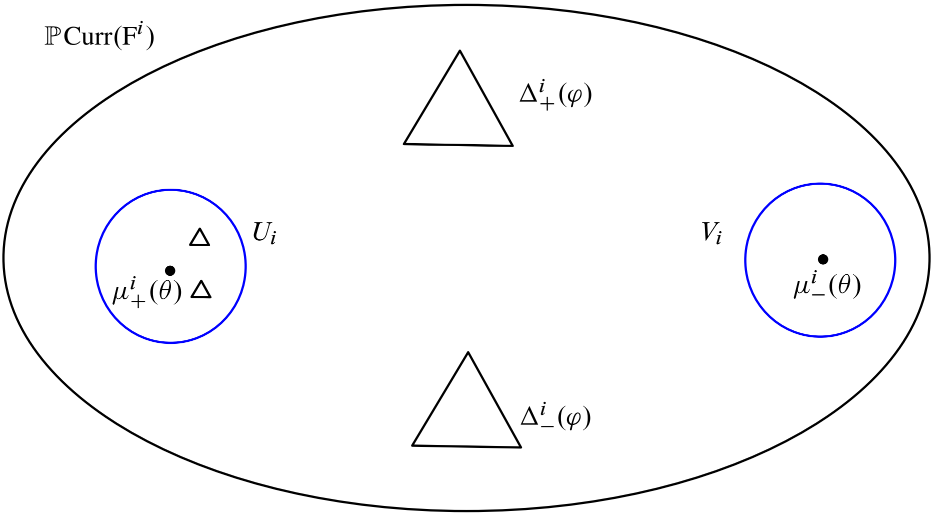

Figure 1: Dynamics on $\mathbb{P}\mathrm{Curr}(\mathrm{F}^i)$

Since fully irreducible and atoroidal elements are precisely the loxodromic isometries of the *cosurface graph* [15], invoking [10, Theorem 5.1] we can find a single $\theta \in \mathcal{H}$ such that for each $i = 1, \dots, s$, the restriction $\theta|_{\mathrm{F}^i}$ is fully irreducible and atoroidal. Recall that each fully irreducible and atoroidal outer automorphism acts on the space of projectivized geodesic currents with uniform north–south dynamics [30; 35]. Let $[\mu^i_-(\theta)] \in \mathbb{P}\mathrm{Curr}(\mathrm{F}^i)$ and $[\mu^i_+(\theta)] \in \mathbb{P}\mathrm{Curr}(\mathrm{F}^i)$ denote the unstable and stable currents for the restriction $\theta|_{\mathrm{F}^i}$. Since the stabilizer of the set $\{[\mu^i_-(\theta)], [\mu^i_+(\theta)]\}$ is virtually cyclic in $\mathrm{Out}(\mathrm{F}^i)$ [25], using the assumption on $\mathcal{H}$ we can furthermore assume that for the above $\theta \in \mathcal{H}$ it holds that $\theta|_{\mathrm{F}^i}$ and $\varphi|_{\mathrm{F}^i}$ are independent.

Hence, we can find $M > 0$ large enough that $\theta^M(\Delta^i_\pm(\varphi)) \cap \Delta^i_\pm(\varphi) = \varnothing$, where $\Delta^i_\pm(\varphi)$ are the attracting and repelling simplices of $\varphi|_{\mathrm{F}^i}$ in $\mathbb{P}\mathrm{Curr}(\mathrm{F}^i)$.

More precisely, choose open neighborhoods U_i and V_i of $[\mu^i_+(\theta)]$ and $[\mu^i_-(\theta)]$ in $\mathbb{P}\mathrm{Curr}(\mathrm{F}^i)$ which are disjoint from $\Delta^i_\pm(\varphi)$. Pick $M > 0$ such that $\theta^m(\mathbb{P}\mathrm{Curr}(\mathrm{F}^i) \setminus V_i) \subset U_i$ for all $m \geq M$; in particular, $\theta^M(\Delta^i_\pm) \subset U_i$. See Figure 1. In fact, we choose M that works for all minimal $\mathcal{H}$–invariant free factors for suitable open neighborhoods of attracting simplices as there are only finitely many minimal $\mathcal{H}$–invariant free factors.

Now consider the automorphism $\eta = \theta^M \varphi \theta^{-M}$, which is atoroidal since being atoroidal is invariant under conjugacy. Furthermore $\Delta^i_+(\eta) = \theta^M(\Delta^i_+(\varphi))$ and $\Delta^i_-(\eta) = \theta^M(\Delta^i_-(\varphi))$ in $\mathbb{P}\mathrm{Curr}(\mathrm{F}^i)$, and $\Delta_+(\eta) = \theta^M(\Delta_+(\varphi))$ and $\Delta_-(\eta) = \theta^M(\Delta_-(\varphi))$ in $\mathbb{P}\mathrm{Curr}(F_N)$.

Claim *η and φ are independent.*

Let $[\mu^k_+]$ be an extremal point in the attracting simplex $\Delta_+(\varphi)$. We first want to show that $\theta^M([\mu^k_+]) \neq [\mu]$ for any point $[\mu] \in \Delta_\pm(\varphi)$.

Notice that by Proposition 3.3 the point $[\mu^k_+]$ corresponds to some EG stratum H_k in the CT map $f\colon G \to G$ that represents φ.

Let F^k_φ be the unique (minimal) free factor carrying $\mathrm{supp}(\mu^k_+)$ (this support is the *attracting lamination* corresponding to the EG stratum H_k in the sense of Bestvina, Feighn and Handel [5]), and consider a minimal φ–invariant free factor $\mathrm{F}^i_\varphi \sqsubset \mathrm{F}^k_\varphi$. The free factor F^i_φ is contained in some minimal $\mathcal{H}$–invariant free factor F^i as above.

Let $[\mu^i_+] \in \Delta_+$ be the unique geodesic current whose support is carried by F^i_φ. We first observe that by definition $\mathrm{supp}(\mu^i_+) \subset \mathrm{supp}(\mu^k_+)$. Second, the subgroup $\mathcal{H}$ and so the element θ^M preserves F^i; therefore, $\mathrm{supp}(\theta^M \mu^i_+)$ is carried by F^i.

Suppose, for the sake of contradiction, that $\theta^M([\mu^k_+]) = [\mu]$. In that case we have $\mathrm{supp}(\theta^M([\mu^k_+])) = \mathrm{supp}([\mu])$; hence,

$$\mathrm{supp}(\theta^M([\mu^i_+])) \subset \mathrm{supp}(\theta^M([\mu^k_+])) = \mathrm{supp}([\mu]).$$

Only sublaminations of $\mathrm{supp}([\mu])$ that are carried by F^i could possibly come from supports of extremal points of $\Delta^i_\pm(\varphi)$, and since $(\theta^M \Delta^i_+) \cap \Delta^i_\pm = \varnothing$, the above inclusion is not possible; hence, we get a contradiction.

Since the support of any point in $\Delta_+(\eta)$ is a union of supports of extremal points, we get $\Delta_+(\eta) \cap \Delta_\pm(\varphi) = \varnothing$. A symmetric argument finishes the proof. □

We will prove the hyperbolicity of the extension using a classical argument of Bestvina, Feighn and Handel [4] which originates in the work of Mosher [31] as interpreted by Kapovich and Lustig [25].

Proposition 4.4 *Let $\varphi, \psi \in \mathcal{H}$ be two independent atoroidal outer automorphisms. Then there exist $M, N > 0$ such that for any $\mu \in \mathrm{Curr}(F_N)$ and for all $n \geq N$ and $m \geq M$, for at least three out of four elements α in $\{\varphi^n, \varphi^{-n}, \psi^m, \psi^{-m}\}$,*

$$|\alpha\mu|_G \geq 2|\mu|_G.$$

Proof Let U be a sufficiently small open neighborhood of $\Delta_+(\varphi)$, and $M_0 > 0$ be such that for any $\mu \in \mathrm{Curr}(F_N)$ such that $[\mu] \in \Delta_+(\varphi)$ it holds that $|\varphi^n(\mu)|_G \geq 2|\mu|_G$

for all $n \geq M_0$. This can be done, because of the topology of the space of currents, and the fact that for each extremal point $[\mu_+]$ of $\Delta_+(\varphi)$, $\varphi(\mu_+) = \lambda\mu_+$ for some $\lambda > 1$. For the corresponding statement in the fully irreducible case see [25, Lemma 4.12].

We also choose a small neighborhood V of $\Delta_-(\varphi)$, and M_1 so that for each $\mu \in \mathrm{Curr}(F_N)$ such that $[\mu] \in \Delta_-(\varphi)$ it holds that $|\varphi^{-n}(\mu)|_G \geq 2|\mu|_G$ for all $n \geq M$. Let $M' = \max\{M_0, M_1\}$.

Similarly, we choose neighborhoods U' and V' of $\Delta_+(\psi)$ and $\Delta_-(\psi)$, respectively, and a corresponding $N' > 0$.

By Theorem 1.4, there exists an exponent M_+ such that

$$\varphi^n(\mathbb{P}\mathrm{Curr}(F_N) \setminus V) \subset U$$

and

$$\varphi^{-n}(\mathbb{P}\mathrm{Curr}(F_N) \setminus U) \subset V$$

for all $n \geq M_+$.

Similarly, there exists an exponent N_+ such that

$$\psi^n(\mathbb{P}\mathrm{Curr}(F_N) \setminus V') \subset U'$$

and

$$\psi^{-n}(\mathbb{P}\mathrm{Curr}(F_N) \setminus U') \subset V'$$

for all $n \geq N_+$.

Now set $M = M' + M_+$ and $N = N' + N_+$. Let $\mu \in V$. Then the choice of M and N guarantees that $|\varphi^{-n}\mu|_G \geq 2|\mu|_G$, $|\psi^m\mu|_G \geq 2|\mu|_G$ and $|\psi^{-m}\mu|_G \geq 2|\mu|_G$. The other cases can be proved similarly; hence, the proposition follows. □

Proof of Theorem 1.1 Let F^i be a minimal, nontrivial $\mathcal{H}$–invariant free factor, and let φ, η and θ be as in Lemma 4.3. Since θ is fully irreducible, for large M the free factors F^i_φ and $\theta^M F^i_\varphi$ *fill* the free factor F^i. Under this assumption, based on work of Bestvina and Feighn [3], Taylor [33, Theorem 1.3] proved that for some $K > 0$, the group $\langle \varphi^K, \eta^K \rangle$ is isomorphic to a free group of rank 2. (He proves much more but we don't need that here.)

The fact that the corresponding free group extension is hyperbolic now follows from Proposition 4.4 and the Bestvina–Feighn combination theorem; see the proof of [4, Theorem 5.2]. □

We finish the paper with Corollary 1.5 from the introduction:

Corollary 1.5 *Let $\varphi \in \mathrm{Out}(F_N)$ be a fully irreducible and atoroidal outer automorphism. Then, for any atoroidal outer automorphism $\psi \in \mathrm{Out}(F_N)$ (not necessarily fully irreducible) which is not commensurable with φ, there exists an exponent $M > 0$ such that for all $n, m > M$, the subgroup $\Gamma = \langle \varphi^n, \psi^m \rangle < \mathrm{Out}(F_N)$ is purely atoroidal and the corresponding free extension E_Γ is hyperbolic.*

Proof Let φ be as above, and let $[\mu_+(\varphi)]$ and $[\mu_-(\varphi)]$ be the corresponding stable and unstable currents in $\mathbb{P}\mathrm{Curr}(F_N)$. Since ψ is not commensurable with φ, the attracting simplex $\Delta_+(\psi)$, the repelling simplex $\Delta_-(\psi)$, and the stable and unstable currents $[\mu_+(\varphi)]$ and $[\mu_-(\varphi)]$ are all disjoint.

Choose disjoint open neighborhoods of these sets, and choose high enough powers of φ and ψ so that there is a uniform north–south dynamics, which is guaranteed by Theorem 1.4. Then Proposition 4.4, together with Bestvina–Feighn combination theorem, gives the required result. □

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Symmetric spectra model global homotopy theory of finite groups

MARKUS HAUSMANN

We show that the category of symmetric spectra can be used to model global equivariant homotopy theory of finite groups.

55P42, 55P43, 55P91

0. Introduction 1413
1. Symmetric spectra 1416
2. Global model structures 1421
3. Multiplicative properties 1431
4. Equivariant homotopy groups of symmetric spectra 1433
5. Comparison to orthogonal spectra 1442
6. Examples 1444
Appendix. Model structures with respect to families 1449
References 1451

0 Introduction

Equivariant stable homotopy theory deals with the study of equivariant spectra and the cohomology theories they represent. While some of these equivariant theories are specific to a fixed group, many of them are defined in a uniform way for all compact Lie groups simultaneously, for example equivariant K–theory, Borel cohomology, equivariant bordism or equivariant cohomotopy. The idea of global equivariant homotopy theory is to view such a compatible collection of equivariant spectra — ranging through all compact Lie groups — as one "global" object, in particular to capture its full algebraic structure of restrictions, transfer maps and power operations. There have been various approaches to formalizing this idea and to obtain a category of global equivariant spectra, for example in Lewis, May and Steinberger [11, Chapter 2], Greenlees and May [5, Section 5] and Bohmann [2]. Schwede [19; 18] introduced a new approach by

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looking at the well-known category of orthogonal spectra of Mandell, May, Schwede and Shipley [13] from a different point of view: Every orthogonal spectrum X gives rise to a G–orthogonal spectrum X_G for any compact Lie group G by letting G act through its orthogonal representations. The fundamental observation used in [19] is that the G–homotopy type of X_G is not determined by the nonequivariant homotopy type of X, ie a stable equivalence of orthogonal spectra does not necessarily give rise to a G–stable equivalence on underlying G–orthogonal spectra. Taking these G–homotopy types for varying G into account gives rise to a much finer notion of weak equivalence called *global equivalence* and thereby to the global stable homotopy category, which splits each nonequivariant homotopy type into many global variants. A strength of Schwede's approach is that it on the one hand allows many examples (all the theories mentioned above are represented by a single orthogonal spectrum in this sense) and on the other hand is technically easy to work with, since the underlying category is just that of orthogonal spectra.

The purpose of this paper is to show that the category of symmetric spectra introduced by Hovey, Shipley and Smith [9] can also be used to model global equivariant homotopy theory if one takes "global" to mean all *finite* groups instead of all compact Lie groups. Symmetric spectra have the advantage that they can also be based on simplicial sets and are generally more combinatorial, as it is sometimes easier to construct actions of symmetric groups than of orthogonal groups. A main example is Schwede's construction of a model for global equivariant algebraic K–theory [16] (which we recall in Section 6.3), whose output is a symmetric spectrum and usually not an orthogonal spectrum.

Besides the fully global theory of orthogonal spectra, which takes into account all compact Lie groups, Schwede [19] also provides a variant where only a fixed family of groups is considered. In particular, there is a version for the family of finite groups Fin. Then the main result of this paper can be stated as:

Theorem (Theorems 2.17 and 5.3) *There exists a model structure on the category of symmetric spectra of topological spaces or simplicial sets — called the* ***global*** *model structure — which is Quillen equivalent to orthogonal spectra with the* Fin*–global model structure of* [19].

More precisely, the forgetful functor from orthogonal to symmetric spectra is the right adjoint of a Quillen equivalence. The central notion in the global model structure is that of a *global equivalence of symmetric spectra*. The basic idea is the same as for orthogonal spectra: every symmetric spectrum X gives rise to a G–symmetric spectrum X_G for any finite group G by letting G act through its finite G–sets, ie the homomorphisms $G \to \Sigma_n$. In particular, one can define its equivariant homotopy groups. However, unlike for orthogonal spectra, equivariant homotopy groups cannot be used to describe global equivalences — a phenomenon already present for nonequivariant symmetric spectra and for G–symmetric spectra over a fixed finite group G. Instead we make use of the notion of G–stable equivalence introduced in Hausmann [6] and define a map $f\colon X \to Y$ of symmetric spectra to be a global equivalence if for all finite groups G the map $f_G\colon X_G \to Y_G$ is a G–stable equivalence. The more complicated definition of G–stable equivalence and hence global equivalence is the main technical difference to orthogonal spectra. The work in this paper lies in assembling the model structures of [6] for varying G into a global one, for which Proposition 2.13 is central.

The cofibrations in our model structure are the same as in Shipley's flat (or $\mathbb{S}$–) model structure introduced in Shipley [21], which hence forms a left Bousfield localization of ours. This determines the model structure completely; the fibrant objects can be characterized as global equivariant versions of Ω–spectra (Definition 2.12), similarly as for orthogonal spectra. We further show that the global model structure (or a positive version) lifts to the categories of symmetric ring spectra and commutative symmetric ring spectra (called "ultracommutative" in [19]), and more generally to categories of modules, algebras and commutative algebras over a fixed (commutative) symmetric ring spectrum.

While equivariant homotopy groups of symmetric spectra cannot be used to characterize global equivalences, they nevertheless provide an important tool. We describe some of their properties and their functoriality as the group varies. This functoriality turns out to be more involved than for orthogonal spectra, as it interacts nontrivially with the theory of (global equivariant) semistability, ie the relationship between "naive" and derived equivariant homotopy groups of symmetric spectra. When X is globally semistable, its equivariant homotopy groups carry restriction maps along arbitrary group homomorphisms and transfer maps for subgroup inclusions, and the two are related via a double coset formula. This functoriality describes a global version of a Mackey functor that has previously been considered in an algebraic context, such as by Webb [22] (where it is called an "inflation functor") and Lewis [10] ("global $(\varnothing, \infty)$–Mackey functor").

Throughout, we focus on the class of all finite groups, but symmetric spectra can also be used to model global homotopy theory with respect to smaller families of groups, such as abelian finite groups or p–groups for a fixed prime p. In the appendix we give a short treatment of the modifications needed to obtain such a relative theory.

The paper is organized as follows: In Section 1 we recall the definition of symmetric spectra, explain how to evaluate them on finite G–sets (Section 1.2) and introduce global free spectra (Section 1.3). Section 2 starts with the construction of the global level model structure (Proposition 2.5), introduces global equivalences (Definition 2.9) and global Ω–spectra (Definition 2.12), explains the connection between the two (Proposition 2.13) and, finally, contains a proof of the stable global model structure (Theorem 2.17). In Section 3 we construct global model structures on module, algebra and commutative algebra categories. Section 4 deals with equivariant homotopy groups of symmetric spectra. Their definition is given in Section 4.1, their functoriality is explained in Sections 4.3, 4.4 and 4.5 and the properties of globally semistable symmetric spectra are discussed in Section 4.6. In Section 5 we prove that our model structure is Quillen equivalent to Fin–global orthogonal spectra. Section 6 discusses examples of symmetric spectra from the global point of view. Finally, the appendix deals with global homotopy theory of symmetric spectra with respect to a family of finite groups.

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1 Symmetric spectra

1.1 Definition

We begin by recalling the definition of a symmetric spectrum. For easier reading we do not treat the simplicial and topological cases in parallel, but for the definitions and the construction of the model structures concentrate on symmetric spectra of simplicial sets. The translation to symmetric spectra of topological spaces is straightforward; see also Remark 2.18.

We let S^n denote the n–sphere, ie the n–fold smash product of $S^1 := \underline{\Delta}^1/\partial\underline{\Delta}^1$.

Definition 1.1 (symmetric spectrum) A *symmetric spectrum* X of simplicial sets consists of

- a based Σ_n–simplicial set X_n, and
- a based *structure map* $\sigma_n\colon X_n \wedge S^1 \to X_{n+1}$

for all $n \in \mathbb{N}$. This data has to satisfy the condition that for all $n, m \in \mathbb{N}$ the *iterated structure map*

$$\sigma_n^m\colon X_n \wedge S^m \cong (X_n \wedge S^1) \wedge S^{m-1} \xrightarrow{\sigma_n \wedge S^{m-1}} X_{n+1} \wedge S^{m-1} \xrightarrow{\sigma_{n+1} \wedge S^{m-2}} \cdots \xrightarrow{\sigma_{n+m-1}} X_{n+m}$$

is $(\Sigma_n \times \Sigma_m)$–equivariant, with Σ_m acting on S^m by permuting the coordinates.

A *morphism of symmetric spectra* $f\colon X \to Y$ is a sequence of based Σ_n–equivariant maps $f_n\colon X_n \to Y_n$ such that $f_{n+1} \circ \sigma_n^{(X)} = \sigma_{n+1}^{(Y)} \circ (f_n \wedge S^1)$ for all $n \in \mathbb{N}$.

We denote the category of symmetric spectra by Sp^Σ.

Example 1.2 (suspension spectra) Every based simplicial set A gives rise to a suspension symmetric spectrum $\Sigma^\infty A$ whose n^{th} level is $A \wedge S^n$ with Σ_n–action through S^n and structure map the associativity isomorphism $(A \wedge S^n) \wedge S^1 \cong A \wedge S^{n+1}$. For $A = S^0$ this gives the sphere spectrum $\mathbb{S}$.

Remark 1.3 (G–symmetric spectra) Throughout this paper we will often make use of the theory of G–symmetric spectra for a fixed finite group G, by which we simply mean a symmetric spectrum with a G–action.

Definition 1.4 (underlying G–symmetric spectra) Given a symmetric spectrum X, we write X_G for the underlying G–symmetric spectrum obtained by giving X the trivial G–action.

The fact that G acts trivially on X_G means that all the G–equivariance is encoded in the symmetric group actions on the levels of X_G. The homotopical properties of X_G depend on the evaluations on finite G–sets introduced below, which will usually not carry trivial G–action. The "exterior action" of G being trivial corresponds to saying that G acts trivially on the evaluations of X_G on *trivial* G–sets.

1.2 Evaluations

Let G be a finite group and M a finite G–set of cardinality m. We denote by $\mathrm{Bij}(\underline{m}, M)$ the discrete simplicial set of bijections between the sets $\underline{m} = \{1, \dots, m\}$

and M. It possesses a right Σ_m–action by precomposition and a left G–action by postcomposition with the action on M.

Definition 1.5 (evaluation) The *evaluation* of a symmetric spectrum X *on* M is defined as

$$\begin{aligned} X(M) &:= X_m \wedge_{\Sigma_m} \mathrm{Bij}(\underline{m}, M)_+ \\ &:= X_m \wedge \mathrm{Bij}(\underline{m}, M)_+/((\sigma x, f) \sim (x, f\sigma) \mid \sigma \in \Sigma_m), \end{aligned}$$

with G–action through M.

Remark 1.6 This is the special case of an evaluation of a G–symmetric spectrum Y on a finite G–set, in which case G acts diagonally on $Y(M) = Y_m \wedge_{\Sigma_m} \mathrm{Bij}(\underline{m}, M)_+$. If $Y = X_G$ for a symmetric spectrum X, ie if the exterior G–action on Y is trivial, the two evaluations $Y(M)$ and $X(M)$ agree as G–simplicial sets. Hence, $X(M)$ can be thought of as the evaluation of the underlying G–symmetric spectrum X_G on M.

The following are two examples of evaluations:

Example 1.7 Let A be a based simplicial set and M a finite G–set. We denote by S^M the smash product of M copies of S^1 with permutation G–action, generalizing the definition of the Σ_n–permutation sphere S^n. Then the map $(\Sigma^\infty A)(M) \to A \wedge S^M$ that sends a class $[(a \wedge x) \wedge f]$ to $a \wedge f_*(x)$ is a G–isomorphism.

Example 1.8 Let G be the symmetric group Σ_n and M be the natural Σ_n–set $\underline{n}$, with X a symmetric spectrum. Then $X(\underline{n})$ is canonically isomorphic to X_n with the Σ_n–action that is part of the data of the symmetric spectrum X. In contrast, evaluating at $\{1, \dots, n\}$ with *trivial* Σ_n–action yields X_n with trivial action.

Moreover, these evaluations are connected by so-called generalized structure maps: Let G be a finite group, M and N two finite G–sets of cardinalities m and n, respectively, and X a symmetric spectrum. We further choose a bijection $\psi\colon \underline{n} \xrightarrow{\cong} N$.

Definition 1.9 (generalized structure map) The map

$$\sigma_M^N\colon X(M) \wedge S^N \to X(M \sqcup N), \quad ([x \wedge f] \wedge s) \mapsto [\sigma_m^n(x \wedge \psi_*^{-1}(s)) \wedge (f \sqcup \psi)],$$

is called the *generalized structure map* of M and N.

It is straightforward to check that the generalized structure map does not depend on the choice of bijection $\psi\colon \underline{n} \xrightarrow{\cong} N$. Furthermore, it is G–equivariant for the diagonal

G–action on $X(M) \wedge S^N$. Again this is a special case of generalized structure maps for G–symmetric spectra.

1.3 Global free symmetric spectra

For every finite group G and every finite G–set M, the above construction yields functors

$$-(M)\colon \mathrm{Sp}^{\Sigma} \to G\mathcal{S}_*$$

from symmetric spectra to the category of based G–simplicial sets $G\mathcal{S}_*$. These functors have left adjoints F_M^G, which is a consequence of the existence of a left adjoint for the analogous evaluation functor from G–symmetric spectra to based G–simplicial sets.

Here we only give the necessary definitions to construct them; more details can be found in [6, Section 2.4]. Given a finite K–set N for another finite group K, we put

$$\mathbf{\Sigma}(M,N) := \bigvee\nolimits_{\alpha\colon M \hookrightarrow N \text{ injective}} S^{N-\alpha(M)}.$$

This based simplicial set carries a right G–action by precomposition on the indexing wedge and a commuting left K–action for which an element k sends a pair $(\alpha, x \in S^{N-\alpha(M)})$ to the pair $(k \circ \alpha, k \cdot x \in S^{N-k\cdot\alpha(M)})$. Given another finite K–set N', there is a natural $(G^{\mathrm{op}} \times K)$–equivariant map

$$\sigma_N^{N'}\colon \mathbf{\Sigma}(M,N) \wedge S^{N'} \to \mathbf{\Sigma}(M, N \sqcup N'), \quad (\alpha, x) \wedge y \mapsto (\alpha, x \wedge y).$$

Definition 1.10 Let A be a based G–simplicial set and M a finite G–set. Then the *global free symmetric spectrum on A in level M* is defined as $(F_M^G(A))_n := A \wedge_G \mathbf{\Sigma}(M, \underline{n})$ with structure map

$$A \wedge_G \sigma_{\underline{n}}^1\colon (A \wedge_G \mathbf{\Sigma}(M,\underline{n})) \wedge S^1 \to A \wedge_G \mathbf{\Sigma}(M, \underline{n+1}).$$

More generally, if N is a finite K–set, the evaluation $(F_M^G(A))(N)$ is canonically isomorphic to $A \wedge_G \mathbf{\Sigma}(M,N)$ with K–action through N. The generalized structure maps arise by smashing $\sigma_N^{N'}$ with $A \wedge_G -$. Then we have:

Proposition 1.11 *Let M be a finite G–set, A a based G–simplicial set and X a symmetric spectrum. Then the assignment*

$$\begin{gathered}\mathrm{map}_{\mathrm{Sp}^{\Sigma}}(F_M^G(A), X) \longrightarrow \mathrm{map}_G(A, X(M)),\\ (f\colon F_M^G(A) \to X) \longmapsto \big(A \xrightarrow{[-\wedge\{\mathrm{id}_M\}+]} A \wedge_G \Sigma(M,M) \xrightarrow{f(M)} X(M)\big),\end{gathered}$$

is a natural isomorphism.

Here, the expression $\mathrm{map}_{\mathrm{Sp}^\Sigma}(-,-)$ refers to the simplicial set of morphisms between two symmetric spectra, which is recalled in the following subsection.

Proof This follows from [6, Proposition 2.14], since by definition $F_M^G(A)$ is the G–quotient of the free G–symmetric spectrum $\mathscr{F}_M(A)$ and we are mapping into spectra with trivial G–action. □

1.4 Mapping spaces and spectra, smash products and shifts

In this section we quickly recall various point-set constructions for symmetric spectra, which are all introduced in [9].

Example 1.12 ((co)tensoring over based spaces) Every based simplicial set A gives rise to a functor $A\wedge -\colon \mathrm{Sp}^\Sigma \to \mathrm{Sp}^\Sigma$ by smashing each level and structure map with A. It is left adjoint to $\mathrm{map}(A,-)\colon \mathrm{Sp}^\Sigma \to \mathrm{Sp}^\Sigma$, defined via $\mathrm{map}(A,X)_n = \mathrm{map}(A,X_n)$ with structure maps adjoint to $\mathrm{map}(A,X_n) \xrightarrow{\tilde{\sigma}_n} \mathrm{map}(A,\Omega(X_{n+1})) \cong \Omega(\mathrm{map}(A,X_{n+1}))$.

Example 1.13 (geometric realization) Symmetric spectra of simplicial sets and topological spaces are related by the adjunction of geometric realization $|\cdot|$ and singular complex $\mathcal{S}$. Both functors are constructed by applying the space level version levelwise, making use of the fact that $|\cdot|$ commutes with $-\wedge S^1$ and $\mathcal{S}$ commutes with $\Omega(-)$ to obtain structure maps (similarly to the previous example).

Example 1.14 (shifts) For every natural number n there is an endofunctor

$$\mathrm{sh}^n\colon \mathrm{Sp}^\Sigma \to \mathrm{Sp}^\Sigma$$

defined by $\mathrm{sh}^n(X)_m := X_{n+m}$ with Σ_m–action through the last m coordinates and structure maps shifted by n. There is a natural transformation $\alpha_X^n\colon S^n\wedge X \to \mathrm{sh}^n(X)$ given in level m by the composite

$$S^n\wedge X_m \cong X_m\wedge S^n \xrightarrow{\sigma_n^m} X_{m+n} \xrightarrow{X(\tau_{m,n})} X_{n+m} = \mathrm{sh}^n(X)_m,$$

where $\tau_{m,n}$ denotes the permutation in Σ_{m+n} that moves the first m elements $\{1,\dots,m\}$ past the last n elements $\{m+1,\dots,m+n\}$ and preserves the order of both of these subsets.

In fact, via the same formula one can shift along arbitrary finite G–sets M, but the result $\mathrm{sh}^M(X)$ is in general a G–symmetric spectrum with nontrivial G–action.

Example 1.15 (mapping spaces) Given two symmetric spectra X and Y there is a mapping simplicial set $\mathrm{map}_{\mathrm{Sp}^\Sigma}(X,Y)$ whose n–simplices are given by the set of symmetric spectra morphisms from $\underline{\Delta}^n_+ \wedge X$ to Y.

Example 1.16 (internal Hom) Combining this with the shifts above gives internal homomorphism spectra $\mathrm{Hom}(X,Y)$ defined by $\mathrm{Hom}(X,Y)_n := \mathrm{map}_{\mathrm{Sp}^\Sigma}(X, \mathrm{sh}^n Y)$ with Σ_n–action through the first n coordinates in $\mathrm{sh}^n(Y)$ and structure map sending a pair $(f\colon X \to \mathrm{sh}^n(Y),\ x \in S^1)$ to the composite

$$X \xrightarrow{x \wedge f} S^1 \wedge \mathrm{sh}^n(Y) \xrightarrow{\alpha^1_{\mathrm{sh}^n(Y)}} \mathrm{sh}^{n+1}(Y).$$

Example 1.17 (smash product) As shown in [9, Section 2], the category of symmetric spectra carries a symmetric monoidal smash product $\wedge$ with unit $\mathbb{S}$, uniquely characterized up to natural isomorphism by the fact that $- \wedge X$ is left adjoint to $\mathrm{Hom}(X,-)$.

2 Global model structures

In this section we construct global model structures on the category of symmetric spectra, beginning with a level model structure which is, later, left Bousfield localized to obtain a stable version.

2.1 Level model structure

We recall the standard model structure on equivariant simplicial sets:

Definition 2.1 A map $f\colon A \to B$ of based G–simplicial sets is called a

- *G–weak equivalence* if the map $f^H\colon X^H \to Y^H$ is a weak equivalence for all subgroups H of G;
- *G–fibration* if the map $f^H\colon X^H \to Y^H$ is a Kan fibration for all subgroups H of G;
- *G–cofibration* if it is degreewise injective.

It is well known that the above classes assemble to a proper, cofibrantly generated and monoidal model structure on the category of based G–simplicial sets. We make use of it to construct a global level model structure on symmetric spectra:

Definition 2.2 A morphism $f\colon X \to Y$ of symmetric spectra is called a

- *global level equivalence* if each level $f_n\colon X_n \to Y_n$ is a Σ_n–weak equivalence;
- *global level fibration* if each level $f_n\colon X_n \to Y_n$ is a Σ_n–fibration;
- *flat cofibration* if each latching map $\nu_n[f]\colon X_n \cup_{L_n(X)} L_n(Y) \to Y_n$ is a Σ_n–cofibration.

For the definition of latching spaces and maps we refer to [9, Definition 5.2.1] or [6, Section 2.5]. The following gives a different interpretation of global level equivalences and fibrations:

Lemma 2.3 *A morphism $f\colon X \to Y$ of symmetric spectra is a global level equivalence (resp. global level fibration) if and only if for all finite groups G and all finite G–sets M, the map $f(M)^G\colon X(M)^G \to Y(M)^G$ is a weak equivalence (resp. Kan fibration) on G–fixed points.*

Proof Given a finite G–set M, any choice of bijection $\underline{m} \cong M$ defines a homomorphism $\varphi\colon G \to \Sigma_m$ and the G–fixed points $X(M)^G$ are naturally identified with $X_m^{\varphi(G)}$. This translates between the different formulations. □

Remark 2.4 In [6], a morphism $f\colon X \to Y$ of G–symmetric spectra is a G–level equivalence if for all subgroups H of G and all finite H–sets M, the map

$$f(M)^H\colon X(M)^H \to Y(M)^H$$

is a weak equivalence. Hence, a morphism of symmetric spectra is a global level equivalence if and only if it induces a G–level equivalence on underlying G–symmetric spectra for all finite groups G. Furthermore, every flat cofibration of symmetric spectra induces a G–flat cofibration on underlying G–symmetric spectra.

Proposition 2.5 (level model structure) *The global level equivalences, global level fibrations and flat cofibrations define a proper, cofibrantly generated and monoidal model structure on the category of symmetric spectra, called the* ***global level model structure****.*

Proof The existence of the model structure and its properness follows from [6, Proposition 2.22] for G the trivial group, since the strong consistency condition [6, Definition 2.21] is satisfied. Monoidality is a consequence of [6, Corollary 2.30] for each finite group separately. □

Since the suspension spectrum functor from based simplicial sets is a strong monoidal left Quillen functor, the monoidality of the global model structure in particular implies that it is simplicial. Let I and J denote sets of generating cofibrations and acyclic cofibrations, respectively, for the Quillen model structure on simplicial sets. Then sets of generating (acyclic) cofibrations for the global level model structure are given by

$$I_{\mathrm{gl}}^{\mathrm{lev}}=\{F_{\underline{n}}^{H}(i)\,|\,n\in\mathbb{N},\,H\leq\Sigma_n,\,i\in I\}\quad\text{and}\quad J_{\mathrm{gl}}^{\mathrm{lev}}=\{F_{\underline{n}}^{H}(j)\,|\,n\in\mathbb{N},\,H\leq\Sigma_n,\,j\in J\},$$

respectively, where in each case the maps i and j are thought of as maps of H–spaces with trivial action and H acts on $\underline{n}$ via its embedding into Σ_n.

In order to obtain a global model structure on commutative symmetric ring spectra we will also need a positive version of the global level model structure. For this we call a morphism $f\colon X\to Y$ a *positive global level equivalence* (resp. *positive global level fibration*) if $f_n\colon X_n\to Y_n$ is a Σ_n–weak equivalence (resp. Σ_n–fibration) for all $n\geq 1$. Furthermore, a *positive flat cofibration* is a flat cofibration which is an isomorphism in degree 0. Then we have:

Proposition 2.6 (positive level model structure) *The positive global level equivalences, positive global level fibrations and positive flat cofibrations define a proper and cofibrantly generated model structure on the category of symmetric spectra, called the* ***positive global level model structure****.*

Proof As above, this model structure can be obtained via [6, Proposition 2.22]. □

The positive global level model structure satisfies the pushout product axiom but not the unit axiom, so it is not quite monoidal.

2.2 Global equivalences

In order to define the global (stable) equivalences we have to recall the notions of $G\Omega$–spectrum and G–stable equivalence for a fixed finite group G. In comparing to [6], we always use the notions formed with respect to a complete G–set universe $\mathcal{U}_G$. These notions do not depend on a particular choice of such and so we omit it from the notation.

Definition 2.7 ($G\Omega$–spectra) A G–symmetric spectrum X is called a *$G\Omega$–spectrum* if for all subgroups H of G and all finite H–sets M and N, the composite

$$X(M)\xrightarrow{\tilde{\alpha}_M^N}\Omega^N X(M\sqcup N)\to\Omega^N(X(M\sqcup N)^f)$$

is an H–weak equivalence, where $X(M\sqcup N)^f$ is a fibrant replacement of $X(M\sqcup N)$ in the model structure on based H–simplicial sets.

Here, the map $\Omega^N X(M \sqcup N) \to \Omega^N(X(M \sqcup N)^f)$ is used to replace $\Omega^N X(M \sqcup N)$ by the derived loop space to make the property homotopically meaningful. When X is G–level fibrant, the above condition is equivalent to the adjoint structure map $\tilde{\alpha}_M^N$ itself being an H–weak equivalence.

As recalled in Remark 2.4, a map $f\colon X \to Y$ of G–symmetric spectra is a *G–level equivalence* if for all subgroups $H \leq G$ and all finite H–sets M the evaluation $f(M)^H\colon X(M)^H \to Y(M)^H$ is a weak equivalence. We denote the localization of G–symmetric spectra at the G–level equivalences by $\gamma_G\colon G\mathrm{Sp}^\Sigma \to G\mathrm{Sp}^\Sigma[G\text{–level eq.}^{-1}]$.

Definition 2.8 (G–stable equivalence) A morphism $f\colon X \to Y$ of G–symmetric spectra is a *G–stable equivalence* if for all $G\Omega$–spectra Z the map

$$G\mathrm{Sp}^\Sigma[G\text{–level eq.}^{-1}](Y,Z) \xrightarrow{\gamma_G(f)^*} G\mathrm{Sp}^\Sigma[G\text{–level eq.}^{-1}](X,Z)$$

is a bijection.

Now we can define:

Definition 2.9 (global equivalence) A morphism $f\colon X \to Y$ of symmetric spectra is a *global equivalence* if the induced morphism on underlying G–symmetric spectra $f_G\colon X_G \to Y_G$ is a G–stable equivalence for all finite groups G.

Example 2.10 Every global level equivalence is a global equivalence, since it induces a G–level equivalence on underlying G–symmetric spectra for all finite groups G. In fact, every "eventual level equivalence" $f\colon X \to Y$ — in the sense that for every finite group G there exists a finite G–set M such that $f(M \sqcup N)^G\colon X(M \sqcup N)^G \to Y(M \sqcup N)^G$ is a weak equivalence for all finite G–sets N — is a global equivalence. This is easiest to see via Proposition 4.5, since every eventual level equivalence induces an isomorphism on equivariant homotopy groups, which are discussed in Section 4.

We make the definition of a global equivalence more concrete and consider the (underlying G–symmetric spectrum/G–fixed points) adjunction

$$(-)_G\colon \mathrm{Sp}^\Sigma \rightleftarrows G\mathrm{Sp}^\Sigma :(-)^G.$$

By definition, a map f of symmetric spectra is a global equivalence if and only if f_G is a G–stable equivalence for all G. Using the global level model structure on Sp^Σ and the G–flat level model structure on $G\mathrm{Sp}^\Sigma$, the adjunction forms a Quillen pair (since the underlying G–spectrum functor preserves all cofibrations and weak equivalences) and so it can be derived to an adjunction between the homotopy categories

$$\mathbb{L}(-)_G\colon \mathrm{Sp}^\Sigma[\text{global level eq.}^{-1}] \rightleftarrows G\mathrm{Sp}^\Sigma[G\text{–level eq.}^{-1}] :(-)^{\mathbb{R}G},$$

where the functor $(-)_G$ does not really need to be derived as it is homotopical. Using this adjunction and the definition of a G–stable equivalence we see:

Corollary 2.11 *A map $f\colon X \to Y$ of symmetric spectra is a global equivalence if and only if for all finite groups G and all $G\Omega$–spectra Z the map*

$$\mathrm{Sp}^\Sigma[\text{global level eq.}^{-1}](Y, Z^{\mathbb{R}G}) \xrightarrow{\gamma(f)^*} \mathrm{Sp}^\Sigma[\text{global level eq.}^{-1}](X, Z^{\mathbb{R}G})$$

is a bijection.

Here, $\gamma\colon \mathrm{Sp}^\Sigma \to \mathrm{Sp}^\Sigma[\text{global level eq.}^{-1}]$ denotes the localization functor. This may still be unsatisfactory, because the definition is not intrinsic to symmetric spectra as it is not clear which symmetric spectra arise as the derived fixed points of $G\Omega$–spectra. It turns out that these fixed points are again equivariant Ω–spectra, in the following global sense:

Definition 2.12 (global Ω–spectra) A symmetric spectrum X is called a *global Ω–spectrum* if for all finite groups G and all finite G–sets M and N of which M is faithful, the adjoint generalized structure map

$$X(M) \xrightarrow{\tilde{\sigma}_M^N} \Omega^N(X(M \sqcup N)) \to \Omega^N(X(M \sqcup N)^f)$$

is a G–weak equivalence.

Again, the fibrant replacement is there to guarantee that the loop space is derived. We note that every global Ω–spectrum is in particular a nonequivariant Ω–spectrum. In general, a global Ω–spectrum X is not quite a $G\Omega$–spectrum on underlying G–symmetric spectra for nontrivial finite groups G, as there is no faithfulness condition in Definition 2.7. However, every faithful finite G–set N gives rise to a $G\Omega$–replacement $X_G \to \Omega^N(\mathrm{sh}^N(X_G))$ of X_G (up to eventual G–level equivalence), but $\Omega^N(\mathrm{sh}^N(X_G))$ has nontrivial exterior G–action and thus does not underlie a symmetric spectrum. It is usually not possible to replace a symmetric spectrum by a globally equivalent symmetric spectrum whose underlying G–symmetric spectra are $G\Omega$–spectra for all finite groups G at once (the most prominent exception being the Eilenberg–Mac Lane spectrum $\mathrm{H}\mathbb{Z}$ for the constant global functor $\underline{\mathbb{Z}}$ discussed in [19, Construction 5.3.8]).

As promised, we have:

Proposition 2.13 *The derived fixed points $Z^{\mathbb{R}G}$ of a $G\Omega$–spectrum Z form a global Ω–spectrum.*

Proof As remarked above, we can use a G–flat fibrant replacement Z^f of Z to compute its right derived fixed points. We now recall from [6, Section 2.6] what it means for a G–symmetric spectrum to be G–flat fibrant. Given two groups G and K we let $\mathcal{F}^{G,K}$ denote the family of subgroups of $G \times K$ whose intersection with $\{e\} \times K$ is trivial. Every such subgroup is of the form $\{(h, \varphi(h)) \mid h \in H\}$ for a unique subgroup H of G and group homomorphism $\varphi\colon H \to K$. Then the fact that Z^f is G–flat fibrant means that each level Z^f_n is $(G\times\Sigma_n)$–fibrant and in addition cofree with respect to the family $\mathcal{F}^{G,\Sigma_n}$, ie the map $Z^f_n \to \mathrm{map}(E\mathcal{F}^{G,\Sigma_n}_+, Z^f_n)$ is a $(G\times\Sigma_n)$–weak equivalence, where $E\mathcal{F}^{G,\Sigma_n}$ is a universal space for $\mathcal{F}^{G,\Sigma_n}$ (see [6, Section 1.3 and Definition 2.18]).

We now show that $(Z^f)^G$ forms a global Ω–spectrum. Let K be a finite group and M and N be finite K–sets of which M faithful (and of cardinality m). We consider the evaluation $Z^f(M) = Z^f_m \wedge_{\Sigma_m} (\mathrm{Bij}(\underline{m}, M)_+)$ and give it a $(G\times K)$–action by letting G act through Z^f_m and K through M. Likewise, we obtain a $(G\times K)$–action on $Z^f(M \sqcup N)$ and hence also on $\Omega^N(Z^f(M \sqcup N))$.

We claim the following:

(i) The map $\tilde{\sigma}^N_M\colon Z^f(M) \to \Omega^N(Z^f(M\sqcup N))$ is an $\mathcal{F}^{G,K}$–weak equivalence, ie it induces a weak equivalence on all fixed points for subgroups in the family $\mathcal{F}^{G,K}$.

(ii) Both $Z^f(M)$ and $\Omega^N(Z^f(M \sqcup N))$ are $\mathcal{F}^{G,K}$–cofree.

Together these imply that $\tilde{\sigma}^N_M\colon Z^f(M) \to \Omega^N(Z^f(M\sqcup N))$ is a $(G\times K)$–weak equivalence, as every $\mathcal{F}^{G,K}$–weak equivalence between $\mathcal{F}^{G,K}$–cofree $(G\times K)$–simplicial sets is a $(G\times K)$–weak equivalence. In particular, the induced map on G–fixed points $(\tilde{\sigma}^N_M)^G\colon (Z^f)^G(M) \to \Omega^N((Z^f)^G(M\sqcup N))$ is a K–weak equivalence, which proves the proposition.

Hence, it remains to show the claims; we begin with the first one. We let H be a subgroup of G and $\varphi\colon H \to K$ a group homomorphism. Then the composite $H \to K \to \Sigma_M$ defines an H–action on M (and likewise on N), which we denote by $\varphi^*(M)$. Pulling back $Z^f(M)$ and $Z^f(M\sqcup N)$ along the graph of φ yields the H–simplicial sets $Z^f(\varphi^*(M))$ and $Z^f(\varphi^*(M \sqcup N))$. In other words, we have to check whether the adjoint structure map $\tilde{\sigma}_m^n\colon Z^f(\varphi^*(M)) \to \Omega^{\varphi^*(N)}(Z^f(\varphi^*(M \sqcup N)))$ induces a weak equivalence on H–fixed points, but this is the case since Z^f is a $G\Omega$–spectrum.

The second claim follows from the observation that when restricting $E\mathcal{F}^{G,\Sigma_m}$ along $\mathrm{id} \times \psi$ for an injective group homomorphism $\psi\colon K \to \Sigma_m$ one obtains a model for $E\mathcal{F}^{G,K}$. This finishes the proof. □

It will be a consequence of Theorem 2.17 that global Ω–spectra are precisely the local objects with respect to the class of global equivalences. In other words, one could alternatively characterize global equivalences as those morphisms that induce bijections on all morphism sets into global Ω–spectra in the global level homotopy category.

2.3 Stable model structure

In this section we introduce the global stable model structure on symmetric spectra. We begin by constructing a global Ω–spectrum replacement functor up to natural global equivalence.

For this we let G be a finite group, M and N two finite G–sets and define

$$\lambda_M^N\colon F^G_{M\sqcup N}(S^N) \to F^G_M(S^0)$$

to be adjoint to the embedding $S^N \hookrightarrow \mathbf{\Sigma}(M, M \sqcup N)/G = (F^G_M(S^0))(M \sqcup N)$ associated to the inclusion $M \hookrightarrow M \sqcup N$ (see Section 1.3 for the definition of $\mathbf{\Sigma}(-,-)$ and global free symmetric spectra). Under the adjunction isomorphism, λ_M^N represents the adjoint generalized structure map on G–fixed points,

$$\begin{aligned}\mathrm{map}_{\mathrm{Sp}^\Sigma}(F^G_M(S^0), X) \cong X(M)^G& \\ &\xrightarrow{(\tilde{\sigma}_M^N)^G} (\Omega^N X(M \sqcup N))^G \cong \mathrm{map}_{\mathrm{Sp}^\Sigma}(F^G_{M\sqcup N}(S^N), X).\end{aligned}$$

The morphisms λ_M^N are usually not cofibrations, so we factor them as

$$F^G_{M\sqcup N}(S^N) \xrightarrow{\overline{\lambda}_M^N} \mathrm{Cyl}(\lambda_M^N) \xrightarrow{r_M^N} F^G_M(S^0)$$

via the levelwise mapping cylinder $\mathrm{Cyl}(-)$. It is a formal consequence, as explained in the proof of [9, Lemma 3.4.10], that $\overline{\lambda}_M^N$ is a flat cofibration, since the global level model structure is simplicial. Finally, we define

$$J^{\mathrm{st}}_{\mathrm{gl}} = \{i \,\square\, \overline{\lambda}_M^N \mid i \in I,\ G \text{ finite},\ M \text{ and } N \text{ finite } G\text{–sets with } M \text{ faithful}\} \cup J^{\mathrm{lev}}_{\mathrm{gl}},$$

where I is a set of generating cofibrations of the Quillen model structure on based simplicial sets. The notation $f \square g$ stands for the pushout product $(A \wedge Y) \cup_{A \wedge X} (B \wedge X) \to (B \wedge Y)$ of a map $f\colon A \to B$ of based simplicial sets with a morphism $g\colon X \to Y$ of symmetric spectra. More precisely, we only include $i \square \overline{\lambda}_M^N$ for a chosen system of representatives of isomorphism classes of triples (G, M, N) to ensure that $J_{\mathrm{gl}}^{\mathrm{st}}$ is a set. Then we have:

Proposition 2.14 *For a symmetric spectrum X the following are equivalent:*

- *X is a level fibrant global Ω–spectrum.*
- *X has the right lifting property with respect to the set $J_{\mathrm{gl}}^{\mathrm{st}}$.*

Proof We already know that X is global level fibrant if and only if it has the right lifting property with respect to $J_{\mathrm{gl}}^{\mathrm{st}}$. By adjunction, X has the right lifting property with respect to $\{i \square \overline{\lambda}_M^N\}_{i \in I}$ if and only if

$$\mathrm{map}_{\mathrm{Sp}^\Sigma}(\overline{\lambda}_M^N, X)\colon \mathrm{map}_{\mathrm{Sp}^\Sigma}(\mathrm{Cyl}(\lambda_M^N), X) \to \mathrm{map}_{\mathrm{Sp}^\Sigma}(F_{M \sqcup N}^G(S^N), X)$$

has the right lifting property with respect to the set I. Since the global level model structure is simplicial, this map is always a Kan fibration. Hence, it has the right lifting property with respect to I if and only if it is a weak homotopy equivalence. Since r_M^N is a homotopy equivalence of symmetric spectra, this in turn is equivalent to

$$\mathrm{map}_{\mathrm{Sp}^\Sigma}(F_M^G(S^0), X) \xrightarrow{\mathrm{map}_{\mathrm{Sp}^\Sigma}(\lambda_M^N, X)} \mathrm{map}_{\mathrm{Sp}^\Sigma}(F_{M \sqcup N}^G(S^N), X)$$

being a weak homotopy equivalence. As remarked above, this map can be identified with the G–fixed points of the adjoint generalized structure map $\tilde{\sigma}_M^N$ of X, which finishes the proof. □

Corollary 2.15 *If M is faithful, then λ_M^N is a global equivalence.*

Proof This follows from Propositions 2.13 and 2.14 and the fact that $F_{M \sqcup N}^G(S^N)$ and $F_M^G(S^0)$ are flat. □

Since the global level model structure is simplicial, it follows that every morphism in $J_{\mathrm{gl}}^{\mathrm{st}}$ is a flat cofibration. Furthermore, all domains and codomains of morphisms in $J_{\mathrm{gl}}^{\mathrm{st}}$ are small with respect to countably infinite sequences of flat cofibrations. So we can apply the small object argument (see [4, Section 7.12]) to obtain a functor $Q\colon \mathrm{Sp}^\Sigma \to \mathrm{Sp}^\Sigma$ with image in global Ω–spectra and a natural relative $J_{\mathrm{gl}}^{\mathrm{st}}$–cell complex $q\colon \mathrm{id} \to Q$. Since every morphism in $J_{\mathrm{gl}}^{\mathrm{st}}$ is a flat cofibration and global equivalence, it follows from [6, Proposition 4.2] applied to each finite group separately that every relative $J_{\mathrm{gl}}^{\mathrm{st}}$–cell complex is a global equivalence. In particular, the morphisms $q_X\colon X \to QX$ are always global equivalences. This also implies that Q preserves global equivalences by 2-out-of-3. Before we use these properties to construct the global stable model structure we need one more lemma:

Lemma 2.16 *Every global equivalence between global Ω–spectra is a global level equivalence.*

Proof Let $f\colon X \to Y$ be a global equivalence of global Ω–spectra. We have to show that each f_n is a Σ_n–weak equivalence. For this we again denote by $\underline{n}$ the tautological Σ_n–set and consider the commutative diagram of Σ_n–symmetric spectra

$$\begin{tikzcd}
X_{\Sigma_n} \arrow[r, "\alpha^{\underline{n}}_{X_{\Sigma_n}}"] \arrow[d, "f_{\Sigma_n}"'] & \Omega^{\underline{n}}(\mathrm{sh}^{\underline{n}}\, X_{\Sigma_n}) \arrow[d, "\Omega^{\underline{n}}(\mathrm{sh}^{\underline{n}} f_{\Sigma_n})"] \\
Y_{\Sigma_n} \arrow[r, "\alpha^{\underline{n}}_{Y_{\Sigma_n}}"'] & \Omega^{\underline{n}}(\mathrm{sh}^{\underline{n}}\, Y_{\Sigma_n})
\end{tikzcd}$$

Since X and Y are global Ω–spectra the horizontal arrows $\alpha^{\underline{n}}_{X_{\Sigma_n}}$ and $\alpha^{\underline{n}}_{Y_{\Sigma_n}}$ induce Σ_n–weak equivalences on all evaluations at faithful Σ_n–sets. In particular, using Example 2.10 we see that they are both Σ_n–stable equivalences and so $\Omega^{\underline{n}}(\mathrm{sh}^{\underline{n}} f_{\Sigma_n})$ is also a Σ_n–stable equivalence. Furthermore, since $\underline{n}$ is a faithful Σ_n–set, the Σ_n–symmetric spectra $\Omega^{\underline{n}}(\mathrm{sh}^{\underline{n}}\, X_{\Sigma_n})$ and $\Omega^{\underline{n}}(\mathrm{sh}^{\underline{n}}\, Y_{\Sigma_n})$ are $\Sigma_n\Omega$–spectra. This implies that $\Omega^{\underline{n}}(\mathrm{sh}^{\underline{n}} f_{\Sigma_n})$ is even a Σ_n–level equivalence by the Yoneda lemma. In particular, it induces a Σ_n–weak equivalence when evaluated on $\underline{n}$ and hence so do f_{Σ_n} and f (again using that the horizontal arrows induce Σ_n–weak equivalences on all faithful evaluations). This finishes the proof. □

Finally, a morphism of symmetric spectra is called a *(positive) global fibration* if it has the right lifting property with respect to all morphisms that are (positive) flat cofibrations and global equivalences. Then we have:

Theorem 2.17 (global model structures) *The global equivalences, (positive) global fibrations and (positive) flat cofibrations define a proper, cofibrantly generated and monoidal model structure on the category of symmetric spectra, called the* ***(positive) global stable model structure****.*

Moreover, the fibrant objects of the (positive) global stable model structure are precisely the (positive) global Ω–spectra.

Here, a symmetric spectrum is called a *positive global Ω–spectrum* if it satisfies the condition of Definition 2.12 in all cases except possibly for $G=\{e\}$ and $M=\varnothing$.

Proof Both model structures are obtained via left Bousfield localization at the respective global level model structures. We apply [3, Theorem 9.3] with respect to the global Ω–spectrum replacement functor Q and the natural global equivalence $q\colon \mathrm{id}\to Q$ just constructed. By Lemma 2.16, a morphism between global Ω–spectra is a global equivalence if and only if it is a (positive) global level equivalence, so the global equivalences agree with the Q–equivalences in the sense of Bousfield's theorem.

It remains to check axioms (A1)–(A3) of [3, Section 9.2]. Axiom (A1) requires that every (positive) global level equivalence be a global equivalence, which is Example 2.10. For a symmetric spectrum X, the morphisms $q_{QX}, Qq_X\colon QX\to QQX$ are global equivalences between global Ω–spectra, and hence global level equivalences by Lemma 2.16, implying axiom (A2). For (A3) we are given a pullback square

$$\begin{tikzcd}
V \arrow[r, "k"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
W \arrow[r, "h"'] & Y
\end{tikzcd}$$

where f is a (positive) global level fibration, h is a global equivalence and X and Y are (positive) global Ω–spectra. We have to show that g is also a global equivalence. This is even true without any hypothesis on X and Y, as follows by applying the dual version of [6, Proposition 4.2] for every finite group G.

Monoidality of the model structures is again implied by the respective monoidality of the G–flat model structures [6, Proposition 6.1]. Finally, the statement about the fibrant objects is a consequence of the characterization of the fibrations in the localized model structure given in [3, Theorem 9.3] and the fact that X is a (positive) global Ω–spectrum if and only if the map $q_X\colon X\to QX$ is a (positive) global level equivalence. □

The generating cofibrations are the same as for the respective level model structures. In the nonpositive case, the generating acyclic cofibrations are given by $J^{\mathrm{st}}_{\mathrm{gl}}$; for the positive version one has to take out those maps that are not positive flat cofibrations (ie those involving a spectrum of the form $F^{\{e\}}_{\varnothing}(-)$). Finally, we note:

Remark 2.18 As written at the beginning of Section 1, analogs of all results of this section also hold for symmetric spectra of topological spaces: There is a global level model structure where the weak equivalences (fibrations) are the morphisms $f\colon X \to Y$ such that $f_n^H\colon X_n^H \to Y_n^H$ is a weak homotopy-equivalence (resp. Serre fibration) for all $n \in \mathbb{N}$ and all subgroups H of Σ_n. The global stable model structure is obtained by left Bousfield localization at the global equivalences, which can be defined as in Definition 2.9 or alternatively be characterized as those morphisms which become global equivalences after applying the singular complex functor. The geometric realization/singular complex adjunction gives a Quillen equivalence between the topological and the simplicial version of the model structures.

3 Multiplicative properties

We have seen in Theorem 2.17 that the global model structure is monoidal, ie that it satisfies the pushout product and unit axioms. In this section we construct global model structures on categories of modules, algebras and commutative algebras by further checking that the monoid and strong commutative monoid axioms hold. In all cases, the properties follow directly from the respective ones for G–symmetric spectra, since the functor $(-)_G$ is strong symmetric monoidal and commutes with all limits and colimits.

3.1 Model structure on module and algebra categories

Given a model structure on symmetric spectra, a map of modules or algebras is called a weak equivalence or fibration if its underlying morphism of symmetric spectra is so. We say that the given model structure lifts to the category of modules or algebras if these two classes define a model structure.

Theorem 3.1 *For every symmetric ring spectrum R the positive and nonpositive global stable model structures lift to the category of R–modules. If R is commutative, these model structures are again monoidal.*

Theorem 3.2 *For every commutative symmetric ring spectrum R the positive and nonpositive global stable model structures lift to the category of R–algebras. Moreover, every cofibration of R–algebras whose source is cofibrant as an R–module is a cofibration of R–modules.*

Both theorems are obtained via the results of [20], which show that it suffices to prove that the monoid axiom (stated below) holds. The main ingredient is the following:

Proposition 3.3 (flatness) (i) *Smashing with a flat symmetric spectrum preserves global equivalences.*

(ii) *Smashing with an arbitrary symmetric spectrum preserves global equivalences between flat symmetric spectra.*

Proof This is a direct consequence of [6, Proposition 6.2]. □

For any symmetric spectrum Y we denote by $\{J_{\mathrm{gl}}^{\mathrm{st}} \wedge Y\}_{\mathrm{cell}}$ the class of morphisms obtained via (transfinite) compositions and pushouts from morphisms of the form $j \wedge Y$, where j lies in $J_{\mathrm{gl}}^{\mathrm{st}}$.

Corollary 3.4 (monoid axiom) *Every morphism in* $\{J_{\mathrm{gl}}^{\mathrm{st}} \wedge Y\}_{\mathrm{cell}}$ *is a global equivalence.*

Proof Again, this follows directly from the monoid axiom for the G–flat stable model structure on G–symmetric spectra [6, Proposition 6.4]. □

By [20, Theorem 4.1], this implies Theorems 3.1 and 3.2.

3.2 Model structure on commutative algebra categories

The positive global model structure also lifts to the category of commutative symmetric ring spectra (or, more generally, commutative algebras over a commutative symmetric ring spectrum). We note that this is a very strong form of equivariant commutativity, which induces norm maps and power operations on equivariant homotopy groups. For this reason commutative symmetric (or orthogonal) ring spectra are called "ultracommutative" in [19] when they are considered from the point of view of global homotopy.

Theorem 3.5 *For every commutative symmetric ring spectrum R the positive global model structure lifts to the category of commutative R–algebras.*

Moreover, the underlying R–module map of a positive flat cofibration of commutative R–algebras $X \to Y$ is a positive flat cofibration of R–modules if X is (not necessarily positive) flat as an R–module. In particular, the symmetric spectrum underlying a positive flat commutative symmetric ring spectrum is flat.

The part about positive flat cofibrations is merely a restating of Shipley's result [21, Proposition 4.1], since the cofibrations in the positive flat nonequivariant and the positive global model structure on commutative algebras are the same.

In order to prove Theorem 3.5 we make use of results of [23]. For this we recall that given a morphism $f\colon X \to Y$ of symmetric spectra, the n–fold pushout product $f^{\square n}$ is defined inductively via $f^{\square n} := f \square f^{\square (n-1)}$.

Proposition 3.6 (strong commutative monoid axiom) *Let $f\colon X \to Y$ be a morphism of symmetric spectra. Then*:

(i) *If f is a (positive) flat cofibration, then $f^{\square n}/\Sigma_n$ is again a (positive) flat cofibration.*

(ii) *If f is a positive flat cofibration and global equivalence, then so is $f^{\square n}/\Sigma_n$.*

Proof This follows immediately from [6, Proposition 6.22]. □

Applying [23, Theorem 3.2] (and [21, Proposition 4.1] for the part on cofibrations), we obtain Theorem 3.5.

4 Equivariant homotopy groups of symmetric spectra

In this section we study equivariant homotopy groups of symmetric spectra. We say that a countable G–set for a finite group G is a *complete G–set universe* if it allows an embedding of every finite G–set. Then for every symmetric spectrum X, every finite group G, every complete G–set universe $\mathcal{U}_G$ and every integer n, we define an abelian group $\pi_n^{G,\mathcal{U}_G}(X)$. Any two complete G–set universes are isomorphic, which will imply that $\pi_n^{G,\mathcal{U}_G}(X)$ only depends on the choice of $\mathcal{U}_G$ up to natural isomorphism. However, unlike for orthogonal spectra this isomorphism of homotopy groups is not canonical: it is affected by the choice of isomorphism of G–set universes. Hence, for arbitrary symmetric spectra X it is misleading to simply write $\pi_n^G(X)$. This phenomenon also affects the functoriality of $\pi_n^{G,\mathcal{U}_G}(X)$ in group homomorphisms, which we discuss in Section 4.3.

All this is tied to the fact that equivariant homotopy groups of symmetric spectra are not homotopical, ie global equivalences generally do not induce isomorphisms on them. If one works with the derived versions (ie replacing $\pi_n^{G,\mathcal{U}_G}(X)$ by $\pi_n^{G,\mathcal{U}_G}(QX)$) these problems disappear and one obtains the same properties as for homotopy groups of orthogonal spectra. In Section 4.6 we discuss criteria to detect for which symmetric spectra the “naive” equivariant homotopy groups are already derived.

4.1 Definition and global $\underline{\pi}_*$–isomorphisms

Given a finite group G and a complete G–set universe $\mathcal{U}_G$, we denote by $s_G(\mathcal{U}_G)$ the poset of finite G–subsets of $\mathcal{U}_G$, partially ordered by inclusion.

Definition 4.1 Let $n \in \mathbb{Z}$ be an integer. Then the n^{th} G–equivariant homotopy group $\pi_n^{G,\mathcal{U}_G}(X)$ of a symmetric spectrum of spaces X (with respect to $\mathcal{U}_G$) is defined as

$$\pi_n^{G,\mathcal{U}_G}(X) := \operatorname*{colim}_{M \in s_G(\mathcal{U})} [S^{n\sqcup M}, X(M)]^G.$$

The connecting maps in the colimit system are given by the composites

$$[S^{n\sqcup M}, X(M)]^G \xrightarrow{(-)\wedge S^{N-M}} [S^{n\sqcup M\sqcup(N-M)}, X(M)\wedge S^{N-M}]^G \xrightarrow{(\sigma_M^{N-M})_*} [S^{n\sqcup N}, X(N)]^G$$

for every inclusion $M \subseteq N$. The last step implicitly uses the homeomorphism $X(M\sqcup(N-M)) \cong X(N)$ induced from the canonical isomorphism $M\sqcup(N-M)\cong N$.

To clarify what this exactly means for negative n we choose an isometric G–embedding $i\colon \mathbb{R}^\infty \hookrightarrow (\mathbb{R}^{(\mathcal{U}_G)})^G$ and only index the colimit system over those G–sets M in $s_G(\mathcal{U})$ for which $\mathbb{R}^M$ contains $i(\mathbb{R}^{-n})$. In this case the corresponding term is given by $[S^{M-i(\mathbb{R}^{-n})}, X(M)]^G$, the expression $M - i(\mathbb{R}^{-n})$ denoting the orthogonal complement of $i(\mathbb{R}^{-n})$ in $\mathbb{R}^M$. Since the space of embeddings $\mathbb{R}^\infty \hookrightarrow (\mathbb{R}^{(\mathcal{U}_G)})^G$ is contractible, the definition only depends on this choice up to *canonical* isomorphism and so we leave it out of the notation. As long as $S^{n\sqcup M}$ has at least two trivial coordinates, the set $[S^{n\sqcup M}, X(M)]^G$ carries a natural abelian group structure and hence so does $\pi_n^{G,\mathcal{U}_G}(X)$.

For a symmetric spectrum of simplicial sets we put $\pi_n^{G,\mathcal{U}_G}(X) := \pi_n^{G,\mathcal{U}_G}(|X|)$.

Definition 4.2 A morphism $f\colon X \to Y$ of symmetric spectra is called a *global $\underline{\pi}_*$–isomorphism* if for all finite groups G, all integers $n \in \mathbb{Z}$ and every complete G–set universe $\mathcal{U}_G$, the induced map $\pi_n^{G,\mathcal{U}_G}(f)\colon \pi_n^{G,\mathcal{U}_G}(X) \to \pi_n^{G,\mathcal{U}_G}(Y)$ is an isomorphism.

In fact it suffices to require an isomorphism for a single choice of complete G–set universe $\mathcal{U}_G$ for each finite group G, since any two are noncanonically isomorphic.

Remark 4.3 The definition of $\pi_*^{G,\mathcal{U}_G}(X)$ agrees with that of $\pi_*^{G,\mathcal{U}_G}(X_G)$ in Section 3 of [6]. Hence, a morphism of symmetric spectra is a global $\underline{\pi}_*$–isomorphism if and only if it is a $\underline{\pi}_*^{\mathcal{U}_G}$–isomorphism on underlying G–symmetric spectra for every finite group G.

The following is immediate from the definition:

Example 4.4 Every global level equivalence is a global $\underline{\pi}_*$–isomorphism.

Every global level equivalence is also a global equivalence, as we remarked in Example 2.10. It is not obvious from the definition that this is true for arbitrary global $\underline{\pi}_*$–isomorphisms, but it follows by applying [6, Theorem 3.36] for each finite group G:

Proposition 4.5 *Every global $\underline{\pi}_*$–isomorphism is a global equivalence.*

4.2 Properties

We now collect some properties of equivariant homotopy groups and global $\underline{\pi}_*$–isomorphisms, all implied by their respective versions for G–symmetric spectra. For this we let $C(f)$ denote the levelwise mapping cone of a morphism $f\colon X \to Y$ of symmetric spectra, $i(f)\colon Y \to C(f)$ the inclusion into the cone and $q(f)\colon C(f) \to S^1 \wedge X$ its cofiber. Dually, we let $H(f)$ stand for the levelwise homotopy fiber, $p(f)\colon H(f) \to X$ the projection and $j(f)\colon \Omega(Y) \to H(f)$ its fiber.

Proposition 4.6 *Let G be a finite group and $\mathcal{U}_G$ a complete G–set universe. Then the following hold:*

(i) *For every symmetric spectrum of spaces X the unit $X \to \Omega(S^1 \wedge X)$ and the counit $S^1 \wedge (\Omega X) \to X$ are global $\underline{\pi}_*$–isomorphisms. In particular, there are natural isomorphisms*

$$\pi_{n+1}^{G,\mathcal{U}_G}(S^1 \wedge X) \cong \pi_n^{G,\mathcal{U}_G}(X) \cong \pi_{n-1}^{G,\mathcal{U}_G}(\Omega X).$$

(ii) *For every morphism $f\colon X \to Y$ of symmetric spectra of spaces the sequences*

$$\cdots \to \pi_n^{G,\mathcal{U}_G}(X) \xrightarrow{f_*} \pi_n^{G,\mathcal{U}_G}(Y) \xrightarrow{i(f)_*} \pi_n^{G,\mathcal{U}_G}(C(f)) \xrightarrow{q(f)_*} \pi_{n-1}^{G,\mathcal{U}_G}(X) \to \cdots$$

and

$$\cdots \to \pi_{n+1}^{G,\mathcal{U}_G}(Y) \xrightarrow{j(f)_*} \pi_n^{G,\mathcal{U}_G}(H(f)) \xrightarrow{p(f)_*} \pi_n^{G,\mathcal{U}_G}(X) \xrightarrow{f_*} \pi_n^{G,\mathcal{U}_G}(Y) \to \cdots$$

are exact. Furthermore, the natural morphism $S^1 \wedge H(f) \to C(f)$ is a global $\underline{\pi}_*$–isomorphism.

(iii) *For every family $(X_i)_{i\in I}$ of symmetric spectra, the canonical map*

$$\bigoplus_{i\in I}(\pi_n^{G,\mathcal{U}_G}(X_i)) \to \pi_n^{G,\mathcal{U}_G}\left(\bigvee_{i\in I} X_i\right)$$

is an isomorphism of abelian groups. If I *is finite, the natural morphism* $\bigvee_{i\in I} X_i \to \prod_{i\in I} X_i$ *is a global* $\underline{\pi}_*$*–isomorphism.*

(iv) *Smashing with a flat symmetric spectrum preserves global* $\underline{\pi}_*$*–isomorphisms.*

In the second item we have implicitly used the isomorphisms of item (i) to obtain the boundary maps.

Proof These are Propositions 3.6 and 6.2 in [6]. □

This proposition also has a simplicial analog, for which in item (i) and the second long exact sequence in (ii) the constructions Ω and $H(-)$ need to be replaced by their derived versions.

4.3 Functoriality

An important feature of global homotopy theory of orthogonal spectra is that their equivariant homotopy groups enjoy a rich functoriality in the group, they form a so-called *global functor*. In short, every group homomorphism $\varphi\colon G \to K$ induces a restriction map $\varphi^*\colon \pi_*^K(X) \to \pi_*^G(X)$ (depending only on its conjugacy class) and for every subgroup $H \le G$, there is a transfer homomorphism $\mathrm{tr}_H^G\colon \pi_*^H(X) \to \pi_*^G(X)$. Moreover, restrictions and transfers are related by a double coset formula.

While the transfer homomorphism works similarly for symmetric spectra, a complication arises when one tries to construct restriction maps. To explain this, we let X be a symmetric spectrum, $\varphi\colon G \to K$ a homomorphism of finite groups and $x \in \pi_0^{K,\mathcal{U}_K}(X)$ an element represented by a K–map $f\colon S^M \to X(M)$ for a finite K–subset M of $\mathcal{U}_K$. Restricting all the actions along φ and making use of the equalities $\varphi^*(S^M) = S^{\varphi^*(M)}$ and $\varphi^*(X(M)) = X(\varphi^*(M))$, we can think of f as a G–map $S^{\varphi^*(M)} \to X(\varphi^*(M))$. In order for this to represent an element $\varphi^*(x)$ in $\pi_0^{G,\mathcal{U}_G}(X)$ we have to choose an embedding of $\varphi^*(M)$ into $\mathcal{U}_G$, but such an embedding is not canonical and — unlike for orthogonal spectra — the outcome is in general affected by the choice one makes. One might try to get around this by using the restricted universe $\varphi^*(\mathcal{U}_K)$ instead of $\mathcal{U}_G$, but this only works if φ is injective because otherwise $\varphi^*(\mathcal{U}_G)$ is not complete.

This issue can be resolved by carrying an embedding $\varphi^*(\mathcal{U}_K) \hookrightarrow \mathcal{U}_G$ around as an additional datum with respect to which one forms the restriction, as we now explain.

4.4 Restriction maps

Let $\mathrm{Fin}_{\mathcal{U}}$ denote the category of pairs $(G, \mathcal{U}_G)$ of a finite group G together with a complete G–set universe $\mathcal{U}_G$, in which a morphism (φ, α) from $(G, \mathcal{U}_G)$ to $(K, \mathcal{U}_K)$ is a group homomorphism $\varphi\colon G \to K$ and a G–equivariant embedding $\alpha\colon \varphi^*(\mathcal{U}_K) \hookrightarrow \mathcal{U}_G$.

Now we let X be a symmetric spectrum and $(\varphi\colon G \to K,\ \alpha\colon \varphi^*(\mathcal{U}_K) \hookrightarrow \mathcal{U}_G)$ a morphism in $\mathrm{Fin}_{\mathcal{U}}$. Further, let x be an element of $\pi_0^{K,\mathcal{U}_K}(X)$ represented by a K–map $f\colon S^M \to X(M)$ with $M \subseteq \mathcal{U}_K$. Then we define $(\varphi, \alpha)^*(x) \in \pi_0^{G,\mathcal{U}_G}(X)$ as the class of the composite

$$S^{\alpha(M)} \xrightarrow{S^{(\alpha|_M)^{-1}}} S^M \xrightarrow{f} X(M) \xrightarrow{X(\alpha|_M)} X(\alpha(M)).$$

This class does not depend on the chosen representative f and hence we obtain a restriction map

$$(\varphi, \alpha)^*\colon \pi_0^{K,\mathcal{U}_K}(X) \to \pi_0^{G,\mathcal{U}_G}(X).$$

The following is straightforward:

Proposition 4.7 *For every symmetric spectrum X the assignment*

$$(G, \mathcal{U}_G) \mapsto \pi_0^{G,\mathcal{U}_G}(X),$$
$$\big(G \xrightarrow{\varphi} K,\ \varphi^*(\mathcal{U}_K) \xhookrightarrow{\alpha} \mathcal{U}_G\big) \longmapsto \big((\varphi, \alpha)^*\colon \pi_0^{K,\mathcal{U}_K}(X) \to \pi_0^{G,\mathcal{U}_G}(X)\big),$$

defines a contravariant functor $\underline{\pi}_0(X)$ from $\mathrm{Fin}_{\mathcal{U}}$ *to abelian groups.*

Using the suspension isomorphisms $\pi_n^{G,\mathcal{U}_G}(X) \cong \pi_0^{G,\mathcal{U}_G}(\Omega^n(X))$ for $n \geq 0$ as well as $\pi_n^{G,\mathcal{U}_G}(X) \cong \pi_0^{G,\mathcal{U}_G}(S^{-n} \wedge X)$ for $n < 0$, we obtain natural $\mathrm{Fin}_{\mathcal{U}}^{\mathrm{op}}$–functors $\underline{\pi}_n(X)$ for all $n \in \mathbb{Z}$.

We note the following special cases of operations obtained this way:

(i) Every subgroup inclusion $i_H^G\colon H \leq G$ gives rise to a restriction homomorphism

$$(i_H^G)^*\colon \pi_0^{G,\mathcal{U}_G}(X) \to \pi_0^{H,(i_H^G)^*(\mathcal{U}_G)}(X).$$

by applying the above construction to the morphism $(i_H^G, \mathrm{id})\colon (H, (i_H^G)^*(\mathcal{U}_G)) \to (G, \mathcal{U}_G)$ in $\mathrm{Fin}_{\mathcal{U}}$.

