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On negative-definite cobordisms among lens spaces of type $(m,1)$ and uniformization of four-orbifolds

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Connected sums of lens spaces which smoothly bound a rational homology ball are classified by P Lisca. In the classification, there is a phenomenon that a connected sum of a pair of lens spaces $L(a,b)\,\#\,L(a,-b)$ appears in one of the typical cases of rational homology cobordisms. We consider smooth negative-definite cobordisms among several disjoint union of lens spaces and a rational homology 3–sphere to give a topological condition for the cobordism to admit the above "pairing" phenomenon. By using Donaldson theory, we show that if $1/m$ has a certain minimality condition concerning the Chern–Simons invariants of the boundary components, then any $L(m,1)$ must have a counterpart $L(m,-1)$ in negative-definite cobordisms with a certain condition only on homology. In addition, we show an existence of a reducible flat connection through which the pair is related over the cobordism. As an application, we give a sufficient condition for a closed smooth negative-definite 4–orbifold with two isolated singular points whose neighborhoods are homeomorphic to the cones over lens spaces $L(m,1)$ and $L(m,-1)$ to admit a finite uniformization.

57R18, 57R57; 57M05, 57R90

1 Introduction

Lisca [25] classified all connected sums of lens spaces which smoothly bound rational homology 4–balls. In fact, all of them are connected sums of several pairs of lens spaces in the list of five typical families of components such as $L(a,b)\,\#\,L(a,-b)$ for coprime a and b. In particular, if a connected sum of s copies of $L(m,1)$ and t copies of $L(m,-1)$ bounds a rational homology ball then s must be equal to t. In this classification, a crucial role was played by Donaldson's theorem on intersection forms of definite closed smooth 4–manifolds. Motivated by this result, we focus on the formation of pairs $L(a,b)\,\#\,L(a,-b)$ on the boundaries of compact oriented smooth 4–manifolds W such that the dimension $b_2^+(W)$ of a maximal positive subspace for the cup product (intersection form) on the second real cohomology $H^2(W)$ of W is zero, called *negative-definite cobordisms*.

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To be precise, we consider the following problem:

Problem 1.1 *Which negative-definite cobordisms between a disjoint union of lens spaces and a rational homology 3–sphere admit a pair of lens spaces $L(a,b)\#L(a,-b)$?*

In this paper, we show the following theorem:

Theorem 1.2 *Let Y be a disjoint union of lens spaces $Y_i = L(a_i, b_i)$ for $i = 1, \dots, s$ and a Brieskorn homology 3–sphere $\Sigma(p,q,r)$. Suppose Y bounds a compact oriented negative-definite smooth 4–manifold W such that*

(1) $Y_1 = L(m,1)$ *with* $m = \max\{a_i\} > pqr$ *and* $m \neq 4$,

(2) *the homomorphism*

$$i_*\colon H_1(Y;\mathbb{Z}) \to H_1(W;\mathbb{Z})$$

induced by the inclusion $i\colon Y \hookrightarrow W$ *is surjective, and*

(3) *there is no* $e \in H^2(W;\mathbb{Z})$ *with rational self-intersection number* $e^2 = -1/m$.

Then there exists $i \neq 1$ *such that* $Y_i = L(m,-1)$.

Note that this theorem does not exclude the possibility $s = 1$, in which case we immediately see such a cobordism W does not exist and $s > 1$ must hold. Here we remark that this Theorem 1.2 is a special case of our main Theorem 4.1, which is a generalization of Theorem 2.1 in [16] and Theorem 6.1 in [14], both due to M Furuta.

To prove our main result, we make use of a topology of the moduli space of SU(2)–instantons on an oriented smooth 4–manifold with cylindrical ends.

Remark 1.3 (1) The condition $m \neq 4$ in Theorem 1.2 is necessary. In fact, without this condition, the complement W of a tubular neighborhood of a non-singular quadratic curve in $\overline{\mathbb{C}P^2}$ would give a counterexample.

(2) The existence of an $L(m,-1)$ does not necessarily imply the existence of an $L(m,1)$ in negative-definite cobordisms stated in Theorem 1.2, since $L(3,-1)$ obtained by plumbing of linear chain $[-2,-2]$ of length 2 with weight -2 would give a counterexample.

Our main result, Theorem 4.1, and Theorem 6.1 in [14] also state existence of a *reducible* flat connection on such negative-definite cobordisms as in Theorem 1.2. As an application we consider a uniformization problem of 4–orbifolds. The notion of orbifolds is firstly introduced by I Satake [30] as the name "V–manifold" in his study on Siegel's modular varieties and is also studied by W Thurston [33] as orbifolds in view of his geometrization program on 3–manifolds. A typical example is a quotient space $X = M/G$ of a manifold M with respect to a properly discontinuous action of a (finite) group G, in which case M is called a (finite) uniformization of the orbifold X. Note that there exist orbifolds which do not admit uniformization; these are called "bad orbifolds".

In this paper, we consider a version of the "uniformization problem" proposed by M Kato [20] in our context. In the complex analytic category, this problem also has its origin in "Fenchel's problem" raised by M Namba [26] as a higher-dimensional generalization of Fenchel's conjecture about F–groups, solved by S Bundgaard and J Nielsen [5] and R Fox [13].

Problem 1.4 (see the uniformization problem [20]) *Let X be a smooth orbifold with singular locus ΣX. Give a good condition on $(X, \Sigma X)$ for the existence of a finite uniformization (G, M) of (X, Σ).*

Here a "good" condition means a condition without referring to the fundamental group $\pi_1(X - \Sigma X)$ of $X - \Sigma X$.

For a $\mathbb{Z}_m$–action on the 4–sphere $S^4 = \mathbb{C}^2 \cup \{\infty\}$ defined by $\zeta \cdot (z, w) = (\zeta z, \zeta^{-1} w)$ on $\mathbb{C}^2 \subset S^4$ for $\zeta = e^{2\pi i/m} \in \mathbb{Z}_m \subset U(1)$, the quotient space $X = S^4/\mathbb{Z}_m$ is a 4–dimensional orbifold which has a singular point $p_1 = 0$ whose neighborhood U_1 is homeomorphic to the cone $cL(m, -1)$ over the lens space $L(m, -1)$, and there is also a singular point $p_2 = \infty$ whose neighborhood U_2 is homeomorphic to the cone $cL(m, 1)$ over the lens space $L(m, 1)$. Conversely, we have the following:

Theorem 1.5 *Let X be a closed negative-definite smooth 4–orbifold with two isolated singular points p_1 and p_2, each with a neighborhood U_i homeomorphic to the cone $cL(a_i, b_i)$ over the lens space $L(a_i, b_i)$ for $i = 1, 2$. Suppose $H_1(X; \mathbb{Z}) = \{0\}$ and that there is no $e \in H^2(X; \mathbb{Z})$ with $e^2 = -1/m$ for $m = \max\{a_1, a_2\} \neq 4$. If $U_1 \approx cL(m, -1)$ then $U_2 \approx cL(m, 1)$ and there exists a smooth 4–manifold $\widetilde{X}$ with a smooth $\mathbb{Z}_m$–action such that $\widetilde{X}/\mathbb{Z}_m \cong X$ as smooth orbifolds.*

This paper is organized as follows. In Section 2, we make further remarks on our main result, Theorem 4.1, and its backgrounds in comparison with other works. In Section 3, we recall basic definitions and results concerning the moduli space of instantons over 4–manifolds with cylindrical ends. Here we give a setting of moduli space of instantons and recall basic facts on counting reducible instantons, virtual dimension of the moduli space, Uhlenbeck compactness and Chern–Simons invariants. In particular, to describe the ends of the moduli space, we discuss the moduli space of $\mathbb{Z}_m$–invariant instantons over the cylinder $\mathbb{R} \times S^3$, C Taubes' gluing result and discuss a holonomy perturbation to deform the ends of the moduli space. In Section 4, we give the full statement of our main result, Theorem 4.1, and give a proof of the theorem. We also discuss several examples to demonstrate necessity of the conditions in the theorem. In Section 5, we prove Theorem 1.5 on a cyclic branched covering of a negative-definite smooth 4–orbifold. In fact, we deduce Theorem 5.1 as a corollary of Theorem 4.1, which gives a cyclic branched covering of a smooth negative-definite 4–orbifold with boundary and with isolated singular points whose neighborhoods are homeomorphic to the cones over lens spaces $L(a_i, b_i)$. The branched covering space is an orbifold itself which locally uniformizes only a pair of neighborhoods homeomorphic to the cones over lens spaces $L(m, 1)$ and $L(m, -1)$.

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2 Further remarks on the main theorem and background

In this section, we make further remarks on the main theorem and its background. In addition to Problem 1.1, the main theorem partially answers the following problem:

Problem 2.1 *If there exists a negative-definite cobordism between a disjoint union Y of several lens spaces $L(a_i, b_i)$ and a rational homology sphere Σ, what kind of constraint for the homomorphism $i_*\colon H_1(Y; \mathbb{Z}) \to H_1(W; \mathbb{Z})$ induced by the inclusion $i\colon Y \hookrightarrow W$ can be expected?*

Theorem 4.1 tells us that the pair $L(m, 1) \# L(m, -1)$ is related through a $U(1)$ flat connection on the cobordism W, which implies the existence of a homomorphism $H_1(W; \mathbb{Z}) \to U(1)$ whose restrictions are the standard inclusion up to complex conjugation on $L(m, \pm 1)$ and trivial on the other components of the boundary.

One typical example is the plumbed homology 3–sphere $\Sigma = \Sigma(\Gamma)$ associated to a graph Γ by taking the connected sum $W = ([0,1] \times L(m,1)) \# P(\Gamma)$ of the cylinder $[0,1] \times L(m,1)$ and the plumbing $P(\Gamma)$ with negative-definite intersection form. In this case, the homomorphism $\chi\colon H_1(W;\mathbb{Z}) = H_1(L(m,1);\mathbb{Z}) \to U(1)$ is given by the inclusion $\mathbb{Z}_m \hookrightarrow U(1)$.

In [14], Furuta developed an abstract perturbation method to prove Donaldson's theorem for intersection forms on closed smooth definite 4–manifolds X with $H_1(X;\mathbb{Z}) = \{0\}$. As an application, he strengthened Theorem 2.1 of [16], which made an assumption on the fundamental group of the 4–manifold (orbifold), to prove Theorem 6.1 of [14], which only makes an assumption on the first homology and proved an existence of an abelian representation. One major difference between our Theorem 4.1 and [14, Theorem 6.1] is that our theorem allows for a general rational homology 3–spheres as components of Y which are not necessarily spherical space forms. One consequence of this is that we cannot apply Theorem 2.1 of [16] to prove our theorem by simply coning off the boundary components to obtain closed orbifold.

Theorem 4.1 states that even if there is another rational homology 3–sphere Σ on the boundary, the pair $L(m,1) \# L(m,-1)$ appears provided m is greater than a positive number $N(\Sigma)$ associated to Σ. In fact, we may choose any number $N(\Sigma)$ in the statement of Theorem 4.1 such that $1/N(\Sigma)$ is less than or equal to the minimum Chern–Simons invariant $\tau(\Sigma)$ of Σ, and if Σ is the Brieskorn homology 3–sphere $\Sigma(p,q,r)$ then we can take $N(\Sigma) = pqr$ [15; 12]. By using a method developed by P Kirk and E Klassen [22], the Chern–Simons invariant can be calculated for many 3–manifolds including graph manifolds. In fact, the Chern–Simons invariants of a large class of manifolds including Seifert fibered 3–manifolds are explicitly calculated by D Auckly [4]. Here we mention the following remarks:

Remark 2.2 (1) Theorem 4.1 does not tell us whether $\chi \circ (i_j)_*$ for $j \in \{1, i\}$ is the inclusion $\rho\colon \mathbb{Z}_m \hookrightarrow U(1)$ or its complex conjugate. We shall discuss this issue and uniformization of 4–orbifolds with more than two isolated singular points elsewhere.

(2) The condition $m > N(\Sigma)$ cannot be removed. In fact, the singular D^2–bundle $\widehat{W} = \Sigma \times_{S^1} D^2$ associated with the Seifert fibration $\Sigma \to S^2$ of negative rational Euler number with the cone over lens spaces removed would give a counterexample.

(3) Theorem 1.2 is a special case of the main result, Theorem 4.1, in which we treat the case that the boundary components are not necessarily lens spaces but "positive" (or "negative") spherical space forms. In this case we obtain a not necessarily reducible flat connection relating the pair of positive and negative spherical space forms.

(4) Theorems 1.2 and 4.1 can be extended to the case of lens spaces $L(a,b)$ with a sufficiently greater than b in place of $L(m,1)$. We shall treat this elsewhere.

To prove an existence of a (*reducible*) flat connection over an oriented smooth 4–manifold with cylindrical ends under a certain condition on homology, we make use of the "sliding end" or "bubbling" phenomenon of SU(2)–instantons. The author does not know how to deduce our theorem from usual applications of the Seiberg–Witten theory or Heegaard Floer theory since we do not know how to extract information on flat SU(2) connections in these theories at least as directly as in the Donaldson theory.

There are several areas in which the Donaldson theory works effectively while the Heegaard Floer theory or Seiberg–Witten theory alone would be difficult to approach. It is known that the Seifert fibered integral homology 3–spheres $\Sigma(p,q,pqk-1)$ for $k=1,2,\dots$ are linearly independent in the homology cobordism group θ^H_3 of homology 3–spheres by a result of Furuta [15] and R Fintushel and R Stern [12]. For the $\mathbb{Z}/2$–homology cobordism group, M Hedden and Kirk [18] showed that $d/(d^k-1)$ surgeries on the right-handed (p,q) torus knot are linearly independent. Hedden and Kirk [19] also showed that the Whitehead doubles of $(2,2^n-1)$ torus knots are independent in the smooth knot concordance group. This result is generalized by J Pinzón-Caicedo [27] to infinite collections of satellite knots. Note that positivity of the R–invariant introduced by Fintushel and Stern in [10] as the virtual dimension of the moduli space of instantons plays a crucial role in these arguments. On Ozsváth and Szabó's Heegaard Floer theory, S Kim and C Livingston [21] proved that $\mathbb{Z}^\infty \subset \operatorname{Coker}\big(\bigoplus_p \theta^3_{\mathbb{Z}[1/p]} \to \theta^3_{\mathbb{Q}}\big)$ by using the d–invariant. Recently, M Stoffregen [32] investigated behaviors of Manolescu invariants α,β,γ on Seiberg–Witten Floer homology under connected sums and used also recent result on Heegaard Floer homology to show that $\Sigma(p,2p-1,2p+1)$ for odd p with $p\geq 3$ are linearly independent in θ^H_3. However, this technique does not detect the family $\Sigma(p,q,pqk-1)$.

Our result is in a sense opposite to that of A Casson and C Gordon [6], P Gilmer and Livingston [17] and Fintushel and Stern [11]. In fact, their arguments give a necessary condition for characters $\chi_i\colon H_1(Y_i;\mathbb{Z}) \to \mathbb{Z}_m \subset U(1)$ of spherical space forms Y_i to extend over a rational homology cobordism W, which implies a necessary condition for an existence of a $\mathbb{Z}_m$–covering $\widetilde{W} \to W$ extending the $\mathbb{Z}_m$–coverings $\widetilde{Y}_i \to Y_i$ associated to the character ρ_i. On the other hand, our result gives a sufficient condition for Y_i and their negative-definite cobordism W with a character $\chi_1\colon H_1(Y_1;\mathbb{Z}) \to \mathbb{Z}_m \subset U(1)$ to extend over $H_1(W;\mathbb{Z})$, which implies a sufficient condition for the existence of $\mathbb{Z}_m$–covering $\widetilde{W} \to W$ extending $\widetilde{Y}_1 \to Y_1$.

In the Casson–Gordon argument, D Ruberman [28] extended their result for spherical space forms to rational space forms, that is, rational homology 3–spheres divided by finite group actions, using the moduli space of instantons on 4–manifolds with cylindrical ends. To extend our result for lens spaces to one for general rational homology 3–spheres such as rational space forms, we need to know the moduli space on the cylinder over rational space forms which corresponds to the end of the moduli space. This deserves further study.

For a simply connected smooth 4–manifold X_0 with boundary the Poincaré homology 3–sphere $\Sigma(2,3,5)$ and negative-definite intersection form $Q_{X_0} = E_8$, N Anvari [2] gave a necessary condition for a free $\mathbb{Z}_p$–action on $\Sigma(2,3,5)$ to extend smoothly to X_0 by using the isotropy data of isolated fixed points. This result is proved by considering the $\mathbb{Z}_p$–equivariant moduli space of instantons over the 4–manifold with a cylindrical end. Furthermore, Anvari and I Hambleton [3] gave infinite families of Brieskorn homology 3–spheres with free $\mathbb{Z}_p$–actions which can be extended locally linearly but cannot smoothly to smooth contractible 4–manifolds. To study relationships with their results, we will consider a negative-definite cobordism between the quotient $\Sigma(a_1,\dots,a_n)/\mathbb{Z}_p$ and a disjoint union of lens spaces.

H Sasahira [29] defined Floer homology for lens spaces by treating problems on reducible connections via Furuta's method using twisted Dirac operators and derived a gluing formula by Fukaya's method using loops. To give an interpretation of our arguments in terms of Floer homology for lens spaces will be a subject of our future work.

3 Preliminaries on instantons over 4–manifolds with cylindrical ends

In this section, we recall basic definitions and results on analysis concerning instantons over 4–manifolds with cylindrical ends. We discuss a new feature which was not treated in [16], a perturbation of the moduli space called holonomy perturbation introduced by Donaldson [7], which we use to prove a key theorem, discussed in Section 4.

