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Truncated Heegaard Floer homology and knot concordance invariants

LINH TRUONG

We construct a sequence of smooth concordance invariants $\nu_n(K)$ defined using truncated Heegaard Floer homology. The invariants generalize the concordance invariants ν of Ozsváth and Szabó and ν^+ of Hom and Wu. We exhibit an example in which the gap between two consecutive elements in the sequence ν_n can be arbitrarily large. We also prove that the sequence ν_n contains more concordance information than τ, ν, ν', ν^+ and $\nu^{+'}$.

57M25, 57M27, 57R58

1 Introduction

Two knots, K_0 and K_1, in S^3 are *smoothly concordant* if there is a smooth proper embedding of a cylinder $S^1\times[0,1]$ into $S^3\times[0,1]$ with boundary $S^1\times\{0\}=K_0$ and $S^1\times\{1\}=-K_1$. The Heegaard Floer homology package of Ozsváth and Szabó has led to a wealth of smooth concordance invariants. A survey on this topic can be found in Hom [6].

Two concordance invariants motivate this article:

(1) $\nu(K)$, defined by Ozsváth and Szabó [14] using maps on Heegaard Floer homology $\widehat{\mathrm{HF}}$ induced by the two-handle cobordism corresponding to integral surgery along K.

(2) $\nu^+(K)$, defined by Hom and Wu [7] using maps induced by surgery on HF^+. Hom and Wu showed that $\nu^+(K)$ produces arbitrarily better four-ball genus bounds than $\nu(K)$.

We construct a sequence of concordance invariants $\nu_n(K)$ for $n\in\mathbb{Z}$ which are defined using maps induced by surgery on the truncated Heegaard Floer homology HF^n. The invariants $\nu_n(K)$ generalize $\nu(K)$ and $\nu^+(K)$, as $\nu_1(K)=\nu(K)$ and $\nu_n(K)=\nu^+(K)$ for n sufficiently large. The properties of $\nu_n(K)$ are stated below.

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Theorem 1.1 *The knot invariants $\nu_n(K)$, where $n \in \mathbb{Z}$, satisfy the following properties:*

- $\nu_n(K)$ *is a concordance invariant.*
- $\nu_1(K) = \nu(K)$.
- $\nu_n(K) \le \nu_{n+1}(K)$.
- *For sufficiently large n, $\nu_n(K) = \nu^+(K)$.*
- $\nu_n(-K) = -\nu_{-n}(K)$, *where $-K$ is the mirror of K.*
- $\nu_n(K) \le g_4(K)$.

By an extension of the large integer surgery formulas to truncated Floer homology (see Propositions 3.1 and 3.2) the invariants $\nu_n(K)$ can be computed from the $\mathbb{Z}\oplus\mathbb{Z}$–filtered knot Floer chain complex $\mathrm{CFK}^\infty(K)$.

Homologically thin knots are a special class of knots whose knot Floer homology lies in a single $\delta = A-M$ grading, where A is the Alexander grading and M is the Maslov grading. We prove that $\nu_n(K)$ of thin knots only depends on $\tau(K)$.

Proposition 4.1 *Let K be a homologically thin knot with $\tau(K) = \tau$.*

(i) *If $\tau = 0$, $\nu_n(K) = 0$ for all n.*

(ii) *If $\tau > 0$,*

$$\nu_n(K) = \begin{cases} 0 & \text{for } n \le -\tfrac{1}{2}(\tau+1), \\ \tau + 2n + 1 & \text{for } -\tfrac{1}{2}\tau \le n \le -1, \\ \tau & \text{for } n \ge 0. \end{cases}$$

(iii) *If $\tau < 0$,*

$$\nu_n(K) = \begin{cases} \tau & \text{for } n \le 0, \\ \tau + 2n - 1 & \text{for } 1 \le n \le -\tfrac{1}{2}\tau, \\ 0 & \text{for } n \ge \tfrac{1}{2}(-\tau+1). \end{cases}$$

The computation of $\nu_n(K)$ for thin knots illustrates that the gap between $\nu_n(K)$ and $\nu_{n+1}(K)$ can be more than one. In fact, the gap between $\nu_n(K)$ and $\nu_{n+1}(K)$ can be arbitrarily big.

Theorem 1.2 *Let $T_{p,p+1}$ denote the $(p,p+1)$–torus knot. For $p > 3$,*

$$\nu_{-1}(T_{p,p+1}) - \nu_{-2}(T_{p,p+1}) = p.$$

Finally, we show that the sequence ν_n contains more concordance information than τ, ν, ν', ν^+ and $\nu^{+'}$.

Proposition 1.3 *There exist knots K and K' with equal τ, ν, ν', ν^+ and $\nu^{+'}$ invariants, but $\nu_n(K) \neq \nu_n(K')$ for some $n \in \mathbb{Z}$.*

Organization of the paper In Section 2 we review the constructions of the concordance invariants $\nu(K)$ and $\nu^+(K)$. In Section 3 we define the invariants $\nu_n(K)$ and prove their properties: monotonicity, stabilization and behavior under mirroring. In Section 4 we compute $\nu_n(K)$ for special families of knots and compare them to $\nu(K)$ and $\nu^+(K)$. In Section 5 we pose some questions about the concordance invariants $\nu_n(K)$.

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2 Background on the invariants $\nu(K)$ and $\nu^+(K)$

A four-dimensional cobordism equipped with a Spin^c structure between two three-manifolds induces a map on the Heegaard Floer homology groups [12]. In particular, for a knot K in S^3, the 2–handle attachment cobordism from $S^3_N(K)$ or $S^3_{-N}(K)$ to S^3 induces maps

(1) $\hat{v}_{s,*}\colon \widehat{\mathrm{HF}}(S^3_N(K), \mathfrak{s}_s) \to \widehat{\mathrm{HF}}(S^3)$, $\quad \hat{v}'_{s,*}\colon \widehat{\mathrm{HF}}(S^3) \to \widehat{\mathrm{HF}}(S^3_{-N}(K), \mathfrak{s}_s)$,

(2) $v^+_{s,*}\colon \mathrm{HF}^+(S^3_N(K), \mathfrak{s}_s) \to \mathrm{HF}^+(S^3)$, $\quad v^{+'}_{s,*}\colon \mathrm{HF}^+(S^3) \to \mathrm{HF}^+(S^3_{-N}(K), \mathfrak{s}_s)$,

(3) $v^-_{s,*}\colon \mathrm{HF}^-(S^3_N(K), \mathfrak{s}_s) \to \mathrm{HF}^-(S^3)$, $\quad v^{-'}_{s,*}\colon \mathrm{HF}^-(S^3) \to \mathrm{HF}^-(S^3_{-N}(K), \mathfrak{s}_s)$,

where $\mathfrak{s}_s$ denotes the restriction to $S^3_N(K)$ or $S^3_{-N}(K)$ of a Spin^c structure $\mathfrak{t}$ on the corresponding 2–handle cobordism such that

$$\langle c_1(\mathfrak{t}), [\hat{F}]\rangle + N = 2s, \quad \langle c_1(\mathfrak{t}), [\hat{F}]\rangle - N = 2s,$$

where $\hat{F}$ is a capped-off Seifert surface for K. These cobordism maps on $\widehat{\mathrm{HF}}$ and HF^+ play a key role in defining the concordance invariants ν and ν^+.