(ii) Every surjective group homomorphism $\varphi\colon G \twoheadrightarrow K$ gives rise to a restriction homomorphism

$$(\varphi, (- \circ \varphi))^*\colon \pi_0^{K,\mathbb{N}^K}(X) \to \pi_0^{G,\mathbb{N}^G}(X),$$

where $\mathbb{N}^G$ denotes the complete G–set universe of functions from G to the natural numbers (and likewise for K) and $(-\circ\varphi)$ denotes the induced injective map by precomposing with φ.

(iii) Every pair of a subgroup $i_H^G\colon H\leq G$ and an element $g\in G$ induces a conjugation homomorphism

$$c_g^*\colon \pi_0^{H,(i_H^G)^*(\mathcal{U}_G)}(X)\to\pi_0^{gHg^{-1},(i_{gHg^{-1}}^G)^*(\mathcal{U}_G)}(X)$$

by applying the above construction to the morphism

$$(g^{-1}(-)g,g\cdot-)\colon (gHg^{-1},(i_{gHg^{-1}}^G)^*(\mathcal{U}_G))\to(H,(i_H^G)^*(\mathcal{U}_G)).$$

(iv) Every injective G–equivariant self-map $\alpha\colon\mathcal{U}_G\hookrightarrow\mathcal{U}_G$ gives rise to an endomorphism

$$\alpha\cdot-\colon\pi_0^{G,\mathcal{U}_G}(X)\to\pi_0^{G,\mathcal{U}_G}(X)$$

via $(\mathrm{id},\alpha)^*$. This defines an additive natural left action of the monoid $\mathrm{Inj}_G(\mathcal{U}_G,\mathcal{U}_G)$ on $\pi_0^{G,\mathcal{U}_G}(X)$.

Any morphism in $\mathrm{Fin}_{\mathcal{U}}$ can be written as a composite of those of type (i), (ii) and (iv). The first three should be seen as genuine global equivariant operations which survive to the global homotopy category, whereas nontriviality of the $\mathrm{Inj}_G(\mathcal{U}_G,\mathcal{U}_G)$–action implies that the morphism $X\to QX$ is not a global $\underline{\pi}_*$–isomorphism and hence the $\pi_n^{G,\mathcal{U}_G}(X)$ are not derived (see Proposition 4.13). In the nonequivariant case $(\{e\},\mathbb{N})$, this action was examined in [17], the equivariant version $(G,\mathcal{U}_G)$ in [6].

We also included the conjugation maps above because they allow a cleaner description of the double coset formula in Proposition 4.10. They have the following property:

Lemma 4.8 *All inner conjugations c_g^* act as the identity on $\pi_0^{G,\mathcal{U}_G}(X)$.*

Proof Let $x\in\pi_0^{G,\mathcal{U}_G}(X)$ be an arbitrary element, represented by a G–map $f\colon S^M\to X(M)$ for some finite $M\subseteq\mathcal{U}_G$. Then, by definition, $c_g^*(x)$ is the class represented by the composite

$$S^{\varphi^*(M)}\xrightarrow{g^{-1}\cdot-}S^M\xrightarrow{f}X(M)\xrightarrow{X(g\cdot-)}X(M).$$

The map $X(g\cdot-)\colon X(M)\to X(M)$ is equal to multiplication by g. So, since f is G–equivariant, this composite equals f and hence $c_g^*(x)=c_g^*([f])=[f]=x$, which proves the claim. □

Remark 4.9 The category $\mathrm{Fin}_{\mathcal{U}}$ comes with a forgetful functor to the category Fin of finite groups. The functor is surjective on objects and morphisms, but it does *not* have a section. In fact, for any nontrivial finite group G, there do not exist two lifts of the homomorphisms $i\colon \{e\} \to G$ and $p\colon G \to \{e\}$ such that their composite is the identity. This is because the second component of any preimage $(p\colon G \to \{e\},\, p^*(\mathcal{U}_{\{e\}}) \hookrightarrow \mathcal{U}_G)$ is never surjective, since the G–set universe $p^*(\mathcal{U}_{\{e\}})$ is trivial. Hence, the second component of the composite is also not surjective, in particular not the identity. There are symmetric spectra X for which $(\mathrm{id}_{\{e\}}, \alpha\colon \mathcal{U}_{\{e\}} \hookrightarrow \mathcal{U}_{\{e\}})$ does not act surjectively on $\pi_0^{\{e\},\mathcal{U}_{\{e\}}}(X)$ for every α which is not surjective (this is the case in Section 4.7); hence, this shows that there is in general no way to turn the $\mathrm{Fin}_{\mathcal{U}}^{\mathrm{op}}$–functor $\underline{\pi}_0(X)$ into a $\mathrm{Fin}^{\mathrm{op}}$–functor.

4.5 Transfer maps

The assignment $(G, \mathcal{U}_G) \mapsto \pi_0^{G,\mathcal{U}_G}(X)$ has more structure than that of a $\mathrm{Fin}_{\mathcal{U}}^{\mathrm{op}}$–functor: it also allows *transfer maps* of the form

$$\mathrm{tr}_H^G\colon \pi_0^{H,(i_H^G)^*(\mathcal{U}_G)}(X) \to \pi_0^{G,\mathcal{U}_G}(X)$$

for a subgroup H of G and the restricted (complete) H–set universe $(i_H^G)^*(\mathcal{U}_G)$. The construction and properties of these transfer maps are similar to those for orthogonal spectra, so we will be brief (see [19, Constructions 3.2.7 and 3.2.22]).

Transfer maps are based on the following construction: Let $M \subseteq \mathcal{U}_G$ be a G–subset which contains a copy of G/H. By thickening up the embedding $G/H \hookrightarrow M \hookrightarrow \mathbb{R}^M$ we obtain another G–embedding $G \ltimes_H D(\mathbb{R}^M) \hookrightarrow \mathbb{R}^M$, where $D(-)$ denotes the closed unit disc. Collapsing everything outside the image of the interior of $G \ltimes_H D(\mathbb{R}^M)$ to a point yields a map $p_H^G\colon S^M \to G \ltimes_H S^M$, the "Thom–Pontryagin collapse map".

Now let X be a symmetric spectrum of spaces and $x \in \pi_0^{H,i^*(\mathcal{U}_G)}(X)$ an element represented by an H–map $f\colon S^M \to X(M)$. Without loss of generality we can assume that M is in fact a G–subset of $\mathcal{U}_G$ and allows a G–embedding of G/H. Then the transfer $\mathrm{tr}_H^G(x) \in \pi_0^{G,\mathcal{U}_G}(X)$ is defined as the class of the composite

$$S^M \xrightarrow{p_H^G} G \ltimes_H S^M \xrightarrow{G \ltimes_H f} G \ltimes_H X(M) \xrightarrow{\mu} X(M),$$

where μ is the action map (which uses that $X(M)$ is a G–space).

Proposition 4.10 *The transfer maps* tr_H^G *do not depend on the choice of embedding* $G/H \hookrightarrow \mathcal{U}_G$*. They are additive and functorial in subgroup inclusions. Furthermore, they are related to the restriction maps by the following formulas:*

(i) *For every morphism* $(\varphi\colon G \twoheadrightarrow K,\ \alpha\colon \varphi^*(\mathcal{U}_K) \hookrightarrow \mathcal{U}_G)$ *in* $\mathrm{Fin}_{\mathcal{U}}$ *with surjective* φ *and every subgroup* $i\colon L \leq K$*, the relation*

$$(\varphi,\alpha)^* \circ \mathrm{tr}_L^K = \mathrm{tr}_{\varphi^{-1}(L)}^G \circ (\varphi|_{\varphi^{-1}(L)}\colon \varphi^{-1}(L) \to L, \alpha)^*$$

holds as maps $\pi_0^{L,(i_L^K)^*(\mathcal{U}_K)}(X) \to \pi_0^{G,\mathcal{U}_G}(X)$.

(ii) *For every pair of subgroups* $H, J \leq G$ *the double coset formula*

$$(i_J^G)^* \circ \mathrm{tr}_H^G = \sum_{[g] \in J\backslash G/H} \mathrm{tr}_{J\cap gHg^{-1}}^J \circ c_g^* \circ (i_{g^{-1}Jg\cap H}^H)^*$$

holds.

Proof See [19, Proposition 3.2.32, Theorem 3.4.9 and Example 3.4.11] for orthogonal spectra. □

Since every morphism (φ,α) in $\mathrm{Fin}_{\mathcal{U}}$ can be written as the composite of a morphism of type (i) and a subgroup inclusion as in (ii), these two can be combined to give a general formula describing the interaction between restrictions and transfers. Again, the definition of the transfer maps is extended to $\underline{\pi}_n(X)$ via the suspension isomorphisms.

4.6 Semistability

In these terms, a Fin–*global functor* in the sense of [19] (or, equivalently, an *inflation functor* in the sense of [22]) can be described as a $\mathrm{Fin}_{\mathcal{U}}^{\mathrm{op}}$–functor with transfers satisfying the relations of Lemma 4.8 and Proposition 4.10 and for which the $\mathrm{Fin}_{\mathcal{U}}^{\mathrm{op}}$–part factors through $\mathrm{Fin}^{\mathrm{op}}$, ie for which the action of an element (φ,α) does not depend on the α (see [19, Theorem 4.2.6ff]). This leads to the following definition:

Definition 4.11 (global semistability) A symmetric spectrum X is called *globally semistable* if the $\mathrm{Fin}_{\mathcal{U}}^{\mathrm{op}}$–functor $\underline{\pi}_n(X)$ factors through a $\mathrm{Fin}^{\mathrm{op}}$–functor for every $n \in \mathbb{Z}$.

Then the previous discussion implies:

Proposition 4.12 *If* X *is globally semistable, the homotopy groups* $\pi_*^{G,\mathcal{U}_G}(X)$ *only depend on* $\mathcal{U}_G$ *up to canonical isomorphism* (hence they can be denoted by $\pi_*^G(X)$) *and the collection* $\underline{\pi}_*(X) = \{\pi_*^G(X)\}_{G \text{ finite}}$ *naturally forms a* Fin*–global functor.*

The class of globally semistable symmetric spectra includes a lot of examples and is closed under many operations, as the following proposition shows. For (i) we recall from [6, Definition 3.22] (and the remark preceding it) that a G–symmetric spectrum X is called G–semistable if the $\mathrm{Inj}_H(\mathcal{U}_H, \mathcal{U}_H)$–action on $\pi_n^{H,\mathcal{U}_H}(X)$ is trivial for all $n \in \mathbb{Z}$ and all subgroups $H \leq G$.

Proposition 4.13 *The following hold:*

- (i) *A symmetric spectrum is globally semistable if and only the underlying G–symmetric spectrum is G–semistable for every finite group G.*
- (ii) *Global Ω–spectra are globally semistable.*
- (iii) *Every symmetric spectrum underlying an orthogonal spectrum is globally semistable.*
- (iv) *Every symmetric spectrum X for which every homotopy group $\pi_n^{G,\mathcal{U}_G}(X)$ is a finitely generated abelian group is globally semistable.*
- (v) *A symmetric spectrum is globally semistable if and only if the morphism $q_X\colon X \to QX$ is a global $\underline{\pi}_*$–isomorphism, in other words if and only if the map from the naive to the derived equivariant homotopy groups is an isomorphism.*
- (vi) *A morphism between globally semistable symmetric spectra is a global equivalence if and only if it is a global $\underline{\pi}_*$–isomorphism.*

Proof For (i), the "only if" part is clear. The other direction follows from the fact that given a group homomorphism $\varphi\colon G \to K$ and G–embeddings $\alpha_1, \alpha_2\colon \varphi^*(\mathcal{U}_K) \hookrightarrow \mathcal{U}_G$, there exist $\beta_1, \beta_2 \in \mathrm{Inj}_G(\mathcal{U}_G, \mathcal{U}_G)$ such that $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$.

Using (i), items (iv) and (vi) follow from [6, Corollaries 3.24 and 3.37]. Moreover, every global Ω–spectrum can be replaced by a $G\Omega$–spectrum up to eventual level equivalence (as explained after Definition 2.12), in particular up to $\underline{\pi}_*^{\mathcal{U}_G}$–isomorphism. Hence, Lemma 3.23 of [6] implies (ii). If $q_X\colon X \to QX$ is a global $\underline{\pi}_*$–isomorphism, then X is globally semistable, since we just argued that QX is globally semistable. If in turn X is assumed to be globally semistable, we know that the global equivalence $q_X\colon X \to QX$ must be a global $\underline{\pi}_*$–isomorphism by (vi). This gives (v) and also (iii), since every orthogonal spectrum allows a global $\underline{\pi}_*$–isomorphism to a global Ω–spectrum (see [19]), so we are done. □

4.7 Example

We close this section with an example of a symmetric spectrum which is not globally semistable, the free symmetric spectrum $F_1^{\{e\}}S^1$. There is a natural G–isomorphism $(F_1^{\{e\}}S^1)(M) \cong M_+ \wedge S^M$, which implies that

$$\begin{aligned}\pi_0^{G,\mathcal{U}_G}(F_1^{\{e\}}S^1) &\cong \operatorname*{colim}_{M\subseteq \mathcal{U}_G}[S^M, M_+ \wedge S^M]^G \\ &\cong \operatorname*{colim}_{M\subseteq \mathcal{U}_G}[S^M, (\mathcal{U}_G)_+ \wedge S^M]^G \\ &\cong \pi_0^{G,\mathcal{U}_G}(\Sigma_+^\infty(\mathcal{U}_G)),\end{aligned}$$

with G acting on $\mathcal{U}_G$. The tom Dieck splitting shows that this is a free abelian group with basis $\{\mathrm{tr}_H^G(x)\}$, where (H,x) runs through representatives of G–conjugacy classes of pairs of a subgroup H of G and an H–fixed point x of $(i_H^G)^*(\mathcal{U}_G)$.

Focusing on those basis elements that are not a transfer from a proper subgroup, we see:

Corollary 4.14 *The* $\mathrm{Fin}_{\mathcal{U}}^{\mathrm{op}}$*–functor* $\underline{\pi}_0(F_1^{\{e\}}(S^1))$ *contains the subfunctor*

$$(G,\mathcal{U}_G) \mapsto \mathbb{Z}[(\mathcal{U}_G)^G],$$

$$\bigl(\varphi\colon G \to K,\ \alpha\colon \varphi^*(\mathcal{U}_K) \hookrightarrow \mathcal{U}_G\bigr) \longmapsto \bigl(\mathbb{Z}[(\mathcal{U}_K)^K] \hookrightarrow \mathbb{Z}[(\varphi^*(\mathcal{U}_K))^G] \xrightarrow{\mathbb{Z}[\alpha]} \mathbb{Z}[(\mathcal{U}_G)^G]\bigr).$$

This determines the whole $\mathrm{Fin}_{\mathcal{U}}^{\mathrm{op}}$–functor structure on $\underline{\pi}_0(F_1^{\{e\}}S^1)$ via Proposition 4.10. The action of a morphism (φ,α) in $\mathrm{Fin}_{\mathcal{U}}$ very much depends on the α and hence $F_1^{\{e\}}(S^1)$ is not globally semistable.

5 Comparison to orthogonal spectra

In this section we show that global homotopy theory of symmetric spectra is equivalent to Fin–global homotopy theory of orthogonal spectra in the sense of [19]. For this we quickly recall the relevant definitions in the orthogonal context.

Definition 5.1 (orthogonal spectra) An orthogonal spectrum is a collection of based $O(n)$–spaces $\{X_n\}_{n\in\mathbb{N}}$ with structure maps $X_n \wedge S^1 \to X_{n+1}$ whose iterates $X_n \wedge S^m \to X_{n+m}$ are $(O(n)\times O(m))$–equivariant.

An orthogonal spectrum X can be evaluated on G–representations V via the formula $X_n \wedge_{O(\dim(V))} \mathrm{L}(\mathbb{R}^{\dim(V)}, V)_+$, with G–acting through V (where $\mathrm{L}(\mathbb{R}^{\dim(V)}, V)$ denotes the space of linear isometries). Again, these are connected by G–equivariant generalized structure maps of the form $X(V) \wedge S^W \to X(V \oplus W)$.

Every orthogonal spectrum X has an underlying symmetric spectrum of spaces $U(X)$ by restricting the $O(n)$–action on X_n to a Σ_n–action along the embedding as permutation matrices. The resulting restriction functor $U\colon \mathrm{Sp}^O \to \mathrm{Sp}^\Sigma$ has a left adjoint L, formally obtained via a left Kan extension (see [13, Sections I.3 and III.23] for details). Note that, since the "underlying G–spectrum" functors $(-)_G$ both for symmetric and orthogonal spectra are given on the point-set level by equipping a spectrum with trivial action, it follows that they commute with U and L.

Example 5.2 For a finite G–set M there is a natural G–homeomorphism

$$U(X)(M) \cong X(\mathbb{R}^M)$$

induced by linearizing a bijection $m \xrightarrow{\cong} M$ to a linear isometry $\mathbb{R}^m \xrightarrow{\cong} \mathbb{R}^M$.

The linearization $\mathbb{R}^{(\mathcal{U}_G)}$ of a complete G–set universe $\mathcal{U}_G$ is a complete G–representation universe. Moreover, the poset of G–subrepresentations of the form $\mathbb{R}^M$ is cofinal inside the poset of all finite-dimensional G–subrepresentations of $\mathbb{R}^{(\mathcal{U}_G)}$. As a consequence, the equivariant homotopy groups of an orthogonal spectrum as defined in [19, Section 3.1] are isomorphic to those of the underlying symmetric spectrum defined in Section 4. Combining this with Proposition 4.13 we see that for a morphism of orthogonal spectra $f\colon X \to Y$ the following are equivalent:

- f is a Fin–equivalence in the sense of [19, Definition 4.3.14].
- $U(f)$ is a global $\underline{\pi}_*$–isomorphism of symmetric spectra.
- $U(f)$ is a global equivalence of symmetric spectra.

Around this notion of equivalence, Schwede defines the Fin–global model structure on orthogonal spectra [19, Theorem 4.3.17]. We have:

Theorem 5.3 *The adjunction*

$$L\colon \mathrm{Sp}^\Sigma \rightleftarrows \mathrm{Sp}^O :U$$

is a Quillen equivalence for the global model structure on symmetric spectra of spaces and the Fin*–global model structure on orthogonal spectra.*

Proof The Fin–cofibrations of orthogonal spectra are given by those morphisms which have the left lifting property with respect to all morphisms $f\colon X \to Y$ such that $f(V)^G\colon X(V)^G \to Y(V)^G$ is an acyclic Serre fibration for all inner product spaces V and finite subgroups G of $O(V)$. Using the G–homeomorphism on evaluations of Example 5.2, we see that the underlying morphism of symmetric spectra of any such f

is an acyclic fibration in the global level model structure of Section 2. Hence, by adjunction, L takes flat cofibrations to Fin–cofibrations. Applying [6, Theorem 7.5] for every finite group G we see that L furthermore sends flat cofibrations which are also global equivalences to Fin–equivalences; hence, L becomes a left Quillen functor for the stable model structures and thus (L, U) a Quillen pair.

Hence, it remains to show that the adjunction induces an equivalence between the homotopy categories. Since U preserves and reflects weak equivalences, it suffices to show that for every flat symmetric spectrum X the morphism $X \to U(L(X))$ is a global equivalence. But, since the underlying G–symmetric spectrum X_G of a flat symmetric spectrum X is G–flat, this follows from [6, Theorem 7.5]. □

6 Examples

Every orthogonal spectrum can be restricted to a symmetric spectrum, so all examples in [19] also give examples for symmetric spectra and their global behavior. In this section we list some constructions of symmetric spectra (from the point of view of global homotopy theory) that do not arise from orthogonal spectra.

6.1 Suspension spectra of $\boldsymbol{I}$–spaces

There is an unstable analog of symmetric spectra, called $\boldsymbol{I}$–spaces. Again, these were previously considered as a model for unstable nonequivariant homotopy theory (see for example [14; 15; 12]). They come with a Day convolution product, the commutative monoids over which model E_∞–spaces.

The category of $\boldsymbol{I}$–spaces can also be used as a model for unstable global homotopy theory. We quickly describe this point of view without giving proofs. The resulting homotopy theory is equivalent to the category of orthogonal spaces with the Fin–global model structure of [19, Theorem 1.4.8].

Let $\boldsymbol{I}$ denote the category of finite sets and injective maps.

Definition 6.1 An $\boldsymbol{I}$*–space* is a functor from $\boldsymbol{I}$ to the category of simplicial sets.

Let A be an $\boldsymbol{I}$–space. By functoriality, if a finite set M comes equipped with an action of a finite group G, the evaluation $A(M)$ becomes a G–space. Every injection of G–sets $M \hookrightarrow N$ induces a G–equivariant map $A(M) \to A(N)$. Analogously to the stable case, one can show that there is a level model structure on $\boldsymbol{I}$–spaces, where the weak equivalences and fibrations are those morphisms that become G–weak equivalences and G–fibrations on $-(M)$ for all finite groups G and finite G–sets M, respectively.

An $\boldsymbol{I}$–space A is called *static* if for every injection $M \hookrightarrow N$ of faithful finite G–sets the induced map $A(M)^G \to A(N)^G$ is a weak equivalence. A morphism of $\boldsymbol{I}$–spaces is a *global equivalence* if it induces bijections on all hom–sets into static $\boldsymbol{I}$–spaces in the level homotopy category. Together with the level cofibrations, these form the global model structure for $\boldsymbol{I}$–spaces.

For a static $\boldsymbol{I}$–space A, the evaluation $A(M)$ at a faithful finite G–set M should be thought of as the *G–space underlying A*. By the definition of static, its G–homotopy type does not depend on the choice of M. The G–space underlying an arbitrary $\boldsymbol{I}$–space A is not as easy to describe directly, but it can be defined by first replacing by a globally equivalent static $\boldsymbol{I}$–space QA and then taking the underlying G–space of QA. In this sense a global equivalence can be interpreted as a morphism that induces equivalences on all underlying G–spaces.

Every $\boldsymbol{I}$–space A gives rise to a suspension symmetric spectrum of spaces $\Sigma^\infty_+ A$. Its n^{th} level is given by $A(\underline{n})_+ \wedge S^n$ with diagonal Σ_n–action, the structure map

$$(A(\underline{n})_+ \wedge S^n) \wedge S^1 \to A(\underline{n+1})_+ \wedge S^{n+1}$$

is the smash product of the induced map $A(\underline{n} \hookrightarrow \underline{n+1})$ with the associativity isomorphism $S^n \wedge S^1 \cong S^{n+1}$. This construction is left adjoint to Ω^∞: $\mathrm{Sp}^\Sigma_{\boldsymbol{T}} \to \boldsymbol{I}$–spaces defined by $(\Omega^\infty(X))(M) := \Omega^M X(M)$. Since Ω^∞ turns level fibrant global Ω–spectra into static $\boldsymbol{I}$–spaces, it is not hard to see that the adjunction $(\Sigma^\infty_+, \Omega^\infty)$ becomes a Quillen pair for the respective global model structures.

Let A be a cofibrant static $\boldsymbol{I}$–space. One can show that the G–homotopy type of the underlying G–symmetric spectrum $(\Sigma^\infty_+ A)_G$ is that of the suspension spectrum of the underlying G–space of A in the sense described above. Hence, suspension spectra of $\boldsymbol{I}$–spaces assemble various equivariant suspension spectra into one global object.

Remark 6.2 In all of the above one can alternatively consider functors from $\boldsymbol{I}$ to the category of topological spaces. Then the analogous statements hold.

Example 6.3 (global classifying spaces) Let G be a finite group and M a finite G–set. This data gives rise to an $\boldsymbol{I}$–space $\boldsymbol{I}(M,-)/G$ whose evaluation on a finite set N is the set of injective maps from M to N, modulo the G–action by precomposition.

Giving a morphism from $\boldsymbol{I}(M,-)/G$ to an $\boldsymbol{I}$–space A is equivalent to picking a G–fixed point in the evaluation $A(M)$. So — by definition of the notion of global equivalence — the global homotopy type of $\boldsymbol{I}(M,-)/G$ is the same for all *faithful* G–sets M. The $\boldsymbol{I}$–spaces $\boldsymbol{I}(M,-)/G$ for faithful M are called *global classifying spaces of* G. Given another finite group K, the K–space underlying $\boldsymbol{I}(M,-)/G$ is a classifying space for principal G–bundles in K–spaces; see [19, Proposition 1.1.26].

Ranging through all finite groups G, the suspension spectra of global classifying spaces of finite groups (which are isomorphic to global free spectra of the form $F_M^G S^M$) form a set of compact generators of the triangulated Fin–global stable homotopy category.

6.2 Ultracommutative localizations

Let $A \subseteq \mathbb{Q}$ be a subring, $M(A,1)$ a Moore space for A in degree 1 and $i\colon S^1 \to M(A,1)$ a map inducing the inclusion $\mathbb{Z} \hookrightarrow A$ on first homology. We define a symmetric spectrum MA via $MA_n = M(A,1)^{\wedge n}$ with permutation Σ_n–action and structure map

$$M(A,1)^{\wedge n} \wedge S^1 \xrightarrow{\mathrm{id}\wedge i} M(A,1)^{\wedge (n+1)}.$$

The associativity homeomorphisms $M(A,1)^{\wedge n} \wedge M(A,1)^{\wedge m} \cong M(A,1)^{\wedge (n+m)}$ together with the equality $S^0 = M(A,1)^{\wedge 0}$ give MA the structure of an ultracommutative symmetric ring spectrum.

To determine the global homotopy type of MA we note that the map $M(A,1)\wedge S^1 \xrightarrow{\mathrm{id}\wedge i} M(A,1)^{\wedge 2}$ is a weak equivalence of spaces, since $A\otimes\mathbb{Z} \to A\otimes A$ is an isomorphism. So, given a subgroup $H \leq \Sigma_n$, the map

$$\begin{aligned} M(A,1)\wedge (S^n)^H &\cong M(A,1)\wedge S^{\wedge(\underline{n}/H)} \\ &\xrightarrow{(\mathrm{id}\wedge i^{\wedge(\underline{n}/H)})} M(A,1)\wedge M(A,1)^{\wedge(\underline{n}/H)} \cong M(A,1)\wedge (M(A,1)^{\wedge n})^H \end{aligned}$$

is also a weak equivalence. In other words, the morphism $\Sigma^\infty(M(A,1)) \to \mathrm{sh}\, MA$ adjoint to the identity of $M(A,1)$ is a global level equivalence. The same argument also shows that $\alpha_{MA}\colon S^1 \wedge MA \to \mathrm{sh}\, MA$ is a positive global level equivalence. So we find that MA is globally equivalent to a desuspension of the suspension spectrum of $M(A,1)$ and hence its global homotopy type is that of the homotopy colimit of the sequence

$$\mathbb{S} \xrightarrow{\cdot n_1} \mathbb{S} \xrightarrow{\cdot n_2} \mathbb{S} \xrightarrow{\cdot n_3} \cdots,$$

where the n_i range through the elements of $\mathbb{Z}$ that become inverted in A. Thus, the (derived) smash product $- \wedge MA$ computes the A–localization in the global homotopy category. On equivariant homotopy groups it has the effect of tensoring with A.

In particular, the ultracommutative structure on MA can be used to see that arithmetic localizations of ultracommutative symmetric ring spectra are again ultracommutative symmetric ring spectra, which is not a priori clear and does not hold in general for equivariant localizations (see [8], in particular Section 4.1).

Remark 6.4 The construction of MA above works more generally for any based space X together with a based map $S^1 \to X$. This gives a functor from the category of based spaces under S^1 to ultracommutative ring spectra, which is left adjoint to sending an ultracommutative ring spectrum Z to the unit map $S^1 \to Z_1$. The latter is a right Quillen functor for the positive global model structure and the usual Quillen model structure on spaces under S^1, turning the adjunction into a Quillen pair. In fact, the adjunction is already a Quillen pair if one uses the nonequivariant positive projective model structure on commutative symmetric ring spectra (as constructed in [13, Theorem 15.1]). This implies that the ultracommutative ring spectra that arise through this construction are multiplicatively left-induced from nonequivariant commutative ring spectra in the sense of appendix.

6.3 Global algebraic K–theory

In [16] Schwede introduces a symmetric spectrum model for global (projective or free) algebraic K–theory of a ring R. Below we summarize the free version. In fact we give a slight variation of that of [16], as we explain in Remark 6.5.

Let R be a discrete ring. Each level is the realization of a bisimplicial set $kR(M)_{n,m}$, which we now explain. A $(0,m)$–simplex of $kR(M)$ is represented by a finite unordered labeled configuration $(W_1,\dots,W_k;x_1,\dots,x_k)$ of the following kind:

- The x_i are m–simplices of S^M.
- The W_i are finitely generated free submodules of the polynomial ring $R[M]$ with variable set M such that their sum is direct and the inclusion $W_1 \oplus \cdots \oplus W_k \hookrightarrow R[M]$ allows an R–linear splitting.

These configurations are considered up to the equivalence relation that a labeled point (W_i,x_i) can be left out if either W_i is zero or x_i the basepoint, and that if two x_i are equal, they can be replaced by a single one with label the sum of the previous labels. The Σ_M–action is the diagonal one through its actions on S^M and $R[M]$.

General (n,m)–simplices are given by similar equivalence classes of configurations, where instead of a single free submodule W_i, each m–simplex x_i carries an n–chain of R–module isomorphisms $(W_{i_0} \xrightarrow{\cong} W_{i_1} \xrightarrow{\cong} \cdots \xrightarrow{\cong} W_{i_n})$ such that for every $0 \le j \le n$ the tuple $(W_{1_j}, \dots, W_{k_j})$ satisfies the conditions above. The simplicial structure maps in the first direction are the usual ones from the nerve; the ones in the second direction are induced by S^M. The spectrum structure maps $kR(M) \wedge S^N \to kR(M \sqcup N)$ are given by smashing the configurations with an element of S^N and leaving the labels unchanged.

In [16] Schwede shows the following:

- The symmetric spectrum kR is globally semistable.
- Its G–fixed point spectrum represents the direct sum K–theory of $R[G]$–lattices, ie $R[G]$–modules that are finitely generated free as R–modules. In particular, the equivariant homotopy groups $\pi_*^G(kR)$ are the K–groups of $R[G]$–lattices.
- If R is commutative, the smash product of modules gives kR the structure of an ultracommutative symmetric ring spectrum.

If R satisfies dimension invariance, the spectrum kR comes with a natural filtration: Let $kR^n(M)$ be the subspace of $kR(M)$ of those configurations $(W_1, \dots, W_k; x_1, \dots, x_k)$ where the sum of the R–ranks of the W_i is at most n, and similarly for higher simplices. These subspaces are closed under the simplicial and spectrum structure and thus define a symmetric subspectrum kR^n. This gives a filtration

$$* = kR^0 \to kR^1 \to \cdots \to kR = \operatorname*{colim}_{n \in \mathbb{N}} kR^n.$$

The underlying nonequivariant filtration is studied by Arone and Lesh in [1], where they call it the *modified stable rank filtration* of algebraic K–theory. In joint work with Dominik Ostermayr [7], we extend some of their results to the global context to show that the subquotients kR^n/kR^{n-1} are globally equivalent to suspension spectra of certain $\boldsymbol{I}$–spaces associated to the lattice of nontrivial direct sum decompositions of R^n. This can be used to give an algebraic description of the Fin–global functors $\pi_0^G(kR^n)$.

Remark 6.5 The version of kR we described here differs slightly from that in [16]. There the tuple $(W_1, \dots, W_k)$ has to satisfy the additional property that for every monomial $t = \prod_{m \in M} m^{i_m} \in R[M]$ there is at most one i such that W_i contains an element whose t–component is nontrivial (which in that setup in particular guarantees that the sum of the W_i is direct). The inclusion from the kR in [16] to the one above is a global level equivalence.

Appendix Model structures with respect to families

In this appendix we explain how to construct model structures with respect to global families of finite groups. For every such family we define two model structures, a projective and a flat one, both useful for constructing derived adjunctions. In the case of the family of trivial groups (where the homotopy category is the nonequivariant stable homotopy category) the projective model structure equals the one in [9, Section 5.1] and the flat model structure is the one introduced in [21]. For the global family of all finite groups the two model structures coincide.

Definition A.1 (global family) A *global family* is a nonempty class of finite groups which is closed under subgroups, quotients and isomorphism.

Let $\mathcal{F}$ be a global family.

Definition A.2 A morphism $f\colon X \to Y$ of symmetric spectra is called

- an *$\mathcal{F}$–level equivalence* if $f_n^H\colon X_n^H \to Y_n^H$ is a weak equivalence for all subgroups $H \leq \Sigma_n$ which lie in $\mathcal{F}$;
- a *projective $\mathcal{F}$–level fibration* if $f_n^H\colon X_n^H \to Y_n^H$ is a Kan fibration for all subgroups $H \leq \Sigma_n$ which lie in $\mathcal{F}$;
- a *projective $\mathcal{F}$–cofibration* if each latching map $\nu_n[f]\colon X_n \cup_{L_n(X)} L_n(Y) \to Y_n$ is a Σ_n–cofibration with relative isotropy in $\mathcal{F}$;
- a *flat $\mathcal{F}$–level fibration* if it has the right lifting property with respect to all flat cofibrations (as defined in Definition 2.2) that are also $\mathcal{F}$–level equivalences.

Then the following two propositions can again be obtained via [6, Proposition 2.22]:

Proposition A.3 *The classes of $\mathcal{F}$–level equivalences, projective $\mathcal{F}$–level fibrations and projective $\mathcal{F}$–cofibrations define a cofibrantly generated, proper and monoidal model structure on the category of symmetric spectra.*

Proposition A.4 *The classes of $\mathcal{F}$–level equivalences, flat $\mathcal{F}$–level fibrations and flat cofibrations define a cofibrantly generated, proper and monoidal model structure on the category of symmetric spectra.*

From the point of view of $\mathcal{F}$–global homotopy theory we have to remember the G–homotopy type of a symmetric spectrum for all groups G in $\mathcal{F}$, which leads to the following definition of stable equivalence:

Definition A.5 ($\mathcal{F}$–global equivalences) A morphism $f\colon X \to Y$ is called an *$\mathcal{F}$–global equivalence* if it is a G–stable equivalence (in the sense of Definition 2.8) for all groups $G \in \mathcal{F}$.

A morphism of symmetric spectra is called a *projective* (*flat*) *$\mathcal{F}$–fibration* if it has the left lifting property with respect to all morphisms that are projective $\mathcal{F}$–cofibrations (respectively flat cofibrations) and $\mathcal{F}$–equivalences. Then we have:

Proposition A.6 *The classes of $\mathcal{F}$–global equivalences, projective $\mathcal{F}$–fibrations and projective $\mathcal{F}$–cofibrations determine a cofibrantly generated, proper and monoidal model structure on the category of symmetric spectra, called the* ***projective $\mathcal{F}$–global stable model structure****.*

Proposition A.7 *The classes of $\mathcal{F}$–global equivalences, flat $\mathcal{F}$–fibrations and flat cofibrations determine a cofibrantly generated, proper and monoidal model structure on the category of symmetric spectra, called the* ***flat $\mathcal{F}$–global stable model structure****.*

Each of these model structures can be obtained via a left Bousfield localization of the respective level model structure. For example, this can be done by applying the small object argument to the subset of those maps $i \,\square\, \overline{\lambda}_M^N$ used in Section 2.3 that are associated to a finite group $G \in \mathcal{F}$ and finite G–sets M and N (of which M is faithful). It follows that a symmetric spectrum is fibrant in either of the $\mathcal{F}$–global model structures if and only if it is fibrant in the respective level model structure and in addition an $\mathcal{F}$–global Ω–spectrum, ie if it satisfies the condition in Definition 2.12 for all $G \in \mathcal{F}$ (instead of for all finite G). The flat $\mathcal{F}$–global model structure can also be obtained by left Bousfield localizing the full global model structure.

Since every projective $\mathcal{F}$–cofibration is a flat cofibration, the $\mathcal{F}$–global model structure and the flat $\mathcal{F}$–global model structure are Quillen equivalent via the identity adjunction. Furthermore, the same proof as that of Theorem 5.3 applies to show that the projective $\mathcal{F}$–model structure is Quillen equivalent to the $\mathcal{F}$–global model structure on orthogonal spectra as introduced in [19, Theorem 4.3.17].

Let $\mathcal{F}' \subseteq \mathcal{F}$ be an inclusion of global families of finite groups. Then, by definition, every $\mathcal{F}$–global equivalence is an $\mathcal{F}'$–global equivalence and hence the localization $\mathrm{Sp}^\Sigma \to \mathrm{Sp}^\Sigma[\mathcal{F}\text{–global eq.}^{-1}]$ factors uniquely through a functor

$$\mathrm{Sp}^\Sigma[\mathcal{F}\text{–global eq.}^{-1}] \to \mathrm{Sp}^\Sigma[\mathcal{F}'\text{–global eq.}^{-1}].$$

This functor has both a left and a right adjoint (both fully faithful) obtained by deriving the identity adjunction with respect to the projective and flat model structures, respectively. In particular, this defines two functors from the nonequivariant stable homotopy category to the global stable homotopy category. It can be shown [19, Example 4.5.19 and Proposition 4.5.8] that the right adjoint gives rise to Borel theories, whereas the image of the left adjoint is given by symmetric spectra with constant geometric fixed points.

Finally, both the projective $\mathcal{F}$–global stable model structure and the flat $\mathcal{F}$–global stable model structure lift to categories of modules over a symmetric ring spectrum and algebras over a commutative symmetric ring spectrum. There exist positive versions of both model structures which lift to the category of commutative algebras over a commutative symmetric ring spectrum. These allow the construction of "multiplicative" change-of-family functors, but there is a caveat: a positive projective $\mathcal{F}$–cofibrant commutative symmetric ring spectrum is in general not projective $\mathcal{F}$–cofibrant as a symmetric spectrum if $\mathcal{F}$ is not the family of all finite groups. As a consequence, the underlying symmetric spectrum of a left-induced ultracommutative symmetric ring spectrum is in general not left-induced.

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Representing the deformation ∞–groupoid

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Our goal is to introduce a smaller, but equivalent version of the deformation ∞–groupoid associated to a homotopy Lie algebra. In the case of differential graded Lie algebras, we represent it by a universal cosimplicial object.

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1 Introduction

The *fundamental principle of deformation theory*, due to Deligne, Grothendieck and many others and recently formalized and proved in the context of ∞–categories by Pridham and Lurie, states that:

> *Every deformation problem in characteristic* 0 *is encoded in the space of Maurer–Cartan elements of a differential graded Lie algebra.*

Therefore, one is naturally led to the study of Maurer–Cartan elements of differential graded Lie algebras and, more generally, homotopy Lie algebras.

In order to encode the Maurer–Cartan elements, gauge equivalences between them, and higher relations between gauge equivalences, Hinich [14] introduced the Deligne–Hinich ∞–groupoid. It is a Kan complex associated to any complete $\mathscr{L}_\infty$–algebra modeling the space of its Maurer–Cartan elements. Since it is a very big object, E Getzler introduced in [11] a smaller but weakly equivalent Kan complex $\gamma_\bullet$ which, however, is more difficult to manipulate. In this paper, we introduce another simplicial set associated to any $\mathscr{L}_\infty$–algebra, for which we prove the following nice properties:

1. it is weakly equivalent to the Deligne–Hinich ∞–groupoid,
2. it is a Kan complex,
3. it is contained in the Getzler ∞–groupoid $\gamma_\bullet$, and
4. if we restrict to the category of complete dg Lie algebras, there is an explicit cosimplicial dg Lie algebra $\mathfrak{mc}_\bullet$ representing this object.

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The cosimplicial dg Lie algebra $\mathfrak{mc}_\bullet$ was already introduced in the work of Buijs, Félix, Murillo and Tanré [4] in the context of rational homotopy theory. We show here that it plays a key role in deformation theory.

Results coming from operad theory play a crucial role throughout the paper, especially in the second part. In particular, we use the explicit formulas for the ∞–morphisms induced by the homotopy transfer theorem given by Loday and Valette [17] and various theorems proven by Robert-Nicoud [18].

Shortly after the appearance of the present article, Buijs, Murillo, Félix and Tanré gave an alternative proof of Corollary 5.3 in [5]. Their proof doesn't rely on general operadic results, but rather on explicit combinatorial computations.

The author was made aware by Marco Manetti in a private conversation that many of the results of this article are already present in the unpublished PhD thesis [1] of his student Ruggero Bandiera (now also appeared in [2]). We acknowledge this, but we consider that the present article remains interesting in that the methods used to prove the results are different. In particular, in view of Bandiera's results, Sections 3 and 4 can be interpreted as an alternative construction of the Getzler ∞–groupoid $\gamma_\bullet$ with new proofs of its properties.

Structure of the paper

In Section 2 we give a short review of the Deligne groupoid, the Deligne–Hinich ∞–groupoid and the main theorems in this context. In Section 3 we state and prove our main theorem, giving a new simplicial set encoding the Maurer–Cartan space of $\mathscr{L}_\infty$–algebras. Next, in Section 4, we study some properties of this object. In particular, we prove that it is a Kan complex, and that it is "small" in a precise sense. Finally, we focus on the special case of dg Lie algebras in Section 5, showing that our Kan complex is represented by a cosimplicial dg Lie algebra in this situation.

Notation and conventions

We work over a fixed field $\mathbb{K}$ of characteristic 0.

We abbreviate "differential graded" by dg, and sometimes omit it completely. All algebras are differential graded unless stated otherwise.

Since we work with differential forms, we adopt the cohomological convention. Therefore, we work over cochain complexes, and Maurer–Cartan elements of dg Lie and $\mathscr{L}_\infty$–algebras (ie homotopy Lie algebras) are in degree 1, not -1. All cochain complexes are $\mathbb{Z}$–graded.

We use the letter s to denote a formal element of degree 1. If $C_\bullet$ is a cochain complex, then $sC_\bullet$ denotes the suspension of $C_\bullet$, which is sometimes written as $C_\bullet[1]$.

We sometimes denote the identity maps by 1.

By a filtered $\mathscr{L}_\infty$–algebra we mean a pair $(\mathfrak{g}, F_\bullet\mathfrak{g})$ where $\mathfrak{g}$ is an $\mathscr{L}_\infty$–algebra and $F_\bullet\mathfrak{g}$ is a descending filtration of $\mathfrak{g}$ such that $F_1\mathfrak{g} = \mathfrak{g}$ and

(1) for all $n \geq 1$, we have $d_\mathfrak{g}(F_n\mathfrak{g}) \subseteq F_n\mathfrak{g}$,

(2) for all $k \geq 2$ and $n_1, \dots, n_k \geq 1$ we have

$$\ell_k(F_{n_1}\mathfrak{g}, \dots, F_{n_k}\mathfrak{g}) \subseteq F_{n_1+\cdots+n_k}\mathfrak{g},$$

and

(3) the $\mathscr{L}_\infty$–algebra $\mathfrak{g}$ is complete with respect to the filtration, ie

$$\mathfrak{g} \cong \varprojlim_n \mathfrak{g}/F_n\mathfrak{g}$$

as $\mathscr{L}_\infty$–algebras.

When the context is clear, we write $\mathfrak{g}^{(n)} := \mathfrak{g}/F_n\mathfrak{g}$. For details about (filtered) $\mathscr{L}_\infty$–algebras and the definitions and basic properties about (filtered) ∞–morphisms we refer the reader to V A Dolgushev and C L Rogers [7].

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2 The deformation ∞–groupoid

An object of fundamental interest in deformation theory is the *Deligne groupoid* $\mathrm{Del}(\mathfrak{g})$ associated to a complete dg Lie algebra $\mathfrak{g}$. There is a higher generalization of the Deligne groupoid in the form of the Deligne–Hinich ∞–groupoid. It is a simplicial set with nice properties and whose 1–truncation gives back the Deligne groupoid. It was introduced in [14] and then studied in depth and further generalized in [11].

2.1 The Deligne groupoid

Let $\mathfrak{g}$ be a dg Lie algebra. Then we can associate a groupoid $\mathrm{Del}(\mathfrak{g})$ to $\mathfrak{g}$, called the *Deligne groupoid*, as follows. The objects of the Deligne groupoid are the *Maurer–Cartan elements* of $\mathfrak{g}$, ie the degree 1 elements $\alpha \in \mathfrak{g}^1$ satisfying the Maurer–Cartan equation

$$d\alpha + \tfrac{1}{2}[\alpha, \alpha] = 0.$$

Definition 2.1 The set of Maurer–Cartan elements of $\mathfrak{g}$ is denoted by $\mathrm{MC}(\mathfrak{g})$.

We have the set of objects of $\mathrm{Del}(\mathfrak{g})$; we still need to define its morphisms. To an element $\lambda \in \mathfrak{g}^0$, one can associate a "vector field" by sending $\alpha \in \mathfrak{g}^1$ to

$$d\lambda + [\lambda, \alpha] \in \mathfrak{g}^1.$$

It is tangent to the Maurer–Cartan locus, in the sense that if $\alpha(t)$ is the flow of λ, that is,

$$\frac{d}{dt}\alpha(t) = d\lambda + [\lambda, \alpha(t)]$$

with $\alpha(0) \in \mathrm{MC}(\mathfrak{g})$, then $\alpha(t) \in \mathrm{MC}(\mathfrak{g})$ for all t, whenever it exists. We say that two Maurer–Cartan elements $\alpha_0, \alpha_1 \in \mathrm{MC}(\mathfrak{g})$ are *gauge equivalent* if there exists such a flow $\alpha(t)$ such that $\alpha(i) = \alpha_i$ for $i = 0, 1$. The Deligne groupoid is the groupoid associated to this equivalence relation, which means that the morphisms are

$$\mathrm{Del}(\mathfrak{g})(\alpha_0, \alpha_1) := \{\lambda \in \mathfrak{g}^0 \mid \text{the flow of } \lambda \text{ starting at } \alpha_0 \text{ gives } \alpha_1 \text{ at time } 1\}.$$

For further reference, see for example [13].

The assignment of the Deligne groupoid to a dg Lie algebra is functorial and has a good homotopical behavior: it sends filtered quasi-isomorphisms to equivalences of groupoids, as can be seen by the Goldman–Millson theorem, which was first proven in [13], and then generalized for example in [22].

2.2 Generalization: the deformation ∞–groupoid

Let $\mathfrak{g}$ be a nilpotent $\mathscr{L}_\infty$–algebra. The Maurer–Cartan equation can be generalized to

$$dx+\sum_{n\geq 2}\frac{1}{n!}\ell_n(x,\dots,x)=0$$

for $x\in\mathfrak{g}^1$. Again, we denote by $\mathrm{MC}(\mathfrak{g})$ the set of all elements satisfying this equation.

Remark 2.2 The condition that $\mathfrak{g}$ be nilpotent is sufficient to make it so that the left-hand side of the Maurer–Cartan equation is well defined.

2.2.1 The Deligne–Hinich ∞–groupoid

Definition 2.3 The *Sullivan algebra* is the simplicial dg commutative algebra

$$\Omega_n:=\mathbb{K}[t_0,\dots,t_n,dt_0,\dots,dt_n]\Big/\left(\sum_{i=0}^n t_i=1,\sum_{i=0}^n dt_i=0\right)$$

with $|t_i|=0$ and endowed with the unique differential satisfying $d(t_i)=dt_i$.

This object was introduced by Sullivan in the context of rational homotopy theory [21]. At level n, it is the algebra of polynomial differential forms on the standard geometric n–simplex. Now let $\mathfrak{g}$ be a nilpotent $\mathscr{L}_\infty$–algebra. Then tensoring $\mathfrak{g}$ with Ω_n gives us back a nilpotent $\mathscr{L}_\infty$–algebra, of which we can consider the Maurer–Cartan elements.

Definition 2.4 The *Deligne–Hinich ∞–groupoid* is the simplicial set

$$\mathrm{MC}_\bullet(\mathfrak{g}):=\mathrm{MC}(\mathfrak{g}\otimes\Omega_\bullet).$$

This association is natural in $\mathfrak{g}$, and thus defines a functor

$$\mathrm{MC}_\bullet\colon\{\text{nilpotent }\mathscr{L}_\infty\text{–algebras}\}\to\mathsf{sSet}.$$

We will rather consider the following slight generalization: Let $(\mathfrak{g},F_\bullet\mathfrak{g})$ be a filtered $\mathscr{L}_\infty$–algebra; then

$$\mathfrak{g}\cong\varprojlim_n\mathfrak{g}/F_n\mathfrak{g}$$

is the limit of a sequence of nilpotent $\mathscr{L}_\infty$–algebras. Thus, we can define

$$\mathrm{MC}_\bullet(\mathfrak{g}):=\varprojlim_n\mathrm{MC}_\bullet(\mathfrak{g}/F_n\mathfrak{g}).$$

Notice that the elements in $\mathrm{MC}_\bullet(\mathfrak{g})$ in this case are not polynomials with coefficients in $\mathfrak{g}$ anymore, but rather power series with some "vanishing at infinity" conditions. We state all the following results in this setting.

Theorem 2.5 *Let either*

- [11, Proposition 4.7] $\mathfrak{g}$ *and* $\mathfrak{h}$ *be nilpotent* $\mathscr{L}_\infty$*–algebras and* $\Phi\colon \mathfrak{g}\to\mathfrak{h}$ *be a surjective strict morphism of* $\mathscr{L}_\infty$*–algebras, or*
- [20, Theorem 2] $\mathfrak{g}$ *and* $\mathfrak{h}$ *be filtered* $\mathscr{L}_\infty$*–algebras and* $\Phi\colon \mathfrak{g}\rightsquigarrow\mathfrak{h}$ *be a filtered* ∞*–morphism that induces a surjection at every level of the filtrations.*

Then

$$\mathrm{MC}_\bullet(\Phi)\colon \mathrm{MC}_\bullet(\mathfrak{g})\to\mathrm{MC}_\bullet(\mathfrak{h})$$

is a fibration of simplicial sets. In particular, for any filtered $\mathscr{L}_\infty$*–algebra* $\mathfrak{g}$*, the simplicial set* $\mathrm{MC}_\bullet(\mathfrak{g})$ *is a Kan complex.*

This result was originally proven by Hinich [14, Theorem 2.2.3] for strict surjections between nilpotent dg Lie algebras concentrated in positive degrees, and then generalized by Getzler and by Rogers to the version stated above.

Generalizing the Goldman–Millson theorem, Dolgushev and Rogers [7, Theorem 2.2] proved that the Deligne–Hinich ∞–groupoid behaves well with respect to homotopy theory: it sends filtered quasi-isomorphisms of filtered $\mathscr{L}_\infty$–algebras to weak equivalences.

2.2.2 Basic forms, Dupont's contraction and Getzler's functor $\gamma_\bullet$ The Sullivan algebra has a subcomplex $C_\bullet$ linearly spanned by the *basic forms*

$$\omega_I := k!\sum_{j=1}^{k}(-1)^j t_{i_j}\,dt_{i_0}\cdots\widehat{dt_{i_j}}\cdots dt_{i_k}\in\Omega_n$$

for $I=\{i_0<i_1<\cdots<i_k\}\subseteq\{0,\dots,n\}$. This is in fact the (co)cellular complex for the standard geometric n–simplex Δ^n. In order to prove a simplicial version of the de Rham theorem, J L Dupont [10] introduced a homotopy retraction

$$\begin{tikzcd} \Omega_\bullet \arrow[loop left, "h_\bullet"] \arrow[r, shift left, "p_\bullet"] & C_\bullet \arrow[l, shift left, "i_\bullet"] \end{tikzcd}$$

where all the maps are simplicial. Homotopy retraction means that we have

$$p_\bullet i_\bullet = 1 \quad\text{and}\quad 1-i_\bullet p_\bullet = dh_\bullet + h_\bullet d.$$

Moreover, the maps satisfy the *side conditions*

$$h_\bullet i_\bullet = 0,\quad p_\bullet h_\bullet = 0\quad\text{and}\quad h_\bullet^2 = 0.$$

A homotopy retraction satisfying the side conditions is called a *contraction*.

This contraction will be a fundamental ingredient in the rest of the paper. As the Deligne–Hinich ∞–groupoid is always a big object, Getzler defined the following subobject:

Definition 2.6 The *Getzler ∞–groupoid* is the subsimplicial set $\gamma_\bullet(\mathfrak{g})$ of the Deligne–Hinich ∞–groupoid $\mathrm{MC}_\bullet(\mathfrak{g})$ given by

$$\gamma_n(\mathfrak{g}) := \{\alpha \in \mathrm{MC}_n(\mathfrak{g}) \mid h_n\alpha = 0\}.$$

Theorem 2.7 [11] *The simplicial set $\gamma_\bullet(\mathfrak{g})$ is a Kan complex, and it is weakly equivalent to the Deligne–Hinich ∞–groupoid $\mathrm{MC}_\bullet(\mathfrak{g})$.*

A part of the definition of $h_\bullet$ and $p_\bullet$ which we will need in what follows is the (formal) integration of a form in the Sullivan algebra over a simplex, which is given by

$$\int_{\Delta^n} t_1^{a_1} \cdots t_n^{a_n}\, dt_1 \cdots dt_n := \frac{a_1! \cdots a_n!}{(a_1 + \cdots + a_n + n)!}.$$

It corresponds to the usual integration when working over $\mathbb{K} = \mathbb{R}$.

Remark 2.8 We have

$$\int_{\Delta^p} \omega_I = 1$$

for $p + 1 = |I|$, where Δ^p is the subsimplex of Δ^n with vertices indexed by I.

Definition 2.9 A form $\alpha \in \gamma_n(\mathfrak{g})$ is said to be *thin* if

$$\int_{\Delta^n} \alpha = 0.$$

Theorem 2.10 [11] *For every horn in $\gamma_\bullet(\mathfrak{g})$, there exists a unique thin simplex filling it.*

Remark 2.11 The existence of a set of thin simplices such that every horn has a unique thin filler is what is meant by Getzler when he speaks of an ∞–groupoid. We use the term simply to mean Kan complex (for example when speaking of the Deligne–Hinich ∞–groupoid).

3 Main theorem

In this section, we give a reminder on the homotopy transfer theorem for commutative and for $\mathscr{L}_\infty$–algebras, before going on to state and prove the main theorem of the article.

3.1 Reminder on the homotopy transfer theorem

Let V and W be cochain complexes, and suppose that we have a retraction

$$\begin{tikzcd} V \arrow[loop left, "h"] \arrow[r, shift left, "p"] & W, \arrow[l, shift left, "i"] \end{tikzcd}$$

that is, we have

$$ip-1=dh+hd$$

and $pi=1$. Furthermore, we can always suppose that

$$h^2=0,\quad hi=0\quad\text{and}\quad ph=0;$$

see for example [15, page 365]. The homotopy transfer theorem tells us that we can coherently transfer algebraic structures from V to W. More precisely, the specific cases of interest to us are the following ones:

Theorem 3.1 (homotopy transfer theorem for commutative algebras) *Suppose V is a commutative algebra. There is a $\mathscr{C}_\infty$–algebra structure on W such that p and i extend to ∞–quasi-isomorphisms p_∞ and i_∞ of $\mathscr{C}_\infty$–algebras between V and W endowed with the respective structures.*

Theorem 3.2 (homotopy transfer theorem for $\mathscr{L}_\infty$–algebras) *Suppose V is an $\mathscr{L}_\infty$–algebra. There is an $\mathscr{L}_\infty$–algebra structure on W such that p and i extend to ∞–quasi-isomorphisms p_∞ and i_∞ of $\mathscr{L}_\infty$–algebras between V and W endowed with the respective structures.*

For details on this theorem, see for example [17, Section 10.3], where it is proven in the general context of algebras over operads. See also [17, Sections 10.3.5–10.3.6] for the explicit formulas for the ∞–morphisms p_∞ and i_∞.

3.2 Statement of the main theorem

Let $\mathfrak{g}$ be a complete $\mathscr{L}_\infty$–algebra. The Dupont contraction induces a contraction

$$\begin{tikzcd} \mathfrak{g}\otimes\Omega_\bullet \arrow[loop left, "1\otimes h_\bullet"] \arrow[r, shift left, "1\otimes p_\bullet"] & \mathfrak{g}\otimes C_\bullet \arrow[l, shift left, "1\otimes i_\bullet"] \end{tikzcd}$$

of $\mathfrak{g}\otimes\Omega_\bullet$ to $\mathfrak{g}\otimes C_\bullet$. Applying the homotopy transfer theorem to this contraction, we obtain a simplicial $\mathscr{L}_\infty$–algebra structure on $\mathfrak{g}\otimes C_\bullet$. We also know that we can extend the maps $1\otimes p_\bullet$ and $1\otimes i_\bullet$ to simplicial ∞–morphisms of simplicial $\mathscr{L}_\infty$–algebras $(1\otimes p_\bullet)_\infty$ and $(1\otimes i_\bullet)_\infty$. Notice that these ∞–morphisms are indeed simplicial because they are given by sums of compositions of copies of $1\otimes i_\bullet$, $1\otimes p_\bullet$, $1\otimes h_\bullet$ and the brackets of $\mathfrak{g}\otimes\Omega_\bullet$, all of which respect the simplicial structure. We denote by $P_\bullet$ and $I_\bullet$ the induced maps on Maurer–Cartan elements. We will also use the notation

$$(1\otimes r_\bullet)_\infty := (1\otimes i_\bullet)_\infty(1\otimes p_\bullet)_\infty,$$

and we dub $R_\bullet$ the map induced by $(1\otimes r_\bullet)_\infty$ on Maurer–Cartan elements.

Theorem 3.3 *Let $\mathfrak{g}$ be a filtered $\mathscr{L}_\infty$–algebra. The maps $P_\bullet$ and $I_\bullet$ are inverse to each other in homotopy, and thus provide a weak equivalence*

$$\mathrm{MC}_\bullet(\mathfrak{g})\simeq\mathrm{MC}(\mathfrak{g}\otimes C_\bullet)$$

of simplicial sets which is natural in $\mathfrak{g}$.

Remark 3.4 The simplicial $\mathscr{L}_\infty$–algebra $\mathfrak{g}\otimes C_\bullet$ has the advantage of being quite a bit smaller than $\mathfrak{g}\otimes\Omega_\bullet$, since C_n is finite-dimensional for each n. The price to pay is that the algebraic structure is much more convoluted.

3.3 Proof of the main theorem

The rest of this section is dedicated to the proof of this result. We begin with the following lemma:

Lemma 3.5 *We have*

$$P_\bullet I_\bullet = \mathrm{id}_{\mathrm{MC}(\mathfrak{g}\otimes C_\bullet)}.$$

Proof This is because $(1\otimes p_\bullet)_\infty(1\otimes i_\bullet)_\infty$ is the identity — see for example Theorem 5 of [9] — and the functoriality of the Maurer–Cartan functor MC. □

Therefore, it is enough to prove that the map

$$R_\bullet = I_\bullet P_\bullet\colon \mathrm{MC}_\bullet(\mathfrak{g})\to\mathrm{MC}_\bullet(\mathfrak{g})$$

is a weak equivalence. The idea is to use the same methods as in [7]. The situation is however slightly different, as the map $R_\bullet$ is not of the form $\Phi\otimes 1_{\Omega_\bullet}$, and thus Theorem 2.2 of [7] cannot be directly applied. The first, easy step is to understand what happens at the level of the zeroth homotopy group.

Lemma 3.6 *The map*

$$\pi_0(R_\bullet)\colon \pi_0\mathrm{MC}_\bullet(\mathfrak{g}) \to \pi_0\mathrm{MC}_\bullet(\mathfrak{g})$$

is a bijection.

Proof We have $\Omega_0 = C_0 = \mathbb{K}$, and the maps i_0 and p_0 both are the identity of $\mathbb{K}$. Therefore, the map R_0 is the identity of $\mathrm{MC}_0(\mathfrak{g})$, and thus obviously induces a bijection on π_0. □

For the higher homotopy groups, we start with a simplified version of Proposition 2.4 of [7], which gives in some sense the base for an inductive argument. If the $\mathscr{L}_\infty$–algebra $\mathfrak{g}$ is abelian, ie all of its brackets vanish, then so do the brackets at all levels of $\mathfrak{g}\otimes\Omega_\bullet$. In this case, the Maurer–Cartan elements are exactly the cocycles of the underlying cochain complex, and therefore $\mathrm{MC}_\bullet(\mathfrak{g})$ is a simplicial vector space.

Lemma 3.7 *If the $\mathscr{L}_\infty$–algebra $\mathfrak{g}$ is abelian, then $R_\bullet$ is a weak equivalence of simplicial vector spaces.*

Proof Recall that the Moore complex of a simplicial vector space $V_\bullet$ is defined by

$$\mathcal{M}(V_\bullet)_n := s^n V_n$$

endowed with the differential

$$\partial := \sum_{i=0}^{n} (-1)^i d_i,$$

where the maps d_i are the face maps of the simplicial set $V_\bullet$. It is a standard result that

$$\pi_0(V_\bullet) = H_0(\mathcal{M}(V_\bullet)), \quad \pi_i(V_\bullet, v) \cong \pi_i(V_\bullet, 0) = H_i(\mathcal{M}(V_\bullet))$$

for all $i \geq 1$ and $v \in V_0$, and that a map of simplicial vector spaces is a weak equivalence if and only if it induces a quasi-isomorphism between the respective Moore complexes [12, Corollary 2.5, Section III.2].

In our case,

$$V_\bullet := \mathrm{MC}_\bullet(\mathfrak{g}) = \mathcal{Z}^1(\mathfrak{g}\otimes\Omega_\bullet)$$

is the simplicial vector space of 1–cocycles of $\mathfrak{g}\otimes\Omega_\bullet$. As in [7], it can be proven that the map

$$\mathcal{M}(1\otimes p_\bullet)\colon \mathcal{M}(\mathcal{Z}^1(\mathfrak{g}\otimes\Omega_\bullet)) \to \mathcal{M}(\mathcal{Z}^1(\mathfrak{g}\otimes C_\bullet))$$

is a quasi-isomorphism. But, as the bracket vanishes, this is exactly $P_\bullet$. Now

$$\mathcal{M}(1\otimes p_\bullet)\mathcal{M}(1\otimes i_\bullet)=1_{\mathcal{M}(\mathscr{Z}^1(\mathfrak{g}\otimes\Omega_\bullet))},$$

which implies that $\mathcal{M}(1\otimes i_\bullet)$ also is a quasi-isomorphism. It follows that $R_\bullet$ is a weak equivalence, concluding the proof. □

Now we basically follow the structure of [7, Section 4]. We define a filtration of $\mathfrak{g}\otimes\Omega_\bullet$ by

$$F_k(\mathfrak{g}\otimes\Omega_\bullet):=(F_k\mathfrak{g})\otimes\Omega_\bullet.$$

We write

$$(\mathfrak{g}\otimes\Omega_\bullet)^{(k)}:=\mathfrak{g}\otimes\Omega_\bullet/F_k(\mathfrak{g}\otimes\Omega_\bullet)=\mathfrak{g}^{(k)}\otimes\Omega_\bullet.$$

The composite $(1\otimes i_\bullet)(1\otimes p_\bullet)$ induces an endomorphism $(1\otimes i_\bullet)^{(k)}(1\otimes p_\bullet)^{(k)}$ of $(\mathfrak{g}\otimes\Omega_\bullet)^{(k)}$. All the ∞–morphisms coming into play obviously respect this filtration, and moreover $1\otimes h_\bullet$ passes to the quotients, so that we have

$$1_{(\mathfrak{g}\otimes\Omega_\bullet)^{(k)}}-(1\otimes i_\bullet)^{(k)}(1\otimes p_\bullet)^{(k)}=d(1\otimes h_\bullet)^{(k)}+(1\otimes h_\bullet)^{(k)}d$$

for all k, which shows that $(1\otimes r_\bullet)_\infty$ is a filtered ∞–quasi-isomorphism.

The next step is to reduce the study of the homotopy groups with arbitrary basepoint to the study of the homotopy groups with basepoint $0\in\mathrm{MC}_0(\mathfrak{g})$.

Lemma 3.8 *Let $\alpha\in\mathrm{MC}(\mathfrak{g})$, and let $\mathfrak{g}^\alpha$ be the $\mathscr{L}_\infty$–algebra obtained by twisting $\mathfrak{g}$ by α, that is, the $\mathscr{L}_\infty$–algebra with the same underlying graded vector space, but with differential*

$$d^\alpha(x):=dx+\sum_{n\geq 2}\frac{1}{(n-1)!}\ell_n(\alpha,\dots,\alpha,x)$$

and brackets

$$\ell^\alpha(x_1,\dots,x_m):=\sum_{n\geq m}\frac{1}{(n-m)!}\ell_n(\alpha,\dots,\alpha,x_1,\dots,x_m).$$

Let

$$\mathrm{Shift}_\alpha\colon \mathrm{MC}_\bullet(\mathfrak{g}^\alpha)\to\mathrm{MC}_\bullet(\mathfrak{g})$$

be the isomorphism of simplicial sets induced by the map given by

$$\beta\in\mathfrak{g}\mapsto\alpha+\beta\in\mathfrak{g}^\alpha.$$

Then the following diagram commutes:

$$\begin{tikzcd}
\mathrm{MC}_\bullet(\mathfrak{g}^\alpha) \arrow[r, "\mathrm{Shift}_\alpha"] \arrow[d, "R_\bullet^\alpha"'] & \mathrm{MC}_\bullet(\mathfrak{g}) \arrow[d, "R_\bullet"] \\
\mathrm{MC}_\bullet(\mathfrak{g}^\alpha) \arrow[r, "\mathrm{Shift}_\alpha"] & \mathrm{MC}_\bullet(\mathfrak{g})
\end{tikzcd}$$

where

$$R_\bullet^\alpha(\beta) := \sum_{k\geq 1}(1\otimes r_\bullet)_k^\alpha(\beta^{\otimes k})$$

and

$$(1\otimes r_\bullet)_k^\alpha(\beta_1\otimes\cdots\otimes\beta_k) := \sum_{j\geq 0}\frac{1}{j!}(1\otimes r_\bullet)_{k+j}(\alpha^{\otimes j}\otimes\beta_1\otimes\cdots\otimes\beta_k)$$

is the twist of $(1\otimes r_\bullet)_\infty$ *by the Maurer–Cartan element* α. *Here, we identified* $\alpha\in\mathfrak{g}$ *with* $\alpha\otimes 1\in\mathfrak{g}\otimes\Omega_\bullet$.

Proof The proof in [8, Lemma 4.3] goes through mutatis mutandis. □

Remark 3.9 The $\mathscr{L}_\infty$–algebra $\mathfrak{g}^\alpha$ in Lemma 3.8 is endowed with the same filtration as $\mathfrak{g}$.

Now we proceed by induction to show that $R^{(k)}$ is a weak equivalence from $\mathrm{MC}_\bullet(\mathfrak{g}^{(k)})$ to itself for all $k\geq 2$. As the $\mathscr{L}_\infty$–algebra $(\mathfrak{g}\otimes\Omega_\bullet)^{(2)}$ is abelian, the base step of the induction is given by Lemma 3.7.

Lemma 3.10 *Let* $m\geq 2$. *Suppose that*

$$R_\bullet^{(k)}\colon \mathrm{MC}(\mathfrak{g}^{(k)})\to\mathrm{MC}(\mathfrak{g}^{(k)})$$

is a weak equivalence for all $2\leq k\leq m$. *Then* $R_\bullet^{(m+1)}$ *is also a weak equivalence.*

Proof The zeroth homotopy set π_0 has already been taken care of in Lemma 3.6. Thanks to Lemma 3.8, it is enough to prove that $R_\bullet^{(m+1)}$ induces isomorphisms of homotopy groups π_i based at 0 for all $i\geq 1$.

Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \frac{F_m(\mathfrak{g}\otimes\Omega_\bullet)}{F_{m+1}(\mathfrak{g}\otimes\Omega_\bullet)} \arrow[r] \arrow[d] & (\mathfrak{g}\otimes\Omega_\bullet)^{(m+1)} \arrow[r] \arrow[d, "(1\otimes r_\bullet)_\infty^{(m+1)}"'] & (\mathfrak{g}\otimes\Omega_\bullet)^{(m)} \arrow[r] \arrow[d, "(1\otimes r_\bullet)_\infty^{(m)}"] & 0 \\
0 \arrow[r] & \frac{F_m(\mathfrak{g}\otimes\Omega_\bullet)}{F_{m+1}(\mathfrak{g}\otimes\Omega_\bullet)} \arrow[r] & (\mathfrak{g}\otimes\Omega_\bullet)^{(m+1)} \arrow[r] & (\mathfrak{g}\otimes\Omega_\bullet)^{(m)} \arrow[r] & 0
\end{tikzcd}$$

where the leftmost vertical arrow is given by the linear term $(1\otimes i_\bullet)(1\otimes p_\bullet)$ of $(1\otimes r_\bullet)_\infty$ since all higher terms vanish, as can be seen by the explicit formulas for the ∞–quasi-isomorphisms induced by the homotopy transfer theorem given in [17, Sections 10.3.5–10.3.6]. Therefore, it is a weak equivalence as the $\mathscr{L}_\infty$–algebras in question are abelian. The first term in each row is the fiber of the next map, which is surjective. By Theorem 2.5, we know that applying the MC functor makes the horizontal maps on the right into fibrations of simplicial sets, while the objects we obtain on the left are easily seen to be the fibers. Taking the long sequence in homotopy and using the five lemma, we see that all we are left to do is to prove that $R_\bullet^{(m+1)}$ induces an isomorphism on π_1. Notice that it is necessary to prove this, as the long sequence is exact everywhere except on the level of π_0.

The long exact sequence of homotopy groups (truncated on both sides) reads

$$\pi_2\mathrm{MC}_\bullet(\mathfrak{g}^{(m)})\xrightarrow{\partial}\pi_1\mathrm{MC}_\bullet\left(\frac{F_m\mathfrak{g}}{F_{m+1}\mathfrak{g}}\right)\to\pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m+1)})\\ \to\pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m)})\xrightarrow{\partial}\pi_0\mathrm{MC}_\bullet\left(\frac{F_m\mathfrak{g}}{F_{m+1}\mathfrak{g}}\right),$$

where in the higher homotopy groups we left the basepoint implicit (as it is always 0). The map

$$\partial\colon\pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m)})\to\pi_0\mathrm{MC}_\bullet\left(\frac{F_m\mathfrak{g}}{F_{m+1}\mathfrak{g}}\right)=H^1(F_{m+1}\mathfrak{g}/F_m\mathfrak{g})$$

is seen to be the obstruction to lifting an element of $\pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m)})$ to an element of $\pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m+1)})$ (for example [12, Lemma 7.3]).

The map $\pi_1(R_\bullet^{(m+1)})$ is surjective Let $y\in\pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m+1)})$ and denote by $\bar{y}$ its image in $\pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m)})$. By the induction hypothesis, there exists a unique $\bar{x}\in\pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m)})$ which is mapped to $\bar{y}$ under $R_\bullet^{(m)}$. As $\bar{y}$ is the image of y, we have $\partial(\bar{y})=0$, and this implies that $\partial(\bar{x})=0$, too. Therefore, there exists $x\in\pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m+1)})$ mapping to $\bar{x}$. Denote by y' the image of x under $R_\bullet^{(m+1)}$. Then $y'y^{-1}$ is in the kernel of the map

$$\pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m+1)})\to\pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m)}).$$

By exactness of the long sequence and the fact that $R_\bullet$ induces an automorphism of $\pi_1\mathrm{MC}_\bullet(F_{m+1}\mathfrak{g}/F_m\mathfrak{g})$, there exists an element $z\in\pi_1(\mathrm{MC}_\bullet(F_{m+1}\mathfrak{g}/F_m\mathfrak{g}))$ mapping to $y'y^{-1}$ under the composite

$$\pi_1\mathrm{MC}_\bullet\left(\frac{F_{m+1}\mathfrak{g}}{F_m\mathfrak{g}}\right)\xrightarrow{R_\bullet}\pi_1\mathrm{MC}_\bullet\left(\frac{F_{m+1}\mathfrak{g}}{F_m\mathfrak{g}}\right)\to\pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m+1)}).$$

Let x' be the image of z in $\pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m+1)})$; then $(x')^{-1}x$ maps to y under $R_\bullet^{(m+1)}$. This proves the surjectivity of the map $\pi_1(R_\bullet^{(m+1)})$.

The map $\pi_1(R_\bullet^{(m+1)})$ is injective Assume $x, x' \in \pi_1\mathrm{MC}_\bullet(\mathfrak{g}^{(m+1)})$ map to the same element under $R_\bullet^{(m+1)}$. Then $x(x')^{-1}$ maps to the neutral element 0 under $R_\bullet^{(m+1)}$. It follows that there is a $z \in \pi_1\mathrm{MC}_\bullet(F_{m+1}\mathfrak{g}/F_m\mathfrak{g})$ mapping to $x(x')^{-1}$, which must be such that its image w is itself the image of some $\widetilde{w} \in \pi_2\mathrm{MC}_\bullet(\mathfrak{g}^{(m)})$ under the map ∂. But, by the induction hypothesis and the exactness of the long sequence, this implies that z is in the kernel of the next map, and thus that $x(x')^{-1}$ is the identity element. Therefore, the map $\pi_1(R_\bullet^{(m+1)})$ is injective.

This ends the proof of the lemma. □

Finally, we can conclude the proof of Theorem 3.3.

Proof of Theorem 3.3 Lemma 3.10, together with all that we have said before, shows that $R_\bullet^{(m)}$ is a weak equivalence for all $m \geq 2$. Therefore, we have the commutative diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\mathrm{MC}_\bullet(\mathfrak{g}^{(4)}) \arrow[r, "\sim"] \arrow[d] & \mathrm{MC}_\bullet(\mathfrak{g}^{(4)}) \arrow[d] \\
\mathrm{MC}_\bullet(\mathfrak{g}^{(3)}) \arrow[r, "\sim"] \arrow[d] & \mathrm{MC}_\bullet(\mathfrak{g}^{(3)}) \arrow[d] \\
\mathrm{MC}_\bullet(\mathfrak{g}^{(2)}) \arrow[r, "\sim"] & \mathrm{MC}_\bullet(\mathfrak{g}^{(2)})
\end{tikzcd}$$

where all objects are Kan complexes, all horizontal arrows are weak equivalences and all vertical arrows are (Kan) fibrations by Theorem 2.5. It follows that the collection of horizontal arrows defines a weak equivalence between fibrant objects in the model category of towers of simplicial sets; see [12, Section VI.1]. The functor from towers of simplicial sets to simplicial sets given by taking the limit is right adjoint to the constant tower functor, which trivially preserves cofibrations and weak equivalences. Thus, the constant tower functor is a left Quillen functor, and it follows that the limit functor is a right Quillen functor. In particular, it preserves weak equivalences between fibrant objects. Applying this to the diagram above proves that $R_\bullet$ is a weak equivalence. □

Remark 3.11 As an anonymous referee pointed out, there is an alternative, shorter proof of the fact that the map $R_\bullet$ induces a bijection on all higher homotopy groups: A Berglund [3, Theorem 1.1] gave an explicit group isomorphism

$$\mathrm{B}\colon H_n(\mathfrak{g}) \to \pi_{n+1}\mathrm{MC}_\bullet(\mathfrak{g}), \quad n \geq 0,$$

for any complete $\mathscr{L}_\infty$–algebra $\mathfrak{g}$. One can use this map together with the explicit formula for the map $R_\bullet$ derived from the homotopy transfer theorem to immediately derive the result.

In [19] an alternative proof of Berglund's theorem will be given which relies on the results of the present article. It is therefore important to have a demonstration of Theorem 3.3 which does not depend on it.

4 Properties and comparison

Theorem 3.3 shows that the simplicial set $\mathrm{MC}(\mathfrak{g}\otimes C_\bullet)$ is a new model for the Deligne–Hinich ∞–groupoid. This section is dedicated to the study of some properties of this object. We start by showing that it is a Kan complex, then we give some conditions on the differential forms representing its simplices. We show how we can use it to rectify cells of the Deligne–Hinich ∞–groupoid, which provides an alternative, simpler proof of [7, Lemma B.2]. Finally we compare it with Getzler's functor $\gamma_\bullet$, proving that our model is contained in Getzler's. Independent results by Bandiera [1; 2] imply that the two models are actually isomorphic.

