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Algebraic & Geometric Topology 19 (2019) 1881–1901

Truncated Heegaard Floer homology and knot concordance invariants

LINH TRUONG

We construct a sequence of smooth concordance invariants $\nu_n(K)$ defined using truncated Heegaard Floer homology. The invariants generalize the concordance invariants ν of Ozsváth and Szabó and ν^+ of Hom and Wu. We exhibit an example in which the gap between two consecutive elements in the sequence ν_n can be arbitrarily large. We also prove that the sequence ν_n contains more concordance information than τ, ν, ν', ν^+ and $\nu^{+'}$.

57M25, 57M27, 57R58

1 Introduction

Two knots, K_0 and K_1, in S^3 are *smoothly concordant* if there is a smooth proper embedding of a cylinder $S^1 \times [0,1]$ into $S^3 \times [0,1]$ with boundary $S^1 \times \{0\} = K_0$ and $S^1 \times \{1\} = -K_1$. The Heegaard Floer homology package of Ozsváth and Szabó has led to a wealth of smooth concordance invariants. A survey on this topic can be found in Hom [6].

Two concordance invariants motivate this article:

(1) $\nu(K)$, defined by Ozsváth and Szabó [14] using maps on Heegaard Floer homology $\widehat{\mathrm{HF}}$ induced by the two-handle cobordism corresponding to integral surgery along K.

(2) $\nu^+(K)$, defined by Hom and Wu [7] using maps induced by surgery on HF^+. Hom and Wu showed that $\nu^+(K)$ produces arbitrarily better four-ball genus bounds than $\nu(K)$.

We construct a sequence of concordance invariants $\nu_n(K)$ for $n \in \mathbb{Z}$ which are defined using maps induced by surgery on the truncated Heegaard Floer homology HF^n. The invariants $\nu_n(K)$ generalize $\nu(K)$ and $\nu^+(K)$, as $\nu_1(K) = \nu(K)$ and $\nu_n(K) = \nu^+(K)$ for n sufficiently large. The properties of $\nu_n(K)$ are stated below.

Published: 16 August 2019 DOI: 10.2140/agt.2019.19.1881

Theorem 1.1 *The knot invariants $\nu_n(K)$, where $n \in \mathbb{Z}$, satisfy the following properties:*

- $\nu_n(K)$ *is a concordance invariant.*
- $\nu_1(K) = \nu(K)$.
- $\nu_n(K) \le \nu_{n+1}(K)$.
- *For sufficiently large* n, $\nu_n(K) = \nu^+(K)$.
- $\nu_n(-K) = -\nu_{-n}(K)$, *where* $-K$ *is the mirror of* K.
- $\nu_n(K) \le g_4(K)$.

By an extension of the large integer surgery formulas to truncated Floer homology (see Propositions 3.1 and 3.2) the invariants $\nu_n(K)$ can be computed from the $\mathbb{Z}\oplus\mathbb{Z}$–filtered knot Floer chain complex $\mathrm{CFK}^\infty(K)$.

Homologically thin knots are a special class of knots whose knot Floer homology lies in a single $\delta = A - M$ grading, where A is the Alexander grading and M is the Maslov grading. We prove that $\nu_n(K)$ of thin knots only depends on $\tau(K)$.

Proposition 4.1 *Let* K *be a homologically thin knot with* $\tau(K) = \tau$.

(i) *If* $\tau = 0$, $\nu_n(K) = 0$ *for all* n.

(ii) *If* $\tau > 0$,

$$\nu_n(K) = \begin{cases} 0 & \text{for } n \le -\frac{1}{2}(\tau+1), \\ \tau + 2n + 1 & \text{for } -\frac{1}{2}\tau \le n \le -1, \\ \tau & \text{for } n \ge 0. \end{cases}$$

(iii) *If* $\tau < 0$,

$$\nu_n(K) = \begin{cases} \tau & \text{for } n \le 0, \\ \tau + 2n - 1 & \text{for } 1 \le n \le -\frac{1}{2}\tau, \\ 0 & \text{for } n \ge \frac{1}{2}(-\tau+1). \end{cases}$$

The computation of $\nu_n(K)$ for thin knots illustrates that the gap between $\nu_n(K)$ and $\nu_{n+1}(K)$ can be more than one. In fact, the gap between $\nu_n(K)$ and $\nu_{n+1}(K)$ can be arbitrarily big.

Theorem 1.2 *Let* $T_{p,p+1}$ *denote the* $(p, p+1)$*–torus knot. For* $p > 3$,

$$\nu_{-1}(T_{p,p+1}) - \nu_{-2}(T_{p,p+1}) = p.$$

Finally, we show that the sequence ν_n contains more concordance information than τ, ν, ν', ν^+ and $\nu^{+'}$.