3.1 Instantons over 4–manifolds with cylindrical ends

Let Y be a disjoint union of a finite number of closed oriented 3–manifolds Y_i indexed by $i \in I$ and W be a compact oriented smooth 4–manifold with boundary Y. Let X be a smooth oriented 4–manifold obtained by gluing W and half-infinite cylinders $\mathbb{R}_+ \times Y_i$ with product orientation along the boundaries $\{0\} \times Y_i$, where $\mathbb{R}_+ = \{t \in \mathbb{R} \mid t \geq 0\}$. In what follows we simply describe such a manifold X as a "4–manifold with cylindrical ends". Let (P, α) be an adapted SU(2)–bundle over X, that is, a pair of a principal SU(2)–bundle P over X and a set $\alpha = \{\alpha_i\}$ of flat SU(2)–connections α_i on $P|_{\mathbb{R}_+ \times Y_i}$. We may assume that α_i is a pullback $p_i^* \beta_i$ by the projection $p_i \colon \mathbb{R}_+ \times Y_i \to Y_i$ of a flat connection β_i on an SU(2)–bundle Q_i over Y_i under an identification $P|_{\mathbb{R}_+ \times Y_i} \cong p_i^* Q_i$ using parallel transport in the direction $\mathbb{R}_+$ with respect to the connection α_i. We do not distinguish α_i and β_i in the sequel. Two adapted bundles (P, α) and (P', α') are equivalent if and only if there exists a bundle isomorphism $h \colon P \to P'$ such that $\alpha_i|_{[r,\infty) \times Y_i} = h^* \alpha_i'|_{[r,\infty) \times Y_i}$ for some $r > 0$ for any $i \in I$. Let θ be the trivial flat connection (the product connection) on the product bundle $X \times \mathrm{SU}(2) \to X$. Then the trivial flat connection θ_i on the restriction $(X \times \mathrm{SU}(2))|_{\mathbb{R}_+ \times Y_i}$ over each ends $\mathbb{R}_+ \times Y_i \subset X$ is induced by θ and we call the adapted bundle isomorphic to $(X \times \mathrm{SU}(2), \{\theta_i\})$ the trivial adapted bundle. The second Chern number of (P, α) is defined to be

$$c_2(P, \alpha) = \frac{1}{8\pi^2} \int_X \mathrm{Tr}(F_A \wedge F_A),$$

where A is a smooth connection on P with $A|_{[r,\infty) \times Y_i} = \alpha_i|_{[r,\infty) \times Y_i}$ for some $r > 0$ for any $i \in I$, and it does not depend on a choice of such connections A. If we fix SU(2)–bundles Q_i over Y_i and flat connections β_i on Q_i, then the isomorphism classes of adapted bundles (P, α) with a bundle isomorphism $h_i \colon P|_{\mathbb{R}_+ \times Y_i} \cong p_i^* Q_i$ satisfying $\alpha_i = h_i^* \beta_i$ are classified by $c_2(P, \alpha)$ and if (P', α) is another such bundle then $c_2(P', \alpha) - c_2(P, \alpha) \in \mathbb{Z}$.

Now we identify the gauge equivalence classes of the flat connection α_i with the conjugacy classes of representation of $\pi_1(Y_i)$ induced by the holonomy of α_i and use the same symbol α_i. Let $\operatorname{ad} \alpha_i$ be the adjoint representation of α_i and place the non-degeneracy condition $H^1(Y_i; \operatorname{ad} \alpha_i) = \{0\}$ on each flat connection α_i. Note that if Y_i is a spherical space form S^3/G for some finite group G acting on S^3 with a flat connection α_i corresponding to a representation $\rho \colon G \to \mathrm{SU}(2)$, then $H^1(S^3/G; \operatorname{ad} \rho) = (H^1(S^3; \mathbb{R}) \otimes \mathfrak{su}(2))^{\operatorname{ad} \rho} = \{0\}$, and that if Y_i is a rational homology 3–sphere Σ and

α_i is a trivial flat connection θ, then $H^1(\Sigma; \mathrm{ad}\,\theta) = H^1(\Sigma;\mathbb{R}) \otimes \mathfrak{su}(2) = \{0\}$, so they satisfy the non-degeneracy condition.

Fix a Riemannian metric g_i on each Y_i and choose a metric g on X which matches the product metrics over ends $g|_{\mathbb{R}_+\times Y_i} = dt^2 + g_i$. Let D_A be the deformation operator associated to a connection A on P which restricts to α_i on $[r,\infty)\times Y_i$ for each i for some $r > 0$,

$$D_A = -d_A^* \oplus d_A^+ \colon \Omega^1(\mathfrak{g}_P) \to \Omega^0(\mathfrak{g}_P) \oplus \Omega^2_+(\mathfrak{g}_P),$$

and let L_{α_i} be the selfadjoint elliptic operator over Y_i such that

$$D_A = \frac{d}{dt} + L_{\alpha_i}$$

over $[r,\infty)\times Y_i$. Let $\delta > 0$ be a positive real number less than the absolute values of the eigenvalues of the selfadjoint operator L_{α_i} for any $i \in I$. Fix a smooth function $w\colon X \to \mathbb{R}$ satisfying $w \geq \frac{1}{2}$ over X and $w(t,y) = e^{\delta t}$ on each end $(t,y) \in \mathbb{R}_+ \times Y_i$ and define a weighted Sobolev norm $\|\phi\|_{L^{2,\delta}_k} = \|w\phi\|_{L^2_k}$ for an integer $k \geq 0$. Fix an integer $l > 2$. Take a reference smooth connection A_0 on P with $A_0|_{\mathbb{R}_+\times Y_i} = \alpha_i$. Let $\mathcal{A}(P,\alpha)$ be the space of $L^{2,\delta}_l$–connections on P and $\mathcal{G}(P,\alpha)$ be the group of $L^{2,\delta}_{l+1,\mathrm{loc}}$–gauge transformations on P defined as follows:

$$\mathcal{A}(P,\alpha) = \{A_0 + a \mid a \in L^{2,\delta}_l(\textstyle\bigwedge^1(\mathfrak{g}_P))\},$$
$$\mathcal{G}(P,\alpha) = \{g \in \Gamma(P \times_{\mathrm{Ad}} G) \mid \nabla_{A_0} g \in L^{2,\delta}_l(\textstyle\bigwedge^1(P \times_{\mathrm{Ad}} G))\}.$$

Then $\mathcal{G}(P,\alpha)$ is a Hilbert Lie group acting smoothly on the Hilbert manifold $\mathcal{A}(P,\alpha)$ by $g\cdot A = A - d_A g\cdot g^{-1}$ for $A \in \mathcal{A}(P,\alpha)$, $g \in \mathcal{G}(P,\alpha)$ to define the quotient topological space $\mathcal{B}(P,\alpha) = \mathcal{A}(P,\alpha)/\mathcal{G}(P,\alpha)$. Now the condition that $\nabla_{A_0} g \in L^{2,\delta}_l$ with weighted norm forces any gauge transformation $g \in \mathcal{G}(P,\alpha)$ to have a uniform limit along each cylinder $\mathbb{R}_+ \times Y_i$. Let $\mathrm{ev}\colon \mathcal{G}(P,\alpha) \to \Gamma_\alpha$ be the evaluation homomorphism to the stabilizer $\Gamma_\alpha = \prod_{i\in I} \Gamma_{\alpha_i}$ of α and define the based gauge group $\mathcal{G}_0(P,\alpha)$ to be the kernel of the map ev. Then $\mathcal{G}_0(P,\alpha)$ acts on $\mathcal{A}(P,\alpha)$ freely and the quotient

$$\widetilde{\mathcal{B}}(P,\alpha) = \mathcal{A}(P,\alpha)/\mathcal{G}_0(P,\alpha)$$

is a Hilbert manifold with local model

$$T_{[A]}\widetilde{\mathcal{B}}(P,\alpha) = \{A + a \mid a \in L^{2,\delta}_l(\textstyle\bigwedge^1(\mathfrak{g}_P)),\ d_A^{*,\delta} a = 0\},$$

where $d_A^{*,\delta} = w^{-2} \cdot d_A^* \cdot w^2$. The full quotient $\mathcal{B}(P,\alpha)$ is obtained by dividing $\widetilde{\mathcal{B}}(P,\alpha)$ with the extra symmetry $\mathcal{G}(P,\alpha)/\mathcal{G}_0(P,\alpha) \cong \Gamma_\alpha$,

$$p\colon \widetilde{\mathcal{B}}(P,\alpha) \to \widetilde{\mathcal{B}}(P,\alpha)/\Gamma_\alpha = \mathcal{B}(P,\alpha).$$

A connection A is called an anti-self-dual (ASD) connection or simply an "instanton" if and only if the curvature F_A is anti-self-dual, $F_A^+ = \frac{1}{2}(F_A + \star F_A) = 0$, where $\star$ is the Hodge star operator with respect to the metric g. Then we have a $\mathcal{G}(P,\alpha)$–equivariant smooth map

$$\Phi\colon \mathcal{A}(P,\alpha) \to L^{2,\delta}_{l-1}\bigl(\textstyle\bigwedge^2_+(\mathfrak{g}_P)\bigr), \qquad \Phi(A) = F_A^+ \quad \text{for } A \in \mathcal{A}(P,\alpha).$$

Let $\mathcal{M}(P,\alpha)$ be the moduli space of instantons on (P,α),

$$\mathcal{M}(P,\alpha) = \{A \in \mathcal{A}(P,\alpha) \mid F_A^+ = 0\}/\mathcal{G}(P,\alpha),$$

which can be seen as the quotient of the framed moduli space $\widetilde{\mathcal{M}}(P,\alpha) = p^{-1}(\mathcal{M}(P,\alpha))$ divided by the free Γ_α–action. By non-degeneracy condition $H^1(Y_i; \operatorname{ad}\alpha_i) = \{0\}$, any instanton A over (P,α) decays exponentially to the flat connection α_i up to gauge transformation toward the ends $\mathbb{R}_+ \times Y_i$. In particular, for any connection B which is equal to α_i over ends, since $|F_A|$ decays exponentially along ends, $\|F_A\|^2_{L^2(X)} = \int_X |F_A|^2 \operatorname{vol}_X = \int_X |F_B|^2 \operatorname{vol}_X = 8\pi^2 c_2(P,\alpha)$.

3.2 Enumeration of reducible instantons for SU(2)

If a connection A on the adapted SU(2)–bundle (P,α) has the isotropy subgroup $\mathcal{G}(P,\alpha)_A$ of the gauge group $\mathcal{G}(P,\alpha)$ isomorphic to $\{\pm 1\}$ then A is said to be *irreducible*; otherwise, A is *reducible*. In particular, *reducible instantons*, which are reducible connections on (P,α) whose curvature 2–forms are anti-self-dual, correspond to the singular points of the moduli space $\mathcal{M}(P,\alpha)$.

For later discussion, we need to count reducible instantons for adapted SU(2)–bundles over 4–manifolds with cylindrical ends. Hedden and Kirk [18] described enumeration of reducible instantons in the case that the gauge group is SO(3).

Proposition 3.1 *Let (P,α) be an adapted* SU(2)*–bundle over a* 4*–manifold X with cylindrical ends $\mathbb{R}_+ \times Y_j$ for $j \in I$, where Y_j are rational homology* 3*–spheres. Suppose that the* SU(2) *flat connections α_j are all reducible, so that the corresponding representation $\alpha_j\colon \pi_1(Y_j) \to U(1)$ is determined up to complex conjugation. Then* *there exists a one-to-one correspondence between the set of all reducible instantons $\mathcal{M}_{\mathrm{red}}(P,\alpha) \subset \mathcal{M}(P,\alpha)$ on (P,α) and the set*

$$C(P,\alpha) = \{e \in H^2(X;\mathbb{Z}) \mid e^2 = -c_2(P,\alpha),\ i_j^* e = \pm e(L_j) \text{ for } j \in I\}/\{\pm 1\},$$

where $i_j\colon Y_j \hookrightarrow X$ is the inclusion and L_j is the flat S^1–bundle over Y_j associated with α_j. Note that $e(L_j)$ is determined up to sign because the representation $\alpha_j\colon \pi_1(Y_j) \to U(1)$ is determined up to complex conjugation.

Proof The proof goes essentially the same as in the SO(3) case discussed by Hedden and Kirk [18]. In our SU(2) case, the primary obstruction of extending gauge transformations from the boundary Y to X gives an element of $H^2\big(X,Y;\pi_1(\mathrm{SU}(2))\big) = \{0\}$. □

3.3 Perturbation on a compact part of the moduli space

In this subsection, we state the following proposition, used in the discussion below. Let X be a smooth oriented 4–manifold with cylindrical ends and (P,α) be an adapted SU(2)–bundle over X. Fix a Riemannian metric on X with product ends.

Let l be an integer satisfying $l > 2$ and consider the space $\mathcal{B}(P,\alpha)$ of the gauge equivalence classes of $L^{2,\delta}_l$–connections on (P,α). We let $\mathcal{B}^*(P,\alpha)$ be the irreducible part of $\mathcal{B}(P,\alpha)$,

$$\mathcal{B}^*(P,\alpha) = \mathcal{A}^*(P,\alpha)/\mathcal{G}(P,\alpha), \quad \mathcal{A}^*(P,\alpha) = \{A \in \mathcal{A}(P,\alpha) \mid \mathcal{G}(P,\alpha)_A = \{\pm 1\}\},$$

and let $\widetilde{\mathcal{B}}^*(P,\alpha) = p^{-1}(\mathcal{B}^*(P,\alpha))$, where $\mathcal{G}(P,\alpha)_A$ is the stabilizer of A in $\mathcal{G}(P,\alpha)$.

Let τ be a $\mathcal{G}(P,\alpha)$–equivariant smooth map $\mathcal{A}(P,\alpha) \to L^{2,\delta}_l\big(\bigwedge^2_+(\mathfrak{g}_P)\big)$ followed by compact embedding $L^{2,\delta}_l \subset L^{2,\delta}_{l-1}$, and consider the $\mathcal{G}(P,\alpha)$–equivariant map defined by the formula

$$\Phi_\tau(A) = F^+_A + \tau(A) \quad \text{for } A \in \mathcal{A}(P,\alpha).$$

Its derivative

$$d^+_A + (D\tau)_A\colon \operatorname{Ker} d^{*,\delta}_A \cap L^{2,\delta}_l\big(\textstyle\bigwedge^1(\mathfrak{g}_P)\big) \to L^{2,\delta}_{l-1}\big(\textstyle\bigwedge^2_+(\mathfrak{g}_P)\big)$$

is a Fredholm operator for each $A \in \mathcal{A}(P,\alpha)$.

Then we have the following:

Proposition 3.2 (see Furuta [16, Proposition 4.2]) *Let X be a 4–manifold with cylindrical ends and with $b_1(X)=0$ and $b_2^+(X)=0$, and let $\mathcal{M}(P,\alpha)$ be the moduli space of instantons on an adapted* $\mathrm{SU}(2)$*–bundle (P,α). Suppose there is a compact set K in $\mathcal{M}(P,\alpha)$ such that*

$$(D\Phi_\tau)_A\colon \operatorname{Ker} d_A^{*,\delta}\cap L_l^{2,\delta}\big(\textstyle\bigwedge^1(\mathfrak{g}_P)\big)\to L_{l-1}^{2,\delta}\big(\textstyle\bigwedge_+^2(\mathfrak{g}_P)\big)$$

is surjective for any $A\in p^{-1}(\mathcal{M}(P,\alpha)-K)$. Then there exists an open neighborhood U of K in $\mathcal{B}(P,\alpha)$ and a smooth $\mathcal{G}(P,\alpha)$–equivariant map $\sigma\colon \mathcal{A}(P,\alpha)\to L_l^{2,\delta}\big(\bigwedge_+^2(\mathfrak{g}_P)\big)$ supported in $p^{-1}(U)$ such that the following properties hold:

(1) *For any $[A]\in\mathcal{M}^{\sigma+\tau}(P,\alpha):=p((\Phi_\tau+\sigma)^{-1}(\{0\}))$, the derivative of $\Phi_\tau+\sigma$ at A,*

$$D(\Phi_\tau+\sigma)_A\colon \operatorname{Ker} d_A^{*,\delta}\cap L_l^{2,\delta}\big(\textstyle\bigwedge^1(\mathfrak{g}_P)\big)\to L_{l-1}^{2,\delta}\big(\textstyle\bigwedge_+^2(\mathfrak{g}_P)\big),$$

is surjective.

(2) *The irreducible part $\mathcal{M}^{*\sigma+\tau}(P,\alpha)=\mathcal{M}^{\sigma+\tau}(P,\alpha)\cap\mathcal{B}^*(P,\alpha)$ is a finite-dimensional smooth manifold and the tangent space $T_{[A]}\widetilde{\mathcal{M}}^{*\sigma+\tau}(P,\alpha)$ of the framed moduli space $\widetilde{\mathcal{M}}^{*\sigma+\tau}(P,\alpha)=\widetilde{\mathcal{M}}^{\sigma+\tau}(P,\alpha)\cap\widetilde{\mathcal{B}}^*(P,\alpha)$ is isomorphic to the first cohomology,*

$$H_A^1=\operatorname{Ker} d_A^+/\operatorname{Im} d_A\cong \operatorname{Ker} d_A^+\cap\operatorname{Ker} d_A^{*,\delta}\subset L_l^{2,\delta},$$

of the elliptic complex

$$0\to L_{l+1}^{2,\delta}\big(\textstyle\bigwedge^0(\mathfrak{g}_P)\big)\xrightarrow{d_A} L_l^{2,\delta}\big(\textstyle\bigwedge^1(\mathfrak{g}_P)\big)\xrightarrow{d_A^+} L_{l-1}^{2,\delta}\big(\textstyle\bigwedge_+^2(\mathfrak{g}_P)\big)\to 0.$$

(3) *For any $[A]\in\mathcal{M}_{\mathrm{red}}(P,\alpha)$ we have $\sigma(A)=0$, and there exists a neighborhood N of $[A]$ in $\mathcal{M}^{\sigma+\tau}(P,\alpha)$ with a homeomorphism $\varphi\colon N\to c\mathbb{C}P^d$ to the cone $c\mathbb{C}P^d$ over $\mathbb{C}P^d$ for some d with $\varphi([A])$ the vertex of $c\mathbb{C}P^d$ whose restriction*

$$\varphi|_{N-\{[A]\}}\colon N-\{[A]\}\to c\mathbb{C}P^d-\{h([A])\}$$

is a C^∞–diffeomorphism.

(4) *$\mathcal{M}^{\sigma+\tau}(P,\alpha)\cap\overline{U}$ is compact.*

Proof Since the second cohomology H_A^2 of the complex has finite dimension at each instanton A, a perturbation can be constructed over a local slice at A. Since $\mathcal{A}(P,\alpha)$ is a Hilbert manifold, we can use a C^∞ cut-off function $\beta\colon \mathcal{B}(P,\alpha)\to\mathbb{R}$ defined by using the $L_l^{2,\delta}$–norm $\|a\|_{L_l^{2,\delta}}$, so that we can take a smooth partition of unity subordinated to an open covering of the compact subset $K \subset \mathcal{M}(P,\alpha) \subset \mathcal{B}(P,\alpha)$ consisting of local slices at each A to construct a perturbation $\sigma\colon \mathcal{A}(P,\alpha) \to L_l^{2,\delta}\big(\bigwedge_+^2(\mathfrak{g}_P)\big)$. This is a standard argument; see for example [24] and for the $L_l^{2,\delta}$–norm see [8]. □

3.4 Equivariant instantons over $\mathbb{R} \times S^3$

Next we consider equivariant instantons over $\mathbb{R} \times S^3$ for the standard sphere S^3. In what follows, we use an identification $\mathbb{C}^2 \cong \mathbb{H}$ given by $(z,w) \mapsto z + jw$, so that the left multiplication $x \mapsto ax$ for $x \in \mathbb{H}$ with $a \in \mathbb{H}$ is $\mathbb{C}$–linear, to identify $\mathrm{SU}(2) = \mathrm{Sp}(1)$. We fix an orientation $o(\mathbb{H}) = 1 \wedge i \wedge j \wedge k$ of $\mathbb{H}$. By noting that

$$x = x_0 + ix_1 + jx_2 + kx_3 = x_0 + ix_1 + j(x_2 - ix_3) \in \mathbb{H},$$

we see that the orientation $o(\mathbb{H})$ is opposite to that induced from the standard one of $\mathbb{C}$ given by $o(\mathbb{C}) = 1 \wedge i$ and so the identification $\mathbb{C} \oplus \mathbb{C} \cong \mathbb{H}$ defined by $(z,w) \mapsto z + j\overline{w}$ gives an orientation-preserving diffeomorphism.