4.1 Properties of $\mathrm{MC}_\bullet(\mathfrak{g}\otimes C_\bullet)$

The following proposition is the analogue to Theorem 2.5 for our model:

Proposition 4.1 *Let $\mathfrak{g},\mathfrak{h}$ be two filtered $\mathscr{L}_\infty$–algebras and suppose that $\phi\colon \mathfrak{g}\to\mathfrak{h}$ is a morphism of $\mathscr{L}_\infty$–algebras inducing a fibration of simplicial sets under the functor* $\mathrm{MC}_\bullet$ (see for example Theorem 2.5 for possible sufficient conditions). *Then the induced morphism*

$$\mathrm{MC}(\phi\otimes\mathrm{id}_{C_\bullet})\colon \mathrm{MC}(\mathfrak{g}\otimes C_\bullet)\to\mathrm{MC}(\mathfrak{h}\otimes C_\bullet)$$

is also a fibration of simplicial sets. In particular, for any filtered $\mathscr{L}_\infty$–algebra $\mathfrak{g}$, the simplicial set $\mathrm{MC}(\mathfrak{g}\otimes C_\bullet)$ *is a Kan complex.*

Proof By assumption, the morphism

$$\mathrm{MC}_\bullet(\phi)\colon \mathrm{MC}_\bullet(\mathfrak{g})\to\mathrm{MC}_\bullet(\mathfrak{h})$$

is a fibration of simplicial set, and by Lemma 3.5 the following diagram exhibits $\mathrm{MC}(\phi\otimes\mathrm{id}_{C_\bullet})$ as a retract of $\mathrm{MC}_\bullet(\phi)$:

$$\begin{tikzcd}
\mathrm{MC}(\mathfrak{g}\otimes C_\bullet) \arrow[r, "I_\bullet"] \arrow[d, "\mathrm{MC}(\phi\otimes\mathrm{id}_{C_\bullet})"'] & \mathrm{MC}_\bullet(\mathfrak{g}) \arrow[r, "P_\bullet"] \arrow[d, "\mathrm{MC}_\bullet(\phi)"'] & \mathrm{MC}(\mathfrak{g}\otimes C_\bullet) \arrow[d, "\mathrm{MC}(\phi\otimes\mathrm{id}_{C_\bullet})"] \\
\mathrm{MC}(\mathfrak{g}\otimes C_\bullet) \arrow[r, "I_\bullet"'] & \mathrm{MC}_\bullet(\mathfrak{g}) \arrow[r, "P_\bullet"'] & \mathrm{MC}(\mathfrak{g}\otimes C_\bullet)
\end{tikzcd}$$

As the class of fibrations is closed under retracts, this concludes the proof. □

We consider the composite $R_\bullet = I_\bullet P_\bullet$, which is not the identity.

Definition 4.2 We call the morphism

$$R_\bullet\colon \mathrm{MC}_\bullet(\mathfrak{g}) \to \mathrm{MC}_\bullet(\mathfrak{g})$$

the *rectification map*.

The following result is a wide generalization of [7, Lemma B.2], as well as a motivation for the name "rectification map" for $R_\bullet$:

Proposition 4.3 *We consider an element*

$$\alpha := \alpha_1(t_0,\dots,t_n) + \cdots \in \mathrm{MC}_n(\mathfrak{g}),$$

where the dots indicate terms in $\mathfrak{g}^{1-k}\otimes\Omega_n^k$ *with* $1\le k\le n$*. Then* $\beta := R_\bullet(\alpha)\in\mathrm{MC}_n(\mathfrak{g})$ *is of the form*

$$\beta = \beta_1(t_0,\dots,t_n) + \cdots + \xi\otimes\omega_{0\dots n},$$

where the dots indicate terms in $\mathfrak{g}^{1-k}\otimes\Omega_n^k$ *with* $1\le k\le n-1$*, where* ξ *is an element of* $\mathfrak{g}^{1-n}$*, and where* α_1 *and* β_1 *agree on the vertices of* Δ^n*. In particular, if* $\alpha\in\mathrm{MC}_1(\mathfrak{g})$*, then* $\beta = F(\alpha)\in\mathrm{MC}_1(\mathfrak{g})$ *is of the form*

$$\beta = \beta_1(t) + \lambda dt$$

for some $\lambda\in\mathfrak{g}^0$*, and satisfies*

$$\beta_1(0) = \alpha_1(0) \quad \textit{and} \quad \beta_1(1) = \alpha_1(1),$$

so that λ *gives a gauge equivalence between* $\alpha_1(0)$ *and* $\alpha_1(1)$*.*

Remark 4.4 As $R_\bullet$ is a projector, this proposition in fact gives information on the form of all the elements of $\mathrm{MC}(\mathfrak{g}\otimes C_\bullet)$.

Proof First notice that the map $R_\bullet$ commutes with the face maps and is the identity on 0–simplices, thus evaluation of the part of β in $\mathfrak{g}^1 \otimes \Omega_n^0$ at the vertices gives the same result as evaluation at the vertices of α_1. Next, we notice that β is in the image of $I_\bullet$. We use the explicit formula for $(1 \otimes i_n)_\infty$ of [17, Section 10.3.5]: the operator acting on arity $k \geq 2$ is given, up to signs, by the sum over all rooted trees with $1 \otimes i_n$ put at the leaves, the brackets ℓ_n of the corresponding arity at all vertices, and $1 \otimes h$ at the inner edges and at the root. But the $1 \otimes h$ at the root lowers the degree of the part of the form in Ω_n by 1, and thus we cannot get something in $\mathfrak{g}^{1-n} \otimes \Omega_n^n$ from these terms. The only surviving term is therefore the one coming from $(1 \otimes i_n)(P_\bullet(\alpha))$, given by $\xi \otimes \omega_{0\ldots n}$ for some $\xi \in \mathfrak{g}^{1-n}$. □

4.2 Comparison with Getzler's ∞–groupoid $\gamma_\bullet$

Finally, we compare the simplicial set $\mathrm{MC}(\mathfrak{g} \otimes C_\bullet)$ with Getzler's Kan complex $\gamma_\bullet(\mathfrak{g})$. We start with an easy result that follows directly from our approach, before presenting Bandiera's result that these two simplicial sets are actually isomorphic.

Lemma 4.5 *We have*

$$I_\bullet \mathrm{MC}(\mathfrak{g} \otimes C_\bullet) \subseteq \gamma_\bullet(\mathfrak{g}).$$

Proof We have $h_\bullet i_\bullet = 0$. Therefore, by the explicit formula for $(i_\bullet)_\infty$ given in [17, Section 10.3.5], we have $h_\bullet(\beta) = 0$ for any $\beta \in \mathfrak{g} \otimes \Omega_\bullet$ in the image of $I_\bullet$. Thus,

$$h_\bullet(\mathrm{MC}(\mathfrak{g} \otimes C_\bullet)) = h_\bullet I_\bullet P_\bullet(\mathrm{MC}_\bullet(\mathfrak{g})) = 0,$$

which proves the claim. □

In his thesis [1], Bandiera proves the following:

Theorem 4.6 [1, Theorem 2.3.3 and Proposition 5.2.7] *The map*

$$(P_\bullet, 1 \otimes h_\bullet)\colon \mathrm{MC}_\bullet(\mathfrak{g}) \to \mathrm{MC}(\mathfrak{g} \otimes C_\bullet) \times (\mathrm{Im}(1 \otimes h_\bullet) \cap (\mathfrak{g} \otimes \Omega_\bullet)^1)$$

is bijective. In particular, its restriction to $\gamma_\bullet(\mathfrak{g}) = \ker(1 \otimes h_\bullet) \cap \mathrm{MC}_\bullet(\mathfrak{g})$ *gives an isomorphism of simplicial sets*

$$P_\bullet\colon \gamma_\bullet(\mathfrak{g}) \to \mathrm{MC}(\mathfrak{g} \otimes C_\bullet).$$

Remark 4.7 Thanks to our approach, we immediately have an inverse for the map $P_\bullet$: it is of course the map $I_\bullet$.

As a consequence of Bandiera's result and of Proposition 4.3, we can partially characterize the thin elements of $\gamma_\bullet(\mathfrak{g})$.

Lemma 4.8 *For each $n \geq 1$, the thin elements contained in $\gamma_n(\mathfrak{g})$ are those with no term in $\mathfrak{g}^{1-n} \otimes \Omega_n^n$.*

Proof By Proposition 4.3 and Theorem 4.6, we know that if $\alpha \in \gamma_n(\mathfrak{g})$, then α is of the form

$$\alpha = \cdots + \xi \otimes \omega_{0\dots n}$$

for some $\xi \in \mathfrak{g}^{1-n}$, where the dots indicate terms in $\mathfrak{g}^{1-k} \otimes \Omega_n^k$ for $0 \leq k \leq n-1$, which will give zero after integration. Integrating, we get

$$\int_{\Delta^n} \alpha = \xi \otimes \int_{\Delta^n} \omega_{0\dots n} = \xi \otimes 1.$$

Therefore, α is thin if and only if $\xi = 0$. □

5 The case of Lie algebras

In this section, we focus on the case where $\mathfrak{g}$ is actually a dg Lie algebra. In this situation, we are able to represent the functor $\mathrm{MC}(\mathfrak{g} \otimes C_\bullet)$ by a cosimplicial dg Lie algebra. The main tools used here are results from [18].

5.1 Reminder on the complete cobar construction

What we explain here is a special case of [17, Sections 11.1–11.3], namely where we take $\mathscr{P} = \mathrm{Lie}$ and only consider the canonical twisting morphism $\pi\colon \mathrm{BLie} \to \mathrm{Lie}$, where BLie is the bar construction of the operad Lie encoding Lie algebras. In fact, we consider a slight variation on the material presented there, as we remove the conilpotency condition on coalgebras but additionally add the requirement that algebras be complete. See also [18, Section 6.2].

Let X be a dg BLie–coalgebra. The *complete cobar construction* of X is the complete dg Lie algebra

$$\widehat{\Omega}_\pi X := (\widehat{\mathrm{Lie}}(X), d := d_1 + d_2),$$

where

$$\widehat{\mathrm{Lie}}(X) := \prod_{n \geq 1} \mathrm{Lie}(n) \otimes_{\mathbb{S}_n} X^{\otimes n}$$

and where the differential d is composed by the following two parts:

(1) The differential $-d_1$ is the unique derivation extending the differential d_X of X.

(2) The differential $-d_2$ is the unique derivation extending the composite

$$X \xrightarrow{\Delta_X} \widehat{\mathrm{BLie}}(X) \xrightarrow{\pi\circ 1_X} \widehat{\mathrm{Lie}}(X).$$

Notice that as X is not assumed to be conilpotent, the decomposition map Δ_X really lands in the product

$$\widehat{\mathrm{BLie}}(X) := \prod_{n\geq 0} (\mathrm{BLie}(n) \otimes X^{\otimes n})^{\mathbb{S}_n}$$

and not the direct sum. Thus, it is necessary to consider the free complete Lie algebra over X. Also, there is a passage from invariants to coinvariants that is left implicit here, as the decomposition map lands in invariants, but the elements of the complete free Lie algebra $\widehat{\mathrm{Lie}}(X)$ are coinvariants. This introduces factors of the form $1/n!$ when computing explicit formulas for d_2.

The complete cobar construction $\widehat{\Omega}_\pi$ defines a functor from dg BLie–coalgebras to complete dg Lie algebras.

5.2 Representing $\mathrm{MC}(\mathfrak{g}\otimes C_\bullet)$

Using the Dupont contraction, the homotopy transfer theorem produces the structure of a simplicial $\mathscr{C}_\infty$–algebra to $C_\bullet$. As the underlying cochain complex C_n is finite-dimensional for each n, it follows that its dual is a cosimplicial $\mathrm{B}(\mathscr{S}\otimes\mathrm{Lie})$–coalgebra. Therefore, the desuspension $sC_\bullet^\vee$ is a cosimplicial BLie–coalgebra, and we can take its complete cobar construction.

Definition 5.1 We denote this cosimplicial dg Lie algebra by $\mathfrak{mc}_\bullet := \widehat{\Omega}_\pi(sC_\bullet^\vee)$.

Theorem 5.2 *Let $\mathfrak{g}$ be a complete dg Lie algebra. There is a canonical isomorphism*

$$\mathrm{MC}(\mathfrak{g}\otimes C_\bullet) \cong \mathrm{hom}_{\mathsf{dgLie}}(\mathfrak{mc}_\bullet, \mathfrak{g}).$$

It is natural in $\mathfrak{g}$.

Proof By [18, Theorem 5.1], the $\mathscr{L}_\infty$–algebra structure we have on $\mathfrak{g}\otimes C_\bullet$ is the same as the structure that we obtain on the tensor product of the dg Lie algebra $\mathfrak{g}$ with the simplicial $\mathscr{C}_\infty$–algebra $C_\bullet$ by using [18, Theorem 3.4] with $\mathscr{P}=\mathscr{Q}=\mathrm{Lie}$ and $\Psi=\mathrm{id}_{\mathrm{Lie}}$. Therefore, we can apply [18, Corollary 6.6], which gives the desired isomorphism. □

With this form for $\mathrm{MC}(\mathfrak{g}\otimes C_\bullet)$, Theorem 3.3 reads as follows:

Corollary 5.3 *Let $\mathfrak{g}$ be a complete dg Lie algebra. There is a weak equivalence of simplicial sets*

$$\mathrm{MC}_\bullet(\mathfrak{g}) \simeq \hom_{\mathsf{dgLie}}(\mathfrak{mc}_\bullet, \mathfrak{g}),$$

natural in $\mathfrak{g}$.

We can completely characterize the first levels of the cosimplicial dg Lie algebra $\mathfrak{mc}_\bullet$. Recall from [16] the Lawrence–Sullivan algebra: it is the unique free complete dg Lie algebra generated by two Maurer–Cartan elements in degree 1 and a single element in degree 0 such that the element in degree 0 is a gauge between the two generating Maurer–Cartan elements.

Proposition 5.4 *The first two levels of the cosimplicial dg Lie algebra $\mathfrak{mc}_\bullet$ are as follows:*

1. *The dg Lie algebra $\mathfrak{mc}_0$ is isomorphic to the free dg Lie algebra with a single Maurer–Cartan element as the only generator.*
2. *The dg Lie algebra $\mathfrak{mc}_1$ is isomorphic to the Lawrence–Sullivan algebra.*

Proof For (1), we have $\Omega_0 \cong \mathbb{K} \cong C_0$, both p_0 and i_0 are the identity, and $h_0 = 0$. It follows that, as a complete graded free Lie algebra, $\mathfrak{mc}_0$ is given by

$$\mathfrak{mc}_0 = \widehat{\mathrm{Lie}}(s\mathbb{K}).$$

We denote the generator by $\alpha := s1^\vee$. It has degree 1. Let $\mathfrak{g}$ be any complete dg Lie algebra; then a morphism

$$\phi\colon \mathfrak{mc}_0 \to \mathfrak{g}$$

is equivalent to the Maurer–Cartan element

$$\phi(\alpha)\otimes 1 \in \mathrm{MC}(\mathfrak{g}\otimes C_\bullet) \cong \mathrm{MC}(\mathfrak{g}).$$

Conversely, through P_0, every Maurer–Cartan element of $\mathfrak{g}$ induces a morphism $\mathfrak{mc}_0 \to \mathfrak{g}$. As this is true for any dg Lie algebra $\mathfrak{g}$, it follows that α is a Maurer–Cartan element.

To prove (2), we start by noticing that

$$C_1 := \mathbb{K}\omega_0 \oplus \mathbb{K}\omega_1 \oplus \mathbb{K}\omega_{01}$$

with ω_0 and ω_1 of degree 0 and ω_{01} of degree 1. Writing $\alpha_i := s\omega_i^\vee$ and $\lambda := s\omega_{01}^\vee$, we have

$$\mathfrak{mc}_1 = \widehat{\mathrm{Lie}}(\alpha_0, \alpha_1, \lambda)$$

as a graded Lie algebra. Let $\mathfrak{g}$ be any dg Lie algebra; then a morphism

$$\phi\colon \mathfrak{mc}_1 \to \mathfrak{g}$$

is equivalent to a Maurer–Cartan element

$$\phi(\alpha_0) \otimes \omega_0 + \phi(\alpha_1) \otimes \omega_1 + \phi(\lambda) \otimes \omega_{01} \in \mathrm{MC}(\mathfrak{g} \otimes C_1);$$

see [18, Sections 6.3–6.4]. Applying I_1, as in the proof of Proposition 4.3 we obtain

$$I_1(\phi(\alpha_0) \otimes \omega_0 + \phi(\alpha_1) \otimes \omega_1 + \phi(\lambda) \otimes \omega_{01}) = a(t_0, t_1) + \phi(\lambda) \otimes \omega_{01} \in \mathrm{MC}_1(\mathfrak{g})$$

with $a(1,0) = \phi(\alpha_0)$ and $a(0,1) = \phi(\alpha_1)$. The Maurer–Cartan equation for

$$a(t_0, t_1) + \phi(\lambda) \otimes \omega_{01}$$

then shows that $\phi(\lambda)$ is a gauge from $\phi(\alpha_0)$ to $\phi(\alpha_1)$. Conversely, if we are given the data of two Maurer–Cartan elements of $\mathfrak{g}$ and a gauge equivalence between them, then this data gives us a Maurer–Cartan element of $\mathfrak{g} \otimes \Omega_1$. Applying P_1 then gives back a nontrivial morphism $\mathfrak{mc}_1 \to \mathfrak{g}$. As this is true for any $\mathfrak{g}$, it follows that $\mathfrak{mc}_1$ is isomorphic to the Lawrence–Sullivan algebra. □

Remark 5.5 Alternatively, one could write down explicitly the differentials for both $\mathfrak{mc}_0$ (which is straightforward) and $\mathfrak{mc}_1$ (with the help of [6, Proposition 19]). An explicit description of $\mathfrak{mc}_\bullet$ is made difficult by the fact that one needs to know the whole $\mathscr{C}_\infty$–algebra structure on $C_\bullet$ in order to write down a formula for the differential.

5.3 Relations to rational homotopy theory

The cosimplicial dg Lie algebra $\mathfrak{mc}_\bullet$ has already made its appearance in the literature not long ago, in [4], in the context of rational homotopy theory, where it plays the role of a Lie model for the geometric n–simplex. With the goal of simplifying comparison and interaction between our work and theirs, we provide here a short review and a dictionary between our vocabulary and the notation used in [4]:

This paper	[4]
$\mathfrak{mc}_\bullet$	$\mathfrak{L}_\bullet$ or $\mathfrak{L}_{\Delta^\bullet}$
$\Omega_\bullet$	$A_{\mathrm{PL}}(\Delta^\bullet)$
B_ι	Quillen functor $\mathcal{C}$
$\hom_{\mathsf{dgLie}}(\mathfrak{mc}_\bullet, -)$	$\langle - \rangle$
$\hom_{\mathsf{dgCom}}(-, \Omega_\bullet)$	$\langle - \rangle_S$

Remark 5.6 The fact that the cosimplicial dg Lie algebra $\mathfrak{mc}_\bullet$ is isomorphic to $\mathfrak{L}_\bullet$ is immediate from [4, Definition 2.1 and Theorem 2.8].

The following theorem has nonempty intersection with our results. We say a dg Lie algebra is of *finite type* if it is finite-dimensional in every degree and if its degrees are bounded either above or below.

Theorem 5.7 [4, Theorem 8.1] *Let $\mathfrak{g}$ be a dg Lie algebra of finite type with $H^n(\mathfrak{g}, d) = 0$ for all $n > 0$. Then there is a homotopy equivalence of simplicial sets*

$$\hom_{\mathsf{dgLie}}(\mathfrak{mc}_\bullet, \mathfrak{g}) \simeq \hom_{\mathsf{dgCom}}(\mathrm{B}_\iota(s\mathfrak{g})^\vee, \Omega_\bullet).$$

We can easily recover an analogous result, which works on complete dg Lie algebras of finite type such that $\mathfrak{g}^{-1} = 0$, but without restrictions on the cohomology, using our main theorem and some results of [18].

Proposition 5.8 *Let $\mathfrak{g}$ be a complete dg Lie algebra of finite type such that $\mathfrak{g}^{-1} = 0$. Then there is a weak equivalence of simplicial sets*

$$\hom_{\mathsf{dgLie}}(\mathfrak{mc}_\bullet, \mathfrak{g}) \simeq \hom_{\mathsf{dgCom}}(\mathrm{B}_\iota(s\mathfrak{g})^\vee, \Omega_\bullet).$$

Proof The proof is given by the sequence of equivalences

$$\begin{aligned} \hom_{\mathsf{dgCom}}(\mathrm{B}_\iota(s\mathfrak{g})^\vee, \Omega_\bullet) &\cong \hom_{\mathsf{dgCom}}(\widehat{\Omega}_\pi(s^{-1}\mathfrak{g}^\vee), \Omega_\bullet) \\ &\cong \mathrm{MC}(\mathfrak{g} \otimes \Omega_\bullet) \\ &\simeq \hom_{\mathsf{dgLie}}(\mathfrak{mc}_\bullet, \mathfrak{g}). \end{aligned}$$

In the first line we used the natural isomorphism

$$\mathrm{B}_\iota(s\mathfrak{g})^\vee \cong \widehat{\Omega}_\pi(s^{-1}\mathfrak{g}^\vee).$$

Notice that the assumptions on $\mathfrak{g}$ make it so that $\mathfrak{g}^\vee$ is a $\mathrm{Lie}^\vee$–coalgebra. In the second line we used a slight generalization of [18, Corollary 6.6] for $\mathscr{Q} = \mathscr{P} = \mathrm{Com}$ and Ψ the identity morphism of Com. Notice that here the assumption that $\mathfrak{g}^{-1} = 0$ makes it so that

$$\hom_{\mathsf{dgCom}}(\widehat{\Omega}_\pi(s^{-1}\mathfrak{g}^\vee), \Omega_\bullet) \cong \hom(s^{-1}\mathfrak{g}^\vee, \Omega_\bullet)^0$$

even though $\Omega_\bullet$ is not complete. Finally, in the third line we used our Corollary 5.3. □

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Classifying spaces from Ore categories with Garside families

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We describe how an Ore category with a Garside family can be used to construct a classifying space for its fundamental group(s). The construction simultaneously generalizes Brady's classifying space for braid groups and the Stein–Farley complexes used for various relatives of Thompson's groups. It recovers the fact that Garside groups have finite classifying spaces.

We describe the categories and Garside structures underlying certain Thompson groups. The indirect product of categories is introduced and used to construct new categories and groups from known ones. As an illustration of our methods we introduce the group *braided* T and show that it is of type F_∞.

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1. Categories generalizing monoids 1481
2. Fundamental examples 1486
3. Finiteness properties of fundamental groups of Ore categories 1491
4. The indirect product of two categories 1500
5. Examples: categories constructed by indirect products 1505
6. Examples: finiteness properties 1514
7. Sketch of further examples 1519

References 1522

There are many results establishing finiteness properties of Thompson groups. The proofs typically follow the same blueprint, due to Brown, Stein and Farley, to reduce the problem to its technical core, which is then solved individually; examples are due to Brown [16], Stein [36], Farley [23], Bux, Fluch, Marschler, Witzel and Zaremsky [24; 18; 43], Martínez-Pérez, Matucci and Nucinkis [31] and Belk and Forrest [3]. This fact is well known to experts but it is not apparent when looking at the articles. The reason is that the proofs are phrased using very different language. The present article provides a uniform formalization of the common ("blueprint") part of the mentioned

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proofs. The result is a theorem that reduces a statement about finiteness properties of Thompson groups to its technical core, which is about connectivity of certain complexes.

In formalizing the blueprint it is fruitful to employ the language of categories, not because any sophisticated category theory would be used, but because categories are flexible enough to model posets, monoids, complexes and other objects that occur in the constructions. A single category encodes at the same time the Thompson group (its fundamental group) as well as the complex for it to act on (a subcomplex of the realization).

In formulating the proof categorically we find that the assumptions that make it work are established concepts in the (recent) literature — see Dehornoy, Digne, Godelle, Krammer and Michel [22]; the key notions are those of an Ore category and of a Garside family (see Section 1 for definitions). An Ore category not only has the property that elements of its fundamental group can be written as a fraction of two morphisms (eg "tree diagrams"), it also gives rise to a contractible space for it to act on. A Garside family of morphisms (eg "elementary splits") is what is needed to make the Quillen trick work and reduce to the smaller Stein–Farley complex. In the abstract formulation our results apply not only to Thompson groups but also to Garside groups such as the braid groups BRAID_n and possibly to entirely different examples.

The main results are given in Section 3 in greater generality (see Observation 1.7 for the relationship between a Garside map and a Garside family).

Theorem A *Let $\mathcal{C}$ be a small right-Ore category that is factor-finite and admits a right-Garside map Δ, and let $* \in \mathrm{Ob}(\mathcal{C})$. There is a contractible simplicial complex X on which $G = \pi_1(\mathcal{C}, *)$ acts. The space is covered by the G–translates of compact subcomplexes K_x for $x \in \mathrm{Ob}(\mathcal{C})$. Every stabilizer is isomorphic to a finite-index subgroup of the automorphism group $\mathcal{C}^\times(x,x)$ for some $x \in \mathrm{Ob}(\mathcal{C})$.*

Taking $\mathcal{C}$ to be a Garside monoid and Δ to be the Garside element, one immediately recovers the known fact that Garside groups, and braid groups in particular, have finite classifying spaces; see Charney, Meier and Whittlesey [20]. In fact, if $\mathcal{C}$ is taken to be the dual braid monoid, the quotient $G\backslash X$ is precisely Brady's classifying space for BRAID_n [10].

In the case of Thompson's group F the complex in Theorem A is the Stein–Farley complex. The action is not cocompact in this case because $\mathcal{C}$ has infinitely many objects. In order to obtain cocompact actions on highly connected spaces, we employ Morse theory.

Theorem B *Let $\mathcal{C}$, Δ, $*$ be as in Theorem A and let $\rho\colon \mathrm{Ob}(\mathcal{C}) \to \mathbb{N}$ be a height function such that $\{x \in \mathrm{Ob}(\mathcal{C}) \mid \rho(x) \le n\}$ is finite for every $n \in \mathbb{N}$. Assume that*

- (STAB) *$\mathcal{C}^\times(x,x)$ is of type F_n for all x,*
- (LK) *there exists an $N \in \mathbb{N}$ such that $|E(x)|$ is $(n-1)$–connected for all x with $\rho(x) \ge N$.*

Then $\pi_1(\mathcal{C},*)$ is of type F_n.

The complexes $|E(x)|$ depend on $\mathcal{C}$ and Δ and are described in Section 3.4. Establishing condition (LK) is what we referred to as the technical core of the problem in the beginning.

Theorem B provides a general scheme for proving that an (eligible) group is of type F_∞: first describe the category, second analyze the complexes $|E(x)|$, and then apply the theorem. This scheme will be illustrated in Section 5 (describe the category) and Section 6 (analyze the complexes, apply the theorem) on the examples of Thompson's groups F, T and V, their braided versions and some other groups. To our knowledge this is the first time that Garside structures are studied in connection with Thompson groups. In the process we define the Thompson group BT, *braided* T, and prove (see Theorem 6.7):

Theorem C *The braided Thompson group BT is of type F_∞.*

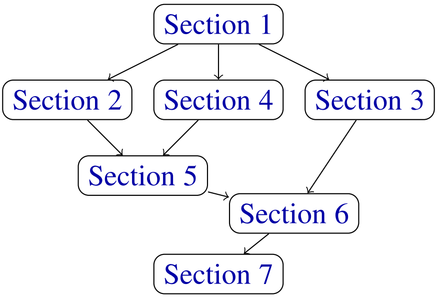
Although braided versions of V — see Dehornoy [21] and Brin [15] — and F — see Brady, Burillo, Cleary and Stein [11] — exist in the literature, our main merit is to be able to define braided T. The fact that it is F_∞ then follows from Theorem B and results from [18]. To explain the issue of defining BT we need to digress a bit (see also Remark 5.11). The category underlying Thompson's group F is the category of forests, where a morphism $m \leftarrow n$ is a rooted forest with m roots and n leaves (see Section 2). The categories underlying Thompson's groups T and V are obtained by adding in the cyclic groups $(\mathbb{Z}/n\mathbb{Z})_{n\in\mathbb{N}}$ respectively the symmetric groups $(\mathrm{SYM}_n)_{n\in\mathbb{N}}$. The categories underlying the braided groups BF, BT and BV are obtained from the forest category by adding in, for each n, the preimage under the map $\mathrm{BRAID}_n \to \mathrm{SYM}_n$ of the trivial group, the cyclic group and the full symmetric group, respectively.

When Brin first introduced BV, he avoided using categories by starting with the monoid of forests with infinitely many roots and leaves and added in the braid group on infinitely many strands $\lim_n \mathrm{BRAID}_n$. He then described which elements of the resulting infinite-strand group should belong to BV by hand. The reason that this workaround is not viable for BT, or actually even for T, is simply that the finite cyclic groups $(\mathbb{Z}/n\mathbb{Z})_{n\in\mathbb{N}}$ do not have enough inclusions into each other and therefore no limiting object (nor do their preimages in BRAID_n).

Formally, the process of adding groups to the forest monoid mentioned in the last paragraph is formation of the *indirect product* (or Zappa–Szép product) $\mathcal{F} \bowtie \mathcal{G}$, where $\mathcal{F}$ is the forest category and $\mathcal{G}$ is the category containing the groups in question. We introduce the indirect product of categories in Section 4.

The applications of Theorem B are somewhat similar to those of Thumann's results [37], so we should clarify how they compare. Basically, Theorem B applies to more general situations but has less power built in. Thumann's framework is restricted to symmetric or braided operads but the connectivity proofs from [18] verifying condition (LK) are already included. Our results apply to more general settings such as the ones discussed in Sections 5.3 and 5.5, and in particular to groups that are not of type F_∞, but leave the work of checking (LK) to the user.

The article is organized as follows. The basic notions are introduced in Section 1. The underlying structures for braid groups and Thompson's group F are described in Section 2. Section 3 contains the main construction and the proofs of Theorems A and B. The indirect product of categories is introduced in Section 4 and is used in Section 5 to construct the categories underlying Thompson's groups and their braided versions. In Section 6, Theorem B is applied to the examples from Section 5 to deduce finiteness properties, among them Theorem C. In Section 7 we briefly sketch how further Thompson groups fit into our framework. Since the results about finiteness properties and the indirect product may be of independent interest, we include the following leitfaden:



This article arose out of the introduction to the author's Habilitation thesis [40], which in addition covers Thompson groups arising from matrix groups via cloning systems; see [43] and Section 5.3. More recently our results were used in proving that for every n

there exists a simple group that is of type F_{n-1} but not of type F_n; see Skipper, Witzel and Zaremsky [35].

1 Categories generalizing monoids

We start by collecting basic notions of categories regarding them as generalizations of monoids. Our exposition is based on [22, Chapter II], where the perspective is similar. The main difference is notational; see Remark 1.1 below.

A monoid may be regarded as (the set of morphisms) of a category with a single object. For us categories will play the role of generalized monoids where the multiplication is only partially defined. In particular, all categories in this chapter will be small. The requirement that they be locally small is important and taking them to be small is convenient; for example, it allows us to talk about morphisms of categories as maps of sets.

Let $\mathcal{C}$ be a category. Notationally, we follow [22] in denoting the set of morphisms of $\mathcal{C}$ by $\mathcal{C}$ as well (thinking of them as elements), while the objects are denoted by $\mathrm{Ob}(\mathcal{C})$. The identity at x will be denoted by 1_x. If f is a morphism from y to x, we call y the *source* and x the *target* of f. Our notation for composition is the familiar one for functions, that is, if f is a morphism from y to x and g is a morphism from z to y, then fg exists and is a morphism from z to x. If $x, y \in \mathrm{Ob}(\mathcal{C})$ then the set of morphisms from y to x is denoted by $\mathcal{C}(x, y)$, the set of morphisms from y to any object is denoted by $\mathcal{C}(-, y)$ and the set of morphisms from any object to x is denoted by $\mathcal{C}(x, -)$. This may be slightly unusual but renders the following intuitive expression valid:

$$f \in \mathcal{C}(x, y),\ g \in \mathcal{C}(y, z) \implies fg \in \mathcal{C}(x, z).$$

The corresponding diagram is

$$\begin{tikzcd} x & y \arrow[l, "f"'] & z \arrow[l, "g"'] \arrow[ll, bend left, "fg"] \end{tikzcd}$$

When we write an expression involving a product of morphisms, the requirement that this product exists is usually an implicit condition of the expression. Thus, $fg = h$ means that the source of f is the target of g and that the equality holds.

Remark 1.1 The net effect of the various differences in notation is that our formalism is consistent with [22], only the meaning of source/target, from/to and the direction of arrows are switched. The reason for this decision is that some of our morphisms will be group elements which we want to act from the left.

1.1 Groupoids

A morphism $f \in \mathcal{C}(x, y)$ is *invertible* if there is an *inverse*, namely a morphism $g \in \mathcal{C}(y, x)$ such that $fg = 1_x$ and $gf = 1_y$. The set of invertible morphisms in $\mathcal{C}(x, y)$ is denoted by $\mathcal{C}^\times(x, y)$ and the set of all invertible morphisms by $\mathcal{C}^\times$. A *groupoid* is a category $\mathcal{G}$ in which every morphism is invertible: $\mathcal{G} = \mathcal{G}^\times$. Just as every monoid naturally maps to a group, every category naturally maps to a groupoid; see [22, Section 3.1]:

Theorem 1.2 *For every category $\mathcal{C}$ there is a groupoid $\mathcal{G}pd(\mathcal{C})$ and a morphism $\iota\colon \mathcal{C} \to \mathcal{G}pd(\mathcal{C})$ with the following universal property: if $\varphi\colon \mathcal{C} \to \mathcal{G}$ is a morphism to a groupoid then there is a unique morphism $\widehat{\varphi}\colon \mathcal{G}pd(\mathcal{C}) \to \mathcal{G}$ such that $\varphi = \widehat{\varphi} \circ \iota$.*

The groupoid $\mathcal{G}pd(\mathcal{C})$ and the morphism ι are determined by $\mathcal{C}$ uniquely up to unique isomorphism.

We call $\mathcal{G}pd(\mathcal{C})$ the *enveloping groupoid* of $\mathcal{C}$. The morphism ι is a bijection on objects but it is not typically injective (on morphisms). One way to think about the enveloping groupoid is as the fundamental groupoid of $\mathcal{C}$:

The *nerve* of $\mathcal{C}$ is the simplicial set whose k–simplices are diagrams

$$x_0 \xleftarrow{f_1} x_1 \xleftarrow{f_2} x_2 \leftarrow \cdots \leftarrow x_{k-1} \xleftarrow{f_k} x_k$$

in $\mathcal{C}$. The i^{th} face is obtained by deleting x_i and replacing f_i and f_{i+1} by $f_i f_{i+1}$ and the j^{th} degenerate coface is obtained by introducing 1_{x_j} between f_j and f_{j+1}.

Proposition 1.3 [33, Proposition 1] *The groupoid $\mathcal{G}pd(\mathcal{C})$ is canonically isomorphic to the fundamental groupoid of the realization of the nerve of $\mathcal{C}$.*

In particular, the fundamental group of $\mathcal{C}$ in an object x is just the set of endomorphisms of $\mathcal{G}pd(\mathcal{C})$ in x: $\pi_1(\mathcal{C}, x) = \mathcal{G}pd(\mathcal{C})(x, x)$.

1.2 Noetherianity conditions

If $fg = h$ then we say that f is a *left-factor* of h and that h is a *right-multiple* of f. It is a *proper left-factor* or *proper right-multiple* if g is not invertible. We say that f is a *(proper) factor* of h if $efg = h$ (and one of e and g is not invertible).

The category $\mathcal{C}$ is *Noetherian* if there is no infinite sequence $f_0, f_1, \ldots$ such that f_{i+1} is a proper factor of f_i. It is said to be *strongly Noetherian* if there exists a map $\delta\colon \mathcal{C} \to \mathbb{N}$

that satisfies $\delta(fg) \geq \delta(f) + \delta(g)$ and, for $f \in \mathcal{C}$ noninvertible, $\delta(f) \geq 1$. Clearly, a strongly Noetherian category is Noetherian. See [22, Sections II.2.3 and II.2.4] for a detailed discussion.

We call a *height function* a map $\rho\colon \mathrm{Ob}(\mathcal{C}) \to \mathbb{N}$ such that $\rho(x) = \rho(y)$ if $\mathcal{C}(x,y)$ contains an invertible morphism and $\rho(x) < \rho(y)$ if $\mathcal{C}(x,y)$ contains a noninvertible morphism. Note that the existence of a height function implies strong Noetherianity by taking $\delta(f) = \rho(y) - \rho(x)$ if $f \in \mathcal{C}(x,y)$.

We say that $\mathcal{C}$ is *factor-finite* if every morphism in $\mathcal{C}$ has only finitely many factors up pre- and postcomposition by invertibles. This condition implies strong Noetherianity (see [22, Proposition 2.48]).

1.3 Ore categories

Two elements $g, h \in \mathcal{C}(x,-)$ have a *common right-multiple* d if there exist elements $e, f \in \mathcal{C}$ with $ge = hf = d$. It is a *least common right-multiple* if every other common right-multiple is a right-multiple of d. We say that $\mathcal{C}$ *has common right-multiples* if any two elements with the same target have a common right-multiple. We say that it *has conditional least common right-multiples* if any two elements that have a common right-multiple have a least common right-multiple. We say that it *has least common right-multiples* if any two elements with the same target have a least common right-multiple. We say that $\mathcal{C}$ is *left-cancellative* if $ef = eh$ implies $f = h$ for all $e, f, g \in \mathcal{C}$. All of these notions have obvious analogues with left and right interchanged. A category is *cancellative* if it is left-cancellative and right-cancellative.

Lemma 1.4 *If $\mathcal{C}$ is cancellative and $f \in \mathcal{C}$ has a left-inverse or right-inverse then it is invertible.*

Proof Let $f \in \mathcal{C}(x,y)$ and assume that there is an $e \in \mathcal{C}(y,x)$ that is a left-inverse for f, that is, $ef = 1_y$. Then $fef = f$ and canceling f on the right shows that e is also a right-inverse. The other case is symmetric. □

Lemma 1.5 *Let $\mathcal{C}$ be strongly Noetherian. Then $\mathcal{C}$ has least common right-multiples if and only if it has greatest common left-factors.*

Proof Suppose that $\mathcal{C}$ has least common right-multiples and let $f, g \in \mathcal{C}(x,-)$. Let s and t be common left-factors of f and g and let r be a least common right-multiple of s and t. Then, since f and g are common right-multiples of s and t, they are right-multiples of r, meaning that r is a common left-factor. If s and t are not right-multiples of each other then $\delta(r) > \delta(s), \delta(t)$ and an induction on $\delta(r) \leq \delta(f), \delta(g)$ over the common left-factors of f and g produces a greatest common left-factor. The other direction is analogous. $\square$

We say that $\mathcal{C}$ is right/left-Ore if it is cancellative and has common right/left-multiples.

Theorem 1.6 *A category $\mathcal{C}$ that is right-Ore embeds in a groupoid $\mathcal{G}$ such that every element $h \in \mathcal{G}$ can be written as $h = fg^{-1}$ with $f, g \in \mathcal{C}$.*

The groupoid $\mathcal{G}$ in the theorem is called the *Ore localization* $\mathcal{O}re(\mathcal{C})$ of $\mathcal{C}$. Using the universal property, it is not hard to see that it coincides with the enveloping groupoid of $\mathcal{C}$.

The fundamental group of an Ore category has a particularly easy description. In general, an element of $\pi_1(\mathcal{C}, x)$ is represented by a sequence $f_0 g_1^{-1} f_1 \cdots f_{n-1} g_n^{-1}$ with $f_i, g_i \in \mathcal{C}(x_i, -)$ and $f_j, g_{j+1} \in \mathcal{C}(-, y_j)$. But if $\mathcal{C}$ has common right-multiples, then $g_1^{-1} f_1$ can be rewritten as $f_1' g_1'^{-1}$ and so the sequence can be shortened to $(f_0 f_1')(g_2 g_1')^{-1} f_2 \cdots f_{n-1} g_n^{-1}$. Iterating this argument, we find that every element of $\pi_1(\mathcal{C}, x)$ is of the form fg^{-1} with $f, g \in \mathcal{C}(x, -)$.

1.4 Presentations

We introduce presentations for categories. This is analogous to the situation for monoids and we will be brief. See [22, Section II.1.4] for details.

A (small) *precategory* $\mathcal{S}$ consists of a set of objects $\mathrm{Ob}(\mathcal{S})$ and a set of morphisms $\mathcal{S}$. As for categories, each morphism has a *source* and a *target* that are objects and it is a morphism from the source to its target. The set of morphisms from y to x is denoted by $\mathcal{S}(y, x)$. The monoidal aspects of a category are missing in a precategory: it does not have identities or a composition.

Given a precategory $\mathcal{S}$ there exists a free category $\mathcal{S}^*$ generated by $\mathcal{S}$. It has the universal property that if $\phi\colon \mathcal{S} \to \mathcal{C}$ is a morphism of precategories and $\mathcal{C}$ is a category, then ϕ uniquely factors through $\mathcal{S} \to \mathcal{S}^*$. One can construct $\mathcal{S}^*$ to have the same objects as $\mathcal{S}$ and have morphisms finite words in $\mathcal{S}$ that are composable.

A *relation* is a pair $r = s$ of morphisms in $\mathcal{S}^*$ with the same source and target. If $\phi\colon \mathcal{S}^* \to \mathcal{C}$ is a morphism, the relation *holds* in $\mathcal{C}$ if $\phi(r) = \phi(s)$. A *presentation* consists of a precategory $\mathcal{S}$ and a family of relations $\mathcal{R}$ in $\mathcal{S}^*$. The category it presents is denoted by $\langle \mathcal{S} \mid \mathcal{R} \rangle$.

It has the universal property that if $\phi\colon \mathcal{S} \to \mathcal{C}$ is a morphism of precategories and $\mathcal{C}$ is a category in which all relations in $\mathcal{R}$ hold then ϕ uniquely factors through $\mathcal{S} \to \langle \mathcal{S} \mid \mathcal{R} \rangle$. One can construct $\langle \mathcal{S} \mid \mathcal{R} \rangle$ by quotienting $\mathcal{S}^*$ by the symmetric, transitive closure of the relations.

1.5 Garside families

The following notions are at the core of [22]. We will sometimes be needing the notions with the reverse order. What in [22] is referred to as a Garside family in a left-cancellative category will be called a left-Garside family here to avoid confusion in categories that are left- and right-cancellative.

Let $\mathcal{C}$ be a left-cancellative category and let $\mathcal{S} \subseteq \mathcal{C}$ be a set of morphisms. We denote by $\mathcal{S}^\sharp$ the set $\mathcal{C}^\times \cup \mathcal{S}\mathcal{C}^\times$ of morphisms that are invertible or left-multiples of invertibles by elements of $\mathcal{S}$. We say that $\mathcal{S}^\sharp$ is *closed under (left/right-)factors* if every (left/right-)factor of an element in $\mathcal{S}^\sharp$ is again in $\mathcal{S}^\sharp$. An element $s \in \mathcal{S}$ is an $\mathcal{S}$*–head* of $f \in \mathcal{C}$ if s is a left-factor of f and every left-factor of f in $\mathcal{S}$ is a left-factor of s [22, Definition IV.1.10]. The set $\mathcal{S}$ is a *left-Garside family* if $\mathcal{S}^\sharp$ generates $\mathcal{C}$, is closed under right-factors and every noninvertible element of $\mathcal{C}$ admits and $\mathcal{S}$–head [22, Proposition IV.1.24]. If $\mathcal{S}$ is a left-Garside family then $\mathcal{C}^\times\mathcal{S} \subseteq \mathcal{S}^\sharp$, so in fact $\mathcal{S}^\sharp = \mathcal{C}^\times \cup \mathcal{C}^\times\mathcal{S}\mathcal{C}^\times$ [22, Proposition III.1.39].

All notions readily translate to right-Garside families, except that the head is called an $\mathcal{S}$*–tail* if $\mathcal{S}$ is a right-Garside family. Note that $\mathcal{S}^\sharp$ is defined as $\mathcal{C}^\times \cup \mathcal{C}^\times\mathcal{S}$ when $\mathcal{S}$ is (regarded as) a right-Garside family.

We will be interested in Garside families that are closed under factors. We describe two situations where this is the case.

Let $\mathcal{C}$ be left-cancellative and consider a map $\Delta\colon \mathrm{Ob}(\mathcal{C}) \to \mathcal{C}$ with $\Delta(x) \in \mathcal{C}(x,-)$. We write

$$\mathrm{Div}(\Delta) = \{g \in \mathcal{C} \mid gh = \Delta(x) \text{ for some } x \in \mathrm{Ob}(\mathcal{C}),\ h \in \mathcal{C}\},$$

$$\widetilde{\mathrm{Div}}(\Delta) = \{h \in \mathcal{C} \mid gh = \Delta(x) \text{ for some } x \in \mathrm{Ob}(\mathcal{C}),\ g \in \mathcal{C}\},$$

for the families of left- and right-factors of morphisms in the image of Δ. Such a map is a *right-Garside map* if $\mathrm{Div}(\Delta)$ generates $\mathcal{C}$, if $\widetilde{\mathrm{Div}}(\Delta) \subseteq \mathrm{Div}(\Delta)$, and if, for every $g \in \mathcal{C}(x,-)$, the elements g and $\Delta(x)$ admit a greatest common left-factor. If Δ is a right-Garside map then $\mathrm{Div}(\Delta)$ is a left-Garside family closed under left-factors and thus under factors [22, Proposition V.1.20]. We note the following for future reference:

Observation 1.7 *Let $\mathcal{C}$ be a left-cancellative, factor-finite category and let Δ be a right-Garside map. Then $\mathcal{S} := \operatorname{Div}(\Delta)$ is a left-Garside family closed under factors and $\mathcal{S}(x,-)$ is finite for every $x \in \operatorname{Ob}(\mathcal{C})$.*

Let $\mathcal{C}$ be right-Ore. A right-Garside family is *strong* if for $s,t \in \mathcal{S}^\sharp$ there exist $s',t' \in \mathcal{S}^\sharp$ such that $st' = ts'$ is a least common right-multiple of s and t [22, Definition 2.29]. If $\mathcal{S}$ is a strong right-Garside family then $\mathcal{S}^\sharp$ is also closed under left-factors and thus is closed under factors [22, Proposition 1.35].

2 Fundamental examples

2.1 Thompson's group F and the category $\mathcal{F}$

Our description of Thompson's groups is not the standard one, which can be found in [19]. An element of Thompson's group F is given by a pair (T_+, T_-) of finite rooted binary trees with the same number of leaves, say n. If we add a caret to the i^{th} leaf $(1 \le i \le n)$ of T_+, that is we make it into an inner vertex with two leaves below it, we obtain a tree T'_+ on $n+1$ vertices. If we also add a caret to the i^{th} leaf of T_- we obtain another tree T'_-. We want to regard (T'_+, T'_-) as equivalent to (T_+, T_-) so we take the reflexive, symmetric, transitive closure of the operation just described and write the equivalence class by $[T_+, T_-]$. Thompson's group F is the set of equivalence classes $[T_+, T_-]$.

In order to define the product of two elements $[T_+, T_-]$ and $[S_+, S_-]$, we note that we can add carets to both tree pairs to get representatives $[T'_+, T'] = [T_+, T_-]$ and $[T', T'_-] = [S_+, S_-]$, where the second tree of the first element and the first tree of the second element are the same. Therefore, multiplication is completely defined by declaring that $[T'_+, T'] \cdot [T', T'_-] = [T'_+, T'_-]$. It is easy to see that $[T,T]$ is the neutral element for any tree T and that $[T_+, T_-]^{-1} = [T_-, T_+]$.

We have defined the group F in such a way that a categorical description imposes itself; see [2]. We define $\mathcal{F}$ to be the category whose objects are positive natural numbers and whose morphisms $m \leftarrow n$ are binary forests on m roots with n leaves. Multiplication of a forest $E \in \mathcal{F}(\ell, m)$ and a forest $F \in \mathcal{F}(m,n)$ is defined by identifying the leaves of E with the roots of F and taking EF to be the resulting tree. Pictorially this corresponds to stacking the two forests on top of each other (see Figure 1).

Proposition 2.1 *The category $\mathcal{F}$ is strongly Noetherian and right-Ore. In fact, it has least common right-multiples and greatest common left-factors.*

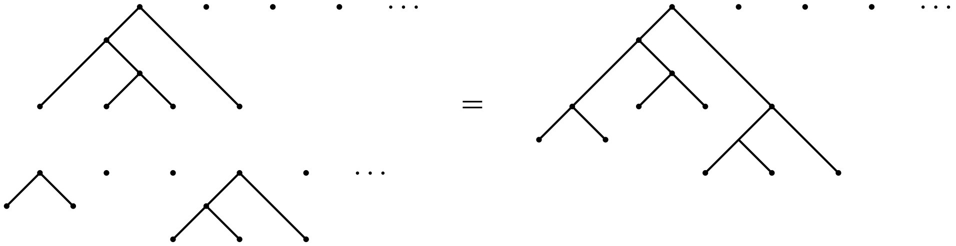


Figure 1: Multiplication of forests (taken from [42])

Proof The identity map $\rho\colon \mathbb{N} = \mathrm{Ob}(\mathcal{F}) \to \mathbb{N}$ is a height function on $\mathcal{F}$. Thus, $\mathcal{F}$ is strongly Noetherian.

The least common right-multiple of two forests in $\mathcal{F}(m,-)$ is their union (regarding both forests as subforests of the leafless binary forest on m roots). The greatest common left-factor is their intersection. Left-cancellativity means that given a forest $f \in \mathcal{F}(m,\ell)$ and a left-factor $a \in \mathcal{F}(m,n)$, the forest in $b \in \mathcal{F}(n,\ell)$ with $f = ab$ is unique. Indeed, it is the forest obtained from f by removing a and turning the leaves of a into roots. Right-cancellativity means that a is uniquely determined if $f = ab$. To see this, we identify the leaves of f with the leaves of b. Now the common predecessor in f of a set of leaves of a tree of b is a leaf of a and every leaf of a arises in that way. □

The proposition together with the remark at the end of Section 1.3 shows that every element of $\pi_1(\mathcal{F},1)$ is represented by fg^{-1} where $f, g \in \mathcal{F}(1,-)$ are binary trees. Cancellativity ensures that $fg^{-1} = f'g'^{-1}$ if and only if there exist h and h' such that $fh = f'h'$ and $gh = g'h'$. Comparing this description with our definition of F we see:

Proposition 2.2 *Thompson's group F is isomorphic to $\pi_1(\mathcal{F},1)$.*

Later on it will be convenient to have a presentation for $\mathcal{F}$. The shape of the relations will not come as a surprise to the reader familiar with Thompson's groups. A proof can be found in [40].

Proposition 2.3 *The category $\mathcal{F}$ has a presentation with morphisms $\lambda_i^n\colon n \leftarrow n+1$ for $1 \le i \le n$ as generators subject to the relations*

$$\lambda_i^n \lambda_j^{n+1} = \lambda_j^n \lambda_{i+1}^{n+1} \quad \text{for } 1 \le j < i \le n. \tag{2-1}$$

Every morphism in $\mathcal{F}(m,n)$ can be written in a unique way as $\lambda_{i_m}^m \cdots \lambda_{i_{n-1}}^{n-1}$ with $(i_j)_j$ nondecreasing.

Remark 2.4 The relations (2-1) reflect a commutation phenomenon: for any forest, adding a caret to the i^{th} leaf and then to the j^{th} leaf has the same effect as doing it the other way around. That it does not algebraically look like a commutation relation is due to the fact the index of the right one of the two leaves has changed when adding the left caret. This is inevitable in the present setup because the i^{th} leaf has no identity as a particular vertex in the infinite rooted binary tree but simultaneously represents all i^{th} leaves of trees with n leaves. A larger category in which the relations are algebraically commutation relations will appear in Section 5.5.

Note that since $\mathcal{F}$ is connected, the fundamental groups at different objects are isomorphic. This corresponds to the elementary fact that the tree pair (T_+, T_-) representing an element of F can always be chosen so that T_+ and T_- contain an arbitrary fixed subtree.

The most convenient way to exhibit a Garside family in $\mathcal{F}$ is by describing a right-Garside map: for every $n \in \mathbb{N} = \mathrm{Ob}(\mathcal{F})$ let $\Delta(n)$ be the forest where every tree is a single caret.

Proposition 2.5 *The map* $\Delta\colon \mathrm{Ob}(\mathcal{F}) \to \mathcal{F}$ *is a right-Garside map.*

Proof The family $\mathrm{Div}(\Delta)$ consists of morphisms where every forest is either a single caret or trivial. Every forest can be built of from these, for example by adding one caret at a time. This shows that $\mathrm{Div}(\Delta)$ generates $\mathcal{F}$. The family $\widetilde{\mathrm{Div}}(\Delta)$ also consists of morphisms where every forest is either a single caret or trivial with the additional condition that the total number of leaves is even and the left leaf of every caret has an odd index. In particular, $\widetilde{\mathrm{Div}}(\Delta) \subseteq \mathrm{Div}(\Delta)$. If $g \in \mathcal{F}(x, -)$ then g and $\Delta(x)$ have a greatest common left-factor by Proposition 2.1. □

With Observation 1.7 we get:

Corollary 2.6 *The category* $\mathcal{F}$ *admits a left-Garside family* $\mathcal{S}$ *that is closed under factors such that* $\mathcal{S}(x, -)$ *is finite for every* $x \in \mathcal{F}$.

Remark 2.7 The family $\mathrm{Div}(\Delta)$ is in fact a right- as well as a left-Garside family. It is strong as a right-Garside family but not as a left-Garside family.

If instead of rooted binary trees one takes rooted n–ary trees ($n \geq 2$) in the description above, one obtains the category $\mathcal{F}_n$. Everything is analogous to $\mathcal{F}$ but the new aspect that occurs for $n > 2$ is that the category is no longer connected: the number of leaves of an n–ary tree with r roots will necessarily be congruent to r modulo $n-1$; hence, there is no morphism in $\mathcal{F}_n$ connecting objects that are not congruent modulo $n-1$. As a consequence, the point at which the fundamental group is taken does matter and we obtain $n-1$ different groups for each category. It turns out, however, that the fundamental groups are in fact isomorphic independently of the basepoint [16, Proposition 4.1] and are denoted by

$$F_{n,\infty} = \pi_1(\mathcal{F}_n, 1).$$

The groups $F_{n,\infty}$ are the smallest examples of the *Higman–Thompson groups* introduced by Higman [27]. As we will see later, the fundamental groups of the different components are nonisomorphic in the categories for the larger Higman–Thompson groups.

2.2 Braid groups

The *braid group* on n strands, introduced by Artin [1], is the group given by the presentation

$$\text{BRAID}_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ if } 1 \leq i \leq n-2 \rangle. \tag{2-2}$$

Its elements, called *braids*, can be conveniently depicted as braid diagrams as in Figure 2, illustrating a physical interpretation as braids on n strands. The first relations are *commutation relations*; the second are *braid relations*. The group BRAID_n arise as the fundamental group of the configuration space of n unordered points in the disc and as the mapping class group of the n–punctured disc; see [7; 29] for more details.

What is known as Garside theory today arose out of Garside's study of braid groups [25]. In this classical case, the category $\mathcal{C}$ has a single object and thus is a monoid. Specifically, a *Garside monoid* is a monoid M with an element $\Delta \in M$, called a *Garside element*, such that:

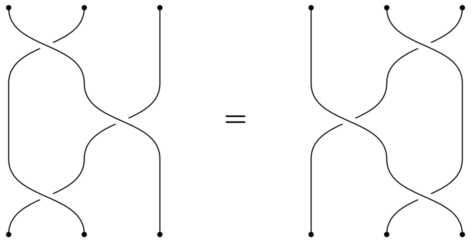


Figure 2: Diagrams illustrating the braid relation

(i) M is cancellative and has least common right- and left-multiples and greatest common right- and left-factors.

(ii) The left- and right-factors of Δ coincide, they are finite in number, and they generate M.

(iii) There is a map $\delta\colon M \to \mathbb{N}$ such that $\delta(fg) \geq \delta(f) + \delta(g)$ and $\delta(g) > 0$ if $g \neq 1$.

A *Garside group* is the group of fractions of a Garside monoid. Among the main features of Garside groups is that they have solvable word problem and conjugacy problem.

Note that a Garside monoid, regarded as a category with one object is, by definition, left- and right-Ore and strongly Noetherian. Moreover, the family of factors of Δ is a left- and right-Garside family.

To see that braid groups are in fact Garside groups, consider the *braid monoid* BRAID_n^+. It is obtained by interpreting the presentation (2-2) as a monoid presentation. It is a nontrivial consequence of Garside's work that the obvious map $\mathrm{BRAID}_n^+ \to \mathrm{BRAID}_n$ is injective, so that the braid monoid can be regarded as a subset of the braid groups. Its elements are called *positive braids* and are characterized by the property that left strands always overcrosses the right strand. The element Δ in BRAID_n^+ is the braid that performs a full half twist and is characterized by the fact that every strand crosses every other strand precisely once; see Figure 3. Its (left- or right-) factors are the braids where every strand crosses every other strand at most once. The function δ is simply the number of crossings, which is the same as length as a word in the generators. Now BRAID_n^+ is a Garside monoid with Garside element Δ; see [22, Section I.1.2, Proposition IX.1.29]. Its group of fractions is BRAID_n, which is therefore a Garside group.

It was noted by Birman, Ko and Lee [8] that there is in fact another monoid BRAID_n^{*+}, called the *dual braid monoid*, that also admits a Garside element Δ^* and has BRAID_n as its group of fractions; see also [22, Section I.1.3]. This monoid is in many ways better behaved than BRAID_n^+. Brady [10] used the dual braid monoid to construct a finite classifying space for the braid group.

Note that adding the relations σ_i^2 to the presentation (2-2) results in a presentation for the symmetric group SYM_n. In particular, there is a surjective homomorphism $\pi\colon \mathrm{BRAID}_n \to \mathrm{SYM}_n$ that takes σ_i to the transposition $s_i := (i\ i+1)$.

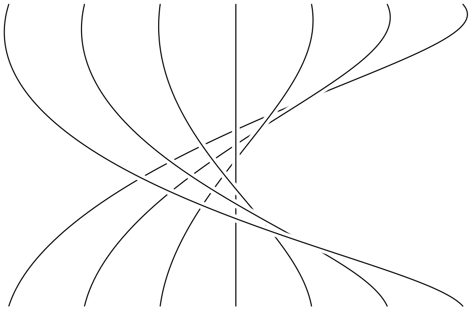


Figure 3: The element Δ in BRAID_7^+

The symmetric group is a finite Coxeter group and the braid group is its corresponding *Artin group*. For every Coxeter system (W, S) there exists an Artin group A_W obtained analogously and a morphism $\pi\colon A_W \to W$. Whenever W is finite, the Artin group A_W again contains a Garside monoid and a dual Garside monoid; see [12; 5].

3 Finiteness properties of fundamental groups of Ore categories

A classifying space for a group G is a CW complex B whose fundamental group is G and whose universal cover $X = \widetilde{B}$ is contractible. Since G acts freely on X with quotient $B = G\backslash X$, one can equivalently say that a classifying space is the quotient of a contractible CW complex by a free G–action. Our goal in this section is to construct "good" classifying spaces for fundamental groups of Ore categories. The best classifying spaces are compact ones; they have finitely many cells so we also refer to them as *finite*. If G admits a finite classifying space, we say that it is of type F. If a finite classifying space does not exist, we aim at classifying spaces with weaker finiteness properties. We start by constructing an action on a contractible space.

3.1 Contractible spaces from Ore categories with Garside families

Let $\mathcal{C}$ be a category that is right-Ore and strongly Noetherian. Let $\mathcal{S}$ be a left- or right-Garside family such that $\mathcal{S}^\sharp$ is closed under factors. Let $* \in \mathrm{Ob}(\mathcal{C})$ be a base object. Our goal is to construct a contractible space X on which $\pi_1(\mathcal{C}, *)$ acts with good finiteness properties of the stabilizers as well as the quotient. In the whole discussion $\mathcal{C}$ can be replaced by the component of $*$ in $\mathcal{C}$, so all assumptions only need to be made for objects and morphisms in that component.

We put $\mathcal{E} = \mathcal{S}^\sharp$ and recall that $\mathcal{E} = \mathcal{C}^\times \cup \mathcal{C}^\times \mathcal{S} \mathcal{C}^\times$. We call the elements of $\mathcal{E}$ *elementary*. Let $\delta\colon \mathcal{C} \to \mathbb{N}$ be a map that witnesses strong Noetherianity. Note that if $f \in \mathcal{C}(x, y)$ and $g \in \mathcal{C}^\times(-, x)$ and $h \in \mathcal{C}^\times(y, -)$ are invertible then

$$-\delta(g^{-1}) + \delta(f) - \delta(h^{-1}) \geq \delta(gfh) \geq \delta(g) + \delta(f) + \delta(h),$$

so $\delta(f) = \delta(gfh)$ and δ is invariant under pre- and postcomposition by invertibles.

We define the set $P = \mathcal{O}re(\mathcal{C})(*, -)/\mathcal{C}^\times$, that is, elements of P are equivalence classes $\overline{a}$ of elements $a \in \mathcal{O}re(\mathcal{C})(*, -)$ modulo the equivalence relation that $\overline{a} = \overline{a}'$ if there exists a $g \in \mathcal{C}^\times$ with $ag = a'$. We define a relation $\leq$ on P by declaring $\overline{a} \leq \overline{b}$ if there exists an $f \in \mathcal{C}$ with $af = b$.

Lemma 3.1 *The relation $\leq$ is a partial order on P in which any two elements have a common upper bound. In particular, the realization $|P|$ is contractible.*

Proof Note first that whether $f = a^{-1}b$ lies in $\mathcal{C}$ is independent of the representatives. Reflexivity and transitivity are clear. If $\overline{a} \leq \overline{b} \leq \overline{a}$ then there exist $f, h \in \mathcal{C}$ and $g \in \mathcal{C}^\times$ such that $af = b$ and $bh = ag$, showing that fh is a unit. In particular, f has a right-inverse and h has a left-inverse, so f and h are units by Lemma 1.4. This shows $\overline{a} = \overline{b}$.

For any $a \in \mathcal{O}re(\mathcal{C})$ there is an $f \in \mathcal{C}$ such that $af \in \mathcal{C}$. Since $\mathcal{C}$ has common right-multiples, it follows that for any two elements $a_1, a_2 \in \mathcal{O}re(\mathcal{C})$ there exist $f_1, f_2 \in \mathcal{C}$ with $a_1 f_1 = a_2 f_2$. □

We define a second, more restrictive relation $\preceq$ on P by declaring that $\overline{a} \preceq \overline{b}$ if there exists an $e \in \mathcal{E}$ with $ae = b$. Note that this relation will typically not be transitive. However, if $\overline{a} \preceq \overline{b}$ and $\overline{a} \leq \overline{c} \leq \overline{b}$ then $\overline{a} \preceq \overline{c} \preceq \overline{b}$ because $\mathcal{E}$ is closed under factors. The complex $X \subseteq |P|$ consists of those chains in $|P|$ that are chains with respect to $\preceq$. In particular, P is the vertex set of X.

Proposition 3.2 *The complex X is contractible.*

Proof Note that X is a subspace of $|P|$ containing all the vertices. One can obtain $|P|$ from X by gluing in (realizations of) intervals $[\overline{a}, \overline{b}]$ not yet contained in X. To organize the gluing, note the following: if $[\overline{c}, \overline{d}]$ is a proper subinterval of $[\overline{a}, \overline{b}]$ with $f = a^{-1}b \in \mathcal{C}$ and $h = c^{-1}d \in \mathcal{C}$, then h is a proper factor of f. To an interval $[\overline{a}, \overline{b}]$ with $f = a^{-1}b$ we assign the height $\hat{\delta}([\overline{a}, \overline{b}]) = \delta(f)$. Note that this is well defined, because any other representative f' will differ from f only by invertibles and δ is invariant under pre- and postcomposition by invertibles. Note also that proper subintervals have strictly smaller $\widehat{\delta}$–value. We can therefore glue in the intervals with increasing value of $\widehat{\delta}$ and be sure that when we glue in an interval, any proper subinterval is already glued in.

For any $n \in \mathbb{N}$ let $|P|_{\widehat{\delta}<n}$ be the subcomplex of $|P|$ consisting of X and intervals of $\widehat{\delta}$–value $< n$. If X was not contractible, there would be a sphere in X that could not be contracted in X but in $|P|$. The contraction would be compactly supported, and hence use simplices supported on finitely many simplices. It therefore suffices to show that the inclusion $X \to |P|_{\widehat{\delta}<n}$ is a homotopy equivalence for all $n \in \mathbb{N}$.

For $n = 0$ this is clear, so assume $n > 0$. Then

$$|P|_{\widehat{\delta}<n} = |P|_{\widehat{\delta}<n-1} \cup \bigcup_{\widehat{\delta}([\overline{a},\overline{b}])=n-1} |[\overline{a},\overline{b}]|.$$

The intervals that are glued in meet only in $|P|_{\widehat{\delta}<n-1}$ and they are glued in along $|[\overline{a},\overline{b})| \cup |(\overline{a},\overline{b}]|$. This is a suspension of $|(\overline{a},\overline{b})|$ and so it suffices to show that the open interval is contractible.

If $\mathcal{S}$ is a left-Garside family, every element h of $\mathcal{C}$, and every left-factor of f in particular, has an $\mathcal{S}$–head $\mathrm{head}(g)$. We define the map $\theta\colon [\overline{a},\overline{b}] \to [\overline{a},\overline{b}]$ by $\overline{ah} \mapsto \overline{a\,\mathrm{head}(h)}$. Note that $\theta(\overline{b}) < \overline{b}$ because otherwise $[\overline{a},\overline{b}]$ is already contained in $|P|$. Note also that $\theta(\overline{c}) > \overline{a}$ for $\overline{c} > \overline{a}$ because the head of a noninvertible is not invertible. This shows that θ restricts to a map $(\overline{a},\overline{b}) \to (\overline{a},\overline{b})$ with $\overline{c} \geq \theta(\overline{c}) \leq \theta(\overline{b})$ and we can apply [34, Section 1.5] to see that $|(\overline{a},\overline{b})|$ is contractible.

If $\mathcal{S}$ is a right-Garside family, θ is defined by $\overline{bh^{-1}} \mapsto \overline{b\,\mathrm{tail}(h)^{-1}}$. For the same reasons as above, θ restricts to a map $(\overline{a},\overline{b}) \to (\overline{a},\overline{b})$ with $\overline{c} \leq \theta(\overline{c}) \geq \theta(\overline{a})$ and we can again apply [34, Section 1.5]. □

There is an obvious action of $\pi_1(\mathcal{C},*)$ on X which is given by precomposition: if $g \in \pi_1(\mathcal{C},*) = \mathcal{O}re(\mathcal{C})(*,*)$ and $a \in \mathcal{O}re(\mathcal{C})(*,-)$ then $g\overline{a} = \overline{ga}$ and the relations $\leq$ and $\preceq$ are clearly preserved under this action.

Next we want to look at stabilizers and weak fundamental domains. These will be particularly well behaved with an additional assumption. We say that $\mathcal{S}$ is *(right-)locally finite* if for every object $x \in \mathrm{Ob}(\mathcal{C})$ the set $\mathcal{S}(x,-)$ is finite up to pre- and postcomposition by invertibles. Local finiteness of $\mathcal{S}$ does *not* imply that X is locally finite but does imply:

Observation 3.3 *Assume that $\mathcal{S}$ is locally finite. For every $\overline{a} \in P$ there are only finitely many $\overline{b} \in P$ with $\overline{a} \preceq \overline{b}$. In particular, there are only finitely many simplices for which $\overline{a}$ is $\preceq$–minimal.*

Lemma 3.4 *Every simplex-stabilizer of the action of $\pi_1(\mathcal{C}, *)$ on X is isomorphic to a subgroup of $\mathcal{C}^\times(x,x)$ for some $x \in \mathrm{Ob}(\mathcal{C})$. If $\mathcal{S}$ is locally finite, the subgroup has finite index.*

Proof Let $\overline{a}$ be a vertex in X with $a \in \mathcal{O}re(\mathcal{C})(*,x)$ and suppose that $g \in \pi_1(\mathcal{C}, *)$ fixes $\overline{a}$, that is, $\overline{a} = g\overline{a} = \overline{ga}$. Then $a^{-1}ga \in \mathcal{C}^\times(x,x)$. This shows that the stabilizer of $\overline{a}$ is conjugate to $\mathcal{C}^\times(x,x)$. If $\mathcal{S}$ is locally finite then Observation 3.3 implies that the stabilizer of an arbitrary simplex has finite index in a vertex stabilizer. □

Corollary 3.5 *If $\mathcal{C}^\times(x,x) = \{1_x\}$ for every object $x \in \mathrm{Ob}(\mathcal{C})$ then the action of $\pi_1(\mathcal{C}, *)$ on X is free. If $\mathcal{C}^\times(x,x)$ is finite then the action is proper.*

Now let us pick, for every $x \in \mathrm{Ob}(\mathcal{C})$, a morphism $f_x \in \mathcal{O}re(\mathcal{C})(*,x)$ arbitrarily and let $K_x \subseteq X$ be the union of the realizations of the intervals $[\overline{f_x}, \overline{f_x e}]$ with $e \in \mathcal{E}(x,-)$.

Lemma 3.6 *The complex X is covered by the $\pi_1(\mathcal{C}, *)$–translates of the complexes K_x for $x \in \mathrm{Ob}(\mathcal{C})$. If $\mathcal{S}$ is locally finite then each K_x is compact.*

Proof If $\sigma = \{f \prec fe_1 \prec \cdots \prec fe_k\}$ is a simplex in X with $f \in \mathcal{O}re(\mathcal{C})(*,x)$ and $e_1, \dots, e_k \in \mathcal{E}(x,-)$, then $f_x f^{-1} \in \pi_1(\mathcal{C}, *)$ and $f_x f^{-1} K_x$ contains σ. The second statement is clear. □

The ideal special case is:

Corollary 3.7 *If $\mathcal{C}$ has no nonidentity invertible morphisms and has only finitely many objects and if $\mathcal{S}$ is locally finite, then $\pi_1(\mathcal{C}, *)$ has a finite classifying space.*

Proof Under the assumption, the action of $\pi_1(\mathcal{C}, *)$ is free by Corollary 3.5 and cocompact by Lemma 3.6. The quotient is then a finite classifying space. □

In particular, we recover the main result of [20]:

Corollary 3.8 *Every Garside group G has a finite classifying space.*

In the case of the dual braid monoid, the complex we constructed is precisely the *dual Garside complex* constructed by Brady [10].

3.2 Finiteness properties

Topological finiteness properties of a group G were introduced by Wall [38; 39] and are conditions on how finite a classifying space for G can be chosen. A group is said to be *of type* F_n if it admits a classifying space B whose n–skeleton $B^{(n)}$ has finitely many cells. Equivalently a group is of type F_n if it acts freely on a contractible space X such that the action on $X^{(n)}$ is cocompact. It is clear that type F_n implies type F_m for $m < n$ and one defines the finiteness length $\phi(G)$ to be the supremal n for which G is of type F_n. If $\phi(G) = \infty$ then G is said to be of type F_∞.

In low dimensions, these properties have familiar descriptions: a group is of type F_1 if and only if it is finitely generated, and it is of type F_2 if and only if it is finitely presented.

Given a group G, in order to study its finiteness properties, one needs to let G act on a highly connected space X. If the action is free, then the low-dimensional skeleta of $G\backslash X$ are those of a classifying space. A useful result is Brown's criterion, which says that one does not have to look at free actions; see [16, Propositions 1.1, 3.1]:

Theorem 3.9 *Let G act cocompactly on an $(n-1)$–connected CW complex X. If the stabilizer of every p–cell of X is of type F_{n-p} then G is of type F_n.*

The full version of Brown's criterion also gives a way to decide that a group is not of type F_n. We formulate it here only to explain why we will not be able to apply it:

Theorem 3.10 *Let G act on an $(n-1)$–connected CW complex X and assume that the stabilizer of every p–cell of X is of type F_{n-p}. If G is of type F_n then, for every cocompact subspace Y and any basepoint $* \in Y$, there exists a cocompact subspace $Z \supseteq Y$ such that the maps $\pi_k(Y,*) \to \pi_k(Z,*)$ induced by inclusion have trivial image for $k \leq n-1$.*

Theorem 3.10 can be used to show that a group is not of type F_n if this is visible in the topology of X. On the other hand, if the stabilizers have bad finiteness properties, we cannot decide whether G has good finiteness properties or not: in that case we are looking at the wrong action.

3.3 Combinatorial Morse theory

In order to study connectivity properties of spaces and apply Brown's criterion we will be using combinatorial Morse theory as introduced by Bestvina and Brady [6]. Here we give the most basic version used in Section 3.4.

Let X be the realization of an abstract simplicial complex, regarded as a CW complex. A *Morse function* is a function $\rho\colon X^{(0)} \to \mathbb{N}$ with the property that $\rho(v) \neq \rho(w)$ if v is adjacent to w. For $n \in \mathbb{N}$ the sublevel set $X_{\rho<n}$ is defined to be the full subcomplex of X supported on vertices v with $\rho(v) < n$. The *descending link* $\operatorname{lk}^{\downarrow} v$ of a vertex v is the full subcomplex of $\operatorname{lk} v$ of those vertices w with $\rho(w) \leq \rho(v)$ and the *descending star* $\operatorname{st}^{\downarrow}$ is defined analogously. That ρ is a Morse function implies that the inequality $\rho(w) \leq \rho(v)$ is strict for the descending link and for the descending star is not strict only when $w = v$. In particular, the descending star is the cone over the descending link.

The goal of combinatorial Morse theory is to compare the connectivity properties of sublevel sets to each other and to those of X. The tool to do so is a basic lemma, called the Morse lemma:

Lemma 3.11 *Let ρ be a Morse function on X. Let $m \leq n \leq \infty$ and assume that for every vertex v with $m \leq \rho(v) < n$ the descending link of v is $(d-1)$–connected. Then the pair $(X_{\rho<n}, X_{\rho<m})$ is d–connected, that is, $\pi_k(X_{\rho<m} \to X_{\rho<n})$ is an isomorphism for $k < d$ and an epimorphism for $k = d$.*

Proof The basic observations are that

$$X_{\rho<m+1} = X_{\rho<m} \cup \bigcup_{\rho(v)=m} \operatorname{st}^{\downarrow} v,$$

that $\operatorname{st}^{\downarrow} v \cap \operatorname{st}^{\downarrow} v' \subseteq X_{\rho<m}$ for $\rho(v) = m = \rho(v')$, and that $\operatorname{st}^{\downarrow} v \cap X_{\rho<m} = \operatorname{lk}^{\downarrow} v$. As a consequence (using compactness of spheres) it suffices to study the extension $Y := X_{\rho<m} \cup_{\operatorname{lk}^{\downarrow} v} \operatorname{st}^{\downarrow} v$ for an individual vertex v with $\rho(v) = m$.