Proposition 1.3 *There exist knots K and K' with equal τ, ν, ν', ν^+ and $\nu^{+'}$ invariants, but $\nu_n(K) \neq \nu_n(K')$ for some $n \in \mathbb{Z}$.*

Organization of the paper In Section 2 we review the constructions of the concordance invariants $\nu(K)$ and $\nu^+(K)$. In Section 3 we define the invariants $\nu_n(K)$ and prove their properties: monotonicity, stabilization and behavior under mirroring. In Section 4 we compute $\nu_n(K)$ for special families of knots and compare them to $\nu(K)$ and $\nu^+(K)$. In Section 5 we pose some questions about the concordance invariants $\nu_n(K)$.

Acknowledgements The author thanks her advisors, Peter Ozsváth and Zoltán Szabó, for their guidance. The author also thanks Adam Levine for carefully reading the thesis version and for corrections. The author thanks the anonymous referee for helpful comments.

This work was partially supported by an NSF postdoctoral fellowship, DMS-1606451.

2 Background on the invariants $\nu(K)$ and $\nu^+(K)$

A four-dimensional cobordism equipped with a Spin^c structure between two three-manifolds induces a map on the Heegaard Floer homology groups [12]. In particular, for a knot K in S^3, the 2–handle attachment cobordism from $S^3_N(K)$ or $S^3_{-N}(K)$ to S^3 induces maps

(1) $\hat{v}_{s,*}\colon \widehat{\mathrm{HF}}(S^3_N(K), \mathfrak{s}_s) \to \widehat{\mathrm{HF}}(S^3)$, $\quad \hat{v}'_{s,*}\colon \widehat{\mathrm{HF}}(S^3) \to \widehat{\mathrm{HF}}(S^3_{-N}(K), \mathfrak{s}_s)$,

(2) $v^+_{s,*}\colon \mathrm{HF}^+(S^3_N(K), \mathfrak{s}_s) \to \mathrm{HF}^+(S^3)$, $\quad v^{+'}_{s,*}\colon \mathrm{HF}^+(S^3) \to \mathrm{HF}^+(S^3_{-N}(K), \mathfrak{s}_s)$,

(3) $v^-_{s,*}\colon \mathrm{HF}^-(S^3_N(K), \mathfrak{s}_s) \to \mathrm{HF}^-(S^3)$, $\quad v^{-'}_{s,*}\colon \mathrm{HF}^-(S^3) \to \mathrm{HF}^-(S^3_{-N}(K), \mathfrak{s}_s)$,

where $\mathfrak{s}_s$ denotes the restriction to $S^3_N(K)$ or $S^3_{-N}(K)$ of a Spin^c structure $\mathfrak{t}$ on the corresponding 2–handle cobordism such that

$$\langle c_1(\mathfrak{t}), [\hat{F}]\rangle + N = 2s, \quad \langle c_1(\mathfrak{t}), [\hat{F}]\rangle - N = 2s,$$

where $\hat{F}$ is a capped-off Seifert surface for K. These cobordism maps on $\widehat{\mathrm{HF}}$ and HF^+ play a key role in defining the concordance invariants ν and ν^+.

Definition 2.1 [14, Section 9] The concordance invariant $\nu(K)$ is defined as

$$\nu(K) = \min\{s \in \mathbb{Z} \mid \hat{v}_{s,*} \text{ is surjective}\}.$$

Definition 2.2 The concordance invariant $\nu'(K)$ is defined as

$$\nu'(K) = \max\{s \in \mathbb{Z} \mid \hat{v}'_{s,*} \text{ is injective}\}.$$

For a rational homology 3–sphere Y with a Spin^c structure $\mathfrak{s}$, $\mathrm{HF}^+(Y,\mathfrak{s})$ can be decomposed as the direct sum of two groups: the first group is the image of $\mathrm{HF}^\infty(Y,\mathfrak{s}) \cong \mathbb{F}[U,U^{-1}]$ in $\mathrm{HF}^+(Y,\mathfrak{s})$, which is isomorphic to $\mathfrak{T}^+ = \mathbb{F}[U,U^{-1}]/U\mathbb{F}[U]$; the second group is $\mathrm{HF}_{\mathrm{red}}(Y,\mathfrak{s}) = \mathrm{HF}^+(Y,\mathfrak{s})/\mathfrak{T}^+$. That is,

$$\mathrm{HF}^+(Y,\mathfrak{s}) = \mathfrak{T}^+ \oplus \mathrm{HF}_{\mathrm{red}}(Y,\mathfrak{s}).$$

Definition 2.3 [7] The concordance invariant ν^+ is defined as

$$\nu^+(K) = \min\{s \mid v^+_{s,*}\colon \mathrm{HF}^+(S^3_N(K),\mathfrak{s}_s) \to \mathrm{HF}^+(S^3) \text{ sends } 1 \text{ to } 1\},$$

where 1 denotes the lowest-graded generator in the subgroup $\mathfrak{T}^+$ of the homology, and N is sufficiently large that the integer surgery formula holds.