Definition 2.1 [14, Section 9] The concordance invariant $\nu(K)$ is defined as

$$\nu(K) = \min\{s \in \mathbb{Z} \mid \hat{v}_{s,*} \text{ is surjective}\}.$$

Definition 2.2 The concordance invariant $\nu'(K)$ is defined as

$$\nu'(K) = \max\{s \in \mathbb{Z} \mid \hat{v}'_{s,*} \text{ is injective}\}.$$

For a rational homology 3–sphere Y with a Spin^c structure $\mathfrak{s}$, $\mathrm{HF}^+(Y,\mathfrak{s})$ can be decomposed as the direct sum of two groups: the first group is the image of $\mathrm{HF}^\infty(Y,\mathfrak{s}) \cong \mathbb{F}[U,U^{-1}]$ in $\mathrm{HF}^+(Y,\mathfrak{s})$, which is isomorphic to $\mathfrak{T}^+ = \mathbb{F}[U,U^{-1}]/U\mathbb{F}[U]$; the second group is $\mathrm{HF}_{\mathrm{red}}(Y,\mathfrak{s}) = \mathrm{HF}^+(Y,\mathfrak{s})/\mathfrak{T}^+$. That is,

$$\mathrm{HF}^+(Y,\mathfrak{s}) = \mathfrak{T}^+ \oplus \mathrm{HF}_{\mathrm{red}}(Y,\mathfrak{s}).$$

Definition 2.3 [7] The concordance invariant ν^+ is defined as

$$\nu^+(K) = \min\{s \mid v^+_{s,*}\colon \mathrm{HF}^+(S^3_N(K),\mathfrak{s}_s) \to \mathrm{HF}^+(S^3) \text{ sends } 1 \text{ to } 1\},$$

where 1 denotes the lowest-graded generator in the subgroup $\mathfrak{T}^+$ of the homology, and N is sufficiently large that the integer surgery formula holds.

Equivalently, Hom [6] defines the invariant $\nu^-(K)$ in terms of the map

$$v^-_{s,*}\colon \mathrm{HF}^-(S^3_N K,\mathfrak{s}_s) \to \mathrm{HF}^-(S^3).$$

Definition 2.4 [6] The concordance invariant

$$\nu^-(K) = \min\{s \in \mathbb{Z} \mid v^-_{s,*} \text{ is surjective}\}$$

is equal to $\nu^+(K)$.

Hom and Wu show that

$$\tau(K) \le \nu(K) \le \nu^+(K)$$

and $\nu^+(K) \ge 0$. In addition, ν^+ gives a lower bound on the four-ball genus, $\nu^+(K) \le g_4(K)$. Furthermore, Hom and Wu provide a family of knots where $\nu^+(K)$ is an arbitrarily better bound on $g_4(K)$ than $\tau(K)$.

The concordance invariants ν and ν^+ are easily computable from $\mathrm{CFK}^\infty(K)$ via the large integer surgery formulas. Let CX denote the subgroup of $\mathrm{CFK}^\infty(K)$ generated by elements $[x, i, j]$ that lie in filtration level $(i, j) \in X \subset \mathbb{Z} \oplus \mathbb{Z}$. Consider the chain maps

$$\hat{v}_s\colon C\{\max(i, j-s) = 0\} \to C\{i = 0\},$$
$$v_s^+\colon C\{\max(i, j-s) \geq 0\} \to C\{i \geq 0\},$$

defined by taking the quotient by $C\{i < 0,\ j = s\}$ or $C\{i < 0,\ j \geq s\}$, respectively, followed by the inclusions. The large integer surgery formula of Ozsváth and Szabó [11] asserts that the maps $\hat{v}_s$ and v_s^+ induce the maps from (1) and (2). Similarly, consider the chain maps

$$\hat{v}'_s\colon C\{i = 0\} \to C\{\min(i, j-s) = 0\},$$
$$v_s^{+'}\colon C\{i \geq 0\} \to C\{\min(i, j-s) \geq 0\},$$

consisting of quotienting by $C\{i = 0,\ j \leq s\}$ followed by the inclusion. Ozsváth and Szabó [11] show that these maps induce the maps from (1) and (2).

We introduce a concordance invariant $\nu^{+'}$, so that the pair ν^+ and $\nu^{+'}$ is the HF^+ analogue to the pair ν and ν'.

Definition 2.5 The concordance invariant $\nu^{+'}$ is defined as

$$\nu^{+'}(K) = \max\{s \in \mathbb{Z} \mid v_{s,*}^{+'}\colon \mathrm{HF}^+(S^3) \to \mathrm{HF}^+(S^3_{-N}(K), \mathfrak{s}_s) \text{ is injective}\},$$

where $-N$ is sufficiently negative that the (negative) large integer surgery formula holds.

We prove a mirroring property which relates $\nu^{+'}(K)$ to the invariant $\nu^+(-K)$ of the mirror of K:

Lemma 2.6 $$\nu^{+'}(K) = -\nu^+(-K).$$

Proof Recall the symmetry of CFK^∞ under mirroring [11, Section 3.5],

$$\mathrm{CFK}^\infty(-K) \simeq \mathrm{CFK}^\infty(K)^*,$$

where $\mathrm{CFK}^\infty(K)^*$ is the dual complex $\mathrm{Hom}_{\mathbb{F}[U,U^{-1}]}(\mathrm{CFK}^\infty(K), \mathbb{F}[U, U^{-1}])$. Therefore,

$$v_{-s,*}^{+'}\colon \mathrm{HF}^+(S^3) \to \mathrm{HF}^+(S^3_{-N}(K), \mathfrak{s}_{-s}) \text{ is injective}$$
$$\iff v_{s,*}^-\colon \mathrm{HF}^-(S^3_N(-K), \mathfrak{s}_s) \to \mathrm{HF}^-(S^3) \text{ is surjective},$$

which implies the result. □

It follows from the above lemma that the invariant $\nu^{+'}$ exhibits properties similar to ν^+,

$$\nu^{+'}(K) \le \nu'(K) \le \tau(K) \le \nu(K) \le \nu^+(K)$$

and $\nu^{+'}(K) \le 0$. In addition, the absolute value of $\nu^{+'}(K)$ gives a lower bound on the four-ball genus:

Theorem 2.7 $$|\nu^{+'}(K)| \le g_4(K).$$

Proof This follows from the fact that $\nu^+(K) \le g_4(K)$ and Lemma 2.6. □

3 The concordance invariants $\nu_n(K)$

The construction of the concordance invariants $\nu_n(K)$ uses truncated Heegaard Floer homology $\mathrm{HF}^n(Y,\mathfrak{s})$, described in [9; 13]. $\mathrm{HF}^n(Y,\mathfrak{s})$ is the homology of the kernel $\mathrm{CF}^n(Y,\mathfrak{s})$ of the multiplication map

$$U^n\colon \mathrm{CF}^+(Y,\mathfrak{s}) \to \mathrm{CF}^+(Y,\mathfrak{s}),$$

where $n \in \mathbb{Z}_+$. The two-handle cobordism from $S^3_N K$ or $S^3_{-N} K$, respectively, to S^3 induces a map on the truncated Floer chain complex,

$$v^n_s\colon \mathrm{CF}^n(S^3_N K, \mathfrak{s}_s) \to \mathrm{CF}^n(S^3), \quad v^{-n}_s\colon \mathrm{CF}^n(S^3) \to \mathrm{CF}^n(S^3_{-N} K, \mathfrak{s}_s),$$

and on the truncated Floer homology,

$$v^n_{s,*}\colon \mathrm{HF}^n(S^3_N K, \mathfrak{s}_s) \to \mathrm{HF}^n(S^3), \quad v^{-n}_{s,*}\colon \mathrm{HF}^n(S^3) \to \mathrm{HF}^n(S^3_{-N} K, \mathfrak{s}_s),$$

where $\mathfrak{s}_s$ denotes the restriction to $S^3_N(K)$ or $S^3_{-N}(K)$, respectively, of a Spin^c structure $\mathfrak{t}$ on the corresponding 2–handle cobordism such that

$$\langle c_1(\mathfrak{t}), [\hat{F}]\rangle + N = 2s, \quad \langle c_1(\mathfrak{t}), [\hat{F}]\rangle - N = 2s,$$

where $\hat{F}$ is a capped-off Seifert surface for K. These cobordism maps on $\widehat{\mathrm{HF}}$ and HF^+ play a key role in defining the concordance invariants ν and ν^+.

We extend the large integer surgery formula of Ozsváth and Szabó to truncated Heegaard Floer homology:

Proposition 3.1 (large negative integer surgery formula for HF^n) *Consider the subquotient complex*

$$\mathrm{CFK}^{-n}(S^3, K, m) = C\{0 \le \min(i, j-m) \le n-1\}$$

of $\mathrm{CFK}^+(S^3, K, m)$ *generated by* $[\boldsymbol{x}, i, j]$ *with* $0 \le \min(i, j-m) \le n-1$. *For each* $m \in \mathbb{Z}$, *there is an integer* $N(m)$ *such that for all* $p \ge N(m)$, *the map* Φ *of Ozsváth and Szabó induces isomorphisms in the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{CFK}^{-n}(S^3, K, m) & \longrightarrow & \mathrm{CFK}^+(S^3, K, m) & \xrightarrow{U^n} & \mathrm{CFK}^+(S^3, K, m) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \Phi(n)} & & \big\downarrow{\scriptstyle \Phi^+} & & \big\downarrow{\scriptstyle \Phi^+} & & \\ 0 & \longrightarrow & \mathrm{CF}^n(S^3_{-p}K, [m]) & \longrightarrow & \mathrm{CF}^+(S^3_{-p}K, [m]) & \xrightarrow{U^n} & \mathrm{CF}^+(S^3_{-p}K, [m]) & \longrightarrow & 0 \end{array}$$

Proof The map Φ^+ is an isomorphism of chain complexes by Theorem 4.1 of [11]. By the five lemma, so is $\Phi(n)$. □

Proposition 3.2 (large positive integer surgery formula for HF^n) *Consider the subquotient complex*

$$\mathrm{CFK}^{n}(S^3, K, m) = C\{0 \le \max(i, j-m) \le n-1\}$$

of $\mathrm{CFK}^+(S^3, K, m)$ *generated by* $[\boldsymbol{x}, i, j]$ *with* $0 \le \max(i, j-s) \le n-1$. *For each* $m \in \mathbb{Z}$, *there is an integer* $N(m)$ *such that for all* $p \ge N(m)$, *the map* Ψ *of Ozsváth and Szabó induces isomorphisms in the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{CF}^{n}(S^3_{p}K, [m]) & \longrightarrow & \mathrm{CF}^+(S^3_{p}K, [m]) & \xrightarrow{U^n} & \mathrm{CF}^+(S^3_{p}K, [m]) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \Psi(n)} & & \big\downarrow{\scriptstyle \Psi^+} & & \big\downarrow{\scriptstyle \Psi^+} & & \\ 0 & \longrightarrow & \mathrm{CFK}^n(S^3, K, m) & \longrightarrow & {}^b\mathrm{CFK}^+(S^3, K, m) & \xrightarrow{U^n} & {}^b\mathrm{CFK}^+(S^3, K, m) & \longrightarrow & 0 \end{array}$$

Proof The map Ψ^+ is an isomorphism of chain complexes by Theorem 4.4 of [11]. By the five lemma, so is $\Psi(n)$. □

Notation We sometimes use the notation, for $n > 0$,

$$A^n_m = C\{0 \le \max(i, j-m) \le n-1\}, \quad A^+_m = C\{0 \le \max(i, j-m)\},$$
$$A^{-n}_m = C\{0 \le \min(i, j-m) \le n-1\}, \quad A^{+'}_m = C\{0 \le \min(i, j-m)\}$$

and

$$B^n = C\{0 \le i \le n-1\},$$
$$B^+ = C\{0 \le i\}.$$

The cobordism maps on truncated Heegaard Floer groups lead us to define concordance invariants ν_n.

Definition 3.3 For $n > 0$, define

$$\nu_n(K) = \min\{s \in \mathbb{Z} \mid v_s^n\colon \mathrm{CF}^n(S^3_N(K), s) \to \mathrm{CF}^n(S^3) \text{ induces a surjection on homology}\},$$

and for $n < 0$, define

$$\nu_n(K) = \max\{s \in \mathbb{Z} \mid v_s^n\colon \mathrm{CF}^{-n}(S^3) \to \mathrm{CF}^{-n}(S^3_{-N}(K), s) \text{ induces an injection on homology}\},$$

where N is sufficiently large that the Ozsváth–Szabó large integer surgery formula of [11] holds. For $n = 0$, we define $\nu_0(K) = \tau(K)$.

Remark 3.4 For $n = \pm 1$, these invariants are already known as $\nu_1(K) = \nu(K)$ and $\nu_{-1}(K) = \nu'(K)$.

Proposition 3.5 *$\nu_n(K)$ is a concordance invariant.*

Proof Suppose K_1 is concordant to K_2. Then $S^3_N(K_1)$ is homology cobordant to $S^3_N(K_2)$. This implies that there exists a (smooth, connected, oriented) cobordism W from $S^3_N(K_1)$ to $S^3_N(K_2)$ with $H_i(W, \mathbb{Q}) = 0$ for $i = 1, 2$.

The map

$$\mathrm{HF}^n(S^3_N(K_1), s) \to \mathrm{HF}^n(S^3)$$

induced by the cobordism obtained by adding a two-handle along K_1 factors through $\mathrm{HF}^n(S^3_N(K_2), s)$. So, if it is surjective, then the map

$$\mathrm{HF}^n(S^3_N(K_2), s) \to \mathrm{HF}^n(S^3)$$

is also surjective. So we get that $\nu_n(K_1) \geq \nu_n(K_2)$. The same argument with K_1 and K_2 switched shows the inequality $\nu_n(K_2) \geq \nu_n(K_1)$. Therefore, $\nu_n(K_1) = \nu_n(K_2)$.