In this situation, $\pi_k(Y, X_{\rho<m}) \cong \pi_k(\operatorname{st}^{\downarrow} v, \operatorname{lk}^{\downarrow} v)$ for $k \leq d$. This can be seen by separately looking at π_1 and H_* (where excision holds) and applying Hurrewicz's theorem [26, Theorem 4.37]. The statement now follows from the long exact homotopy/homology sequence for the pair $(\operatorname{st}^{\downarrow} v, \operatorname{lk}^{\downarrow} v)$. □

3.4 Finiteness properties of fundamental groups of Ore categories

We take up the construction from Section 3.1. So $\mathcal{C}$ is again a right-Ore category, $\mathcal{S}$ is a left- or right-Garside family closed under factors, and $* \in \operatorname{Ob}(\mathcal{C})$ is a base object. More than requiring strong Noetherianity, we now need a height function $\rho\colon \operatorname{Ob}(\mathcal{C}) \to \mathbb{N}$.

We use these data and assumptions to provide a criterion to prove finiteness properties for the fundamental group.

We need to introduce one further space construction. It is another variant of the nerve construction. For $x \in \mathrm{Ob}(\mathcal{C})$ let $E(x)$ be the set of equivalence classes in $a \in \mathcal{E}(-,x) \smallsetminus \mathcal{E}^\times(x,x)$ modulo the equivalence relation that $\overline{a} = \overline{a}'$ if there exists a $g \in \mathcal{C}^\times$ with $ga = a'$. We define a relation $\le$ on $E(x)$ by declaring $\overline{a} \le \overline{b}$ if there is an $f \in \mathcal{C}$ with $fa = b$. Note that if g and f as above exist, they lie in $\mathcal{E}$, so the description can be formulated purely in terms of $\mathcal{E}$. As in Lemma 3.1 one sees that $\le$ is a partial order on $E(x)$; however, it is usually not contractible.

Theorem 3.12 *Let $\mathcal{C}$ be a right-Ore category and let $* \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{S}$ be a locally finite left- or right-Garside family that is closed under factors. Let $\rho\colon \mathrm{Ob}(\mathcal{C}) \to \mathbb{N}$ be a height function such that $\{x \in \mathrm{Ob}(\mathcal{C}) \mid \rho(x) \le n\}$ is finite for every $n \in \mathbb{N}$. Assume*

- (STAB) *$\mathcal{C}^\times(x,x)$ is of type F_n for all x,*
- (LK) *there exists an $N \in \mathbb{N}$ such that $|E(x)|$ is $(n-1)$–connected for all x with $\rho(x) \ge N$.*

(If ρ is unbounded on the component of $*$ then it suffices if (STAB) holds for every x with $\rho(x)$ beyond a fixed bound.)

Then $\pi_1(\mathcal{C},*)$ is of type F_n.

Remark 3.13 Recall that $\mathcal{C}$ can be replaced by the component of $*$ in $\mathcal{C}$, so all assumptions need to be made only for that component.

Proof We take X to be the complex constructed in Section 3. Assume first that (STAB) holds for all $x \in \mathrm{Ob}(\mathcal{C})$.

For a vertex $\overline{a} \in X$ with $a \in \mathcal{O}re(\mathcal{C})(*,x)$ we define $\rho(\overline{a}) = \rho(x)$. This is a $\pi_1(\mathcal{C},*)$–invariant Morse function, which we think of as height. For $n \in \mathbb{N}$ we consider the subcomplex $X_{\rho<n}$ supported on vertices of height $< n$.

We want to see that every $X_{\rho<n}$ is $\pi_1(\mathcal{C},*)$–cocompact. To do so we note that $\pi_1(\mathcal{C},*)$ acts transitively on vertices $\overline{a}$ with $a \in \mathcal{O}re(\mathcal{C})(*,x)$: indeed, if $\overline{b}$ is another such then $ba^{-1} \in \pi_1(\mathcal{C},*)$ takes $\overline{a}$ to $\overline{b}$. It follows from the assumption on ρ that there are only finitely many vertices $\overline{a}$ with $\rho(\overline{a}) < n$ up to the $\pi_1(\mathcal{C},*)$–action. Cocompactness now follows from Observation 3.3.

Stabilizers are of type F_n by Lemma 3.4 because finiteness properties are inherited by finite-index subgroups.

Let N be large enough that all the $x \in \mathrm{Ob}(\mathcal{C})$ for which the nerve of $|E(x)|$ is not $(n-1)$–connected have $\rho(x) < N$. We have just seen that $\pi_1(\mathcal{C},*)$ acts on $X_{\rho<N}$ cocompactly with stabilizers of type F_n, so once we show that $X_{\rho<N}$ is $(n-1)$–connected,

we are done by Theorem 3.9. We want to apply the Morse lemma (Lemma 3.11), so let us look at the descending link of a vertex $\overline{b}$ of X, where $b \in \mathcal{C}(*, x)$. The vertices in the descending link are the $\overline{a}$ that are comparable with $\overline{b}$ and have $\rho(\overline{a}) < \rho(\overline{b})$. The condition on the height shows that a cannot be a right-multiple of b but has to be a left-factor. Thus, $a^{-1}b \in \mathcal{E}(-, x)$ and the descending link of $\overline{b}$ is the realization of $\{\overline{a} \mid a \prec b\}$. We see that the map $\mathcal{E}(-, x) \setminus \mathcal{E}(x, x) \to \{\overline{a} \mid a \prec b\}$ that takes f to $\overline{af^{-1}}$ is an order-reversing surjection. The definition of $E(x)$ is made so that the induced map $E(x) \to \{\overline{a} \mid a \prec b\}$ is well defined and an order-reversing bijection. Since $|E(x)|$ is $(n-1)$–connected by assumption, this completes the proof in the case that (STAB) holds for all x.

If (STAB) only holds for x with $\rho(x) \geq M$, let $*'$ be in the component of $*$ satisfying $\rho(*') > M$. Since $\mathcal{C}$ is Ore, one sees that

$$\pi_1(\mathcal{C}, *) = \pi_1(\mathcal{C}, *') = \pi_1(\mathcal{C}_{\rho \geq M}, x_0),$$

where $\mathcal{C}_{\rho \geq M}$ is obtained from $\mathcal{C}$ by removing objects y with $\rho(y) < M$. Moreover, local finiteness of $\mathcal{S}$ implies that the complexes $E(y)$ for $\mathcal{C}$ and for $\mathcal{C}_{\rho \geq r}$ are the same for y in the component of $*'$ once $\rho(y)$ is large enough. One can therefore consider $\mathcal{C}_{\rho \geq M}$ instead of $\mathcal{C}$, with the effect that the groups $\mathcal{C}^{\times}(x, x)$ only need to be of type F_n when $\rho(x) \geq M$. □

Corollary 3.14 *Let $\mathcal{C}$, $\mathcal{S}$, ρ and $*$ be as in the theorem. If $\mathcal{C}^{\times}(x, x)$ is of type F_∞ for every x and the connectivity of $|E(x)|$ tends to infinity for $\rho(x) \to \infty$, then $\pi_1(\mathcal{C}, *)$ is of type F_∞.*

The construction of X uses two important ideas. One is the passage from $|P|$ to X, which is due to Stein; see [36, Theorem 1.5]. The other is to take P to consist of $\mathcal{C}^{\times}$–equivalence classes and goes back to [18]. Apart from these ideas the main difficulty in proving that $\pi_1(\mathcal{C}, *)$ is of type F_n lies in establishing the connectivity properties of the complexes $|E(x)|$. This problem depends individually on the concrete setup and we will see various examples later.

3.5 Example: F is of type F_∞

As a first illustration of the results in this section we reprove a result due to Brown and Geoghegan [17]:

Proposition 3.15 *Thompson's group F is of type F_∞.*

We have seen in Proposition 2.1 that $\mathcal{F}$ is right-Ore and admits a height function and by Corollary 2.6 it has a locally finite left-Garside family that is closed under factors. Moreover, $\mathcal{F}^{\times}(x,x) = \{1_x\}$ for every x, so (STAB) is satisfied as well. It only remains to verify (LK). Although things are not always as easy, we remark that this is the typical situation: property (LK) is where one actually needs to show something.

To understand the complexes $|E(n)|$ we first need to unravel the definition. Recall that a *matching* of a graph Γ is a set of edges $M \subseteq E(\Gamma)$ that are pairwise disjoint. Matchings are ordered by containment and we denote the poset of matchings by $\mathcal{M}(\Gamma)$. In fact, since every subset of a matching is again a matching, $\mathcal{M}(\Gamma)$ is (the face poset of) a simplicial complex, the *matching complex*. We denote by L_n the *linear graph* on n vertices $\{1,\dots,n\}$, so its edges are $\{i,i+1\}$ for $1 \le i < n$.

Lemma 3.16 *The poset $E_{\mathcal{F}}(n)$ is isomorphic to $\mathcal{M}(L_n)$.*

Proof Let $f \in \mathcal{E}_{\mathcal{F}}(-,n)$, so f is an element of $E_{\mathcal{F}}(n)$. We identify the roots of f with the vertices of the linear graph L_n on the vertices $\{1,\dots,n\}$. Every caret of f connects two of these roots and thus corresponds to an edge of L_n. All these edges are disjoint, so the resulting subgraph M_f of L_n is a matching. It is clear that, conversely, every matching of L_n arises in a unique way from an elementary forest.

If $h \le f$ then h is a left-multiple of f, that is, f can be obtained from h by adding carets to some roots of h that do not have carets yet. On the level of graphs this means that M_f is obtained from M_h by adding edges so that $M_h \le M_f$ in the poset of matchings. □

Remark 3.17 In particular, $E_{\mathcal{F}}(n)$ is (the face poset of) a simplicial complex. The realization as a poset is the barycentric subdivision of the realization as a simplicial complex, and in particular both are homeomorphic. So there is no harm in working with the coarser cell structure where elements of $E_{\mathcal{F}}(n)$ are simplices rather than vertices. This fact applies in most of our cases.

Matching complexes of various graphs have been studied intensely and their connectivity properties can be verified in various ways [9]. In fact, for linear and cyclic graphs the precise homotopy type is known [30, Proposition 11.16].

Rather than using the known optimal connectivity bounds we use the opportunity to introduce a criterion due to Belk and Forrest [3, Theorem 4.9] that is particularly well suited to verifying that the connectivity of the spaces $E(x)$ tends to infinity in easier cases. We need to introduce some notation.

An abstract simplicial complex X is *flag* if every set of pairwise adjacent vertices forms a simplex. A simplex σ in a simplicial flag complex is called a *k–ground* for $k \in \mathbb{N}$ if every vertex of X is connected to all but at most k vertices of σ. The complex is said to be *(n,k)–grounded* if there is an n–simplex that is a k–ground.

Theorem 3.18 [3, Theorem 4.9] *For $m,k \in \mathbb{N}$ every (mk,k)–grounded flag complex is $(m-1)$–connected.*

The reference requires $m,k \geq 1$ but it is clear that every $(0,k)$–grounded flag complex is nonempty, and every $(0,0)$–grounded flag complex is a cone and therefore contractible.

Using Theorem 3.18 we verify:

Lemma 3.19 *For every $n \in \mathbb{N}$ let Γ_n be a subgraph of K_n containing L_n. The connectivity of $\mathcal{M}(\Gamma_n)$ goes to infinity as n goes to infinity.*

Proof Consider the matchings of L_n that use only the edges $\{2i-1, 2i\}$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. They form an $\left(\lfloor \frac{n}{2} \rfloor - 1\right)$–simplex σ in $\mathcal{M}(\Gamma_n)$. If $v = \{j,k\}$ is any edge of Γ_n, so a vertex of $\mathcal{M}(\Gamma_n)$, then there are at most 2 vertices of σ that v is not connected to: one is $\{j-1, j\}$ or $\{j, j+1\}$, the other is $\{k-1,k\}$ or $\{k,k+1\}$. This shows that $\mathcal{M}(\Gamma_n)$ is $\left(\lfloor \frac{n}{2} \rfloor - 1, 2\right)$–grounded, so by Theorem 3.18 it is $\left(\lfloor \frac{n}{4} \rfloor - 1\right)$–connected. □

Proof of Proposition 3.15 We want to apply Corollary 3.14. The only thing left to check is condition (LK). This follows from Lemmas 3.16 and 3.19. □

4 The indirect product of two categories

The construction introduced in this section will help us to produce more interesting examples. It is usually called the Zappa–Szép product in the literature of groups and monoids; see [14]. The Zappa–Szép product naturally generalizes the semidirect product in the same way as the semidirect product generalizes the direct product. We think that such a basic construction should have a simpler name and therefore call it the *indirect product*.

For motivation, let M be a monoid (or group) whose multiplication we denote by $\circ$ and suppose that M decomposes uniquely as $M = A \circ B$. By this we mean that A and B are submonoids of M such that every element $m \in M$ can be written in a unique way as $m = a' \circ b'$ with $a' \in A$ and $b' \in B$. In particular, if $b \in B$ and $a \in A$, the product $m = b \circ a$ can be rewritten as $b \circ a = a' \circ b'$. This allows us to formally define maps $B \times A \to A$, $(b,a) \mapsto b \cdot a := a'$, and $B \times A \to B$, $(b,a) \mapsto b^a := b'$, so that

$$b \circ a = (b \cdot a) \circ b^a.$$

These maps turn out to be actions of monoids on sets. If both actions are trivial then M is a direct product, if one of the actions is trivial then M is a semidirect product, and in general it is an indirect product.

We therefore start by introducing the appropriate notion of actions of categories.

4.1 Actions

Let $\mathcal{C}$ be a category and let $(X_m)_{m \in \mathrm{Ob}(\mathcal{C})}$ be a family of sets, one for each object of $\mathcal{C}$. We say that a left action of $\mathcal{C}$ on $(X_m)_m$ is a family of maps

$$\mathcal{C}(n,m) \times X_m \to X_n, \quad (f,s) \mapsto f \cdot s,$$

satisfying $1_m \cdot s = s$ for all $m \in \mathrm{Ob}(\mathcal{C})$ and $s \in X_m$ and $fg \cdot s = f \cdot (g \cdot s)$ whenever fg is defined. A right action is defined analogously. An action is said to be *injective* if $f \cdot x = f \cdot y$ implies $x = y$. Note that actions of groupoids are always injective.

In our examples the family $(X_m)_m$ itself will consist of morphisms of a category with the same objects as $\mathcal{C}$. We have to bear in mind, however, that the action is on these as sets and does not preserve products.

4.2 The indirect product

Let $\mathcal{C}$ be a category and let $\mathcal{F}$ and $\mathcal{G}$ be subcategories. We say that $\mathcal{C}$ is an *internal indirect product* $\mathcal{F} \bowtie \mathcal{G}$ if every $h \in \mathcal{C}$ can be written in a unique way as $h = fg$ with $f \in \mathcal{F}$ and $g \in \mathcal{G}$. Note that this means in particular that $\mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathcal{F}) = \mathrm{Ob}(\mathcal{G})$. Given elements $f \in \mathcal{F}(x,-)$ and $g \in \mathcal{G}(-,x)$ there exist then unique elements $f' \in \mathcal{F}$ and $g' \in \mathcal{G}$ such that $gf = f'g'$; see Figure 4 (left). In this situation we define $g \cdot f$ to be f' and g^f to be g'.

The following properties are readily verified — see Figure 4 (center and right) — the last four hold whenever one of the sides is defined:

(IP1) $1_x \cdot f = f$ for $f \in \mathcal{F}(x,-)$.

(IP2) $g^{1_y} = g$ for $g \in \mathcal{G}(-,y)$.

$$\begin{tikzcd}
\bullet & \bullet \arrow[l, "g"'] \\
\bullet \arrow[u, dashed, "g\cdot f"] & \bullet \arrow[u, "f"'] \arrow[l, dashed, "g^{f}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
x & x \arrow[l, "1_x"'] \\
y \arrow[u, "f"] & y \arrow[u, "f"'] \arrow[l, "1_y"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet & \bullet \arrow[l, "g_1"'] & \bullet \arrow[l, "g_2"'] \\
\bullet \arrow[u, "g_1\cdot(g_2\cdot f)"] & \bullet \arrow[u, "g_2\cdot f"'] \arrow[l, "g_1^{g_2\cdot f}"] & \bullet \arrow[u, "f"'] \arrow[l, "g_2^{f}"]
\end{tikzcd}$$

Figure 4: The indirect product

(IP3) $(g_1g_2)\cdot f = g_1\cdot(g_2\cdot f)$.

(IP4) $g^{f_1f_2} = (g^{f_1})^{f_2}$.

(IP5) $1_x^f = 1_y$ for $f\in\mathcal{F}(x,y)$.

(IP6) $g\cdot 1_y = 1_z$ for $g\in\mathcal{G}(z,x)$.

(IP7) $(g_1g_2)^f = g_1^{(g_2\cdot f)}g_2^f$.

(IP8) $g\cdot(f_1f_2) = (g\cdot f_1)(g_2^{f_1}\cdot f_2)$.

The first four relations say that the map $(g,f)\mapsto g\cdot f$ is an left action of $\mathcal{G}$ on the sets $(\mathcal{F}(x,-))_x$ and that $(g,f)\mapsto g^f$ is a right action of $\mathcal{F}$ on the sets $(\mathcal{G}(-,y))_y$. The next two relations say that identity elements are taken to identity elements, while the last two are cocycle conditions. We call actions satisfying (IP1)–(IP8) *indirect product actions*.

Now assume that conversely categories $\mathcal{F}$ and $\mathcal{G}$ with $\mathrm{Ob}(\mathcal{F}) = \mathrm{Ob}(\mathcal{G})$ are given together with indirect product actions of $\mathcal{F}$ and $\mathcal{G}$ on each other. Then the *external indirect product* $\mathcal{C} = \mathcal{F}\bowtie\mathcal{G}$ is defined to have objects $\mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathcal{F}) = \mathrm{Ob}(\mathcal{G})$ and morphisms

$$\mathcal{C} = \bigcup_{x\in\mathrm{Ob}(\mathcal{C})} \{(f,g)\mid f\in\mathcal{F}(-,x), g\in\mathcal{G}(x,-)\}.$$

Composition is defined by

$$(f_1,g_1)(f_2,g_2) = (f_1(g_1\cdot f_2), g_1^{f_2}g_2). \tag{4-1}$$

Lemma 4.1 *The external indirect product $\mathcal{F}\bowtie\mathcal{G}$ is well defined. It is naturally isomorphic to the internal indirect product of the copies of $\mathcal{F}$ and $\mathcal{G}$ inside $\mathcal{F}\bowtie\mathcal{G}$.*

Proof That the identity morphisms $(1_x,1_x)$ behave as they should is easily seen using relations (IP1), (IP2), (IP5) and (IP6). To check associativity we verify the four

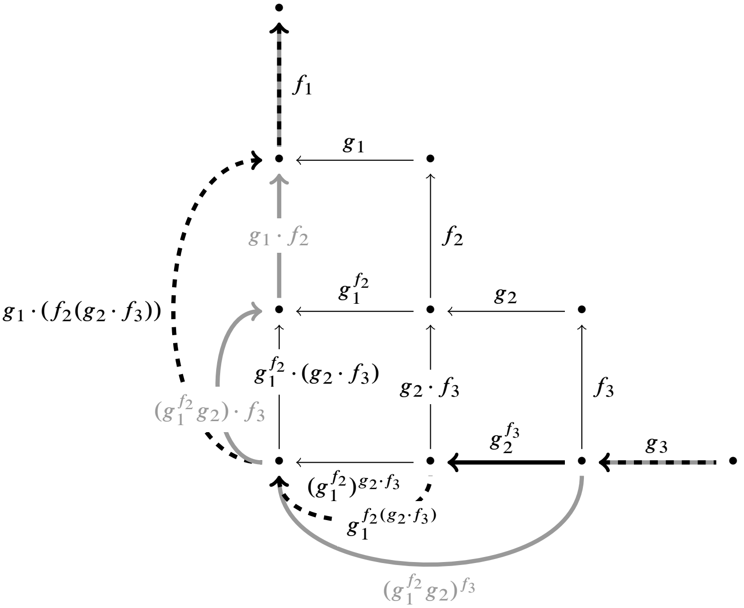


Figure 5: Associativity in $\mathcal{F} \bowtie \mathcal{G}$. The thick dashed and gray paths are the components of $(f_1, g_1)((f_2, g_2)(f_3, g_3))$ and $((f_1, g_1)(f_2, g_2))(f_3, g_3)$, respectively.

equations

$$g_1^{f_2(g_2 \cdot f_3)} \overset{\text{(IP4)}}{=} (g_1^{f_2})^{g_1 \cdot f_3}, \tag{4-2}$$

$$(g_1^{f_2} g_2) \cdot f_3 \overset{\text{(IP3)}}{=} g_1^{f_2} \cdot (g_2 \cdot f_3), \tag{4-3}$$

$$g_1^{f_2(g_2 \cdot f_3)} g_2^{f_3} \overset{\text{(4-2)}}{=} (g_1^{f_2})^{g_1 \cdot f_3} g_2^{f_3} \overset{\text{(IP7)}}{=} (g_1^{f_2} g_2)^{f_3}, \tag{4-4}$$

$$(g_1 \cdot f_2)((g_1^{f_2} g_2) \cdot f_3) \overset{\text{(4-3)}}{=} (g_1 \cdot f_2)(g_1^{f_2} \cdot (g_2 \cdot f_3)) \overset{\text{(IP8)}}{=} g_1 \cdot (f_2(g_2 \cdot f_3)); \tag{4-5}$$

see Figure 5.

The categories $\mathcal{F}$ and $\mathcal{G}$ naturally embed into the external indirect product $\mathcal{F} \bowtie \mathcal{G}$ as $f \mapsto (f, 1_y)$ for $f \in \mathcal{F}(-, y)$ and $g \mapsto (1_x, g)$ for $g \in \mathcal{G}(x, -)$. Any morphism of $\mathcal{F} \bowtie \mathcal{G}$ decomposes as $(f, g) = (f, 1_y)(1_y, g)$ and it is clear from (4-1) that the respective actions on each other are the ones used to define $\mathcal{F} \bowtie \mathcal{G}$. □

If the action of $\mathcal{G}$ on $\mathcal{F}$ is trivial then the indirect product is a *semidirect product* $\mathcal{F} \ltimes \mathcal{G}$. Similarly, if the action of $\mathcal{F}$ on $\mathcal{G}$ is trivial then it is a semidirect product $\mathcal{F} \rtimes \mathcal{G}$. Finally, if both actions are trivial then the indirect product is in fact a *direct product* $\mathcal{F} \times \mathcal{G}$.

We close the section by collecting facts that ensure that an indirect product is Ore.

Lemma 4.2 *If $\mathcal{F}$ and $\mathcal{G}$ are right-cancellative and the action of $\mathcal{F}$ on $\mathcal{G}$ is injective then $\mathcal{F} \bowtie \mathcal{G}$ is right-cancellative. Symmetrically, if $\mathcal{F}$ and $\mathcal{G}$ are left-cancellative and the action of $\mathcal{G}$ on $\mathcal{F}$ is injective, then $\mathcal{F} \bowtie \mathcal{G}$ is left-cancellative.*

Proof If $f_1 g_1 fg = f_2 g_2 fg$ then $f_1(g_1 \cdot f) = f_2(g_2 \cdot f)$ and $g_1^f g = g_2^f g$. Since $\mathcal{G}$ is right-cancellative the latter equation shows that $g_1^f = g_2^f$ and injectivity of the action then implies $g_1 = g_2$. Putting this in the former equation and using right-cancellativity of $\mathcal{F}$ gives $f_1 = f_2$. □

Observation 4.3 *Let $\mathcal{F}$ have common right-multiples and let $\mathcal{G}$ be a groupoid. Then $\mathcal{F} \bowtie \mathcal{G}$ has common right-multiples.*

Proof Let $fg \in \mathcal{F} \bowtie \mathcal{G}$ with $f \in \mathcal{F}$ and $g \in \mathcal{G}$. Since $\mathcal{G}$ is a groupoid, f is both a left-factor and a right-multiple of fg. It follows that common right-multiples exist in $\mathcal{F} \bowtie \mathcal{G}$ because they exist in $\mathcal{F}$. □

Observation 4.4 *Let $\mathcal{F}$ have no nontrivial invertible morphisms and let $\mathcal{G}$ be a groupoid. Then $(\mathcal{F} \bowtie \mathcal{G})^\times = \mathcal{G}$.*

Proposition 4.5 *Let $\mathcal{C} = \mathcal{F} \bowtie \mathcal{G}$, where $\mathcal{F}$ has no nontrivial invertibles and $\mathcal{G}$ is a discrete groupoid.*

(i) *If $\mathcal{F}$ is right-Ore and the action of $\mathcal{F}$ on $\mathcal{G}$ is injective, then $\mathcal{C}$ is right-Ore.*

(ii) *If $\mathcal{F}$ is strongly Noetherian then so is $\mathcal{C}$.*

(iii) *If ρ is a height function on $\mathcal{F}$ then it is a height function on $\mathcal{C}$.*

(iv) *If $\mathcal{S}$ is a left-Garside family in $\mathcal{F}$ then it is a left-Garside family in $\mathcal{C}$.*

(v) *If $\mathcal{S}$ is a right-Garside family in $\mathcal{F}$ then $\mathcal{S}\mathcal{G}$ is a right-Garside family in $\mathcal{C}$.*

Proof Property (i) follows from Lemma 4.2 and Observation 4.3. Properties (ii) and (iv) follow from the fact that for $f \in \mathcal{F}$ and $g \in \mathcal{G}$ the morphisms f and fg are right-multiples by invertibles of each other. Property (iii) follows from $\mathcal{G}$ being discrete (ie every morphism being an endomorphism). Toward (v), it is clear that every right-factor of $\mathcal{S}\mathcal{G}$ is contained in $\mathcal{S}\mathcal{G}$. Moreover, if t is an $\mathcal{S}$–tail for f then tg is a $\mathcal{S}$–tail for fg. □

5 Examples: categories constructed by indirect products

In this section we show how the indirect product can be used to construct new groups. The basic examples are Thompson's groups T and V as well as the braided Thompson groups, which all arise as fundamental groups of categories of the form $\mathcal{F} \bowtie \mathcal{G}$ where $\mathcal{G}$ is an appropriate groupoid. More generally, the groups studied in joint work with Zaremsky [43] are essentially by definition groups that can be obtained in this form. Later we also describe other groups obtained via indirect products.

We will sometimes draw pictures to motivate our definition. In these pictures the up direction always corresponds to left in our notation and down corresponds to right. This is especially relevant for group elements. For example, a permutation $X \leftarrow X$, $g(x) \leftarrow\!\shortmid x$, will be depicted by connecting the point x at the bottom to the point $g(x)$ at the top.

5.1 Thompson's groups T and V

In this section we introduce Thompson's groups T and V as fundamental groups of categories $\mathcal{T}$ and $\mathcal{V}$. The categories will be obtained from $\mathcal{F}$ as indirect products with groupoids and we start by introducing these.

We define $\mathcal{G}_T$ and $\mathcal{G}_V$ to be groupoids whose objects are positive natural numbers with $\mathcal{G}_T(m,n) = \varnothing$ for $m \neq n$. We put $\mathcal{G}_T(n,n) = \mathbb{Z}/n\mathbb{Z}$ and $\mathcal{G}_V(n,n) = \mathrm{SYM}_n$. We want to define $\mathcal{T} = \mathcal{F} \bowtie \mathcal{G}_T$ and $\mathcal{V} = \mathcal{F} \bowtie \mathcal{G}_V$ and have to specify the actions that define these indirect products. That is, given a forest $f \in \mathcal{F}(m,n)$ and a group element $g \in \mathcal{G}(m,m)$ we need to specify how the product gf should be written as $(g \cdot f)g^f$ with $g \cdot f \in \mathcal{F}(m,n)$ and $g^f \in \mathcal{G}(n,n)$ (for $\mathcal{G}$ one of $\mathcal{G}_T$ and $\mathcal{G}_V$).

Since $\mathcal{G}_T$ is contained in $\mathcal{G}_V$, it would suffice to only define the actions for $\mathcal{G}_V$, but we look at the simpler case of $\mathcal{G}_T$ first.

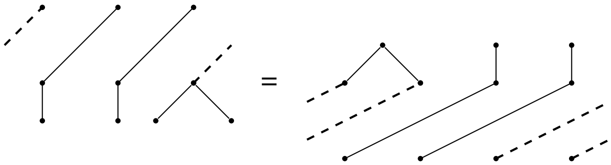


Figure 6: Defining $\mathcal{F} \bowtie \mathcal{G}_T$. The picture shows how to write gf as $(g \cdot f)g^f$ in the case where f is the caret $\lambda_3^3 \in \mathcal{F}(3,4)$ and $g = 1 + \mathbb{Z}/3\mathbb{Z} \in \mathcal{G}_T(3,3)$. The dashed strand gets doubled under the action of f. As a result, $g \cdot f = \lambda_1^3 \in \mathcal{F}(3,4)$ and $g^f = 2 + \mathbb{Z}/4\mathbb{Z} \in \mathcal{G}_T(4,4)$.

We need to rewrite a cyclic permutation followed by a tree as a tree followed by a cyclic permutation. This is illustrated in Figure 6. For $f \in \mathcal{F}(m,n)$ and $g = \ell + \mathbb{Z}/m\mathbb{Z} \in \mathcal{G}_T(m,m)$ the forest $g \cdot f$ is just f with the trees rotated by ℓ to the right. The definition of g^f is more subtle: looking at the figure we see that we have to define it to be $k + \mathbb{Z}/n\mathbb{Z}$, where k is the number of leaves of the first ℓ trees of $g \cdot f$, or equivalently, to be the number leaves of the last ℓ trees of f. Note that this number does not depend on the chosen representative ℓ: if we replace ℓ by $\ell + m$, instead of k we get $k + n$, because we counted every leaf once more. If k_ℓ denotes the number of leaves of the last ℓ trees of f, the sequence $(k_\ell)_{0 \le \ell < m}$ is strictly increasing. This shows:

Observation 5.1 *The action of $\mathcal{F}$ on $\mathcal{G}_T$ is injective.*

Lemma 5.2 *The actions of $\mathcal{F}$ and $\mathcal{G}_T$ on each other are indirect product actions.*

Proof Conditions (IP1), (IP2), (IP3), (IP4), (IP5) and (IP6) are clear.

The condition (IP7) in our setting follows from the fact that the last $k + \ell$ trees of f are the last ℓ trees of f plus the last k trees of $(\ell + m\mathbb{Z}) \cdot f$. Condition (IP8) can be verified by drawing a picture. □

The lemma allows us to define $\mathcal{T} = \mathcal{F} \bowtie \mathcal{G}_T$. Combining Observation 5.1 with Proposition 2.1 and Corollary 2.6 and applying Proposition 4.5 we find:

Corollary 5.3 *The category $\mathcal{T}$ is right-Ore and admits a height function and a left-Garside family $\mathcal{S}$ that is closed under factors such that $\mathcal{S}(x,-)/\mathcal{S}^\times$ is finite for every x.*

The fundamental group $\pi_1(\mathcal{T},1)$ is *Thompson's group T*.

Now we want to define the actions of $\mathcal{F}$ and $\mathcal{G}_V$ on each other. So let $f \in \mathcal{F}(m,n)$ and let $g \in \mathcal{G}_V(m,m)$. The action of $\mathcal{G}_V$ on $\mathcal{F}$ is again as expected: the forest $f' = (g \cdot f)$ is given by the relationship that the $g(j)^{\text{th}}$ tree of f' is the j^{th} tree of f. The permutation $g' = g^f \in \mathcal{G}_V(n,n)$ has the following description. Identify $\{1,\dots,n\}$ with the leaves of f and with the leaves of $(g \cdot f)$. If i is the k^{th} leaf of the j^{th} tree of f then $g'(i)$ is the k^{th} leaf of the $g(j)^{\text{th}}$ tree of $g \cdot f$; see Figure 6.

At this point it becomes clear that working with the actions as described above is virtually impossible. To obtain a more explicit algebraic description, we make use of the presentation of $\mathcal{F}$. Property (IP4) tells us that we know how any element of $\mathcal{F}$ acts as soon as we know how the generators act and property (IP8) tells us that we know how $\mathcal{G}_V$ acts on any element once we know how it acts on the generators of $\mathcal{F}$. It therefore suffices to specify both actions for generators of $\mathcal{F}$. Checking well-definedness then means to check various conditions coming from the relations in $\mathcal{F}$.

So now we consider $g \in \mathcal{G}_V(m,m)$ and $\lambda_i^m \in \mathcal{F}(m,m+1)$ and define the actions on each other. We start again with the easy case,

$$g \cdot \lambda_i = \lambda_{g(i)}. \tag{5-1}$$

Working out g^{λ_i} we have to distinguish four cases depending on the position of a point relative to i and relative to $g(i)$:

$$g^{\lambda_i}(j) = \begin{cases} g(j) & \text{if } j \le i,\ g(j) \le g(i), \\ g(j-1) & \text{if } j > i,\ g(j-1) \le g(i), \\ g(j)+1 & \text{if } j \le i,\ g(j) > g(i), \\ g(j-1)+1 & \text{if } j > i,\ g(j-1) > g(i). \end{cases} \tag{5-2}$$

Since $i = j$ if and only if $g(i) = g(j)$, the inequalities in the second and third case can be taken to be strict.

Lemma 5.4 *The formulas (5-1) and (5-2) define well-defined indirect product actions of $\mathcal{F}$ and $\mathcal{G}_V$ on each other.*

Proof The conditions that involve only the action of $\mathcal{G}_V$, namely (IP1), (IP3) and (IP6), are clear. Condition (IP2) is defined to hold. Verifying conditions (IP5) and (IP7) on the λ_i is straightforward, although in the second case tedious.

Conditions (IP4) and (IP8) should also be defined to hold, but in order for this to be well defined, we need to check them on relations. That is, we need to verify that

$$(g^{\lambda_i})^{\lambda_j} = g^{\lambda_i \lambda_j} = g^{\lambda_j \lambda_{i+1}} = (g^{\lambda_j})^{\lambda_{i+1}}$$

and

$$(g \cdot \lambda_i)(g_2^{\lambda_i} \cdot \lambda_j) = g \cdot (\lambda_i \lambda_j) = g \cdot (\lambda_j \lambda_{i+1}) = (g \cdot \lambda_j)(g_2^{\lambda_j} \cdot \lambda_{i+1})$$

for $j < i$. These are again not difficult but tedious and we skip them here. See [43, Example 2.9] for a detailed verification. □

Thus, we can define $\mathcal{V} = \mathcal{F} \bowtie \mathcal{G}_V$.

Lemma 5.5 *The action of $\mathcal{F}$ on $\mathcal{G}_V$ defined by (5-2) is injective.*

Proof Since by definition $g^{\lambda_{i_1}\cdots\lambda_{i_n}} = (\cdots(g^{\lambda_{i_1}})\cdots)^{\lambda_{i_n}}$, we only need to check that the map $g \mapsto g^{\lambda_i}$ defined in (5-1) is injective. But g can be recovered from g^{λ_i} as follows. Let $\tau_i, \pi_i \colon \mathbb{N} \to \mathbb{N}$ be given by

$$\tau_i(j) := \begin{cases} j & \text{if } j \le i, \\ j+1 & \text{if } j > i, \end{cases} \qquad \pi_i(j) := \begin{cases} j & \text{if } j \le i, \\ j-1 & \text{if } j > i. \end{cases}$$

Then $g(j) = \pi_i(g^{\lambda_i}(\tau_i(j)))$. □

Proposition 2.1, Corollary 2.6 and Proposition 4.5 now imply:

Corollary 5.6 *The category $\mathcal{V}$ is right-Ore and admits a height function and a left-Garside family $\mathcal{S}$ that is closed under factors such that $\mathcal{S}(x,-)/\mathcal{S}^\times$ is finite for every x.*

The fundamental group $\pi_1(\mathcal{V}, 1)$ is *Thompson's group V*.

5.2 The braided Thompson groups

The group BV, called *braided V*, was introduced independently by Brin [15] and Dehornoy [21]. We describe it using our framework, which is similar to Brin's approach.

To define the categories underlying the braided Thompson groups, we define the groupoid $\mathcal{G}_{BV}$ to have as objects natural numbers, and to have morphisms $\mathcal{G}_{BV}(m,n) = \varnothing$ for $m \neq n$, and $\mathcal{G}_{BV}(n,n) = \mathrm{BRAID}_n$. Note that the morphisms $\pi\colon \mathrm{BRAID}_n \to \mathrm{SYM}_n$ define a morphism $\mathcal{G}_{BV} \to \mathcal{G}_V$, which we denote by π as well. We want to define a indirect product $\mathcal{F} \bowtie \mathcal{G}_{BV}$ and need to define actions of $\mathcal{F}$ and $\mathcal{G}_{BV}$ on each other. Our guiding picture is Figure 7.

We define the action of $\mathcal{G}_{BV}$ on $\mathcal{F}$ simply as the action of $\mathcal{G}_V$ composed with π. In particular, $\sigma_i \cdot \lambda_i = \lambda_{i+1}$, $\sigma_i \cdot \lambda_{i+1} = \lambda_i$ and $\sigma_i \cdot \lambda_j = \lambda_j$ for $j \neq i, i+1$. The action

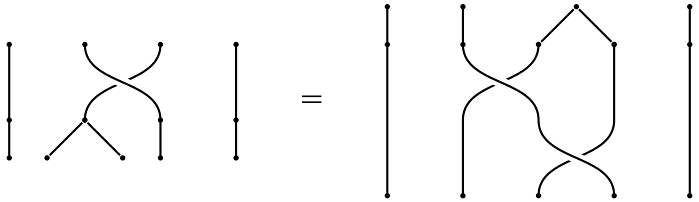


Figure 7: Defining $\mathcal{F} \bowtie \mathcal{G}_{BV}$

of $\mathcal{F}$ on $\mathcal{G}_{BV}$ we only define for generators acting on generators by

$$\sigma_i^{\lambda_j} := \begin{cases} \sigma_{i+1} & \text{if } j < i, \\ \sigma_i\sigma_{i+1} & \text{if } j = i, \\ \sigma_{i+1}\sigma_i & \text{if } j = i+1. \\ \sigma_i & j > i+1. \end{cases}$$

Lemma 5.7 *The formulas above define well-defined indirect product actions of $\mathcal{F}$ and $\mathcal{G}_{BV}$ on each other.*

In the proof we will use the fact that there is a set-theoretic splitting $\iota\colon \mathrm{SYM}_n \to \mathrm{BRAID}_n$ that takes a reduced word $w(s_1, \dots, s_{n-1})$ to the braid $w(\sigma_1, \dots, \sigma_{n-1})$. This map is not multiplicative but if β is a positive word (meaning involving no inverses) of length at most 3 in the σ_i then $\iota\pi(\beta) = \beta$.

Proof As in the proof of Lemma 5.4 most conditions hold by definition but we need to check well-definedness on relations. Namely,

$$(5\text{-}3) \qquad (\sigma_i\sigma_{i+1}\sigma_i)\cdot\lambda_k = \sigma_i\cdot(\sigma_{i+1}\cdot(\sigma_i\cdot\lambda_k)) = \sigma_{i+1}\cdot(\sigma_i\cdot(\sigma_{i+1}\cdot\lambda_k)) = (\sigma_{i+1}\sigma_i\sigma_{i+1})\cdot\lambda_k,$$

$$(5\text{-}4) \qquad (\sigma_i\sigma_{i+1}\sigma_i)^{\lambda_k} = \sigma_i^{(\sigma_{i+1}\sigma_i)\cdot\lambda_k}\sigma_{i+1}^{\sigma_i\cdot\lambda_k}\sigma_i^{\lambda_k} = \sigma_{i+1}^{(\sigma_i\sigma_{i+1})\cdot\lambda_k}\sigma_i^{\sigma_{i+1}\cdot\lambda_k}\sigma_{i+1}^{\lambda_k} = (\sigma_{i+1}\sigma_i\sigma_{i+1})^{\lambda_k},$$

$$(5\text{-}5) \qquad (\sigma_i\sigma_j)\cdot\lambda_k = \sigma_i\cdot(\sigma_j\cdot\lambda_k) = \sigma_i\cdot(\sigma_j\cdot\lambda_k) = (\sigma_j\sigma_i)\cdot\lambda_k,$$

$$(5\text{-}6) \qquad (\sigma_i\sigma_j)^{\lambda_k} = \sigma_i^{\sigma_j\cdot\lambda_k}\sigma_j^{\lambda_k} = \sigma_j^{\sigma_i\cdot\lambda_k}\sigma_i^{\lambda_k} = (\sigma_j\sigma_i)^{\lambda_k},$$

$$(5\text{-}7) \qquad \sigma_i\cdot(\lambda_\ell\lambda_k) = (\sigma_i\cdot\lambda_\ell)(\sigma_i^{\lambda_\ell}\cdot\lambda_k) = (\sigma_i\cdot\lambda_k)(\sigma_i^{\lambda_k}\cdot\lambda_{\ell+1}) = \sigma_i\cdot(\lambda_k\lambda_{\ell+1}),$$

$$(5\text{-}8) \qquad \sigma_i^{\lambda_\ell\lambda_k} = (\sigma_i^{\lambda_\ell})^{\lambda_k} = (\sigma_i^{\lambda_k})^{\lambda_{\ell+1}} = \sigma_i^{\lambda_k\lambda_{\ell+1}}$$

for $i - j \geq 2$, $\ell > k$.

Relations (5-3) and (5-5) follow from Lemma 5.4. For the remaining relations note that $\pi(\beta^{\lambda_k}) = \pi(\beta)^{\lambda_k}$. Now (5-7) follows from Lemma 5.4 as well because

$$(5\text{-}9) \qquad \pi(\sigma_i^{\lambda_\ell})\cdot\lambda_k = \sigma_i^{\lambda_\ell}\cdot\lambda_k \quad \text{and} \quad \pi(\sigma_i^{\lambda_k})\cdot\lambda_{\ell+1} = \sigma_i^{\lambda_k}\cdot\lambda_{\ell+1}.$$

Relation (5-8) follows from Lemma 5.4 by noting that both sides are positive words of length at most 3 and applying ι.

We verify (5-4) by distinguishing cases. The cases $k < i$ and $k > i + 2$ are clear. If $k = i + 1$ then the left-hand side equals $(\sigma_{i+1}\sigma_i)\sigma_{i+2}(\sigma_{i+1}\sigma_i)$ and the right-hand side equals $(\sigma_{i+1}\sigma_{i+2})\sigma_i(\sigma_{i+1}\sigma_{i+2})$. Both are equivalent through two braid relations with intermediate commutator relations. The cases $k = i$ and $k = i + 2$ are symmetric and we only verify $k = i$. The left-hand side equals $\sigma_i(\sigma_{i+1}\sigma_{i+2})(\sigma_i\sigma_{i+1})$ while the right-hand side equals $(\sigma_{i+1}\sigma_{i+2})(\sigma_i\sigma_{i+1})\sigma_{i+2}$. Again these are equivalent through two braid relations with intermediate commutator relations.

Relation (5-6) is left to the reader. □

For future reference we record (5-9), which in the presence of Lemma 5.7 can be formulated as:

Observation 5.8 *The morphism $\pi\colon \mathcal{G}_{BV} \to \mathcal{G}_V$ is equivariant with respect to the $\mathcal{F}$–action in the sense that*

$$\pi(\beta^f) = \pi(\beta)^f$$

for $\beta \in \mathcal{G}_{BV}$ and $f \in \mathcal{F}$.

We define the category $\mathcal{BV}$ to be $\mathcal{F} \bowtie \mathcal{G}_{BV}$ with the above indirect product actions.

Lemma 5.9 *The action of $\mathcal{G}_{BV}$ on $\mathcal{F}$ is injective.*

Proof We only need to check that $\beta \mapsto \beta^{\lambda_i}$ is injective. But β can be recovered from β^{λ_i} by removing the $(i+1)^{\text{st}}$ strand. □

Corollary 5.10 *The category $\mathcal{BV}$ is right-Ore.*

The fundamental group $\pi_1(\mathcal{BV}, 1)$ is the *braided Thompson group BV.*

It is now easy to define braided versions of T and F. We let $\mathcal{G}_{BT}$ and $\mathcal{G}_{BF}$ be the inverse image under $\pi\colon \mathcal{G}_{BV} \to \mathcal{G}_V$ of $\mathcal{G}_T$ and $\mathcal{G}_F$, respectively. Both of these act on $\mathcal{F}$ by restricting the action of $\mathcal{G}_{BV}$, which is the same as to say that they act through π.

The action of $\mathcal{F}$ of $\mathcal{G}_{BV}$ leaves $\mathcal{G}_{BT}$ and $\mathcal{G}_{BF}$ invariant and restricts to actions on these, thanks to Observation 5.8: we know from Section 5.1 that $\mathcal{F}$ leaves $\mathcal{G}_T$ invariant and it is axiomatically required that it leaves the trivial groupoid invariant. Hence, if $\beta \in \mathcal{G}_{BT}$ and $f \in \mathcal{F}$ then $\pi(\beta^f) = \pi(\beta)^f \in \mathcal{G}_T$, so that $\beta^f \in \mathcal{G}_{BT}$, and an analogous reasoning applies for $\beta \in \mathcal{G}_{BF}$.

As a consequence we can define the categories $\mathcal{BT} = \mathcal{F} \bowtie \mathcal{G}_{BT}$ and $\mathcal{BF} = \mathcal{F} \bowtie \mathcal{G}_{BF}$, which are right-Ore. The group $BF = \pi_1(\mathcal{BF}, 1)$ is called *braided F* and was first introduced in [11]. We call the group $BT = \pi_1(\mathcal{BT}, 1)$ *braided T.*

Remark 5.11 The group BT was not introduced before for the following technical reason. Instead of our category $\mathcal{BV}$, Brin [15] used a monoid that can be thought of as a category with a single object ω which represents countably infinitely many strands. This is possible because splitting one of countably infinitely many strands leads to countably infinitely many strands and because braid groups BRAID_n are contained in a braid group $\varinjlim \mathrm{BRAID}_n$ on infinitely many strands. A practical downside of that approach is that the group of fractions of that monoid is too big, so one needs to describe which elements should be elements of BV. A formal downside is that groups like BT or even T cannot be described because $\mathbb{Z}/n\mathbb{Z}$ is not contained in $\mathbb{Z}/(n+1)\mathbb{Z}$, so that the needed limit does not exist.

Despite this formal problem, the main topological ingredient to establishing the finiteness properties of BT has been verified in [18, Section 3.4].

Remark 5.12 Since braid groups are themselves groups of fractions, one can also obtain BV as the fundamental group of the category $\mathcal{F} \bowtie \mathcal{G}_{BV}^{+}$, where $\mathcal{G}_{BV}^{+}(n,n)$ is the monoid of positive (or dual positive) braids rather than the full braid group (and analogous statements hold for BF and BT). This possibility has been noted by several people; see for example the last paragraph of Section 3.1 in [28]. When applying Theorem 3.12, condition (STAB) would become trivial, so verifying condition (LK) will presumably be accordingly harder.

5.3 Groups arising from cloning systems

In [43] Zaremsky and the author have defined (filtered) cloning systems to be the data needed to define indirect product actions of $\mathcal{F}$ and a groupoid on each other. Thus, the groups considered there are by definition fundamental groups of categories $\mathcal{F} \bowtie \mathcal{G}$, where $\mathcal{G}$ is a groupoid. However, the approach follows Brin [15] to construct the groups as subgroups of an indirect product of monoids $\mathcal{F}_\infty \bowtie \mathcal{G}_\infty$. As a consequence it has to deal with technical complications such as the notion of being *properly graded*, as well as practical shortcomings such as being unable to construct (braided) T.

Our categorical approach removes the necessity that the groups $(G_n)_n$ fit into a directed system of groups and therefore the whole discussion goes through without that assumption. Thus, a *cloning system* is given by a sequence $(G_n)_{n\in\mathbb{N}}$ of groups, a sequence $(\rho_n)_{n\in\mathbb{N}}\colon G_n \to S_n$ of morphisms and a family of maps $(\kappa_k^n)_{k\le n}\colon G_n \to G_{n+1}$ such that the following hold for all $k \le n$, $k < \ell$ and $g, h \in G_n$:

(CS1) **Cloning a product** $(gh)\kappa_k^n = (g)\kappa_{\rho(h)k}^n (h)\kappa_k^n$.

(CS2) **Product of clonings** $\kappa_\ell^n \circ \kappa_k^{n+1} = \kappa_k^n \circ \kappa_{\ell+1}^{n+1}$.

(CS3) **Compatibility** $\rho_{n+1}((g)\kappa_k^n)(i) = (\rho_n(g))\varsigma_k^n(i)$ for all $i \neq k, k+1$.

Here ς_k^n describes the action of $\mathcal{F}$ on $\mathcal{G}_V$, so that $(g)\varsigma_k^n(j) = g^{\lambda_k}(j)$ as in (5-1).

Given a cloning system, a groupoid $\mathcal{G}$ is defined by setting $\mathcal{G}(m,n) = \varnothing$ if $m \neq n$ and setting $\mathcal{G}(n,n) = G_n$. Indirect product actions of $\mathcal{F}$ and $\mathcal{G}$ on each other are defined by $g \cdot \lambda_k^n = \lambda_{\rho_n(g)k}^{n+1}$ and $g^{\lambda_k^n} = (g)\kappa_k^n$ for $g \in G_n$. The axioms (CS1), (CS2) and (CS3) ensure that these indeed define indirect product actions.

5.4 The Higman–Thompson groups

In total analogy to Section 5.1 one can define $\mathcal{T}_n = \mathcal{F}_n \bowtie \mathcal{G}_T$ and $\mathcal{V}_n = \mathcal{F}_n \bowtie \mathcal{G}_V$. As mentioned in Section 2 the category $\mathcal{F}_n$ is not connected for $n > 2$ and neither are the categories $\mathcal{T}_n$ and $\mathcal{V}_n$. Thus, it makes sense to define the groups

$$T_{n,r} = \pi_1(\mathcal{T}_n, r), \quad V_{n,r} = \pi_1(\mathcal{V}_n, r)$$

and, unlike the situation of $\mathcal{F}_n$, these groups are generally nonisomorphic for different r; see [27; 32] for a precise statement concerning the $V_{n,r}$. They are the remaining *Higman–Thompson groups*.

5.5 Groups from graph rewriting systems

We now look at indirect products that do not involve $\mathcal{F}$. The corresponding groups have been introduced and described in some detail in [3]. In this section, when we talk about graphs we will take their edges to be directed and allow multiple edges and loops. In particular, every edge has an initial and a terminal vertex. The edge set of a graph G is denoted by $E(G)$ and the vertex set by $V(G)$.

An *edge replacement rule* $e \to R$ consists of a single directed edge e and a finite graph R that contains the two vertices of e (but not e itself). If G is any graph and ε is an edge of G, the edge replacement rule can be *applied to* G *at* ε by removing ε and adding in a copy of R while identifying the initial/terminal vertex of ε with the initial/terminal vertex of e in R. The resulting graph is denoted by $G \lhd \varepsilon$. If δ is another edge of G, then it is also an edge of $G \lhd \varepsilon$ and so the replacement rule can be applied to $G \lhd \varepsilon$ at δ. We regard $G \lhd \varepsilon \lhd \delta$ and $G \lhd \delta \lhd \varepsilon$ as the same graph.

The vagueness inherent in the last sentence can be remedied by declaring that a graph obtained from G by applying the edge replacement rule (possibly many times) has as edges words in $E(G) \times E(R)^*$ and as vertices words in $V(G) \cup (E(G) \times E(R)^* \times V(R))$. For example, the graph $G \lhd \varepsilon \lhd \delta$ would have edges $\zeta \in E(G) \setminus \{\varepsilon, \delta\}$ as well as $\varepsilon\xi$ and $\delta\xi$ for $\xi \in E(R)$ and vertices $v \in V(G)$ as well as εw and δw for $w \in V(R)$.

For every edge replacement rule $e \to R$ we define a category $\mathcal{R}_{e\to R}$ whose objects are finite graphs. In order for the category to be small we will take the graphs to have vertices and edges coming from a fixed countable set, which in addition is closed under attaching words in $E(R)$ and $V(R)$. The category is presented by having generators

$$\lambda_\varepsilon^G \in \mathcal{R}_{e\to R}(G, G \lhd \varepsilon) \quad \text{for } G \text{ a graph and } \varepsilon \text{ an edge of } G$$

subject to the relations

$$\lambda_\delta^G \lambda_\varepsilon^{G\lhd\delta} = \lambda_\varepsilon^G \lambda_\delta^{G\lhd\varepsilon} \quad \text{for } G \text{ a graph and } \delta \text{ and } \varepsilon \text{ distinct edges of } G. \tag{5-10}$$

Lemma 5.13 *For any edge replacement rule $e \to R$ the category $\mathcal{R}_{e\to R}$ is right-Ore.*

Proof Thanks to the relations (5-10) a morphism $\lambda_{\varepsilon_1} \cdots \lambda_{\varepsilon_k}$ in $\mathcal{R}_{e\to R}$ is uniquely determined by its source, its target and the set $\{\varepsilon_1, \dots, \varepsilon_k\}$. The claim now follows by taking differences and unions of these sets of edges. □

As in previous sections, the second ingredient will be a groupoid. Its definition does not depend on the edge replacement rule, except possibly for the foundational issues of choosing universal sets of vertices and edges. We define $\mathcal{G}_{\text{graph}}$ to have as objects finite graphs and as morphisms isomorphisms of graphs.

We define actions of $\mathcal{R}_{e\to R}$ and $\mathcal{G}_{\text{graph}}$ on each other as follows. If $g\colon G \to G'$ is an isomorphism of graphs and $\varepsilon \in E(G)$ is an edge, then

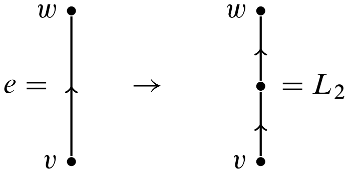
$$g \cdot \lambda_\varepsilon^G = \lambda_{g(\varepsilon)}^{G'}$$

and g^{λ_ε} is the isomorphism $G \lhd \varepsilon \to G' \lhd g(\varepsilon)$ that takes δ to $g(\delta)$ for $\delta \in E(G) \setminus \{\varepsilon\}$ and that takes $\varepsilon\zeta$ to $g(\varepsilon)\zeta$ for $\zeta \in V(R) \cup E(R)$. The following is easy to verify:

Observation 5.14 *The actions of $\mathcal{R}_{e\to R}$ and $\mathcal{G}_{\text{graph}}$ on each other defined above are well-defined indirect product actions. The action of $\mathcal{R}_{e\to R}$ on $\mathcal{G}_{\text{graph}}$ is injective.*

As a consequence we obtain a right-Ore category $\mathcal{RG}_{e\to R} := \mathcal{R}_{e\to R} \bowtie \mathcal{G}_{\text{graph}}$ and for every finite graph G we obtain a potential group $\pi_1(\mathcal{RG}_{e\to R}, G)$.

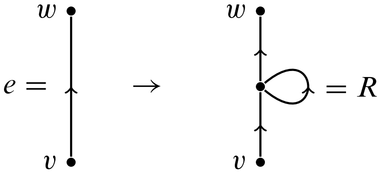
Example 5.15 If we consider the edge replacement rule



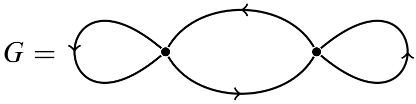
and take L_1 to be the graph consisting of a single edge, then $\pi_1(\mathcal{RG}_{e\to L_2}, L_1)$ is isomorphic to F. Similarly, if C_1 is the graph consisting of a single loop then $\pi_1(\mathcal{RG}_{e\to L_2}, C_1)$ is isomorphic to T. Finally, V arises as $\pi_1(\mathcal{RG}_{e\to D_2}, L_1)$, where the rule $e \to D_2$ replaces an edge by two disconnected edges.

Various fundamental groups of categories arising from graph rewriting systems are described in [3]. Here we will only mention the Basilica–Thompson group, introduced by them in [4].

We consider the replacement rule



and the graph



The *Basilica–Thompson group* is $T_B := \pi_1(\mathcal{RG}_{e\to R}, G)$.

6 Examples: finiteness properties

In this section we give various examples of applications of Theorem 3.12 and Corollary 3.14. In most cases these finiteness properties are known and the proofs involve proving that certain complexes are highly connected. We will see that these complexes always coincide with the complexes $|E(x)|$. As a consequence the connectivity statement from the literature together with Theorem 3.12 gives the result.

6.1 Finiteness properties of Thompson's groups

We start with the categories $\mathcal{T}$ and $\mathcal{V}$. The conditions needed to apply the results from Section 3 have been verified in Corollaries 5.3 and 5.6.

In order to apply Corollary 3.14 two more things are left to verify: that automorphism groups are of type F_∞ and that the connectivity of the simplicial complexes $|E(n)|$ goes to infinity with n. The groups $\mathcal{F}(n,n) = \{1\}$, $\mathcal{T}(n,n) = \mathbb{Z}/n\mathbb{Z}$ and $\mathcal{V}(n,n) = \mathrm{SYM}_n$ are all finite and therefore of type F_∞.

In order to describe the complexes $E(n)$, we need to talk about further graphs. The *cyclic graph* is denoted by C_n, it has the same edges as L_n and additionally $\{1,n\}$. The *complete graph* K_n has all edges $\{i,j\}$ for $1 \leq i < j \leq n$. We describe the complexes $E(n)$ in the case of $\mathcal{V}$ and leave $\mathcal{T}$ to the reader.

Lemma 6.1 *The poset $E_{\mathcal{T}}(n)$ is isomorphic to $\mathcal{M}(C_n)$.*

Lemma 6.2 *There is a poset morphism $E_{\mathcal{V}}(n) \to \mathcal{M}(K_n)$ whose fibers over k–simplices are k–spheres.*

Proof Every element of $(\mathcal{E} \bowtie \mathcal{G}_V)(-,n)$ can be written as a product fg of an elementary forest $f \in \mathcal{E}(-,n)$ and a permutation $g \in \mathcal{G}_V(n,n)$. By definition the vertices of $E(n)$ are these products modulo multiplication by permutations from the left. As in Lemma 3.16 an elementary forest can be interpreted as a matching on L_n. Under this correspondence, the group $\mathcal{G}_V(n,n) = \mathrm{SYM}_n$ acts on the vertices of L_n and the permutations from the left act on components of the matching. Thus, elements of $(\mathcal{E} \bowtie \mathcal{G}_V)(-,n)$ can be described by matchings on the linear graph on $g^{-1}(1), \dots, g^{-1}(n)$ modulo reordering the components of the matching.

The possibility of reordering the vertices of the matching means that any two elements of $\{1,\dots,n\}$ can be connected and so we obtain a map $|E(n)| \to \mathcal{M}(K_n)$ to the matching complex of the *complete* graph on $\{1,\dots,n\}$. This map is clearly surjective.

It is not injective because in $E(n)$ the order of two matched vertices matters while in $\mathcal{M}(K_n)$ it does not. For example, λ_i and $\lambda_i(i\ i{+}1)$ map to the same vertex in $\mathcal{M}(K_n)$. As a result the fiber over a k–simplex is a join of $k+1$ many 0–spheres, ie a k–sphere. □

The fact that the morphism in Lemma 6.2 is not an isomorphism means that we have to do one extra step, namely to apply the following result by Quillen [33, Theorem 9.1].

Rather than giving the general formulation for posets we restrict to face posets of (n–skeleta of) simplicial complexes, to save us some notation.

Theorem 6.3 *Let $n \in \mathbb{N}$ and let $f\colon X \to Y$ be a simplicial map. Assume that Y is $(n-1)$–connected and that for every k–simplex σ of Y the link $\operatorname{lk}\sigma$ is $(n-\dim\sigma-2)$–connected and the fiber $|f^{-1}(\sigma)|$ is $(k-1)$–connected. Then X is $(n-1)$–connected.*

Theorem 6.4 *Thompson's groups T and V are of type F_∞.*

Proof Using Corollary 3.14 we need to show that the connectivity of the complexes $|E(n)|$ goes to infinity as n goes to infinity. We work with the simplicial complexes $E(n)$ instead. In the case of T the complexes are matching complexes by Lemma 6.1 whose connectivity goes to infinity by Lemma 3.19. In the case of V the complexes map to matching complexes with good fibers by Lemma 6.2. Noting that the link of a k–simplex in $\mathcal{M}(K_n)$ is isomorphic to $\mathcal{M}(K_{n-2(k+1)})$, we can apply Theorem 6.3 to see that the connectivity of $E_{\mathcal{V}}$ goes to infinity as well. □

The proof for the Higman–Thompson groups is completely analogous.

6.2 Finiteness properties of braided Thompson groups

We have already seen that $\mathcal{BF}$, $\mathcal{BT}$ and $\mathcal{BV}$ are right-Ore. That they admit a height function and a left-Garside family follows via Proposition 4.5, just as it did for $\mathcal{T}$ and $\mathcal{V}$. The braid groups $\mathcal{BV}^\times(n,n) = \mathcal{G}_{BV}(n,n)$ are of type F by Corollary 3.8 (and hence of type F_∞). Consequently the finite-index subgroups of pure braids $\mathcal{BF}^\times(n,n)$ and of cyclically permuting braids $\mathcal{BT}^\times(n,n)$ are of type F as well.

It remains to understand the complexes $|E(n)|$. For that purpose, we will want to think of braid groups as mapping class groups. Let D be a closed disc with n punctures $p_1,\dots,p_n$, which we can think of as distinguished points in the interior of D. The mapping class group of the n–punctured disc is

$$\operatorname{Homeo}^+(D \setminus \{p_1,\dots,p_n\}, \partial D)/\operatorname{Homeo}^+_0(D \setminus \{p_1,\dots,p_n\}, \partial D),$$

where $\operatorname{Homeo}^+(D \setminus \{p_1,\dots,p_n\}, \partial D)$ is the group of orientation-preserving homeomorphisms of $D \setminus \{p_1,\dots,p_n\}$ that fix ∂D and $\operatorname{Homeo}^+_0(D \setminus \{p_1,\dots,p_n\}, \partial D)$ is the subgroup of homeomorphisms that are isotopic to the identity. It is well known that the mapping class group of the n–punctured disc is isomorphic to the braid group; see for example [29].

With this description in place, we can start to look at the complexes $|E(n)|$. Let $fg \in E(n)$ with $f \in \mathcal{E}(-,n)$ and $g \in \mathcal{G}_{BV}(n,n)$. Regard the n punctures $p_1,\dots,p_n$ as the vertices of an L_n embedded into D. As we have seen before, f corresponds to a matching M_f on L_n, which we now regard as a disjoint selection of the fixed arcs connecting pairs of adjacent punctures. The element g, regarded as a mapping class, acts on M_f and we obtain a set $M_f g$ of disjoint arcs connecting some pairs of punctures. Such a collection of arcs is called an *arc matching* in [18]. Note that if $f \in \mathcal{E}(k,n)$, so that the arc matching consists of $n-k$ arcs, then removing the arcs from the punctured disc results in a k–punctured disc. The action of $\mathcal{G}_{BV}(k,k)$ from the left is just the action of the mapping class group of that k–punctured disc and in particular does nothing to M_f.

For a subgraph Γ of K_n the *arc matching complex* $\mathcal{MA}(\Gamma)$ is the simplicial complex whose k–simplices are sets of pairwise disjoint arcs connecting punctures with the condition that an arc can only connect two punctures if they are connected by an edge in Γ.

Proposition 6.5 *There exist surjective morphisms of simplicial complexes*

(i) $E_{\mathcal{BF}}(n) \to \mathcal{MA}(L_n)$,

(ii) $E_{\mathcal{BT}}(n) \to \mathcal{MA}(C_n)$,

(iii) $E_{\mathcal{BV}}(n) \to \mathcal{MA}(K_n)$,

whose fiber over any k–simplex is the join of k countable infinite discrete sets.

Proof The product $fg \in E(n)$ is taken to the arc matching $M_f g$ as described above. Since $\mathcal{G}_{BF}(n,n)$ takes every puncture to itself, the map (i) maps onto $\mathcal{MA}(L_n)$. Similarly, since $\mathcal{G}_{BT}(n,n)$ cyclically permutes the punctures, the map (ii) maps into $\mathcal{MA}(C_n)$. Surjectivity is clear.

To describe the fibers consider a disc D' containing p_i and p_{i+1} but none of the other punctures and let β be a braid that is arbitrary inside D' but trivial outside. Then $\lambda_i\beta$ maps to the same arc ($=$ vertex of $\mathcal{MA}(K_n)$) irrespective of β. Thus, the fiber over this vertex is the mapping class group of $D' \smallsetminus \{p_i, p_{i+1}\}$ in the case of $\mathcal{BV}$ and is the pure braid group of $D' \smallsetminus \{p_i, p_{i+1}\}$ in the cases of $\mathcal{BF}$ and $\mathcal{BT}$. In either case it is a countable infinite discrete set. □

The connectivity properties of arc matching complexes have been studied in [18]. We summarize Theorem 3.8, Corollary 3.11 and the remark in Section 3.4 from there in the following theorem. It applies to arc matching complexes not only on disks but on arbitrary surfaces with (possibly empty) boundary.

Theorem 6.6 (i) *$\mathcal{MA}(K_n)$ is $(\nu(n)-1)$–connected,*

(ii) *$\mathcal{MA}(C_n)$ is $(\eta(n-1)-1)$–connected,*

(iii) *$\mathcal{MA}(L_n)$ is $(\eta(n)-1)$–connected,*

where $\nu(n) = \lfloor \frac{n-1}{3} \rfloor$ and $\eta(n) = \lfloor \frac{n-1}{4} \rfloor$.

Theorem 6.7 *The braided Thompson groups BF, BT and BV are of type F_∞.*

Proof We want to apply Corollary 3.14. By Proposition 6.5 the complexes $E(n)$ map onto arc matching complexes and we want to apply Theorem 6.3. To do so, we need to observe that the link of a $(k+1)$–simplex on an arc matching complex on a surface with n punctures is an arc matching complex with $n-2k$ punctures, where the k arcs connecting two punctures have been turned into boundary components. Putting these results together shows that the connectivity properties of $E(n)$ go to infinity with n by Theorem 6.6. □

6.3 Absence of finiteness properties

Theorem 3.12 gives a way to prove that certain groups are of type F_n. If the group is not of type F_n, one of the hypotheses fails. We will now discuss to what extent the construction is (un)helpful in proving that the group is not of type F_n, depending on which hypothesis fails.

In the first case the groups $\mathcal{C}^\times(x,x)$ are not of type F_n (even for $\rho(x)$ large). In this case the general part of Brown's criterion, Theorem 3.10, cannot be applied. Thus, the whole construction from Section 3.4 is useless for showing that $\pi_1(\mathcal{G}_\mathcal{C}, *)$ is not of type F_n. An example of this case are the groups $\mathcal{T}(B_*(\mathcal{O}_S))$ treated in [43, Theorem 8.12]. The proof redoes part of the proof that the groups $\mathcal{C}^\times(x,x)$, which are the groups in $B_n(\mathcal{O}_S)$ in this case, are not of type $F_n = F_{|S|}$.

In the second case the complexes $E(x)$ are not (even asymptotically) $(n-1)$–connected. In this case Brown's criterion, Theorem 3.10, can in principle be applied, but not by using just Morse theory. An example of this case is the Basilica–Thompson group from Section 5.5, which is not finitely presented [41], so $n = 2$. A morphism in $\mathcal{RG}_{e\to R} = \mathcal{R}_{e\to R} \bowtie \mathcal{G}_{\text{graph}}$ is declared to be elementary if there are edges $\{e_1, \dots, e_k\}$

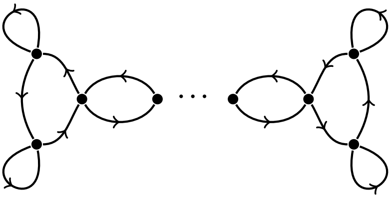


Figure 8: Arbitrarily large graphs H with $E(H)$ not simply connected

of G such that $f = \lambda_{e_1} \cdots \lambda_{e_k}$. The function $\rho\colon \mathcal{R}_{e\to R} \to \mathbb{N}$ is the number of edges of a graph. The basepoint $*$ is the Basilica graph G.

The connectivity assumption of Theorem 3.12 is violated because the $\mathcal{R}\mathcal{G}_{e\to R}$–component of G contains graphs H with arbitrarily many edges for which $E(H)$ is not simply connected. Examples of such graphs are illustrated in Figure 8. In these examples $E(H)$ has four vertices: two vertices v_{ll}, v_{ul} corresponding to the loops on the left and two vertices v_{lr} and v_{ur} corresponding to the loops on the right. The left vertices are connected to the right vertices but not to each other and neither are the right vertices. Thus, $E(H)$ is a circle $v_{\mathrm{ll}}, v_{\mathrm{lr}}, v_{\mathrm{ul}}, v_{\mathrm{ur}}$ and is not simply connected.

Looking into the proof of Theorem 3.12 we can compare directly what the nonsimple connectedness of $E(H)$ tells us and what is needed to apply Brown's criterion (Theorem 3.10) in order to prove that the group is not of type F_n. To apply Theorem 3.10, one needs to show that for every m there is an arbitrarily large n such that, passing from $X_{\rho<m}$ to $X_{\rho<n+1}$, a nontrivial 1–sphere in $X_{\rho<m}$ is filled in. The assumption that $E(H)$ is not simply connected for $\rho(H) = n$ translates via the Morse argument to the statement that when passing from $X_{\rho<n}$ to $X_{\rho<n+1}$ either a nontrivial 1–sphere in $X_{\rho<n}$ is filled in, or a nontrivial 2–sphere is created. The proof in [41] that the Basilica–Thompson group T_B is not finitely presented therefore needs to rule out the second possibility and also show that the 1–sphere that is filled in was nontrivial already in $X_{\rho<m}$.

7 Sketch of further examples

In this final section we sketch two further examples of categories associated to Thompson groups that fit in our framework. This is aimed mainly at experts who already know the groups and we will be brief.

7.1 Brin–Thompson groups

The higher-dimensional versions of V, denoted by sV for $s \geq 1$, were introduced by Brin [13]. If $C = \{0,1\}^\omega$ denotes Cantor space, a morphism in $\mathcal{V}(m,n)$ can be interpreted as a homeomorphism (subject to conditions)

$$\{1,\dots,m\} \times C \leftarrow \{1,\dots,n\} \times C$$

that represents subdividing m copies of C into n copies. The category $s\mathcal{V}$ similarly consists of homeomorphisms

$$\{1,\dots,m\} \times C^s \leftarrow \{1,\dots,n\} \times C^s$$

that represent subdividing m copies of C^s into n copies. See Figure 9 for an example illustrating composition. The Brin–Thompson groups are the groups $sV = \pi_1(s\mathcal{V}, 1)$.

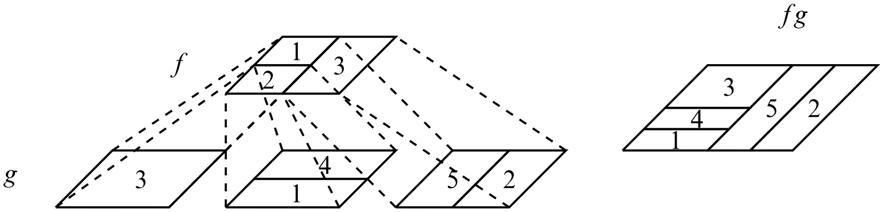


Figure 9: Composition of two morphisms in $2\mathcal{V}$

If one wants to obtain a presentation for $s\mathcal{V}$ whose objects are the natural numbers, one always needs to pick an order for the copies of C^s and the order is changed under relations. The presentation will therefore involve $\mathcal{G}_V$ from the start. Besides that, we take generators

$$\lambda^n_{a,i} \in s\mathcal{V}(n, n+1) \quad \text{for } 1 \leq a \leq s,\ 0 \leq i < n$$

representing the subdivision of the $(i+1)^{\text{st}}$ of n copies of C^s in the a^{th} direction. For each direction these satisfy the familiar relations

$$\lambda^n_{a,i}\lambda^{n+1}_{a,j} = \lambda^n_{a,j}\lambda^{n+1}_{a,i+1} \quad \text{for } 1 \leq a \leq s,\ 0 \leq j < i < n. \tag{7-1}$$

In addition, for two distinct directions we have the relations

$$\lambda^n_{a,i}\lambda^{n+1}_{b,i+1}\lambda^{n+2}_{b,i} = \lambda^n_{b,i}\lambda^{n+1}_{a,i+1}\lambda^{n+2}_{a,i}s_{i+1} \quad \text{for } 1 \leq a < b \leq s,\ 0 \leq i < n \tag{7-2}$$

(recall that s_{i+1} is the transposition $(i+1\ i+2)$).

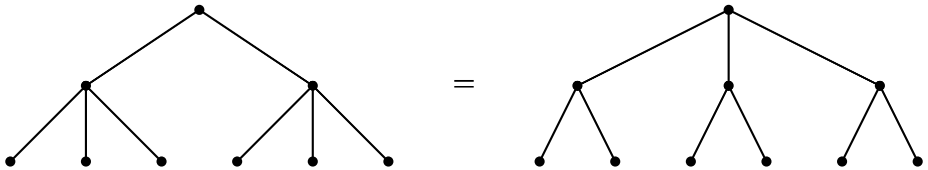


Figure 10: A relation in $\mathcal{F}_{2,3}$, also depicted in [36, page 485]

We claim without proof that $s\mathcal{V}$ has the presentation

$$s\mathcal{V} = \langle \mathcal{G}_V, \lambda^n_{a,i} \mid \text{relations in } \mathcal{G}_V, \text{(7-1)}, \text{(7-2)} \rangle.$$

To apply Theorem B to this setup one needs to verify condition (LK). This verification is the essence of [24].

7.2 Stein–Thompson groups

The idea underlying the Stein–Thompson groups is to allow carets with different arity chosen from a finite set $S = \{n_1, \dots, n_k\}$; see [36]. Thus, the underlying category $\mathcal{F}_S$ may be thought of as generated by $\mathcal{F}_{n_1}, \dots, \mathcal{F}_{n_k}$. There are number-theoretic relations, however. For instance, a tree that has a full layer of n_1–carets followed by a full layer of n_2–carets is the same as one with a full layer of n_2–carets followed by a full layer of n_1–carets; see Figure 10. We refrain from writing down a presentation but we should point out that the perspective taken in [36] is fairly close to ours. This does not include the F_∞–proof as Stein's space is carefully tailored to provide more precise homological information.

The categories $\mathcal{T}_S = \mathcal{F}_S \bowtie \mathcal{G}_T$ and $\mathcal{V}_S = \mathcal{F}_S \bowtie \mathcal{G}_V$ arise as indirect products in a straightforward manner.

It is clear that the categories are right-Ore and admit a height function.

A Garside family consists of the family of forests $\mathcal{S}$ where along any path from root to leaf at most one n_i–caret is met for any i. The maximal such tree $\Delta(x) \in \mathcal{F}_S(x, -)$ is the one that has a full layer of n_i–carets for each i. Any two elements in $\mathcal{S}(x, -)$ have a least common right multiple by [36, Proposition 1.2]. This together with the height function implies the existence of $\mathcal{S}$–heads.

The rest of the proof that the groups are of type F_∞ is completely analogous to that for Thompson's groups and the Higman–Thompson groups in Section 5.1.

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The Lannes–Zarati homomorphism and decomposable elements

NGÔ A TUẤN

Let X be a pointed CW–complex. The generalized conjecture on spherical classes states that *the Hurewicz homomorphism* $H\colon \pi_*(Q_0X) \to H_*(Q_0X)$ *vanishes on classes of* $\pi_*(Q_0X)$ *of Adams filtration greater than* 2. Let $\varphi_s^M\colon \mathrm{Ext}_{\mathcal{A}}^s(M,\mathbb{F}_2) \to (\mathbb{F}_2 \otimes_{\mathcal{A}} R_sM)^*$ denote the s^{th} Lannes–Zarati homomorphism for the unstable $\mathcal{A}$–module M. When $M = \widetilde{H}^*(X)$, this homomorphism corresponds to an associated graded of the Hurewicz map. An algebraic version of the conjecture states that the s^{th} Lannes–Zarati homomorphism, φ_s^M, vanishes in any positive stem for $s > 2$ and for any unstable $\mathcal{A}$–module M.