Definition 3.3 Let $(\widetilde{Q}_0, \theta_0)$ be an adapted SU(2)–bundle over the cylinder $\mathbb{R} \times S^3$ defined by $\widetilde{Q}_0 = \mathbb{R} \times S^3 \times \mathrm{Sp}(1)$ with a pair $\theta_0 = \{\theta_0^-, \theta_0^+\}$ consisting of the trivial flat connection $\theta_0^\pm$ over $\pm(r,\infty) \times S^3$ for some $r > 0$ with respect to the trivialization

$$\tau_\pm\colon \widetilde{Q}_0|_{\pm(r,\infty)\times S^3} = \pm(r,\infty) \times S^3 \times \mathrm{Sp}(1) \to \pm(r,\infty) \times S^3 \times \mathrm{Sp}(1),$$
$$\tau_\pm(t,y,q) = (t,y,y^{(\pm 1+1)/2}q) \quad \text{for } (t,y,q) \in \pm(r,\infty) \times S^3 \times \mathrm{Sp}(1).$$

Then $(\widetilde{Q}_0, \theta_0)$ admits an SU(2)×SU(2)–action

$$g \cdot (t,y,q) = (t, e_+ y e_-^{-1}, e_- q)$$

for

$$(t,y,q) \in \mathbb{R} \times S^3 \times \mathrm{Sp}(1), \quad g = (e_+, e_-) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)$$

to define an (SU(2)×SU(2))–equivariant adapted SU(2)–bundle over $\mathbb{R} \times S^3$ with $c_2(\widetilde{Q}_0, \theta_0) = 1$. Note that under the trivialization $\tau_\pm$, the SU(2)×SU(2)–action has the form

$$g \cdot (t,y,q) = (t, e_+ y e_-^{-1}, \alpha_0^\pm(g) q), \quad \text{where } \alpha_0^\pm(g) = e_\pm,$$

for

$$(t,y,q) \in \pm(r,\infty) \times S^3 \times \mathrm{Sp}(1) \quad \text{and} \quad g = (e_+, e_-) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1).$$

If the action of a finite subgroup $\widetilde{G}$ of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on $S^3 \subset \mathbb{H}$ is free, we call $\widetilde{G}$ a pseudofree subgroup of $\mathrm{SU}(2) \times \mathrm{SU}(2)$. Consider the quotient bundle $Q_0 = \widetilde{Q}_0/\widetilde{G}$

over $\mathbb{R} \times S^3/\widetilde{G}$ with the flat connection $\alpha_0^{\pm}$ on $Q_0|_{\pm(r,\infty)\times S^3/\widetilde{G}}$ induced by the trivial flat connection $\theta_0^{\pm}$ on $\pm(r,\infty) \times S^3 \times \mathrm{Sp}(1)$. The adapted bundle (Q_0, α_0) with $\alpha_0 = \{\alpha_0^-, \alpha_0^+\}$ and $c_2(Q_0, \alpha_0) = 1/|\widetilde{G}|$ is called the adapted SU(2)–bundle associated with a pseudofree subgroup $\widetilde{G}$ of $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

Remark 3.4 Let $Y = S^3/G$ be the 3–dimensional spherical space form obtained by taking a quotient of S^3 by a finite group $G \subset \mathrm{SO}(4)$ acting freely on $S^3 \subset \mathbb{R}^4$ by matrix multiplication. The action $(e_+, e_-) \cdot x = e_+ x e_-^{-1}$ for $x \in \mathbb{H}$ of $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ on $\mathbb{H}$ induces the projection $\pi\colon \mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SU}(2) \times_{\{\pm 1\}} \mathrm{SU}(2) \cong \mathrm{SO}(4)$. If we can take a lift $\widetilde{G}$ of G, which means that $\widetilde{G}$ is the image of an injective homomorphism $\iota\colon G \to \mathrm{SU}(2) \times \mathrm{SU}(2)$ with $\pi \circ \iota = \mathrm{id}_{\mathrm{SO}(4)}$, then $\widetilde{G}$ is a pseudofree subgroup of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ acting on $S^3 \subset \mathbb{H}$ as above.

Then we have the following proposition:

Proposition 3.5 (see Furuta [16, Lemmas 5.1 and 5.2]) *Let (Q_0, α_0) be the adapted SU(2)–bundle over $\mathbb{R} \times S^3/\widetilde{G}$ associated to a pseudofree subgroup $\widetilde{G}$ of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ as in Definition 3.3. Then the moduli space $\mathcal{M}(Q_0, \alpha_0)$ is regular and diffeomorphic to $\mathbb{R}$.*

Proof Let $f\colon S^4 - \{0, \infty\} \to \mathbb{R} \times S^3$ be the $\widetilde{G}$–equivariant conformal equivalence. Then $\widetilde{G}$–invariant instantons on the $\widetilde{G}$–equivariant adapted bundle $(\widetilde{Q}_0, \theta_0)$ over $\mathbb{R} \times S^3$ with finite energy $c_2(\widetilde{Q}_0, \theta_0) = 1$ have the pullback over $S^4 - \{0, \infty\}$ and extends across $\{0, \infty\}$ uniquely by Uhlenbeck's removable singularity theorem [34] to obtain an $\widetilde{G}$–invariant instantons on a $\widetilde{G}$–equivariant SU(2)–bundle $\widetilde{Q}$ over S^4 with $c_2(\widetilde{Q}) = 1$. This correspondence induces a diffeomorphism $\mathcal{M}_{\widetilde{G}}(\widetilde{Q}_0, \theta_0) \to \mathcal{M}_{\widetilde{G}}(\widetilde{Q})$ between moduli spaces of $\widetilde{G}$–invariant instantons. Then take the quotient by $\widetilde{G}$ to obtain a diffeomorphism $\mathcal{M}(Q_0, \alpha) \to \mathcal{M}(Q)$ and the assertion follows from the fact that $\mathcal{M}(Q)$ is regular and diffeomorphic to $\mathbb{R}$ by Lemmas 5.1 and 5.2 in [16]. □

Now we introduce a notion of positivity and negativity of spherical space forms, which was introduced by Furuta in the context of orbifolds [16].

Definition 3.6 (see [16, Definition 2.1]) A closed 3–manifold Y is called a positive (resp. negative) spherical space form — positive (resp. negative) for short — if $Y \cong S^3/G$ with a finite group $G \subset \mathrm{Sp}(1)$ acting on $S^3 \subset \mathbb{H}$ by the left (resp. right) multiplication,

$$g \cdot y = gy \quad (\text{resp. } g \cdot y = yg^{-1}) \qquad \text{for } y \in S^3 \subset \mathbb{H} \text{ and } g \in G \subset \mathrm{Sp}(1).$$

Definition 3.7 (see [16, Section 5, before Lemma 5.3]) If $Y = S^3/G$ is a positive (resp. negative) spherical space form and G is the corresponding subgroup of $\mathrm{Sp}(1)$, then G can be embedded into the group $\mathrm{SO}(4) = \mathrm{Sp}(1) \times_{\{\pm 1\}} \mathrm{Sp}(1)$ and we regard G as a subgroup of $\mathrm{SO}(4)$ by the identification

$$G \subset \mathrm{Sp}(1) \subset \mathrm{Sp}(1) \times_{\{\pm 1\}} \{\pm 1\} \subset \mathrm{SO}(4) = \mathrm{Sp}(1) \times_{\{\pm 1\}} \mathrm{Sp}(1)$$

or

$$G \subset \mathrm{Sp}(1) \subset \{\pm 1\} \times_{\{\pm 1\}} \mathrm{Sp}(1) \subset \mathrm{SO}(4) = \mathrm{Sp}(1) \times_{\{\pm 1\}} \mathrm{Sp}(1),$$

respectively, and call G a positive (resp. negative) subgroup of $\mathrm{SO}(4)$. In this case, we can take a lift

$$\iota_{\mathrm{pos}} \colon G \subset \mathrm{Sp}(1) \to \widetilde{G} = G \times \{1\} \subset \mathrm{Sp}(1) \times \mathrm{Sp}(1)$$

or

$$\iota_{\mathrm{neg}} \colon G \subset \mathrm{Sp}(1) \to \widetilde{G} = \{1\} \times G \subset \mathrm{Sp}(1) \times \mathrm{Sp}(1),$$

respectively, to define a G–action on the adapted bundle $(\widetilde{Q}_0, \theta_0)$ by composing with ι_{pos} (resp. ι_{neg}). We denote the representation corresponding to the inclusion $G \subset \mathrm{Sp}(1)$ by $\rho \colon G \hookrightarrow \mathrm{Sp}(1)$ acting on fibers $\mathrm{Sp}(1)$ by left multiplication. Then we call the quotient adapted bundle (Q_0, α_0) obtained by taking the quotient of $(\widetilde{Q}_0, \theta_0)$ by G the adapted $\mathrm{SU}(2)$–bundle over $\mathbb{R} \times Y$ of $c_2(Q_0, \alpha_0) = 1/m$ associated with the positive (resp. negative) subgroup G_1.

Remark 3.8 The $\mathrm{SO}(4){=}\mathrm{SU}(2){\times_{\{\pm 1\}}}\mathrm{SU}(2)$–action on $S^3 = \mathbb{H} \cup \{\infty\}$ is given by

$$[e_+, e_-] \cdot x = e_+ x e_-^{-1}, \quad x \in S^3,\ [e_+, e_-] \in \mathrm{Sp}(1) \times_{\{\pm 1\}} \mathrm{Sp}(1).$$

Note that we have an identification $\mathbb{H} \ni x = z + jw \mapsto (z, w) \in \mathbb{C} \oplus \mathbb{C}$, so that $\mathrm{Sp}(1)$ is identified with $\mathrm{SU}(2)$ under the left multiplication. If $G = \mathbb{Z}_m \subset U(1) \subset \mathrm{Sp}(1) \subset \mathrm{Sp}(1) \times_{\{\pm 1\}} \{\pm 1\}$, where $U(1) = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$ and $[e_+, e_-] = [\zeta, 1] \in G$, then

$$[\zeta, 1] \cdot x = \zeta(z + jw) = \zeta z + j\zeta^{-1} w.$$

On the other hand, the orientation $o(\mathbb{C}^2)$ induced from the standard one $o(\mathbb{C}) = 1 \wedge i$ is opposite to the orientation $o(\mathbb{H}) = 1 \wedge i \wedge j \wedge k$ of

$$\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k = \mathbb{C} \oplus j\mathbb{C} \cong \mathbb{C}^2,$$

so that $\iota \colon \mathbb{C}^2 \ni (z, w) \mapsto z + j\overline{w} \in \mathbb{H}$ gives the orientation-preserving diffeomorphism and

$$(\iota^{-1} \circ [\zeta, 1] \circ \iota)(z, w) = (\zeta z, \zeta w),$$

implying $S^3/G \cong L(m,1)$ is positive. Similarly if $G = \mathbb{Z}_m \subset U(1) \subset \mathrm{Sp}(1) \subset \{\pm 1\} \times_{\{\pm 1\}} \mathrm{Sp}(1)$ and $[e_+, e_-] = [1, \zeta] \in G$ then

$$[1,\zeta]\cdot x = (z + jw)\zeta^{-1} = \zeta^{-1} z + j\zeta^{-1} w$$

and

$$(\iota^{-1} \circ [1,\zeta] \circ \iota)(z,w) = (\zeta^{-1} z, \zeta w),$$

so that $S^3/G \cong L(m,-1)$ is negative.

The following proposition is a key to studying behaviors of flat connections after the "sliding ends" phenomenon:

Proposition 3.9 (see [16, Lemma 5.3]) *Let G be a finite group acting on S^3 freely. Let $(\widetilde{Q}_1, \theta_1)$ be a G–equivariant adapted $\mathrm{SU}(2)$–bundle $\widetilde{Q}_1 \to \mathbb{R} \times S^3$ with the trivial flat connections $\theta_1 = \{\theta_1^\pm\}$ over $\mathbb{R}_\pm \times S^3$, $c_2(\widetilde{Q}_1, \theta_1) = 1$, and the G–action on $\widetilde{Q}_1$ defined by representations $\alpha_1^\pm\colon G \to \mathrm{SU}(2)$ with respect to the trivializations of $\theta_1^\pm$. Suppose there exists a G–invariant instanton $\widetilde{A}_1$ on $(\widetilde{Q}_1, \theta_1)$. Then the following properties hold:*

(1) *If $\alpha_1^+ = 1$, then S^3/G is negative and α_1^- is conjugate to the inclusion $\rho\colon G \hookrightarrow \mathrm{Sp}(1)$.*

(2) *If S^3/G is positive and α_1^+ is the inclusion $\rho\colon G \hookrightarrow \mathrm{Sp}(1)$ then $\alpha_1^- = \varepsilon$ and ρ is conjugate to $\varepsilon\rho$ for some homomorphism $\varepsilon\colon G \to \{\pm 1\}$.*

Proof The proof is the same as in [16] for the case of G–equivariant instantons on the G–equivariant $\mathrm{SU}(2)$–bundle $\widetilde{Q}_1$ over S^4 under an identification with G–equivariant instantons on an adapted G–equivariant $\mathrm{SU}(2)$–bundle $(\widetilde{Q}_1, \theta_1)$ over $\mathbb{R}_+ \times S^3$. Note here that the conformal map

$$f\colon \mathbb{R} \times S^3 \to S^4 = \mathbb{H} \cup \{\infty\}, \quad f(t,y) = (e^t y)^{-1},$$

is orientation-preserving with respect to the standard orientation $o(\mathbb{H}) = 1 \wedge i \wedge j \wedge k$. However, the map f sends the positive end $\mathbb{R}_+ \times S^3$ to the punctured unit ball $B_1 - \{0\}$, where $B_1 = \{x \in \mathbb{H} \mid |x| \le 1\}$, and the induced orientation on the boundary $S^3 = \partial B_1$ of $B_1 \subset \mathbb{H}$ is opposite to that of the standard orientation S^3 in $\mathbb{R} \times S^3$. On the other hand, self-dual connections change to anti-self-dual connections if we change the orientation of the 4–manifold. Hence, the "positivity" (resp. "negativity") in our proposition coincides with that of Lemma 5.3 in [16]. □

3.5 The virtual dimension of the moduli space

To calculate the virtual dimension of moduli spaces of instantons over 4–manifolds with cylindrical ends, it is easier to use the addition property for indices [8, Section 3.3]. First we recall:

Definition 3.10 Let (P_k, α_k) for $k = 1, 2$ be an adapted $\mathrm{SU}(2)$–bundle over a 4–manifold X_k with cylindrical ends $\mathbb{R}_+ \times Y_{ki}$ indexed by $i \in I_k$ for a finite set I_k containing 0 and with an identification $Y_{10} = Y = -Y_{20}$ for the $i = 0^{\text{th}}$ component, and a fixed bundle isomorphism $h\colon P_1|_{\mathbb{R}_+\times Y_{10}} \to P_2|_{\mathbb{R}_+\times Y_{20}}$ such that $\alpha_{10} = h^*\alpha_{20}$. For a fixed parameter $T > 0$, we define a gluing $X_1 \#_{(T)} X_2$ of $X_1 - ((2T, \infty) \times Y_{10})$ and $X_2 - ((2T, \infty) \times Y_{20})$ under the identification $(t_1, y_1) \sim (t_2, y_2)$ if $(t_2, y_2) = (2T - t_1, y_1)$ for $(t_1, y_1) \in [T, 2T] \times Y_{10}$, $(t_2, y_2) \in [T, 2T] \times Y_{20}$. The gluing $P_1 \#_{(T)} P_2$ is defined naturally by using the isomorphism $h\colon P_1|_{\mathbb{R}_+\times Y_{10}} \to P_2|_{\mathbb{R}_+\times Y_{20}}$. Set $\alpha_1 \# \alpha_2 = \alpha_1 \cup \alpha_2 - \{\alpha_{10}, \alpha_{20}\}$. Then $(P_1 \#_{(T)} P_2, \alpha_1 \# \alpha_2)$ defines an adapted $\mathrm{SU}(2)$–bundle over $X_1 \#_{(T)} X_2$.