For negative n, that $\nu_n(K)$ is a concordance invariant follows from a similar argument to the above. □

Proposition 3.6 (mirroring property) $\nu_n(-K) = -\nu_{-n}(K)$.

Proof Recall the symmetry of CFK^∞ under mirroring [11, Section 3.5],

$$\mathrm{CFK}^\infty(-K) \simeq \mathrm{CFK}^\infty(K)^*,$$

where $\mathrm{CFK}^\infty(K)^*$ is the dual complex $\mathrm{Hom}_{\mathbb{F}[U,U^{-1}]}(\mathrm{CFK}^\infty(K), \mathbb{F}[U,U^{-1}])$. Letting $C = \mathrm{CFK}^\infty(S^3, K)$ and $n > 0$, the following conditions are equivalent:

- $v^{-n}_{-s,*}(K)\colon \mathrm{HF}^n(S^3) \to \mathrm{HF}^n(S^3_{-N}(K), \mathfrak{s}_{-s})$ is injective.
- $v^{-n}_{-s}(K)\colon C\{0 \le i \le n-1\} \to C\{0 \le \min(i, j+s) \le n-1\}$ is injective on H_*.
- $U^{n-1} v^n_s(-K)\colon C\{-(n-1) \le \max(i, j-s) \le 0\} \to C\{-(n-1) \le i \le 0\}$ is surjective on H_*.
- $v^n_s(-K)\colon C\{0 \le \max(i, j-s) \le n-1\} \to C\{0 \le i \le n-1\}$ is surjective on H_*.
- $v^n_{s,*}(-K)\colon \mathrm{HF}^n(S^3_N(-K), \mathfrak{s}_s) \to \mathrm{HF}^n(S^3)$ is surjective.

Here U^{n-1} is a degree-shifting isomorphism on $\mathrm{CFK}^\infty(K)$. Therefore,

$$\begin{aligned}\nu_n(-K) &= \min(s \in \mathbb{Z} \mid v^n_{s,*}(-K) \text{ is surjective}) \\ &= -\max(-s \in \mathbb{Z} \mid v^{-n}_{-s,*}(K) \text{ is injective}) = -\nu_{-n}(K).\end{aligned}$$

□

Proposition 3.7 (monotonicity) $\nu_n(K) \le \nu_{n+1}(K)$.

Proof It is known that $\nu_{-1}(K) \le \tau(K) \le \nu_1(K)$, so we focus on the two separate cases where $n > 0$ and $n < 0$.

For $n > 0$, consider the commutative diagram

$$\begin{array}{ccc} \mathrm{HF}^{n+1}(S^3_N K, s) & \xrightarrow{v^{n+1}_{s,*}} & \mathrm{HF}^{n+1}(S^3) \\ \big\downarrow{\cdot U} & & \big\downarrow{\cdot U} \\ \mathrm{HF}^{n}(S^3_N K, s) & \xrightarrow{v^{n}_{s,*}} & \mathrm{HF}^{n}(S^3) \end{array}$$

where the vertical maps are given by multiplication by U. The vertical map on the right is surjective. Thus, if $v^{n+1}_{s,*}$ is surjective, then so is $v^n_{s,*}$.

For $n < 0$, consider the commutative diagram

$$\begin{array}{ccc} \mathrm{HF}^{-n}(S^3) & \xrightarrow{v^{n}_{s,*}} & \mathrm{HF}^{-n}(S^3_{-N} K, s) \\ \big\downarrow{i'_a} & & \big\downarrow{i'_b} \\ \mathrm{HF}^{-(n-1)}(S^3) & \xrightarrow{v^{n-1}_{s,*}} & \mathrm{HF}^{-(n-1)}(S^3_{-N} K, s) \end{array}$$

where the vertical maps are induced by inclusion of chain groups. In particular, the left map i'_a is injective on homology. Therefore, if $v^{n-1}_{s,*}(K)$ is injective, then so is $v^n_{s,*}(K)$. We conclude that $\nu_{n-1}(K) \le \nu_n(K)$. □

Proposition 3.8 (boundedness) *$\nu^{+'}(K) \le \nu_n(K) \le \nu^+(K)$ for all n.*

Proof It is known that $\nu(K) \le \nu^+(K)$ from [7]. For $n \ge 1$, consider the commutative diagram

$$\begin{array}{ccc} H_*(A^-_k) & \xrightarrow{j_A} & H_*(A^n_k) \\ \big\downarrow{\scriptstyle v^-_{k,*}} & & \big\downarrow{\scriptstyle v^n_{k,*}} \\ H_*(B^-) & \xrightarrow{j_B} & H_*(B^n) \end{array}$$

The map j_B is surjective, so if $v^-_{k,*}$ is surjective, then so is $v^n_{k,*}$.

For $n \le -1$, consider the commutative diagram

$$\begin{array}{ccc} H_*(B^n) & \xrightarrow{i_B} & H_*(B^+) \\ \big\downarrow{\scriptstyle v^n_{k,*}} & & \big\downarrow{\scriptstyle v^{+'}_{k,*}} \\ H_*(A^n_k) & \xrightarrow{i_A} & H_*(A^{+'}_k) \end{array}$$

The map i_B is injective, so if $v^{+'}_{k,*}$ is injective, then so is $v^n_{k,*}$. □

Proposition 3.9 (stabilization) *For sufficiently large positive n, $\nu_n(K) = \nu^+(K)$ and $\nu_{-n}(K) = \nu^{+'}(K)$.*

Proof Let $C_1 = \mathrm{CF}^-(S^3_N K, s)$ and $C_2 = \mathrm{CF}^-(S^3)$. There is a canonical degree-shifting isomorphism

$$\mathrm{CF}^n(Y, \mathfrak{s}) \cong \mathrm{CF}^-(Y, \mathfrak{s}) \otimes_{\mathbb{F}[U]} \frac{\mathbb{F}[U]}{U^n}.$$

Moreover, the map

$$v^-_{n,s}\colon C_1 \otimes_{\mathbb{F}[U]} \frac{\mathbb{F}[U]}{U^n} \to C_2 \otimes_{\mathbb{F}[U]} \frac{\mathbb{F}[U]}{U^n}$$

is the same as the map $v^n_s\colon \mathrm{CF}^n(S^3_N K, s) \to \mathrm{CF}^n(S^3)$. We show that if $v^-_{s,*}$ is not surjective, then neither is $v^n_{s,*}$ for sufficiently large n. By the universal coefficient theorem,

$$\begin{array}{ccccccc} 0 \to H_*(C_1) \otimes \dfrac{\mathbb{F}[U]}{U^n} & \xrightarrow{i_1} & H_*\left(C_1 \otimes \dfrac{\mathbb{F}[U]}{U^n}\right) & \longrightarrow & \mathrm{Tor}\left(H_*C_1, \dfrac{\mathbb{F}[U]}{U^n}\right) \to 0 \\ \Big\downarrow v^-_{s,*}\otimes \mathrm{id} & & \Big\downarrow v^n_{s,*} & & \Big\downarrow \mathrm{Tor}(v^-_s) \\ 0 \to H_*(C_2) \otimes \dfrac{\mathbb{F}[U]}{U^n} & \xrightarrow{i_2} & H_*\left(C_2 \otimes \dfrac{\mathbb{F}[U]}{U^n}\right) & \longrightarrow & \mathrm{Tor}\left(H_*C_2, \dfrac{\mathbb{F}[U]}{U^n}\right) \to 0 \end{array}$$

where all tensor products are taken over $\mathbb{F}[U]$.