We prove that, for M an unstable $\mathcal{A}$–module of finite type, the s^{th} Lannes–Zarati homomorphism, φ_s^M, vanishes on decomposable elements of the form $\alpha\beta$ in positive stems, where $\alpha \in \mathrm{Ext}_{\mathcal{A}}^p(\mathbb{F}_2,\mathbb{F}_2)$ and $\beta \in \mathrm{Ext}_{\mathcal{A}}^q(M,\mathbb{F}_2)$ with either $p \geq 2$, $q > 0$ and $p+q = s$, or $p = s \geq 2$, $q = 0$ and $\mathrm{stem}(\beta) > s-2$. Consequently, we obtain a theorem proved by Hưng and Peterson in 1998. We also prove that the fifth Lannes–Zarati homomorphism for $\widetilde{H}^*(\mathbb{RP}^\infty)$ vanishes on decomposable elements in positive stems.

55P47, 55Q45, 55S10, 55T15

1 Introduction and statement of results

Let X be a pointed CW–complex. Let $Q_0X = \Omega_0^\infty S^\infty X$ be the basepoint component of $QX = \Omega^\infty S^\infty X$. It is a classical unsolved problem to compute the image of the Hurewicz homomorphisms

$$H\colon \pi_*^S(X) = \pi_*(Q_0X) \to H_*(Q_0X).$$

Here and throughout the paper, homology and cohomology are taken with coefficients in $\mathbb{F}_2$, the field of two elements. The classical conjecture on spherical classes for $X = S^0$ states that the Hopf invariant-one and the Kervaire invariant-one classes are the only elements in $\pi_*^S(S^0) \cong \pi_*(Q_0S^0)$ detected by the Hurewicz homomorphism. Nguyễn H V Hưng states the generalized conjecture on spherical classes as follows (see Hưng and Tuấn [14]).

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Conjecture 1.1 Let X be a pointed CW–complex. Then the Hurewicz homomorphism $H\colon \pi_*(Q_0X) \to H_*(Q_0X)$ vanishes on classes of $\pi_*(Q_0X)$ of Adams filtration greater than 2.

(See Curtis [4], Snaith and Tornehave [21] and Wellington [22] for a discussion with $X = S^0$.)

An algebraic version of this problem goes as follows.

Let $P_s = \mathbb{F}_2[x_1, \dots, x_s]$ be the polynomial algebra on s indeterminates $x_1, \dots, x_s$, each of degree 1. Let the general linear group $\mathrm{GL}_s = \mathrm{GL}(s, \mathbb{F}_2)$ and the mod 2 Steenrod algebra $\mathcal{A}$ both act on P_s in the usual way. The Dickson algebra of s variables, D_s, is the algebra of invariants

$$D_s := \mathbb{F}_2[x_1, \dots, x_s]^{\mathrm{GL}_s}.$$

As the action of $\mathcal{A}$ and that of GL_s on P_s commute with each other, D_s is an algebra over $\mathcal{A}$.

Let M be an unstable $\mathcal{A}$–module. The Singer construction R_sM of M is the D_s–submodule of $P_s \otimes M$ generated by $\mathrm{St}_s\, M$, where St_s denotes the Steenrod homomorphism defined as follows. Given a homogeneous element $z \in M$ of degree $|z|$, we set for convention $\mathrm{St}_0(z) = z$, and define by induction

$$\mathrm{St}_1(x; z) = \sum_{i=0}^{|z|} x^{|z|-i} \otimes \mathrm{Sq}^i(z),$$

$$\mathrm{St}_s(x_1, \dots, x_s; z) = \mathrm{St}_1(x_1; \mathrm{St}_{s-1}(x_2, \dots, x_s; z)).$$

Note that R_sM is an $\mathcal{A}$–submodule of $P_s \otimes M$. (See Lannes and Zarati [16, Definition-Proposition 2.4.1].)

Let us denote by

$$\varphi_s^M\colon \mathrm{Ext}_{\mathcal{A}}^{s,s+i}(M, \mathbb{F}_2) \to (\mathbb{F}_2 \otimes_{\mathcal{A}} R_sM)_i{}^*$$

the s^{th} Lannes–Zarati homomorphism for an unstable $\mathcal{A}$–module M, defined in [16]. Here $(\mathbb{F}_2 \otimes_{\mathcal{A}} R_sM)_i{}^*$ is the $\mathbb{F}_2$–dual of $(\mathbb{F}_2 \otimes_{\mathcal{A}} R_sM)_i$. When $M = \widetilde{H}^*(X)$, this homomorphism corresponds to an associated graded of the Hurewicz map. The proof of this assertion is unpublished, but it is sketched by Lannes [15] and by Goerss [7].

The Hopf invariant-one and the Kervaire invariant-one classes are represented by certain permanent cycles in $\mathrm{Ext}_{\mathcal{A}}^{1,*}(\mathbb{F}_2, \mathbb{F}_2)$ and $\mathrm{Ext}_{\mathcal{A}}^{2,*}(\mathbb{F}_2, \mathbb{F}_2)$, respectively, on which the Lannes–Zarati homomorphisms are nonzero (see Adams [1], Browder [3] and Lannes and Zarati [16]). Hưng stated the so-called algebraic version of the generalized conjecture on spherical classes for $M = \widetilde{H}^*(S^0) = \mathbb{F}_2$ in [9] and for any unstable $\mathcal{A}$–module M in [14].

Conjecture 1.2 (the generalized algebraic spherical class conjecture) The Lannes–Zarati homomorphism

$$\varphi_s^M\colon \operatorname{Ext}_{\mathcal{A}}^{s,s+i}(M,\mathbb{F}_2) \to (\mathbb{F}_2 \otimes_{\mathcal{A}} R_s M)_i{}^*$$

vanishes in any positive stem i for $s > 2$, and for any unstable $\mathcal{A}$–module M.

The conjecture was established for the case $M = \widetilde{H}^*(S^0)$ with $s = 3$, 4 and 5, respectively, in Hưng [10; 11] and Hưng, Quỳnh and Tuấn [13]. That the Lannes–Zarati homomorphism for $M = \widetilde{H}^*(S^0)$ vanishes for $s > 2$ on decomposable elements in $\operatorname{Ext}_{\mathcal{A}}^{s}(\mathbb{F}_2,\mathbb{F}_2)$ was proved in [12]. The conjecture was also established for the case $M = \widetilde{H}^*(\mathbb{R}\mathbb{P}^\infty)$ with $s = 3, 4$ in [14].

One of the main results of the paper is the following theorem:

Theorem 1.3 *Let M be an unstable $\mathcal{A}$–module of finite type. Then the s^{th} Lannes–Zarati homomorphism for M,*

$$\varphi_s^M\colon \operatorname{Ext}_{\mathcal{A}}^{s,s+i}(M,\mathbb{F}_2) \to (\mathbb{F}_2 \otimes_{\mathcal{A}} R_s M)_i^*,$$

vanishes on the elements of the form $\alpha\beta$ in any positive stem i, where $\alpha \in \operatorname{Ext}_{\mathcal{A}}^{p}(\mathbb{F}_2,\mathbb{F}_2)$ and $\beta \in \operatorname{Ext}_{\mathcal{A}}^{q}(M,\mathbb{F}_2)$ with either $p \geq 2$, $q > 0$ and $p+q = s$, or $p = s \geq 2$, $q = 0$ and $\operatorname{stem}(\beta) > s-2$.

Theorem 1.3 gives evidence supporting Conjecture 1.2, in particular providing a result valid for all unstable $\mathcal{A}$–modules of finite type M.

Using Theorem 1.3 for the case $M = \mathbb{F}_2$, we obtain the following theorem, which was first proved in [12]:

Theorem 1.4 (Hưng and Peterson [12]) *The s^{th} Lannes–Zarati homomorphism for $\mathbb{F}_2$,*

$$\varphi_s^{\mathbb{F}_2}\colon \operatorname{Ext}_{\mathcal{A}}^{s,s+i}(\mathbb{F}_2,\mathbb{F}_2) \to (\mathbb{F}_2 \otimes_{\mathcal{A}} D_s)_i^*,$$

vanishes on the decomposable elements in any positive stem i for $s \geq 3$.

In [12], Hưng and Peterson proved Theorem 1.4 by showing that $\varphi_* = \bigoplus_s \varphi_s^{\mathbb{F}_2}$ is a homomorphism of algebras and, more importantly, that the product of the nonunital algebra $\bigoplus_{s>0}(\mathbb{F}_2 \otimes_{\mathcal{A}} D_s)^*$ is trivial, except for the case $(\mathbb{F}_2 \otimes_{\mathcal{A}} D_1)^* \otimes (\mathbb{F}_2 \otimes_{\mathcal{A}} D_1)^* \to (\mathbb{F}_2 \otimes_{\mathcal{A}} D_2)^*$. The methods used to prove Theorem 1.3 are different from the methods of Hưng and Peterson. The important new ingredient is the usage of the chain level representation of the dual of the Lannes–Zarati homomorphism (see Theorem 2.1). Moreover, the advantage of using the chain level representation of the dual of the Lannes–Zarati homomorphism is that the proof of Theorem 1.3 is short and elementary. The proof of Theorem 1.3 is based upon the key Lemma 3.3.

Hưng and the author [14] established a relation between the Lannes–Zarati homomorphisms for $\widetilde{H}^*(\mathbb{RP}^\infty)$ and for $\widetilde{H}^*(S^0)$. The relation comes from the so-called algebraic Kahn–Priddy theorem (see [17, Theorem 1.1]). By using the algebraic Kahn–Priddy theorem, Hưng and Tuấn showed that if $\varphi_{s-1}^{\widetilde{H}^*(\mathbb{RP}^\infty)}$ vanishes in positive stems, then so does $\varphi_s^{\widetilde{H}^*(S^0)}$, for $s \geq 1$ (see [14, Proposition 10.2]). So, Conjecture 1.2 with $M = \widetilde{H}^*(\mathbb{RP}^\infty)$ is interesting. In this paper, by using Theorem 1.3 and the fact that $\varphi_5^{\mathbb{F}_2}$ and $\varphi_4^{\widetilde{H}^*(\mathbb{RP}^\infty)}$ vanish in positive stems (see [13, Theorem 1.4; 14, Theorem 1.8]), we obtain the following proposition:

Proposition 1.5 *The fifth Lannes–Zarati homomorphism for* $\widetilde{H}^*(\mathbb{RP}^\infty)$,

$$\varphi_5^{\widetilde{H}^*(\mathbb{RP}^\infty)}\colon \operatorname{Ext}_{\mathcal{A}}^{5,5+i}(\widetilde{H}^*(\mathbb{RP}^\infty), \mathbb{F}_2) \to (\mathbb{F}_2 \otimes_{\mathcal{A}} R_5 \widetilde{H}^*(\mathbb{RP}^\infty))_i^*,$$

vanishes on the decomposable elements in any positive stem i.

Note that $\operatorname{Ext}_{\mathcal{A}}^*(M, \mathbb{F}_2)$ is a module over $\operatorname{Ext}_{\mathcal{A}}^*(\mathbb{F}_2, \mathbb{F}_2)$ (see Section 2); the notation of the submodule of decomposables is the usual one.

The paper is divided into three sections and organized as follows. Background and references are provided in Section 2. Theorems 1.3 and 1.4 and Proposition 1.5 are proved in Section 3.

2 Background

We start this section by sketching briefly Singer's invariant-theoretic description of the lambda algebra.

Let T_s be the Sylow 2–subgroup of GL_s consisting of all upper triangular $s \times s$ matrices with 1 on the main diagonal. The T_s–invariant ring, $M_s = P_s^{T_s}$, is called the

Mùi algebra. In [19], Mùi shows that $P_s^{T_s}$ is a polynomial algebra

$$P_s^{T_s} = \mathbb{F}_2[V_1, \dots, V_s],$$

on elements V_k of degree 2^{k-1}, where

$$V_i = V_i(x_1, \dots, x_i) = \prod_{a_j \in \mathbb{F}_2} (a_1 x_1 + \dots + a_{i-1} x_{i-1} + x_i).$$

Recall that the Dickson algebra D_s was computed in [5]:

$$D_s = \mathbb{F}_2[Q_{s,0}, \dots, Q_{s,s-1}].$$

Here the Dickson invariant $Q_{s,i}$ of degree $2^s - 2^i$ can inductively be defined by

$$Q_{s,i} = Q_{s-1,i-1}^2 + Q_{s-1,i} V_s,$$

where, by convention, $Q_{s,s} = 1$ and $Q_{s,i} = 0$ for $i < 0$ (see [5; 19]). (For the action of Steenrod algebra on V_i and $Q_{s,i}$, see [8].)

Let $L(s) \subset P_s$ be the multiplicative subset generated by all the nonzero linear forms in P_s. Let $(P_s)_{L(s)}$ be the localization given by inverting all the nonzero linear forms in P_s. Using the results of Dickson [5] and Mùi [19], Singer notes in [20] that

$$\begin{aligned} \Delta_s &:= ((P_s)_{L(s)})^{T_s} = \mathbb{F}_2[V_1^{\pm 1}, \dots, V_s^{\pm 1}], \\ \Gamma_s &:= ((P_s)_{L(s)})^{\mathrm{GL}_s} = \mathbb{F}_2[Q_{s,s-1}, \dots, Q_{s,1}, Q_{s,0}^{\pm 1}]. \end{aligned}$$

Further, he sets

$$v_1 = V_1, v_k = V_k / V_1 \cdots V_{k-1} \quad (k \geq 2),$$

so that

$$V_k = v_1^{2^{k-2}} v_2^{2^{k-3}} \cdots v_{k-1} v_k \quad (k \geq 2).$$

Then, he obtains

$$\Delta_s = \mathbb{F}_2[v_1^{\pm 1}, \dots, v_s^{\pm 1}],$$

with $\deg v_i = 1$ for every i.

Singer defines Γ_s^+ to be the $\mathbb{F}_2$–subspace of $\Gamma_s = D_s[Q_{s,0}^{-1}]$ spanned by all monomials $\gamma = Q_{s,s-1}^{i_{s-1}} \cdots Q_{s,0}^{i_0}$ with $i_{s-1}, \dots, i_1 \geq 0, i_0 \in \mathbb{Z}$, and $i_0 + \deg \gamma \geq 0$. He also shows in [20] that the homomorphism

$$\partial_s \colon \Delta_s \otimes N \to \Delta_{s-1} \otimes N, \quad \partial_s(v_1^{j_1} \cdots v_s^{j_s} \otimes z) = v_1^{j_1} \cdots v_{s-1}^{j_{s-1}} \otimes \mathrm{Sq}^{j_s+1} z,$$

maps $\Gamma_s^+ \otimes N$ to $\Gamma_{s-1}^+ \otimes N$. Here N is an arbitrary left $\mathcal{A}$–module. Moreover, it is a differential on $\Gamma^+ N = \bigoplus_s (\Gamma_s^+ \otimes N)$. He also proves that

$$H_s(\Gamma^+ N) \cong \operatorname{Tor}_s^{\mathcal{A}}(\mathbb{F}_2, N).$$

Let Λ be the (opposite) lambda algebra, in which the product in lambda symbols is written in the order opposite to that used in [2]. It is bigraded by putting $\operatorname{bideg}(\lambda_i) = (1, i)$, where $\lambda_i \in \Lambda^{1,i}$. Singer proves in [20] that the $\mathbb{F}_2$–linear map

$$\ell_s \colon \Gamma_s^+ \to (\Lambda^s)^*, \quad v_1^{j_1} \cdots v_s^{j_s} \mapsto (\lambda_{j_1} \cdots \lambda_{j_s})^*,$$

is an isomorphism for each $s \geq 0$. Here the duality $*$ is taken with respect to the basis of admissible monomials of Λ. Recall that for each $s \geq 1$, a basis for Λ^s is given by the set of admissible monomials

$$\{\lambda_{j_1}\lambda_{j_2} \cdots \lambda_{j_s} \mid 0 \leq j_1,\ j_1 \leq 2j_2,\ \ldots,\ j_{s-1} \leq 2j_s\},$$

while Λ^0 is spanned by the unit (see [20]).

Suppose N is a left $\mathcal{A}$–module which is finitely generated in every degree. Let N^* be the $\mathbb{F}_2$–dual of N which is a right $\mathcal{A}$–module by transposing the left $\mathcal{A}$–module on N. The tensor product $\Lambda \otimes N^*$ is bigraded by

$$(\Lambda \otimes N^*)^{s,t} = \sum_k \Lambda^{s,t-k} \otimes N_k^*.$$

For any sequence $I = (i_1, \ldots, i_s)$ of nonnegative integers, we write λ_I to denote $\lambda_{i_1} \cdots \lambda_{i_s} \in \Lambda$. For $m^* \in N^*$, we write $\lambda_I m^*$ to denote $\lambda_I \otimes m^* \in \Lambda \otimes N^*$ and let $m^* = 1m^*$. So $\Lambda \otimes N^*$ is a bigraded differential left Λ–module with the action of Λ on it given by

$$\lambda_J(\lambda_I m^*) = \lambda_J \lambda_I m^*,$$

where J is a sequence of nonnegative integers. Moreover, the differential of $\Lambda \otimes N^*$ is given by

$$\delta(\lambda_I m^*) = \delta(\lambda_I) m^* + \sum_{j \geq 0} \lambda_I \lambda_j m^* \operatorname{Sq}^{j+1}.$$

(For the differential δ on the lambda algebra, see [2; 18].) Then $\operatorname{Ext}_{\mathcal{A}}^{s,s+t}(N, \mathbb{F}_2) = H^{s,t}(\Lambda \otimes N^*, \delta)$ (see [2; 18]). By means of the differential, one recognizes that the left action of Λ on $\Lambda \otimes N^*$ induces a left action of $\operatorname{Ext}_{\mathcal{A}}^{*,*} := \operatorname{Ext}_{\mathcal{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ on $\operatorname{Ext}_{\mathcal{A}}^{*,*}(N, \mathbb{F}_2)$. Hence, the latter becomes a left $\operatorname{Ext}_{\mathcal{A}}^{*,*}$–module.

In the remaining part of this section, we recall some results used to prove the main results in this paper.

Theorem 2.1 (Hưng and Tuấn [14]) *Let M be an unstable $\mathcal{A}$–module. Then, for any $s \geq 0$, the map*

$$(\widetilde{\varphi_s^M})^*\colon R_sM \to \Gamma_s^+ M, \quad c\,\mathrm{St}_s(z) \mapsto c\,Q_{s,0}^{|z|} \otimes z,$$

for $c \in D_s$ and a homogeneous element z of degree $|z|$ in M, is a chain-level representation of the dual of the Lannes–Zarati homomorphism

$$(\varphi_s^M)^*\colon (\mathbb{F}_2 \otimes_{\mathcal{A}} R_sM)_i \to \mathrm{Tor}^{\mathcal{A}}_{s,s+i}(\mathbb{F}_2, M).$$

This map is natural with respect to $\mathcal{A}$–homomorphisms of unstable $\mathcal{A}$–modules.

As R_sM is a free D_s–module (see [16, Definition-Proposition 2.4.1]), the map is well defined.

An element in D_s is called $\mathcal{A}$–decomposable if it is in $\bar{\mathcal{A}}D_s$, where $\bar{\mathcal{A}}$ denotes the augmentation ideal of the Steenrod algebra $\mathcal{A}$.

Giambalvo and Peterson showed in [6] a sufficient condition for a monomial in D_s to be $\mathcal{A}$–decomposable as follows:

Theorem 2.2 (see [6, Corollary 4.8]) *Let $s \geq 2$ and assume that $I = (i_0, \dots, i_{s-1})$ is a s–tuple of nonnegative integers and $Q^I = Q_{s,0}^{i_0} \cdots Q_{s,s-1}^{i_{s-1}} \in D_s$ with $i_0 > s-2$. Then Q^I is $\mathcal{A}$–decomposable.*

3 On the vanishing of the Lannes–Zarati homomorphism on decomposable elements

The goal of this section is to prove Theorems 1.3 and 1.4 and Proposition 1.5.

In [20], Singer defines an algebra isomorphism $\psi_{p,q}\colon \Delta_s \to \Delta_p \otimes \Delta_q$ by

$$\psi_{p,q}(v_i) = \begin{cases} v_i \otimes 1 & \text{if } 1 \leq i \leq p, \\ 1 \otimes v_{i-p} & \text{if } p+1 \leq i \leq s, \end{cases}$$

for any pair of nonnegative integers p and q for which $p+q=s$. Here we understand $\Delta_0 = \mathbb{F}_2$, $\psi_{s,0}(x) = x \otimes 1$ and $\psi_{0,s}(x) = 1 \otimes x$. Then, he shows that

$$\psi_{p,q}(Q_{s,i}) = \sum_{j \geq 0} Q_{p,0}^{2^q - 2^j} Q_{p,i-j}^{2^j} \otimes Q_{q,j} \tag{3.0.1}$$

for each i with $0 \le i < s$. Suppose $c = Q_{s,0}^{t_0} \cdots Q_{s,s-1}^{t_{s-1}} \in D_s$; then

$$\begin{aligned}
\psi_{p,q}(c) &= \prod_{i=0}^{s-1} \psi_{p,q}(Q_{s,i})^{t_i} && \text{(since } \psi_{p,q} \text{ is an algebra isomorphism)} \\
&= \prod_{i=0}^{s-1} \Bigl(\sum_{j=0}^{\min\{i,q\}} Q_{p,0}^{2^q-2^j} Q_{p,i-j}^{2^j} \otimes Q_{q,j} \Bigr)^{t_i} && \text{(by (3.0.1))} \\
&= \prod_{i=0}^{s-1} \sum_{\substack{|\alpha_i|=t_i \\ d_i=1}} Q^{\alpha_i} && \text{(by the binomial theorem),}
\end{aligned}$$

where

$$d_i = \frac{t_i!}{k_0^{(i)}! \cdots k_{\min\{i,q\}}^{(i)}!}, \quad \alpha_i = (k_0^{(i)}, \dots, k_{\min\{i,q\}}^{(i)}), \quad |\alpha_i| = k_0^{(i)} + \cdots + k_{\min\{i,q\}}^{(i)}$$

and

$$Q^{\alpha_i} = \prod_{j=0}^{\min\{i,q\}} (Q_{p,0}^{2^q-2^j} Q_{p,i-j}^{2^j} \otimes Q_{q,j})^{k_j^{(i)}}.$$

So, for $c \in D_s$, we have $\psi_{p,q}(c) = \sum Q^I \otimes Q^J$ with $Q^I \in D_p$ and $Q^J \in D_q$.

Lemma 3.1 *Suppose $c \in D_s$ and $\psi_{p,q}(c) = \sum Q^I \otimes Q^J$, $p+q=s$. Then each Q^I has the form $Q_{p,0}^{i_0} \cdots Q_{p,p-1}^{i_{p-1}}$, where $i_1 = n_1 + 2m_1, \dots, i_{p-1} = n_{p-1} + 2m_{p-1}$ and $i_0 \ge (2^q-1)(n_1 + \cdots + n_{p-1})$ for $n_1, \dots, n_{p-1}, m_1, \dots, m_{p-1}$ nonnegative integers.*

Proof Suppose $c = Q_{s,0}^{t_0} \cdots Q_{s,s-1}^{t_{s-1}} \in D_s$. From the above calculation, we see that each Q^I has the form $Q_{p,0}^{i_0} \cdots Q_{p,p-1}^{i_{p-1}}$ with

$$\begin{aligned}
i_0 &\ge (2^q-1)(k_0^{(0)} + k_0^{(1)} + \cdots + k_0^{(s-1)}), \\
i_1 &= k_0^{(1)} + 2^1 k_1^{(2)} + \cdots + 2^q k_q^{(q+1)}, \\
i_2 &= k_0^{(2)} + 2^1 k_1^{(3)} + \cdots + 2^q k_q^{(q+2)}, \\
&\vdots \\
i_{p-1} &= k_0^{(p-1)} + 2^1 k_1^{(p)} + \cdots + 2^q k_q^{(s-1)}.
\end{aligned}$$

Set $n_i = k_0^{(i)}$ and $m_i = \sum_{j=1}^{q} 2^{j-1} k_j^{(i+j)}$ for $1 \le i \le p-1$. Then $i_1 = n_1 + 2m_1, \dots, i_{p-1} = n_{p-1} + 2m_{p-1}$ and $i_0 \ge (2^q-1)(n_1 + \cdots + n_{p-1})$.

The lemma follows. □

Suppose N is an $\mathcal{A}$–module of finite type. By ambiguity of notation, the following $\mathbb{F}_2$–linear map is also denoted by the same notation as the isomorphism $\ell_s\colon \Gamma_s^+ \to (\Lambda^s)^*$ (see [20, page 689]):

$$\ell_s\colon \Gamma_s^+ \otimes N \to (\Lambda^s \otimes N^*)^*, \quad v_1^{j_1} \cdots v_s^{j_s} \otimes z \mapsto \langle z, \cdot\rangle \langle \ell_s(v_1^{j_1} \cdots v_s^{j_s}), \cdot \rangle.$$

This map is an $\mathbb{F}_2$–isomorphism for each $s \geq 0$.

The following lemma was first proved for $N = \mathbb{F}_2$ by Singer in [20, page 689].

Lemma 3.2 *The diagram*

$$\begin{tikzcd}
\Gamma_s^+ \otimes N \arrow[r, "\ell_s"] \arrow[d, "\partial"] & (\Lambda^s \otimes N^*)^* \arrow[d, "\delta^*"] \\
\Gamma_{s-1}^+ \otimes N \arrow[r, "\ell_{s-1}"] & (\Lambda^{s-1} \otimes N^*)^*
\end{tikzcd}$$

commutes for $s \geq 1$. Here, N is an $\mathcal{A}$–module of finite type.

Proof Use an argument similar to the proof of [20, Proposition 8.2]. □

Suppose N is an $\mathcal{A}$–module of finite type. Let $\langle \cdot, \cdot \rangle$ be the usual dual paring $\mathrm{Tor}_s^{\mathcal{A}}(\mathbb{F}_2, N) \otimes \mathrm{Ext}_{\mathcal{A}}^s(N, \mathbb{F}_2) \to \mathbb{F}_2$. We note that this dual paring is induced in homology by the dual paring $(\Gamma_s^+ \otimes N) \otimes (\Lambda^s \otimes N^*) \to \mathbb{F}_2$ that allows us to identify $\Gamma_s^+ \otimes N$ with the dual of $\Lambda^s \otimes N^*$, as mentioned in Lemma 3.2. We also denote by $\langle \cdot, \cdot \rangle$ the dual paring $(\mathbb{F}_2 \otimes_{\mathcal{A}} R_s M) \otimes (\mathbb{F}_2 \otimes_{\mathcal{A}} R_s M)^* \to \mathbb{F}_2$ for M an unstable $\mathcal{A}$–module.

Let N be an $\mathcal{A}$–module. Suppose α is an element in $\mathrm{Ext}_{\mathcal{A}}^{s,t}(N, \mathbb{F}_2)$. Then, $\mathrm{stem}(\alpha)$ is given by $\mathrm{stem}(\alpha) = t - s$.

Lemma 3.3 *Let M be an unstable $\mathcal{A}$–module of finite type. Let $c\,\mathrm{St}_s(z)$ be an element of $R_s M$ for $c \in D_s$ and a homogeneous element z of degree $|z|$ in M. Then, for $\alpha \in \mathrm{Ext}_{\mathcal{A}}^p(\mathbb{F}_2, \mathbb{F}_2)$ and $\beta \in \mathrm{Ext}_{\mathcal{A}}^q(M, \mathbb{F}_2)$, $p > 0$, $q \geq 0$ and $p + q = s$,*

$$\langle [c\,\mathrm{St}_s(z)], \varphi_s^M(\alpha\beta) \rangle = \sum_{\substack{|Q^I Q_{p,0}^{2^q|z|}| = \mathrm{stem}(\alpha) \\ |Q^J \mathrm{St}_q(z)| = \mathrm{stem}(\beta)}} \langle [Q^I Q_{p,0}^{2^q|z|}], \varphi_p^{\mathbb{F}_2}(\alpha) \rangle \langle [Q^J \mathrm{St}_q(z)], \varphi_q^M(\beta) \rangle.$$

Here Q^I and Q^J appear in $\psi_{p,q}(c) = \sum Q^I \otimes Q^J$.

Proof Suppose $\alpha = [x] \in \mathrm{Ext}^p_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ and $\beta = [y] \in \mathrm{Ext}^q_{\mathcal{A}}(M, \mathbb{F}_2)$, where x is a cycle in Λ^p and y is a cycle in $\Lambda^q \otimes M^*$. Then we have

$$\begin{aligned}
&\langle [c\,\mathrm{St}_s(z)], \varphi^M_s(\alpha\beta)\rangle \\
&\qquad = \langle (\varphi^M_s)^*([c\,\mathrm{St}_s(z)]), \alpha\beta\rangle \\
&\qquad = \langle [c\,Q^{|z|}_{s,0} \otimes z], \alpha\beta\rangle && \text{(by Theorem 2.1)} \\
&\qquad = \langle c\,Q^{|z|}_{s,0} \otimes z, xy\rangle \\
&\qquad = \langle \psi_{p,q}(c\,Q^{|z|}_{s,0}) \otimes z, x \otimes y\rangle && \text{(see [20, page 688])} \\
&\qquad = \Big\langle \sum Q^I Q^{2^q|z|}_{p,0} \otimes Q^J Q^{|z|}_{q,0} \otimes z, x \otimes y \Big\rangle && (\text{since } \psi_{p,q}(Q_{s,0}) = Q^{2^q}_{p,0} \otimes Q_{q,0}) \\
&\qquad = \sum \langle Q^I Q^{2^q|z|}_{p,0}, x\rangle \langle Q^J Q^{|z|}_{q,0} \otimes z, y\rangle.
\end{aligned}$$

We note that $Q^I Q^{2^q|z|}_{p,0}$ and $Q^J Q^{|z|}_{q,0} \otimes z$ are cycles in Γ^+_p and $\Gamma^+_q \otimes M$, respectively. So, we get

$$\begin{aligned}
&\langle [c\,\mathrm{St}_s(z)], \varphi^M_s(\alpha\beta)\rangle \\
&\quad = \sum \langle [Q^I Q^{2^q|z|}_{p,0}], \alpha\rangle \langle [Q^J Q^{|z|}_{q,0} \otimes z], \beta\rangle \\
&\quad = \sum \langle (\varphi^{\mathbb{F}_2}_p)^*[Q^I Q^{2^q|z|}_{p,0}], \alpha\rangle \langle (\varphi^M_q)^*[Q^J\,\mathrm{St}_q(z)], \beta\rangle && \text{(by Theorem 2.1)} \\
&\quad = \sum_{\substack{|Q^I Q^{2^q|z|}_{p,0}| = \mathrm{stem}(\alpha) \\ |Q^J \mathrm{St}_q(z)| = \mathrm{stem}(\beta)}} \langle (\varphi^{\mathbb{F}_2}_p)^*[Q^I Q^{2^q|z|}_{p,0}], \alpha\rangle \langle (\varphi^M_q)^*[Q^J\,\mathrm{St}_q(z)], \beta\rangle \\
&\quad = \sum_{\substack{|Q^I Q^{2^q|z|}_{p,0}| = \mathrm{stem}(\alpha) \\ |Q^J \mathrm{St}_q(z)| = \mathrm{stem}(\beta)}} \langle [Q^I Q^{2^q|z|}_{p,0}], \varphi^{\mathbb{F}_2}_p(\alpha)\rangle \langle [Q^J\,\mathrm{St}_q(z)], \varphi^M_q(\beta)\rangle. && \square
\end{aligned}$$

We recall the following lemma, which was first proved in [12]. We give a proof to make the paper self-contained.

Lemma 3.4 (Hưng and Peterson [12]) *Let $c = Q^{i_0}_{p,0} \cdots Q^{i_{p-1}}_{p,p-1} \in D_p$ with $i_0 > 0$. If $i_m \equiv 0 \pmod 2$ for some $m > 0$, then c is $\mathcal{A}$–decomposable.*

Proof We prove this by induction on the smallest $m > 0$ with $i_m \equiv 0 \pmod 2$. If $m = 1$, then $\mathrm{Sq}^1(Q^{i_0-1}_{p,0} Q^{i_1+1}_{p,1} \cdots Q^{i_{p-1}}_{p,p-1}) = c$. For the induction step,

$$\mathrm{Sq}^{2^{m-1}}(Q^{i_0}_{p,0} \cdots Q^{i_{m-1}-1}_{p,m-1} Q^{i_m+1}_{p,m} \cdots Q^{i_{p-1}}_{p,p-1}) = c + \sum Q^K,$$

where each Q^K has the form $Q^{k_0}_{p,0} \cdots Q^{k_{p-1}}_{p,p-1}$ with $k_{m-1} \equiv 0 \pmod 2$ and $k_0 > 0$. □

The following repeats Theorem 1.3 from the introduction:

Theorem 3.5 *Let M be an unstable $\mathcal{A}$–module of finite type. Then the s^{th} Lannes–Zarati homomorphism for M*

$$\varphi_s^M\colon \operatorname{Ext}_{\mathcal{A}}^{s,s+i}(M,\mathbb{F}_2)\to(\mathbb{F}_2\otimes_{\mathcal{A}}R_sM)_i^*$$

vanishes on the elements of the form $\alpha\beta$ in any positive stem i, where $\alpha\in\operatorname{Ext}_{\mathcal{A}}^p(\mathbb{F}_2,\mathbb{F}_2)$ and $\beta\in\operatorname{Ext}_{\mathcal{A}}^q(M,\mathbb{F}_2)$ with either $p\geq 2$, $q>0$ and $p+q=s$, or $p=s\geq 2$, $q=0$ and $\operatorname{stem}(\beta)>s-2$.

Proof We will show that $\varphi_s^M(\alpha\beta)=0$, where $\alpha\in\operatorname{Ext}_{\mathcal{A}}^p(\mathbb{F}_2,\mathbb{F}_2)$, $\beta\in\operatorname{Ext}_{\mathcal{A}}^q(M,\mathbb{F}_2)$ with either $p\geq 2$, $q>0$ and $p+q=s$, or $p=s\geq 2$, $q=0$ and $\operatorname{stem}(\beta)>s-2$.

Case 1 ($p\geq 2$, $q>0$) By Lemma 3.3, for any $c\operatorname{St}_s(z)\in R_sM$ with $c\in D_s$, a homogeneous element $z\in M$ of degree $|z|$ and $\psi_{p,q}(c)=\sum Q^I\otimes Q^J$ with $Q^I\in D_p$ and $Q^J\in D_q$, we have

$$\langle[c\operatorname{St}_s(z)],\varphi_s^M(\alpha\beta)\rangle=\sum_{\substack{|Q^IQ_{p,0}^{2^q|z|}|=\operatorname{stem}(\alpha)\\ |Q^J\operatorname{St}_q(z)|=\operatorname{stem}(\beta)}}\langle[Q^IQ_{p,0}^{2^q|z|}],\varphi_p^{\mathbb{F}_2}(\alpha)\rangle\langle[Q^J\operatorname{St}_q(z)],\varphi_q^M(\beta)\rangle.$$

We see that $\psi_{p,q}(cQ_{s,0}^{|z|})=\sum Q^IQ_{p,0}^{2^q|z|}\otimes Q^JQ_{q,0}^{|z|}$. So, by Lemma 3.1, $Q^IQ_{p,0}^{2^q|z|}$ has the form $Q_{p,0}^{i_0}\cdots Q_{p,p-1}^{i_{p-1}}$, where $i_0\geq(2^q-2^0)(n_1+\cdots+n_{p-1})$, $i_1=n_1+2m_1,\ldots,$ $i_{p-1}=n_{p-1}+2m_{p-1}$. We will prove that $Q^IQ_{p,0}^{2^q|z|}$ is $\mathcal{A}$–decomposable.

If $i_0=0$, then $0\geq(2^q-2^0)(n_1+\cdots+n_{p-1})$. So, it implies that $n_1=\cdots=n_{p-1}=0$. We get

$$Q^IQ_{p,0}^{2^q|z|}=Q_{p,1}^{2m_1}\cdots Q_{p,p-1}^{2m_{p-1}}=\operatorname{Sq}^{(2^p-2^1)m_1+\cdots+(2^p-2^{p-1})m_{p-1}}(Q_{p,1}^{m_1}\cdots Q_{p,p-1}^{m_{p-1}}).$$

Hence, $Q^IQ_{p,0}^{2^q|z|}\in\bar{\mathcal{A}}D_p$

If $i_0>0$ and one of the nonnegative integers $n_1,\ldots,n_{p-1}$ is even, then by Lemma 3.4 we have $Q^IQ_{p,0}^{2^q|z|}=Q_{p,0}^{i_0}\cdots Q_{p,p-1}^{i_{p-1}}\in\bar{\mathcal{A}}D_p$.

If $i_0>0$ and all of the nonnegative integers $n_1,\ldots,n_{p-1}$ are odd, then

$$i_0\geq(2^q-2^0)(n_1+\cdots+n_{p-1})\geq p-1>p-2.$$

Hence, by Theorem 2.2, we obtain $Q^IQ_{p,0}^{2^q|z|}=Q_{p,0}^{i_0}\cdots Q_{p,p-1}^{i_{p-1}}\in\bar{\mathcal{A}}D_p$.

So, we get $\langle[Q^IQ_{p,0}^{2^q|z|}],\varphi_p^{\mathbb{F}_2}(\alpha)\rangle=\langle[0],\varphi_p^{\mathbb{F}_2}(\alpha)\rangle=0$. We conclude that $\varphi_s^M(\alpha\beta)=0$.

Case 2 ($p = s \geq 2$, $q = 0$ and $\mathrm{stem}(\beta) > s-2$) By Lemma 3.3, for any $c\,\mathrm{St}_s(z) \in R_s M$ with $c \in D_s$, a homogeneous element $z \in M$ of degree $|z|$ and $\psi_{s,0}(c) = c \otimes 1$, we have

$$\langle [c\,\mathrm{St}_s(z)], \varphi_s^M(\alpha\beta)\rangle = \langle [c\,Q_{s,0}^{|z|}], \varphi_s^M(\alpha)\rangle \langle [z], \beta\rangle.$$

If $|z| \neq \mathrm{stem}(\beta)$, then $\langle [z], \beta\rangle = 0$. So $\langle [c\,\mathrm{St}_s(z)], \varphi_s^M(\alpha\beta)\rangle = 0$.

If $|z| = \mathrm{stem}(\beta)$, then $|z| > s-2$ (since $\mathrm{stem}(\beta) > s-2$). By Theorem 2.2, we have $c\,Q_{s,0}^{|z|} \in \bar{\mathcal{A}} D_s$. We conclude that $\langle [c\,Q_{s,0}^{|z|}], \varphi_s^M(\alpha)\rangle = \langle [0], \varphi_s^M(\alpha)\rangle = 0$. Hence, $\langle [c\,\mathrm{St}_s(z)], \varphi_s^M(\alpha\beta)\rangle = 0$.

The theorem is proved. □

Consequently, when $M = \mathbb{F}_2$, we obtain the following theorem, which was first proved by Hưng and Peterson in [12]. Recall that Hưng and Peterson proved this theorem by showing that $\varphi_* = \bigoplus_s \varphi_s^{\mathbb{F}_2}$ is a homomorphism of algebras and, more importantly, that the product of the nonunital algebra $\bigoplus_{s>0}(\mathbb{F}_2 \otimes_{\mathcal{A}} D_s)^*$ is trivial, except for the case $(\mathbb{F}_2 \otimes_{\mathcal{A}} D_1)^* \otimes (\mathbb{F}_2 \otimes_{\mathcal{A}} D_1)^* \to (\mathbb{F}_2 \otimes_{\mathcal{A}} D_2)^*$.

The following repeats Theorem 1.4 from the introduction:

Theorem 3.6 (Hưng and Peterson [12]) *The s^{th} Lannes–Zarati homomorphism*

$$\varphi_s^{\mathbb{F}_2}\colon \mathrm{Ext}_{\mathcal{A}}^{s,s+i}(\mathbb{F}_2, \mathbb{F}_2) \to (\mathbb{F}_2 \otimes_{\mathcal{A}} D_s)_i^*$$

vanishes on the decomposable elements in any positive stem i for $s \geq 3$.

Proof We must show that $\varphi_s^{\mathbb{F}_2}(\alpha\beta) = 0$, where $\alpha \in \mathrm{Ext}_{\mathcal{A}}^p(\mathbb{F}_2, \mathbb{F}_2)$ and $\beta \in \mathrm{Ext}_{\mathcal{A}}^q(\mathbb{F}_2, \mathbb{F}_2)$ with $p > 0$, $q > 0$ and $p + q = s$. Since the algebra $\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ is commutative, we have left to consider the case $p \geq 2$ and $q > 0$. In this case, by Theorem 3.5, we have $\varphi_s^{\mathbb{F}_2}(\alpha\beta) = 0$.

The theorem is proved. □

For brevity, $\widetilde{H}^*(\mathbb{RP}^\infty)$ will be denoted by $\widetilde{P}$. The following repeats Proposition 1.5 from the introduction:

Proposition 3.7 *The fifth Lannes–Zarati homomorphism for $\widetilde{P}$,*

$$\varphi_5^{\widetilde{P}}\colon \mathrm{Ext}_{\mathcal{A}}^{5,5+i}(\widetilde{P}, \mathbb{F}_2) \to (\mathbb{F}_2 \otimes_{\mathcal{A}} R_5 \widetilde{P})_i^*,$$

vanishes on the decomposable elements in any positive stem i.

Proof We must prove that $\varphi_5^{\widetilde{P}}(\alpha\beta)=0$, where $\alpha\in\mathrm{Ext}_{\mathcal{A}}^{p}(\mathbb{F}_2,\mathbb{F}_2)$ and $\beta\in\mathrm{Ext}_{\mathcal{A}}^{q}(\widetilde{P},\mathbb{F}_2)$ with $p>0$, $q\geq 0$ and $p+q=5$. We will consider the following three cases:

Case 1 $(p\geq 2,\ q>0)$ By Theorem 3.5, we have $\varphi_5^{\widetilde{P}}(\alpha\beta)=0$.

Case 2 $(p=5,\ q=0)$ Then, for any $c\,\mathrm{St}_5(z)\in R_5\widetilde{P}$ with $c\in D_5$ and a homogeneous element $z\in\widetilde{P}$ of degree $|z|$, we have $\psi_{5,0}(c)=c\otimes 1$, and

$$\begin{aligned}\langle[c\,\mathrm{St}_5(z)],\varphi_5^{\widetilde{P}}(\alpha\beta)\rangle &= \langle[c\,Q_{5,0}^{|z|}],\varphi_5^{\mathbb{F}_2}(\alpha)\rangle\langle[\mathrm{St}_0(z)],\varphi_0^{\widetilde{P}}(\beta)\rangle \quad \text{(by Lemma 3.3)}\\ &=0,\end{aligned}$$

where the last equality follows from the fact that $\varphi_5^{\mathbb{F}_2}(\alpha)=0$ (see [13, Theorem 1.4]).

Case 3 $(p=1,\ q=4)$ Then, for any $c\,\mathrm{St}_5(z)\in R_5\widetilde{P}$ with $c\in D_5$, a homogeneous element $z\in\widetilde{P}$ of degree $|z|$ and $\psi_{1,4}(c)=\sum Q^I\otimes Q^J$ with $Q^I\in D_1$ and $Q^J\in D_4$, we have

$$\begin{aligned}\langle[c\,\mathrm{St}_5(z)],\varphi_5^{\widetilde{P}}(\alpha\beta)\rangle &= \sum_{\substack{|Q^I Q_{1,0}^{2^4|z|}|=\mathrm{stem}(\alpha)\\ |Q^J\mathrm{St}_4(z)|=\mathrm{stem}(\beta)}} \langle[Q^I Q_{1,0}^{2^4|z|}],\varphi_1^{\mathbb{F}_2}(\alpha)\rangle\langle[Q^J\mathrm{St}_4(z)],\varphi_4^{\widetilde{P}}(\beta)\rangle\\ &\hspace{20em}\text{(by Lemma 3.3)}\\ &=0,\end{aligned}$$

where the last equality follows from the fact that $\varphi_4^{\widetilde{P}}(\beta)=0$ (see [14, Theorem 1.8]).

The proposition is completely proved. □

Remark 3.8 From the proof of Proposition 3.7, and Theorem 3.5, we can see that for $s\geq 3$, and for any unstable $\mathcal{A}$–module M of finite type, if $\varphi_s^{\mathbb{F}_2}$ and φ_{s-1}^{M} vanish in positive stems, then φ_s^{M} vanishes on the decomposable elements in positive stems.

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Homotopy theory of unital algebras

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We provide an extensive study of the homotopy theory of types of algebras with units, for instance unital associative algebras or unital commutative algebras. To this purpose, we endow the Koszul dual category of curved coalgebras, where the notion of quasi-isomorphism barely makes sense, with a model category structure Quillen equivalent to that of unital algebras. To prove such a result, we use recent methods based on presentable categories. This allows us to describe the homotopy properties of unital algebras in a simpler and richer way. Moreover, we endow the various model categories with several enrichments which induce suitable models for the mapping spaces and describe the formal deformations of morphisms of algebras.

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Introduction		1541
1.	Preliminaries	1546
2.	Operads, cooperads, algebras and coalgebras	1554
3.	Enrichment	1565
4.	Bar–cobar adjunctions	1572
5.	Homotopy theory of algebras over an operad	1577
6.	Model structures on coalgebras over a cooperad	1581
7.	The universal model structure	1589
8.	Examples	1600
Appendix		1614
References		1617

Introduction

Among the various types of algebras, some of them include units, like the ubiquitous unital associative algebras and unital commutative algebras or the unital Batalin–Vilkovisky algebras, which arose in mathematical physics. When working with a chain complex carrying such an algebraic structure, like the de Rham algebra of differential

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manifolds, one would like to understand the properties that this algebraic data satisfies up to quasi-isomorphisms. The purpose of the present paper is to develop a framework which allows one to prove the homotopical properties carried by types of algebras with units, that is, their properties up to quasi-isomorphisms.

In order to work with types of algebras in a general way, one needs a precise notion which encodes these ones. This is achieved by the concept of an operad. Operads are generalizations of associative algebras which encode some types of algebras (associative, commutative, Lie, Batalin–Vilkovisky, …) in a way that a representation of an operad $\mathscr{P}$ is a chain complex together with a structure of algebra of the type encoded by $\mathscr{P}$.

Further, one of the most common and powerful tool to study homotopical algebra — that is to study categories with a notion of weak equivalences — is the model category structure introduced by Daniel Quillen, which makes the manipulation of weak equivalences easier by means of other maps, called cofibrations and fibrations, respectively. Hinich proved in [15] that the category of algebras over an operad carries a model structure whose weak equivalences are quasi-isomorphisms and whose fibrations are surjections. In a purely theoretical perspective, this model structure describes all the homotopical data of this category. However, the cofibrant objects are not easy to handle; they are the retracts of free algebras whose generators carry a particular filtration.

Hinich [16] embedded the category of differential graded (dg) Lie algebras into the category of dg cocommutative coalgebras. From the model structure of the category of dg Lie algebras he obtained a model structure on the category of dg cocommutative coalgebras which is Quillen equivalent to the first one. In this new model category, any object is cofibrant. Moreover, this context allows one to build an obstruction theory for the existence of the algebra structures and the algebra morphisms. So this new context of dg cocommutative coalgebras is more suitable to study the homotopy theory of dg Lie algebras than the category of dg Lie algebras itself. With a similar perspective, Lefevre and Hasegawa embedded the category of nonunital dg associative algebras into the category dg coassociative coalgebras, shown to be Quillen equivalent to the first one; see [18]. Vallette generalized these results to all types of algebras encoded by any operad satisfying a technical condition: that it is an augmented operad. Augmented operads are related to the dual notion of conilpotent cooperads by an adjunction called the operadic bar–cobar adjunction $\Omega \dashv B$. Vallette embedded the category of algebras over an augmented operad $\mathscr{P}$ into category of coalgebras over a cooperad $\mathscr{P}^{¡}$ called the Koszul dual of $\mathscr{P}$. He transferred the model structure on the category of $\mathscr{P}$–algebras to the category of $\mathscr{P}^{¡}$–coalgebras and got again a Quillen equivalence between these two model categories; see [26].

However the operads describing types algebras with units do not satisfy the technical condition to be augmented. To extend the result of Vallette to categories of algebras over any operad, one first needs to modify the operadic bar–cobar adjunction. Inspired by the work of Hirsh and Millès [17], we introduce an adjunction à la bar–cobar relating dg operads to curved conilpotent cooperads:

$$\begin{tikzcd}\text{curved conilpotent cooperads} \arrow[r, shift left, "\Omega_u"] & \text{dg operads.} \arrow[l, shift left, "B_c"]\end{tikzcd}$$

Moreover, any morphism of dg operads f from a cobar construction $\Omega_u\mathscr{C}$ of a curved conilpotent cooperad $\mathscr{C}$ to an operad $\mathscr{P}$ comes equipped with an adjunction $\Omega_f \dashv B_f$ relating $\mathscr{P}$–algebras to $\mathscr{C}$–coalgebras,

$$\begin{tikzcd}\mathscr{C}\text{–coalgebras} \arrow[r, shift left, "\Omega_f"] & \mathscr{P}\text{–algebras.} \arrow[l, shift left, "B_f"]\end{tikzcd}$$

The model structure of $\mathscr{P}$–algebras can be transferred to the category of $\mathscr{C}$–coalgebras along this adjunction.

Theorem 82 *Let $\alpha\colon \mathscr{C} \to \mathscr{P}$ be an operadic twisting morphism and let $\Omega_\alpha \dashv B_\alpha$ be the bar–cobar adjunction between $\mathscr{P}$–algebras and $\mathscr{C}$–coalgebras induced by α. There exists a model structure on the category of $\mathscr{C}$–coalgebras whose cofibrations (resp. weak equivalences) are morphisms whose image under Ω_α is a cofibration (resp. weak equivalence). With this model category structure, the adjunction $\Omega_\alpha \dashv B_\alpha$ is a Quillen adjunction.*

To prove this theorem, we use new techniques coming from category theory. Specifically, we utilize a theorem of Bayeh, Hess, Karpova, Kędziorek, Riehl and Shipley [3] involving presentable categories.

We study in detail the particular case where the morphism of operads f from $\Omega_u\mathscr{C}$ to $\mathscr{P}$ is a quasi-isomorphism, for instance if f is the identity ι of $\Omega_u\mathscr{C}$. In this case, the Quillen adjunction $\Omega_\iota \dashv B_\iota$ is a Quillen equivalence. We show that the fibrant $\mathscr{C}$–coalgebras are the images of the $\Omega_u\mathscr{C}$–algebras under the functor B_ι. So, switching from the category of $\Omega_u\mathscr{C}$–algebras to the category of $\mathscr{C}$–coalgebras by the functor B_ι amounts to introducing new morphisms between $\Omega_u\mathscr{C}$–algebras. These new morphisms can be built using obstruction methods. Moreover, any $\Omega_u\mathscr{C}$–algebra becomes cofibrant in this new context.

This article also deals with enrichments of the category of $\mathscr{P}$–algebras for any differential graded operad $\mathscr{P}$, and of the category of $\mathscr{C}$–coalgebras for any curved cooperad $\mathscr{C}$. These two categories are enriched in simplicial sets in a way that recovers the mapping spaces. Further, they are tensored, cotensored and enriched in cocommutative coalgebras. These cocommutative coalgebras encode the formal deformations of morphisms of algebras over an operad. Indeed, for any two algebras $\mathcal{A}$ and $\mathcal{A}'$ over an operad $\mathscr{P}$, the atoms of their mapping cocommutative coalgebra $\{\mathcal{A}, \mathcal{A}'\}$ — that is, the closed elements $e \in \{\mathcal{A}, \mathcal{A}'\}_0$ such that $\Delta(e) = e \otimes e$ — are exactly the morphisms of $\mathscr{P}$–algebras from $\mathcal{A}$ to $\mathcal{A}'$. Moreover, if $\mathcal{A}$ is cofibrant, the maximal coaugmented conilpotent subcoalgebra of $\{\mathcal{A}, \mathcal{A}'\}$ that contains an atom f is the bar construction of the Lie algebra that controls the formal deformations of the morphism f. In the context of nonsymmetric operads and nonsymmetric cooperads, this enrichment can be extended to all coassociative coalgebras. These coassociative coalgebras encode in single objects both the mapping spaces and the deformation of morphisms.

Finally, we apply the framework developed here to concrete operads like the operad $u\mathcal{A}s$ of unital associative algebras and the operad $u\mathcal{C}om$ of unital commutative algebras. For these two operads, the process of curved Koszul duality developed in [17] relates the curved cooperads $u\mathcal{A}s^{\text{¡}}$ and $u\mathcal{C}om^{\text{¡}}$ to the operads $u\mathcal{A}s$ and $u\mathcal{C}om$, respectively. We show that the category of $u\mathcal{A}s^{\text{¡}}$–coalgebras and the category of $u\mathcal{C}om^{\text{¡}}$–coalgebras are equivalent to the category of curved conilpotent coassociative coalgebras and the category of curved conilpotent Lie coalgebras, respectively.

Layout

The article is organized as follows. In Section 1, we recall several notions about category theory, and homological algebra. In Section 2, we recall the notions of operads, cooperads, algebras over an operad and coalgebras over a cooperad. We also prove some results, as the presentability of the category of coalgebras over a curved cooperad, that we will need in the sequel. Section 3 deals with enrichments of the category of algebras over an operad and of the category of coalgebras over a curved cooperad; specifically, we study enrichments over simplicial sets, cocommutative coalgebras and coassociative coalgebras. In Section 4, we introduce an adjunction à la bar–cobar between operads and curved cooperads related to a notion of twisting morphism. We use it to define an adjunction between $\mathscr{P}$–algebras and $\mathscr{C}$–coalgebras for a twisting morphism from a curved cooperad $\mathscr{C}$ to an operad $\mathscr{P}$. In Section 5, we recall the projective model structure on the category of algebras over an operad.

We describe models for the mapping spaces and we show that the enrichment over cocommutative coalgebras encodes deformations of morphisms. Section 6 transfers the projective model structure on $\mathscr{P}$–algebras along the previous adjunction to obtain a model structure on $\mathscr{C}$–coalgebras and a Quillen adjunction. Section 7 deals with these model structures in the case where the operad $\mathscr{P}$ is the cobar construction $\Omega_u\mathscr{C}$ of $\mathscr{C}$. In particular, the adjunction induced is a Quillen equivalence. Finally, in Section 8, we apply the formalism developed in the previous sections to study the examples of unital associative algebras and unital commutative algebras.

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Conventions and notation

- We work over a field $\mathbb{K}$. Note that no further assumption is needed when working with nonsymmetric operads. However, when dealing with homotopy results that concern all operads and cooperads and their algebras and coalgebras, we will assume the characteristic of the field to be zero.
- The category of $\mathbb{Z}$–graded $\mathbb{K}$–modules, that is, functors from the discrete category $\mathbb{Z}$ to the category of $\mathbb{K}$–vector spaces, is denoted by gMod. The category of chain complexes, that is, $\mathbb{Z}$–graded $\mathbb{K}$–modules equipped with a degree -1 square-zero map, is denoted by dgMod. They are endowed with their usual closed symmetric monoidal structures. The internal hom is denoted by $[\,,]$. The category of chain complexes is also endowed with its projective model structure, where the weak equivalences are the quasi-isomorphisms and where the fibrations are the degreewise surjections. The degree of a homogeneous element x of a graded $\mathbb{K}$–module or a chain complex is denoted by $|x|$.
- For any integer n, let D^n be the chain complex generated by one element in degree n and its boundary in degree $n-1$. Let S^n be the chain complex generated by a cycle in degree n.

- The category of simplicial set is denoted by sSet. It is endowed with its Kan–Quillen model structure; see Goerss and Jardine [13, I.11.3].

- A diagram of the form

$$\begin{tikzcd} \mathsf{C} \arrow[r, shift left, "L"] & \mathsf{D} \arrow[l, shift left, "R"] \end{tikzcd}$$

means that the functor R is right adjoint to the functor L.

- For any graded $\mathbb{K}$–module $\mathcal{V}$ endowed with a filtration $(F_n\mathcal{V})_{n\in\mathbb{N}}$, the graded complex associated to this filtration is denoted by $G\mathcal{V}$. In other words,

$$G\mathcal{V} = \bigoplus_n G_n\mathcal{V}$$

where $G_n\mathcal{V} = F_n\mathcal{V}/F_{n-1}\mathcal{V}$. If $\mathcal{V}$ is a chain complex such that $(F_n\mathcal{V})_{n\in\mathbb{N}}$ is a filtration of chain complexes, that is, $d(F_n\mathcal{V}) \subset F_n\mathcal{V}$ for any integer n, then $G\mathcal{V}$ inherits the structure of a chain complex.

1 Preliminaries

In this first section, we recall some categorical concepts like the presentability and the notions of enrichment, tensoring and cotensoring. Moreover, we describe several notions of coalgebras, like coassociative coalgebras and cocommutative coalgebras, that have been extensively studied in [11] and [16], respectively. More specifically, the category of coassociative coalgebras admits a model structure related by a Quillen adjunction to the category of simplicial sets; the category of conilpotent cocommutative coalgebras admits a model structure Quillen equivalent to the projective model structure on Lie algebras. Finally, we describe the Sullivan polynomial algebras.

1.1 Presentable categories

Definition 1 (presentable category) Let C be a cocomplete category. An object X of C is called *compact* if for any filtered diagram $F\colon I \to \mathsf{C}$ the map

$$\operatorname{colim}(\hom_{\mathsf{C}}(X, F)) \to \hom_{\mathsf{C}}(X, \operatorname{colim} F)$$

is an isomorphism. The category C is said to be *presentable* if there exists a set of compact objects such that any object of C is the colimit of a filtered diagram involving only these compact objects.

The following proposition is a classical result of category theory:

Proposition 2 [1] *A functor* $L\colon \mathsf{C} \to \mathsf{D}$ *between presentable categories is a left adjoint if and only if it preserves colimits.*

1.2 Tensoring, cotensoring and enrichment

In this section, we recall the definition of tensored-cotensored-enriched category over a monoidal category. See [5] for the original reference.

Definition 3 (action, coaction) Let $(\mathsf{E}, \otimes, \mathcal{I})$ be a monoidal category and let C be a category.

- An enrichment of C over E is a bifunctor $[-,-]\colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{E}$ together with functorial morphisms

  $$\gamma_{X,Y,Z}\colon [Y,Z] \otimes [X,Y] \to [X,Z], \quad \upsilon_X\colon \mathcal{I} \to [X,X]$$

  for any objects X, Y and Z of C and which are composition and unit in terms of the commutative diagrams

  $$\begin{tikzcd}
  {[Y,Z] \otimes [X,Y] \otimes [V,X]} \arrow[r, "\gamma_{X,Y,Z} \otimes \mathrm{Id}"] \arrow[d, "\mathrm{Id} \otimes \gamma_{V,X,Y} \otimes \mathrm{Id}"'] & {[X,Z] \otimes [V,X]} \arrow[d, "\gamma_{V,X,Z}"] \\
  {[Y,Z] \otimes [V,Y]} \arrow[r, "\gamma_{V,Y,Z}"'] & {[V,Z]}
  \end{tikzcd}$$

  $$\begin{tikzcd}
  {[X,Y] \otimes [X,X]} \arrow[dr] & {[X,Y]} \arrow[l, "\mathrm{Id} \otimes \upsilon_X"'] \arrow[r, "\upsilon_Y \otimes \mathrm{Id}"] \arrow[d, "\mathrm{Id}"] & {[Y,Y] \otimes [X,Y]} \arrow[dl] \\
  & {[X,Y]} &
  \end{tikzcd}$$

- A right action of E on C is a functor

  $$- \triangleleft -\colon \mathsf{C} \times \mathsf{E} \to \mathsf{C}$$

  together with functorial isomorphisms

  $$\begin{cases} X \triangleleft (\mathcal{A} \otimes \mathcal{B}) \simeq (X \triangleleft \mathcal{A}) \triangleleft \mathcal{B}, \\ X \triangleleft \mathcal{I} \simeq X, \end{cases}$$

for any $X \in \mathsf{C}$ and any $\mathcal{A}, \mathcal{B} \in \mathsf{E}$; these functors are compatible with the monoidal structure of E in terms of the commutative diagrams

$$\begin{tikzcd}
((X \triangleleft \mathcal{A}) \triangleleft \mathcal{B}) \triangleleft \mathcal{C} \arrow[r] \arrow[d] & (X \triangleleft (\mathcal{A} \otimes \mathcal{B})) \triangleleft \mathcal{C} \arrow[r] & X \triangleleft ((\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}) \arrow[d] \\
(X \triangleleft \mathcal{A}) \triangleleft (\mathcal{B} \otimes \mathcal{C}) \arrow[rr] & & X \triangleleft (\mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C}))
\end{tikzcd}$$

$$\begin{tikzcd}
(X \triangleleft \mathcal{I}) \triangleleft \mathcal{A} \arrow[rr] \arrow[dr] & & X \triangleleft (\mathcal{I} \otimes \mathcal{A}) \arrow[dl] \\
& X \triangleleft \mathcal{A} &
\end{tikzcd}$$

- A left coaction of E on C is a functor

$$\langle -, - \rangle \colon \mathsf{E}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{C}$$

together with functorial isomorphisms

$$\begin{cases} \langle \mathcal{A} \otimes \mathcal{B}, X \rangle \simeq \langle \mathcal{A} \langle \mathcal{B}, X \rangle \rangle, \\ \langle \mathcal{I}, X \rangle \simeq X, \end{cases}$$

which satisfy the commutative duals of the diagrams above.

Definition 4 (category tensored-cotensored-enriched over a monoidal category) Let E be a monoidal category and let C be a category. We say that C is *tensored-cotensored-enriched* over E if there exist three functors

$$\{-, -\} \colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{E}, \quad - \triangleleft - \colon \mathsf{C} \times \mathsf{E} \to \mathsf{C}, \quad \langle -, - \rangle \colon \mathsf{E}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{C},$$

together with functorial isomorphisms

$$\hom_{\mathsf{C}}(X \triangleleft \mathcal{A}, Y) \simeq \hom_{\mathsf{E}}(\mathcal{A}, \{X, Y\}) \simeq \hom_{\mathsf{C}}(X, \langle \mathcal{A}, Y \rangle)$$

for any $X, Y \in \mathsf{C}$, any $\mathcal{A}, \mathcal{B} \in \mathsf{E}$ and where $\mathcal{I}$ is the monoidal unit of E, such that $- \triangleleft -$ defines a right action of E on C.

The axioms and terminology of these notions are justified by the following proposition:

Proposition 5 *If the category C is tensored-cotensored-enriched over E, then it is enriched in the usual sense and the functor $\langle -, - \rangle$ is a left coaction in the sense of Definition 3.*

Proof Suppose that the category C is tensored-cotensored-enriched over E. On the one hand, let us define the composition relative to the enrichment $\{-, -\}$. For any objects

X and Y of C, the identity morphism of $\{X,Y\}$ defines a morphism $X \triangleleft \{X,Y\} \to Y$. So, for any objects X, Y and Z, we have a map

$$X \triangleleft (\{X,Y\} \otimes \{Y,Z\}) \simeq (X \triangleleft \{X,Y\}) \triangleleft \{Y,Z\} \to Y \triangleleft \{Y,Z\} \to Z$$

and hence a map $\{X,Y\} \otimes \{Y,Z\} \to \{X,Z\}$. Thus is defined the composition. The coherence diagrams of Definition 3 ensure us that the composition is associative and gives us a unit. On the other hand, let us show that the functor $\langle -,- \rangle$ is a left coaction. For any $X, Y \in \mathsf{C}$ and any $\mathcal{A}, \mathcal{B} \in \mathsf{E}$, we have functorial isomorphisms

$$\begin{aligned}\hom_{\mathsf{C}}(X, \langle \mathcal{A} \otimes \mathcal{B}, Y\rangle) &\simeq \hom_{\mathsf{C}}(X \triangleleft (\mathcal{A} \otimes \mathcal{B}), Y) \simeq \hom_{\mathsf{C}}((X \triangleleft \mathcal{A}) \triangleleft \mathcal{B}, Y)\\ &\simeq \hom_{\mathsf{C}}(X \triangleleft \mathcal{A}, \langle \mathcal{B}, Y\rangle) \simeq \hom_{\mathsf{C}}(X, \langle \mathcal{A} \langle \mathcal{B}, Y\rangle\rangle).\end{aligned}$$

By the Yoneda lemma, this gives us a functorial isomorphism $\langle \mathcal{A} \otimes \mathcal{B}, Y\rangle \simeq \langle \mathcal{A}\langle \mathcal{B}, Y\rangle\rangle$. This functorial isomorphism satisfies the coherence conditions of Definition 3 because the functorial isomorphism $X \triangleleft (\mathcal{A} \otimes \mathcal{B}) \simeq (X \triangleleft \mathcal{A}) \triangleleft \mathcal{B}$ satisfies the coherence conditions of the same definition. □

Proposition 6 *Let E be a presentable monoidal category and let C be a presentable category.*

- *Suppose that there exists a right action $- \triangleleft -$ of E on C and that for any $\mathcal{A} \in \mathsf{E}$ and for any $X \in \mathsf{C}$, the functors $X \triangleleft - \colon \mathsf{E} \to \mathsf{C}$ and $- \triangleleft \mathcal{A} \colon \mathsf{C} \to \mathsf{C}$ preserve colimits. Then C is tensored-cotensored-enriched over E.*
- *Suppose that there exists a left coaction $\langle -,- \rangle$ of E on C and that there exists a functor*

  $$- \triangleleft - \colon \mathsf{C} \times \mathsf{E} \to \mathsf{C}$$

  together with a functorial isomorphism

  $$\hom_{\mathsf{C}}(X \triangleleft \mathcal{A}, Y) \simeq \hom_{\mathsf{C}}(X, \langle \mathcal{A}, Y\rangle).$$

  Suppose moreover that the functor $\langle -, Y\rangle \colon \mathsf{E}^{\mathrm{op}} \to \mathsf{C}$ sends colimits in E to limits. Then C is tensored-cotensored-enriched over E.

Proof The first point is a direct consequence of Proposition 2. Let us prove the second point. Since E left coacts on C, by the same arguments as in the proof of Proposition 5 we can show that the bifunctor $- \triangleleft -$ is a right action of E on C. Moreover, since the functors $\langle -, Y\rangle$ preserve limits, any functor of the form $X \triangleleft -$ preserves colimits. The result is then a direct consequence of the first point. □

Definition 7 (homotopical enrichment) Let M be a model category and let E be a model category with a monoidal structure. We say that M is homotopically enriched over E if it is enriched over E and if for any cofibration $f\colon X \to X'$ in M and any fibration $g\colon Y \to Y'$ in M, the morphism in E

$$\{X', Y\} \to \{X', Y'\} \times_{\{X,Y'\}} \{X, Y\}$$

is a fibration. Moreover, we require this morphism to be a weak equivalence whenever f or g is a weak equivalence.

This definition implies in particular that the homotopy category $\mathrm{Ho}(\mathsf{M})$ is enriched over the monoidal category $\mathrm{Ho}(\mathsf{E})$.

1.3 Coalgebras

Definition 8 (coalgebras) A *coassociative coalgebra* $\mathscr{C} = (C, \Delta, \epsilon)$ is a chain complex C equipped with a coassociative coproduct $\Delta\colon C \to C \otimes C$ and a counit $\epsilon\colon C \to \mathbb{K}$ such that $\mathrm{Id}_C = (\mathrm{Id}_C \otimes \epsilon)\Delta = (\epsilon \otimes \mathrm{Id}_C)\Delta$. The kernel of the map ϵ is denoted by $\bar{C}$. The coalgebra $\mathscr{C}$ is called cocommutative if $\Delta = \tau\Delta$, where

$$\tau(x \otimes y) = (-1)^{|x||y|} y \otimes x.$$

A *graded* atom is a nonzero element $1 \in C$ such that $\Delta 1 = 1 \otimes 1$. In this context, let us define the map $\bar{\Delta}\colon \bar{C} \to \bar{C} \otimes \bar{C}$ by

$$\bar{\Delta} x := \Delta x - 1 \otimes x - x \otimes 1 \in \bar{C} \otimes \bar{C}.$$

A graded atom 1 is called a *dg atom* if $d1 = 0$. A *conilpotent coalgebra* $\mathscr{C} = (C, \Delta, \epsilon, 1)$ is the data of a coassociative coalgebra (C, Δ, ϵ) together with a graded atom such that, for any $x \in \bar{C}$, there exists an integer n such that

$$\bar{\Delta}^n x := (\mathrm{Id}_C^{\otimes n-1} \otimes \bar{\Delta}) \cdots (\mathrm{Id}_C \otimes \bar{\Delta})\bar{\Delta}(x) = 0.$$

A conilpotent cocommutative coalgebra $\mathscr{C}$ is said to be a *Hinich coalgebra* if 1 is a dg atom. We denote by uCog be the category of coassociative coalgebras and by uCocom the category of cocommutative coalgebras. Let $\mathsf{uNilCocom}$ (resp. $\mathsf{Hinich\text{–}cog}$) be the category whose objects are conilpotent cocommutative coalgebras (resp. Hinich coalgebras) and whose morphisms are morphisms of coalgebras.

Remark 9 The reader may be familiar with the notion of a coaugmented coalgebra. This is actually exactly the data of a coassociative coalgebra together with a dg atom.

Indeed, the data of a dg atom of a coalgebra $\mathscr{C}$ is equivalent to the data of a morphism of dg coalgebras from $\mathbb{K}$ to $\mathscr{C}$.

Any conilpotent coalgebra $\mathscr{C}$ has a canonical filtration, called the coradical filtration,

$$F_n^{\mathrm{rad}}\mathcal{C} := \mathbb{K}\cdot 1 \oplus \{x \in \overline{\mathcal{C}} \mid \overline{\Delta}^{n+1}x = 0\},$$

which is not necessarily stable under the codifferential d.

Proposition 10 *Let f be a morphism of coalgebras between two conilpotent coalgebras $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon, 1)$ and $\mathscr{D} = (\mathcal{D}, \Delta', \epsilon', 1')$. Then $f(1) = 1'$.*

Proof Let $x \in \overline{\mathcal{D}}$ be such that $f(1) = 1' + x$. Since $\Delta f(1) = (f \otimes f)\Delta(1)$, then $\overline{\Delta}x = x \otimes x$. Since there exists an integer n such that $\overline{\Delta}^n(x) = x \otimes \cdots \otimes x = 0$, then $x = 0$. □

Proposition 11 *The categories* uCog, uCocom, $\mathsf{uNilCocom}$ *and* $\mathsf{Hinich\text{–}cog}$ *are presentable. The forgetful functor from* uCog *to the category of chain complexes has a right adjoint called the cofree counital coalgebra functor. The same statement holds for the category* uCocom. *The functor $\mathscr{C} \mapsto \overline{\mathcal{C}}$ from the category* $\mathsf{Hinich\text{–}cog}$ *to the category of chain complexes has a right adjoint. The tensor product of the category of chain complexes induces closed symmetric monoidal structures on the categories* uCog *and* uCocom.

Proof The results are proven in [2, Sections 2.1, 2.2 and 2.5] for the category uCog. The methods used apply mutatis mutandis for the other categories. □

Theorem 12 [11] *The full subcategory* $\mathsf{uCog}^{\geq 0}$ *of* uCog *made up of nonnegatively graded coalgebras admits a model structure whose cofibrations are the monomorphisms and whose weak equivalences are the quasi-isomorphisms.*

The category $\mathsf{Hinich\text{–}cog}$ is related to the category of Lie-algebras by an adjunction, described in [24],

$$\begin{tikzcd}\mathsf{Hinich\text{–}cog} \arrow[r, shift left, "\mathcal{L}"] & \mathsf{Lie\text{–}alg}. \arrow[l, shift left, "\mathcal{C}"]\end{tikzcd}$$

Theorem 13 [16] *Suppose the characteristic of the base field $\mathbb{K}$ is zero. Then there exists a model structure on the category* $\mathsf{Hinich\text{–}cog}$ *whose cofibrations are monomorphisms and whose weak equivalences are morphisms whose image under the functor $\mathcal{L}$*

is a quasi-isomorphism. The class of weak equivalences is contained in the class of quasi-isomorphisms. Moreover, the adjunction $\mathcal{L} \dashv \mathcal{C}$ is a Quillen equivalence when the category of Lie algebras is equipped with its projective model structure whose fibrations (resp. weak equivalences) are surjections (resp. quasi-isomorphisms) (see [15]).

Definition 14 (deformation problems) Let $\mathsf{Artin{-}alg}$ be the category of nonpositively graded local finite-dimensional dg commutative algebras. A deformation problem is a functor from the category $\mathsf{Artin{-}alg}$ to the category of simplicial sets.

Lurie showed in [20] that a suitable infinity-category of deformation problems (called formal moduli problems) is equivalent to the infinity-category of Lie algebras if the characteristic of the base field $\mathbb{K}$ is zero. Therefore, it is equivalent to the infinity-category of Hinich coalgebras. In that perspective, any Hinich coalgebra $\mathscr{C}$ induces a deformation problem as follows:

$$R \mapsto \mathrm{Map}_{\mathsf{Hinich{-}cog}}(R^*, \mathscr{C}) \quad \text{for } R \in \mathsf{Artin{-}alg}.$$

Remark 15 We use Hinich's definition of a deformation problem given in [16]. We do not describe here the homotopy theory of such deformation problems nor a precise link with the work of Lurie, who uses the framework of quasicategories (see [20]). In the sequel, we will only use the fact that, for any morphism of deformation problems $f \colon X \to Y$, if $f(R)$ is a weak equivalence of simplicial sets for any algebra $R \in \mathsf{Artin{-}alg}$, then f is an equivalence of deformation problems.

1.4 Coalgebras and simplicial sets

In this subsection, we describe a Quillen adjunction between the category of simplicial sets and the category of coassociative coalgebras. This adjunction is part of the Dold–Kan correspondence. From a simplicial set X, one can produce a chain complex $\mathrm{DK}(X)$, called the normalized Moore complex. In degree n, $\mathrm{DK}(X)_n$ is the subvector space of $\mathbb{K} \cdot X_n$ which is the intersection of the kernels of the faces $d_0, \ldots, d_{n-1}$. The differential is $(-1)^n d_n$. Moreover, the Alexander–Whitney map makes the functor DK comonoidal. Then the diagonal map $X \to X \times X$ gives to $\mathrm{DK}(X)$ a structure of coalgebras. Thus, we have a functor DK^c from simplicial sets to the category uCog of coassociative coalgebras. This functor DK^c admits a right adjoint N defined by

$$N(\mathcal{C})_n := \hom_{\mathsf{uCog}}(\mathrm{DK}^c(\Delta[n]), \mathcal{C}).$$

Actually, we have the sequence of adjunctions

$$\begin{tikzcd}\mathsf{sSet} \arrow[r, shift left, "\mathrm{DK}^c"] & \mathsf{uCog}^{\geq 0} \arrow[l, shift left, "N"] \arrow[r, shift left, "\mathrm{in}"] & \mathsf{uCog}, \arrow[l, shift left, "\mathrm{tr}"]\end{tikzcd}$$

where in is the embedding of $\mathsf{uCog}^{\geq 0}$ into uCog and where tr is the truncation.

Proposition 16 *The above adjunction between* $\mathsf{uCog}^{\geq 0}$ *and* sSet *is a Quillen adjunction.*

Proof The functor DK^c carries monomorphisms to monomorphisms and weak homotopy equivalences to quasi-isomorphisms; see [13, III.2]. □

1.5 The Sullivan algebras of polynomial forms on standard simplices

Definition 17 (Sullivan polynomial algebras [25]) For any integer $n \in \mathbb{N}$, the n^{th} *algebra of polynomial forms* is the differential graded unital commutative algebra

$$\Omega_n := \mathbb{K}[t_0, \dots, t_n, dt_0, \dots, dt_n]/(\Sigma t_i = 1),$$

where the degree of t_i is zero and where $d_{\Omega_n}(t_i) = dt_i$. In particular, $\sum dt_i = 0$.