In fact, the isomorphism class of $(P_1 \#_{(T)} P_2, \alpha_1 \# \alpha_2)$ does not depend on the parameter T, so we omit the parameter T in the notation $\#_{(T)}$ of the gluing if we do not need to specify it. Then, by additivity of indices of elliptic operators under connected sums, we have the following:

Proposition 3.11 *Let X be a Riemannian 4–manifold with cylindrical ends $\mathbb{R}_+ \times Y_i$ for $i \in \{1, \dots, s+1\}$ and $b_1(X) = 0$, $b_2^+(X) = 0$, where Y_{s+1} is a rational homology 3–sphere and the Y_i are spherical space forms $Y_i \cong S^3/G_i$ for $i \in \{1, \dots, s\}$. Suppose that Y_1 is a positive spherical space form S^3/G_1 with $|G_1| = m$. Let (Q, β) be an adapted $\mathrm{SU}(2)$–bundle over $\mathbb{R} \times Y_1$ of $c_2(Q, \beta) = 1/m$ associated with the positive subgroup G_1 equipped with flat connections $\beta = \{\beta_\pm\}$ on each ends $\mathbb{R}_\pm \times Y_1$, where $\beta_+ = \rho_1$ is the inclusion $\rho_1\colon G_1 \hookrightarrow \mathrm{Sp}(1)$ and $\beta_- = \theta$. Let (P_0, θ) be the trivial adapted $\mathrm{SU}(2)$–bundle $P_0 = X \times \mathrm{SU}(2)$ equipped with trivial connections $\theta = \{\theta_i\}$ on each ends. If $(P, \alpha) = (P_0 \# Q, \theta \# \beta)$ is the connected sum of (P_0, θ) and (Q, β) along Y_1. Then we have the following properties:*

1. *The virtual dimension of the moduli space $\mathcal{M}(P, \alpha)$ is equal to that of $\mathcal{M}(Q, \beta)$.*
2. *If (Q', β') is an adapted $\mathrm{SU}(2)$–bundle over $\mathbb{R} \times Y_1$ of $c_2(Q', \beta') = 0$ equipped with flat connections $\beta' = \{\beta'_\pm\}$ on each ends $\mathbb{R}_\pm \times Y_1$, where β'_+ corresponds to a representation $\varepsilon\colon G_1 \to \{\pm 1\}$ and $\beta'_- = \theta$, and let $(P', \alpha') = (P_0 \# Q', \theta \# \beta')$*

be the adapted bundle obtained by the connected sum of (P_0, θ) *and* (Q', β') *along* Y_1*, then the virtual dimension of the moduli space* $\mathcal{M}(P', \alpha')$ *is equal to* -3*.*

(3) *If* $Y_i = S^3/G_i$ *is a negative spherical space form with* $|G_i| = |G_1|$ *for some* $i \neq 1$ *and let* (Q'', β'') *be an adapted* $\mathrm{SU}(2)$*–bundle over* $\mathbb{R} \times Y_i$ *of* $c_2(Q'', \beta'') = 1/m$ *associated with a negative subgroup* G_i *equipped with flat connections* $\beta'' = \{\beta''_\pm\}$ *on each ends* $\mathbb{R}_\pm \times Y_i$*, where* $\beta''_+ = \theta$ *and* $\beta''_- = \rho_i$ *is the flat connection corresponding to the inclusion* $\rho_i\colon G_i \hookrightarrow \mathrm{Sp}(1)$*. Let* (P'', α'') *be an adapted* $\mathrm{SU}(2)$*–bundle such that the connected sum* $(P'' \# Q'', \alpha'' \# \beta'')$ *of* (P'', α'') *and* (Q'', β'') *along* Y_i *is isomorphic to* (P, α)*. Then the virtual dimension of the moduli space* $\mathcal{M}(P'', \alpha'')$ *is* $-\dim \Gamma_{\rho_i}$*, where* Γ_{ρ_i} *is the stabilizer of* ρ_i *in the gauge transformation group of* $Q''|_{Y_i}$*.*

Proof The virtual dimension of the moduli space $\mathcal{M}(P, \alpha)$ is equal to the index $\mathrm{ind}^+(P, \alpha)$ of the deformation operator $D_A = -d_A^* \oplus d_A^+$ for a connection A on (P, α) acting on sections with positively weighted norms,

$$D_A = -d_A^* \oplus d_A^+\colon L_l^{2,\delta}\big(\textstyle\bigwedge^1(\mathfrak{g}_P)\big) \to L_{l-1}^{2,\delta}\big(\textstyle\bigwedge^0(\mathfrak{g}_P) \oplus \textstyle\bigwedge^2_+(\mathfrak{g}_P)\big),$$

which can be calculated as the index of the operator D_A^{APS} over the 4–manifold W with boundary Y under Atiyah–Patodi–Singer boundary condition (see [8, Section 3.3.2, Proposition 3.19]). The statements (1) and (3) follows from the addition formula for indices under the connected sum $(P, \alpha) = (P', \alpha') \# (Q, \beta)$ along the i^{th} component with flat connection $\alpha'_i = \beta_-$ and $\alpha_i = \beta_+$ [8, Propositions 3.9 and 3.10],

$$\mathrm{ind}^+(P, \alpha) = \mathrm{ind}^+(P', \alpha') + \mathrm{ind}^+(Q, \beta) + \dim \Gamma_{\alpha'_i},$$

where $\Gamma_{\alpha'_i}$ is the stabilizer of α'_i in the gauge transformation group of $P|_{Y_i}$. For the statement (2), we now apply (1) to see $\mathrm{ind}^+(P', \alpha') = \mathrm{ind}^+(Q', \beta')$, and $\mathrm{ind}^+(Q', \beta') = -3$ follows from Proposition 2.6 of [18] by noting that $\rho(Y_1, \theta) = 0$ and $\rho(Y_1, \varepsilon) = 0$ since ε is a nontrivial central representation such that $\mathrm{ad}\,\varepsilon = 1$. □

3.6 Chern–Simons invariants and compactness of the moduli

Instanton moduli spaces on 4–manifolds with product ends in general fail to be compact due to either "bubbling" (when a divergent sequence of instantons has curvatures concentrating to a point in the 4–manifold) or "sliding ends" (when the curvatures recede to infinity along a cylindrical end). These phenomena can be controlled by Chern–Simons invariants.

Here we recall the definitions.

Definition 3.12 (see [15]) Let Y be a closed oriented 3–manifold, and $Q \to Y$ be a principal $\mathrm{SU}(2)$–bundle with a flat connection α on Q. Let X be an oriented smooth 4–manifold with cylindrical ends $\mathbb{R} \times (Y \sqcup Y')$ for some disjoint union Y' of closed oriented 3–manifolds, and $(P, \tilde{\alpha})$ be an adapted $\mathrm{SU}(2)$–bundle over X such that $(P|_{\mathbb{R}_+ \times Y}, \tilde{\alpha}|_{\mathbb{R}_+ \times Y}) \cong (p_Y^* Q, p_Y^* \alpha)$ for the projection $p_Y \colon \mathbb{R}_+ \times Y \to Y$ and $(P|_{\mathbb{R}_+ \times Y'}, \tilde{\alpha}|_{\mathbb{R}_+ \times Y'}) \cong (\mathbb{R}_+ \times Y' \times \mathrm{SU}(2), \theta)$ for the trivial flat connection θ. Then the Chern–Simons invariant of (Y, α) is defined to be

$$\mathrm{cs}(Y, \alpha) \equiv c_2(P, \tilde{\alpha}) \pmod{\mathbb{Z}}.$$

Note that $\mathrm{cs}(Y, \alpha)$ does not depend on the choice of the adapted bundle $(P, \tilde{\alpha}) \to X$ as above. In fact, if we take another $(P', \tilde{\alpha}') \to X$ then the bundle $P \# P'$ over $X \cup (-X)$ obtained by gluing along ends gives $c_2(P, \tilde{\alpha}) - c_2(P', \tilde{\alpha}') = c_2(P \# P') \in \mathbb{Z}$.

Definition 3.13 [18] Let Q be an $\mathrm{SU}(2)$–bundle over a closed oriented 3–manifold Y and let $\mathbb{R} \times Q \to \mathbb{R} \times Y$ be the pullback by the projection $\mathbb{R} \times Y \to Y$. Let $\alpha_\pm$ be flat connections on Q viewed as flat connection on the ends $\mathbb{R}_\pm \times Y$ of the cylinder $\mathbb{R} \times Y$. Then the relative Chern–Simons invariant $\mathrm{cs}(Y, \alpha_-, \alpha_+)$ is defined to be

$$\mathrm{cs}(Y, \alpha_-, \alpha_+) = c_2(\mathbb{R} \times Q, \{\alpha_-, \alpha_+\}) \in \mathbb{R}.$$

Note that $\mathrm{cs}(Y, \alpha_-, \alpha_+) - \mathrm{cs}(Y, g^* \alpha_-, \alpha_+) \in \mathbb{Z}$ for any $g \in \mathrm{Aut}(Q)$, and $\mathrm{cs}(Y, -, \alpha_+)$ descends to a locally constant $\mathbb{R}/\mathbb{Z}$–valued function on the space $\mathcal{R}(Y, \mathrm{SU}(2))$ of all conjugacy classes of $\mathrm{SU}(2)$–representations of the fundamental group $\pi_1(Y)$ of Y. Since $\mathcal{R}(Y, \mathrm{SU}(2))$ is compact, the set of all values $\mathrm{cs}(Y, \gamma, \alpha) \pmod{\mathbb{Z}}$ for all flat connections α and γ on Q is finite.

Definition 3.14 [18] Let $b \colon \mathbb{R}/\mathbb{Z} \to (0, 1]$ be the obvious bijection. For a rational homology 3–sphere Y and a flat connection α on a $\mathrm{SU}(2)$–bundle Q over Y, we define its minimum Chern–Simons invariant $\tau(Y, \alpha)$ by

$$\tau(Y, \alpha) = \min\{b(\mathrm{cs}(Y, \gamma, \alpha)) \in (0, 1] \mid \gamma \text{ a flat connection on } Q\}.$$

Then we have the following compactness result:

Proposition 3.15 *Let X be a 4–manifold with cylindrical ends $\mathbb{R}_+ \times Y_i$ for $i \in I$ and (P, α) be an adapted $\mathrm{SU}(2)$–bundle over X with*

$$0 < c_2(P, \alpha) = \epsilon = \min\{\tau(Y_i, \alpha_i) \mid i \in I\} \le \tfrac{1}{2}.$$

Suppose that $H^1(Y_i;\mathrm{ad}\,\gamma)=\{0\}$ *for any flat* $\mathrm{SU}(2)$*–connection* γ *over each component* Y_i *with* $\mathrm{cs}(Y_i,\gamma)\equiv\epsilon \pmod{\mathbb{Z}}$*. Fix a Riemannian metric on* X *with product ends. Let* $\mathcal{M}^\sigma(P,\alpha)$ *be the moduli space of* σ*–instantons* A *on* (P,α)*, that is, connections* A *satisfying the equation* $F_A^+ + \sigma(A)=0$*, where* $\sigma\colon \mathcal{A}(P,\alpha)\to L_l^{2,\delta}\big(\bigwedge_+^2(\mathfrak{g}_P)\big)$ *is a smooth perturbation such that*

(1) $\|\sigma(A)\|^2_{L^2(X)} < 4\pi^2\epsilon$,

(2) *there exists a compact set* $\Omega\subset W = X-\coprod_i(\mathbb{R}_+\times Y_i)$ *such that*

  (a) $\operatorname{supp}\sigma(A)\subset\Omega$ *for all* $A\in\mathcal{A}(P,\alpha)$*, and*

  (b) $\sigma(A)=0$ *if* $\|F_A\|^2_{L^2(\Omega)} > 4\pi^2$.

Then, for any sequence $\{[A_n]\}$ *in* $\mathcal{M}^\sigma(P,\alpha)$*, there exists a subsequence* $\{n'\}$ *of* $\{n\}$ *and a sequence of gauge transformations* $\{g_{n'}\}$ *such that one of the following holds:*

(1) $\{g_{n'}^*A_{n'}\}$ *converges to a* σ*–instanton* A' *on the bundle* (P,α) *over* X *in* C^∞_{loc}*–topology.*

(2) $\{g_{n'}^*A_{n'}\}$ *converges to a* σ*–instanton* A' *on a limiting adapted bundle* (P',α') *over* X *in* C^∞_{loc}*–topology, and there exists a unique* $i\in I$ *and a sequence* $\{T_{n'}\}$ *with* $T_{n'}>0$ *and* $\lim_{n'\to\infty}T_{n'}=+\infty$ *such that the sequence of the pullback connections* $\{c^*_{T_{n'}}(g^*_{n'}A_{n'}|_{\mathbb{R}_+\times Y_i})\}$ *by the translation* $c_{T_{n'}}$ *with* $T_{n'}$ *over* $[-T_{n'},\infty)\times Y_i$ *converges to an instanton* B'_i *on an adapted bundle* (Q'_i,β'_i) *over the cylinder* $\mathbb{R}\times Y_i$ *in* C^∞_{loc}*–topology such that*

$$c_2(P',\alpha')=0,\quad c_2(Q'_i,\beta'_i)=\epsilon,$$

where

$$\alpha'_j=\alpha_j \quad \text{for } j\neq i,\qquad \alpha'_i={\beta'_i}^-,\qquad {\beta'_i}^+=\alpha_i.$$

Proof Each term of the sequence of σ–instanton $\{[A_n]\}\subset\mathcal{M}^\sigma(P,\alpha)$ satisfies the perturbed equation $F^+_{A_n}+\sigma(A_n)=0$. Since the support of the perturbation σ is contained in a compact subset $\Omega\subset W$ away from the cylindrical ends $\mathbb{R}_+\times Y_i$, we see as in the proof of Theorem 5.4 in [8] that the sequence $\{[A_n]\}$ has a weak chain convergent subsequence. By the condition $\|\sigma(A_n)\|^2_{L^2}<4\pi^2\epsilon$, we have

$$\|F_{A_n}\|^2_{L^2}=8\pi^2\epsilon+2\|\sigma(A_n)\|^2_{L^2}<8\pi^2\epsilon+2\cdot 4\pi^2\epsilon=2\cdot 8\pi^2\epsilon\le 8\pi^2,$$

and since $\sigma(A)=0$ for those connections A with $\|F_A\|^2_{L^2(\Omega)}>4\pi^2$, we see that the perturbation term vanishes for connections with the energy concentrating on Ω. Therefore, we can apply Uhlenbeck's removable singularity theorem [34] to see that the bubbling does not occur. Since $\tau(Y_i,\alpha_i) \geq \epsilon$ for all $i \in I$, the sequence has a chain-convergent subsequence [8, Section 5.1] with a limit given by π–instanton A' on a limiting adapted bundle (P',α') with $\lim_{n\to\infty} \|\sigma(A_n)\|_{L^2} = \|\sigma(A')\|_{L^2} < 4\pi^2\epsilon$ and at most "one-term chain" from α'_i to α_i with $\tau(Y_i,\alpha_i) = \epsilon$ over some $\mathbb{R}\times Y_i$ (possibly empty). If the limit has a one-term chain, an instanton B'_i is formed on some limiting bundle (Q'_i,β'_i) over $\mathbb{R}\times Y_i$ with $\beta'^{-}_i = \alpha'_i$ and $\beta'^{+}_i = \alpha_i$ and satisfies the conservation of the topological charge

$$\epsilon = c_2(P,\alpha) = c_2(P',\alpha') + c_2(Q'_i,\beta'_i) = c_2(P',\alpha') + \epsilon,$$

so that $c_2(P',\alpha') = 0$, which corresponds to (2). The other cases correspond to (1). □

3.7 Perturbation on ends of the moduli space

In the following discussion, we need a perturbation which works even after bubbling or sliding ends occur. In [7], Donaldson used a holonomy of the connections to perturb (holonomy perturbation) the moduli space of flat connections after bubbling to prove his Theorem A for smooth negative-definite 4–manifolds with nontrivial fundamental groups. Here we use a variant of the formulation given by Kronheimer in [23]. It is easier in our case since we know that the bubbling does not occur.

Let X be a 4–manifold with cylindrical ends $\mathbb{R}_+\times Y_i$ for $i \in I$ and (P',α') be an adapted $\mathrm{SU}(2)$–bundle over X. Fix a Riemannian metric on X with product ends. Let B be a smooth embedded ball in X and $q\colon S^1\times B \to X$ be a smooth submersion with $q(1,x) = x$ for all $x \in B$. Let $\omega \in \Omega^2_+(X)$ be a self-dual 2–form with $\operatorname{supp}\omega \subset B$. For a smooth connection A in $P' \to X$, we denote by $\mathrm{Hol}_{q_x}(A) \in \mathrm{Aut}(P'_x)$ the holonomy of A around the loop $q_x\colon S^1 \to X$ given by $q_x(z) = q(z,x)$. Then we have a holonomy map $\mathrm{Hol}_q(A)\colon B \to \Gamma(\mathfrak{gl}_{P'}|_B)$, where $\mathfrak{gl}_{P'} = P' \times_{\mathrm{Ad}} \mathfrak{gl}_2$ with $\mathrm{Ad}(g)a = gag^{-1}$ for $g \in \mathrm{SU}(2)$ and $a \in \mathfrak{gl}_2$, and define $\omega \otimes \mathrm{Hol}_q(A) \in \Omega^2_+(\mathfrak{gl}_2)$ by extending by zero.

Let $\psi\colon \mathfrak{gl}_{P'} \to \mathfrak{su}_{P'} = \mathfrak{g}_{P'}$ be the orthogonal bundle projection and $\rho_\mu\colon \mathbb{R} \to [0,1]$ be a smooth cut-off function satisfying $\rho_\mu(x) = 0$ for $|x| \leq \mu$, $\rho_\mu(x) = 1$ for $|x| \geq 2\mu$ with $0 \leq |\rho'_\mu(x)| \leq 1/\mu$ for some fixed sufficiently small μ which we specify later, in the proof of Proposition 3.16. Let $\psi_\mu\colon \mathfrak{gl}_{P'} \to \mathfrak{g}_{P'}$ be the adjoint equivariant cut-off of the map ψ,

$$\psi_\mu(g) = \rho_\mu(|\psi(g)|)\psi(g) \quad \text{for } g \in \mathfrak{gl}_{P'},$$

and denote the induced maps by $\psi, \psi_\mu\colon \Gamma(\mathfrak{gl}_{P'}|_B) \to \Gamma(\mathfrak{g}_{P'}|_B)$, respectively. Now define

$$V^\mu_{q,\omega}(A) = \omega \otimes (\psi_\mu \circ \mathrm{Hol}_q(A)) \in \Omega^2_+(\mathfrak{g}_{P'}).$$

Then, taking $l > 2$, the map $V^{\mu}_{q,\omega}$ extends to a smooth map of Hilbert manifolds (see [23])

$$V^{\mu}_{q,\omega}\colon \mathcal{A}_{L^{2,\delta}_l}(P',\alpha') \to L^{2,\delta}_l\big(\textstyle\bigwedge^2_+(\mathfrak{g}_{P'})\big).$$

For the reference smooth connection A_0 there are constants K_n, depending only on q, A_0 and the cut-off function ρ_μ, such that the n^{th} derivative

$$D^n V^{\mu}_{q,\omega}|_A\colon L^{2,\delta}_l\big(\textstyle\bigwedge^1(\mathfrak{g}_{P'})\big)^n \to L^{2,\delta}_l\big(\textstyle\bigwedge^2_+(\mathfrak{g}_{P'})\big)$$

satisfies

$$\|D^n V^{\mu}_{q,\omega}|_A(a_1,\dots,a_n)\|_{L^{2,\delta}_{l,A_0}} \le K_n \|\omega\|_{C^l} \prod_{i=1}^n \|a_i\|_{L^{2,\delta}_{l,A_0}},$$

which follows from Proposition 7 in [23] since $V^{\mu}_{q,\omega}(A)$ and $D^n V^{\mu}_{q,\omega}|_A$ are compactly supported in $\operatorname{supp}\omega$, and $L^{2,\delta}_l$–norm and L^2_l–norm are equivalent on $\operatorname{supp}\omega$.

Let $\pi = (\{B_\nu, q_\nu, \omega_\nu, \varepsilon_\nu\}_{\nu=1}^N, \mu)$ be a finite collection consisting of

(1) a 4–ball B_ν in X, a submersion $q_\nu\colon S^1 \times B_\nu \to X$,

(2) a self-dual 2–form $\omega_\nu \in \Omega^2_+(X)$ with $\operatorname{supp}\omega_\nu \subset B_\nu$,

(3) coefficients $\varepsilon_\nu \in \mathbb{R}$, and

(4) a cut-off parameter $\mu > 0$.