We note the following facts:

- For a rational homology 3–sphere Y, $\mathrm{HF}^-(Y,\mathfrak{s})/\{U\text{–torsion}\} = \mathfrak{T}^- = \mathbb{F}[X]$. So $H_*(C_1) = \mathfrak{T}^- \oplus \left(\bigoplus \mathbb{F}[U]/U^{m_i}\right)$.
- $\mathfrak{T}^- \otimes \mathbb{F}[U]/U^n = \mathbb{F}[U]/U^n$ and $\mathbb{F}[U]/U^{m_i} \otimes \mathbb{F}[U]/U^n = \mathbb{F}[U]/U^{m_i}$. So $H_*(C_1) \otimes \mathbb{F}[U]/U^n = \mathbb{F}[U]/U^n \oplus \left(\bigoplus \mathbb{F}[U]/U^{m_i}\right)$.
- $\mathrm{Tor}(\mathbb{F}[U]/U^m, \mathbb{F}[U]/U^n) = \mathbb{F}[U]/U^m$ if $m < n$.
- $H_*(C_2 \otimes \mathbb{F}[U]/U^n) = \mathbb{F}[U]/U^n$.

Assume n is sufficiently large that $m_i < n$ for all m_i. So the above Tor groups are $n-1$ torsion.

If v^-_s is not surjective, we can further choose n sufficiently large that the image of $v^-_{s,*} \otimes \mathrm{id}$ is $n-1$ U–torsion. By commutativity of the diagram, the image of $v^n_{s,*} \circ i_1$ is $n-1$ $U-$torsion.

Suppose $\xi \in H_*(C_1 \otimes \mathbb{F}[U]/U^n)$ is such that $v^n_{s,*}(\xi)$ is an element of order n. Then, since the short exact sequence in the universal coefficient theorem splits, $\xi = \alpha + \beta$, where $\alpha \in H_*(C_1) \otimes \mathbb{F}[U]/U^n$ and $\beta \in \mathrm{Tor}(H_*C_1, \mathbb{F}[U]/U^n)$. But

$$U^{n-1} \cdot v^n_{s,*}(\alpha+\beta) = v^n_{s,*}(U^{n-1}\alpha) + v^n_{s,*}(U^{n-1}\beta) = 0.$$

Since $H_*(C_2 \otimes \mathbb{F}[U]/U^n) = \mathbb{F}[U]/U^n$, the invariant $v^n_{s,*}$ is not surjective. Therefore, for sufficiently large n, $\nu_n(K) = \nu^+(K)$.

Finally, by the mirroring property, $\nu_n(K) = \nu^{+'}(K)$ for sufficiently large negative integers n. □

The fact that $\nu_n(K)$ are not concordance homomorphisms from $\mathcal{C}$ to $\mathbb{Z}$ can easily be seen. Note $\nu_n(K)$ is not additive under connected sum of knots. For $n = 1$, just consider two knots with $\varepsilon(K)=\varepsilon(K')=-1$. Then

$$\nu(K)=\tau(K)+1 \quad \text{and} \quad \nu(K')=\tau(K')+1,$$

but

$$\nu(K\,\#\,K')=\tau(K\,\#\,K')+1=\tau(K)+\tau(K')+1<\nu(K)+\nu(K').$$

4 Computations

Knot Floer homology groups can be easily computed for certain special families of knots. For example, homologically thin knots are knots with $\widehat{\mathrm{HFK}}(K)$ supported in a single δ–grading, where $\delta=A-M$. If the homology is supported on the diagonal $\delta=-\frac{1}{2}\sigma(K)$, where $\sigma(K)$ denotes the knot signature, then we say the knot is σ–thin. The class of σ–thin knots contains as a proper subset all quasialternating knots, and in particular all alternating knots. The following theorem shows that $\nu_n(K)$ of thin knots only depends on $\tau(K)$:

Proposition 4.1 *Let K be a homologically thin knot with $\tau(K)=\tau$.*

(i) *If $\tau=0$, $\nu_n(K)=0$ for all n.*

(ii) *If $\tau>0$,*

$$\nu_n(K)=\begin{cases}0 & \text{for } n\le -\frac{1}{2}(\tau+1),\\ \tau+2n+1 & \text{for } -\frac{1}{2}\tau\le n\le -1,\\ \tau & \text{for } n\ge 0.\end{cases}$$

(iii) *If $\tau<0$,*

$$\nu_n(K)=\begin{cases}\tau & \text{for } n\le 0,\\ \tau+2n-1 & \text{for } 1\le n\le -\frac{1}{2}\tau,\\ 0 & \text{for } n\ge \frac{1}{2}(-\tau+1).\end{cases}$$

Proof In [15, Theorem 4] Petkova constructs model complexes for $\mathrm{CFK}^\infty(K)$ of homologically thin knots. She shows the model chain complex contains a direct summand (called the "staircase") isomorphic to

$$\begin{cases}\mathrm{CFK}^\infty(T_{2,2\tau+1}) & \text{if } \tau(K)>0,\\ \mathrm{CFK}^\infty(T_{2,2\tau-1}) & \text{if } \tau(K)\le 0.\end{cases}$$

The "staircase" summand supports $H_*(\mathrm{CFK}^\infty(K))$; that is,

$$H_*(\mathrm{CFK}^\infty(K))=H_*(\mathrm{CFK}^\infty(T_{2,2\tau-1})).$$

The maps induced on homology by v_s^n (or $v_{n,s}^-$) will thus only depend on the "staircase" summand and not the acyclic summands. Thus, $\nu_n(T_{2,2\tau\pm 1}) = \nu_n(K)$.

Without loss of generality, assume $\tau(K) > 0$. The chain complex $\mathrm{CFK}^-(T_{2,2\tau+1})$ is generated over $\mathbb{F}[U]$ by generators $\{z_p\}_{p=1}^{2\tau+1}$ with U–filtration levels i and Alexander filtration levels j specified (for all $1 \le p \le 2\tau+1$) by

$$j(z_p) = \begin{cases} \tau - \frac{1}{2}(p-1) & \text{if } p \text{ odd,} \\ \tau - \frac{1}{2}(p-2) & \text{if } p \text{ even,} \end{cases} \qquad i(z_p) = \begin{cases} \frac{1}{2}(p-1) & \text{if } p \text{ odd,} \\ \frac{1}{2}p & \text{if } p \text{ even,} \end{cases}$$

and differential

$$\partial z_p = \begin{cases} 0 & \text{if } p \text{ odd,} \\ z_{p-1} + z_{p+1} & \text{if } p \text{ even.} \end{cases}$$

The above complex with generators $\{z_p\}_{p=1}^{2\tau+1}$ and given differential maps forms the generating "staircase" complex C_τ, and $\mathrm{CFK}^\infty(T_{2,2\tau+1}) = C_\tau \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}]$. The U–action lowers i and j by one.