Any map of finite ordinals $\phi\colon [n] \to [m]$ defines a morphism of differential graded unital commutative algebra

$$\Omega(\phi)\colon \Omega_m \to \Omega_n, \quad t_i \mapsto \sum_{\phi(j)=i} t_j.$$

Therefore, the collection $\{\Omega_n\}_{n\in\mathbb{N}}$ defines a simplicial differential graded commutative algebra. Moreover, one can extend this construction to a contravariant functor $\Omega_\bullet$ from simplicial sets to differential graded unital commutative algebras such that $\Omega_{\Delta[n]} = \Omega_n$. This functor is part of an adjunction

$$\begin{tikzcd}\mathsf{sSet} \arrow[r, shift left, "\Omega_\bullet"] & \mathit{uCom}-\mathsf{alg}^{\mathrm{op}}. \arrow[l, shift left]\end{tikzcd}$$

Proposition 18 [6, Chapter 8] *When the characteristic of the field* $\mathbb{K}$ *is zero, the category* uCom–alg *of differential graded unital commutative algebras admits a projective model structure where fibrations (resp. weak equivalences) are degreewise surjections (resp. quasi-isomorphisms). In that context, the adjunction between simplicial sets and* uCom–alg *is a Quillen adjunction.*

2 Operads, cooperads, algebras and coalgebras

The purpose of this section is to recall the definitions of operads, cooperads, algebras over an operad and coalgebras over a cooperad that we will use in the sequel; we refer the reader to [19]. Moreover, we prove that the category of coalgebras over a curved cooperad is presentable.

2.1 Operads and cooperads

We recall here the definitions of operads and cooperads. We refer to [19; 17].

Definition 19 (symmetric modules) Let $\mathbb{S}$ be the groupoid whose objects are integers $n \in \mathbb{N}$ and whose morphisms are

$$\begin{cases} \hom_{\mathbb{S}}(n,m) = \varnothing & \text{if } n \neq m, \\ \hom_{\mathbb{S}}(n,n) = \mathbb{S}_n & \text{otherwise.} \end{cases}$$

A graded $\mathbb{S}$–module (resp. dg $\mathbb{S}$–module) is a presheaf on $\mathbb{S}$ valued in the category of graded $\mathbb{K}$–modules (resp. chain complexes). The name $\mathbb{S}$–module will refer both to graded $\mathbb{S}$–modules and dg $\mathbb{S}$–modules. We say that a $\mathbb{S}$–module $\mathcal{V}$ is reduced if $\mathcal{V}(0) = \{0\}$.

The category of $\mathbb{S}$–modules has a monoidal structure which is as follows: for any $\mathbb{S}$–modules $\mathcal{V}$ and $\mathcal{W}$, and for any $n \geq 1$,

$$(\mathcal{V} \circ \mathcal{W})(n) := \bigoplus_{k \geq 1} \mathcal{V}(k) \otimes_{\mathbb{S}_k} \Bigg(\bigoplus_{X_1 \sqcup \cdots \sqcup X_k = \{1,\dots,n\}} \mathcal{W}(\# X_1) \otimes \cdots \otimes \mathcal{W}(\# X_k) \Bigg),$$

where $\# X_i$ is the cardinal of the set X_i. For $n = 0$,

$$(\mathcal{V} \circ \mathcal{W})(0) := \mathcal{V}(0) \oplus \Bigg(\bigoplus_{k \geq 1} \mathcal{V}(k) \otimes_{\mathbb{S}_k} (\mathcal{W}(0) \otimes \cdots \otimes \mathcal{W}(0)) \Bigg).$$

The monoidal unit is given by the $\mathbb{S}$–module $\mathcal{I}$ which is $\mathbb{K}$ in arity 1 and $\{0\}$ in other arities.

Notation
- For any dg $\mathbb{S}$–module $\mathcal{V}$, we will denote by $\mathcal{V}^{\mathrm{grad}}$ the underlying graded $\mathbb{S}$–module.
- Let $f \colon \mathcal{V} \to \mathcal{V}'$ and $g \colon \mathcal{W} \to \mathcal{W}'$ and $h \colon \mathcal{W} \to \mathcal{W}'$ be three morphisms of $\mathbb{S}$–modules. Then we denote by $f \circ (g; h)$ the map from $\mathcal{V} \circ \mathcal{W}$ to $\mathcal{V}' \circ \mathcal{W}'$ defined

as follows:

$$f \circ (g; h) := \sum_{i+j=n-1} f \otimes_{\mathbb{S}_n} (g^{\otimes i} \otimes h \otimes g^{\otimes j}).$$

In the case where g is the identity, we use the notation $f \circ' h$:

$$f \circ' h := f \circ (\mathrm{Id}; h).$$

- For any two graded $\mathbb{S}$–modules (resp. dg $\mathbb{S}$–modules) $\mathcal{V}$ and $\mathcal{W}$, we denote by $[\mathcal{V}, \mathcal{W}]$ the graded $\mathbb{K}$–module (resp. chain complex)

  $$[\mathcal{V}, \mathcal{W}]_n := \prod_{\substack{k \geq 0 \\ l \in \mathbb{N}}} \hom_{\mathbb{K}[\mathbb{S}_n]}(\mathcal{V}(k)_l, \mathcal{W}(k)_{l+n}).$$

  In that context morphisms of chain complexes from X to $[\mathcal{V}, \mathcal{W}]$ are in one-to-one correspondence with morphisms of $\mathbb{S}$–modules from the aritywise tensor product $X \otimes \mathcal{V}$ to $\mathcal{W}$.

Proposition 20 [19, Chapter 6] *If the characteristic of the field* $\mathbb{K}$ *is zero, then the operadic Künneth*

$$H(\mathcal{V} \circ \mathcal{W}) \simeq H(\mathcal{V}) \circ H(\mathcal{W})$$

holds for any dg $\mathbb{S}$*–modules* $\mathcal{V}$ *and* $\mathcal{W}$*, where* H *denotes the homology.*

Definition 21 (operads) A graded operad $\mathscr{P} = (\mathcal{P}, \gamma, 1)$ (resp. dg operad) is a monoid in the category of graded $\mathbb{S}$–modules (resp. dg $\mathbb{S}$–modules). We denote by Operad the category of dg operads.

Example 22 For any graded $\mathbb{K}$–module (resp. chain complex) $\mathcal{V}$, $\mathrm{End}_{\mathcal{V}}$ is the graded operad (resp. dg operad) defined by

$$\mathrm{End}_{\mathcal{V}}(n) := \hom(\mathcal{V}^{\otimes n}, \mathcal{V}).$$

The composition in the operad $\mathrm{End}_{\mathcal{V}}$ is given by the composition of morphisms of graded $\mathbb{K}$–modules (resp. chain complexes).

A degree k derivation d on a graded operad $\mathscr{P} = (\mathcal{P}, \gamma, 1)$ is the data of degree k maps $d \colon \mathcal{P}(n) \to \mathcal{P}(n)$ which commute with the action of $\mathbb{S}_n$ and such that

$$d\gamma = \gamma(d \circ \mathrm{Id} + \mathrm{Id} \circ' d).$$

Proposition 23 [19, Chapter 5] *The forgetful functor from operads to $\mathbb{S}$–modules has a left adjoint called the free operad functor and denoted by $\mathbb{T}$. For any $\mathbb{S}$–module $\mathcal{V}$, $\mathbb{T}\mathcal{V}$ is the $\mathbb{S}$–module made up of trees whose vertices are filled with elements of $\mathcal{V}$ with coherent arity. The composition is given by the grafting of trees.*

There is a one-to-one correspondence between the degree k derivation on the graded free operad $\mathbb{T}\mathcal{V}$ and the degree k maps from $\mathcal{V}$ to $\mathbb{T}\mathcal{V}$. Indeed, from such a map u one can produce the derivation D_u such that, for any tree T labeled by elements of $\mathcal{V}$,

$$D_u(T) := \sum_v \mathrm{Id} \otimes \cdots \otimes u(v) \otimes \cdots \otimes \mathrm{Id},$$

where the sum is taken over the vertices of the tree T.

Definition 24 (cooperads) A cooperad $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon)$ is a comonoid in the category of $\mathbb{S}$–modules. We denote by $\bar{\mathcal{C}}$ the kernel of the morphism $\epsilon : \mathcal{C} \to \mathcal{I}$. A cooperad $\mathscr{C}$ is said to be coaugmented if it is equipped with a morphism of cooperads $\mathcal{I} \to \mathscr{C}$. In this case, we denote by 1 the image of the unit of $\mathbb{K}$ into $\mathcal{C}(1)$. A coaugmented cooperad $\mathscr{C}$ is said to be conilpotent if the process of successive decomposition stabilizes in finite time for any element. A precise definition is given in [19, Section 5.8.6].

The forgetful functor from conilpotent cooperads to $\mathbb{S}$–modules which sends $\mathscr{C}$ to $\bar{\mathcal{C}}$ has a right adjoint sending $\mathcal{V}$ to the tree module $\mathbb{T}(\mathcal{V})$ with the decomposition given by the degrafting of trees. We denote it by $\mathbb{T}^c(\mathcal{V})$. We also denote by $\delta\colon \mathscr{C} \to \mathbb{T}^c(\bar{\mathcal{C}})$ the counit of the adjunction. Any conilpotent cooperad is equipped with a filtration, called the coradical filtration,

$$F_n^{\mathrm{rad}}\mathcal{C}(m) := \{p \in \mathcal{C}(m) \mid \delta(p) \in \mathbb{T}^{\leq n}(\bar{\mathcal{C}})(m)\},$$

where the symbol $\mathbb{T}^{\leq n}$ denotes the trees with at most n vertices. In particular, $F_0^{\mathrm{rad}}\mathcal{C} = \mathcal{I}$.

Notation Let $\mathcal{C}$ be coaugmented cooperad and m be an integer. We denote by Δ_m the composite map

$$\Delta_m\colon \mathcal{C} \xrightarrow{\Delta} \mathcal{C} \circ \mathcal{C} \to \mathbb{T}(\bar{\mathcal{C}}) \twoheadrightarrow \mathbb{T}^m(\bar{\mathcal{C}}).$$

A degree k coderivation on a cooperad $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon)$ is a degree k map d of $\mathbb{S}$–modules from $\mathcal{C}$ to $\mathcal{C}$ such that

$$\Delta d = (d \circ \mathrm{Id} + \mathrm{Id} \circ' d)\Delta.$$

If the cooperad is coaugmented, we also require that $d(1) = 0$. Let $\mathbb{T}^c(\mathcal{V})$ be a cofree conilpotent cooperad. There is a one-to-one correspondence between degree k coderivations on $\mathbb{T}^c(\mathcal{V})$ and degree k maps from $\overline{\mathbb{T}}(\mathcal{V})$ to $\mathcal{V}$. Indeed, such a map u is uniquely extended by the following coderivation D_u, defined on any tree T labeled by elements of $\mathcal{V}$ as follows:

$$D_u(T) := \sum_{T' \subset T} \mathrm{Id} \otimes \cdots \otimes u(T') \otimes \cdots \otimes \mathrm{Id},$$

where the sum is taken on the subtrees T' of T.

Definition 25 (curved cooperads) A curved cooperad $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon, 1, d, \theta)$ is a coaugmented graded cooperad equipped with a degree -2 map of graded $\mathbb{S}$–modules $\theta\colon \mathcal{C} \to \mathcal{I}$ and a degree -1 coderivation d such that

$$d^2 = (\theta \otimes \mathrm{Id} - \mathrm{Id} \otimes \theta)\Delta_2, \quad \theta d = 0.$$

A morphism of curved cooperads is a morphism of cooperads $\phi\colon \mathscr{C} \to \mathscr{D}$ which commutes with the coderivations and such that $\theta_{\mathscr{C}} = \theta_{\mathscr{D}}\phi$. We denote by cCoop the category of curved conilpotent cooperads.

The coradical filtration of a conilpotent cooperad has the following property with respect to the decomposition map:

Lemma 26 *Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon, 1)$ be a conilpotent cooperad. Then*

$$\Delta(F_n^{\mathrm{rad}}\mathcal{C}) \subset \sum_{p_0+\cdots+p_k \le n} (F_{p_0}^{\mathrm{rad}}\mathcal{C})(k) \otimes_{\mathbb{S}_k} (F_{p_1}^{\mathrm{rad}}\mathcal{C} \otimes \cdots \otimes F_{p_k}^{\mathrm{rad}}\mathcal{C}).$$

Proof It suffices to prove the result for cofree cooperads. Indeed, any conilpotent cooperad $\mathscr{C}$ is equipped with a map $\delta\colon \mathcal{C} \to \mathbb{T}^c(\bar{\mathcal{C}})$ such that $F_n^{\mathrm{rad}}\mathcal{C} = \delta^{-1}(F_n^{\mathrm{rad}}\mathbb{T}^c(\bar{\mathcal{C}}))$. □

Lemma 27 *Let $\mathscr{C} = \mathbb{T}^c(\mathcal{V})$ be a cofree conilpotent graded cooperad equipped with a degree -2 map $\theta\colon \mathbb{T}(\mathcal{V})(1) \twoheadrightarrow \mathcal{V}(1) \to \mathbb{K}$. Let $\phi\colon \overline{\mathbb{T}}\mathcal{V} \to \mathcal{V}$ be a degree -1 map and let D_ϕ be the corresponding coderivation on $\mathscr{C}$. Then the triple $(\mathbb{T}^c\mathcal{V}, D_\phi, \theta)$ is a curved cooperad if and only if ϕ satisfies the equation*

$$\phi D_\phi = (\theta \otimes \pi_{\mathcal{V}} - \pi_{\mathcal{V}} \otimes \theta)\Delta_2,$$

where $\pi_{\mathcal{V}}$ is the projection $\mathbb{T}(\mathcal{V}) \to \mathcal{V}$.

Proof If $(\mathbb{T}^c\mathcal{V}, D_\phi, \theta)$ is a curved cooperad, then $\phi D_\phi = \pi_\mathcal{V} D_\phi^2 = (\theta\otimes\pi_\mathcal{V} - \pi_\mathcal{V}\otimes\theta)\Delta_2$. Conversely, suppose that $\phi D_\phi = (\theta\otimes\pi_\mathcal{V} - \pi_\mathcal{V}\otimes\theta)\Delta_2$. For any tree T labeled by elements of $\mathcal{V}$, one can prove that

$$D_\phi^2(T) = \sum_{T'\subset T} \mathrm{Id}\otimes(\phi D_\phi(T'))\otimes\mathrm{Id}.$$

Actually, it is the sum over every arity 1 vertex v of

- $\pm\theta(v)(T-v)$ if v is the bottom vertex or a top vertex;
- $\pm(\theta(v)(T-v) - \theta(v)(T-v)) = 0$ otherwise.

Hence, $(\mathbb{T}^c\mathcal{V}, D_\phi, \theta)$ is a curved cooperad. □

There exist notions of $\mathbb{N}$–modules, nonsymmetric operads, nonsymmetric cooperads and their morphisms, defined for instance in [19, Section 5.9]. We will speak about the nonsymmetric context to refer to these ones. Notice that the operadic Künneth formula holds in the nonsymmetric context without the assumption that the characteristic of the field $\mathbb{K}$ is zero.

2.2 Modules and algebras over an operad

Definition 28 (algebras over an operad) Let $\mathscr{P} = (\mathcal{P}, \gamma, 1)$ be an operad. A *$\mathscr{P}$–module* $\mathscr{A} = (\mathcal{A}, \gamma_\mathscr{A})$ is a left module in the category of $\mathbb{S}$–module, that is, an $\mathbb{S}$–module $\mathcal{A}$ equipped with a map $\gamma_\mathscr{A}\colon \mathcal{P}\circ\mathcal{A}\to\mathcal{A}$ such that the following diagrams commute:

$$\begin{tikzcd}
\mathcal{P}\circ\mathcal{P}\circ\mathcal{A} \arrow[r, "\mathrm{Id}\circ\gamma_\mathscr{A}"] \arrow[d, "\gamma\circ\mathrm{Id}"'] & \mathcal{P}\circ\mathcal{A} \arrow[d, "\gamma_\mathscr{A}"] \\
\mathcal{P}\circ\mathcal{A} \arrow[r, "\gamma_\mathscr{A}"'] & \mathcal{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{I}\circ\mathcal{A} \arrow[r, "1\circ\mathrm{Id}"] \arrow[rr, bend right, "\mathrm{Id}"] & \mathcal{P}\circ\mathcal{A} \arrow[r, "\gamma_\mathcal{A}"] & \mathcal{A}
\end{tikzcd}$$

A morphism of $\mathscr{P}$–modules from $\mathscr{A}$ to $\mathscr{B} = (\mathcal{B}, \gamma_\mathscr{B})$ is a morphism of $\mathbb{S}$–modules $f\colon \mathcal{A}\to\mathcal{B}$ such that $\gamma_\mathscr{B}(\mathrm{Id}\circ f) = f\gamma_\mathscr{A}$. A *$\mathscr{P}$–algebra* is a $\mathscr{P}$–module $\mathcal{A}$ concentrated in arity 0. We denote by $\mathscr{P}$–alg the category of $\mathscr{P}$–algebras.

The forgetful functor from the category of $\mathscr{P}$–modules to the category of $\mathbb{S}$–modules has a left adjoint given by

$$\mathcal{V}\mapsto\mathcal{P}\circ\mathcal{V}.$$

The images of this left adjoint functor are called the free $\mathscr{P}$–modules.

Definition 29 (ideal) An *ideal* of a $\mathscr{P}$–module $\mathscr{A}$ is a sub-$\mathbb{S}$–module $\mathcal{B} \subset \mathcal{A}$ such that, for any $p \in \mathcal{P}(n)$ and $(x_i)_{i=1}^n \in \mathcal{A}(k_i)$ with $n \geq 1$,

$$\gamma_{\mathscr{A}}(p \otimes_{\mathbb{S}_n} (x_1 \otimes \cdots \otimes x_n)) \in \mathcal{B}(k_1 + \cdots k_n)$$

whenever one of the x_i is in $\mathcal{B}$ (for $n \geq 1$). Then the quotient $\mathcal{A}/\mathcal{B}$ has an induced structure of $\mathscr{P}$–module.

Definition 30 (derivation) Let $\mathscr{P}$ be a graded operad and let $\mathscr{A}$ be a $\mathscr{P}$–module. Suppose that the graded operad $\mathscr{P}$ is equipped with a degree k derivation $d_{\mathscr{P}}$. Then a *derivation* of $\mathscr{A}$ is a degree k map $d_{\mathscr{A}}$ from $\mathcal{A}$ to $\mathcal{A}$ such that

$$d_{\mathscr{A}} \gamma_{\mathscr{A}} = \gamma_{\mathscr{A}}(d_{\mathscr{P}} \circ \mathrm{Id}_{\mathscr{A}} + \mathrm{Id} \circ' d_{\mathscr{A}}).$$

Let $\mathscr{P}$ be a graded operad equipped with a degree k derivation $d_{\mathscr{P}}$. There is a one-to-one correspondence between the derivations of a free $\mathscr{P}$–module $\mathscr{A} = \mathcal{P} \circ \mathcal{V}$ and the degree k maps $\mathcal{V} \to \mathcal{P} \circ \mathcal{V}$. Indeed, any such map $u\colon \mathcal{V} \to \mathcal{P} \circ \mathcal{V}$ is uniquely extended by the derivation

$$D_u = d_{\mathscr{P}} \circ \mathrm{Id} + \mathrm{Id} \circ (i; u),$$

where i denotes the canonical inclusion map $\mathcal{V} \to \mathcal{P} \circ \mathcal{V}$.

2.3 Comodules and coalgebras over a cooperad

Definition 31 (comodules and coalgebras over a cooperad) Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon)$ be a cooperad. A *$\mathscr{C}$–comodule* $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ is a left $\mathscr{C}$–comodule in the category of $\mathbb{S}$–modules, that is a $\mathbb{S}$–module $\mathcal{D}$ together with a morphism $\Delta_{\mathscr{D}}\colon \mathcal{D} \to \mathcal{C} \circ \mathcal{D}$ such that the following diagrams commute:

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "\Delta_{\mathscr{D}}"] \arrow[d, "\Delta_{\mathscr{D}}"'] & \mathcal{C} \circ \mathcal{D} \arrow[d, "\mathrm{Id} \circ \Delta_{\mathscr{D}}"] \\
\mathcal{C} \circ \mathcal{D} \arrow[r, "\Delta_{\mathscr{C}} \circ \mathrm{Id}"'] & \mathcal{C} \circ \mathcal{C} \circ \mathcal{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{D} \arrow[r, "\Delta_{\mathscr{D}}"] \arrow[rr, bend right, "\mathrm{Id}"'] & \mathcal{C} \circ \mathcal{D} \arrow[r, "\epsilon \circ \mathrm{Id}"] & \mathcal{D}
\end{tikzcd}$$

A *$\mathscr{C}$–coalgebra* is a $\mathscr{C}$–comodule concentrated in arity 0.

Remark 32 Our notion of $\mathscr{C}$–coalgebra actually recovers a notion sometimes called in the literature conilpotent $\mathscr{C}$–coalgebra; see [19, 5.4.8].

Let $\mathscr{C}$ be a coaugmented cooperad. Then the forgetful functor from the category of $\mathscr{C}$–comodules to the category of $\mathbb{S}$–modules has a right adjoint which sends $\mathcal{V}$ to $\mathscr{C} \circ \mathcal{V}$. The images of the right adjoint are called the cofree $\mathscr{C}$–comodules.

Definition 33 (coderivation) Let $\mathscr{C}$ be a graded cooperad and let $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ be a $\mathscr{C}$–comodule. Suppose that $\mathscr{C}$ is equipped with a degree k coderivation $d_{\mathscr{C}}$. A *coderivation* on $\mathscr{D}$ is a degree k map $d_{\mathscr{D}}$ from $\mathcal{D}$ to $\mathcal{D}$ such that

$$\Delta_{\mathscr{D}} d_{\mathscr{D}} = (d_{\mathscr{C}} \circ \mathrm{Id} + \mathrm{Id} \circ' d_{\mathscr{D}})\Delta_{\mathscr{D}}.$$

Let $\mathscr{C}$ be a cooperad equipped with a degree k coderivation and let $\mathcal{V}$ be a graded $\mathbb{K}$–module. Then there is a one-to-one correspondence between the coderivations on the $\mathscr{C}$–coalgebra $\mathscr{C} \circ \mathcal{V}$ and the degree k maps $\mathscr{C} \circ \mathcal{V} \to \mathcal{V}$. Indeed, any such map u induces the coderivation

$$D_u := (d_{\mathscr{C}} \circ \mathrm{Id}_{\mathcal{V}}) + (\mathrm{Id} \circ (\pi; u))(\Delta_{\mathscr{C}} \circ \mathrm{Id}_{\mathcal{V}}),$$

where $\pi = \epsilon \circ \mathrm{Id}\colon \mathscr{C} \circ \mathcal{V} \to \mathcal{V}$.

Definition 34 (comodules and coalgebras over a curved cooperad) Let $\mathscr{C}$ be a curved cooperad. A *$\mathscr{C}$–comodule* is a graded $\mathscr{C}^{\mathrm{grad}}$–comodule $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ together with a coderivation $d_{\mathcal{D}}$ such that

$$d_{\mathcal{D}}^2 = (\theta_{\mathscr{C}} \circ \mathrm{Id})\Delta_{\mathscr{D}}.$$

Moreover, a *$\mathscr{C}$–coalgebra* is a $\mathscr{C}$–comodule concentrated in arity 0.

Proposition 35 *Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon, 1, d, \theta)$ be a conilpotent curved cooperad and let $\mathcal{V}$ be a graded $\mathbb{S}$–module. There is a one-to-one correspondence between the degree -1 maps $\phi\colon \mathcal{C} \circ \mathcal{V} \to \mathcal{V}$ such that*

$$\phi D_\phi := \phi(\mathrm{Id} \circ (\pi; \phi))(\Delta_{\mathscr{C}} \circ \mathrm{Id}_{\mathcal{V}}) + \phi(d_{\mathcal{C}} \circ \mathrm{Id}_{\mathcal{V}}) = \theta \circ \mathrm{Id}_{\mathcal{V}}$$

and the structures of $\mathscr{C}$–comodule (where $\mathscr{C}$ is considered as a curved cooperad) *on the graded cofree comodule $\mathscr{C}^{\mathrm{grad}} \circ \mathcal{V}$.*

Proof A structure of $\mathscr{C}$–comodule on $\mathscr{C}^{\mathrm{grad}} \circ \mathcal{V}$ amounts to the data of a degree -1 coderivation D_ϕ such that $D_\phi^2 = (\theta \circ \mathrm{Id}_{\mathscr{C}} \circ \mathrm{Id}_{\mathcal{V}})(\Delta_{\mathscr{C}} \circ \mathrm{Id}_{\mathcal{V}})$. Given this equality, $\phi D_\phi = \theta \circ \mathrm{Id}_{\mathcal{V}}$. Conversely, suppose that $\phi D_\phi = \theta \circ \mathrm{Id}$. We have

$$\begin{aligned} D_\phi^2 = (d_{\mathscr{C}}^2 \circ \mathrm{Id}_{\mathcal{C}}) + (\mathrm{Id} \circ (\pi; \phi))(\Delta \circ \mathrm{Id})(d_{\mathscr{C}} \circ \mathrm{Id}_{\mathcal{V}}) + (d_{\mathscr{C}} \circ (\pi; \phi))(\Delta \circ \mathrm{Id}) \\ + (\mathrm{Id} \circ (\pi; \phi))(\Delta \circ \mathrm{Id})(\mathrm{Id} \circ (\pi; \phi))(\Delta \circ \mathrm{Id}). \end{aligned}$$

On the one hand,

$$\begin{aligned}(\mathrm{Id}\circ(\pi;\phi))(\Delta\circ\mathrm{Id})(\mathrm{Id}\circ(\pi;\phi))(\Delta\circ\mathrm{Id})+(\mathrm{Id}\circ(\pi;\phi))(\Delta\circ\mathrm{Id})(d_{\mathscr{C}}\circ\mathrm{Id}_{\mathcal{C}})&\\ +(d_{\mathscr{C}}\circ(\pi;\phi))(\Delta\circ\mathrm{Id})&\\ =(\mathrm{Id}\circ(\pi;\phi))(\mathrm{Id}\circ' D_{\phi})(\Delta\circ\mathrm{Id})&\\ =(\mathrm{Id}\circ(\pi;\phi D_{\phi}))(\Delta\circ\mathrm{Id})&\\ =\big((\mathrm{Id}\circ(\epsilon;\theta))\Delta\big)\circ\mathrm{Id}.&\end{aligned}$$

On the other hand,

$$(d_{\mathscr{C}}^2\circ\mathrm{Id})=((\theta\circ\mathrm{Id})\Delta)\circ\mathrm{Id}+\Big(\Big(\sum\mathrm{Id}\otimes_{\mathbb{S}}(\epsilon^{\otimes i}\otimes\theta\otimes\epsilon^{\otimes j})\Big)\Delta\Big)\circ\mathrm{Id}.$$

Hence, $D_{\phi}^2=((\theta\circ\mathrm{Id})\Delta)\circ\mathrm{Id}_{\mathcal{V}}$. □

Definition 36 (coradical filtration) Any $\mathscr{C}$–coalgebra $\mathscr{D}=(\mathcal{D},\Delta_{\mathscr{D}})$ over a conilpotent cooperad $\mathscr{C}$ admits a filtration called the *coradical filtration* and defined as follows:

$$F_n^{\mathrm{rad}}\mathcal{D}:=\{x\in\mathcal{D}\mid\Delta_{\mathscr{D}}(x)\in(F_n^{\mathrm{rad}}\mathcal{C})\circ\mathcal{D}\}.$$

Proposition 37 *Let $\mathscr{C}$ be a conilpotent cooperad and let $\mathscr{D}$ be a $\mathscr{C}$–coalgebra. For any integer n,*

$$\Delta_{\mathscr{D}}(F_n^{\mathrm{rad}}\mathcal{D})\subset\sum_{i_0+i_1+\cdots+i_k=n}(F_{i_0}^{\mathrm{rad}}\mathcal{C})(k)\otimes_{\mathbb{S}_k}(F_{i_1}^{\mathrm{rad}}\mathcal{D}\otimes\cdots\otimes F_{i_k}^{\mathrm{rad}}\mathcal{D}).$$

Lemma 38 *Let $\mathcal{V}$ and $\mathcal{W}$ be two graded $\mathbb{K}$–modules equipped with filtrations $(F_n\mathcal{V})_{n\in\mathbb{N}}$ and $(F_n\mathcal{W})_{n\in\mathbb{N}}$, and let $\phi\colon\mathcal{V}\to\mathcal{W}$ be an injection such that $F_n\mathcal{V}=\phi^{-1}(F_n\mathcal{W})$ for any integer n. Then there exists a map $\psi\colon\mathcal{W}\to\mathcal{V}$ such that $\psi\phi=\mathrm{Id}$ and $\psi(F_n\mathcal{W})=F_n\mathcal{V}$ for any $n\in\mathbb{N}$.*

Proof For an integer $n\geq -1$, suppose that we have built a subgraded $\mathbb{K}$–module $\mathcal{U}_n$ of $F_n\mathcal{W}$ such that $F_m\mathcal{W}=\phi(F_m\mathcal{V})\oplus(\mathcal{U}_n\cap F_m\mathcal{W})$ for any $m\leq n$. Let $\mathcal{U}_n'$ be a subgraded $\mathbb{K}$–module of $F_{n+1}\mathcal{W}$ that is an algebraic complement to $\phi(F_{n+1}\mathcal{V})\oplus\mathcal{U}_n$. Then let $\mathcal{U}_{n+1}:=\mathcal{U}_n\oplus\mathcal{U}_n'$. Finally, let $\mathcal{U}:=\mathrm{colim}\,\mathcal{U}_n$. We define ψ by

$$\psi=\begin{cases}\phi^{-1} & \text{on } \phi(\mathcal{V}),\\ 0 & \text{on } \mathcal{U}.\end{cases}$$

□

Proof of Proposition 37 The map $\Delta_{\mathscr{D}}\colon\mathcal{D}\to\mathcal{C}\circ\mathcal{D}$ is actually a morphism of $\mathscr{C}$–coalgebras such that $\Delta_{\mathscr{D}}^{-1}(F_n^{\mathrm{rad}}\mathcal{C}\circ\mathcal{D})=F_n^{\mathrm{rad}}\mathcal{D}$. By Lemma 38, there exists a map of graded $\mathbb{K}$–modules $\nabla\colon \mathcal{C}\circ\mathcal{D}\to\mathcal{D}$ such that $\nabla\Delta_{\mathscr{D}}=\mathrm{Id}_{\mathcal{D}}$ and $\nabla(F_n\mathcal{C}\circ\mathcal{D})=F_n\mathcal{D}$. Then the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathcal{C}\circ\mathcal{D} \arrow[d, "\Delta\circ\mathrm{Id}"] & \\
\mathcal{C}\circ\mathcal{D} \arrow[r, "\mathrm{Id}\circ\Delta"'] \arrow[rr, bend right, "\mathrm{Id}"'] & \mathcal{C}\circ\mathcal{C}\circ\mathcal{D} \arrow[r, "\mathrm{Id}\circ\nabla"'] & \mathcal{C}\circ\mathcal{D}
\end{tikzcd}$$

By Lemma 26, we know that

$$(\Delta\circ\mathrm{Id})\Delta(F_n^{\mathrm{rad}}\mathcal{D})\subset\sum_{i_0+\cdots+i_k=n}F_{i_0}^{\mathrm{rad}}\mathcal{C}(k)\otimes_{\mathbb{S}_k}(F_{i_1}^{\mathrm{rad}}\mathcal{C}\circ\mathcal{D}\otimes\cdots\otimes F_{i_k}^{\mathrm{rad}}\mathcal{C}\circ\mathcal{D}).$$

Moreover, we know that

$$(\mathrm{Id}\circ\nabla)(F_{i_0}^{\mathrm{rad}}\mathcal{C}(k)\otimes_{\mathbb{S}_k}(F_{i_1}^{\mathrm{rad}}\mathcal{C}\circ\mathcal{D}\otimes\cdots\otimes F_{i_k}^{\mathrm{rad}}\mathcal{C}\circ))\subset F_{i_0}^{\mathrm{rad}}\mathcal{C}(k)\otimes_{\mathbb{S}_k}(F_{i_1}^{\mathrm{rad}}\mathcal{D}\otimes\cdots\otimes F_{i_k}^{\mathrm{rad}}\mathcal{D}).$$

So, we have

$$\begin{aligned}\Delta(F_n^{\mathrm{rad}}\mathcal{D})&=(\mathrm{Id}\circ\nabla)(\Delta\circ\mathrm{Id})\Delta(F_n^{\mathrm{rad}}\mathcal{D})\\&\subset\sum_k\sum_{i_0+\cdots+i_k=n}F_{i_0}^{\mathrm{rad}}\mathcal{C}(k)\otimes_{\mathbb{S}_k}(F_{i_1}^{\mathrm{rad}}\mathcal{D}\otimes\cdots\otimes F_{i_k}^{\mathrm{rad}}\mathcal{D}).\end{aligned}$$ □

2.4 Presentability

This subsection deals with the presentability of the category of algebras over an operad and the presentability of the category of coalgebras over a conilpotent curved cooperad.

Theorem 39 [8, Lemma 5.2] *Let $\mathscr{P}$ be a dg-operad. Then the category $\mathscr{P}$–*alg *of $\mathscr{P}$–algebras is presentable.*

The essence of the last theorem is that any $\mathscr{P}$–algebra is the colimit of a filtered diagram of finitely presented $\mathscr{P}$–algebras.

Theorem 40 *Let $\mathscr{C}$ be a conilpotent curved cooperad. The category $\mathscr{C}$–*cog *of $\mathscr{C}$–coalgebras is presentable.*

The essence of this theorem is that any $\mathscr{C}$–coalgebra is the colimit of a filtered diagram of finite-dimensional $\mathscr{C}$–coalgebras. Since the category of $\mathscr{C}$–coalgebras does not seem to be comonadic over a known presentable category, we cannot use the same kind of arguments as in the proof of [8, Lemma 5.2].

Lemma 41 *The category $\mathscr{C}$–cog is cocomplete.*

Proof The colimit of a diagram of $\mathscr{C}$–coalgebras is its colimit in the category of graded $\mathbb{K}$–modules, together with the obvious decomposition map and coderivation map. □

Lemma 42 *For any $\mathscr{C}$–coalgebra $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ and any finite-dimensional subgraded $\mathbb{K}$–module $\mathcal{V} \subset \mathcal{C}$, there exists a finite-dimensional sub-$\mathscr{C}$–coalgebra $\mathscr{E}$ of $\mathscr{D}$ which contains $\mathcal{V}$.*

Proof Let us prove the result by induction on the coradical filtration of $\mathscr{D}$. Suppose first that $\mathcal{V} \subset F_0\mathcal{D}$. Then $\mathcal{V} + d\mathcal{V}$ is a sub-$\mathscr{C}$–coalgebra of $\mathscr{D}$. Then suppose that, for any finite-dimensional subgraded $\mathbb{K}$–module $\mathcal{W} \in F_n^{\mathrm{rad}}\mathcal{D}$, there exists a finite-dimensional sub-$\mathscr{C}$–coalgebra $\mathscr{E}$ of $F_n^{\mathrm{rad}}\mathscr{D}$ which contains $\mathcal{W}$. Consider now a finite-dimensional subgraded $\mathbb{K}$–module $\mathcal{V} \subset F_{n+1}\mathcal{D}$. By Proposition 37, for any element $x \in F_{n+1}^{\mathrm{rad}}\mathcal{D}$, $\Delta_{\mathscr{D}}(x) - 1 \otimes x \in \mathcal{C} \circ F_n^{\mathrm{rad}}\mathcal{D}$. Since we are working with conilpotent $\mathscr{C}$–coalgebras, there exists a finite-dimensional subgraded $\mathbb{K}$–module $\mathcal{V}(x)$ of $F_n^{\mathrm{rad}}\mathcal{D}$ such that $\Delta_{\mathscr{D}}(x) - 1 \otimes x \in \mathcal{C} \circ \mathcal{V}(x)$. Let $(e_i)_{i=1}^k$ be a linearly free family of elements of $\mathcal{V}$ such that $\mathcal{V} = \mathcal{V} \cap F_n^{\mathrm{rad}}\mathcal{D} \oplus \bigoplus_{i=1}^k \mathbb{K}.e_i$. By the induction hypothesis, let $\mathcal{E}$ be a finite-dimensional sub-$\mathscr{C}$–coalgebra of $\mathscr{D}$ which contains

$$\mathcal{V} \cap F_n^{\mathrm{rad}}\mathcal{D} \oplus \sum \mathcal{V}(e_i) + \mathcal{V}(d_{\mathcal{D}}e_i).$$

Then the sum

$$\mathcal{E} + \sum_i (\mathbb{K}.e_i \oplus \mathbb{K}.d_{\mathcal{D}}e_i)$$

is a finite-dimensional sub-$\mathscr{C}$–coalgebra of $\mathscr{D}$ which contains $\mathcal{V}$. □

Finally, we show that a finite-dimensional $\mathscr{C}$–coalgebra is a compact object.

Proposition 43 *A finite-dimensional $\mathscr{C}$–coalgebra is a compact object.*

We need the following technical lemma:

Lemma 44 *Let $D: I \to \mathscr{C}$–cog be a filtered diagram. Let $x \in D(i)$ for an object i of I. If the image of x in colim D is zero, then there exists an object i' of I and a map $\phi: i \to i'$ such that $D(\phi)(x) = 0$.*

Proof The colimit of the diagram D is the cokernel of the map

$$g\colon \bigoplus_{f\colon j\to j'} D(j) \to \bigoplus_{i\in Ob(I)} D(i)$$

such that for any morphism $f\colon j \to j'$ of I, the morphism g sends $x \in D(j)$ to $x - D(f)(x)$. Let $x \in D(i)$ whose image in $\operatorname{colim} D$ is zero. Then there exists an element $y = \sum y_f$ of $\bigoplus_{f\colon j\to j'} D(j)$ such that $g(y) = x$. Let i' be a cocone in I of the finite diagram made up of the morphisms f such that $y_f \neq 0$. Then the image in $D(i')$ of $\sum y_f$ is the same as the image in $D(i')$ of $\sum D(f)(y_f)$. Hence, the image of x in $D(i')$ is zero. □

Proof of Proposition 43 Let $D\colon I \to \mathscr{C}\text{–cog}$ be a filtered diagram and let $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ be a finite-dimensional $\mathscr{C}$–coalgebra. We have to show that the canonical map

$$\operatorname{colim}(\hom_{\mathscr{C}\text{–cog}}(\mathscr{D}, D)) \to \hom_{\mathscr{C}\text{–cog}}(\mathscr{D}, \operatorname{colim} D)$$

is bijective.

- Let us first show that it is surjective. Let $f\colon \mathscr{D} \to \operatorname{colim} D$ be a map of $\mathscr{C}$–coalgebra and let $\mathcal{D}'$ be the image of f inside $\operatorname{colim} D$ which is also a sub-$\mathscr{C}$–coalgebra of $\operatorname{colim} D$. Let $\{e_a\}_{a=1}^n$ be a basis of the graded $\mathbb{K}$–module $\mathcal{D}'$. Since the diagram D is filtered, there exists an object i of I and for each a an element $x_a \in D(i)$ whose image in $\operatorname{colim} D$ is e_a. Let $\mathscr{E}$ be the smallest sub-$\mathscr{C}$–coalgebra of $D(i)$ which contains all the x_a and let $\mathscr{E}'$ be the image of $\mathscr{E}$ in $\operatorname{colim} D$. Notice that $\mathscr{E}'$ contains $\mathcal{D}'$ and that the map $\mathscr{E} \to \mathscr{E}'$ is surjective. By Lemma 44 and since $\mathscr{E}$ is finite-dimensional, there exists an object i' and a map $\phi\colon i \to i'$ such that the map $\mathscr{E}'' := D(\phi)(\mathscr{E}) \to \mathscr{E}'$ is an isomorphism of $\mathscr{C}$–coalgebras. So let $\mathscr{D}''$ be the sub-$\mathscr{C}$–coalgebra of $\mathscr{E}''$ which is the image of $\mathcal{D}'$ through the inverse isomorphism $\mathscr{E}' \to \mathscr{E}''$. Hence, the map $\mathscr{D} \to \mathscr{D}' \to \operatorname{colim} D$ factors through the map $\mathscr{D} \to \mathscr{D}' \simeq \mathscr{D}'' \to D(i')$ and so the canonical map $\operatorname{colim}(\hom_{\mathscr{C}\text{–cog}}(\mathscr{D}, D)) \to \hom_{\mathscr{C}\text{–cog}}(\mathscr{D}, \operatorname{colim} D)$ is surjective.

- Let us show that it is injective. Let

$$f \in \hom_{\mathscr{C}\text{–cog}}(\mathscr{D}, D(i)) \quad \text{and} \quad g \in \hom_{\mathscr{C}\text{–cog}}(\mathscr{D}, D(j))$$

be two maps whose images in $\hom_{\mathscr{C}\text{–cog}}(\mathscr{D}, \operatorname{colim} D)$ are the same; it is denoted by h. Since the category I is filtered, there exists an object k together with maps $\phi\colon i \to k$ and $\psi\colon j \to k$. Then $D(\phi)f(\mathscr{D}) + D(\psi)g(\mathscr{D})$ is a finite-dimensional sub-$\mathscr{C}$–coalgebra of $D(k)$ whose image in colim D is $h(\mathscr{D})$. As in the previous point (by Lemma 44), there exists a map $\zeta\colon k \to k'$ in I such that the map

$$u\colon D(\zeta)(D(\phi)f(\mathscr{D}) + D(\psi)g(\mathcal{C})) \to h(\mathscr{D})$$

is an isomorphism. Since the dimension (as a graded $\mathbb{K}$–module) of $D(\zeta)D(\phi)f(\mathscr{D})$ and the dimension of $D(\zeta)D(\psi)g(\mathscr{D})$ are both greater than the dimension of $h(\mathscr{D})$, we must have

$$D(\zeta)(D(\phi)f(\mathscr{D}) + D(\psi)g(\mathscr{D})) = D(\zeta)D(\phi)f(\mathscr{D}) = D(\zeta)D(\psi)g(\mathscr{D}).$$

In this context, we have

$$D(\zeta)D(\phi)f = u^{-1}h = D(\zeta)D(\psi)g.$$

Hence, f and g represent the same element of $\operatorname{colim}(\hom_{\mathscr{C}\text{–cog}}(\mathscr{D}, D))$. □

Proof of Theorem 40 The isomorphisms classes of finite-dimensional $\mathscr{C}$–coalgebras form a set. By Proposition 43, any finite-dimensional $\mathscr{C}$–coalgebra is a compact object of the category $\mathscr{C}$–cog. Moreover, any $\mathscr{C}$–coalgebra is the colimit of the diagram of its finite-dimensional sub-$\mathscr{C}$–coalgebras (with inclusions between them); this is a filtered diagram (and even a directed set). Hence, the category $\mathscr{C}$–cog is presentable. □

3 Enrichment

This section deals with several enrichments of the category of algebras of an operad and of the category of coalgebras of a curved conilpotent cooperad. Specifically, we prove that both the category of algebras over an operad and the category of coalgebras over a curved conilpotent cooperad are tensored, cotensored and enriched over cocommutative coalgebras and enriched over simplicial sets. In the nonsymmetric context, algebras over an operad and coalgebras over a curved conilpotent cooperad are tensored, cotensored and enriched over coassociative coalgebras.

3.1 Enrichment over coassociative coalgebras and cocommutative coalgebras

We show in this subsection that the category of algebras over an operad and the category of coalgebras over a curved conilpotent cooperad are tensored-cotensored-enriched (see Definition 4) over the category uCocom of counital cocommutative coalgebras.

Moreover, in the nonsymmetric context, they are tensored-cotensored-enriched over the category uCog of coassociative coalgebras. We will use these enrichments in the sequel to describe deformations of morphisms and mapping spaces, respectively.

3.1.1 Enrichment of $\mathscr{P}$–algebras over coalgebras Let $\mathscr{P} = (\mathcal{P}, \gamma, 1)$ be a dg operad. For any counital cocommutative coalgebra $\mathscr{C} = (\mathcal{C}, \Delta_{\mathscr{C}}, \epsilon)$ and any $\mathscr{P}$–algebra $\mathscr{A} = (\mathcal{A}, \gamma_{\mathscr{A}})$, the chain complex $[\mathcal{C}, \mathcal{A}]$ has a canonical structure of $\mathscr{P}$–algebra as follows.

- For any $p \in \mathcal{P}(n)$ $(n \geq 1)$, and for any $f_1, \dots, f_n \in [\mathcal{C}, \mathcal{A}]$ and any $x = \mathcal{C}$,
$$\gamma_{[\mathcal{C},\mathcal{A}]}(p \otimes_{\mathbb{S}_n} (f_1 \otimes \cdots \otimes f_n))(x) = \gamma_{\mathscr{A}}(p \otimes -)(f_1 \otimes \cdots \otimes f_n)\Delta_{\mathscr{C}}^{n-1}(x)$$
- For any $p \in \mathcal{P}(0)$,
$$\gamma_{[\mathcal{C},\mathcal{A}]}(p) = \gamma_{\mathscr{A}}(p)\epsilon_{\mathcal{C}}$$

The chain complex $[\mathcal{C}, \mathcal{A}]$ together with its structure of $\mathscr{P}$–algebra is denoted by $[\mathscr{C}, \mathscr{A}]$.

Lemma 45 *The assignment $\mathscr{C}, \mathscr{A} \mapsto [\mathscr{C}, \mathscr{A}]$ defines a left coaction* (see Definition 3) *of the category* uCocom *of counital cocommutative coalgebras on the category* $\mathscr{P}$–alg *of $\mathscr{P}$–algebras.*

Proof The construction is functorial covariantly with respect to $\mathscr{P}$–algebras and contravariantly with respect to counital cocommutative coalgebras. Moreover, for any counital cocommutative coalgebras $\mathscr{C}$ and $\mathscr{D}$, and any $\mathscr{P}$–algebra $\mathscr{A}$ there is an isomorphism of chain complexes
$$\rho_{\mathscr{C},\mathscr{D},\mathscr{A}}\colon [\mathcal{C} \otimes \mathcal{D}, \mathcal{A}] \to [\mathcal{C}, [\mathcal{D}, \mathcal{A}]]$$
such that $\rho_{\mathscr{C},\mathscr{D},\mathscr{A}}(f)(x)(y) = f(x \otimes y)$. This is a morphism of $\mathscr{P}$–algebras which is functorial in $\mathscr{C}$, $\mathscr{D}$ and $\mathscr{A}$, and it satisfies the coherence conditions of Definition 3. □

One can define a left adjoint to the functor $[\mathscr{C}, -]$ as follows. Let $\mathscr{A} \lhd \mathscr{C}$ be the quotient of the free $\mathscr{P}$–algebra $\mathcal{P} \circ (\mathcal{A} \otimes \mathcal{C})$ by the ideal I generated by the relations
$$\gamma_{\mathscr{A}}(p \otimes_{\mathbb{S}_n} (y_1 \otimes \cdots \otimes y_n)) \otimes x \sim \sum (-1)^{\sum_{i<j} |x_{(i)}||y_j|} p \otimes_{\mathbb{S}_n} ((y_1 \otimes x_{(1)}) \otimes \cdots \otimes (y_n \otimes x_{(n)})),$$
$$\gamma_{\mathscr{A}}(p) \otimes x \sim \epsilon(x) p \quad \text{for any } p \in \mathcal{P}(0),$$
with $\Delta^{n-1}(x) = \sum x_{(1)} \otimes \cdots \otimes x_{(n)}$.

Theorem 46 *The category of $\mathscr{P}$–algebras is tensored-cotensored-enriched over the category* uCocom *of counital cocommutative coalgebras. The right action is given by the functor $- \triangleleft -$ and the left coaction is given by the functor $[-,-]$. We denote the enrichment by $\{-,-\}$.*

Proof Since the functor $[-,-]$ defines a coaction of the category of counital cocommutative coalgebras on the category of $\mathscr{P}$–algebras, since the functor $[-,\mathcal{A}]$ sends colimits to limits and since the functor $[\mathscr{C},-]$ is left adjoint to the functor $- \triangleleft \mathscr{C}$, we can conclude by Proposition 6. □

Let us describe $\{\mathscr{A},\mathscr{A}'\}$ for two $\mathscr{P}$–algebras $\mathscr{A}$ and $\mathscr{A}'$. This is the maximal subcoalgebra of the cofree cocommutative coalgebra $F([\mathcal{A},\mathcal{A}'])$ such that the following diagram commutes:

$$\begin{tikzcd}
\{\mathscr{A},\mathscr{A}'\} \arrow[rrr] \arrow[d, "{(\epsilon,\mathrm{Id},\Delta,\dots)}"'] & & & {[\mathcal{A},\mathcal{A}']} \arrow[d, "{[\gamma_{\mathcal{A}},\mathrm{Id}]}"] \\
\prod_{n\geq 0}\{\mathscr{A},\mathscr{A}'\}^{\otimes n}/\mathbb{S}_n \arrow[r] & \prod_{n\geq 0}[\mathcal{A},\mathcal{A}']^{\otimes n}/\mathbb{S}_n \arrow[r] & {[\mathcal{P}\circ\mathcal{A},\mathcal{P}\circ\mathcal{A}']} \arrow[r, "{[\mathrm{Id},\gamma_{\mathcal{A}'}]}"'] & {[\mathcal{P}\circ\mathcal{A},\mathcal{A}']}
\end{tikzcd}$$

where the map $\prod_{n\geq 0}[\mathcal{A},\mathcal{A}']^{\otimes n}/\mathbb{S}_n \to [\mathcal{P}\circ\mathcal{A},\mathcal{P}\circ\mathcal{A}']$ sends $f_1\otimes\cdots\otimes f_n$ to

$$\mathrm{Id}_{\mathcal{P}(n)}\otimes_{\mathbb{S}_n}(f_1\otimes\cdots\otimes f_n),$$

and where the map $\{\mathscr{A},\mathscr{A}'\}\to[\mathcal{A},\mathcal{A}']$ is the composition

$$\{\mathscr{A},\mathscr{A}'\}\to F([\mathcal{A},\mathcal{A}'])\to[\mathcal{A},\mathcal{A}'].$$

3.1.2 Enrichment of $\mathscr{C}$–coalgebras over coalgebras

Let $\mathscr{C}=(\mathcal{C},\Delta,\epsilon,1,d,\theta)$ be a curved conilpotent cooperad.

For any $\mathscr{C}$–coalgebra $\mathscr{D}=(\mathcal{D},\Delta_{\mathscr{D}})$ and any counital cocommutative coalgebra $\mathscr{E}=(\mathcal{E},\Delta_{\mathscr{E}},\epsilon)$, the tensor product $\mathcal{D}\otimes\mathcal{E}$ has a structure of $\mathscr{C}$–coalgebra given by

$$\mathcal{D}\otimes\mathcal{E}\xrightarrow{\bigoplus_n \Delta_n\otimes\Delta^{n-1}}\bigoplus_n(\mathcal{C}(n)\otimes_{\mathbb{S}_n}\mathcal{D}^{\otimes n})\otimes\mathcal{E}^{\otimes n}\to\bigoplus_n\mathcal{C}(n)\otimes_{\mathbb{S}_n}(\mathcal{D}\otimes\mathcal{E})^{\otimes n}.$$

Theorem 47 *The category $\mathscr{C}$–*cog *of $\mathscr{C}$–coalgebras is tensored-cotensored-enriched over the category of cocommutative counital coalgebras. The right action is given by the construction $-\otimes-$. We denote the left coaction by $\langle -,-\rangle$ and the enrichment by $\{-,-\}$.*

Proof The assignment $\mathscr{D}, \mathscr{E} \mapsto \mathscr{D} \otimes \mathscr{E}$ defines a right action of the category of counital cocommutative coalgebras on the category of $\mathscr{C}$–coalgebras. Moreover, the functor $\mathscr{D} \otimes -$ and the functor $- \otimes \mathscr{E}$ preserve colimits. We conclude by Proposition 6. □

If $\mathscr{D}$ and $\mathscr{D}'$ are two $\mathscr{C}$–coalgebras, then the cocommutative counital hom coalgebra $\{\mathscr{D}, \mathscr{D}'\}$ is the final subcoalgebra of the cofree counital cocommutative coalgebra $F([\mathcal{D}, \mathcal{D}'])$ over the chain complex $[\mathcal{D}, \mathcal{D}']$ such that the following diagram, built in a similar way as its counterpart for algebras, commutes:

$$\begin{tikzcd}
\{\mathscr{D}, \mathscr{D}'\} \arrow[rrr] \arrow[d, "(\epsilon, \mathrm{Id}, \Delta, \dots)"'] & & & {[\mathcal{D}, \mathcal{D}']} \arrow[d] \\
\prod_{n \geq 0} \{\mathscr{D}, \mathscr{D}'\}^{\otimes n}/\mathbb{S}_n \arrow[r] & \prod_{n \geq 0} [\mathcal{D}, \mathcal{D}']^{\otimes n}/\mathbb{S}_n \arrow[r] & {[\mathcal{C} \circ \mathcal{D}, \mathcal{C} \circ \mathcal{D}']} \arrow[r] & {[\mathcal{D}, \mathcal{C} \circ \mathcal{D}']}
\end{tikzcd}$$

3.1.3 Morphisms are atoms

Proposition 48 *For any two $\mathscr{P}$–algebras $\mathscr{A}$ and $\mathscr{A}'$, the dg atoms of the cocommutative coalgebra $\{\mathscr{A}, \mathscr{A}'\}$ are the morphisms of $\mathscr{P}$–algebras from $\mathscr{A}$ to $\mathscr{A}'$. Similarly, for any two $\mathscr{C}$–coalgebras $\mathscr{D}$ and $\mathscr{D}'$, the dg atoms of the cocommutative coalgebra $\{\mathscr{D}, \mathscr{D}'\}$ are the morphisms of $\mathscr{C}$–coalgebras from $\mathscr{D}$ to $\mathscr{D}'$.*

Proof We have

$$\hom_{\mathsf{uCocom}}(\mathbb{K}, \{\mathscr{A}, \mathscr{A}'\}) \simeq \hom_{\mathscr{P}\text{–}\mathsf{alg}}(\mathscr{A} \triangleleft \mathbb{K}, \mathscr{A}') \simeq \hom_{\mathscr{P}\text{–}\mathsf{alg}}(\mathscr{A}, \mathscr{A}'). \qquad \square$$

3.1.4 Nonsymmetric context In the nonsymmetric context, we can get rid of the cocommutativity condition.

Proposition 49

- *If $\mathscr{P}$ is a nonsymmetric operad, then the category of $\mathscr{P}$–algebras is tensored-cotensored-enriched over the category uCog of counital coassociative coalgebras.*
- *If $\mathscr{C}$ is a nonsymmetric conilpotent curved cooperad, then the category of $\mathscr{C}$–coalgebras is tensored-cotensored-enriched over the category uCog of counital coassociative coalgebras.*

We denote by $\{-,-\}^{\mathrm{ns}}$ these two enrichments over counital coassociative coalgebras.

Proof The proof is similar to the proofs of Theorems 46 and 47. □

The inclusion functor $\mathsf{uCocom} \hookrightarrow \mathsf{uCog}$ is a left adjoint (since it preserves colimits). Let R be its right adjoint. It sends any counital coassociative coalgebra to its final cocommutative subcoalgebra.

Proposition 50 *For any $\mathscr{P}$–algebras $\mathscr{A}$ and $\mathscr{A}'$, the cocommutative coalgebra $\{\mathscr{A}, \mathscr{A}'\}$ is the final cocommutative subcoalgebra $R(\{\mathscr{A}, \mathscr{A}'\}^{\mathrm{ns}})$ of $\{\mathscr{A}, \mathscr{A}'\}^{\mathrm{ns}}$. Similarly, for any $\mathscr{C}$–coalgebras $\mathscr{D}$ and $\mathscr{D}'$, the cocommutative coalgebra $\{\mathscr{D}, \mathscr{D}'\}$ is the final cocommutative subcoalgebra $R(\{\mathscr{D}, \mathscr{D}'\}^{\mathrm{ns}})$ of $\{\mathscr{D}, \mathscr{D}'\}^{\mathrm{ns}}$.*

Proof For any cocommutative coalgebra $\mathscr{E}$, we have

$$\begin{aligned}\hom_{\mathsf{uCocom}}(\mathscr{E}, \{\mathscr{A}, \mathscr{A}'\}) \simeq \hom_{\mathscr{P}\text{-}\mathsf{alg}}(\mathscr{A} \triangleleft \mathscr{E}, \mathscr{A}') &\simeq \hom_{\mathsf{uCog}}(\mathscr{E}, \{\mathscr{A}, \mathscr{A}'\}^{\mathrm{ns}}) \\ &\simeq \hom_{\mathsf{uCocom}}(\mathscr{E}, R(\{\mathscr{A}, \mathscr{A}'\}^{\mathrm{ns}})).\end{aligned}$$

Since these isomorphisms are functorial, $R(\{\mathscr{A}, \mathscr{A}'\}^{\mathrm{ns}})$ is isomorphic to $\{\mathscr{A}, \mathscr{A}'\}$. □

3.2 Simplicial enrichment

In this section, we recall the fact that the Sullivan polynomials forms algebras allow one to enrich the category of algebras over an operad. See for instance [16].

3.2.1 General case Let A be a differential graded unital commutative $\mathbb{K}$–algebra. The category of dg A–modules is equipped with a tensor product

$$M \otimes_A N = \operatorname{colim}(M \otimes A \otimes N \rightrightarrows M \otimes N),$$

where the two maps are given by the action of A on M and on N, respectively. The functor $A \otimes -\colon \mathsf{dgMod} \to \mathsf{dgMod}_A$ is strong symmetric monoidal. Hence, it induces several functors:

- from operads to operads enriched in A–modules,
- from cooperads to cooperads enriched in A–modules,
- from $\mathscr{P}$–algebras (in the category of $\mathbb{K}$–modules) to $A \otimes \mathscr{P}$–algebras (in the category of A–modules),
- from $\mathscr{C}$–coalgebras (in the category of $\mathbb{K}$–modules) to $A \otimes \mathscr{C}$–coalgebras (in the category of A–modules).

Applying this to the case of the Sullivan algebras of polynomial forms on standard simplices leads us to the following proposition:

Proposition 51 *Let $\mathscr{P}$ be a dg operad and let $\mathscr{C}$ be a curved conilpotent cooperad. The category of $\mathscr{P}$–algebras and the category of $\mathscr{C}$–coalgebras are enriched in simplicial sets as follows:*

$$\mathrm{HOM}(\mathscr{A}, \mathscr{A}')_n := \hom_{\mathscr{P}\text{–alg}}(\mathscr{A}, \Omega_n \otimes \mathscr{A}') \simeq \hom_{\Omega_n \otimes \mathscr{P}\text{–alg}}(\Omega_n \otimes \mathscr{A}, \Omega_n \otimes \mathscr{A}'),$$
$$\mathrm{HOM}(\mathscr{D}, \mathscr{D}')_n := \hom_{\Omega_n \otimes \mathscr{C}\text{–cog}}(\Omega_n \otimes \mathscr{D}, \Omega_n \otimes \mathscr{D}').$$

Proof The only point that needs to be cleared up is the simplicial structure on $\mathrm{HOM}(\mathscr{D}, \mathscr{D}')$. Let $\phi\colon [m] \to [n]$ be a map between finite ordinals. We want to define $\phi^*\colon \mathrm{HOM}(\mathscr{D}, \mathscr{D}')_n \to \mathrm{HOM}(\mathscr{D}, \mathscr{D}')_m$. An element of $\mathrm{HOM}(\mathscr{D}, \mathscr{D}')_n$ is a morphism of graded $\mathbb{K}$–modules f from $\mathcal{D}$ to $\Omega_n \otimes \mathcal{D}'$ such that $f d_{\mathcal{D}} = (d_{\Omega_n} \otimes \mathrm{Id}_{\mathcal{D}'} + \mathrm{Id}_{\Omega_n} \otimes d_{\mathcal{D}'})$ and such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "f"] \arrow[d, "\Delta"'] & & \Omega_n \otimes \mathcal{D}' \arrow[d, "\mathrm{Id} \otimes \Delta"] \\
\mathcal{C} \circ \mathcal{D} \arrow[r, "\mathrm{Id} \circ f"'] & \mathcal{C} \circ (\Omega_n \otimes \mathcal{D}') \arrow[r] & \Omega_n \otimes (\mathcal{C} \circ \mathcal{D}')
\end{tikzcd}$$

where the map $\mathcal{C} \circ (\Omega_n \otimes \mathcal{D}') \to \Omega_n \otimes (\mathcal{C} \circ \mathcal{D}')$ is the map

$$x \otimes_{\mathbb{S}_k} ((a_1 \otimes x_1) \otimes \cdots \otimes (a_k \otimes x_k))$$
$$\mapsto (-1)^{|x|(\sum |a_i|)} (-1)^{\sum_{i>j} |a_i||x_j|} (a_1 \cdots a_k) \otimes (x \otimes_{\mathbb{S}_k} (x_1 \otimes \cdots \otimes x_k)).$$

Then $\phi^*(f) = (\Omega[\phi] \otimes \mathrm{Id}) f$ where $\Omega[\phi]\colon \Omega_n \to \Omega_m$ is the structural map induced by ϕ. □

Proposition 52 *For any simplicial set X which is the colimit of a finite diagram of simplices $\Delta[n]$ and for any $\mathscr{P}$–algebras $\mathscr{A}$ and $\mathscr{A}'$, we have*

$$\hom_{\mathsf{sSet}}(X, \mathrm{HOM}(\mathscr{A}, \mathscr{A}')) \simeq \hom_{\mathscr{P}\text{–alg}}(\mathscr{A}, \Omega_X \otimes \mathscr{A}').$$

Proof It suffices to notice that the functor from commutative algebras to $R \otimes \mathscr{P}$–algebras $R \mapsto R \otimes \mathscr{A}'$ preserves finite limits. □

Remark 53 The enrichment of the category of $\mathscr{P}$–algebras and of the category of $\mathscr{C}$–coalgebras over simplicial sets that we described above is a part of a more general enrichment over functors from the category of unital commutative algebras to simplicial sets:

$$R \mapsto (\hom_{\mathscr{P}\text{–alg}}(\mathscr{A}, \Omega_n \otimes R \otimes \mathscr{B}))_{n \in \mathbb{N}},$$
$$R \mapsto (\hom_{\Omega_n \otimes R \otimes \mathscr{C}\text{–cog}}(\Omega_n \otimes R \otimes \mathscr{D}, \Omega_n \otimes R \otimes \mathscr{D}'))_{n \in \mathbb{N}}.$$

3.2.2 Nonsymmetric context In the nonsymmetric context, we can use some associative algebras instead of the commutative Sullivan algebras to define a simplicial mapping spaces. However, this does not define an enrichment any more. Let Λ_n be the linear dual of the Dold–Kan coalgebra over the standard simplex,

$$\Lambda_n := \mathrm{DK}^c(\Delta[n])^*.$$

This defines a simplicial unital associative algebra.

Further, let $\mathscr{P}$ be a nonsymmetric dg operad. For any $\mathscr{P}$–algebra $\mathscr{A} = (\mathcal{A}, \gamma_{\mathscr{A}})$, and for any associative algebra A, $A \otimes \mathcal{A}$ has a canonical structure of a $\mathscr{P}$–algebra.

Definition 54 (nonsymmetric simplicial mapping spaces of algebras over an operad) For any two $\mathscr{P}$–algebras $\mathscr{A}$ and $\mathscr{B}$, let $\mathrm{HOM}^{\mathrm{ns}}(\mathscr{A}, \mathscr{B})$ be the simplicial set

$$\mathrm{HOM}^{\mathrm{ns}}(\mathscr{A}, \mathscr{B})_n := \hom_{\mathscr{P}\text{-}\mathsf{alg}}(\mathscr{A}, \Lambda_n \otimes \mathscr{B}).$$

Let $\mathscr{C}$ be a nonsymmetric curved conilpotent cooperad. For any associative algebra A and for any two $\mathscr{C}$–coalgebras $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ and $\mathscr{E} = (\mathcal{E}, \Delta_{\mathcal{E}})$, we denote by $\hom_{A,\mathscr{C}}(\mathscr{D}, \mathscr{E})$ the set of morphisms of graded $\mathbb{K}$–modules f from $\mathcal{D}$ to $A \otimes \mathcal{E}$ which commute with the coderivations and such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "f"] \arrow[d, "\Delta"'] & & A \otimes \mathcal{E} \arrow[d, "\mathrm{Id}_A \otimes \Delta_{\mathscr{E}}"] \\
\mathcal{C} \circ_{\mathrm{ns}} \mathcal{D} \arrow[r] & \mathcal{C} \circ_{\mathrm{ns}} (A \otimes \mathcal{E}) \arrow[r] & A \otimes (\mathcal{C} \circ_{\mathrm{ns}} \mathcal{E})
\end{tikzcd}$$

Definition 55 (nonsymmetric simplicial mapping spaces of coalgebras over a curved cooperad) For any two $\mathscr{C}$–coalgebras $\mathscr{D}$ and $\mathscr{D}'$, let $\mathrm{HOM}^{\mathrm{ns}}(\mathscr{D}, \mathscr{D}')_n$ be the simplicial set

$$\mathrm{HOM}^{\mathrm{ns}}(\mathscr{D}, \mathscr{D}')_n := \hom_{\Lambda_n,\mathscr{C}}(\mathscr{D}, \mathscr{D}').$$

These simplicial sets are related to the enrichments over coassociative coalgebras that we described above.

Proposition 56 *For any two $\mathscr{P}$–algebras $\mathscr{A}$ and $\mathscr{B}$ and for any two $\mathscr{C}$–coalgebras $\mathscr{D}$ and $\mathscr{D}'$, we have isomorphisms*

$$\mathrm{HOM}^{\mathrm{ns}}(\mathscr{A}, \mathscr{B}) \simeq N(\{\mathscr{A}, \mathscr{B}\}^{\mathrm{ns}}),$$
$$\mathrm{HOM}^{\mathrm{ns}}(\mathscr{D}, \mathscr{D}') \simeq N(\{\mathscr{D}, \mathscr{D}'\}^{\mathrm{ns}}).$$

Proof The proof for $\mathscr{P}$–algebras is straightforward. Let us prove the result for the $\mathscr{C}$–coalgebras. A morphism of graded $\mathbb{K}$–modules f from $\mathcal{D}$ to $\Lambda_n \otimes \mathcal{D}'$ is equivalent to a morphism from $\mathcal{D} \otimes \mathrm{DK}^c(\Delta[n])$ to $\mathcal{D}'$. In that context, f belongs to $\hom_{\Lambda_n,\mathscr{C}}(\mathscr{D}, \mathscr{D}')$ if and only if the corresponding morphism from $\mathcal{D} \otimes \mathrm{DK}^c(\Delta[n])$ to $\mathcal{D}'$ is a morphism of $\mathscr{C}$–coalgebras. So

$$\begin{aligned} \mathrm{HOM}^{\mathrm{ns}}(\mathscr{A}, \mathscr{B})_n := \hom_{\Lambda_n,\mathscr{C}}(\mathscr{D}, \mathscr{D}') &\simeq \hom_{\mathscr{C}\text{–}\mathsf{cog}}(\mathscr{D} \otimes \mathrm{DK}^c(\Delta[n]), \mathscr{D}') \\ &\simeq \hom_{\mathsf{uCog}}(\mathrm{DK}^c(\Delta[n]), \{\mathscr{D}, \mathscr{D}'\}^{\mathrm{ns}}) \\ &\simeq \hom_{\mathsf{sSet}}(\Delta[n], N(\{\mathscr{D}, \mathscr{D}'\}^{\mathrm{ns}})). \end{aligned}$$ □

Remark 57 Beware! The construction $\mathscr{A}, \mathscr{B} \mapsto \mathrm{HOM}^{\mathrm{ns}}(\mathscr{A}, \mathscr{B})$ does not define an enrichment. This comes from the fact that the nerve functor $N\colon \mathsf{uCog} \to \mathsf{sSet}$ is not monoidal.

4 Bar–cobar adjunctions

The usual bar–cobar adjunction relates nonunital algebras to noncounital conilpotent coalgebras; see [19, Chapter 2]. It can be extended to nonunital operads and conilpotent cooperads; see [12]. Further, as a direct consequence of work of Hirsh and Millès [17], there exists an adjunction à la bar–cobar relating unital algebras with curved conilpotent coalgebras. We extend it to operads and curved conilpotent cooperads.

The bar–cobar adjunction $\Omega_u \dashv B_c$ is a tool to compute resolutions of operads. But it has other aspects: any morphism of operads from the cobar construction $\Omega_u\mathscr{C}$ of a curved conilpotent cooperad $\mathscr{C}$ to an operad $\mathscr{P}$ gives rise to a new adjunction à la bar cobar between $\mathscr{C}$–coalgebras and $\mathscr{P}$–algebras.

4.1 Operadic bar–cobar construction

The usual operadic bar–cobar adjunction (see [19, Chapter 6]) relates augmented operads to differential graded conilpotent cooperads. The bar construction $B\mathscr{P}$ of an operad $\mathscr{P}$ does use the augmentation of $\mathscr{P}$ as it is the graded cofree cooperad on the suspension of $\overline{\mathscr{P}}$. If $\mathscr{P}$ is not augmented, one can try to add an element to $\mathscr{P}$ whose boundary is the unit of $\mathscr{P}$ and try the same computation. This is the new bar construction; its output is no longer a differential graded cooperad but a curved cooperad.

The new curved bar functor B_c has also a left adjoint Ω_u whose formula looks like the usual operadic cobar functor. Again, as in [19, Chapter 6], this adjunction is related to a notion of twisting morphism.

Definition 58 (operadic bar construction) The *operadic bar construction* of a dg operad $\mathscr{P} = (\mathcal{P}, \gamma_{\mathscr{P}}, 1)$ is the curved conilpotent cooperad $B_c\mathscr{P} = (\mathbb{T}^c(s\mathcal{P} \oplus \mathbb{K}\cdot v), D, \theta)$, where $s\mathcal{P}$ is the suspension of the $\mathbb{S}$–module $\mathcal{P}$ and where v is an arity 1, degree 2 element. It is equipped with the coderivation D which extends the following map from $\overline{\mathbb{T}}(s\mathcal{P} \oplus \mathbb{K}\cdot v)$ to $s\mathcal{P} \oplus \mathbb{K}\cdot v$:

$$\mathbb{T}(s\mathcal{P} \oplus v) \to \mathbb{T}^{\leq 2}(s\mathcal{P} \oplus v) \to s\mathcal{P} \oplus v, \qquad \begin{aligned} sx &\mapsto -sd_{\mathcal{P}}x, \\ sx \otimes sy &\mapsto (-1)^{|x|} s\gamma_{\mathscr{P}}(x \otimes y), \\ v &\mapsto s1. \end{aligned}$$

It has the curvature map

$$\theta\colon \overline{\mathbb{T}}(s\mathcal{P} \oplus v) \to s\mathcal{P} \oplus \mathbb{K}\cdot v \to \mathbb{K}\cdot v \to \mathbb{K}, \quad v \mapsto 1.$$

Proposition 59 *The map θ is actually a curvature for the coderivation, that is, $D^2 = (\theta \otimes \mathrm{Id} - \mathrm{Id} \otimes \theta)\Delta_2$.*

Proof Let π be the projection from $B_c\mathscr{P}$ to $s\mathcal{P}$. By Lemma 27, it suffices to prove that $\pi D^2 = (\theta \otimes \pi - \pi \otimes \theta)\Delta_2$. This is a straightforward calculation. □

Definition 60 (operadic cobar construction) The *operadic cobar construction* of a curved conilpotent cooperad $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon, 1, \theta)$ is the dg operad $\Omega_u\mathscr{C} = (\mathbb{T}s^{-1}\mathcal{C}, D)$, where D is the degree -1 derivation

$$s^{-1}x \mapsto \theta(x)1 - s^{-1}dx - \sum (-1)^{|x_{(1)}|} s^{-1}x_{(1)} \otimes s^{-1}x_{(2)},$$

where $\Delta_2(x) = \sum x_1 \otimes x_2$.

Proposition 61 *The derivation D squares to zero.*

Proof It suffices to prove the result for any element of the form $s^{-1}x$, which is a straightforward calculation. □

Definition 62 (operadic twisting morphism) Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon, 1, d, \theta)$ be a curved conilpotent cooperad and let $\mathscr{P} = (\mathcal{P}, \gamma_{\mathscr{P}}, 1_{\mathscr{P}}, d)$ be a dg operad. An *operadic twisting morphism* from $\mathscr{C}$ to $\mathscr{P}$ is a degree -1 map of $\mathbb{S}$–modules (or $\mathbb{N}$–modules in the nonsymmetric case)

$$\alpha\colon \bar{\mathcal{C}} \to \mathcal{P}$$

such that

$$\partial(\alpha)+\gamma(\alpha\otimes\alpha)\Delta_2=\Theta,$$

where $\Theta(x)=\theta(x)1_{\mathscr{P}}$ for any $x\in\mathscr{C}$. We denote by $\mathrm{Tw}(\mathscr{C},\mathscr{P})$ the set of operadic twisting morphisms from $\mathscr{C}$ to $\mathscr{P}$.

Proposition 63 *We have the functorial isomorphisms*

$$\hom_{\mathsf{Operad}}(\Omega_u\mathscr{C},\mathscr{P})\simeq\mathrm{Tw}(\mathscr{C},\mathscr{P})\simeq\hom_{\mathsf{cCoop}}(\mathscr{C},B_c\mathscr{P}).$$

Proof Proving the existence of the functorial isomorphism $\hom_{\mathsf{Operad}}(\Omega_u\mathscr{C},\mathscr{P})\simeq\mathrm{Tw}(\mathscr{C},\mathscr{P})$ is similar to the proof of [17, Theorem 3.4.1]. Let us show that we have a functorial isomorphism $\mathrm{Tw}(\mathscr{C},\mathscr{P})\simeq\hom_{\mathsf{cCoop}}(\mathscr{C},B_c\mathscr{P})$. Let $\alpha\colon\mathscr{C}\to\mathscr{P}$ be an operadic twisting morphism. We obtain a degree zero map from $\bar{\mathcal{C}}$ to $s\mathcal{P}\oplus\mathbb{K}\cdot v$ as follows:

$$\bar{\mathcal{C}}\to s\mathcal{P}\oplus\mathbb{K}\cdot v,\quad c\mapsto s\alpha(x)+\theta_{\mathscr{C}}(x).$$

This induces a morphism of graded cooperads $f_\alpha\colon\mathscr{C}\to B_c\mathscr{P}=\mathbb{T}^c(s\mathcal{P}\oplus\mathbb{K}\cdot v)$ such that $\theta_{\mathscr{C}}=\theta_{B_c\mathscr{P}}f_\alpha$. Since $\partial(\alpha)+\gamma_{\mathscr{P}}(\alpha\otimes\alpha)\Delta_2=\Theta$, the morphism f_α commutes with the coderivations and so is a morphism of curved cooperads. Conversely, from any morphism of curved cooperads f from $\mathscr{C}$ to $B_c\mathscr{P}$, one obtains a twisting morphism as follows:

$$\mathscr{C}\xrightarrow{f}B_c\mathscr{P}\twoheadrightarrow s\mathscr{P}\to\mathscr{P}.$$

The two constructions that we described are inverse one to another. □

Hence, the functors Ω_u and B_c realize an adjunction between the category of dg operads and the category of curved conilpotent cooperads,

$$\begin{tikzcd}\mathsf{cCoop} \arrow[r, shift left, "\Omega_u"] & \mathsf{Operad}. \arrow[l, shift left, "B_c"]\end{tikzcd}$$

4.2 Twisted products

Let $\alpha\colon\overline{\mathscr{C}}\to\mathscr{P}$ be an operadic twisting morphism.