Then we call π a perturbation data and define a smooth map of Hilbert manifolds

$$V_\pi\colon \mathcal{A}(P',\alpha') \to L^{2,\delta}_l\big(\textstyle\bigwedge^2_+(\mathfrak{g}_{P'})\big), \quad A \mapsto V_\pi(A) = \sum_{\nu=1}^N \varepsilon_\nu V^{\mu}_{q_\nu,\omega_\nu}(A).$$

The perturbed moduli space $\mathcal{M}^\pi(P',\alpha')$ with perturbation data π is defined to be

$$\mathcal{M}^\pi(P',\alpha') = \{A \in \mathcal{A}(P',\alpha') \mid F^+_A + V_\pi(A) = 0\}/\mathcal{G}(P',\alpha').$$

Now we have the following:

Proposition 3.16 *Let $\mathcal{P}'$ be a finite set of nontrivial adapted bundles (P',α') with $c_2(P',\alpha') = 0$ over a 4–manifold X with cylindrical ends. Suppose there is no reducible connection in the bundle (P',α') and $\operatorname{ind}^+(P',\alpha') < 0$ for all $(P',\alpha') \in \mathcal{P}'$. Then there exists a perturbation data $\pi = (\{B_\nu, q_\nu, \omega_\nu, \varepsilon_\nu\}_{\nu=1}^N, \mu)$ such that*

$$\mathcal{M}^\pi(P',\alpha') = \varnothing$$

for all $(P',\alpha') \in \mathcal{P}'$. If there is no reducible flat connection on the trivial bundle $(X \times \mathrm{SU}(2), \theta_0)$ except for the trivial flat connection θ then $\mathcal{M}^\pi(X \times \mathrm{SU}(2), \theta_0) = \{[\theta]\}$.

Proof The proof is essentially the same as that of Proposition 2.6 in [7]. Since the second homology H^2_A of the Atiyah–Hitchin–Singer complex is finite-dimensional for each flat connection A, we can take a finite number of mutually disjoint embeddings $q_\nu\colon S^1 \times B_\nu \to X$ and self-dual 2–forms ω_ν with support inside 4–balls B_ν such that $\{\omega_\nu \otimes (\psi \circ \mathrm{Hol}_{q_\nu}(A))\}_\nu$ span the space H^2_A for each *irreducible* flat connection A, where $\psi\colon \mathfrak{gl}_{P'} \to \mathfrak{su}_{P'}$ is the adjoint equivariant map defined above. Since the space of all flat connections is compact and the trivial flat connection is isolated for $b_1(X) = 0$ and $b_2^+(X) = 0$, the irreducible part V is compact. Then we can take a finite collection $\{\omega_\nu \otimes (\psi \circ \mathrm{Hol}_{q_\nu}(A))\}_\nu$ to span H^2_A for all irreducible A, so that there is no solution on an open neighborhood U of V in $\mathcal{B}(P,\alpha)$. Note that the Uhlenbeck compactness [35] holds with small holonomy perturbation due to the fact that (i) there exist constants $\varepsilon_0 > 0$ and $C > 0$ such that for any sufficiently small ball B_x centered at each point x such that $\|F_A\|_{L^2(B_x)} < \varepsilon_0$, there exists a connection matrix $\widetilde{A}$ with respect to a trivialization such that $d^*\widetilde{A} = 0$ and

$$\|\widetilde{A}\|_{L^2_1(B_x)} \le C\,\|F_A\|_{L^2(B_x)},$$

and (ii) for each $s \ge 2$, there exists a polynomial $R_s(Z,W) \in \mathbb{R}[Z,W]$ and an interior domain $x \in D_s \Subset B_x$ such that if $\|\widetilde{A}\|_{L^2_1(B_x)} < \varepsilon_1$ and $|\varepsilon| := \max\{|\varepsilon_\nu|\}$ then

$$\|\widetilde{A}\|_{L^2_s(D_s)} \le R_s(\varepsilon_1, |\varepsilon|).$$

Now take $\mu > 0$ to be

$$\mu = \tfrac{1}{2}\min\bigl\{\min_{x\in B_\nu} |\psi \circ \mathrm{Hol}_{q_{\nu,x}}(A)| \mid \nu \in \{1,\dots,N\},\ [A] \in \mathcal{M}^*(P',\alpha'),\ (P',\alpha') \in \mathcal{P}'\bigr\}.$$

Note also that the trivial flat connection is the isolated solution during the holonomy perturbation. Then we see that there is no solution outside U for sufficiently small perturbations. Now by definition of μ we have $\psi_\mu \circ \mathrm{Hol}_{q_\nu}(A) = \psi \circ \mathrm{Hol}_{q_\nu}(A)$ for $[A] \in \mathcal{M}^*(P',\alpha')$ and the perturbation V_π has the required properties. □

Then, by using Propositions 3.15 and 3.16, we obtain:

Proposition 3.17 *Let X be a 4–manifold with cylindrical ends $\mathbb{R}_+ \times Y_i$ for $i \in I$. Let (P,α) be an adapted $\mathrm{SU}(2)$–bundle over X with*

$$0 < c_2(P,\alpha) = \epsilon = \min\{\tau(Y_i,\alpha_i) \mid i \in I\} \le \tfrac{1}{2}.$$

Let $\mathcal{P}'$ be the finite set of all possible limiting adapted $\mathrm{SU}(2)$–bundles (P',α') over X which come from (P,α) satisfying

(1) $c_2(P',\alpha') = 0$,

(2) $\operatorname{ind}^+(P',\alpha') < 0$,

(3) *there is no reducible instantons on* (P',α')*, that is,* $\mathcal{M}_{\mathrm{red}}(P',\alpha') = \varnothing$.

Then there exists a holonomy perturbation

$$V_\pi\colon \mathcal{A}(P,\alpha) \to L^{2,\delta}_l\big(\textstyle\bigwedge^2_+(\mathfrak{g}_P)\big), \quad V_\pi(A) = \sum_{\nu=1}^{N} \rho_\nu(A)\varepsilon_\nu V^\mu_{q_\nu,\omega_\nu}(A),$$

with perturbation data $\pi = (\{B_\nu, q_\nu, \omega_\nu, \varepsilon_\nu\}_{\nu=1,\dots,N}, \mu)$ *satisfying* $q_\nu(S^1 \times B_\nu) \cap (\mathbb{R}_+ \times Y_i) = \varnothing$ *for all* $i \in I$, $\nu \in \{1,\dots,N\}$ *and* $\mathcal{G}(P,\alpha)$*–invariant smooth functions*

$$\rho_\nu\colon \mathcal{A}(P,\alpha) \to \mathbb{R} \quad \text{for } \nu \in \{1,\dots,N\}$$

such that $\mathcal{M}^\pi(P',\alpha') = \varnothing$ *for all adapted bundle* $(P',\alpha') \in \mathcal{P}'$ *and if* $(X \times \mathrm{SU}(2), \theta)$ *admits no nontrivial reducible flat connection then* $\mathcal{M}^\pi(X \times \mathrm{SU}(2), \theta) = \{[\theta]\}$.

Proof By Proposition 3.16, we can take a perturbation $\pi = (\{B_\nu, q_\nu, \omega_\nu, \varepsilon_\nu\}, \mu)$ so that $\mathcal{M}^\pi(P',\alpha') = \varnothing$ for all $(P',\alpha') \in \mathcal{P}'$ and if $(X \times \mathrm{SU}(2), \theta)$ admits no reducible flat connection then $\mathcal{M}^\pi(X \times \mathrm{SU}(2), \theta) = \{[\theta]\}$. Here, by taking B_ν sufficiently small and $q_x\colon S^1 \times \{x\} \to X$ in general position, we may assume $q_\nu(S^1 \times B_\nu)$ are inside $W = X - \coprod_{i\in I}(\mathbb{R}_+ \times Y_i)$ and they are all disjoint and take $\lambda > 0$ to be less than a half of the distances between them. Then we can take a λ–neighborhood U_ν of the image $q_\nu(S^1 \times B_\nu)$ and set a $\mathcal{G}(P,\alpha)$–invariant function

$$E_\nu(A) = \frac{1}{8\pi^2}\int_{U_\nu} |F_A(y)|^2 \operatorname{vol}_X(y)$$

to define a $\mathcal{G}(P,\alpha)$–invariant smooth map $\rho_\nu\colon \mathcal{A}(P,\alpha) \to \mathbb{R}$ by composing with a smooth cut-off function. Then the assertion follows from Propositions 3.16 and 3.15. □

3.8 Taubes' gluing instantons

In our discussion, it is crucial to construct an end of the instanton moduli space and in particular the moduli space need to be nonempty. We use a result of Taubes concerning gluing of instantons.

Let (P_k, α_k) for $k = 1,2$ be an adapted SU(2)–bundle over a 4–manifold X_k with cylindrical ends $\mathbb{R}_+ \times Y_{ki}$ indexed by $i \in I_k$ for a finite set I_k containing 0 and with an identification $Y_{10} = Y = -Y_{20}$ for the $i = 0^{\text{th}}$ component, and a fixed bundle isomorphism $h\colon P_1|_{\mathbb{R}_+\times Y_{10}} \to P_2|_{\mathbb{R}_+\times Y_{20}}$ such that $\alpha_{10} = h^*\alpha_{20}$. Let A_k be an instanton on (P_k,α_k). For a fixed parameter $T>0$, we defined an adapted SU(2)–bundle $(P_1\#_{(T)}P_2,\alpha_1\#\alpha_2)$ over the glued 4–manifold $X_1\#_{(T)}X_2$ in Definition 3.10. If $X_1=X=W\cup\coprod_i(\mathbb{R}_+\times Y_{1i})$ and $X_2=\mathbb{R}\times Y$, $\alpha_2=\{\alpha_{20},\alpha_{21}\}$ then $X_1\#_{(T)}X_2=X\#_{(T)}(\mathbb{R}\times Y)$ is diffeomorphic to $X_1=X$. Now we identify $Q_{11}=Q_{21}$, $\alpha_{11}=\alpha_{21}$ and $\mathbb{R}\times Q_{21}\cong P_2$ using the parallel translation, $P_1\#_{(T)}P_2$ is isomorphic to P_1 and $\alpha_1\#\alpha_2=(\alpha_1-\{\alpha_{10}\})\cup\{\alpha_{21}\}$. Therefore, we have an isomorphism as adapted bundles

$$(P_1\#_{(T)}P_2,\alpha_1\#\alpha_2)\cong(P_1,(\alpha_1-\{\alpha_{10}\})\cup\{\alpha_{21}\}).$$

Fix a Riemannian metric on X_k for $k=1,2$ with product ends. Let $\mathcal{M}'(P_2,\alpha_2)$ be the (translation-)reduced moduli space, that is, the moduli space of instantons A on the adapted SU(2)–bundle (P_2,α_2) over $\mathbb{R}\times Y$ with the center of mass zero,

$$\int_{-\infty}^{\infty} t\,|F_A|^2\,\mathrm{vol}_{\mathbb{R}\times Y}=0,$$

so that $\mathcal{M}(P_2,\alpha_2)=\mathbb{R}\times\mathcal{M}'(P_2,\alpha_2)$ under the identification $(t,A)\mapsto c_t^*A$ using translation c_t by t. We denote by $\Gamma_{\alpha_{10}}$ the stabilizer of α_{10} in the gauge transformation group of $P_k|_{Y_{k0}}$, and let $\widetilde{\mathcal{M}}(P_k,\alpha_k)$ be the framed moduli space of instantons on (P_k,α_k) with respect to the stabilizer $\Gamma_{\alpha_{10}}$, so that $\widetilde{\mathcal{M}}(P_k,\alpha_k)/\Gamma_{\alpha_{10}}\cong\mathcal{M}(P_k,\alpha_k)$. Then Taubes' gluing result is the following:

Theorem 3.18 (see [8, Section 4.4]) *Let A_k be an instanton on (P_k,α_k) for $k=1,2$, $N_k\subset\mathcal{M}(P_k,\alpha_k)$ a precompact neighborhood around $[A_k]$ whose closure consists of irreducible and regular points and $\widetilde{N}_k=p_k^{-1}(N_k)$ the inverse image of the projection $p_k\colon\widetilde{\mathcal{M}}(P_k,\alpha_k)\to\mathcal{M}(P_k,\alpha_k)=\widetilde{\mathcal{M}}(P_k,\alpha_k)/\Gamma_{\alpha_{10}}$. Then there exists a map*

$$\tau_T\colon E=\widetilde{N}_1\times_{\Gamma_{\alpha_{10}}}\widetilde{N}_2\to\mathcal{M}(P_1\#_{(T)}P_2,\alpha_1\#\alpha_2)$$

which is a diffeomorphism onto a neighborhood of the moduli space of instantons on the adapted bundle $(P_1\#_{(T)}P_2,\alpha_1\#\alpha_2)$ over the gluing $X_1\#_{(T)}X_2$ for sufficiently large $T>0$.

In particular, if $X_2=\mathbb{R}\times Y_2$, then $(P_1\#_{(T)}P_2,\alpha_1\#\alpha_2)\cong(P_1,(\alpha_1-\{\alpha_{10}\})\cup\{\alpha_{21}\})$. Let $N_2'\subset\mathcal{M}'(P_2,\alpha_2)$ be a precompact neighborhood around $[A_2]$ whose closure consists of irreducible and regular points and $\widetilde{N}_2'=p_2^{-1}(N_2')$ be the inverse image of the projection $p_2\colon\widetilde{\mathcal{M}}'(P_2,\alpha_2)\to\mathcal{M}'(P_2,\alpha_2)=\widetilde{\mathcal{M}}'(P_2,\alpha_2)/\Gamma_{\alpha_{10}}$, and suppose N_1 is isolated and regular in $\mathcal{M}(P_1,\alpha_1)$; then

$$\tau\colon E=\widetilde{N}_1\times_{\Gamma_{\alpha_{10}}}\widetilde{N}_2'\times(T,\infty)\to\mathcal{M}(P_1,(\alpha_1-\{\alpha_{10}\})\cup\{\alpha_{21}\})$$

gives a diffeomorphism onto an end of the moduli space for sufficiently large $T>0$.

Proof This is now a standard result and follows by taking the function spaces with $l > 2$,

$$U = L^{2,\delta}_{l-1}\big(\textstyle\bigwedge^2_+(\mathfrak{g}_{P_{\#(T)}})\big), \quad V = L^{2,\delta}_{l}\big((\textstyle\bigwedge^0 \oplus \bigwedge^1)(\mathfrak{g}_{P_{\#(T)}})\big)$$

in the proof of Theorem 4.17 in [8] and an argument for gluing reducible instantons [8, Section 4.4.1; 9, Theorem 7.2.62]. □

We apply Theorem 3.18 to show:

Theorem 3.19 *Suppose $l > 2$. Let X be a Riemannian 4–manifold with cylindrical ends $\mathbb{R}_+ \times Y_i$ for $i \in I$ having the following properties:*

1. *$b_1(X) = 0$ and $b_2^+(X) = 0$,*
2. *$i_*\colon H_1(Y;\mathbb{Z}) \to H_1(X;\mathbb{Z})$ is surjective, where $i\colon Y = \coprod_{i \in I} Y_i \hookrightarrow X$ is the inclusion,*
3. *all Y_i are rational homology 3–spheres and $Y_1 = S^3/G_1$ is positive.*

Let (P,α) be the adapted bundle obtained by gluing of the trivial bundle $(X\times \mathrm{SU}(2), \theta_0)$ over X and the adapted bundle (Q,β) over $\mathbb{R}\times Y_1$ of $c_2(Q,\beta) = 1/m$ associated with the positive subgroup G_1 of $\mathrm{SO}(4) = \mathrm{Sp}(1) \times_{\{\pm 1\}} \mathrm{Sp}(1)$ with $\beta_- = \theta$ and $\beta_+ = \rho_1$ along Y_1, where ρ_1 corresponds to the inclusion $\rho_1\colon G_1 \hookrightarrow \mathrm{Sp}(1)$. Let

$$V_\pi\colon \mathcal{A}(P,\alpha) \to L^{2,\delta}_{l-1}\big(\textstyle\bigwedge^2_+(\mathfrak{g}_P)\big)$$

be a holonomy perturbation with perturbation data $\pi = (\{B_\nu, q_\nu, \omega_\nu, \varepsilon_\nu\}, \mu)$ such that the perturbed moduli space $\mathcal{M}^\pi(X \times \mathrm{SU}(2), \theta_0)$ consists only of the gauge equivalence class $[\theta]$ of the trivial flat connection θ on $X \times \mathrm{SU}(2)$. Then the perturbed moduli space $\mathcal{M}^\pi(P,\alpha)$ of the bundle (P,α) has at least one end component.

Proof Note that the trivial flat connection θ over rational homology 3–sphere Y_i is non-degenerate, $H^1(Y_i, \mathrm{ad}_\theta) = \{0\}$. Since $H^1_\theta = H^1(X;\mathbb{R}) \otimes \mathfrak{g} = \{0\}$ and $H^2_\theta = H^2_+(X;\mathbb{R}) \otimes \mathfrak{g} = \{0\}$, the moduli space $\mathcal{M}(X \times \mathrm{SU}(2), \theta_0)$ has an isolated point $[\theta]$ corresponding to the gauge equivalence class of the trivial flat connection θ on $X \times \mathrm{SU}(2)$ and $[\theta]$ is a regular point. Now the set of all reducible instantons $\mathcal{M}_{\mathrm{red}}(X \times \mathrm{SU}(2), \theta_0)$ has a one-to-one correspondence with the set

$$\{\chi \in \mathrm{Hom}(H_1(X;\mathbb{Z}), U(1)) \mid \chi \circ (i_j)_* = 0 \text{ for } j \in I\}/_{\chi\sim\bar{\chi}},$$

where $i_j\colon Y_j \hookrightarrow X$ is the inclusion. Since $i_*\colon H_1(Y;\mathbb{Z}) \to H_1(X;\mathbb{Z})$ is surjective we see that $\mathcal{M}_{\mathrm{red}}(X \times \mathrm{SU}(2), \theta_0) = \{[\theta]\}$. Note that the virtual dimension of the moduli space $\mathcal{M}(X \times \mathrm{SU}(2), \theta_0)$ is

$$\operatorname{vir}\dim \mathcal{M}(X \times \mathrm{SU}(2), \theta_0) = -3(1 - b_1(X) + b_2^+(X)) = -3 < 0.$$

Let $\pi = (\{B_\nu, q_\nu, \omega_\nu, \varepsilon_\nu\}_{\nu=1}^N, \mu)$ be a perturbation data such that the perturbed moduli space $\mathcal{M}^\pi(X \times \mathrm{SU}(2), \theta_0)$ with respect to the holonomy perturbation

$$V_\pi \colon \mathcal{A}(P, \alpha) \to L_l^{2,\delta}\big(\textstyle\bigwedge_+^2(\mathfrak{g}_P)\big)$$

consists only of the gauge equivalence class $[\theta]$ of trivial flat connection θ. Let us denote $\mathcal{A}_{L_l^{2,\delta}}(P, \alpha)$ simply by $\mathcal{A}$, and so on. Let $\mathrm{Hol}_q(A) \in L_l^2(\mathfrak{gl}_P|_B)$ be the holonomy function over B of an $L_l^{2,\delta}$–connection A along the family of loops $q \colon S^1 \times B \to X$. Since $L^2_{l,\mathrm{loc}} \subset C^0$, $\mathrm{Hol}_q(A)$ is continuous on B, so that we have definite values $(\mathrm{Hol}_q(A))(x)$ for each $x \in B$, which we denote by $\mathrm{Hol}_{q_x}(A)$. Let $f_q \colon \mathcal{A} \to \mathbb{R}$ be the $\mathcal{G}$–invariant continuous function

$$f_q(A) = \max_{x \in B} |\psi \circ \mathrm{Hol}_{q_x}(A)|.$$

We denote the induced map $f_q \colon \mathcal{B} = \mathcal{A}/\mathcal{G} \to \mathbb{R}$ by the same symbol by abuse of notation. Let U be an open set in $\mathcal{B}$ defined by

$$U = \bigcap_{\nu=1}^{N} f_{q_\nu}^{-1}((-\infty, \mu)).$$

Let $\widetilde{\mathcal{B}}$ be the space of gauge equivalence classes of framed connections on (P, α) with respect to $\Gamma_{\alpha_1} = \Gamma_\theta$ and $p \colon \widetilde{\mathcal{B}} \to \mathcal{B} = \widetilde{\mathcal{B}}/\Gamma_\theta$ be the basepoint fibration and take the intersection $\widetilde{N}_1$ of the open set $\widetilde{U} = p^{-1}(U)$ in $\widetilde{\mathcal{B}}$, and let $\widetilde{\mathcal{M}}^\pi(X \times \mathrm{SU}(2), \theta_0)$ be the perturbed framed moduli space.