Computation of $\nu^+(T_{2,2\tau+1})$ and $\nu_n(T_{2,2\tau+1})$ for $n > 0$ Since

$$C\{i < 0,\ j \ge \tau\} = 0,$$

the map v_τ^+ is the same as the inclusion

$$C\{0 \le i \le n-1,\ j \le \tau + n - 1\} \to C\{0 \le i \le n-1\} = B^n.$$

Moreover, the generator with the highest Alexander grading in $C\{0 \le i \le n-1\}$ is $U^{n-1} z_1$, with

$$j(U^{n-1} z_1) = \tau + n - 1.$$

Thus, $C\{0 \le i \le n-1,\ j > \tau + n - 1\} = 0$. That is, the inclusion v_τ^+ is an isomorphism of chain complexes, so $\nu^+(K) = \tau(K)$. Therefore, $\nu_n(K) = \tau(K)$ for all $n \ge 0$.

Computation of $\nu^{+'}(T_{2,2\tau+1})$ The homology of B^+ is generated by $\{[U^{-i} z_1]\}$ for all $i \ge 0$. The subquotient complex $A_0^{+'}$ contains $U^{-i} C_\tau$ for all $i \ge 0$, and the homology of $U^{-i} C_\tau$ is generated by the class $[U^{-i} z_1]$. Therefore, $v_{0,*}^{+'}[U^{-i} z_1] \ne 0$ in $H_*(A_0^{+'})$, and $v_{0,*}^{+'}$ is injective. So $\nu^{+'}(T_{2,2\tau+1}) \ge 0$. But since $\nu^{+'}(K) \le 0$ for any knot K, we conclude $\nu^{+'}(T_{2,2\tau+1}) = 0$.

Computation of $\nu_n(T_{2,2\tau+1})$ for $-\frac{1}{2}\tau \le n \le -1$ Consider the subquotient complex A_k^n where $k = \tau + 2n + 1$. For each $1 \le p \le 2\tau + 1$,

$$\min(i(z_p), j(z_p) - k) = \begin{cases} \min\left(\frac{1}{2}(p-1), -\frac{1}{2}(p-1) - 2n - 1\right) & \text{if } p \text{ is odd,} \\ \min\left(\frac{1}{2}p, -\frac{1}{2}(p-2) - 2n - 1\right) & \text{if } p \text{ is even} \end{cases}$$

$$= \begin{cases} \frac{1}{2}(p-1) & \text{if } p \text{ is odd and } p \le -2n-1, \\ -\frac{1}{2}(p-1)-2n-1 & \text{if } p \text{ is odd and } p > -2n, \\ \frac{1}{2}p & \text{if } p \text{ is even and } p \le -2n, \\ -\frac{1}{2}(p-2)-2n-1 & \text{if } p \text{ is even and } p > -2n. \end{cases}$$

Using these formulas, it is straightforward to check that A^n_k contains z_p for $1 \le p \le -2n-1$ but $z_{-2n} \notin A^n_k$. Therefore, $[z_1] \ne 0$ in A^n_k. Similarly, for $1 \le i \le -n-1$, A^n_k contains $U^{-i} z_p$ for $1 \le p \le -2(n+i)-1$ but $z_{-2(n+i)} \notin A^n_k$. Therefore, $[U^{-i} z_1] \ne 0$ in $H_*(A^n_k)$. Since $H_*(B^n)$ is generated by $[U^{-i} z_1]$ for $0 \le i \le -n-1$, v^n_k is injective on homology.

To check that $\nu_n(T_{2,2\tau+1}) = \tau + 2n + 1$, consider the subquotient complex $A^n_{\tau+2n+2}$. For each $1 \le p \le 2\tau+1$,

$$\min(i(z_p), j(z_p)-k) = \begin{cases} \min\bigl(\frac{1}{2}(p-1), -\frac{1}{2}(p-1)-2n-2\bigr) & \text{if } p \text{ is odd}, \\ \min\bigl(\frac{1}{2}p, -\frac{1}{2}(p-2)-2n-2\bigr) & \text{if } p \text{ is even} \end{cases}$$

$$= \begin{cases} \frac{1}{2}(p-1) & \text{if } p \text{ is odd and } p \le -2n-1, \\ -\frac{1}{2}(p-1)-2n-2 & \text{if } p \text{ is odd and } p > -2n, \\ \frac{1}{2}p & \text{if } p \text{ is even and } p \le -2n-2, \\ -\frac{1}{2}(p-2)-2n-2 & \text{if } p \text{ is even and } p \ge -2n. \end{cases}$$

Using the above, it is straightforward to check that $A^n_{\tau+2n+2}$ contains z_p for $1 \le p \le -4n-2$ but $z_{-4n-1} \notin A^n_{\tau+2n+2}$. Therefore, $[z_1] = 0$ in $H_*(A^n_{\tau+2n+2})$. Thus, $\nu_n(T_{2,2\tau+1}) = \tau + 2n + 1$.

Computation of $\nu_n(T_{2,2\tau+1})$ for $n \le -\frac{1}{2}(\tau+1)$ Consider $A^{n'}_0$, where

$$n' = \begin{cases} -\frac{1}{2}(\tau+1) & \text{if } \tau \text{ is odd}, \\ -\frac{1}{2}\tau - 1 & \text{if } \tau \text{ is even}. \end{cases}$$

For each $1 \le p \le 2\tau+1$,

$$\min(i(z_p), j(z_p)-0) = \begin{cases} \min\bigl(\frac{1}{2}(p-1), \tau - \frac{1}{2}(p-1)\bigr) & \text{if } p \text{ is odd}, \\ \min\bigl(\frac{1}{2}p, \tau - \frac{1}{2}(p-2)\bigr) & \text{if } p \text{ is even} \end{cases}$$

$$= \begin{cases} \frac{1}{2}(p-1) & \text{if } p \text{ is odd and } p \le \tau+1, \\ \tau - \frac{1}{2}(p-1) & \text{if } p \text{ is odd and } p > \tau+1, \\ \frac{1}{2}p & \text{if } p \text{ is even and } p \le \tau+1, \\ \tau - \frac{1}{2}(p-2) & \text{if } p \text{ is even and } p > \tau+1. \end{cases}$$

These computations show that $v^{n'}_0$ is injective on homology:

- If τ is odd, $A^{n'}_0$ contains z_p for $1 \le p \le -2n-1$ but $z_{-2n} \notin A^{n'}_0$. Similarly, for $1 \le i \le -n-1$, $A^{n'}_0$ contains $U^{-i} z_p$ for $1 \le p \le -2(n+i)-1$ but

$z_{-2(n+i)} \notin A_0^{n'}$. Therefore, $[U^{-i} z_1] \neq 0$ in $H_*(A_0^{n'})$ for $0 \leq i \leq -n-1$. So $v_0^{n'}$ is injective on homology.

- If τ is even, $A_0^{n'}$ contains all z_p for $1 \leq p \leq 2\tau+1$. Furthermore, for $1 \leq i \leq -n-1$, $A_0^{n'}$ contains $U^{-i} z_p$ for $1 \leq p \leq -2(n+i)-1$ but $z_{-2(n+i)} \notin A_0^{n'}$. Therefore, $[U^{-i} z_1] \neq 0$ in $H_*(A_0^{n'})$ for $0 \leq i \leq -n-1$. So $v_0^{n'}$ is injective on homology.