Definition 64 (twisted $\mathscr{P}$–module) For any $\mathscr{C}$–comodule $\mathscr{D}$, let $\mathscr{P}\circ_\alpha\mathscr{D}$ be the free $\mathscr{P}^{\mathrm{grad}}$–module $\mathcal{P}\circ\mathcal{D}$ equipped with the unique derivation which extends the map

$$\mathcal{D}\to\mathcal{P}\circ\mathcal{D},\quad x\mapsto d_{\mathcal{D}}(x)-(\alpha\circ\mathrm{Id})\Delta(x).$$

Definition 65 (twisted $\mathscr{C}$–comodule) For any $\mathscr{P}$–module $\mathscr{A}$, let $\mathscr{C} \circ_\alpha \mathscr{A}$ be the cofree $\mathscr{C}^{\mathrm{grad}}$–comodule $\mathcal{C} \circ \mathcal{A}$ equipped with a unique coderivation which extends the map

$$\mathcal{C} \circ \mathcal{A} \to \mathcal{A}, \quad x \mapsto (d_{\mathcal{A}}(\epsilon_{\mathscr{C}} \circ \mathrm{Id}) + \gamma_{\mathscr{A}}(\alpha \circ \mathrm{Id}))(x).$$

Proposition 66 *The derivation of $\mathscr{P} \circ_\alpha \mathscr{D}$ squares to zero. Hence, $\mathscr{P} \circ_\alpha \mathscr{D}$ is a dg $\mathscr{P}$–module. Similarly, the coderivation of $\mathscr{C} \circ_\alpha \mathscr{A}$ squares to $(\theta \circ \mathrm{Id})\Delta$. Hence, $\mathscr{C} \circ_\alpha \mathscr{A}$ is a $\mathscr{C}$–comodule.*

Proof To prove the first point, it suffices to show that $\pi D^2 = 0$, which is a straightforward calculation. To prove the second point, it suffices to show that $\pi D^2 = (\theta \circ \mathrm{Id})$, which is a straightforward calculation. □

Definition 67 (twisting morphism relative to an operadic twisting morphism) For any $\mathscr{C}$–comodule $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ and any $\mathscr{P}$–module $\mathscr{A} = (\mathcal{A}, \gamma_{\mathscr{A}})$ an α–twisting morphism from $\mathscr{D}$ to $\mathscr{A}$ is a degree 0 map $\phi\colon \mathcal{D} \to \mathcal{A}$ such that

$$\partial(\phi) + \gamma_{\mathcal{A}}(\alpha \circ \phi)\Delta_{\mathcal{C}} = 0.$$

We denote by $\mathrm{Tw}_\alpha(\mathscr{D}, \mathscr{A})$ the set of α–twisting morphisms from $\mathscr{D}$ to $\mathscr{A}$.

Proposition 68 *There are functorial isomorphisms*

$$\hom_{\mathscr{P}\text{–mod}}(\mathscr{P} \circ_\alpha \mathscr{D}, \mathscr{A}) \simeq \mathrm{Tw}_\alpha(\mathscr{D}, \mathscr{A}) \simeq \hom_{\mathscr{C}\text{–comod}}(\mathscr{D}, \mathscr{C} \circ_\alpha \mathscr{A})$$

for any $\mathscr{C}$–comodule $\mathscr{D}$ and any $\mathscr{P}$–module $\mathscr{A}$.

Proof The proof is similar to [19, Proposition 11.3.2]. □

4.3 Bar–cobar adjunction for algebras over an operad and coalgebras over a cooperad

Following [19, Chapter 11], we call the previous functors the bar construction for $\mathscr{P}$–algebras and the cobar construction for $\mathscr{C}$–coalgebras, respectively.

Definition 69 (bar construction and cobar construction relatives to an operadic twisting morphism) Let $\alpha\colon \overline{\mathscr{C}} \to \mathscr{P}$ be an operadic twisting morphism. The α–bar construction is the functor from $\mathscr{P}$–algebras to $\mathscr{C}$–coalgebras defined by

$$B_\alpha \mathscr{A} := \mathscr{C} \circ_\alpha \mathscr{A}.$$

The α–cobar construction is the functor from $\mathscr{C}$–coalgebras to $\mathscr{P}$–algebras defined by

$$\Omega_\alpha \mathscr{D} := \mathscr{P} \circ_\alpha \mathscr{D}.$$

We already know, by Proposition 68, that Ω_α is left adjoint to B_α. Moreover, this adjunction is enriched over cocommutative coalgebras and simplicial sets.

Proposition 70 *The functors Ω_α and B_α induce functorial isomorphisms of counital cocommutative coalgebras and of simplicial sets*

$$\{\Omega_\alpha \mathscr{D}, \mathscr{A}\} \simeq \{\mathcal{C}, B_\alpha \mathscr{A}\}, \quad \mathrm{HOM}(\Omega_\alpha \mathscr{D}, \mathscr{A}) \simeq \mathrm{HOM}(\mathscr{D}, B_\alpha \mathscr{A})$$

for any $\mathscr{C}$–coalgebra $\mathscr{D}$ and any $\mathscr{P}$–algebra $\mathscr{A}$;

Lemma 71 *We have a functorial isomorphism*

$$\mathrm{Tw}_\alpha(\mathscr{D} \otimes \mathscr{E}, \mathscr{A}) \simeq \mathrm{Tw}_\alpha(\mathscr{D}, [\mathscr{E}, \mathscr{A}])$$

for any $\mathscr{C}$–coalgebra $\mathscr{D}$, any $\mathscr{P}$–algebra $\mathscr{A}$ and any counital cocommutative coalgebra $\mathscr{E}$.

Proof The set of morphisms of graded $\mathbb{K}$–modules from $\mathcal{D} \otimes \mathcal{E}$ to $\mathcal{A}$ is in bijection with the set of morphisms of graded $\mathbb{K}$–modules from $\mathcal{D}$ to $[\mathcal{E}, \mathcal{A}]$. This bijection and its inverse preserve α–twisting morphisms. □

Lemma 72 *We have a functorial isomorphism*

$$\mathrm{Tw}_\alpha(\mathscr{D}, R \otimes \mathscr{A}) \simeq \hom_{R\otimes\mathscr{C}\text{–cog}}(R \otimes \mathscr{D}, R \otimes B_\alpha \mathscr{A})$$

for any $\mathscr{C}$–coalgebra $\mathscr{D}$, any $\mathscr{P}$–algebra $\mathscr{A}$ and any dg unital commutative algebra R.

Proof Let us first denote by π the map from $R \otimes (\mathcal{C} \circ \mathcal{A})$ to $R \otimes \mathcal{A}$ defined by the formula

$$\pi = \mathrm{Id} \otimes (\epsilon \circ \mathrm{Id}).$$

As we have already seen, a morphism in $\hom_{R\otimes\mathscr{C}\text{–cog}}(R \otimes \mathscr{D}, R \otimes B_\alpha \mathscr{A})$ is equivalent to the data of a map $f \colon \mathcal{D} \to R \otimes (\mathcal{C} \circ \mathcal{A})$ which satisfies some conditions; see the proof of Proposition 51. On the one hand, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "f"] \arrow[d, "\Delta"'] & & R \otimes (\mathcal{C} \circ \mathcal{A}) \arrow[d, "\mathrm{Id}\otimes\Delta"] \\
\mathcal{C} \circ \mathcal{D} \arrow[r, "\mathrm{Id}\circ f"'] & \mathcal{C} \circ (R \otimes (\mathcal{C} \circ \mathcal{A})) \arrow[r] & R \otimes (\mathcal{C} \circ \mathcal{C} \circ \mathcal{A})
\end{tikzcd}$$

This implies that f is the composition

$$\mathcal{D} \xrightarrow{\Delta} \mathcal{C} \circ \mathcal{D} \xrightarrow{\mathrm{Id} \circ f} \mathcal{C} \circ (R \otimes (\mathcal{C} \circ \mathcal{A})) \to R \otimes (\mathcal{C} \circ \mathcal{C} \circ \mathcal{A}) \xrightarrow{\mathrm{Id} \otimes (\mathrm{Id} \circ \epsilon \circ \mathrm{Id})} R \otimes (\mathcal{C} \circ \mathcal{A}).$$

Then, exchanging the action of ϵ with the exchange between $\mathcal{C}$ and R, we see that f is the composition

$$\mathcal{D} \xrightarrow{\Delta} \mathcal{C} \circ \mathcal{D} \xrightarrow{\mathrm{Id} \circ (\pi f)} \mathcal{C} \circ (R \otimes \mathcal{A}) \to R \otimes (\mathcal{C} \circ \mathcal{A}).$$

On the other hand, $\partial(f) = 0$. Given the relation between f and πf just above, a straightforward calculation shows that this is equivalent to the fact that πf is a twisting morphism. □

Proof of Proposition 70 On the one hand, for any cocommutative coalgebra $\mathscr{E}$, we have

$$\begin{aligned} \hom_{\mathsf{uCocom}}(\mathscr{E}, \{\Omega_\alpha \mathscr{D}, \mathscr{A}\}) &\simeq \hom_{\mathscr{P}\text{-}\mathsf{alg}}(\Omega_\alpha \mathscr{D}, [\mathscr{E}, \mathscr{A}]) \\ &\simeq \mathrm{Tw}_\alpha(\mathscr{D}, [\mathscr{E}, \mathscr{A}]) \simeq \mathrm{Tw}_\alpha(\mathscr{D} \otimes \mathscr{E}, \mathscr{A}) \\ &\simeq \hom_{\mathscr{C}\text{-}\mathsf{cog}}(\mathscr{D} \otimes \mathscr{E}, B_\alpha \mathscr{A}) \\ &\simeq \hom_{\mathsf{uCocom}}(\mathscr{E}, \{\mathscr{D}, B_\alpha \mathscr{A}\}). \end{aligned}$$

On the other hand, by Lemma 72 and Proposition 68 we have functorial isomorphisms

$$\hom_{\mathscr{P}\text{-}\mathsf{alg}}(\Omega_\alpha \mathscr{D}, R \otimes \mathscr{A}) \simeq \mathrm{Tw}_\alpha(\mathscr{D}, R \otimes \mathscr{A}) \simeq \hom_{R \otimes \mathscr{C}\text{-}\mathsf{cog}}(R \otimes \mathscr{D}, R \otimes B_\alpha \mathscr{A})$$

for any dg unital commutative algebra R. Taking $R = \Omega_n$ gives us a natural isomorphism of simplicial sets $\mathrm{HOM}(\Omega_\alpha \mathscr{D}, \mathscr{A}) \simeq \mathrm{HOM}(\mathscr{D}, B_\alpha \mathscr{A})$. □

5 Homotopy theory of algebras over an operad

In this section, we recall a result of Hinich, stating that for any dg operad $\mathscr{P}$, the category of $\mathscr{P}$–algebras admits a projective model structure whose weak equivalences are quasi-isomorphisms (see [15; 4]). Moreover, we show that the simplicial enrichment of the category of $\mathscr{P}$–algebras that we described above gives models for the mapping spaces. Finally, we show that the enrichment over cocommutative coalgebras introduced in Section 3 encodes deformation of morphisms of $\mathscr{P}$–algebras.

5.1 Model structure on algebras over an operad

We recall here results about model structures on the category of algebras over an operad.

Definition 73 (right induced model structures) Consider the adjunction

$$\begin{tikzcd} \mathsf{C} \arrow[r, shift left, "L"] & \mathsf{D} \arrow[l, shift left, "R"] \end{tikzcd}$$

Suppose that C admits a cofibrantly generated model structure. We say that D admits a model structure *right induced* by the adjunction $L \dashv R$ if it admits a model structure whose weak equivalences (resp. fibrations) are the morphisms f such that $R(f)$ is a weak equivalence (resp. a fibration) and whose generating cofibrations (resp. generating acyclic cofibrations) are the images under L of the generating cofibrations (resp. generating acyclic cofibrations) of C.

Definition 74 (admissible operad) An operad $\mathscr{P}$ is said to be *admissible* if the category of $\mathscr{P}$–algebras admits a projective model structure, that is, a model structure right induced by the adjunction

$$\begin{tikzcd} \mathsf{dgMod} \arrow[r, shift left, "\mathscr{P}\circ -"] & \mathscr{P}\text{–}\mathsf{alg} \arrow[l, shift left] \end{tikzcd}$$

whose right adjoint is the forgetful functor.

Theorem 75 [15] *Any nonsymmetric operad is admissible. When the characteristic of the field* $\mathbb{K}$ *is zero, any operad is admissible.*

5.2 Mapping spaces

The simplicial enrichments of the category of $\mathscr{P}$–algebras described above give us models for the mapping spaces.

Proposition 76 *Suppose that the characteristic of the field* $\mathbb{K}$ *is zero. Let* $\mathscr{P}$ *be a dg operad. The assignment* $\mathscr{A}, \mathscr{A}' \mapsto \mathrm{HOM}(\mathscr{A}, \mathscr{A}')$ *defines a homotopical enrichment of the category of* $\mathscr{P}$*–algebras over the category of simplicial sets. Moreover, for any cofibrant* $\mathscr{P}$*–algebra* $\mathscr{A}$ *and any* $\mathscr{P}$*–algebra* $\mathscr{A}'$*, the simplicial set* $\mathrm{HOM}(\mathscr{A}, \mathscr{A}')$ *is a model of the mapping space* $\mathrm{Map}(\mathscr{A}, \mathscr{A}')$.

Remark 77 The characteristic zero assumption is not necessary in the nonsymmetric context.

Proof Let $f\colon \mathscr{A} \to \mathscr{A}'$ and $g\colon \mathscr{B} \to \mathscr{B}'$ be a cofibration and a fibration of $\mathscr{P}$–algebras, respectively. Let $h\colon X \to Y$ be a monomorphism of simplicial sets which is a generating cofibration or acyclic cofibration for the Kan–Quillen model structure.

Then X and Y are colimits of finite diagrams made up of simplices $\Delta[n]$. Consider a square

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \mathrm{HOM}(\mathscr{A}', \mathscr{B}) \arrow[d] \\
Y \arrow[r] & \mathrm{HOM}(\mathscr{A}', \mathscr{B}') \times_{\mathrm{HOM}(\mathscr{A}, \mathscr{B}')} \mathrm{HOM}(\mathscr{A}, \mathscr{B})
\end{tikzcd}$$

By Proposition 52, it induces the square

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d] & \Omega_Y \otimes \mathscr{B} \arrow[d] \\
\mathcal{A}' \arrow[r] & \Omega_Y \otimes \mathscr{B}' \times_{\Omega_X \otimes \mathscr{B}'} \Omega_X \otimes \mathscr{B}
\end{tikzcd}$$

which has a lifting whenever f, g or h is a weak equivalence; indeed, by Proposition 18, the map $\Omega_Y \to \Omega_X$ is a fibration and it is an acyclic fibration whenever h is an acyclic cofibration. Further, to prove that $\mathrm{HOM}(\mathscr{A}, \mathscr{A}')$ is a model of the mapping space $\mathrm{Map}(\mathscr{A}, \mathscr{A}')$, it suffices to notice that $\{\Omega_n \otimes \mathscr{A}'\}_{n \in \mathbb{N}}$ is a Reedy fibrant resolution of the constant simplicial $\mathscr{P}$–algebra $\mathscr{A}'$. □

5.3 Deformation theory of morphisms of algebras over an operad

We know that the category of $\mathscr{P}$–algebras is enriched over the category uCocom of cocommutative coalgebras. In this subsection, we show that for any $\mathscr{P}$–algebras $\mathscr{A}$ and $\mathscr{B}$, the cocommutative coalgebra $\{\mathscr{A}, \mathscr{B}\}$ encodes the deformation theory of morphisms from $\mathscr{A}$ to $\mathscr{B}$. We suppose in this subsection that the field $\mathbb{K}$ is algebraically closed.

Any morphism of $\mathscr{P}$–algebras $f \colon \mathscr{A} \to \mathscr{B}$ defines a deformation problem $\mathrm{Def}(f)$.

$$\begin{aligned}
&\mathsf{Artin\text{–}alg} \to \mathsf{sSet}, \\
&\qquad R \mapsto \mathrm{Map}(\mathscr{A}, \mathscr{B} \otimes R) \times^h_{\mathrm{Map}(\mathscr{A}, \mathscr{B})} \{f\} \simeq \mathrm{HOM}(\mathscr{A}, \mathscr{B} \otimes R) \times_{\mathrm{HOM}(\mathscr{A}, \mathscr{B})} \{f\}.
\end{aligned}$$

The following theorem is a direct consequence of a result by Chuang, Lazarev and Mannan [7, Theorem 2.9]. It is proven in the appendix.

Theorem 78 *Suppose that the base field $\mathbb{K}$ is algebraically closed and that its characteristic is zero. Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon)$ be a dg cocommutative coalgebra and let A be its set of graded atoms. There exists a unique decomposition $\mathscr{C} \simeq \bigoplus_{a \in A} \mathscr{C}_a$, where $\mathscr{C}_a$ is a subcoalgebra of $\mathscr{C}$ which contains a and which belongs to the category* $\mathsf{uNilCocom}$. *Moreover, a morphism of dg cocommutative coalgebras $f \colon \bigoplus_{a \in A} \mathscr{C}_a \to \bigoplus_{b \in B} \mathscr{D}_b$ is the data of a function $\phi \colon A \to B$ and of a morphism $f_a \colon \mathscr{C}_a \to \mathscr{D}_{\phi(a)}$ for any $a \in A$.*

We know from Proposition 48 that a morphism f of $\mathscr{P}$–algebras from $\mathscr{A}$ to $\mathscr{B}$ is a dg atom of the cocommutative coalgebra $\{\mathscr{A},\mathscr{B}\}$. Applying Theorem 78 to the cocommutative coalgebra $\{\mathscr{A},\mathscr{B}\}$, we obtain the conilpotent cocommutative coalgebra $\{\mathscr{A},\mathscr{B}\}_f$. This is in particular a Hinich coalgebra which encodes a deformation problem $R \mapsto \mathrm{Map}(R^*, \{\mathscr{A},\mathscr{B}\}_f)$. We show in the next proposition that this deformation problem is $\mathrm{Def}(f)$.

Theorem 79 *Suppose that $\mathscr{A}$ is a cofibrant $\mathscr{P}$–algebra. Then the deformation problem induced by the conilpotent cocommutative coalgebra $\{\mathscr{A},\mathscr{B}\}_f$ is* $\mathrm{Def}(f)$.

Lemma 80 *If $\mathscr{A}$ is a cofibrant $\mathscr{P}$–algebra, the simplicial Hinich coalgebra*

$$\{\mathscr{A},\Omega_n\otimes\mathscr{B}\}_f$$

is a Reedy fibrant replacement of the constant simplicial Hinich coalgebra $\{\mathscr{A},\mathscr{B}\}_f$.

Proof Let $g\colon X\to Y$ be a monomorphism of simplicial sets which are finite colimits of standard simplices $\Delta[n]$. Let $h\colon \mathscr{C}_1\to\mathscr{C}_2$ be a monomorphism of Hinich coalgebras. Consider the square

$$\begin{tikzcd}
\mathscr{C}_1 \arrow[r] \arrow[d] & \{\mathscr{A},\Omega_Y\otimes\mathscr{B}\}_f \arrow[d, "\{\mathscr{A},\Omega[g]\otimes\mathscr{B}\}"] \\
\mathscr{C}_2 \arrow[r] & \{\mathscr{A},\Omega_X\otimes\mathscr{B}\}_f
\end{tikzcd}$$

Any morphism of cocommutative coalgebras from a conilpotent cocommutative coalgebra $\mathscr{C}$ to $\{\mathscr{A},\mathscr{B}\}$ such that the atom of $\mathscr{C}$ targets the atom f of $\{\mathscr{A},\mathscr{B}\}$ is a morphism from $\mathscr{C}$ to $\{\mathscr{A},\mathscr{B}\}_f$. So, lifting the previous square amounts to lifting the square of $\mathscr{P}$–algebras

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & {[\mathscr{C}_2,\Omega_Y\otimes\mathscr{B}]} \arrow[d] \\
\mathscr{A} \arrow[r] & {[\mathscr{C}_1,\Omega_Y\otimes\mathscr{B}]\times_{[\mathscr{C}_1,\Omega_X\otimes\mathscr{B}]}[\mathscr{C}_2,\Omega_Y\otimes\mathscr{B}]}
\end{tikzcd}$$

This is possible whenever, g or h is a weak equivalence, since any weak equivalence of Hinich coalgebras is in particular a quasi-isomorphism. So, in particular, any face map $\{\mathscr{A},\Omega_{n+1}\otimes\mathscr{B}\}\to\{\mathscr{A},\Omega_n\otimes\mathscr{B}\}$ is an acyclic fibration of Hinich coalgebras and, for any integer $n\in\mathbb{N}$, the morphism $\{\mathscr{A},\Omega_n\otimes\mathscr{B}\}\to\{\mathscr{A},\Omega_{\partial\Delta[n]}\otimes\mathscr{B}\}$ is a fibration. □

Proof of Theorem 79 By Lemma 80, the deformation problem induced by the Hinich coalgebra $\{\mathscr{A},\mathscr{B}\}_f$ is equivalent to the deformation problem

$$R \in \mathsf{Artin\text{–}alg} \mapsto (\hom_{\mathsf{Hinich\text{–}cog}}(R^*, \{\mathscr{A}, \Omega_n \otimes \mathscr{B}\}_f))_{n\in\mathbb{N}}.$$

We have

$$\begin{aligned}\hom_{\mathsf{Hinich\text{–}cog}}&(R^*, \{\mathscr{A}, \Omega_n \otimes \mathscr{B}\}_f)\\ &\simeq \hom_{\mathsf{uCocom}}(R^*, \{\mathscr{A}, \Omega_n \otimes \mathscr{B}\}) \times_{\hom_{\mathsf{uCocom}}(\mathbb{K},\{\mathscr{A},\Omega_n\otimes\mathscr{B}\})} \{f\}\\ &\simeq \hom_{\mathscr{P}\text{–}\mathsf{alg}}(\mathscr{A} \triangleleft R^*, \Omega_n \otimes \mathscr{B}) \times_{\hom_{\mathscr{P}\text{–}\mathsf{alg}}(\mathscr{A},\Omega_n\otimes\mathscr{B})} \{f\}\\ &\simeq \hom_{\mathscr{P}\text{–}\mathsf{alg}}(\mathscr{A}, R\otimes\Omega_n \otimes \mathscr{B}) \times_{\hom_{\mathscr{P}\text{–}\mathsf{alg}}(\mathscr{A},\Omega_n\otimes\mathscr{B})} \{f\}.\end{aligned}$$

Since the simplicial sets

$$(\hom_{\mathscr{P}\text{–}\mathsf{alg}}(\mathscr{A}, R\otimes\Omega_n \otimes \mathscr{B}))_{n\in\mathbb{N}} \quad\text{and}\quad (\hom_{\mathscr{P}\text{–}\mathsf{alg}}(\mathscr{A}, \Omega_n \otimes \mathscr{B}))_{n\in\mathbb{N}}$$

are Kan complexes and models of $\mathrm{Map}(\mathscr{A}, R\otimes\mathscr{B})$ and $\mathrm{Map}(\mathscr{A},\mathscr{B})$, respectively, and since the map between them is a fibration, the simplicial set

$$(\hom_{\mathscr{P}\text{–}\mathsf{alg}}(\mathscr{A}, R\otimes\Omega_n \otimes \mathscr{B}) \times_{\hom_{\mathscr{P}\text{–}\mathsf{alg}}(\mathscr{A},\Omega_n\otimes\mathscr{B})} \{f\})_{n\in\mathbb{N}}$$

is a model of the homotopy pullback $\mathrm{Map}(\mathscr{A}, R\otimes\mathscr{B}) \times^h_{\mathrm{Map}(\mathscr{A},\mathscr{B})} \{f\}$. □

6 Model structures on coalgebras over a cooperad

In this section, we show that, for any operadic twisting morphism $\alpha\colon \mathscr{C} \to \mathscr{P}$, the projective model structure on the category of $\mathscr{P}$–algebras can be transferred through the cobar construction functor Ω_α to the category of $\mathscr{C}$–coalgebras. This result is in the vein of similar results by Hinich [16], Lefevre and Hasegawa [18], Vallette [26] and Positselski [23]. However, we use a new method for the proof that uses the presentability of the category of algebras over an operad and of the category of coalgebras over a curved conilpotent cooperad; specifically, we use a theorem proved by Bayeh, Hess, Karpova, Kedziorek, Riehl and Shipley [3; 14].

6.1 Model structure induced by a twisting morphism

Definition 81 (left induced model structures) Consider the adjunction

$$\begin{tikzcd} \mathsf{C} \arrow[r, shift left, "L"] & \mathsf{D} \arrow[l, shift left, "R"] \end{tikzcd}$$

Suppose that D admits a model structure. We say that C admits a model structure left induced by the adjunction $L \dashv R$ if it admits a model structure whose weak equivalences (resp. cofibrations) are the morphisms f such that $L(f)$ is a weak equivalence (resp. a cofibration).

Here is the main theorem of the present article:

Theorem 82 *Let $\mathscr{P}$ be a dg operad, let $\mathscr{C}$ be a curved conilpotent cooperad and let α be an operadic twisting morphism between them. Suppose that the characteristic of the base field $\mathbb{K}$ is zero. We know that the category of $\mathscr{P}$–algebras admits a projective model structure. Then the category of $\mathscr{C}$–coalgebras admits a model structure left induced by the adjunction $\Omega_\alpha \dashv B_\alpha$. We call it the α–model structure. In the nonsymmetric context, we can drop the assumption that the characteristic of the field $\mathbb{K}$ is zero.*

To prove this theorem, we will use the following result:

Theorem 83 [3, Theorem 2.23; 14, Theorem 2.2.1] *Consider an adjunction*

$$\begin{tikzcd} \mathsf{C} \arrow[r, shift left, "L"] & \mathsf{M} \arrow[l, shift left, "R"] \end{tikzcd}$$

between presentable categories. Suppose that M is endowed with a cofibrantly generated model structure. Then there exists a left induced model structure on C if the morphisms which have the right lifting property with respect to left induced cofibrations are left induced weak equivalences. In particular, this is true if the category C has a cofibrant replacement functor, and if any cofibrant object has a cylinder.

From now on, a *weak equivalence* (resp. *cofibration*) of $\mathscr{C}$–coalgebras is a morphism whose image under Ω_α is a weak equivalence (resp. cofibration). An *acyclic cofibration* is a morphism which is both a cofibration and a weak equivalence. A *fibration* is a morphism which has the right lifting property with respect to all acyclic cofibrations and an *acyclic fibration* is a morphism which is both a fibration and a weak equivalence. Here is the proof.

Proof of Theorem 82 Proposition 84 ensures us that the cofibrations of the category of $\mathscr{C}$–coalgebras are the monomorphisms. Hence, any object is cofibrant. Then Proposition 90 provides us with a cylinder for any object if the characteristic of $\mathbb{K}$ is zero. In the nonsymmetric context, Proposition 93 provides us with a cylinder. We conclude by Theorem 83. □

6.2 Cofibrations

Proposition 84 *The class of cofibrations of $\mathscr{C}$–coalgebras is the class of monomorphisms.*

Lemma 85 *Let $f\colon \mathscr{D} \to \mathscr{E}$ be a monomorphism of $\mathscr{C}$–coalgebras such that $\Delta(\mathcal{E}) \subset \mathcal{C} \circ f(\mathcal{D})$. Then f is a cofibration.*

Proof We can decompose the graded $\mathbb{K}$–module $\mathcal{E}$ as $\mathcal{E} = \mathcal{D} \oplus \mathcal{F}$. The coderivation $d_{\mathcal{E}}$ corresponds then to the matrix

$$\begin{pmatrix} d_{\mathcal{D}} & \phi \\ 0 & d_{\mathcal{F}} \end{pmatrix}$$

Consider the diagram of $\mathscr{P}$–algebras

$$\begin{tikzcd}
\mathcal{P} \circ (s^{-1}\mathcal{F}) \arrow[r] \arrow[d, hook] & \Omega_\alpha \mathcal{D} \\
\mathcal{P} \circ (s^{-1}\mathcal{F} \oplus \mathcal{F}) &
\end{tikzcd}$$

where the horizontal map sends $s^{-1}x$ to $\phi(x) + (\alpha \circ \mathrm{Id})\Delta(x)$. The fact that it commutes with derivations is given by the fact that the derivation of $\Omega_\alpha \mathcal{E}$ squares to zero. Moreover, $s^{-1}\mathcal{F} \oplus \mathcal{F}$ is endowed with the differential $d(s^{-1}x + y) = -s^{-1}d_{\mathcal{F}}x + s^{-1}y + d_{\mathcal{F}}y$. The vertical map is a cofibration since it is the image under the left Quillen functor $\mathcal{P} \circ (-)$ of a cofibration, and f is the pushout of this vertical map along the horizontal map. Hence, f is a cofibration. □

Proof of Proposition 84 Let $f\colon \mathscr{D} \to \mathscr{E}$ be a cofibration. Then $\Omega_\alpha(f)$ is a monomorphism. Since the following square is commutative, f is a monomorphism:

$$\begin{tikzcd}
\Omega_\alpha \mathscr{D} \arrow[r] & \Omega_\alpha \mathscr{E} \\
\mathscr{D} \arrow[r] \arrow[u] & \mathscr{E} \arrow[u]
\end{tikzcd}$$

Conversely, if f is a monomorphism, then, it can be decomposed into the transfinite composition of the maps $f_n = \mathscr{D} + F^{\mathrm{rad}}_{n-1}\mathscr{E} \to \mathscr{D} + F^{\mathrm{rad}}_{n}\mathscr{E}$. Since the maps f_n satisfy the conditions of Lemma 85, they are cofibrations. So f is a cofibration. □

6.3 Filtered quasi-isomorphism

Definition 86 (filtered quasi-isomorphism) Let $\mathscr{D}$ and $\mathscr{E}$ be two $\mathscr{C}$–coalgebras. A morphism of $\mathscr{C}$–coalgebras f from $\mathscr{D}$ to $\mathscr{E}$ is said to be a *filtered quasi-isomorphism* if the induced morphisms between the graded complexes relative to the coradical filtrations are quasi-isomorphisms, that is, if for any integer n, the morphism from $G_n^{\mathrm{rad}}\mathscr{D}$ to $G_n^{\mathrm{rad}}\mathscr{E}$ is a quasi-isomorphism.

Proposition 87 *If the characteristic of $\mathbb{K}$ is zero, a filtered quasi-isomorphism is a weak equivalence of $\mathscr{C}$–coalgebras. The characteristic zero assumption is not necessary in the nonsymmetric context.*

We will use the following classical result:

Theorem 88 [21, Theorem XI.3.4] *Let $f\colon A \to B$ be a map of filtered chain complexes. Suppose that the filtrations are bounded below and exhaustive. If, for any integer n, the map $G_n A \to G_n B$ is a quasi-isomorphism, then f is a quasi-isomorphism.*

Proof of Proposition 87 Consider the filtration on $\Omega_\alpha \mathscr{D}$ (resp. $\Omega_\alpha \mathscr{E}$)

$$F_n \Omega_\alpha \mathscr{D} = \mathcal{P}(0) \oplus \sum_{\substack{k \geq 1 \\ p_1 + \cdots + p_k = n}} \mathcal{P}(k) \otimes_{\mathbb{S}_k} (F_{p_1}^{\mathrm{rad}} \mathcal{D} \otimes \cdots \otimes F_{p_k}^{\mathrm{rad}} \mathcal{D}).$$

It is clear that $\Omega_\alpha(f)$ sends $F_n \Omega_\alpha \mathscr{D}$ to $F_n \Omega_\alpha \mathscr{E}$ for any integer n. Moreover, we have

$$G_n \Omega_\alpha \mathscr{D} = \sum_{\substack{k \geq 1 \\ p_1 + \cdots + p_k = n}} \mathcal{P}(k) \otimes_{\mathbb{S}_k} (G_{p_1}^{\mathrm{rad}} \mathcal{D} \otimes \cdots \otimes G_{p_k}^{\mathrm{rad}} \mathcal{D}).$$

Then, by the operadic Künneth formula, $G_n(\Omega_\alpha(f))\colon G_n \Omega_\alpha \mathscr{D} \to G_n \Omega_\alpha \mathscr{E}$ is a quasi-isomorphism for any $n \in \mathbb{N}$. Hence, by Theorem 88, $\Omega_u(f)$ is a quasi-isomorphism. □

Remark 89 The coradical filtration is not the only filtration whose notion of filtered quasi-isomorphism gives us weak equivalences. An exhaustive filtration $(F_n \mathcal{D})_{n \in \mathbb{N}}$ is called *admissible* if

$$\begin{aligned} \Delta(F_n \mathcal{D}) &\subset \sum_{p_0 + p_1 + \cdots + p_k = n} F_0^{\mathrm{rad}} \mathcal{C} \otimes_{\mathbb{S}_k} (F_{p_1} \mathcal{D} \otimes \cdots \otimes F_{p_k} \mathcal{D}), \\ d(F_n \mathcal{D}) &\subset F_n \mathcal{D}, \\ d^2(F_n \mathcal{D}) &\subset F_{n-1} \mathcal{D}. \end{aligned}$$

Using similar arguments as in the proof just above, we can prove that a filtered quasi-isomorphism with respect to two admissible filtrations is a weak equivalence.

6.4 Cylinder object

Proposition 90 *Let $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ be a $\mathscr{C}$–coalgebra. Let $\mathscr{A} = (\mathcal{A}, \gamma_{\mathscr{A}})$ be a cylinder of $\Omega_\alpha(\mathscr{D})$ such that the structural map $p\colon \mathscr{A} \to \Omega_\alpha(\mathscr{D})$ is an acyclic fibration. The diagram*

$$\begin{tikzcd}
B_\alpha\Omega_\alpha(\mathscr{D}\oplus\mathscr{D}) \arrow[r] & B_\alpha(\mathscr{A}) \arrow[r, "B_\alpha p"] & B_\alpha\Omega_\alpha(\mathscr{D}) \\
\mathscr{D}\oplus\mathscr{D} \arrow[r] \arrow[u] & \mathscr{E} \arrow[r] \arrow[u] & \mathscr{D} \arrow[u]
\end{tikzcd}$$

where $\mathscr{E} := B_\alpha(\mathscr{A}) \times_{B_\alpha(\Omega_\alpha\mathscr{D})} \mathscr{D}$ provides us with a cylinder $\mathscr{E} = (\mathcal{E}, \Delta_{\mathscr{E}})$ for the $\mathscr{C}$–coalgebra $\mathscr{D}$.

Lemma 91 *The pullback $\mathscr{E}$ is the final subgraded $\mathscr{C}^{\mathrm{grad}}$–coalgebra of $B_\alpha\mathscr{A}$ whose image in $B_\alpha\Omega_\alpha\mathscr{D}$ is in the image of the morphism $\mathscr{D} \to B_\alpha\Omega_\alpha\mathscr{D}$.*

Proof Let $\mathscr{F} = (\mathcal{F}, \Delta_{\mathscr{F}})$ be the final subgraded $\mathscr{C}^{\mathrm{grad}}$–coalgebra of $B_\alpha\mathscr{A}$ whose image in $B_\alpha\Omega_\alpha\mathscr{D}$ is in the image of the morphism $\mathscr{D} \to B_\alpha\Omega_\alpha\mathscr{D}$. Proving that $\mathscr{F}$ is the underlying $\mathscr{C}^{\mathrm{grad}}$–coalgebra of $\mathscr{E}$ amounts to proving that $\mathcal{F}$ is stable under the coderivation D of $B_\alpha\mathscr{A}$. We prove it by induction on the coradical filtration of $\mathcal{F}$. First, by the maximality property of $\mathscr{F}$, $F_0^{\mathrm{rad}}\mathcal{F}$ is stable under D. Then suppose that $F_n^{\mathrm{rad}}\mathcal{F}$ is stable under D for an integer $n \geq 0$. Let x be an element of $F_{n+1}^{\mathrm{rad}}\mathcal{F}$. On the one hand, $B_\alpha(p)D(x) = D(B_\alpha(p)(x))$. Since $B_\alpha(p)(x)$ is in the image of $\mathscr{D}$ and since this image is stable under the coderivation of $B_\alpha\Omega_\alpha\mathscr{D}$, then $B_\alpha(p)D(x)$ is in the image of $\mathscr{D}$. On the other hand, we have

$$\Delta(D(x)) = (d_{\mathcal{C}} \circ \mathrm{Id} + \mathrm{Id} \circ' D)\Delta(x).$$

So, since $\Delta(x) - 1_{\mathscr{C}} \otimes x \in \mathcal{C} \circ (F_n^{\mathrm{rad}}\mathcal{F})$, and since $F_n^{\mathrm{rad}}\mathcal{F}$ is stable under D by the inductivity assumption,

$$\Delta(D(x)) - 1_{\mathscr{C}} \otimes D(x) = (d_{\mathcal{C}} \circ \mathrm{Id} + \mathrm{Id} \circ' D)(\Delta(x) - 1_{\mathscr{C}} \otimes x) \in \mathscr{C} \circ (F_n^{\mathrm{rad}}\mathcal{F}).$$

By these two points, $\mathcal{F} + \mathbb{K} \cdot D(x)$ is a subgraded $\mathscr{C}^{\mathrm{grad}}$–coalgebra of $B_\alpha(\mathscr{A})$ whose image in $B_\alpha\Omega_\alpha\mathscr{D}$ is in the image of $\mathscr{D}$. By the maximality property of $\mathscr{F}$, then, $D(x) \in \mathcal{F}$. So, $F_{n+1}^{\mathrm{rad}}\mathcal{F}$ is stable under D. Hence, by induction $\mathcal{F}$ is stable under D. □

To prove Proposition 90, we will show that the pullback map $\mathscr{E} \to \mathscr{D}$ is a filtered quasi-isomorphism. Since $\Omega_\alpha \mathscr{D}$ is a cofibrant $\mathscr{P}$–algebra, there exists a right inverse $q\colon \Omega_\alpha \mathscr{D} \to \mathscr{A}$ to the acyclic fibration $p\colon \mathscr{A} \to \Omega_\alpha \mathscr{D}$. Then let us decompose $\mathcal{A}$ as $\mathcal{A} = \Omega_\alpha \mathscr{D} \oplus K$. The chain complex K is acyclic. So let $h\colon K \to K$ be a degree 1 map such that $\partial(h) = \mathrm{Id}_K$. It can be extended to a map

$$B_\alpha \mathscr{A} \twoheadrightarrow \mathcal{A} \twoheadrightarrow K \to \mathcal{A}, \quad x \mapsto h(x).$$

The zero map is a coderivation on the graded cooperad $\mathscr{C}^{\mathrm{grad}}$. Then let D_h be the degree 1 coderivation of $(B_\alpha \mathscr{A})^{\mathrm{grad}}$ relative to the zero coderivation on $\mathscr{C}^{\mathrm{grad}}$ whose projection on $\mathcal{A}$ is h. In other words, $D_h = \mathrm{Id}_{\mathscr{C}} \circ' h$.

Lemma 92 *The sub-$\mathscr{C}$–coalgebra $\mathscr{E}$ of $B_\alpha \mathscr{A}$ is stable under D_h.*

Proof By Lemma 91, it suffices to prove that the final subgraded $\mathscr{C}^{\mathrm{grad}}$–coalgebra of $B_\alpha \mathscr{A}$ whose image in $B_\alpha \Omega_\alpha \mathscr{D}$ lies inside $\mathscr{D}$ is stable under D_h. To that purpose, we use the same arguments as in the proof of Lemma 91 and the fact that $B_\alpha(p) D_h = 0$. □

Proof of Proposition 90 Since the map $\mathscr{D} \oplus \mathscr{D} \to \mathscr{E}$ is a monomorphism and so a cofibration, it suffices to show that the map $\mathscr{E} \to \mathscr{D}$ is a weak equivalence. We show that it is a filtered quasi-isomorphism. Let $n \in \mathbb{N}$; let us show that the map $G_n \mathcal{E} \to G_n \mathcal{D}$ is a quasi-isomorphism. To that purpose, consider the filtration on $B_\alpha \mathscr{A}$

$$F'_k B_\alpha \mathscr{A} := \sum_{i \le k} \mathcal{C} \otimes_{\mathbb{S}} (K^{\otimes i} \otimes (\Omega_\alpha \mathscr{D})^{\otimes j}).$$

This filtration is stable under the coderivations d and D_h and it induces a filtration on $G_n^{\mathrm{rad}} \mathcal{E}$. It is clear that the morphism $G'_0 G_n^{\mathrm{rad}} \mathcal{E} \to G_n^{\mathrm{rad}} \mathcal{D}$ is an isomorphism. Moreover, for any integer $k \ge 1$, $\partial(D_h) = k.\mathrm{Id}$ on $G'_k G_n^{\mathrm{rad}} \mathcal{E}$. Since the characteristic of $\mathbb{K}$ is zero, $G'_k G_n^{\mathrm{rad}} \mathcal{E}$ is acyclic. By Theorem 88, the map $G_n \mathcal{E} \to G_n \mathcal{D}$ is a quasi-isomorphism. □

Proposition 93 *In the nonsymmetric context, $\mathscr{D} \otimes \mathrm{DK}^c(\Delta[1])$ provides us with a cylinder for the $\mathscr{C}$–coalgebra $\mathscr{D}$.*

Proof Since $G_n^{\mathrm{rad}}(\mathcal{D} \otimes \mathrm{DK}^c(\Delta[1])) = G_n^{\mathrm{rad}}(\mathcal{D}) \otimes \mathrm{DK}^c(\Delta[1])$ and since the map $\mathrm{DK}^c(\Delta[1]) \to \mathbb{K}$ is a quasi-isomorphism, $\mathcal{D} \otimes \mathrm{DK}^c(\Delta[1]) \to \mathcal{D}$ is a filtered quasi-isomorphism and so a weak equivalence. □

6.5 Enrichment in coalgebras the nonsymmetric context

Proposition 94 *In the nonsymmetric context, the assignment* $\mathscr{D}, \mathscr{D}' \mapsto \{\mathscr{D}, \mathscr{D}'\}^{\mathrm{ns}}$ *defines a homotopical enrichment of the category of* $\mathscr{C}$*–coalgebras together with its* α*–model structure over the category of counital coassociative coalgebras.*

Proof Let $f\colon \mathscr{D} \to \mathscr{D}'$ be a cofibration of $\mathscr{C}$–coalgebras, let $g\colon \mathscr{E} \to \mathscr{E}'$ be a fibration of $\mathscr{C}$–coalgebras and let $h\colon X \to Y$ be a cofibration (ie a monomorphism) of counital coassociative coalgebras. Consider the square

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \{\mathscr{D}', \mathscr{E}\}^{\mathrm{ns}} \arrow[d] \\
Y \arrow[r] & \{\mathscr{D}', \mathscr{E}'\} \times_{\{\mathscr{D}, \mathscr{E}'\}} \{\mathscr{D}, \mathscr{E}\}
\end{tikzcd}$$

It induces a square

$$\begin{tikzcd}
\mathscr{D}' \otimes X \amalg_{\mathscr{D} \otimes X} \mathscr{D} \otimes Y \arrow[r] \arrow[d] & \mathscr{E} \arrow[d] \\
\mathscr{D}' \otimes Y \arrow[r] & \mathscr{E}'
\end{tikzcd}$$

The left vertical map is a monomorphism and so a cofibration.

- If the morphism $g\colon \mathscr{E} \to \mathscr{E}'$ is an acyclic fibration, then the square has a lifting.
- Suppose that the morphism $h\colon X \to Y$ is an acyclic cofibration. Then the morphism $\mathscr{D} \otimes X \to \mathscr{D} \otimes Y$ is a filtered quasi-isomorphism and a cofibration, so it is an acyclic cofibration. Hence, its pushout $\mathscr{D}' \otimes X \to \mathscr{D}' \otimes X \amalg_{\mathscr{D} \otimes X} \mathscr{D} \otimes Y$ is also an acyclic cofibration. Moreover, the map $\mathscr{D}' \otimes X \to \mathscr{D}' \otimes Y$ is a filtered quasi-isomorphism and so a weak equivalence. So, by the 2-out-of-3 rule, the morphism $\mathscr{D}' \otimes X \amalg_{\mathscr{D} \otimes X} \mathscr{D} \otimes Y \to \mathscr{D}' \otimes Y$ is a weak equivalence. Since it is a cofibration, it is an acyclic cofibration and the square has a lifting.
- Suppose that the morphism $f\colon \mathscr{D} \to \mathscr{D}'$ is an acyclic cofibration. Then the morphism $\mathscr{D} \otimes X \to \mathscr{D}' \otimes X$ is an acyclic cofibration. This is a consequence of the fact that $\Omega_\alpha(\mathscr{D} \otimes X) = (\Omega_\alpha \mathscr{D}) \triangleleft X$, and that for any fibration of $\mathscr{P}$–algebras $\mathscr{A} \to \mathscr{A}'$, the morphism $[X, \mathscr{A}] \to [X, \mathscr{A}']$ is also a fibration. Then the same arguments as in the previous point show us that the morphism $\mathscr{D}' \otimes X \amalg_{\mathscr{D} \otimes X} \mathscr{D} \otimes Y \to \mathscr{D}' \otimes Y$ is an acyclic cofibration and so the square has a lifting. □

6.6 Changing operads and cooperads

In this subsection, we explore how the left induced model structure on coalgebras over a curved conilpotent cooperad is modified when we change the underlying operadic twisting morphism. This is inspired by [8], where a similar study is done in the context of augmented dg operads and dg conilpotent cooperads.

Recall first that a morphism of dg operads $f\colon \mathscr{P} \to \mathscr{Q}$ induces an adjunction between their categories of algebras

$$\begin{tikzcd} \mathscr{P}\text{–alg} \arrow[r, shift left, "f_!"] & \mathscr{Q}\text{–alg} \arrow[l, shift left, "f^*"] \end{tikzcd}$$

whose right adjoint f^* sends a $\mathscr{Q}$–algebra $\mathscr{A}$ to the same underlying chain complex. This adjunction is a Quillen adjunction with respect to the projective model structures; see [4]. Similar things happen for coalgebras over curved conilpotent cooperads.

Proposition 95 *Let $f\colon \mathscr{C} \to \mathscr{D}$ be a morphism of curved conilpotent cooperads. It induces an adjunction between their categories of coalgebras,*

$$\begin{tikzcd} \mathscr{C}\text{–cog} \arrow[r, shift left, "f_*"] & \mathscr{D}\text{–cog}, \arrow[l, shift left, "f^!"] \end{tikzcd}$$

whose left adjoint f_* sends a $\mathscr{C}$–coalgebra $\mathscr{E}$ to the same underlying graded $\mathbb{K}$–module.

Proof Let $\mathscr{E} = (\mathcal{E}, \Delta, d)$ be a $\mathscr{C}$–coalgebra. It has a structure of $\mathscr{D}$–coalgebra defined by the composite map

$$\mathcal{E} \xrightarrow{\Delta} \mathscr{C} \circ \mathcal{E} \xrightarrow{f \circ \mathrm{Id}} \mathscr{D} \circ \mathcal{E}.$$

This defines the functor f_*. Since it preserves colimits and since the category of $\mathscr{C}$–coalgebras and the category of $\mathscr{D}$–coalgebras are presentable, f_* has a right adjoint by Proposition 2. □

Further, let us fix a dg operad $\mathscr{P}$. The canonical operadic twisting morphism $\pi\colon B_c\mathscr{P} \to \mathscr{P}$ is universal in the sense that any operadic twisting morphism α from a curved conilpotent cooperad $\mathscr{C}$ to $\mathscr{P}$ is equivalent to a morphism of curved cooperads f from $\mathscr{C}$ to $B_c\mathscr{P}$; then $\alpha = \pi f$. In that context, the cobar functor Ω_α can be decomposed as $\Omega_\alpha = \Omega_\pi f_*$, and the α–model structure on the category of $\mathscr{C}$–coalgebras is the model structure left induced by the π–model structure on the category of $B_c\mathscr{P}$–coalgebras.

On the other hand, let us fix a curved conilpotent cooperad $\mathscr{C}$. The canonical operadic twisting morphism $\iota\colon \mathscr{C} \to \Omega_u\mathscr{C}$ is universal in the sense that any operadic twisting morphism $\alpha\colon \mathscr{C} \to \mathscr{P}$ is equivalent to the data of a morphism of operads f from $\Omega_u\mathscr{C}$ to $\mathscr{P}$; then $\alpha = f\iota$. A direct consequence of the following proposition is that the model structure on $\mathscr{C}$–coalgebras induced by the universal operadic twisting morphism $\iota\colon \mathscr{C} \to \Omega_u\mathscr{C}$ is universal in the sense that any α–model structure is a left Bousfield localization of this ι–model structure.

Proposition 96 *Let $\alpha\colon \mathscr{C} \to \mathscr{P}$ be an operadic twisting morphism and let $f\colon \mathscr{P} \to \mathscr{Q}$ be a morphism of dg operads. The $(f\alpha)$–model structure on the category of $\mathscr{C}$–coalgebras is the left Bousfield localization of the α–model structure with respect to $(f\alpha)$–weak equivalences. Moreover, if the Quillen adjunction $f_! \dashv f^*$ is a Quillen equivalence, the $(f\alpha)$–model structure coincides with the α–model structure.*

Proof The cofibrations of the α–model structure and the cofibrations of the $(f\alpha)$–model structure are both the monomorphisms. Moreover, the functor $f_!$ is a left Quillen adjoint functor. So, for any α–weak equivalence g, since $\Omega_\alpha(g)$ is a weak equivalence between cofibrant objects, $\Omega_{(f\alpha)}(g) = f_!\Omega_\alpha(g)$ is a weak equivalence. So the α–weak equivalences are in particular $(f\alpha)$–weak equivalences. So is proven the fact that the $(f\alpha)$–model structure is a left Bousfield localization of the α–model structure. Suppose now that the adjunction $f_! \dashv f^*$ is a Quillen equivalence. Then, for any $\mathscr{C}$–coalgebra $\mathscr{E}$, the morphism

$$\Omega_\alpha\mathscr{E} \to f^* f_!\Omega_\alpha\mathscr{E} = f^*\Omega_{f\alpha}\mathscr{E}$$

is a quasi-isomorphism. Since the functor f^* is the identity on the underlying chain complexes, the commutative square

$$\begin{tikzcd}
f^*\Omega_{(f\alpha)}\mathscr{E} \arrow[r, "f^*\Omega_{(f\alpha)}(g)"] & f^*\Omega_{(f\alpha)}\mathscr{E}' \\
\Omega_\alpha\mathscr{E} \arrow[r, "\Omega_\alpha(g)"'] \arrow[u] & \Omega_\alpha\mathscr{E}' \arrow[u]
\end{tikzcd}$$

ensures that a morphism $g\colon \mathscr{E} \to \mathscr{E}'$ of $\mathscr{C}$–coalgebras is a α–weak equivalence if and only if it is an $(f\alpha)$–weak equivalence. □

7 The universal model structure

In the previous section, we studied model structures on categories of coalgebras over a curved conilpotent cooperad which are induced by an operadic twisting morphism α.

In this section, we investigate the particular case where the operadic twisting morphism is the universal twisting morphism $\iota\colon \mathscr{C} \to \Omega_u\mathscr{C}$ for any curved conilpotent cooperad $\mathscr{C}$. This model structure is universal in the sense that, for any operadic twisting morphism $\alpha\colon \mathscr{C} \to \mathscr{P}$, the α–model structure on the category of $\mathscr{C}$–coalgebras is obtained from the ι–model structure by Bousfield localization. We will show that the adjunction $\Omega_\iota \dashv B_\iota$ is a Quillen equivalence, that the fibrant $\mathscr{C}$–coalgebras in the ι–model structure are the images of the $\Omega_u\mathscr{C}$–algebras under the functor B_ι, and we will describe the cofibrations, the weak equivalences and the fibrations between them. Moreover, we will prove that the enrichment of $\mathscr{C}$–coalgebras over simplicial sets that we described above computes the mapping spaces expected by the model structure.

We suppose here that the characteristic of the field $\mathbb{K}$ is zero. This assumption is not necessary in the nonsymmetric context.

7.1 Quillen equivalence

Theorem 97 *The adjunction $\Omega_\iota \dashv B_\iota$ relating $\mathscr{C}$–coalgebras to $\Omega_u\mathscr{C}$–algebras is a Quillen equivalence.*

Proof Let us show that for any $\Omega_u\mathscr{C}$–algebra $\mathscr{A} = (\mathcal{A}, \gamma_{\mathscr{A}})$, the map $\Omega_\iota B_\iota \mathscr{A} = \Omega_u\mathscr{C} \circ_\iota \mathscr{C} \circ_\iota \mathscr{A} \to \mathscr{A}$ is a quasi-isomorphism. The coradical filtration of $\mathscr{C}$ induces a filtration on $\Omega_u\mathscr{C}$,

$$\begin{aligned} &F_0\Omega_u\mathscr{C} := \mathbb{K}.1,\\ &F_n\Omega_u\mathscr{C} := \mathbb{K}.1 \oplus \sum_{\substack{i_1+\ldots+i_k=n\\ k\geq 1}} s^{-1}F^{\mathrm{rad}}_{i_1}\overline{\mathcal{C}} \otimes \cdots \otimes s^{-1}F^{\mathrm{rad}}_{i_k}\overline{\mathcal{C}} \quad \text{for } n \geq 1. \end{aligned}$$

It induces a filtration on $\Omega_u\mathscr{C} \circ_\iota \mathscr{C}$ and on $\Omega_u\mathscr{C} \circ_\iota \mathscr{C} \circ_\iota \mathscr{A}$,

$$F_n(\Omega_u\mathscr{C}\circ_\iota\mathscr{C}) := F_n\Omega_u\mathscr{C}(0) \oplus \sum_{\substack{i_0+\cdots+i_k=n\\ k\geq 1}} F_{i_0}(\Omega_u\mathscr{C})(k) \otimes_{\mathbb{S}_k} (F^{\mathrm{rad}}_{i_1}\mathcal{C} \otimes \cdots \otimes F^{\mathrm{rad}}_{i_k}\mathcal{C}),$$

$$F_n(\Omega_u\mathscr{C}\circ_\iota\mathscr{C}\circ_\iota\mathscr{A}) := F_n(\Omega_u\mathscr{C}\circ_\iota\mathscr{C})\circ\mathcal{A}.$$

Then $G(\Omega_u\mathscr{C} \circ_\iota \mathscr{C}) = \Omega_u(G\mathscr{C}) \circ_{G\iota} G\mathscr{C}$. By [19, Lemma 6.5.14], the map

$$\Omega_u(G\mathscr{C}) \circ_{G\iota} G\mathscr{C} \to \mathcal{I}$$

is a quasi-isomorphism. So, the map

$$G(\Omega_u\mathscr{C} \circ_\iota \mathscr{C} \circ_\iota \mathcal{A}) \to G\mathcal{A}$$

is a quasi-isomorphism (here $G\mathcal{A}$ is the graded complex corresponding to the constant filtration $F_n\mathcal{A} = \mathcal{A}$). Hence, by Theorem 88, the map $\Omega_u\mathscr{C} \circ_\iota \mathscr{C} \circ_\iota \mathscr{A} \to \mathscr{A}$ is a quasi-isomorphism. Since the model structure on $\mathscr{C}$–coalgebras is transferred from the model structure on $\Omega_u\mathscr{C}$–algebras, then the fact that the counit $\Omega_\iota B_\iota\mathscr{A} \to \mathscr{A}$ is a weak equivalence for any algebra $\mathscr{A}$ ensures us that the Quillen adjunction $\Omega_\iota \dashv B_\iota$ is a Quillen equivalence. □

7.2 Fibrant objects

The purpose of this subsection is to describe the fibrant objects of the ι–model structure.

Definition 98 (quasicofree $\mathscr{C}$–coalgebras) A $\mathscr{C}$–coalgebra is said to be *quasicofree* if its underlying $\mathscr{C}^{\mathrm{grad}}$–coalgebra is cofree, that is, isomorphic to a coalgebra of the form $\mathscr{C}^{\mathrm{grad}} \circ \mathcal{V}$. A morphism of quasicofree $\mathscr{C}$–coalgebras $F\colon \mathscr{C} \circ \mathcal{V} \to \mathscr{C} \circ \mathcal{W}$ (together with choices of cogenerators $\mathcal{V}$ and $\mathcal{W}$) is said to be *strict* if there exists a map $f\colon \mathcal{V} \to \mathcal{W}$ such that $F = \mathrm{Id} \circ f$.

Proposition 99 *The functor B_ι is an embedding of the category of $\Omega_u\mathscr{C}$–algebras into the category of $\mathscr{C}$–coalgebras whose essential image is spanned by quasicofree $\mathscr{C}$–coalgebras. Moreover, a morphism of $\mathscr{C}$–coalgebras $B_\iota\mathscr{A} = \mathscr{C} \circ_\iota \mathscr{A} \to B_\iota\mathscr{A}' = \mathscr{C} \circ_\iota \mathscr{A}'$ is in the image of B_ι if and only if it is strict.*

Proof It is straightforward to prove that the functor B_ι is faithful and conservative. Moreover, it is clear that the images of the functor B_ι are in particular quasicofree $\mathscr{C}$–coalgebras and strict morphisms. Conversely, let $\mathscr{D} := \mathscr{C} \circ \mathcal{A}$ be a quasicofree $\mathscr{C}$–coalgebra. Its coderivation extends the degree -1 map $d_{\mathcal{A}} \oplus \gamma\colon \mathcal{A} \oplus \overline{\mathcal{C}} \circ \mathcal{A} \to \mathcal{A}$. The map γ gives us a degree -1 map from $\overline{\mathcal{C}}$ to the operad $\mathrm{End}_{\mathcal{A}}$, that we denote by α. The coderivation which extends $d_{\mathcal{A}} \oplus \gamma$ squares to $(\theta \circ \mathrm{Id})\Delta$, so α is a twisting morphism and so induces a morphism of operads from $\Omega_u\mathscr{C}$ to $\mathrm{End}_{\mathcal{A}}$, which is an $\Omega_u\mathscr{C}$–algebra structure on $\mathcal{A}$. Then $\mathscr{D} \simeq B_\iota\mathcal{A}$. Further, let $F = \mathrm{Id} \circ f$ be a strict morphism from $B_\iota\mathscr{A}$ to $B_\iota\mathscr{B}$. Since F commutes with the coderivations, f is a morphism of $\Omega_u\mathscr{C}$–algebras. □

Theorem 100 *The fibrant $\mathscr{C}$–coalgebras in the ι–model structure are the quasicofree $\mathscr{C}$–coalgebras* (and so the objects in the essential image of the functor B_ι).

Proof Let $\mathscr{D}$ be a fibrant object. Since the morphism $\mathscr{D} \to B_\iota\Omega_\iota\mathscr{D}$ is an acyclic cofibration, the following square has a lifting:

$$\begin{tikzcd}
\mathscr{D} \arrow[r, "\mathrm{Id}"] \arrow[d] & \mathscr{D} \arrow[d] \\
B_\iota\Omega_\iota\mathscr{D} \arrow[r] & *
\end{tikzcd}$$

Hence, $\mathscr{D}$ is a retract of a quasicofree $\mathscr{C}$–coalgebra. By Lemma 101, it is a quasicofree $\mathscr{C}$–coalgebra. Conversely, a quasicofree $\mathscr{C}$–coalgebra is fibrant since it is isomorphic to the image under B_ι of an $\Omega_u\mathscr{C}$–algebra which is fibrant. □

Lemma 101 *A retract of a cofree graded $\mathscr{C}^{\mathrm{grad}}$–coalgebra is a cofree graded $\mathscr{C}^{\mathrm{grad}}$–coalgebra.*

Proof Let $\mathscr{D} = (\mathcal{D}, \Delta_{\mathscr{D}})$ be a graded $\mathscr{C}^{\mathrm{grad}}$–coalgebra which is a retract of $\mathscr{C} \circ \mathcal{V}$. On the one hand, the following diagram is a retract, that is, the compositions of the horizontal maps give the identity on the bottom and on the top:

$$\begin{tikzcd}
G_n^{\mathrm{rad}}\mathcal{D} \arrow[r] \arrow[d] & G_n^{\mathrm{rad}}(\mathcal{C} \circ \mathcal{V}) \arrow[r] \arrow[d] & G_n^{\mathrm{rad}}\mathcal{D} \arrow[d] \\
(G_n^{\mathrm{rad}}\mathcal{C}) \circ F_0^{\mathrm{rad}}\mathcal{D} \arrow[r] & (G_n^{\mathrm{rad}}\mathcal{C}) \circ F_0^{\mathrm{rad}}(\mathcal{C} \circ \mathcal{V}) \arrow[r] & (G_n^{\mathrm{rad}}\mathcal{C}) \circ F_0^{\mathrm{rad}}\mathcal{D}
\end{tikzcd}$$

Since the middle vertical map is an isomorphism, all the vertical maps are isomorphisms. On the other hand, the map $\epsilon \circ \mathrm{Id}\colon \mathcal{C} \circ \mathcal{V} \to \mathcal{V} = F_0^{\mathrm{rad}}\mathcal{C} \circ \mathcal{V}$ gives us a map $\mathcal{D} \to F_0\mathcal{D}$ and hence a morphism of graded $\mathscr{C}$–coalgebras $f\colon \mathscr{D} \to \mathscr{C} \circ F_0\mathcal{D}$. Let us show that f is an isomorphism. It is clear that the map $F_0\mathcal{D} \to F_0(\mathcal{C} \circ F_0\mathcal{D})$ is an isomorphism. For any integer $n \geq 1$, the following diagram is commutative:

$$\begin{tikzcd}
G_n(\mathcal{D}) \arrow[r, "f"] \arrow[d, "\Delta"'] & G_n(\mathcal{C} \circ F_0\mathcal{D}) \arrow[d, "\Delta"] \\
(G_n\mathcal{C}) \circ F_0\mathcal{D} \arrow[r, "\mathrm{Id}\circ f"'] & (G_n\mathcal{C}) \circ F_0(\mathcal{C} \circ F_0\mathcal{D}) = (G_n\mathcal{C}) \circ F_0\mathcal{D}
\end{tikzcd}$$

Since the vertical maps are isomorphisms and since the bottom horizontal map is an isomorphism, the top horizontal map is also an isomorphism. Hence, the map $Gf\colon G\mathcal{D} \to G(\mathcal{C} \circ F_0\mathcal{D})$ is an isomorphism. By Theorem 88, f is an isomorphism. □

7.3 Cofibrations, fibrations and weak equivalences between fibrant objects

We show here that cofibrations, weak equivalences and fibrations between fibrant $\mathscr{C}$–coalgebras are easily characterized.

Proposition 102 *Let* $\mathscr{A} = (\mathcal{A}, \gamma_{\mathscr{A}})$ *and* $\mathscr{B} = (\mathcal{B}, \gamma_{\mathscr{B}})$ *be two* $\Omega_u\mathscr{C}$*–algebras and let* $F\colon B_\iota\mathscr{A} \to B_\iota\mathscr{B}$ *be a morphism between their bar constructions. We denote by* $f\colon B_\iota\mathscr{A} \to \mathcal{B}$ *its projection* $f = \pi_{\mathcal{B}}F$ *on* $\mathcal{B}$*.*

- *The morphism* F *is a cofibration if and only if the restriction* $f|_{\mathcal{A}}$ *is a monomorphism.*
- *The morphism* F *is a weak equivalence if and only* $f|_{\mathcal{A}}$ *is a quasi-isomorphism.*
- *The morphism* F *is a fibration if and only if* $f|_{\mathcal{A}}$ *is an epimorphism.*

Lemma 103 *The morphism of chain complexes* $\mathcal{A} \to \Omega_\iota B_\iota\mathscr{A}$ *which is the restriction to* $\mathcal{A}$ *of the canonical morphism* $B_\iota\mathscr{A} \to B_\iota\Omega_\iota B_\iota\mathscr{A}$ *is a quasi-isomorphism.*

Proof It is a right inverse of the canonical morphism of $\Omega_u\mathscr{C}$–algebras $\Omega_\iota B_\iota\mathscr{A} \to \mathscr{A}$, which is a quasi-isomorphism. □

Proof of Proposition 102 Note first that $f|_{\mathcal{A}} = F|_{\mathcal{A}}$.

- Suppose that F is a cofibration, ie a monomorphism. Then its restriction $F|_{\mathcal{A}}$ is also a monomorphism. Conversely, suppose that the map $f|_{\mathcal{A}}$ is a monomorphism. We can prove by induction that, for any integer n, the map $F\colon F_n^{\mathrm{rad}}B_\iota\mathscr{A} \to F_n^{\mathrm{rad}}B_\iota\mathscr{B}$ is a monomorphism.

- By Lemma 103, the maps $\mathcal{A} \to \Omega_\iota B_\iota\mathscr{A}$ and $\mathcal{B} \to \Omega_\iota B_\iota\mathscr{B}$ are quasi-isomorphisms. Consider the diagram

$$\begin{tikzcd}
\Omega_\iota B_\iota\mathscr{A} \arrow[r, "\Omega_\iota F"] & \Omega_\iota B_\iota\mathscr{B} \\
\mathcal{A} \arrow[u] \arrow[r, "f|_{\mathcal{A}}"'] & \mathcal{B} \arrow[u]
\end{tikzcd}$$

It ensures that $f|_{\mathcal{A}}$ is a quasi-isomorphism if and only if $\Omega_\iota F$ is a quasi-isomorphism, that is, if and only if F is a weak equivalence.

- Suppose that F is a fibration. Notice first that any chain complex can be considered as a $\mathscr{C}$–coalgebra whose decomposition is given by the map with $\Delta x = 1_{\mathscr{C}} \otimes x$ (it is a coalgebra since $\Delta_{\mathscr{C}}(1_{\mathscr{C}}) \otimes x = 1_{\mathscr{C}} \otimes 1_{\mathscr{C}} \otimes x = 1_{\mathscr{C}} \otimes \Delta x$; the commutation with the derivations and the curvature condition are straightforward to check). Then any square of $\mathscr{C}$–coalgebras as follows has a lifting:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & B_\iota \mathscr{A} \arrow[d] \\ D^n \arrow[r] & B_\iota \mathscr{B} \end{tikzcd}$$

This ensures that the map $f|_{\mathcal{A}}$ is an epimorphism. Conversely, suppose that $f|_{\mathcal{A}}$ is an epimorphism. By Lemma 104, there exists an isomorphism $G\colon B_\iota \mathscr{A}' \to B_\iota \mathscr{A}$ such that FG is in the image of the functor B_ι. If we denote by g the map from $\mathcal{A}'$ to $\mathcal{A}$ which underlies G, then g is an isomorphism by Lemma 105. Then fg is a fibration of $\Omega_u\mathscr{C}$–algebras and so $FG = B_\iota(fg)$ is a fibration. Since G is an isomorphism, F is a fibration. □

Lemma 104 *Let $F\colon B_\iota \mathscr{A} \to B_\iota \mathscr{B}$ be a morphism of $\mathscr{C}$–coalgebras such that the underlying morphism $f\colon \mathcal{A} \to \mathcal{B}$ is surjective. Then there exists an $\Omega_u\mathscr{C}$–algebra $\mathscr{A}'$ and an isomorphism of $\mathscr{C}$–coalgebras $G\colon B_\iota \mathscr{A}' \to B_\iota \mathscr{A}$ such that FG is a strict morphism, that is, in the image of the functor B_ι.*

Proof We build an isomorphism of graded $\mathscr{C}^{\mathrm{grad}}$–coalgebras $G\colon \mathscr{C} \circ \mathcal{A} \to \mathscr{C} \circ \mathcal{A}$ such that FG is a strict morphism, that is, of the form $\mathrm{Id}_{\mathscr{C}} \circ h$. To that purpose we define inductively maps $g_n\colon F_n^{\mathrm{rad}}\mathscr{C} \circ \mathcal{A} \to \mathcal{A}$ such that g_{n-1} is the restriction of g_n to $F_{n-1}^{\mathrm{rad}}\mathscr{C} \circ \mathcal{A}$ and such that we have the equality between maps from $F_n^{\mathrm{rad}}\mathscr{C} \circ \mathcal{A}$ to $\mathcal{A}$

$$fg_n + f(\mathrm{Id} \circ g_{n-1})(\overline{\Delta} \circ \mathrm{Id}) = f\pi_{\mathcal{A}}, \tag{1}$$

where $\pi_{\mathcal{A}} = \epsilon \circ \mathrm{Id}$ is the projection of $\mathscr{C} \circ \mathcal{A}$ on $\mathcal{A}$. First, let us choose $g_0 = \mathrm{Id}_{\mathcal{A}}$. Then suppose that we have built g_n satisfying (1). The map $f\colon \mathcal{A} \to \mathcal{B}$ and the injection of $F_n^{\mathrm{rad}}\mathscr{C} \circ \mathcal{A}$ into $F_{n+1}^{\mathrm{rad}}\mathscr{C} \circ \mathcal{A}$ give us the square

$$\begin{tikzcd} \hom_{\mathsf{gMod}}(F_{n+1}^{\mathrm{rad}}\mathscr{C} \circ \mathcal{A}, \mathcal{A}) \arrow[r] \arrow[d] & \hom_{\mathsf{gMod}}(F_{n+1}^{\mathrm{rad}}\mathscr{C} \circ \mathcal{A}, \mathcal{B}) \arrow[d] \\ \hom_{\mathsf{gMod}}(F_n^{\mathrm{rad}}\mathscr{C} \circ \mathcal{A}, \mathcal{A}) \arrow[r] & \hom_{\mathsf{gMod}}(F_n^{\mathrm{rad}}\mathscr{C} \circ \mathcal{A}, \mathcal{B}) \end{tikzcd}$$

The following map is surjective:

$$\begin{aligned} &\hom_{\mathsf{gMod}}(F_{n+1}^{\mathrm{rad}}\mathscr{C} \circ \mathcal{A}, \mathcal{A}) \\ &\qquad \to \hom_{\mathsf{gMod}}(F_n^{\mathrm{rad}}\mathscr{C} \circ \mathcal{A}, \mathcal{A}) \times_{\hom_{\mathsf{gMod}}(F_n^{\mathrm{rad}}\mathscr{C} \circ \mathcal{A}, \mathcal{B})} \hom_{\mathsf{gMod}}(F_{n+1}^{\mathrm{rad}}\mathscr{C} \circ \mathcal{A}, \mathcal{B}). \end{aligned}$$

So there exists an element of $\hom_{\mathsf{gMod}}(F^{\mathrm{rad}}_{n+1}\mathscr{C} \circ \mathcal{A}, \mathcal{A})$ whose image under this map is the pair $(g_n, f\pi_{\mathcal{A}} - f_{n+1}(\mathrm{Id} \circ g_n)(\overline{\Delta} \circ \mathrm{Id}))$. We can choose this element to be g_{n+1}. Thus, let g be the map from $\mathscr{C} \circ \mathcal{A}$ to $\mathcal{A}$ whose restriction to $F^{\mathrm{rad}}_{n}\mathscr{C} \circ \mathcal{A}$ is g_n for any n. Let G be the map of graded $\mathscr{C}^{\mathrm{grad}}$–coalgebras which extends g. By Lemma 105, the map G is an isomorphism. Let us transfer the coderivation of $B_\iota\mathscr{A}$ to $\mathscr{C} \circ \mathcal{A}$ along the isomorphism G. This gives us a new $\Omega_u\mathscr{C}$–algebra structure on the chain complex $\mathcal{A}$, which we denote by $\mathscr{A}'$. Finally, the morphism FG is the image under the functor B_ι of the morphism of $\Omega_u\mathscr{C}$–algebras $fg_0\colon \mathscr{A}' \to \mathscr{B}'$. □

Lemma 105 *Let $F\colon \mathscr{D} = \mathscr{C} \circ \mathcal{V} \to \mathscr{E} = \mathscr{C} \circ \mathcal{W}$ be a morphism of quasicofree $\mathscr{C}$–coalgebras. Then F is an isomorphism if and only if its underlying map $f\colon \mathcal{V} \to \mathcal{W}$ is an isomorphism.*

Proof Suppose first that F is an isomorphism with inverse G. Let us denote by $g\colon \mathcal{W} \to \mathcal{V}$ the map underlying G. Then the map g is inverse to f and so f is an isomorphism. Conversely, suppose that f is an isomorphism. A straightforward induction shows that F is both injective and surjective. □

7.4 Mapping spaces and deformation theory

Proposition 106 *For any cofibrant $\mathscr{C}$–coalgebra $\mathscr{D}$ and any fibrant $\mathscr{C}$–coalgebra $\mathscr{E}$, the simplicial set $\mathrm{HOM}(\mathscr{D}, \mathcal{E})$ is a Kan complex and is a model for the mapping space $\mathrm{Map}(\mathcal{C}, \mathcal{D})$ expected by the ι–model structure.*

Proof Any fibrant $\mathscr{C}$–coalgebra $\mathscr{E}$ is isomorphic to the image under B_ι of an $\Omega_u\mathscr{C}$–algebra $\mathscr{A}$. So we have

$$\mathrm{HOM}(\mathscr{D}, \mathscr{E}) \simeq \mathrm{HOM}(\mathscr{D}, B_\iota\mathscr{A}) \simeq \mathrm{HOM}(\Omega_\iota\mathscr{D}, \mathscr{A}) \simeq \mathrm{Map}(\Omega_\iota\mathscr{D}, \mathscr{A}) \simeq \mathrm{Map}(\mathscr{D}, B_\iota\mathscr{A}).$$

Further, we know from Proposition 76 that $\mathrm{HOM}(\Omega_\iota\mathscr{D}, \mathscr{A})$ is a Kan complex. □

Corollary 107 *Let $\alpha\colon \mathscr{C} \to \mathscr{P}$ be an operadic twisting morphism. Let us endow the category of $\mathscr{C}$–coalgebras with the α–model structure. For any cofibrant $\mathscr{C}$–coalgebra $\mathscr{D}$ and any fibrant $\mathscr{C}$–coalgebra $\mathscr{E}$, the simplicial set $\mathrm{HOM}(\mathscr{D}, \mathscr{E})$ is a Kan complex and is a model for the mapping space $\mathrm{Map}(\mathscr{D}, \mathscr{E})$.*

Proof It suffices to notice that fibrations and acyclic fibrations in the α–model structure are in particular fibrations and acyclic fibrations in the ι–model structure. Then we can conclude by Proposition 106. □

Let $f\colon \mathscr{D} \to B_\iota\mathscr{A}$ be a morphism of $\mathscr{C}$–coalgebras. We know from Proposition 48 that it is a dg atom of the cocommutative coalgebra $\{\mathscr{D}, B_\iota\mathscr{A}\}$. Consider the Hinich coalgebra $\{\mathscr{D}, B_\iota\mathscr{A}\}_f$ that appears from the decomposition described in Theorem 78.

Proposition 108 *The deformation problem induced by $\{\mathscr{D}, B_\iota\mathscr{A}\}_f$ is equivalent to the deformation problem*

$$R \in \mathsf{Artin–alg} \mapsto (\hom_{R\otimes\Omega_n\otimes\mathscr{C}\text{–cog}}(R \otimes \Omega_n \otimes \mathscr{C}, R \otimes \Omega_n \otimes B_\iota\mathscr{A}))_{n\in\mathbb{N}}.$$

Proof This is a direct consequence of Proposition 70 and Theorem 79. □

7.5 Algebras of the operad $\Omega_u\mathscr{C}$

We have shown above that the adjunction $\Omega_\iota \dashv B_\iota$ is a Quillen equivalence. Moreover, in Proposition 99, we have shown that fibrant $\mathscr{C}$–coalgebras are $\Omega_u\mathscr{C}$–algebras. So switching from the model category of $\Omega_u\mathscr{C}$–algebras to the model category of $\mathscr{C}$–coalgebras amounts to add new morphisms between any two $\Omega_u\mathscr{C}$–algebras. The weak equivalences and the fibrations of $\Omega_u\mathscr{C}$–algebras remain weak equivalences and fibrations, respectively, under this embedding but, in the category of $\mathscr{C}$–coalgebras, any monomorphism is a cofibration. In particular, any object is cofibrant. Subsequently, $\mathscr{C}$–coalgebras provide a convenient framework to study the homotopy theory of $\Omega_u\mathscr{C}$–algebras. For instance, the following proposition provides a tool to decide whether or not two $\Omega_u\mathscr{C}$–algebras are equivalent.