Since $V_\pi(A) = 0$ for $[A] \in \widetilde{U}$, $\widetilde{N}_1$ is equal to the intersection of $\widetilde{U}$ and the unperturbed framed moduli space $\widetilde{\mathcal{M}}(X \times \mathrm{SU}(2), \theta_0)$ consisting only of the $\mathrm{SU}(2)$–conjugate orbit of the trivial flat connection θ, the fiber $p^{-1}([\theta])$ is a single point $\{[\theta]\}$ and hence

$$\widetilde{N}_1 = \widetilde{U} \cap \widetilde{\mathcal{M}}^\pi(X \times \mathrm{SU}(2), \theta_0) = \widetilde{U} \cap \widetilde{\mathcal{M}}(X \times \mathrm{SU}(2), \theta_0) = \{[\theta]\}.$$

On the other hand, the moduli space $\mathcal{M}(Q, \beta)$ of the adapted bundle (Q, β) over $\mathbb{R} \times Y_1$ of $c_2(Q, \beta) = 1/m$ associated with the positive subgroup G_1 with $\beta_- = \theta$ and $\beta_+ = \rho$ which corresponds to the standard representation $\rho \colon G \hookrightarrow \mathrm{SU}(2)$ is diffeomorphic to the moduli space $\mathcal{M}(\widetilde{Q})^{G_1}$ of G_1–invariant instantons on a G_1–equivariant $\mathrm{SU}(2)$–bundle $\widetilde{Q}$ over S^4 and hence is diffeomorphic to $\mathbb{R}$ and the corresponding reduced moduli space is a point, $\mathcal{M}'(Q, \beta) = \{\mathrm{pt}\}$. Note that all instantons on (Q, β) are irreducible. The space $\mathcal{M}(Q,\beta)$ is regular; see Proposition 3.5. Since the point in $\mathcal{M}'(Q,\beta)$ is irreducible, its framed moduli space $\widetilde{\mathcal{M}}'(Q,\beta)$ is $\mathrm{SU}(2)$, so that we may take its precompact neighborhood $\widetilde{N}_2'$ to be $\widetilde{\mathcal{M}}'(Q,\beta)$ itself. Now the stabilizer of the flat connection θ is $\Gamma_\theta=\mathrm{SU}(2)$. Then, by Theorem 4.17 in [8] and the argument for the reducible case [8, Section 4.4.1], the Taubes' gluing map

$$\tau\colon E=\widetilde{N}_1\times_{\Gamma_\theta}\widetilde{N}_2'\times(T,\infty)\to U\cap\mathcal{M}\big((X\times\mathrm{SU}(2))\,\#\,Q,\theta_0\,\#\,\beta\big)=U\cap\mathcal{M}^\pi(P,\alpha)$$

gives a diffeomorphism onto an end of the perturbed moduli space $\mathcal{M}^\pi(P,\alpha)$ of π–instantons contained in U for a sufficiently long "neck" $T\gg 0$. Then the assertion follows from the diffeomorphism

$$\begin{aligned}E=\widetilde{N}_1\times_{\Gamma_\theta}\widetilde{N}_2'\times(T,\infty)&=\{[\theta]\}\times_{\Gamma_\theta}\widetilde{\mathcal{M}}'(Q,\beta)\times(T,\infty)\\ &\cong\{[\theta]\}\times\widetilde{\mathcal{M}}'(Q,\beta)/\Gamma_\theta\times(T,\infty)\\ &\cong\{[\theta]\}\times\mathrm{SU}(2)/\mathrm{SU}(2)\times(T,\infty)\cong(T,\infty).\end{aligned}$$ □

4 Main theorem

In this section, we state and prove Theorem 4.1 by using the preliminary result of Donaldson theory discussed in the previous section.

4.1 Statement of the main theorem

Our theorem is stated with condition only on the homology groups without referring to the fundamental groups itself, which is our major deviation from a theorem in [16].

Theorem 4.1 *Let X be an oriented smooth 4–manifold with cylindrical ends $\mathbb{R}_+\times Y_i$ for $i\in\{1,\dots,s+t\}$, where Y_i for $i\in\{1,\dots,s\}$ are spherical space forms $Y_i\cong S^3/G_i$ and $Y_{s+k}=\Sigma_k$ for $k\in\{1,\dots,t\}$ are rational homology 3–spheres. Suppose that X satisfies the following conditions:*

(1) $b_1(X)=0$, $b_2^+(X)=0$.

(2) $\mathrm{Cok}(H_1(Y;\mathbb{Z})\to H_1(X;\mathbb{Z}))=\{0\}$ *and* $Y=\coprod_{i=1}^{s+t}Y_i$.

Set $m=\max\{|G_i|\mid i\in\{1,\dots,s\}\}$. Then there exists a positive constant $N(\Sigma)>0$, depending only on $\Sigma=\coprod_{k=1}^{t}\Sigma_k$, such that if $m>N(\Sigma)$ the following property holds: Let Y_1 be a positive spherical space form satisfying the following conditions:

(1) $|G_1|=m$.

(2) *If G_1 is cyclic, then we assume that*

$$\kappa = \#\left\{ e \in H^2(X) \;\middle|\; e^2 = -\frac{1}{|G_1|},\ i_1^* e = \pm e(L_\rho),\ i_j^* e = 0 \text{ for } j \neq 1 \right\} \Big/ \{\pm 1\}$$

is even, where $i_j\colon Y_j \hookrightarrow X$ is the inclusion and $L_\rho = S^3 \times_{G_1} S^1 \to Y_1$ is the S^1–bundle associated with the inclusion $\rho\colon G_1 \hookrightarrow S^1 \subset \mathrm{SU}(2)$.

(3) *The intersection of the images of the two maps*

$$\mathrm{Hom}(H_1(X;\mathbb{Z}), S^1) \to \mathrm{Hom}(G_1, S^1),$$
$$\mathrm{Hom}(G_1/N_1, \{\pm 1\}) \to \mathrm{Hom}(G_1, S^1)$$

is $\{1\}$, where N_1 is the normal subgroup of G_1 generated by elements $g \in G_1$ of $\operatorname{ord} g \neq 4$.

Then there exists a negative spherical space form Y_i with $i \neq 1$ such that the following holds:

(1) *$|G_i| = m$ and there exists a representation*

$$\chi\colon \pi_1(X, x_0) \to \mathrm{SU}(2)$$

such that the induced map $\chi \circ (i_j)_*\colon G_j \to \mathrm{SU}(2)$ is conjugate to the inclusion homomorphism $G_j \hookrightarrow \mathrm{SU}(2)$ if $j \in \{1, i\}$ and is the trivial homomorphism $G_j \to \{1\}$ for any $j \notin \{1, i\}$, where $(i_j)_*\colon \pi_1(Y_j, y_j) \to \pi_1(X, x_0)$ is induced by the inclusion $i_j\colon Y_j \hookrightarrow X$ and a path from $y_j \in Y_j$ to $x_0 \in X$.

(2) *If all G_j with $|G_j| = m$ are cyclic, then $G_i = \mathbb{Z}_m$ and there exists a character*

$$\chi\colon H_1(X;\mathbb{Z}) \to U(1)$$

such that the induced map $\chi \circ (i_j)_*\colon G_j \to U(1)$ is the inclusion $\mathbb{Z}_m \hookrightarrow U(1)$ up to complex conjugation for $j \in \{1, i\}$ and is trivial $G_j \to \{1\}$ for $j \notin \{1, i\}$, where $(i_j)_*\colon H_1(Y_j;\mathbb{Z}) \to H_1(X;\mathbb{Z})$ is induced by the inclusion $i_j\colon Y_j \hookrightarrow X$.

Remark 4.2 The condition (3) for the group G_1 in Theorem 4.1 is automatically satisfied if G_1 is not isomorphic to $\mathbb{Z}_4$ or to any binary dihedral groups [16, Lemma 2.1; 14, Section 6].

Remark 4.3 The author does not know whether the character $\chi \circ (i_j)_*\colon G_j \to U(1)$ for $j \in \{1, i\}$ in the statement of Theorem 4.1 is equal to the inclusion $G_j \subset U(1)$ itself or its complex conjugate.

Proof of Theorem 1.2 The lens space $L(a,b)$ is a special case of the spherical space form S^3/G for $G=\mathbb{Z}_a$ and, in particular, $L(m,1)$ (resp. $L(m,-1)$ is the positive (resp. negative) spherical space form as in Remark 3.8. Then $m\neq 4$ implies $G_1/N_1=\{1\}$ in the statement of Theorem 4.1. By the assumption that $\kappa=0$ is even, the assertion follows from Theorem 4.1. □

Example 4.4 Let $Y=L(m,1)$ and $(K,-m)$ be the trivial knot K in S^3 with framing $-m$, so that the $(-m)$–Dehn surgery along K is $Y=\chi_{-m}(K)$, and $W=B^4\cup(D^2\times D^2)$ be the compact oriented 4–manifold obtained by attaching 2–handle $D^2\times D^2$ along K with framing $-m$. Then W is a simply connected negative-definite 4–manifold such that $i_*\colon H_1(Y;\mathbb{Z})=\mathbb{Z}_m\to H_1(W;\mathbb{Z})=\{0\}$ is surjective. On the other hand, $H_2(W;\mathbb{Z})$ is generated by the 2–sphere S obtained by gluing a properly embedded disk in B^4 with boundary K and the core of the 2–handle $\{0\}\times D^2\subset D^2\times D^2$. Then, by taking the disk D in $D^2\times D^2$ with boundary $\partial D=\mu$ the meridian loop of K, the rational intersection pairing is $D\bullet D=-1/m$ and we have

$$\begin{aligned}\kappa&=\#\{aD\in H_2(W,Y;\mathbb{Z})\mid (aD)\bullet(aD)=-1/m,\ \partial_*(aD)=\pm\mu\}/\{\pm1\}\\&=\#\{a\in\mathbb{Z}\mid a^2=1,\ a\equiv\pm1 \pmod m\}/\{\pm1\}=1.\end{aligned}$$

Since κ is odd, we cannot apply Theorem 4.1.

Example 4.5 Let $Y=L(m,-1)=L(m,m-1)$ and $(L,\vec{m})=\bigcup_{i=1}^{m-1}(L_i,-2)$ be a linear chain of $m-1$ components of trivial knots L_i for $i\in\{1,\dots,m-1\}$ of framing -2 and with linking number $\operatorname{lk}(L_i,L_{i+1})=1$ for $i\in\{1,\dots,m-2\}$, so that the corresponding Dehn surgery is $Y=\chi_{\vec{m}}(L)$. Now denote $V_i=D^2\times D^2$ and let $W=B^4\cup\coprod_{i=1}^{m-1}V_i$ be the compact oriented 4–manifold obtained by attaching 2–handles V_i along L_i to the 4–ball B^4 with framing -2 for $i\in\{1,\dots,m-1\}$. Then W is a simply connected negative-definite 4–manifold such that $i_*\colon H_1(Y;\mathbb{Z})=\mathbb{Z}_m\to H_1(W;\mathbb{Z})=0$ is surjective. On the other hand, $H_2(W;\mathbb{Z})$ is generated by the 2–spheres S_i for $i\in\{1,\dots,m-1\}$ obtained by gluing a properly embedded disk in B^4 with boundary L_i and the core of the 2–handle $\{0\}\times D^2\subset V_i=D^2\times D^2$. Then, by taking the disk D_i in $V_i=D^2\times D^2$ with boundary $\partial D_i=\mu_i$ the meridian loop of L_i, the rational intersection pairing

$$H_2(W,Y;\mathbb{Z})\times H_2(W,Y;\mathbb{Z})\to\mathbb{Q}$$

is given by $(D_i\bullet D_j)=Q(W)^{-1}$, where $Q(W)=(S_i\bullet S_j)$ is the intersection matrix of W.

In particular, if $m = 3$ for example, then

$$(D_i \bullet D_j) = \frac{1}{3}\begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix},$$

and we have

$$\begin{aligned}\kappa &= \#\left\{ aD_1 + bD_2 \in H_2(W, Y; \mathbb{Z}) \;\middle|\; \begin{matrix} (aD_1 + bD_2)^2 = -\frac{1}{3}, \\ \partial_*(aD_1 + bD_2) = \pm\mu_1 \end{matrix} \right\} \Big/ \{\pm 1\} \\ &= \#\{(a, b) \in \mathbb{Z}^2 \mid 2(a^2 + ab + b^2) = 1,\ a + 2b \equiv \pm 1 \pmod{m}\}/\{\pm 1\} = 0.\end{aligned}$$

Here, we cannot apply Theorem 4.1 despite κ being even because $L(m, -1)$ is not positive.

4.2 Proof of the main theorem

In this section we give a proof of Theorem 4.1. Let (P, α) be the adapted SU(2)–bundle over X obtained by gluing the trivial adapted bundle $(X \times \mathrm{SU}(2), \theta_0)$ over X and the adapted bundle (Q, β) over $\mathbb{R} \times Y_1$ defined in Definition 3.3 of $c_2(Q, \beta) = 1/|G_1|$ associated with the positive subgroup G_1 of $\mathrm{SO}(4) = \mathrm{SU}(2) \times_{\{\pm 1\}} \mathrm{SU}(2)$ and with $\beta = \{\beta_\pm\}$, where $\beta_- = \theta$ and $\beta_+ = \alpha_1$ along Y_1, where α_1 is the flat connection over Y_1 corresponding to the inclusion $\rho\colon G_1 \hookrightarrow \mathrm{SU}(2)$ defined in Definition 3.7. Then $c_2(P, \alpha) = c_2(Q, \beta) = 1/|G_1|$. Note that $H^1(Y; \mathrm{ad}\,\gamma) = \{0\}$ for any flat connection γ over any spherical space form Y and $H^1(Y_0; \mathrm{ad}\,\theta) = \{0\}$ for the trivial flat connection θ over any $\mathbb{Q}$–homology 3–sphere Y_0. Fix a Riemannian metric g on X with product ends $g|_{(0,\infty)\times Y_i} = dt^2 + g_{Y_i}$. Let $\mathcal{M}(P, \alpha)$ be the moduli space of instantons on P,

$$\mathcal{M}(P, \alpha) = \{A \in \mathcal{A}(P, \alpha) \mid F_A^+ = 0\}/\mathcal{G}(P, \alpha).$$

The proof of Theorem 4.1 consists of a somewhat lengthy chain of lemmas. Here we sketch the main steps of the proof as follows:

(1) The moduli space $\mathcal{M}(P, \alpha)$ has virtual dimension one and the number of reducible instantons is even (Lemma 4.6).

(2) We construct a holonomy perturbation π to obtain the perturbed moduli space $\mathcal{M}^\pi(P, \alpha)$ such that the irreducible part of the moduli spaces $\mathcal{M}^\pi(P', \alpha')$ of the limiting flat bundles $(P', \alpha') \prec (P, \alpha)$ with virtual dimension negative are all empty except for the trivial one, $(X \times \mathrm{SU}(2), \theta_0)$. Taubes' gluing construction works in our setting to see there is at least one end component of $\mathcal{M}^\pi(P, \alpha)$.

(3) Take a sequence $\{[A_n]\}$ in $\mathcal{M}^\pi(P, \alpha)$. Since the energy of π–instantons on (P, α) is nearly $1/m$ no bubbling occurs and the series converges on (P, α) or chain

convergent to a π–instanton $[A']$ on some limiting flat adapted bundle (P',α') and a genuine instanton $[B_i']$ with energy $1/m$ on a limiting bundle (Q_i',β_i') over $\mathbb{R}\times Y_i$ for one $i\in\{1,\dots,s\}$ (sliding end).

(4) If $i\neq 1$ then we see Y_i must be the negative spherical space form $Y_i=S^3/G_i$ with $|G_i|=m$ and the flat connection $\alpha_i=\theta_i$ changes in the limit to the one corresponding to the standard inclusion $\rho_i\colon G_i\hookrightarrow \mathrm{Sp}(1)$ and the virtual dimension of the moduli space of the limiting bundle (P',α') is $-\dim\Gamma_{\rho_i}$. In particular, if G_i is cyclic then $-\dim\Gamma_{\rho_i}=-1$. On the other hand, if $i=1$ then $\alpha_1=\rho$ changes to the flat connection corresponding to the representation conjugate to $\varepsilon\colon G_1\hookrightarrow\{\pm1\}$. In this case the virtual dimension of the moduli space of the limiting bundle (P',α') is -3 and so that ε must be trivial by assumption and the perturbation π. Then we see that the limiting bundle must be trivial $(P',\alpha')\cong(X\times\mathrm{SU}(2),\theta_0)$ (Lemmas 4.7, 4.8 and 4.9).

(5) Suppose there is no choice of the limiting bundle $(P',\alpha')\prec(P,\alpha)$ except for the trivial one, $(X\times\mathrm{SU}(2),\theta_0)$. Then perturbing compact part of $\mathcal{M}^{\pi}(P,\alpha)$, we obtain a smooth 1–manifold $\mathcal{M}^{\pi+\sigma}(P,\alpha)$ with even number of boundary points and with one end, which is impossible. Hence, there exists a nontrivial limiting flat bundle (P',α') corresponding to the representation $\chi\colon\pi_1(X,x_0)\to\mathrm{Sp}(1)$ in the statement of Theorem 4.1. If all G_j with $|G_j|=m$ are cyclic then we use the proof by contradiction above to see there exists some limiting flat bundle (P',α') with nontrivial *reducible* flat connection corresponding to the character $\chi\colon H_1(X;\mathbb{Z})\to S^1$ we are looking for (Lemmas 4.10 and 4.11).