Since $\nu^{+'}(T_{2,2\tau+1}) = 0$ is a lower bound on $\nu_n(T_{2,2\tau+1})$, we conclude that

$$\nu_n(T_{2,2\tau+1}) = 0$$

for all $n \leq -\frac{1}{2}(\tau+1)$. □

We have the following result for strongly quasipositive knots (see [4] for background on strongly quasipositive knots):

Proposition 4.2 *If K is strongly quasipositive, then $\nu_n(K) = \tau(K) = g_4(K) = g(K)$ for all positive n.*

Proof Theorem 1.2 of [4] states that K is strongly quasipositive if and only if $\tau(K) = g_4(K) = g(K)$. The result immediately follows since $\tau(K) \leq \nu_n(K) \leq \nu^+(K) \leq g_4(K)$ for positive n. See also [7, Proposition 3]. □

Example 4.3 Figure 1 (top-left) shows the knot Floer chain complex CFK^∞ of the $(2,9)$–torus knot. The computation of $\nu_{-2}(T_{2,9})$ is shown in Figure 1 (top-right and bottom-left). We have

$$\nu_n(T_{2,9}) = \begin{cases} 4 & \text{for all } n \geq 0, \\ 3 & \text{for } n = -1, \\ 1 & \text{for } n = -2, \\ 0 & \text{for all } n \leq -3. \end{cases}$$

The computation of $\nu_n(K)$ for thin knots shows that the sequence ν_n can increase by more than one at a time, in contrast to the local h–invariants defined by Rasmussen, which jump by at most one [16, Proposition 7.6].

In fact, the gap between $\nu_n(K)$ and $\nu_{n+1}(K)$ can be arbitrarily big. For example, a straightforward (partial) computation of $\nu_n(T_{p,p+1})$ using $\mathrm{CFK}^\infty(T_{p,p+1})$ shows that for $p > 3$,

$$\nu_{-1}(T_{p,p+1}) - \nu_{-2}(T_{p,p+1}) = p.$$

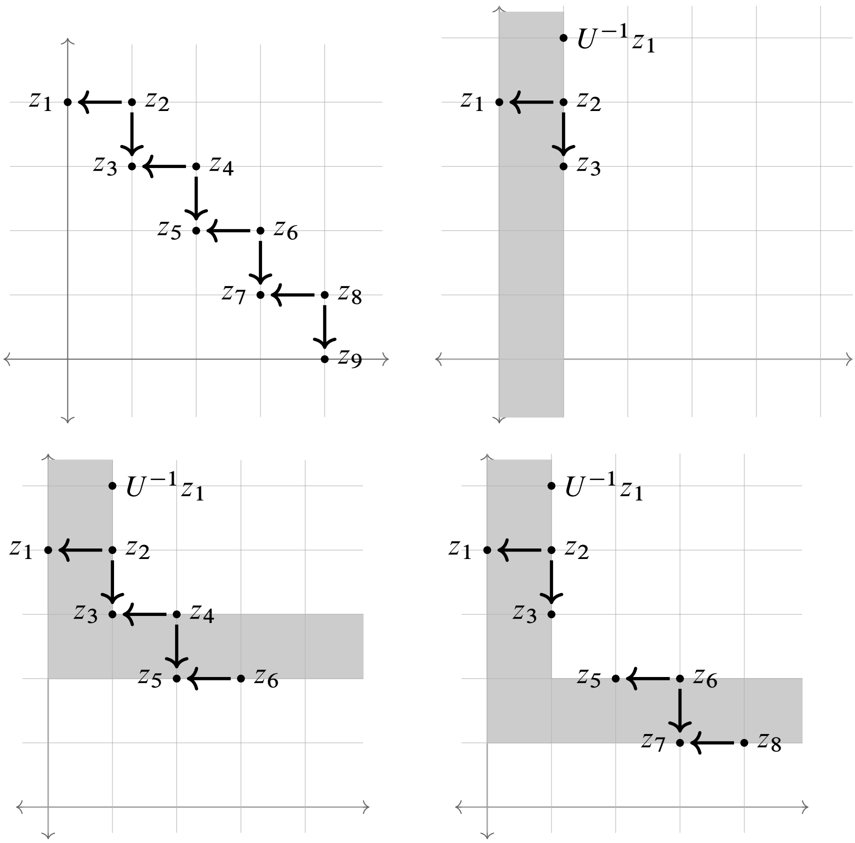

Figure 1: Top-left: Generating complex for CFK^∞ of the (2, 9)-torus knot $T_{2,9}$. Top-right: The classes $[U^{-1}z_1]$ and $[z_1]=[z_3]$ generate $\mathrm{HF}^2(S^3)$. Bottom-left: The classes $[z_1]$ and $[z_3]$ vanish in $\mathrm{HF}^2(S^3_{-N}K,[2])$. Bottom-right: The classes $[z_1]=[z_3]$ and $[U^{-1}z_1]$ survive in $\mathrm{HF}^2(S^3_{-N}K,[1])$.

Theorem 4.4 *Let $T_{p,p+1}$ denote the $(p,p+1)$–torus knot for $p>3$. Let $\tau=\tau(T_{p,p+1})=\frac{1}{2}(p-1)p$. Then*

$$\nu_n(T_{p,p+1})=\begin{cases}\tau & \text{for } n\geq 0,\\ \tau-1 & \text{for } n=-1,\\ \tau-1-p & \text{for } n=-2.\end{cases}$$

Thus, $\nu_{-1}(T_{p,p+1})-\nu_{-2}(T_{p,p+1})=p$.

Proof In [1], Allen shows that the staircase model chain complex for $\mathrm{CFK}^\infty(T_{p,p+1})$ takes the form

$$[1,\ p-1,\ 2,\ p-2,\ \dots,\ j,\ p-j,\ \dots,\ p-1,\ 1],$$

where the indices alternate between the widths of the horizontal and vertical steps. From this staircase description, there exists a (i,j)–filtered basis for $\mathrm{CFK}^\infty(T_{p,p+1})$ consisting of generators $\{b_l\}_{l=0}^{2(p-1)}$ lying in (i,j)–filtrations

$$b_{2m}: \quad \Bigl(\sum_{k=1}^{m} k,\ \tfrac{1}{2}(p-1)p-\sum_{k=1}^{m}(p-k)\Bigr),$$
$$b_{2m+1}: \quad \Bigl(\sum_{k=1}^{m+1} k,\ \tfrac{1}{2}(p-1)p-\sum_{k=1}^{m}(p-k)\Bigr),$$

and differential

$$\partial b_{2m}=0, \quad \partial b_{2m+1}=b_{2m}+b_{2m+2}.$$

The same argument for showing that $\nu^+(T_{2,2\tau+1})=\tau(T_{2,2\tau+1})$ in Proposition 4.1 holds for the knots $T_{p,p+1}$. Moreover, in the terminology of [5], the basis $\{b_l\}_{l=0}^{2(p-1)}$ satisfies:

- b_0 is the vertically distinguished element of a vertically simplified basis.
- b_0 has a unique incoming horizontal arrow (from b_1) (and no outgoing horizontal arrows).