Proposition 109 *Let $\mathscr{A}$ and $\mathscr{B}$ be two $\Omega_u\mathscr{C}$–algebras. There exists a chain of weak equivalences of $\Omega_u\mathscr{C}$–algebras between $\mathscr{A}$ and $\mathscr{B}$*

$$\mathscr{A} = \mathscr{A}_0 \xrightarrow{\sim} \mathscr{A}_1 \xleftarrow{\sim} \cdots \xrightarrow{\sim} \mathscr{A}_{n-1} \xleftarrow{\sim} \mathscr{A}_n = \mathscr{B}$$

if and only if there exists a weak equivalence of $\mathscr{C}$–coalgebras between $B_\iota\mathscr{A}$ and $B_\iota\mathscr{B}$.

Proof Suppose that there exists a chain of weak equivalences from $\mathscr{A}$ to $\mathscr{B}$. Then there exists a chain of weak equivalences between $B_\iota\mathscr{A}$ and $B_\iota\mathscr{B}$. Moreover, the objects of this chain are fibrant and cofibrant. So any morphism of this chain has a homotopical inverse. So there exists a weak equivalence from $B_\iota\mathscr{A}$ to $B_\iota\mathscr{B}$. Conversely, consider a weak equivalence F from $B_\iota\mathscr{A}$ to $B_\iota\mathscr{B}$. Then the following chain of weak equivalences of $\Omega_u\mathscr{C}$–algebras links $\mathscr{A}$ to $\mathscr{B}$:

$$\mathscr{A} \xleftarrow{\sim} \Omega_\iota B_\iota\mathscr{A} \xrightarrow{\Omega_\iota(F)} \Omega_\iota B_\iota\mathscr{B} \xrightarrow{\sim} \mathscr{B}.$$ □

7.6 Koszul morphisms

In this subsection, we study the operadic twisting morphisms $\alpha\colon \mathscr{C} \to \mathscr{P}$ such that the α–model structure on the category of $\mathscr{C}$–coalgebras coincides with the universal ι–model structure that we described above. Let $\alpha\colon \mathscr{C} \to \mathscr{P}$ be an operadic twisting morphism. We denote by $\phi\colon \Omega_u(\mathscr{C}) \to \mathscr{P}$ the morphism of operads induced by α.

Theorem 110 *The following assertions are equivalent:*

(1) *The adjunction*

$$\begin{tikzcd} \Omega_u(\mathscr{C})\text{–}\mathsf{alg} \arrow[r, shift left, "\phi_!"] & \mathscr{P}\text{–}\mathsf{alg} \arrow[l, shift left, "\phi^*"] \end{tikzcd}$$

is a Quillen equivalence.

(2) *The morphism of operads $\phi\colon \Omega_u(\mathscr{C}) \to \mathscr{P}$ is a quasi-isomorphism.*

(3) *The α–model structure coincides with the ι–model structure and $\Omega_\alpha \dashv B_\alpha$ is a Quillen equivalence.*

(4) *For any $\mathscr{P}$–algebra $\mathscr{A}$, the map $\mathscr{P} \circ_\alpha \mathscr{C} \circ_\alpha \mathscr{A} \to \mathscr{A}$ is a quasi-isomorphism, and for any $\mathscr{C}$–coalgebra $\mathscr{D}$, the morphism $\mathscr{D} \to \mathscr{C} \circ_\alpha \mathscr{P} \circ_\alpha \mathscr{D}$ is a ι–equivalence* (it is the case if, for instance, it is a filtered quasi-isomorphism).

(5) *The morphisms of $\mathbb{S}$–modules $\Omega_u(\mathscr{C}) \circ_\iota \mathscr{C} \circ_\iota \Omega_u(\mathscr{C}) \to \mathscr{P} \circ_\alpha \mathscr{C} \circ_\alpha \mathscr{P}$ and $\mathscr{P} \circ_\alpha \mathscr{C} \circ_\alpha \mathscr{P} \to \mathscr{P}$ are quasi-isomorphisms.*

Lemma 111 *Let $f\colon \mathcal{V} \to \mathcal{V}'$ be a morphism of dg $\mathbb{S}$–modules. Suppose that, for any chain complex $\mathcal{W}$* (that is, an $\mathbb{S}$–module concentrated in arity zero), *the morphism $\mathcal{V} \circ \mathcal{W} \to \mathcal{V}' \circ \mathcal{W}$ is a quasi-isomorphism. Then f is a quasi-isomorphism.*

Proof By the operadic Künneth formula, for any graded $\mathbb{K}$–module $\mathcal{W}$, the map $H(\mathcal{V}) \circ \mathcal{W} \to H(\mathcal{V}') \circ \mathcal{W}$ is an isomorphism. So, for any integer n, the map

$$f_n\colon H(\mathcal{V})(n) \otimes_{\mathbb{S}_n} \mathbb{K}^n \to H(\mathcal{V}')(n) \otimes_{\mathbb{S}_n} \mathbb{K}^n$$

is an isomorphism. Let $(e_i)_{i=1}^n$ be a basis of $\mathbb{K}^n$. The map

$$p \in H(\mathcal{V})(n) \mapsto p \otimes (e_1 \otimes \cdots \otimes e_n) \mapsto f_n(p) \otimes (e_1 \otimes \cdots \otimes e_n) \mapsto f_n(p) \in H(\mathcal{V}')(n)$$

is an isomorphism. So, the morphism $H(\mathcal{V}) \to H(\mathcal{V}')$ is an isomorphism. □

Proof of Theorem 110 • Let us first prove the equivalence between (1) and (2). Suppose (2). Let $\mathscr{A}$ be a cofibrant $\Omega_u\mathscr{C}$–algebra and let $\mathscr{B}$ be a $\mathscr{P}$–algebra. Consider a map $f\colon \phi_!(\mathscr{A}) \to \mathscr{B}$ and its adjoint map $g\colon \mathscr{A} \to \phi^*(\mathscr{B})$. The following diagram of $\Omega_u\mathscr{C}$–algebras is commutative:

$$\begin{tikzcd}
\Omega_\iota B_\iota \mathscr{A} \arrow[r] \arrow[d] & \phi^*\phi_!\Omega_\iota B_\iota\mathscr{A} \arrow[d] & \\
\mathscr{A} \arrow[r] \arrow[rr, bend right, "g"'] & \phi^*\phi_!\mathscr{A} \arrow[r, "\phi^*(f)"] & \phi^*\mathscr{B}
\end{tikzcd}$$

The left vertical map is a quasi-isomorphism. Since a left Quillen functor preserves weak equivalences between cofibrant objects and since ϕ^* preserves quasi-isomorphisms, the right vertical map is a quasi-isomorphism. Further, $\phi_!\Omega_\iota B_\iota\mathscr{A}$ is actually $\Omega_\alpha B_\iota\mathscr{A}$. Since the morphism ϕ is a quasi-isomorphism, the map $\Omega_\iota B_\iota\mathscr{A} \to \phi^*\Omega_\alpha B_\iota\mathscr{A}$ is a filtered quasi-isomorphism with respect to the filtrations

$$F_n\Omega_\iota B_\iota\mathscr{A} = \bigoplus_k \Omega_u\mathscr{C}(k) \otimes_{\mathbb{S}_n} \Bigg(\sum_{i_1+\cdots+i_k=n} F^{\mathrm{rad}}_{i_1} B_\iota\mathscr{A} \otimes \cdots \otimes F^{\mathrm{rad}}_{i_k} B_\iota\mathscr{A} \Bigg),$$

$$F_n\phi^*\Omega_\alpha B_\iota\mathscr{A} = \bigoplus_k \mathcal{P}(k) \otimes_{\mathbb{S}_n} \Bigg(\sum_{i_1+\cdots+i_k=n} F^{\mathrm{rad}}_{i_1} B_\iota\mathscr{A} \otimes \cdots \otimes F^{\mathrm{rad}}_{i_k} B_\iota\mathscr{A} \Bigg).$$

Indeed, the resulting map on the graded object $G(\Omega_\iota B_\iota\mathscr{A}) = \Omega_u\mathscr{C} \circ G(B_\iota\mathscr{A})$ is actually $\phi \circ \mathrm{Id}_{G(B_\iota\mathscr{A})}$. So the map $\Omega_\iota B_\iota\mathscr{A} \to \phi^*\Omega_\alpha B_\iota\mathscr{A}$ is a quasi-isomorphism. So, by the 2-out-of-3 rule, the map $\mathscr{A} \to \phi^*\phi_!\mathscr{A}$ is a quasi-isomorphism. Hence, f is a quasi-isomorphism if and only if $\phi^*(f)$ is a quasi-isomorphism, if and only if g is a quasi-isomorphism. So assertion (1) is true. Conversely, suppose (1). Then, for any chain complex (considered as a $\mathscr{C}$–coalgebra) $\mathcal{V}$, the map $\Omega_\iota\mathcal{V} \to \Omega_\alpha\mathcal{V}$ is a quasi-isomorphism. Since the coaction of $\mathscr{C}$ on $\mathcal{V}$ is trivial, we have canonical isomorphisms of chain complexes

$$\Omega_\iota\mathcal{V} \simeq \Omega_u\mathscr{C} \circ \mathcal{V}, \quad \Omega_\alpha\mathcal{V} \simeq \mathcal{P} \circ \mathcal{V}.$$

We can thus apply Lemma 111 which shows that (2) is true.

- Suppose (1) and let us show (3). By Proposition 96, the α–model structure coincides with the ι–model structure. Moreover, since the adjunctions $\phi_! \dashv \phi^*$ and $\Omega_\iota \dashv B_\iota$ are both Quillen equivalences, the adjunction $\phi_!\Omega_\iota \dashv B_\iota\phi^*$, which is $\Omega_\alpha \dashv B_\alpha$, is a Quillen equivalence.

- Suppose (3) and let us show (4). Since $\Omega_\alpha \dashv B_\alpha$ is a Quillen equivalence, $\Omega_\alpha B_\alpha \mathscr{A} \to \mathscr{A}$ is a quasi-isomorphism for any $\mathscr{P}$–algebra $\mathscr{A}$ and $\mathscr{D} \to B_\alpha \Omega_\alpha \mathscr{D}$ is an α–weak equivalence for any $\mathscr{C}$–coalgebra $\mathscr{D}$. Since the α–model structure coincides with the ι–model structure, $\mathscr{D} \to B_\alpha \Omega_\alpha \mathscr{D}$ is a ι–weak equivalence. So (4) is true.

- Suppose (4) and let us show (5). For any $\mathscr{P}$–algebra $\mathscr{A}$, the morphism $\Omega_\alpha B_\alpha \mathscr{A} \to \mathscr{A}$ is a quasi-isomorphism. In particular, this is true for any free $\mathscr{P}$–algebras. So, for any chain complex $\mathcal{V}$, the map $\Omega_\alpha B_\alpha(\mathscr{P} \circ \mathcal{V}) \to \mathscr{P} \circ \mathcal{V}$ is a quasi-isomorphism. This map is actually the morphism of chain complexes

$$(\mathscr{P} \circ_\alpha \mathscr{C} \circ_\alpha \mathscr{P}) \circ \mathcal{V} \to \mathscr{P} \circ \mathcal{V}.$$

Using Lemma 111, we conclude that the map $\mathscr{P} \circ_\alpha \mathscr{C} \circ_\alpha \mathscr{P} \to \mathscr{P}$ is a quasi-isomorphism. Moreover, for any $\mathscr{P}$–algebra $\mathscr{A}$, the following diagram commutes:

$$\begin{tikzcd}
\Omega_u(\mathscr{C}) \circ_\iota \mathscr{C} \circ_\iota \mathscr{A} \arrow[r] \arrow[dr] & \mathscr{P} \circ_\alpha \mathscr{C} \circ_\alpha \mathscr{A} \arrow[d] \\
& \mathscr{A}
\end{tikzcd}$$

Since the composite map and the vertical map are quasi-isomorphisms (because $\Omega_\iota \dashv B_\iota$ and $\Omega_\alpha \dashv B_\alpha$ are Quillen equivalences), by the 2-out-of-3 rule the horizontal map is a quasi-isomorphism. Applying this to free $\mathscr{P}$–algebras and using Lemma 111, we conclude that the map $\Omega_u(\mathscr{C}) \circ_\iota \mathscr{C} \circ_\alpha \mathscr{P} \to \mathscr{P} \circ_\alpha \mathscr{C} \circ_\alpha \mathscr{P}$ is a quasi-isomorphism. Further, for any $\mathscr{C}$–coalgebra $\mathscr{D}$ the following diagram commutes:

$$\begin{tikzcd}
\mathscr{D} \arrow[r] \arrow[dr] & \mathscr{C} \circ_\iota \Omega_u(\mathscr{C}) \circ_\iota \mathscr{D} \arrow[d] \\
& \mathscr{C} \circ_\alpha \mathscr{P} \circ_\alpha \mathscr{D}
\end{tikzcd}$$

By the 2-out-of-3 rule, the vertical map is a ι–weak equivalence. So the map

$$\Omega_u(\mathscr{C}) \circ_\iota \mathscr{C} \circ_\iota \Omega_u(\mathscr{C}) \circ_\iota \mathscr{D} \to \Omega_u(\mathscr{C}) \circ_\iota \mathscr{C} \circ_\alpha \mathscr{P} \circ_\alpha \mathscr{D}$$

is a quasi-isomorphism. Applying this for $\mathscr{C}$–coalgebras which are just chain complexes and using Lemma 111, we obtain that the map $\Omega_u(\mathscr{C}) \circ_\iota \mathscr{C} \circ_\iota \Omega_u(\mathscr{C}) \to \Omega_u(\mathscr{C}) \circ_\iota \mathscr{C} \circ_\alpha \mathscr{P}$ is a quasi-isomorphism.

- Suppose (5) and let us show (2). Using the previous point in the case $\mathscr{P} = \Omega_u \mathscr{C}$ gives us the fact that the map of dg $\mathbb{S}$–modules

$$\Omega_u(\mathscr{C}) \circ_\iota \mathscr{C} \circ_\iota \Omega_u(\mathscr{C}) \to \Omega_u(\mathscr{C})$$

is an aritywise quasi-isomorphism. Then the following square of $\mathbb{S}$–modules is commutative:

$$\begin{tikzcd}
\Omega_u(\mathscr{C}) \circ_\iota \mathscr{C} \circ_\iota \Omega_u(\mathscr{C}) \arrow[r] \arrow[d] & \Omega_u(\mathscr{C}) \arrow[d] \\
\mathscr{P} \circ_\alpha \mathscr{C} \circ_\alpha \mathscr{P} \arrow[r] & \mathscr{P}
\end{tikzcd}$$

Since the left vertical map and the horizontal maps are quasi-isomorphisms, the right vertical map is also a quasi-isomorphism. □

Definition 112 (Koszul morphisms) An operadic twisting morphism $\alpha\colon \mathscr{C} \to \mathscr{P}$ satisfying the properties of Theorem 110 is called a *Koszul morphism*.

In the next section, we will explore Koszul duality, which is a method to produce Koszul morphisms from a presentation of an operad.

8 Examples

The purpose of this section is to apply the general framework described in the previous sections to the case of common nonaugmented operads like the operads $u\mathcal{A}s$ and $u\mathcal{C}om$, whose algebras are the unital associative algebras and the unital commutative algebras, respectively. So, for any of these operads $\mathscr{P}$, one looks after a curved conilpotent cooperad $\mathscr{C}$ together with an operadic twisting morphism α from $\mathscr{C}$ to $\mathscr{P}$ such that the induced morphism of operads from $\Omega_u \mathscr{C}$ to $\mathscr{P}$ is a quasi-isomorphism; that is, α is a Koszul morphism. One can use the universal twisting morphism $B_c \mathscr{P} \to \mathscr{P}$. However, the bar construction is always very big. Instead, one usually tries to produce a subcooperad of $B_c \mathscr{P}$ whose cobar construction will be a resolution of $\mathscr{P}$. The Koszul duality theory is a way to produce such a subcooperad when the operad $\mathscr{P}$ has a quadratic presentation or a quadratic-linear presentation. This construction has been extended to quadratic-linear-constant presentations by Hirsh and Millès in [17], generalizing to operads the curved Koszul duality of algebras developed by Polishchuk and Positselski [22].

8.1 Koszul duality

Koszul duality is a way to build a cooperad $\mathscr{P}^{¡}$ together with a canonical operadic twisting morphism from $\mathscr{P}^{¡}$ to $\mathscr{P}$ out of an operad $\mathscr{P}$ which has a "nice enough" presentation $\mathscr{P} = \mathbb{T}(\mathcal{V})/(\mathcal{R})$. Here, we present the construction of Hirsh and Millès in [17].

Let $\mathscr{P}$ be a graded operad equipped with a presentation $\mathscr{P} = \mathbb{T}(\mathcal{V})/(\mathcal{R})$, where $\mathcal{V}$ is a graded $\mathbb{S}$–module and where $(\mathcal{R})$ is the operadic ideal generated by a subgraded $\mathbb{S}$–module $\mathcal{R}$ of $\mathbb{T}^{\leq 2}(\mathcal{V})$ such that

$$\mathcal{R} \cap (\mathcal{I} \oplus \mathcal{V}) = \{0\},$$
$$(\mathcal{R}) \cap \mathbb{T}^{\leq 2}(\mathcal{V}) = \mathcal{R}.$$

We denote by $q\mathcal{R}$ the projection of $\mathcal{R} \subset \mathbb{T}^{\leq 2}(\mathcal{V})$ onto $\mathbb{T}^{2}(\mathcal{V})$ along $\mathcal{I} \oplus \mathcal{V}$. Moreover, let $q\mathscr{P}$ be the operad

$$q\mathscr{P} := \mathbb{T}(\mathcal{V})/(q\mathcal{R}).$$

This is a quadratic operad. The condition $\mathcal{R} \cap (\mathcal{I} \oplus \mathcal{V}) = \{0\}$ induces a function $\phi = (\phi_0, \phi_1)\colon q\mathcal{R} \to \mathcal{I} \oplus \mathcal{V}$.

Definition 113 (curved cooperad Koszul dual of an operad [17, Section 4.1]) The Koszul dual cooperad $\mathscr{P}^{¡}$ of $\mathscr{P}$ associated to the presentation $\mathscr{P} = \mathbb{T}(\mathcal{V})/(\mathcal{R})$ is the following curved conilpotent cooperad. The underlying graded cooperad is the final graded subcooperad of $\mathbb{T}^{c}(s\mathcal{V})$ such that the composition

$$\mathscr{P}^{¡} \to \mathbb{T}^{c}(s\mathcal{V}) \to \mathbb{T}^{2}(s\mathcal{V})/s^2 q\mathcal{R}$$

is zero. It is equipped with the unique coderivation which extends the map

$$\mathscr{P}^{¡} \twoheadrightarrow s^2 q\mathcal{R} \to s\mathcal{V}, \quad sx \otimes sy \mapsto (-1)^{|x|} s\phi_1(x \otimes y).$$

Its curvature is the degree -2 map

$$\theta\colon \mathscr{P}^{¡} \twoheadrightarrow s^2 q\mathcal{R} \to \mathbb{K}, \quad sx \otimes sy \mapsto (-1)^{|x|} s\phi_0(x \otimes y).$$

Moreover, the map

$$\kappa\colon \mathscr{P}^{¡} \twoheadrightarrow s\mathcal{V} \to \mathcal{V} \hookrightarrow \mathscr{P},$$

is an operadic twisting morphism which induces both a morphism of operads $\Omega_u \mathscr{P}^{¡} \to \mathscr{P}$ and a morphism of curved conilpotent cooperads $\mathscr{P}^{¡} \to B_c \mathscr{P}$.

Remark 114 The coherence of the above definition is proven in [17, Section 4.1].

Definition 115 (Koszul operad) The operad $\mathscr{P}$ (together with the presentation $\mathscr{P} = \mathbb{T}(\mathcal{V})/(\mathcal{R})$) is said to be Koszul if the twisting morphism $\kappa\colon \mathscr{P}^{¡} \to \mathscr{P}$ is Koszul, that is, if the map $\Omega_c \mathscr{P}^{¡} \to \mathscr{P}$ is a quasi-isomorphism.

The following theorem is a powerful tool to show that an operad is Koszul:

Theorem 116 [17, Theorem 4.3.1] *Suppose that the canonical morphism*

$$q\mathscr{P} \circ_\kappa q\mathscr{P}^{¡} \to \mathcal{I}$$

is a quasi-isomorphism. Then $\mathscr{P}$ is Koszul.

8.2 Coalgebras over a Koszul dual

In this subsection, we describe the category of $\mathscr{P}^{¡}$–coalgebras, where $\mathscr{P}^{¡}$ is the Koszul dual of the "quadratic-linear-homogeneous operad" $\mathscr{P}$ defined above. We will need the following definition:

Definition 117 (precoradical filtration) Let $\mathcal{W}$ be a graded $\mathbb{S}$–module and let $\mathcal{C}$ be a graded $\mathbb{K}$–module equipped with a map $\Delta^{(1)}\colon \mathcal{C} \to \mathcal{W} \circ \mathcal{C}$. We define $(F_n^{\text{prad}}\mathcal{C})_{n\in\mathbb{N}}$ to be the following (nonnecessarily exhaustive) filtration on $\mathcal{C}$, called the precoradical filtration:

$$F_0^{\text{prad}}(\mathcal{C}) := \ker(\Delta^{(1)}),$$

$$F_n^{\text{prad}}(\mathcal{C}) := (\Delta^{(1)})^{-1}\Bigl(\mathcal{W}(0) \oplus \sum_{\substack{i_1+\cdots+i_k=n-1\\ k\geq 1}} \mathcal{W}(k) \otimes_{\mathbb{S}_k} (F_{i_1}^{\text{prad}}\mathcal{C} \otimes \cdots \otimes F_{i_k}^{\text{prad}}\mathcal{C})\Bigr) \quad \text{if } n \geq 1.$$

Lemma 118 *Consider a cofree graded conilpotent cooperad $\mathbb{T}^c(\mathcal{W})$. The category of graded coalgebras over $\mathbb{T}^c(\mathcal{W})$ is equivalent to the category of graded $\mathbb{K}$–modules $\mathcal{C}$ equipped with a map $\Delta^{(1)}\colon \mathcal{C} \to \mathcal{W} \circ \mathcal{C}$ such that the precoradical filtration $(F_n^{\text{prad}}\mathcal{C})_{n\in\mathbb{N}}$ is exhaustive. Moreover, under this equivalence, the coradical filtration coincides with the precoradical filtration.*

Proof Let $\mathcal{C}$ be a graded $\mathbb{K}$–module with a map $\Delta^{(1)}\colon \mathcal{C} \to \mathcal{W} \circ \mathcal{C}$ such that the precoradical filtration $(F_n^{\text{prad}}\mathcal{C})_{n\in\mathbb{N}}$ is exhaustive. Then let us define $\Delta_{\mathcal{C}}\colon \mathcal{C} \to \mathbb{T}(\mathcal{W}) \circ \mathcal{C}$ by induction as follows:

$$\begin{cases} \Delta_{\mathcal{C}}(x) := 1 \otimes x & \text{if } x \in F_0^{\text{prad}}\mathcal{C}, \\ \Delta_{\mathcal{C}}(x) := 1 \otimes x + (\mathrm{Id} \circ \Delta_{\mathcal{C}})\Delta^{(1)}(x) & \text{if } x \in F_n^{\text{prad}}\mathcal{C}. \end{cases}$$

This defines a structure of $\mathbb{T}^c(\mathcal{W})$–coalgebra on $\mathcal{C}$. Conversely, let $(\mathcal{C}, \Delta)$ be a graded $\mathbb{T}^c(\mathcal{W})$–coalgebra. We obtain a map $\Delta^{(1)}\colon \mathcal{C} \to \mathcal{W} \circ \mathcal{C}$ by composing Δ with the projection of $\mathbb{T}^c(\mathcal{W})$ onto $\mathcal{W}$. Then the construction we just described recovers Δ from $\Delta^{(1)}$. □

Theorem 119 *Suppose that the characteristic of the field* $\mathbb{K}$ *is zero* (this assumption is not necessary in the nonsymmetric context). *The category of* $\mathscr{P}^{¡}$*–coalgebras is equivalent to the category of graded* $\mathbb{T}^c(s\mathcal{V})$*–coalgebras* (that is graded $\mathbb{K}$–modules $\mathcal{C}$ equipped with a map $\Delta^{(1)}\colon \mathcal{C} \to s\mathcal{V} \circ \mathcal{C}$ such that the precoradical filtration $(F_n^{\mathrm{prad}}\mathcal{C})_{n\in\mathbb{N}}$ is exhaustive) *such that the composite map*

$$\mathcal{C} \xrightarrow{\Delta_2 = (\mathrm{Id} \circ' \Delta^{(1)})\Delta^{(1)}} \mathbb{T}^2(s\mathcal{V}) \circ \mathcal{C} \twoheadrightarrow (\mathbb{T}^2(s\mathcal{V})/s^2 q\mathcal{R}) \circ \mathcal{C}$$

is zero, together with a degree -1 *map* $d_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ *such that*

$$d_{\mathcal{C}}^2 = (\theta \circ \mathrm{Id})\Delta_2, \quad \Delta^{(1)} d_{\mathcal{C}} = (d_{\mathscr{P}^{¡}} \circ \mathrm{Id})\Delta_2 + (\mathrm{Id} \circ' d_{\mathcal{C}})\Delta^{(1)}.$$

Proof Let $\mathcal{C}$ be a graded $\mathbb{T}^c(s\mathcal{V})$–coalgebra together with a degree -1 map $d_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ satisfying the conditions of Theorem 119. For any $x \in \mathcal{C}$, let $\mathcal{C}(x)$ be a finite-dimensional sub-$\mathbb{T}^c(s\mathcal{V})$–coalgebra of $\mathcal{C}$ which contains x. By Lemma 120, the map $\Delta_{\mathcal{C}(x)}\colon \mathcal{C}(x) \to \mathbb{T}^c(s\mathcal{V}) \circ \mathcal{C}(x)$ factorizes through a unique map $\mathcal{C}(x) \to \mathscr{P}^{¡} \circ \mathcal{C}(x)$. Hence, $\mathcal{C}$ has a structure of graded $(\mathscr{P}^{¡})^{\mathrm{grad}}$–coalgebra. Moreover, we can prove by induction on the coradical filtration of $\mathcal{C}$ that $d_{\mathcal{C}}$ is a coderivation. □

Lemma 120 *Let* $\mathcal{C}(x)$ *be the graded* $\mathbb{T}^c(s\mathcal{V})$*–coalgebra defined in the proof of Theorem 119. Then* $\mathcal{C}(x)$ *is a graded* $\mathscr{P}^{¡}$*–coalgebra.*

Proof Remember that $\mathcal{C}(x)$ is a finite-dimensional subgraded $\mathbb{T}^c(s\mathcal{V})$–coalgebra of $\mathcal{C}$. Let $(e_i)_{i=1}^m$ be a basis of $\mathcal{C}(x)$. Then, for any $i \in \{1, \dots, m\}$, let $p_{i,0} \in \overline{\mathbb{T}}(s\mathcal{V})(0)$, and for any integer $k \geq 1$ and for any nondecreasing function s from $\{1, \dots, k\}$ to $\{1, \dots, m\}$, let $p_{i,k,s} \in \overline{\mathbb{T}}(s\mathcal{V})(k)$ be such that

$$\Delta(e_i) = 1 \otimes e_i + p_{i,0} + \sum_{k=0}^{\infty} \sum_{s} p_{i,k,s} \otimes_{\mathbb{S}_k} (e_{s(1)} \otimes \cdots \otimes e_{s(k)}).$$

For any nondecreasing function s from $\{1, \dots, k\}$ to $\{1, \dots, m\}$ and for any $\sigma \in \mathbb{S}_k$, let $\epsilon(s, \sigma)$ be the element of $\mathbb{Z}/2\mathbb{Z}$ such that the structural action of σ on $\mathcal{C}^{\otimes k}$ sends $e_{s(1)} \otimes \cdots \otimes e_{s(k)}$ to $(-1)^{\epsilon(s,\sigma)} e_{s\sigma^{-1}(1)} \otimes \cdots \otimes e_{s\sigma^{-1}(k)}$. Further, let $\mathrm{Inv}(s)$ be the subgroup of $\mathbb{S}_k$ of permutations σ such that $s = s\sigma^{-1}$. Then we can choose $p_{i,k,s}$

such that $p^\sigma_{i,k,s} = (-1)^{\epsilon(s,\sigma)} p_{i,k,s}$ for any $\sigma \in \mathrm{Inv}(s)$. Indeed, if it is not the case, we can replace $p_{i,k,s}$ by

$$\frac{1}{\#\,\mathrm{Inv}(s)} \sum_{\sigma \in \mathrm{Inv}(s)} (-1)^{\epsilon(s,\sigma)} p^\sigma_{i,k,s}.$$

Let $\mathscr{D}$ be the subgraded $\mathbb{S}$–module of $\mathbb{T}(s\mathcal{V})$ generated by 1 and the elements $p_{i,k,s}$. Since $(\Delta \circ \mathrm{Id})\Delta(e_i) = (\mathrm{Id} \circ \Delta)\Delta(e_i)$ for any i, there exists an element of $q_{i,k,s} \in (\mathscr{D} \circ \mathscr{D})(k)$ such that

$$\Delta(p_{i,k,s}) \otimes_{\mathbb{S}_k} (e_{s(1)} \otimes \cdots \otimes e_{s(k)}) = q_{i,k,s} \otimes_{\mathbb{S}_k} (e_{s(1)} \otimes \cdots \otimes e_{s(k)}).$$

Since $p^\sigma_{i,k,s} = (-1)^{\epsilon(s,\sigma)} p_{i,k,s}$ for any $\sigma \in \mathrm{Inv}(s)$,

$$\Delta(p_{i,k,s}) = \frac{1}{\#\,\mathrm{Inv}(s)} \sum_{\sigma \in \mathrm{Inv}(s)} (-1)^{\epsilon(s,\sigma)} q^\sigma_{i,k,s}.$$

So, $\Delta(p_{i,k,s}) \in \mathscr{D} \circ \mathscr{D}$. Hence, $\mathscr{D}$ is a subgraded cooperad of $\mathbb{T}^c(s\mathcal{V})$. Moreover, for any i, $(\pi \circ \mathrm{Id})\Delta(e_i) = 0$, where π is the projection of $\mathbb{T}(s\mathcal{V})$ onto $\mathbb{T}^2(s\mathcal{V})/s^2 q\mathcal{R}$. So, $\pi(p_{i,k,s}) = 0$ for any 3–tuple (i,k,s) and $\pi(p_{i,0}) = 0$ for any i; so $\pi|_{\mathscr{D}} = 0$. Hence, $\mathscr{D} \subset \mathscr{P}^{¡}$. □

8.3 Unital associative algebras up to homotopy

Notation Let $\mathcal{V}$ and $\mathcal{W}$ be two $\mathbb{N}$–modules, and n, p, $i_1, \dots, i_p$ natural integers such that $i_1 + \cdots + i_p = n$. We will usually denote the image of an element

$$x \otimes y_1 \otimes \cdots \otimes y_p \in \mathcal{V}(p) \otimes \mathcal{W}(i_1) \otimes \cdots \otimes \mathcal{W}(i_p)$$

under the inclusion

$$\mathcal{V}(p) \otimes \mathcal{W}(i_1) \otimes \cdots \otimes \mathcal{W}(i_p) \to (\mathcal{V} \circ_{\mathrm{ns}} \mathcal{W})(n)$$

by $x \otimes_{\mathrm{ns}} (y_1 \otimes \cdots \otimes y_p)$.

8.3.1 A presentation of the operad $u\mathcal{A}s$ Let $u\mathcal{A}s$ be the nonsymmetric operad defined by the presentation $u\mathcal{A}s := \mathbb{T}(\mathbb{K}\cdot\mu \oplus \mathbb{K}\cdot\xi)/(\mathcal{R})$, where μ is an arity 2 element and ξ is an arity 0 element. The nonsymmetric module $\mathcal{R} \subset \mathcal{I} \oplus \mathbb{T}^2(\mathbb{K}\cdot\mu \oplus \mathbb{K}\cdot\xi)$ is made up of the relations

$$\begin{cases} \mu \otimes_{\mathrm{ns}} (\xi \otimes 1) - 1, \\ \mu \otimes_{\mathrm{ns}} (1 \otimes \xi) - 1, \\ \mu \otimes_{\mathrm{ns}} (\mu \otimes 1) - \mu \otimes_{\mathrm{ns}} (1 \otimes \mu). \end{cases}$$

Remark 121 Here, the symbol ns stands for the composition product of nonsymmetric modules.

Given this presentation, the Koszul dual $u\mathcal{A}s^{¡}$ is a nonsymmetric curved conilpotent cooperad whose underlying graded cooperad is the final subcooperad of $\mathbb{T}^c(\mathbb{K}\cdot s\mu \oplus \mathbb{K}\cdot\xi)$ such that

$$u\mathcal{A}s^{¡} \cap \mathbb{T}^2(\mathbb{K}\cdot s\mu) = \mathbb{K}\cdot(s\mu \otimes_{\mathrm{ns}} (s\mu \otimes 1) - s\mu \otimes_{\mathrm{ns}} (1 \otimes s\mu)).$$

The coderivation of $u\mathcal{A}s^{¡}$ is zero and the curvature is given by

$$\theta(s\mu \otimes_{\mathrm{ns}} (s\xi \otimes 1)) = \theta(s\mu \otimes_{\mathrm{ns}} (1 \otimes s\xi)) = -1.$$

Remark 122 The Koszul dual curved cooperad $u\mathcal{A}s^{¡}$ of the operad $u\mathcal{A}s$ is described in detail in [17].

8.3.2 Coalgebras over $u\mathcal{A}s^{¡}$

Proposition 123 *The endofunctor of the category of graded $\mathbb{K}$–modules $\mathcal{V} \mapsto s\mathcal{V}$ induces an equivalence between the category of $u\mathcal{A}s^{¡}$–coalgebras and the category of noncounital curved conilpotent coassociative coalgebras.*

Proof The proof relies on the same arguments as the proof of Proposition 129, which will be detailed. □

Remark 124 The map $\mathcal{V} \to s\mathcal{V}$ also induces an equivalence between graded $(u\mathcal{A}s^{¡})^{\mathrm{grad}}$–coalgebras and graded noncounital conilpotent coassociative coalgebras $\mathscr{C}$ equipped with a degree -2 map $\mathscr{C} \to \mathbb{K}$. Moreover, this equivalence sends a cofree graded $(u\mathcal{A}s^{¡})^{\mathrm{grad}}$–coalgebra $u\mathcal{A}s^{¡} \circ \mathcal{V}$ to the cofree conilpotent coalgebra $\overline{\mathbb{T}}(\mathcal{V} \oplus \mathbb{K}\cdot v)$, where $|v| = 2$ with the degree -2 map

$$\overline{\mathbb{T}}(\mathcal{V} \oplus \mathbb{K}\cdot v) \twoheadrightarrow \mathbb{K}\cdot v \to \mathbb{K}, \quad v \mapsto 1.$$

Notation We denote the category of curved conilpotent coassociative coalgebras by cCog. Moreover, we denote the operad $\Omega_u u\mathcal{A}s^{¡}$ by $u\mathscr{A}_\infty$.

8.3.3 The bar–cobar adjunction and $u\mathscr{A}_\infty$–algebras On the one hand, there exists an adjunction relating $u\mathcal{A}s$–algebras to $u\mathcal{A}s^{¡}$–coalgebras which is induced by the operadic twisting morphism $\alpha\colon u\mathcal{A}s^{¡} \to u\mathcal{A}s$. On the other hand, the category of $u\mathcal{A}s^{¡}$–coalgebras is equivalent to the category cCog of curved conilpotent coalgebras. Thus, we obtain a bar–cobar adjunction between unital associative algebras and curved conilpotent coalgebras which is the restriction to arity 1 of the operadic bar–cobar adjunction described in Section 4.1 (with the exception that we can consider noncounital coalgebras instead of coaugmented counital coalgebras). For this reason, we denote this adjunction using the same symbols as in the operadic context, that is, $\Omega_u \dashv B_c$. So we have

$$\Omega_u \mathscr{C} := \overline{\mathbb{T}}(s^{-1}\mathscr{C}), \quad B_c \mathscr{A} := \overline{\mathbb{T}}(s\mathscr{A} \oplus \mathbb{K}\cdot v)$$

for any curved conilpotent coalgebra $\mathscr{C}$ and for any unital algebra $\mathscr{A}$. The derivation of $\Omega_u(\mathscr{C})$ and the coderivation of $B_c(\mathscr{A})$ are defined as in Section 4.1.

The adjunction $\Omega_u \dashv B_c$ is part of a larger picture,

$$\begin{tikzcd} \mathsf{cCog} \arrow[r, shift left, "\Omega_\iota"] & u\mathscr{A}_\infty\text{–}\mathsf{alg} \arrow[l, shift left, "B_\iota"] \arrow[r, shift left, "\phi_!"] & u\mathcal{A}s\text{–}\mathsf{alg}, \arrow[l, shift left, "\phi^*"] \end{tikzcd}$$

where the adjunction $\phi_! \dashv \phi^*$ is induced by the morphism of operads $\phi\colon u\mathscr{A}_\infty \to u\mathcal{A}s$ and where $\Omega_u = \phi_!\Omega_\iota$ and $B_c = B_\iota\phi^*$. We know that a $u\mathscr{A}_\infty$–algebra $\mathscr{A} = (\mathcal{A}, \gamma)$ is the data of a chain complex $\mathcal{A}$ together with a coderivation on the cofree graded $(u\mathcal{A}s^{¡})^{\mathrm{grad}}$–coalgebra $u\mathcal{A}s^{¡} \circ \mathcal{A}$, so that it becomes a $u\mathcal{A}s$–coalgebra. Equivalently, it is the data of a chain complex together with a coderivation on the cofree conilpotent coassociative coalgebra $\overline{\mathbb{T}}(s\mathcal{A}\oplus\mathbb{K}\cdot v)$, so that it becomes a curved conilpotent coalgebra whose curvature θ is given by

$$\overline{\mathbb{T}}(s\mathcal{A} \oplus \mathbb{K}\cdot v) \twoheadrightarrow \mathbb{K}\cdot v \to \mathbb{K}, \quad v \mapsto 1.$$

By Lemma 27, this is equivalent to a degree -1 map

$$\gamma\colon \overline{\mathbb{T}}(s\mathcal{A} \oplus \mathbb{K}\cdot v) \to \mathcal{A},$$

such that, for any $x_1, \cdots, x_n \in (s\mathcal{A} \oplus \mathbb{K}\cdot v)$,

$$\sum_{0\leq i\leq j\leq n} (-1)^{|x_1|+\cdots+|x_{i-1}|}\gamma(x_1 \otimes \cdots \otimes \gamma(x_i \otimes \cdots \otimes x_j) \otimes \cdots x_n) = \begin{cases} 0 & \text{if } n \neq 2, \\ \theta(x_1)x_2 - \theta(x_2)x_1 & \text{if } n = 2. \end{cases}$$

In particular, we have the following:

- A degree zero product
$$\gamma_2\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}.$$
- A degree 1 map
$$\gamma_3\colon \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \to \mathcal{A},$$
whose boundary is the associator of γ_2, that is,
$$\partial(\gamma_3) = \gamma_2(\mathrm{Id} \otimes \gamma_2) - \gamma_2(\gamma_2 \otimes \mathrm{Id}).$$

- An element $1_{\mathscr{A}}$ defined by $\gamma(v) = s1_{\mathscr{A}}$.
- Maps $\gamma_{1,l}\colon \mathcal{A} \to \mathcal{A}$ and $\gamma_{1,r}\colon \mathcal{A} \to \mathcal{A}$ of degree 1 which make $1_{\mathscr{A}}$ a unit up to homotopy, that is,
$$\partial(\gamma_{1,l}) = \gamma_2(1_{\mathscr{A}} \otimes \mathrm{Id}) - \mathrm{Id},$$
$$\partial(\gamma_{1,l}) = \gamma_2(\mathrm{Id} \otimes 1_{\mathscr{A}}) - \mathrm{Id}.$$

8.3.4 The Koszul property and the infinity category of $u\mathscr{A}_\infty$–algebras

Proposition 125 [17, Theorem 6.1.8] *The operad $u\mathcal{A}s$ is Koszul.*

Remark 126 The model structure on curved conilpotent coalgebras that we get by transfer along the adjunction $\Omega_u \dashv B_c$ is the model structure that Positselski described in [23].

There are several ways to describe the infinity-category of $u\mathcal{A}s$–algebras:

- One can take the Dwyer–Kan simplicial localization of the category of $u\mathcal{A}s$–algebras with respect to quasi-isomorphisms as described in [10; 9].
- One can take the simplicial category whose objects are cofibrant-fibrant $u\mathcal{A}s$–algebras and whose spaces of morphisms are
$$\mathrm{Map}(\mathscr{A}, \mathscr{B})_n := \mathrm{HOM}_{u\mathcal{A}s-\mathsf{alg}}(\mathscr{A}, \mathscr{B}).$$
- One can also take the simplicial category whose objects are all $u\mathcal{A}s$–algebras and whose spaces of morphisms are
$$\mathrm{Map}(\mathscr{A}, \mathscr{B}) := \mathrm{HOM}_{u\mathcal{A}s-\mathsf{alg}}(\Omega_u B_c \mathscr{A}, \mathscr{B}) \simeq \mathrm{HOM}_{\mathsf{cCog}}(B_c \mathscr{A}, B_c \mathscr{B}).$$

8.4 Unital commutative algebras up to homotopy

In this section, we assume that the characteristic of the base field $\mathbb{K}$ is zero.

8.4.1 A presentation of the operad $u\mathcal{C}om$ Let $u\mathcal{C}om$ be the operad defined by the presentation $u\mathcal{C}om := \mathbb{T}(\mathbb{K} \cdot \mu \oplus \mathbb{K} \cdot \xi)/(\mathcal{R})$, where μ is an arity 2 element such that $\mu^{(1,2)} = \mu$ and ξ is an arity 0 element. The $\mathbb{S}$–module $\mathcal{R} \subset \mathcal{I} \oplus \mathbb{T}^2(\mathbb{K} \cdot \mu \oplus \mathbb{K} \cdot \xi)$ is generated by the elements
$$\mu \otimes_{\mathbb{S}_2} (\mu \otimes 1) - \mu \otimes_{\mathbb{S}_2} (1 \otimes \mu), \quad \mu \otimes_{\mathbb{S}_2} (\xi \otimes 1) - 1.$$

Remark 127

- Since the action of $\mathbb{S}_2$ on μ is trivial, we have
$$\mu \otimes_{\mathbb{S}_2} (1 \otimes \mu) = (\mu \otimes_{\mathbb{S}_2} (\mu \otimes 1))^{(132)}.$$
- The element $\mu \otimes_{\mathbb{S}_2} (\mu \otimes 1) - \mu \otimes_{\mathbb{S}_2} (1 \otimes \mu)$ is a generator of the $\mathbb{S}_3$–module $\mathcal{R}(3)$. However, it is not a generator of $\mathcal{R}(3)$ as a $\mathbb{K}$–module; one needs to add the element $\mu \otimes_{\mathbb{S}_2} (\mu \otimes 1) - (\mu \otimes_{\mathbb{S}_2} (\mu \otimes 1))^{(2,3)}$.

Given this presentation, the Koszul dual $\mathit{uCom}^{\text{¡}}$ is a curved conilpotent cooperad whose underlying graded cooperad is the final subcooperad of $\mathbb{T}^c(\mathbb{K} \cdot s\mu \oplus \mathbb{K} \cdot \xi)$ such that

$$\mathit{u\mathcal{A}s}^{\text{¡}}(3) \cap \mathbb{T}^2(\mathbb{K} \cdot s\mu)(3) = \mathbb{K}[\mathbb{S}_3] \cdot (s\mu \otimes_{\mathbb{S}_2} (s\mu \otimes 1) - s\mu \otimes_{\mathbb{S}_2} (1 \otimes s\mu)).$$

The coderivation of $\mathit{uCom}^{\text{¡}}$ is zero and the curvature is given by

$$\theta(s\mu \otimes_{\mathbb{S}_2} (s\xi \otimes 1)) = -1.$$

Notation We denote by uCom_∞ the operad $\Omega_u \mathit{uCom}^{\text{¡}}$.

8.4.2 Coalgebras over $\mathit{uCom}^{\text{¡}}$

We will show that the category of $\mathit{uCom}^{\text{¡}}$–coalgebras is equivalent to the category of curved conilpotent Lie coalgebras.

Definition 128 (curved Lie coalgebra) A *curved Lie coalgebra* $\mathscr{C} = (C, \delta, d, \theta)$ is a graded $\mathbb{K}$–module C equipped with an antisymmetric map $\delta\colon C \to C \otimes C$ such that

$$(\delta \otimes \mathrm{Id})\delta = (\mathrm{Id} \otimes \delta)\delta + (\mathrm{Id} \otimes \tau)(\delta \otimes \mathrm{Id})\delta,$$

where τ is the exchange map $\tau(x \otimes y) = (-1)^{|x||y|} y \otimes x$. It is also equipped with a degree -1 map $d\colon C \to C$ which is a coderivation, that is,

$$\delta d = (d \otimes \mathrm{Id} + \mathrm{Id} \otimes d)\delta,$$

and with a degree -2 map $\theta\colon C \to \mathbb{K}$ which is a curvature, that is,

$$d^2 = (\theta \otimes \mathrm{Id} - \mathrm{Id} \otimes \theta)\delta.$$

A curved Lie coalgebra C is said to be *conilpotent* if for any $x \in C$, there exists an integer n such that the element

$$(\mathrm{Id} \otimes \cdots \otimes \delta \otimes \cdots \otimes \mathrm{Id}) \cdots \delta(x)$$

is zero whenever δ appears n times. We denote by $\mathsf{cLieCog}$ the category of curved conilpotent Lie coalgebras.

Proposition 129 *The endofunctor of the category of graded $\mathbb{K}$–modules $\mathcal{V} \mapsto s\mathcal{V}$ induces an equivalence between the category of $u\mathcal{C}om^{¡}$–coalgebras and the category* cLieCog *of curved conilpotent Lie coalgebras.*

Lemma 130 *The category of $u\mathcal{C}om^{¡}$–coalgebras is equivalent to the category whose objects are graded $\mathbb{K}$–modules $\mathcal{C}$ equipped with three maps:*

- *A degree -1 map $\delta\colon \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ which is symmetric is the sense that $\tau\delta = \delta'$, which satisfy the equation*

  $$(\delta \otimes \mathrm{Id})\delta(x) + ((\delta \otimes \mathrm{Id})\delta(x))^{(2,3)} + ((\delta \otimes \mathrm{Id})\delta(x))^{(1,3)} = 0,$$

  and such that for any $x \in \mathcal{C}$, there exists an integer n such that the element

  $$(\mathrm{Id} \otimes \cdots \otimes \delta \otimes \cdots \otimes \mathrm{Id}) \cdots (\mathrm{Id} \otimes \delta)\delta(x)$$

  is zero whenever δ appears at least n times.
- *A degree -1 map $\theta\colon \mathcal{C} \to \mathbb{K}$.*
- *A degree -1 map $d\colon \mathcal{C} \to \mathcal{C}$ such that $\theta d = 0$, such that $\delta d = -(d \otimes \mathrm{Id} + \mathrm{Id} \otimes d)\delta$ and such that $d^2 = -(\theta \otimes \mathrm{Id} + \mathrm{Id} \otimes \theta)\delta = -2(\theta \otimes \mathrm{Id})\delta$.*

The morphisms of this category are the morphisms of graded $\mathbb{K}$–modules which commute with these structure maps.

Proof We apply Theorem 119. A graded $u\mathcal{C}om^{¡}$–coalgebra is a graded $\mathbb{K}$–module $\mathcal{C}$ equipped with maps

$$\begin{aligned} \delta'&\colon \mathcal{C} \to (\mathbb{K} \cdot s\mu) \otimes_{\mathbb{S}_2} (\mathcal{C} \otimes \mathcal{C}),\\ \theta'&\colon \mathcal{C} \to \mathbb{K} \cdot s\xi,\\ d'&\colon \mathcal{C} \to \mathcal{C}, \end{aligned}$$

such that the corresponding precoradical filtration is exhaustive, such that

$$(\mathrm{Id}_{s\mu} \circ' \delta')\delta'(\mathcal{C}) \subset s^2\mathcal{R}(3) \otimes_{\mathbb{S}_3} \mathcal{C}^{\otimes 3} \tag{2}$$

and such that

$$\begin{aligned} \delta' d' &= (\mathrm{Id} \circ' d')\delta',\\ \theta' d' &= 0,\\ d'^2 &= (\theta_{u\mathcal{C}om^{¡}} \circ \mathrm{Id})(\mathrm{Id} \circ' \delta')\delta'. \end{aligned}$$

These maps induce new maps

$$\begin{aligned}\delta\colon \mathcal{C} &\xrightarrow{\delta'} (\mathbb{K}\cdot s\mu)\otimes_{\mathbb{S}_2}(\mathcal{C}\otimes\mathcal{C})\to\mathcal{C}\otimes\mathcal{C},\\ \theta\colon \mathcal{C} &\xrightarrow{\theta'} \mathbb{K}\cdot s\xi\to\mathbb{K},\\ d=d'\colon \mathcal{C}&\to\mathcal{C},\end{aligned}$$

where the degree -1 map $(\mathbb{K}\cdot s\mu)\otimes_{\mathbb{S}_2}(\mathcal{C}\otimes\mathcal{C})\to\mathcal{C}\otimes\mathcal{C}$ sends $s\mu\otimes_{\mathbb{S}_2}(x\otimes y)$ to $\frac{1}{2}(x\otimes y+(-1)^{|x||y|}y\otimes x)$. Then, for any $x\in\mathcal{C}$,

$$\delta'(x)=s\mu\otimes_{\mathbb{S}_2}\delta(x).$$

We know from [19, Section 7.6.3] that the $\mathbb{K}$–module $\mathbb{T}(s\mu)(3)$ has three generators ν_{I}, ν_{II} and ν_{III}, which are obtained from the composite $s\mu\otimes_{\mathbb{S}_2}(s\mu\otimes 1)$ by applying the permutations $\mathrm{Id}\in\mathbb{S}_3$, $(2,3)$ and $(1,3)$, respectively. Moreover, $s^2\mathcal{R}(3)$ is spanned by $\nu_{\mathrm{I}}-\nu_{\mathrm{II}}$ and $\nu_{\mathrm{I}}-\nu_{\mathrm{III}}$. Further, $\mathbb{K}\cdot(\nu_{\mathrm{I}}+\nu_{\mathrm{II}}+\nu_{\mathrm{III}})$ is a complementary sub-$\mathbb{K}[\mathbb{S}_3]$–module of $s^2\mathcal{R}(3)$ in $\mathbb{T}(s\mu)(3)$. Let us denote by π the projection of $\mathbb{T}(s\mu)(3)$ onto $\mathbb{K}\cdot(\nu_{\mathrm{I}}+\nu_{\mathrm{II}}+\nu_{\mathrm{III}})$ along $s^2\mathcal{R}(3)$. Since the action of the group $\mathbb{S}_2$ on $s\mu$ is trivial, we have, for any $x\in\mathcal{C}$,

$$(\mathrm{Id}_{s\mu}\circ'\delta')\delta'(x)=2\nu_{\mathrm{I}}\otimes_{\mathbb{S}_3}((\delta\otimes\mathrm{Id})\delta(x)).$$

Then

$$(\pi\circ\mathrm{Id})(\mathrm{Id}_{s\mu}\circ'\delta')\delta'(x)=\tfrac{2}{3}(\nu_{\mathrm{I}}+\nu_{\mathrm{II}}+\nu_{\mathrm{III}})\otimes_{\mathbb{S}_3}((\delta\otimes\mathrm{Id})\delta(x)).$$

The above condition (2) is equivalent to the fact that $(\nu_{\mathrm{I}}+\nu_{\mathrm{II}}+\nu_{\mathrm{III}})\otimes_{\mathbb{S}_3}((\delta\otimes\mathrm{Id})\delta(x))$ is zero, which is equivalent to

$$(\delta\otimes\mathrm{Id})\delta(x)+((\delta\otimes\mathrm{Id})\delta(x))^{(2,3)}+((\delta\otimes\mathrm{Id})\delta(x))^{(1,3)}=0.$$

The other conditions are equivalent to

$$\begin{aligned}\delta d&=-(d\otimes\mathrm{Id}+\mathrm{Id}\otimes d)\delta,\\ \theta d&=0,\\ d^2&=-(\theta\otimes\mathrm{Id}+\mathrm{Id}\otimes\theta)\delta.\end{aligned}$$

Conversely, from the maps δ, θ and d, one can reconstruct δ', θ' and d' in the obvious way. □

Proof of Proposition 129 We show that the category described in Lemma 130 is equivalent to the category of curved conilpotent Lie coalgebras. Let $\mathscr{C}=(\mathcal{C},\delta,\theta,d)$ be a curved conilpotent Lie coalgebra. Then we can define the maps (δ',θ',d') on $s^{-1}\mathcal{C}$, where δ' is the composite

$$s^{-1}\mathcal{C} \simeq \mathbb{K}\cdot s^{-1}\otimes\mathcal{C} \to \mathbb{K}\cdot s^{-1}\otimes\mathbb{K}\cdot s^{-1}\otimes\mathcal{C}\otimes\mathcal{C} \simeq \mathbb{K}\cdot s^{-1}\otimes\mathcal{C}\otimes\mathbb{K}\cdot s^{-1}\otimes\mathcal{C},$$
$$s^{-1}\otimes x \mapsto s^{-1}\otimes s^{-1}\otimes\delta(x),$$

and where $\theta'(s^{-1}x) = \theta(x)$ and $d'(s^{-1}x) = -s^{-1}dx$ for any $x \in \mathcal{C}$. It is straightforward to prove that these maps satisfy the conditions of Lemma 130. Conversely, from a graded $\mathbb{K}$–module $\mathcal{D}$ and maps (δ, θ, d) as in Lemma 130, one can build a structure of curved conilpotent Lie coalgebra (δ', θ', d') on $s\mathcal{D}$, where δ' is the composite

$$s\mathcal{D} \simeq \mathbb{K}\cdot s\otimes\mathcal{D} \to \mathbb{K}\cdot s\otimes\mathbb{K}\cdot s\otimes\mathcal{D}\otimes\mathcal{D} \simeq \mathbb{K}\cdot s\otimes\mathcal{D}\otimes\mathbb{K}\cdot s\otimes\mathcal{D},$$
$$s\otimes x \mapsto -s\otimes s\otimes\delta(x),$$

and where $\theta'(sx) = \theta(x)$ and $d'(sx) = -sdx$ for any $x \in \mathcal{D}$. It is again straightforward to prove that these maps define actually a structure of curved conilpotent Lie coalgebra. Moreover, these two constructions are inverse to one another. □

8.4.3 The bar–cobar adjunction If we compose the bar–cobar adjunction between $u\mathcal{C}om$–algebras and $u\mathcal{C}om^{\text{¡}}$–coalgebras with the equivalence between $u\mathcal{C}om^{\text{¡}}$–coalgebras and curved conilpotent Lie coalgebras, then we obtain an adjunction $\Omega_C \dashv B_L$ between unital commutative algebras and curved conilpotent Lie coalgebras, which is as follows.

Definition 131 (curved Lie bar construction) Let $\mathscr{A} = (\mathcal{A}, \gamma_{\mathscr{A}}, 1)$ be a unital commutative algebra. Its *curved Lie bar construction* $B_L(\mathscr{A})$ is the following curved conilpotent Lie coalgebra. The underlying graded Lie coalgebra of $B_L(\mathscr{A})$ is

$$B_L(\mathcal{A}) := \mathcal{L}ie^c \circ (s\mathcal{A} \oplus \mathbb{K}\cdot v),$$

where $\mathcal{L}ie^c$ denotes the Lie cooperad which is the linear dual of the Lie operad and where $|v| = 2$. The coderivation of $B_L(\mathscr{A})$ extends the map

$$\mathcal{L}ie^c(s\mathcal{A}\oplus\mathbb{K}v) \twoheadrightarrow s\mathcal{A}\wedge s\mathcal{A}\oplus s\mathcal{A}\oplus\mathbb{K}v \to s\mathcal{A}, \qquad \begin{aligned} sx\wedge sy &\mapsto (-1)^{|x|}s\gamma_{\mathcal{A}}(x\otimes y),\\ v &\mapsto s1,\\ sx &\mapsto -sdx.\end{aligned}$$

The curvature is the map

$$\mathcal{L}ie^c(s\mathcal{A}\oplus\mathbb{K}\cdot v) \twoheadrightarrow \mathbb{K}\cdot v \to \mathbb{K}, \quad v \mapsto 1.$$

Definition 132 (unital commutative cobar construction) Let $\mathscr{C} = (\mathcal{C}, \delta, d_{\mathcal{C}}, \theta)$ be a curved Lie coalgebra. Its *unital commutative cobar construction* $\Omega_C(\mathscr{C})$ is the free unital commutative algebra

$$\Omega_C \mathcal{C} := S(s^{-1}\mathcal{C}) := \bigoplus_{n\in\mathbb{N}} (s^{-1}\mathcal{C})^{\otimes n}/\mathbb{S}_n,$$

whose coderivation extends the map

$$s^{-1}\mathcal{C} \to S(s^{-1}\mathcal{C}), \quad s^{-1}x \mapsto \theta(x)1 - s^{-1}d_{\mathcal{C}}x - \sum (-1)^{|x_1|} s^{-1}x_1 \otimes_{\mathbb{S}_2} s^{-1}x_2,$$

where $\sum x_1 \wedge x_2 = \delta(x)$.

Definition 133 (twisting morphisms) A twisting morphism from a curved conilpotent Lie coalgebra $\mathscr{C}$ to a unital commutative algebra $\mathscr{A}$ is a degree -1 map $\alpha\colon \mathcal{C} \to \mathcal{A}$ such that

$$\partial\alpha + \gamma_{\mathscr{A}}(\alpha \otimes \alpha)\delta_{\mathscr{C}} = \theta(-)1_{\mathscr{A}}.$$

We denote by $\mathrm{Tw}_L(\mathscr{C}, \mathscr{A})$ the set of twisting morphisms from $\mathscr{C}$ to $\mathscr{A}$.

Proposition 134 *We have functorial isomorphisms*

$$\hom_{\mathit{uCom}-\mathsf{alg}}(\Omega_C \mathscr{C}, \mathscr{A}) \simeq \mathrm{Tw}_L(\mathscr{C}, \mathscr{A}) \simeq \hom_{\mathsf{cLieCog}}(\mathscr{C}, B_L \mathscr{A})$$

for any unital commutative algebra $\mathscr{A}$ and any curved conilpotent Lie coalgebra $\mathscr{C}$.

Proof The proof uses the same arguments as the proof of Proposition 63. ☐

The adjunction $\Omega_C \dashv B_L$ is part of a larger picture,

$$\begin{tikzcd} \mathsf{cLieCog} \arrow[r, shift left, "\Omega_\iota"] & \mathit{uCom}_\infty\text{–}\mathsf{alg} \arrow[l, shift left, "B_\iota"] \arrow[r, shift left, "\psi_!"] & \mathit{uCom}\text{–}\mathsf{alg}, \arrow[l, shift left, "\psi^*"] \end{tikzcd}$$

where the adjunction $\psi_! \dashv \psi^*$ is induced by the morphism of operads $\psi\colon \mathit{uCom}_\infty \to \mathit{uCom}$ and where $\Omega_C = \psi_!\Omega_\iota$ and $B_L = B_\iota\psi^*$. We know that a uCom_∞–algebra $\mathscr{A} = (\mathcal{A}, \gamma)$ is the data of a chain complex $\mathcal{A}$ together with a degree -1 map

$$\gamma\colon \mathcal{L}ie^c(s\mathcal{A} \oplus \mathbb{K}\cdot v) \to s\mathcal{A}$$

such that the coderivation of the curved Lie coalgebra $\mathcal{L}ie^c(s\mathcal{A} \oplus \mathbb{K}\cdot v)$ which extends γ squares to $(\theta \otimes \mathrm{Id})\delta$, where θ is given by

$$\mathcal{L}ie^c(s\mathcal{A} \oplus \mathbb{K}\cdot v) \twoheadrightarrow \mathbb{K}\cdot v \to \mathbb{K}, \quad v \mapsto 1.$$

In particular, we have the following:

- A degree zero symmetric product
$$\gamma_2\colon \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}.$$
- A degree 1 map
$$\gamma_{\mathrm{I,II}}\colon \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$$
whose boundary is the associator of γ_2, that is,
$$\partial(\gamma_{\mathrm{I,II}})=\gamma_2(\mathrm{Id}\otimes\gamma_2)-\gamma_2(\gamma_2\otimes\mathrm{Id}).$$
- A degree 1 map
$$\gamma_{\mathrm{I,III}}\colon \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$$
whose boundary is
$$\partial(\gamma_{\mathrm{I,III}})=\gamma_2(\mathrm{Id}\otimes\gamma_2)-\gamma_2(\mathrm{Id}\otimes\gamma_2)(\tau\otimes\mathrm{Id}).$$
- An element $1_{\mathscr{A}}$ defined by $\gamma(v)=s1_{\mathscr{A}}$.
- A degree 1 map $\gamma_u\colon \mathcal{A}\to\mathcal{A}$ which makes $1_{\mathscr{A}}$ a unit up to homotopy:
$$\partial(\gamma_u)=\gamma_2(1_{\mathscr{A}}\otimes\mathrm{Id})-\mathrm{Id}.$$

8.4.4 The Koszul property and the infinity category of uCom_∞–algebras

Theorem 135 *The operad* uCom *is Koszul.*

Proof We know from [17] that $\mathit{quCom}^{¡}\simeq \mathit{Com}^{¡}\circ(\mathcal{I}\oplus\mathbb{K}\cdot s\xi)$. So, we have
$$\mathit{quCom}\circ\mathit{quCom}^{¡}\simeq\mathbb{K}\cdot\xi\oplus\mathit{Com}\circ\mathit{Com}^{¡}\circ(\mathcal{I}\oplus\mathbb{K}\cdot s\xi).$$
We can filter $\mathit{quCom}\circ_\kappa\mathit{quCom}^{¡}$ by the number of ξ and $s\xi$ appearing in the trees. Then the induced graded complex have the form
$$G(\mathit{quCom}\circ_\kappa\mathit{quCom}^{¡})\simeq\mathbb{K}\cdot\xi\oplus(\mathit{Com}\circ_\kappa\mathit{Com}^{¡})\circ(\mathcal{I}\oplus\mathbb{K}\cdot s\xi).$$
We already know by [19, Theorems 7.4.6 and 13.1.7] that the canonical morphism $\mathit{Com}\circ_\kappa\mathit{Com}^{¡}\to\mathcal{I}$ is a weak equivalence. Then the map $G(\mathit{quCom}\circ_\kappa\mathit{quCom}^{¡})\to\mathcal{I}$ may be decomposed as follows:
$$G(\mathit{quCom}\circ_\kappa\mathit{quCom}^{¡})\simeq\mathbb{K}\cdot\xi\oplus(\mathit{Com}\circ_\kappa\mathit{Com}^{¡})\circ(\mathcal{I}\oplus\mathbb{K}\cdot s\xi)\to\mathcal{I}\oplus\mathbb{K}\cdot\xi\oplus\mathbb{K}\cdot s\xi\to\mathcal{I}.$$
All the maps of this composition are quasi-isomorphisms. So, by Theorem 88, the canonical map $\mathit{quCom}\circ_\kappa\mathit{quCom}^{¡}\to\mathcal{I}$ is a quasi-isomorphism. We conclude by Theorem 116. □

There are several ways to describe the infinity category of $uCom$–algebras:

- One can take the Dwyer–Kan simplicial localization of the category of $uCom$–algebras with respect to quasi-isomorphisms as described in [10; 9].
- One can take the simplicial category whose objects are cofibrant-fibrant $uCom$–algebras and whose spaces of morphisms are
$$\mathrm{Map}(\mathscr{A}, \mathscr{B})_n := \mathrm{HOM}_{uCom-\mathsf{alg}}(\mathscr{A}, \mathscr{B}).$$
- One can also take the simplicial category whose objects are all $uCom$–algebras and whose spaces of morphisms are
$$\mathrm{Map}(\mathscr{A}, \mathscr{B}) := \mathrm{HOM}_{uCom-\mathsf{alg}}(\Omega_C B_L \mathscr{A}, \mathscr{B}) \simeq \mathrm{HOM}_{\mathsf{cLieCog}}(B_L \mathscr{A}, B_L \mathscr{B}).$$

Appendix

The purpose of this appendix is to describe the category of dg counital cocommutative coalgebras over an algebraically closed field of characteristic zero in the vein of [7]. In the sequel, dg counital cocommutative coalgebras are simply called cocommutative coalgebras. We suppose that the base field $\mathbb{K}$ is algebraically closed field and of characteristic zero.

Remark 136 The characteristic zero assumption is needed in [7, Theorem 2.9].

We know that the linear dual of a cocommutative coalgebra is a commutative algebra. Moreover, for any cocommutative coalgebra $\mathscr{C}$, the subcoalgebras of $\mathscr{C}$ are in correspondence with the ideals of $\mathscr{C}^*$.

Definition 137 (orthogonal ideals and subcoalgebras) Let $\mathscr{D} = (\mathcal{D}, \Delta, \epsilon)$ be a subcoalgebra of $\mathscr{C}$. The orthogonal of $\mathscr{D}$ is the subchain complex
$$\mathscr{D}^{\perp} := \{f \in \mathscr{C}^* \mid f(x) = 0 \text{ for all } x \in \mathcal{D}, \} \subset \mathscr{C}^*,$$
which is an ideal of $\mathscr{C}^*$. Let I be an ideal of the commutative algebra $\mathscr{C}^*$. The orthogonal of I is the subchain complex $I^{\perp} := \{x \in \mathscr{C} \mid f(x) = 0 \text{ for all } f \in I\} \subset \mathscr{C}$, which is a subcoalgebra of $\mathscr{C}$.

Definition 138 (pseudocompact algebras) A pseudocompact algebra is a dg unital commutative algebra $\mathscr{A}$ together with a set $\{I_u\}_{u \in U}$ of ideals of finite codimension, which is stable under finite intersections and such that

$$\mathscr{A} \simeq \lim \mathscr{A}/I_u.$$

A morphism of pseudocompacts algebras from $(\mathscr{A}, \{I_u\}_{u \in U})$ to $(\mathscr{B}, \{J_v\}_{v \in V})$ is a morphism of algebras $f\colon \mathscr{A} \to \mathscr{B}$ which is continuous with respect to the induced topologies, and is such that for any $v \in V$, there exists a $u \in U$ such that the composite morphism $\mathscr{A} \to \mathscr{B} \to \mathscr{B}/J_v$ factors through $\mathscr{A} \to \mathscr{A}/I_u$. A pseudocompact algebra $\mathscr{A}$ is called local if its underlying graded algebra is local.

Proposition 139 *The linear dual of a cocommutative coalgebra is a pseudocompact algebra. Moreover, the linear dual functor is an antiequivalence between the category of cocommutative coalgebras and the category of pseudocompact algebras.*

Proof It is clear that linear duality induces an antiequivalence between finite-dimensional cocommutative coalgebras and finite-dimensional commutative algebras. The rest is a consequence of the following Proposition 140. □

Proposition 140 [11] *Let $\mathscr{C}$ be a cocommutative coalgebra and let x be an element of $\mathscr{C}$. There exists a finite-dimensional subcoalgebra of $\mathscr{C}$ which contains x. Then $\mathscr{C}$ is the colimit of the filtered diagram of its finite-dimensional subcoalgebras.*

Chuang, Lazarev and Mannan showed that any pseudocompact algebra can be decomposed into a product of local pseudocompact algebras.

Theorem 141 [7, Theorem 2.9] *Any pseudocompact algebra $\mathscr{A}$ is isomorphic to the product of local pseudocompact algebras $\mathscr{A} \simeq \prod_{i \in I} \mathscr{A}_i$. Moreover, a morphism of products of local pseudocompact algebras $f\colon \prod_{i \in I} \mathscr{A}_i \to \prod_{j \in J} \mathscr{B}_j$ is the data of a function $\phi\colon J \to I$ and a morphism $f_j\colon \mathscr{A}_{\phi(j)} \to \mathscr{B}_j$ for any $j \in J$, where $\pi_j f = f_j \pi_{\phi(j)}$* (here π_j and $\pi_{\phi(j)}$ denote the projection of $\prod_{j \in J} \mathscr{B}_j$ onto $\mathscr{B}_j$ and the projection of $\prod_{i \in I} \mathscr{A}_i$ onto $\mathscr{A}_{\phi(j)}$, respectively).

We show that local pseudocompact algebras are linear duals of conilpotent cocommutative coalgebras.

Definition 142 (irreducible coalgebras) A nonzero graded cocommutative coalgebra is said to be *irreducible* if any two nonzero subcoalgebras have a nonzero intersection.

Proposition 143 *A graded cocommutative coalgebra is irreducible if and only if its dual algebra is local.*

Proof Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon)$ be a graded cocommutative coalgebra. We first suppose that it is irreducible. Let M_1 and M_2 be two maximal ideals of the commutative algebra $\mathscr{C}^*$. Since $\mathscr{C}$ is irreducible, the subcoalgebras $M_1^\perp$ and $M_2^\perp$ have a nonzero intersection. So $M_1 + M_2 \subset (M_1^\perp \cap M_2^\perp)^\perp$ is a proper ideal. Since M_1 and M_2 are maximal ideals, $M_1 = M_1 + M_2 = M_2$. So $\mathscr{C}^*$ is local. Conversely, suppose that $\mathscr{C}^*$ is local. We denote by M its maximal ideal. By Lemma 144, M is the kernel of an augmentation $\mathscr{C}^* \to \mathbb{K}$. By the antiequivalence between pseudocompact algebras and cocommutative coalgebras, we obtain a morphism of coalgebras $\mathbb{K} \to \mathscr{C}$, that is, an atom a of $\mathscr{C}$. For any nonzero subcoalgebra $\mathscr{D}$ of $\mathscr{C}$, the orthogonal $\mathscr{D}^\perp$ is contained in M. Thus, $\mathbb{K} \cdot a = M^\perp \subset (\mathscr{D}^\perp)^\perp = \mathscr{D}$. So any nonzero subcoalgebra of $\mathscr{C}$ contains a. Subsequently, $\mathscr{C}$ is irreducible. □

Lemma 144 *Let $\mathscr{A}$ be a graded local pseudocompact algebra. Then the maximal ideal M of $\mathscr{A}$ is the kernel of an augmentation $\mathscr{A} \to \mathbb{K}$.*

Proof Since $\mathscr{A} = (\mathcal{A}, \gamma_{\mathscr{A}}, 1)$ is the inverse limit of finite-dimensional algebras and since M is maximal, M is the kernel of a surjection $\mathscr{A} \to \mathscr{B}$, where $\mathscr{B} = (\mathcal{B}, \gamma_{\mathcal{B}}, 1)$ is a finite-dimensional commutative algebra. Since M is maximal, any nonzero element of $\mathcal{B}$ is invertible. Since the elements in nonzero degrees are nilpotent, $\mathcal{B}$ is concentrated in degree zero. So $\mathcal{B}$ is a finite-dimensional field extension of $\mathbb{K}$. Finally, $\mathcal{B} \simeq \mathbb{K}$ because $\mathbb{K}$ is an algebraically closed field. □

Corollary 145 *A graded cocommutative coalgebra is irreducible if and only if it contains a single atom.*

Proof It is a direct consequence of Proposition 143. □

Proposition 146 *Irreducible graded cocommutative coalgebras are conilpotent graded cocommutative coalgebras.*

Proof Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon)$ be an irreducible graded cocommutative coalgebra. Let x be an element of $\mathcal{C}$ and let $\mathscr{D} = (\mathcal{D}, \Delta, \epsilon)$ be a finite-dimensional subcoalgebra of $\mathscr{C}$ which contains x. The commutative algebra $\mathscr{D}^*$ is local; its maximal ideal is $M := \overline{\mathcal{D}}^*$. Then $\mathscr{D}_0^*$ is also local with maximal ideal M_0. By Nakayama's lemma, M_0 is nilpotent. So, M is nilpotent and so $\mathscr{D}$ is a conilpotent cocommutative coalgebra. □

Corollary 147 *The antiequivalence between the category of pseudocompact algebras and the category* uCocom *of cocommutative coalgebras restricts to an antiequivalence* *between the category of local pseudocompact algebras and the category* uNilCocom *of conilpotent cocommutative coalgebras.*

Proof It is a direct consequence of Propositions 143 and 146. □

Theorem 78 *Let $\mathscr{C} = (\mathcal{C}, \Delta, \epsilon)$ be a dg cocommutative coalgebra over an algebraically closed field of characteristic zero and let A be its set of graded atoms. There exists a unique decomposition $\mathscr{C} \simeq \bigoplus_{a \in A} \mathscr{C}_a$, where $\mathscr{C}_a$ is a subcoalgebra of $\mathscr{C}$ which contains a and which belongs to the category* uNilCocom*. Moreover, a morphism of dg cocommutative coalgebras $f \colon \bigoplus_{a \in A} \mathscr{C}_a \to \bigoplus_{b \in B} \mathscr{D}_b$ is the data of a function $\phi \colon A \to B$ and of a morphism $f_a \colon \mathscr{C}_a \to \mathscr{D}_{\phi(a)}$ for any $a \in A$.*

Proof The only point that needs to be cleared up is that, in the decomposition $\mathscr{C} = \bigoplus_{i \in I} \mathscr{C}_i$, the set I is isomorphic to the set of graded atoms of $\mathscr{C}$. A graded atom of $\mathscr{C}$ is a morphism of graded cocommutative coalgebras from $\mathbb{K}$ to $\mathscr{C}$, that is, a morphism of graded pseudocompact algebras from $\prod_{i \in I} \mathscr{C}_i^*$ to $\mathbb{K}$. So it is the choice of an element of I. □

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ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 19 Issue 3 (pages 1079–1618) 2019

Homotopical intersection theory, III: Multirelative intersection problems 1079
JOHN R KLEIN and BRUCE WILLIAMS

Rips filtrations for quasimetric spaces and asymmetric functions with stability results 1135
KATHARINE TURNER

C^*–algebraic drawings of dendroidal sets 1171
SNIGDHAYAN MAHANTA

Examples of nontrivial contact mapping classes for overtwisted contact manifolds in all dimensions 1207
FABIO GIRONELLA

Uniform exponential growth for CAT(0) square complexes 1229
ADITI KAR and MICHAH SAGEEV

Commensurability invariance for abelian splittings of right-angled Artin groups, braid groups and loop braid groups 1247
MATTHEW C B ZAREMSKY

Occupants in simplicial complexes 1265
STEFFEN TILLMANN

On uniqueness of end sums and 1–handles at infinity 1299
JACK S CALCUT and ROBERT E GOMPF

The topology of arrangements of ideal type 1341
NILS AMEND and GERHARD RÖHRLE

Topological complexity of unordered configuration spaces of surfaces 1359
ANDREA BIANCHI and DAVID RECIO-MITTER

Hyperbolic extensions of free groups from atoroidal ping-pong 1385
CAGLAR UYANIK

Symmetric spectra model global homotopy theory of finite groups 1413
MARKUS HAUSMANN

Representing the deformation ∞–groupoid 1453
DANIEL ROBERT-NICOUD

Classifying spaces from Ore categories with Garside families 1477
STEFAN WITZEL

The Lannes–Zarati homomorphism and decomposable elements 1525
NGÔ A TUẤN

Homotopy theory of unital algebras 1541
BRICE LE GRIGNOU