Now we start the proof with the following:

Lemma 4.6 *The moduli space $\mathcal{M}(P,\alpha)$ has virtual dimension one and the number κ of reducible instantons is even.*

Proof By Proposition 3.11(1), the virtual dimension of the moduli space $\mathcal{M}(P,\alpha)$ is equal to 1. If G_1 is cyclic, then the inclusion $G_1\hookrightarrow\mathrm{Sp}(1)$ comes from the reducible one, $\rho\colon G_1\hookrightarrow S^1$, and all other flat connections α_i are the trivial one. Hence, by Proposition 3.1 there is a one-to-one correspondence between the set $\mathcal{M}_{\mathrm{red}}(P,\alpha)\subset\mathcal{M}(P,\alpha)$ of all reducible instantons on (P,α) and the set

$$C(P,\alpha)=\left\{e\in H^2(X;\mathbb{Z})\;\middle|\; e^2=-\frac{1}{|G_1|},\ \begin{matrix} i_1^*e=\pm e(L_\rho),\\ i_j^*e=0\ \text{for } j\neq 1\end{matrix}\right\}\Big/\{\pm1\},$$

where $i_j\colon Y_j\hookrightarrow X$ is the inclusion and $L_\rho=S^3\times_\rho S^1\to Y_1$. On the other hand, if $G_1\subset\mathrm{Sp}(1)$ is not cyclic then there is no reducible instanton, $\mathcal{M}_{\mathrm{red}}(P,\alpha)=\varnothing$. □

Take any positive constant $N(\Sigma)>0$ satisfying $1/N(\Sigma)\le\min\{\tau(\Sigma_k,\theta)\,|\,k\in\{1,\dots,t\}\}$. Then, for any $m=|G_1|=\max\{|G_i|\}>N(\Sigma)$ the adapted SU(2)–bundle (P,α) over X satisfies

$$0<c_2(P,\alpha)=1/m=\min\{\tau(Y_i,\alpha_i)\mid i\in\{1,\dots,s+t\}\}\le\tfrac{1}{2}.$$

Note that $\tau(Y_{s+k},\alpha_{s+k})\neq 1/m$ for $k\in\{1,\dots,t\}$. Let $\mathcal{R}(Y,\mathrm{SU}(2))$ be the space of all conjugacy classes of SU(2) representations of the fundamental group $\pi_1(Y)$. Now $Y_i=S^3/G_i$ are spherical space form for $i\in\{1,\dots,s\}$ and hence $\#\mathcal{R}(Y_i,\mathrm{SU}(2))\le|G_i|$ is finite. Let $\mathcal{P}_0$ be the set of all adapted bundles (P',α') over X satisfying

(1) $c_2(P',\alpha')=0$,

(2) $\mathrm{ind}^+(P',\alpha')<0$,

(3) there are no reducible instantons on (P',α'), that is, $\mathcal{M}_{\mathrm{red}}(P',\alpha')=\varnothing$, and

(4) $\alpha'_{s+k}=\theta$ for $k\in\{1,\dots,t\}$.

By (1), $c_2(P',\alpha')=0$, the isomorphism class of adapted SU(2)–bundles $(P',\alpha')\in\mathcal{P}_0$ with given flat limit α' is unique. By (4), $\alpha'_{s+k}=\theta$ for $k\in\{1,\dots,t\}$ and $\#\mathcal{R}(Y_i,\mathrm{SU}(2))$ is finite for $i\in\{1,\dots,s\}$, so there are only finitely many possibilities for α'_i, so that the set $\mathcal{P}_0$ is finite. Then take a holonomy perturbation with perturbation data $\pi=(\{B_\nu,q_\nu,\omega_\nu,\varepsilon_\nu\},\mu)$ as in Proposition 3.17 applied to the case $\mathcal{P}'=\mathcal{P}_0$ and $\epsilon=1/m$.

Let $\{[A_n]\}$ be a sequence in $\mathcal{M}^\pi(P,\alpha)$. Then Proposition 3.15 applied to the case $\epsilon=1/m$ implies that, by taking gauge transformation if necessary, A_n converges to a π–instanton A' on the original bundle (P,α) or converges to a π–instanton A' on a limiting bundle (P',α') over X with an instanton B'_i formed on a limiting bundle (Q'_i,β'_i) over $\mathbb{R}\times Y_i$ for one $i\in\{1,\dots,s\}$ satisfying

$$c_2(P',\alpha')=0,\quad c_2(Q'_i,\beta'_i)=\frac{1}{m},$$

where

$$\alpha'_j=\alpha_j\quad\text{for } j\neq i,\qquad \alpha'_i=\beta_i'^{-},\qquad \beta_i'^{+}=\alpha_i.$$

Now if an adapted SU(2)–bundle (P',α') over X is isomorphic to a limiting bundle after "bubbling" or "sliding end" of a sequence $\{[A_n]\}$ of instantons on a bundle (P,α) over X with $c_2(P',\alpha')<c_2(P,\alpha)$ then we denote $(P',\alpha')\prec(P,\alpha)$.

Then, by Proposition 3.11, we have $\operatorname{vir\,dim}\mathcal{M}(P',\alpha')<0$, as follows:

Lemma 4.7 *Let $(P', \alpha') \prec (P, \alpha)$ be the limiting flat bundle. Then either one of the following holds:*

(1) *There exists $i \notin \{1, s+1, \dots, s+t\}$ such that Y_i must be negative and the flat connection α'_i corresponds to a representation conjugate to the standard inclusion $\rho_i\colon G_i \hookrightarrow \mathrm{Sp}(1)$ and $\alpha'_k = 1$ for all $k \neq i$, and*

$$\operatorname{vir}\dim \mathcal{M}(P', \alpha') = -\dim \Gamma_{\rho_i}.$$

In particular, if G_i is cyclic then $\operatorname{vir}\dim \mathcal{M}(P', \alpha') = -1 < 0$.

(2) *The flat connection α'_1 corresponds to a representation conjugate to a representation $\varepsilon\colon G_1 \to \{\pm 1\}$ and $\varepsilon(g) = 1$ for all $g \in G_1$ with* $\operatorname{ord} g \neq 4$ *and $\alpha'_k = 1$ for $k \neq 1$, and in this case* $\operatorname{vir}\dim \mathcal{M}(P', \alpha') = -3 < 0$.

Proof Let B'_i be a limiting instanton formed on a limiting adapted SU(2)–bundle (Q'_i, β'_i) over $\mathbb{R} \times S^3/G_i$ with

$$c_2(Q'_i, \beta'_i) = 1/|G_i| = 1/|G_1|,$$

which corresponds to a G_i–invariant instanton $\widetilde{B}'_i$ on a G_i–equivariant principal SU(2)–bundle $\widetilde{Q}'_i = \widetilde{Q}_0$ over $\mathbb{R} \times S^3$ associated with some G_i–action on $\widetilde{Q}_0$ given by $\beta_i'^{\pm}\colon G_i \to \mathrm{Sp}(1)$.

If $i \notin \{1, s+1, \dots, s+t\}$ then, by definition of the adapted bundle (P, α) at Y_i, we have $\beta_i'^{+} = 1$ and, by Proposition 3.9, Y_i is negative,

$$G_i \subset \{\pm 1\} \times_{\{\pm 1\}} \mathrm{Sp}(1) \subset \mathrm{Sp}(1) \times_{\{\pm 1\}} \mathrm{Sp}(1)$$

and $\beta_i'^{-}$ is conjugate to the inclusion $\rho_i\colon G_i \hookrightarrow \mathrm{Sp}(1)$. Then the limiting flat connection α'_i on the i^{th} component of the limiting adapted bundle (P', α') is conjugate to $\beta_i'^{-}$ and hence to the inclusion ρ_i. Then, by Proposition 3.11(3), we have

$$\operatorname{vir}\dim \mathcal{M}(P', \alpha') = -\dim \Gamma_{\rho_i}.$$

In particular, if G_i is cyclic then $\rho_i\colon G_i = \mathbb{Z}_m \hookrightarrow S^1$ for some S^1 subgroup of Sp(1) and $\Gamma_{\rho_i} \cong S^1$, so that the moduli space of the limiting bundle (P', α') has $\operatorname{vir}\dim \mathcal{M}(P', \alpha') = -1 < 0$.

On the other hand, suppose $i = 1$. Since Y_1 is positive,

$$G_1 \subset \mathrm{Sp}(1) \times_{\{\pm 1\}} \{\pm 1\} \subset \mathrm{Sp}(1) \times_{\{\pm 1\}} \mathrm{Sp}(1)$$

and, by definition of (P,α) to Y_1, we have ${\beta'_1}^+ = \rho_1$. By Proposition 3.9, we have ${\beta'_1}^- = \varepsilon$ and ρ_1 is conjugate to $\varepsilon\rho_1$. Now evaluate $g \in G_1$ and take the trace; we get

$$\operatorname{Tr} g = \operatorname{Tr} \rho_1(g) = \operatorname{Tr}(\varepsilon(g)\rho_1(g)) = \operatorname{Tr}(\varepsilon(g)g) \quad \text{for } g \in G_1.$$

Hence, $\varepsilon(g) = 1$ for $g \in G_1$ with $\operatorname{Tr} g \neq 0$. This condition is equivalent to the condition that $\varepsilon(g) = 1$ for $\operatorname{ord} g \neq 4$. The virtual dimension of the moduli space of the limiting bundle (P',α') is $\operatorname{vir}\dim \mathcal{M}(P',\alpha') = -3 < 0$ by Proposition 3.11(2). □

Lemma 4.8 *If the limiting bundle $(P',\alpha') \prec (P,\alpha)$ admits no reducible flat connection and $\operatorname{ind}^+(P',\alpha') < 0$, then (P',α') belongs to $\mathcal{P}_0$.*

Proof As we mentioned before the statement of Lemma 4.7, the limiting bundle (P',α') satisfies $c_2(P',\alpha') = 0$. Suppose (P',α') admits no reducible flat connection, $\mathcal{M}_{\mathrm{red}}(P',\alpha') = \varnothing$, and the virtual dimension of the moduli space $\mathcal{M}(P',\alpha')$ is negative, $\operatorname{ind}^+(P',\alpha') < 0$. Since $c_2(P,\alpha) = 1/m$, Proposition 3.15 applied to the case $\epsilon = 1/m$ implies that $\alpha'_{s+k} = \theta$ for $Y_{s+k} = \Sigma_k$ for $k \in \{1,\dots,t\}$ with $\tau(\Sigma_k,\theta) \geq 1/N(\Sigma) > 1/m = c_2(P,\alpha)$ and therefore $(P',\alpha') \in \mathcal{P}_0$. □

Lemma 4.9 *The representation $\varepsilon\colon G_1 \to \{\pm 1\}$ in Lemma 4.7(2) must be trivial and in this case $(P',\alpha') \cong (X \times \mathrm{SU}(2), \theta_0)$.*

Proof By Lemma 4.7(2), the virtual dimension of the moduli space $\mathcal{M}(P',\alpha')$ is negative, $\operatorname{ind}^+(P',\alpha') < 0$. Suppose (P',α') admits no reducible flat connection. Then, by Lemma 4.8, we see that $(P',\alpha') \in \mathcal{P}_0$. Since $c_2(P',\alpha') = 0$ and $\alpha'_i \in \mathcal{R}(Y_i, \mathrm{SU}(2))$ with $\#\mathcal{R}(Y_i, \mathrm{SU}(2))$ is finite for $i \in \{1,\dots,s\}$ and $\alpha'_{s+k} = \theta$ for $k \in \{1,\dots,t\}$, we see $\mathcal{P}_0$ is a finite set so that, by Proposition 3.16, we have $\mathcal{M}^\pi(P',\alpha') = \varnothing$. Hence, there is no divergent sequence in $\mathcal{M}^\pi(P,\alpha)$ weakly convergent to a connection in $\mathcal{M}^\pi(P',\alpha')$, which contradicts the assumption that $(P',\alpha') \prec (P,\alpha)$. Therefore, (P',α') admits a reducible flat connection A' over X with $\alpha'_1 = \varepsilon$. Then the corresponding holonomy representation $\operatorname{Hol}(A')$ of A' has a reduction to a circle subgroup S^1 of $\mathrm{Sp}(1)$ and satisfies $\operatorname{Hol}(A') \circ (i_1)_* = \varepsilon$, and we see that

$$\begin{aligned}\varepsilon \in \operatorname{Im}\bigl(\operatorname{Hom}(H_1(X), S^1) \to \operatorname{Hom}(G_1, S^1)\bigr)& \\ \cap \operatorname{Im}\bigl(\operatorname{Hom}(G_1/N_1, \{\pm 1\}) \to \operatorname{Hom}(G_1, S^1)\bigr) &= \{1\},\end{aligned}$$

where

$$N_1 = \langle g \in G_1 \mid \operatorname{ord} g \neq 4 \rangle,$$

and therefore $\varepsilon = 1$. Now we know that $\operatorname{Hol}(A') \circ (i_j)_* = 1$ for all $j \notin \{1, i\}$, so that $\operatorname{Hol}(A') \circ i_* = 1$ for $i_*\colon H_1(Y) \to H_1(X)$. Since $i_*\colon H_1(Y;\mathbb{Z}) \to H_1(X;\mathbb{Z})$ is surjective, the homomorphism $H_1(X;\mathbb{Z}) \to S^1$ induced by $\mathrm{Hol}(A')$ is trivial in all $H_1(X;\mathbb{Z})$, so that the reducible flat connection A' is trivial and therefore $(P',\alpha') \cong (X \times \mathrm{SU}(2), \theta_0)$. □

Lemma 4.10 (1) *There exists a sequence of instantons on (P,α) which weakly converges to some nontrivial flat $\mathrm{SU}(2)$–bundle (P',α'), that is, $(P',\alpha') \prec (P,\alpha)$.*

(2) *If all G_j with $|G_j| = m$ are cyclic then there exists some limiting adapted flat $\mathrm{SU}(2)$–bundle $(P',\alpha') \prec (P,\alpha)$ which supports a nontrivial reducible flat connection.*

Proof (1) By Theorem 3.19 applied to the case $\mathcal{P}' = \mathcal{P}_0$ and $\epsilon = 1/m$, the perturbed moduli space $\mathcal{M}^\pi(P,\alpha)$ has one end component $\mathcal{N} \cong \mathbb{R}_+$ admitting a divergent sequence weakly converging to the trivial flat connection θ on $(X \times \mathrm{SU}(2), \theta_0)$. Suppose there is no other limiting bundle $(P',\alpha') \prec (P,\alpha)$ except for the trivial bundle $(X \times \mathrm{SU}(2), \theta_0)$ then the complement $K = \mathcal{M}^\pi(P,\alpha) \setminus \mathcal{N}$ is compact. In fact, any sequence in $\mathcal{M}^\pi(P,\alpha)$ has a subsequence which converges to an instanton on (P,α) itself or weakly chain convergent to the trivial flat connection θ on $(X \times \mathrm{SU}(2), \theta_0)$ up to gauge transformation, and for the latter case, the subsequence must have a subsequence contained in the component $\mathcal{N}$. Note that $b_1(X) = 0$ and $b_2^+(X) = 0$. Then, by Proposition 3.2, we take a perturbation σ of $\mathcal{M}^\pi(P,\alpha)$ with support an open neighborhood of K such that the perturbed moduli space $\mathcal{M}^{\pi+\sigma}(P,\alpha)$ is a smooth 1–dimensional manifold with one end and even number $(=\kappa)$ of boundary points and this is impossible. Therefore, some (P',α') admits a nontrivial flat connection.

(2) Suppose all G_j with $|G_j| = m$ are cyclic, so that $\Gamma_{\rho_j} \cong S^1$ and suppose that any limiting bundle $(P',\alpha') \prec (P,\alpha)$ admits no nontrivial reducible flat connection over X. If (P',α') is not isomorphic to $(X \times \mathrm{SU}(2), \theta_0)$, then by assumption that there is no reducible on (P',α'), $\mathcal{M}_{\mathrm{red}}(P',\alpha') = \varnothing$, and

$$\mathrm{ind}^+(P',\alpha') = -\dim \Gamma_{\rho_j} = -1 < 0,$$

so, by Lemma 4.8, we have $(P',\alpha') \in \mathcal{P}_0$, and by Proposition 3.16, $\mathcal{M}^\pi(P',\alpha') = \varnothing$. Hence, the limiting bundle (P',α') must be isomorphic to $(X \times \mathrm{SU}(2), \theta_0)$ and the perturbed moduli space $\mathcal{M}^\pi(P,\alpha)$ has at most one end. Then the rest of the argument goes as in (1) to get a contradiction, so that some limiting bundle (P',α') admits a reducible *nontrivial* flat connection. □

Lemma 4.11 (1) *There exists some limiting bundle $(P',\alpha') \prec (P,\alpha)$ with a nontrivial flat connection which induces a representation $\chi\colon \pi_1(X,x_0) \to \mathrm{Sp}(1)$*

such that $\chi \circ (i_j)_*$ *is conjugate to the inclusion* $G_j \hookrightarrow \mathrm{Sp}(1)$ *for* $j = 1, i$ *and is trivial otherwise.*

(2) *If all* G_j *with* $|G_j| = m$ *are cyclic, there is a limiting bundle* $(P', \alpha') \prec (P, \alpha)$ *with a nontrivial reducible flat connection inducing a character* $\chi\colon H_1(X;\mathbb{Z}) \to S^1$ *such that* $\chi \circ (i_j)_*$ *is conjugate in* $\mathrm{Sp}(1)$ *to the inclusion* $G_j \hookrightarrow S^1$ *for* $j = 1, i$ *and is trivial otherwise.*

Proof (1) By Lemmas 4.10, 4.7 and 4.9, there exists a limiting bundle $(P', \alpha') \prec (P, \alpha)$ with a nontrivial flat connection A' on (P', α'), and over each ends $\mathbb{R}_+ \times Y_j$ the flat connection $\lim_{t\to\infty} A'|_{\{t\}\times Y_j} = \alpha'_j$ corresponds to the representations

$$\begin{aligned} \alpha'_1(g) &= e_+, && \text{where } g = [e_+, 1] \in G_1, \\ \alpha'_i(g) &= e_-, && \text{where } g = [1, e_-] \in G_i, \\ \alpha'_j &= 1 && \text{if } j \neq 1, i. \end{aligned}$$

Now the holonomy of A' gives a representation $\chi\colon \pi_1(X, x_0) \to \mathrm{Sp}(1)$ such that the composition $\chi \circ (i_j)_*$ is conjugate to the inclusion $G_j \hookrightarrow \mathrm{Sp}(1)$ for $j \in \{1, i\}$ and is conjugate to the trivial one otherwise.

(2) If all G_j with $|G_j| = m$ are cyclic, there is a limiting bundle $(P', \alpha') \prec (P, \alpha)$ with a nontrivial reducible flat connection A' and the flat connection α'_j over each ends $\mathbb{R}_+ \times Y_j$ as above. Now the holonomy representation of A' has a reduction to a subgroup S^1 of $\mathrm{Sp}(1)$ and hence factors through the Hurewicz homomorphism $\pi_1(X, x_0) \to H_1(X;\mathbb{Z})$ to give a character $\chi\colon H_1(X;\mathbb{Z}) \to S^1$ such that the composition $\chi \circ (i_j)_*$ is $\mathrm{Sp}(1)$–conjugate to the inclusion $G_j \hookrightarrow S^1$ for $j \in \{1, i\}$ and is conjugate to the trivial one otherwise. □

This completes the proof of Theorem 4.1.