We immediately conclude that $\varepsilon(T_{p,p+1})=1$ and $\nu_{-1}(T_{p,p+1})=\tau-1$.

To show $\nu_{-2}(T_{p,p+1})=\tau-p-1$, we observe:

- $A^{-2}_{\tau-p-1}$ contains the generators b_0, b_1 and b_2, but $b_3\notin A^{-2}_{\tau-p-1}$. Therefore, $[b_0]\neq 0$ in $H_*(A^{-2}_{\tau-p-1})$. Moreover, $[U^{-1}b_0]\neq 0$ in $H_*(A^{-2}_{\tau-p-1})$. Thus, $v^{-2}_{\tau-p-1}$ is injective on homology.
- $A^{-2}_{\tau-p}$ contains the generators b_0, b_1, b_2 and b_3, but $b_4\notin A^{-2}_{\tau-p}$. Therefore, $[b_0]=0$ in $H_*(A^{-2}_{\tau-p})$. □

We show the concordance invariants $\{\nu_n(K)\}$ contain more concordance information than the collection $\{\tau,\nu,\nu',\nu^+,\nu^{+'}\}$:

Proposition 4.5 *There exist knots K and K' with equal τ, ν, ν', ν^+ and $\nu^{+'}$ invariants, but $\nu_n(K)\neq\nu_n(K')$ for some $n\in\mathbb{Z}$.*

Proof The torus knot $T_{4,5}$ and the torus knot $T_{2,13}$ share the following invariants in common:

$$\nu^{+'}(T_{4,5})=0=\nu^{+'}(T_{2,13}),$$
$$\nu'(T_{4,5})=5=\nu'(T_{2,13}),$$
$$\tau(T_{4,5})=\nu(T_{4,5})=\nu^+(T_{4,5})=6=\tau(T_{2,13})=\nu(T_{2,13})=\nu^+(T_{2,13}).$$

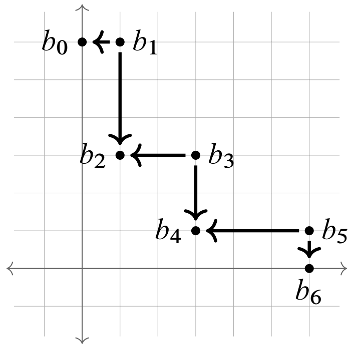

Figure 2: Generating complex for CFK^∞ of the left-handed $(4,5)$–torus knot $T_{4,5}$. $\mathrm{CFK}^\infty(T_{4,5})$ is generated over $\mathbb{F}[U,U^{-1}]$ by the above chain complex. The arrows, representing terms in the differential, are drawn to scale, with lengths of arrows ranging between one and three.

However, the invariants $\nu_n(T_{4,5})$ are different from $\nu_n(T_{2,13})$:

$$\nu_n(T_{4,5}) = \begin{cases} 6 & \text{for } n \geq 0, \\ 5 & \text{for } n = -1, \\ 1 & \text{for } n = -2, \\ 0 & \text{for } n \leq -3, \end{cases} \qquad \nu_n(T_{2,13}) = \begin{cases} 6 & \text{for } n \geq 0, \\ 5 & \text{for } n = -1, \\ 3 & \text{for } n = -2, \\ 1 & \text{for } n = -3, \\ 0 & \text{for } n \leq -4, \end{cases}$$

where $\nu_n(T_{4,5})$ is computed from the knot Floer chain complex $\mathrm{CFK}^\infty(T_{4,5})$ as shown in Figure 2. □

5 Further directions

One question is the effectiveness of $\nu_n(K)$ when compared to other concordance invariants such as $\Upsilon_K(t)$, coming from t–modified knot Floer homology [10], or V_k, coming from surgery formulas of Ozsváth and Szabó [13].

The invariants $\nu_n(K)$ do not define concordance homomorphisms $\mathcal{C} \to \mathbb{Z}$, where $\mathcal{C}$ is the concordance group of knots. This implies that they do not necessarily vanish on knots of finite concordance order. The existence of p–torsion, with $p \neq 2$, in the concordance group $\mathcal{C}$ is an open question. A related conjecture, based on a question of Gordon [3], as phrased in [8] is:

Conjecture 5.1 (Gordon) A knot has order two in $\mathcal{C}$ if and only if $K = -K$ is negative amphichiral.

Recently, Hendricks and Manolescu defined involutive Heegaard Floer concordance invariants $\overline{V}_0$ and $\underline{V_0}$, which detects the nonsliceness of the figure eight knot. The nonsliceness of 4_1 was previously known through classical methods, but this is the first method of detection coming from the Heegaard Floer package. By additivity of τ and the behavior of ε under connected sums, $\tau(K)$ and $\nu(K)$ vanish for all knots K of finite concordance order. This leaves open the cases $\nu_n(K)$ for $n > 1$ and $n < -1$. We pose the question:

Question 5.2 Does there exist a knot K of finite concordance order such that $\nu_n(K) \neq 0$ for some n?

Another question is how the invariants $\nu_n(K)$ behave under connected sum. It is known that $\nu^+(K)$ is subadditive by [2]. That is,

$$\nu^+(K \# L) \leq \nu^+(K) + \nu^+(L).$$

Using mirroring relations and subadditivity of $\nu^+(K)$ shows that $\nu^{+'}(K)$ is superadditive:

Lemma 5.3 *For any two knots K and L,*

$$\nu^{+'}(K \# L) \geq \nu^{+'}(K) + \nu^{+'}(L).$$

Proof By subadditivity of ν^+ and the mirroring relations,

$$\begin{aligned} \nu^+(-K \# -L) &\leq \nu^+(-K) + \nu^+(-L), \\ -\nu^{+'}(K \# L) &\leq -\nu^{+'}(K) + -\nu^{+'}(L), \\ \nu^{+'}(K \# L) &\geq \nu^{+'}(K) + \nu^{+'}(L). \end{aligned}$$ □

As pointed out to the author by Jen Hom, it can also be seen by additivity of τ and the behavior of ε under connected sum that $\nu(K)$ is subadditive. A similar argument shows that $\nu'(K)$ is superadditive. This leads us to ask the following two questions:

Question 5.4 Is $\nu_n(K \# K') \leq \nu_n(K) + \nu_n(K')$ for all positive integers $n \in \mathbb{Z}_+$?

Question 5.5 Is $\nu_n(K \# K') \geq \nu_n(K) + \nu_n(K')$ for all negative integers $n \in \mathbb{Z}_-$?

The next question was posed by Zhongtao Wu:

Question 5.6 (Wu) If $\nu_n(K) = \nu_n(K')$ for all $n \in \mathbb{Z}$, then is $\nu^+(K \# -K') = \nu^+(-K \# K') = 0$?

The condition that $\nu^+(K \# -K') = \nu^+(-K \# K') = 0$ implies that

$$\mathrm{CFK}^\infty(K \# -K') \simeq \mathrm{CFK}^\infty(U) \oplus A,$$

where A is an acyclic complex [6].

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Department of Mathematics, Columbia University
New York, NY, United States

ltruong@math.columbia.edu

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