5 Application to a uniformization problem of 4–orbifolds

In this section, we give an application of Theorem 4.1 to a uniformization problem of smooth 4–orbifolds. Here we recall the notion of orbifolds. For the definitions and basic facts concerning orbifolds see [30; 33]. A smooth orbifold $(X, \mathcal{F})$ is a pair of a Hausdorff space X and a collection $\mathcal{F}$ of $\{\widetilde{U}_\alpha, G_\alpha, \varphi_\alpha\}$ consisting of an open set $\widetilde{U}_\alpha \subset \mathbb{R}^n$, a finite group G_α acting on $\widetilde{U}_\alpha$ and a homeomorphism $\varphi_\alpha\colon U_\alpha \approx \widetilde{U}_\alpha / G_\alpha$ satisfying the property that $\{U_\alpha\}$ is an open covering of X which is closed under finite intersections and if $U_\alpha \subset U_\beta$ then there exists an injective homomorphism $h_{\beta\alpha}\colon G_\alpha \hookrightarrow G_\beta$ and a *smooth* embedding $\lambda_{\beta\alpha}\colon \widetilde{U}_\alpha \hookrightarrow \widetilde{U}_\beta$ equivariant with respect to $h_{\beta\alpha}$ which induces the inclusion $U_\alpha \subset U_\beta$.

An orbifold covering map $\pi\colon (\widetilde{X}, \widetilde{\mathcal{F}}) \to (X, \mathcal{F})$ is a continuous map $\pi\colon \widetilde{X} \to X$ such that each point $x \in X$ has a neighborhood U with a homeomorphism $\varphi\colon U \approx \widetilde{U}/G$ for some $\{\widetilde{U}, G, \varphi\} \in \mathcal{F}$ and each component $V_i \subset p^{-1}(U)$ admits a homeomorphism $\psi_i\colon V_i \approx \widetilde{U}/G_i$ for some subgroup $G_i \subset G$ so that $\{\widetilde{U}, G_i, \psi_i\} \in \widetilde{\mathcal{F}}$ and the natural projection $\widetilde{U}/G_i \to \widetilde{U}/G$ induces π restricted to U. If the covering transformation group $\mathrm{Aut}(\pi)$ of π is isomorphic to a group H then π is called an orbifold H–covering. Suppose a (finite) group G acts on a manifold M properly discontinuously, then M/G has a natural orbifold structure and the natural projection map $\pi\colon M \to M/G$ is an orbifold covering map and M is called a (finite) uniformization of M/G.

Now Theorem 1.5 is a special case of the following:

Theorem 5.1 *Let X be an oriented smooth 4–orbifold with finite isolated singular points $p_1, \dots, p_s$ in the interior of X whose neighborhood U_i of the singular point p_i is homeomorphic to the cone cY_i over a spherical space form $Y_i = S^3/G_i$ for $i \in \{1, \dots, s\}$ and with boundary a disjoint union of rational homology 3–spheres $\{\Sigma_k\}_{k=1}^t$. Suppose that X satisfies the following conditions:*

1. $b_1(X) = 0$ *and* $b_2^+(X) = 0$.
2. $\mathrm{Cok}(i_{\Sigma *}\colon H_1(\Sigma;\mathbb{Z}) \to H_1(X;\mathbb{Z})) = \{0\}$ *and* $\Sigma = \coprod_{k=1}^t \Sigma_k$.

Set $m = \max\{|G_i| \mid i \in \{1, \dots, s\}\}$. Then there exists a positive constant $N(\Sigma) > 0$, depending only on Σ, such that for any $m > N(\Sigma)$ the following property holds:

Let $U_1 \approx cY_1$ be the cone over a negative spherical space form $Y_1 = S^3/G_1$ satisfying the following conditions:

1. $|G_1| = m$.
2. *If G_1 is cyclic, then we assume that*

$$\kappa = \#\left\{ e \in H^2(X - \{p_1\};\mathbb{Z}) \,\middle|\, (\iota e)^2 = -\frac{1}{|G_1|},\ i_1^* e = \pm e(L_\rho),\ i_\Sigma^* e = 0 \right\} \Big/ \{\pm 1\}$$

is even, where ι is the composition

$$\iota\colon H^2(X - \{p_1\};\mathbb{Z}) \to H^2(X - \{p_1\};\mathbb{Q}) \to H^2(X;\mathbb{Q}),$$

$i_1\colon Y_1 \hookrightarrow X$ is the inclusion and $L_\rho = S^3 \times_{G_1} S^1 \to Y_1$ is the S^1–bundle associated with the inclusion $\rho\colon G_1 \hookrightarrow S^1 \subset \mathrm{SU}(2)$.

(3) *G_1 is not isomorphic to $\mathbb{Z}_4$ or a binary dihedral group.*

Then $U_i \approx cY_i$ for some positive spherical space form $Y_i = S^3/G_i$ for $i \neq 1$ with $|G_i| = m$ and the following holds:

(1) *There exists a smooth orbifold finite Galois covering $\pi\colon \widetilde{X} \to X$ such that each connected component V_j of $\pi^{-1}(U_j)$ is homeomorphic to $V_j \approx \mathbb{C}^2/H_j$, where $H_j = \{e\}$ for $j \in \{1,i\}$ and $H_j = G_j$ otherwise, and the natural map $\mathbb{C}^2/H_j \to \mathbb{C}^2/G_j$ induces π restricted to V_j.*

(2) *If $\pi_1(X,x_0)$ is generated by $(i_{\Sigma_k})_{\#}\pi_1(\Sigma_k,z_k)$ for $k \in \{1,\dots,t\}$ with basepoints $x_0 \in X$, $z_k \in \Sigma_k$, where $(i_{\Sigma_k})_{\#}$ is the homomorphism induced by the inclusion $i_{\Sigma_k}\colon \Sigma_k \hookrightarrow X$, there exists $h \in \mathrm{SU}(2)$ and a torsion-free finite-index normal subgroup N' of the group Γ' generated by $G_1 \cup hG_ih^{-1}$ in $\mathrm{SU}(2)$ and a smooth orbifold Γ'/N'–covering $\pi\colon \widetilde{X} \to X$ such that each connected component V_j of $\pi^{-1}(U_j)$ is homeomorphic to $V_j \approx \mathbb{C}^2/H_j$, where $H_j = \{e\}$ for $j \in \{1,i\}$ and $H_j = G_j$ otherwise, and the natural map $\mathbb{C}^2/H_j \to \mathbb{C}^2/G_j$ induces π restricted to V_j.*

(3) *If all G_j with $|G_j| = m$ are cyclic, then $G_i = \mathbb{Z}_m$ and there exists a smooth orbifold $\mathbb{Z}_m$–covering $\pi\colon \widetilde{X} \to X$ such that each connected component V_j of $\pi^{-1}(U_j)$ is homeomorphic to $V_j \approx \mathbb{C}^2/H_j$, where $H_j = \{e\}$ for $j \in \{1,i\}$ and $H_j = \mathbb{Z}_m$ otherwise, and the natural map $\mathbb{C}^2/H_j \to \mathbb{C}^2/\mathbb{Z}_m$ induces π restricted to V_j.*

Proof Let W be a 4–manifold obtained by removing the neighborhoods $U_i \approx cY_i$ of singular points p_i for $i \in \{1,\dots,s\}$. Then W is a compact smooth 4–manifold with boundary Y the disjoint union of spherical space forms $Y_i = -S^3/G_i$ for $i \in \{1,\dots,s\}$ and t–components of rational homology 3–spheres $Y_{s+i} = \Sigma_i$ for $i \in \{1,\dots,t\}$. Since X is negative-definite with respect to the rational intersection pairing $H_2(X;\mathbb{Q}) \times H_2(X;\mathbb{Q}) \to \mathbb{Q}$ as a rational homology manifold, we see $H_2(W;\mathbb{Q}) \cong H_2(X;\mathbb{Q})$ and therefore W has a negative-definite intersection pairing $H_2(W;\mathbb{Q}) \times H_2(W;\mathbb{Q}) \to \mathbb{Q}$. By assumption that $i_{\Sigma *}\colon H_1(\Sigma;\mathbb{Z}) \to H_1(X;\mathbb{Z})$ is surjective, the homomorphism $i_*\colon H_1(Y;\mathbb{Z}) \to H_1(W;\mathbb{Z})$ induced by inclusion $i\colon Y \hookrightarrow W$ is surjective. Note also that by the assumption

$$\kappa = \frac{1}{2}\#\Big\{e \in H^2(W;\mathbb{Z}) \;\Big|\; e^2 = -\frac{1}{m},\ i_1^*e = \pm e(L_\rho),\ i_j^*e = 0 \text{ if } j \neq 1\Big\}$$

is even, $Y_1 = -S^3/G_1$ is a positive spherical space form and $m = |G_1| = \max\{|G_i|\} > N(\Sigma)$ and G_1 is not isomorphic to $\mathbb{Z}_4$ or a binary dihedral group. Then by Theorem 4.1

we see $|G_i| = |G_1|$ and $Y_i = -S^3/G_i$ is a negative spherical space form for some $i \neq 1$ and there exists a representation $\chi\colon \pi_1(W, x_0) \to \mathrm{SU}(2)$ such that $\chi \circ (i_j)_*$ is the inclusion $\rho\colon G_j \hookrightarrow \mathrm{SU}(2)$ up to $\mathrm{SU}(2)$–conjugation for $j \in \{1, i\}$ and is trivial otherwise, where $(i_j)_* = (\overline{\delta}_j)_* \circ (i_j)_\#\colon \pi_1(Y_j, y_j) \to \pi_1(W, x_0)$ is a homomorphism obtained by composing the isomorphism $(\delta_j)_*\colon \pi_1(X, x_0) \to \pi_1(X, y_j)$ induced by a path δ_j from x_0 to y_j in W and the homomorphism $(i_j)_\#\colon \pi_1(Y_j, y_j) \to \pi_1(W, y_j)$ induced from the inclusion $i_j\colon Y_j \hookrightarrow W$.

(1) Now we have a representation $\chi\colon \pi_1(W, x_0) \to \mathrm{SU}(2)$. Then we can apply an argument in the proof of Theorem 1.5.10 in [26] due to Namba. Note that the image $\Gamma = \chi(\pi_1(W, x_0)) \subset \mathrm{SU}(2) \subset \mathrm{GL}(2, \mathbb{C})$ is finitely generated and $\Gamma_j = (\chi \circ i_{j*})(G_j) \subset \mathrm{SU}(2)$ is isomorphic to G_j if $j \in \{1, i\}$ and $\Gamma_j = \{e\}$ otherwise. By a lemma of A Selberg [31, Lemma 8] — or see an elementary proof in [1] due to R C Alperin — there exists a torsion-free normal subgroup N of Γ of finite index. Then we take a quotient $\chi_N\colon \pi_1(W, x_0) \to \Gamma \to \Gamma/N$. Let $\pi\colon \widetilde{W} \to W$ be the Γ/N–covering space associated with the surjective homomorphism $\chi_N\colon \pi_1(W, x_0) \to \Gamma/N$. Since $\chi \circ (i_j)_*$ is conjugate to $\rho\colon G_j \subset \mathrm{SU}(2)$ for $j \in \{1, i\}$, we see $\chi_N \circ (i_j)_*$ is conjugate to the induced homomorphism $\rho_N\colon G_j \subset \Gamma \to \Gamma/N$. Since N is torsion-free, $N \cap \Gamma_j = \{e\}$ and so $\chi_N \circ (i_j)_*$ is conjugate to an injection $G_j \hookrightarrow \Gamma/N$ for $j \in \{1, i\}$. Hence, the group of covering transformations restricted to a single connected component $\widetilde{Y}_j$ of the inverse image $\pi^{-1}(Y_j)$ is isomorphic to the image $\overline{\Gamma}_j$ of Γ_j in Γ/N. For $j \in \{1, i\}$, $\overline{\Gamma}_j$ is isomorphic to G_j, and $\widetilde{Y}_j/G_j \approx S^3/G_j$ so that $\widetilde{Y}_j \approx S^3$, and $\pi^{-1}(Y_j)$ is a disjoint union of $|\Gamma/N|/m$ copies of S^3. For $j \notin \{1, j\}$, $\chi \circ (i_j)_* = 1$ implies $\overline{\Gamma}_j = \{e\}$, so that $\pi^{-1}(Y_j)$ is a disjoint union of $|\Gamma/N|$ copies of $Y_j = S^3/G_j$ for $j \in \{1, \dots, s\} \setminus \{1, i\}$ and $|\Gamma/N|$ copies of $Y_{s+j} = \Sigma_j$ for $j \in \{1, \dots, t\}$. Then the 4–orbifold $\widetilde{X}$ obtained by gluing two B^4 along $\widetilde{Y}_1$, $\widetilde{Y}_i \approx S^3$ and gluing Γ/N copies of cY_j along $Y_j = S^3/G_j$ for each $j \in \{1, \dots, s\} \setminus \{1, i\}$ and the induced map $\pi\colon \widetilde{X} \to X$ satisfies the assertion.

(2) Now set $\chi_j = \chi \circ (i_j)_*$; then, by taking a conjugation if necessary, we may take

$$\chi_1(\pi_1(Y_1, y_1)) = \Gamma_1, \quad \chi_i(\pi_1(Y_i, y_i)) = h\Gamma_i h^{-1},$$
$$\chi_j(\pi_1(Y_j, y_j)) = \{e\} \quad \text{if } j \notin \{1, i\}$$

for some $h \in \mathrm{SU}(2)$. If $\pi_1(X, x_0)$ is generated by $(i_{\Sigma_k})_*\pi_1(\Sigma_k, z_k)$ for $k \in \{1, \dots, t\}$, $\pi_1(W, x_0)$ is generated by $(i_j)_*\pi_1(Y_j, y_j)$ for $j \in \{1, \dots, s\}$ and $(i_{\Sigma_k})_*\pi_1(\Sigma_k, z_k)$ for $k \in \{1, \dots, t\}$ and hence the representation χ induces a surjective homomorphism $\chi\colon \pi_1(W, x_0) \to \Gamma'$, where Γ' is the subgroup of $\mathrm{SU}(2)$ generated by $\Gamma_1 \cup h\Gamma_i h^{-1}$.

Then, by Selberg's lemma again, there exists a torsion-free finite-index normal subgroup N' of Γ', and we take the quotient $\chi_{N'}\colon \pi_1(W,x_0)\to\Gamma'\to\Gamma'/N'$ to the finite group Γ'/N'. Since N' is torsion-free, $N'\cap\Gamma_1=\{e\}$ and $N'\cap h\Gamma_i h^{-1}=\{e\}$, so that the maps $\Gamma'\to\Gamma'/N'$ restricted to Γ_1 and $h\Gamma_i h^{-1}$ are injective. Let $\pi\colon \widetilde{W}\to W$ be the Γ'/N'–covering space associated with the surjective homomorphism $\chi'_N\colon \pi_1(W,x_0)\to\Gamma'/N'$. Since $\chi\circ(i_j)_*\cong\rho$, $\rho\colon G_j\subset \mathrm{SU}(2)$ for $j\in\{1,i\}$, the group of covering transformations restricted to a single connected component $\widetilde{Y}_j$ of the inverse image $\pi^{-1}(Y_j)$ is isomorphic to the image $\overline{\Gamma}_j$ of Γ_j in Γ'/N'. For $j\in\{1,i\}$, $\overline{\Gamma}_j$ is isomorphic to G_j, and $\widetilde{Y}_j/G_j\approx S^3/G_j$, so that $\widetilde{Y}_j\approx S^3$, and $\pi^{-1}(Y_j)$ is a disjoint union of $|\Gamma'/N'|/m$ copies of S^3. For $j\notin\{1,j\}$, $\chi_{N'}\circ(i_j)_*=1$ implies $\overline{\Gamma}'_j=\{e\}$, so that $\pi^{-1}(Y_j)$ is a disjoint union of $|\Gamma'/N'|$ copies of $Y_j=S^3/G_j$ for $j\in\{1,\dots,s\}\setminus\{1,i\}$ and $|\Gamma'/N'|$ copies of $Y_{s+j}=\Sigma_j$ for $j\in\{1,\dots,t\}$. Then the 4–orbifold $\widetilde{X}$ obtained by gluing $|\Gamma'/N'|/m$ copies of B^4 along $\widetilde{Y}_j\approx S^3$ for $j\in\{1,i\}$ and gluing $|\Gamma'/N'|$ copies of cY_j along $Y_j=S^3/G_j$ for each $j\in\{1,\dots,s\}-\{1,i\}$ and the induced map $\pi\colon\widetilde{X}\to X$ satisfies the assertion.

(3) If all G_j with $|G_j|=m$ are cyclic, then by Theorem 4.1 we see $Y_i=L(m,-1)$ for some $i\neq 1$ and there exists a character $\chi\colon H_1(W;\mathbb{Z})\to U(1)$ such that $\chi\circ i_{j*}$ is the inclusion $\rho\colon \mathbb{Z}_m\hookrightarrow U(1)$ up to complex conjugation for $j\in\{1,i\}$ and is trivial otherwise. Since $i_*\colon H_1(Y;\mathbb{Z})\to H_1(W;\mathbb{Z})$ is surjective, we see the character χ induces the surjective homomorphism $\chi\colon H_1(W;\mathbb{Z})\to\mathbb{Z}_m$. Let $\pi\colon\widetilde{W}\to W$ be the $\mathbb{Z}_m$–covering space associated with the surjective homomorphism $\pi_1(W)\to H_1(W;\mathbb{Z})\to\mathbb{Z}_m$ obtained by composing χ with the Hurewicz homomorphism. Since $\chi\circ i_{j*}=\rho^{\pm1}$ and $\rho\colon\mathbb{Z}_m\subset U(1)$ for $j\in\{1,i\}$, the group G_j of covering transformations $g\in\mathbb{Z}_m$ is the cyclic group $\mathbb{Z}_m$ itself and acting on the single connected component $\widetilde{Y}_j$ of the inverse image $\pi^{-1}(Y_j)$ with $\widetilde{Y}_j/\mathbb{Z}_m\approx L(m,\pm1)$, so that $\widetilde{Y}=S^3$. On the other hand, $\chi\circ i_{j*}=1$ for $j\notin\{1,j\}$ implies that $\pi^{-1}(Y_j)$ is a disjoint union of m copies of $Y_j=L(a_j,b_j)$ for $j\in\{1,\dots,s\}\setminus\{1,i\}$ and m copies of $Y_{s+j}=\Sigma_j$ for $j\in\{1,\dots,t\}$. Then the 4–orbifold $\widetilde{X}$ obtained by gluing two B^4 along $\widetilde{Y}_1$, $\widetilde{Y}_i\approx S^3$ and gluing m copies of $cL(a_j,b_j)$ along $Y_j=L(a_j,b_j)$ for each $j\in\{1,\dots,s\}\setminus\{1,i\}$ and the induced map $\pi\colon\widetilde{X}\to X$ satisfies the assertion. □

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