Equivalently, Hom [6] defines the invariant $\nu^-(K)$ in terms of the map

$$v^-_{s,*}\colon \mathrm{HF}^-(S^3_N K,\mathfrak{s}_s) \to \mathrm{HF}^-(S^3).$$

Definition 2.4 [6] The concordance invariant

$$\nu^-(K) = \min\{s \in \mathbb{Z} \mid v^-_{s,*} \text{ is surjective}\}$$

is equal to $\nu^+(K)$.

Hom and Wu show that

$$\tau(K) \le \nu(K) \le \nu^+(K)$$

and $\nu^+(K) \ge 0$. In addition, ν^+ gives a lower bound on the four-ball genus, $\nu^+(K) \le g_4(K)$. Furthermore, Hom and Wu provide a family of knots where $\nu^+(K)$ is an arbitrarily better bound on $g_4(K)$ than $\tau(K)$.

The concordance invariants ν and ν^+ are easily computable from $\mathrm{CFK}^\infty(K)$ via the large integer surgery formulas. Let CX denote the subgroup of $\mathrm{CFK}^\infty(K)$ generated by elements $[x,i,j]$ that lie in filtration level $(i,j) \in X \subset \mathbb{Z} \oplus \mathbb{Z}$. Consider the chain maps

$$\hat{v}_s\colon C\{\max(i,j-s)=0\} \to C\{i=0\},$$
$$v_s^+\colon C\{\max(i,j-s)\geq 0\} \to C\{i\geq 0\},$$

defined by taking the quotient by $C\{i<0,\ j=s\}$ or $C\{i<0,\ j\geq s\}$, respectively, followed by the inclusions. The large integer surgery formula of Ozsváth and Szabó [11] asserts that the maps $\hat{v}_s$ and v_s^+ induce the maps from (1) and (2). Similarly, consider the chain maps

$$\hat{v}'_s\colon C\{i=0\} \to C\{\min(i,j-s)=0\},$$
$$v_s^{+'}\colon C\{i\geq 0\} \to C\{\min(i,j-s)\geq 0\},$$

consisting of quotienting by $C\{i=0,\ j\leq s\}$ followed by the inclusion. Ozsváth and Szabó [11] show that these maps induce the maps from (1) and (2).

We introduce a concordance invariant $\nu^{+'}$, so that the pair ν^+ and $\nu^{+'}$ is the HF^+ analogue to the pair ν and ν'.

Definition 2.5 The concordance invariant $\nu^{+'}$ is defined as

$$\nu^{+'}(K) = \max\{s\in\mathbb{Z} \mid v_{s,*}^{+'}\colon \mathrm{HF}^+(S^3) \to \mathrm{HF}^+(S^3_{-N}(K),\mathfrak{s}_s) \text{ is injective}\},$$

where $-N$ is sufficiently negative that the (negative) large integer surgery formula holds.

We prove a mirroring property which relates $\nu^{+'}(K)$ to the invariant $\nu^+(-K)$ of the mirror of K:

Lemma 2.6 $$\nu^{+'}(K) = -\nu^+(-K).$$

Proof Recall the symmetry of CFK^∞ under mirroring [11, Section 3.5],

$$\mathrm{CFK}^\infty(-K) \simeq \mathrm{CFK}^\infty(K)^*,$$

where $\mathrm{CFK}^\infty(K)^*$ is the dual complex $\mathrm{Hom}_{\mathbb{F}[U,U^{-1}]}(\mathrm{CFK}^\infty(K),\mathbb{F}[U,U^{-1}])$. Therefore,

$$v_{-s,*}^{+'}\colon \mathrm{HF}^+(S^3) \to \mathrm{HF}^+(S^3_{-N}(K),\mathfrak{s}_{-s}) \text{ is injective}$$
$$\iff v_{s,*}^-\colon \mathrm{HF}^-(S^3_N(-K),\mathfrak{s}_s) \to \mathrm{HF}^-(S^3) \text{ is surjective},$$

which implies the result. □

It follows from the above lemma that the invariant $\nu^{+'}$ exhibits properties similar to ν^+,

$$\nu^{+'}(K) \le \nu'(K) \le \tau(K) \le \nu(K) \le \nu^+(K)$$

and $\nu^{+'}(K) \le 0$. In addition, the absolute value of $\nu^{+'}(K)$ gives a lower bound on the four-ball genus:

Theorem 2.7 $$|\nu^{+'}(K)| \le g_4(K).$$

Proof This follows from the fact that $\nu^+(K) \le g_4(K)$ and Lemma 2.6. □

3 The concordance invariants $\nu_n(K)$

The construction of the concordance invariants $\nu_n(K)$ uses truncated Heegaard Floer homology $\mathrm{HF}^n(Y,\mathfrak{s})$, described in [9; 13]. $\mathrm{HF}^n(Y,\mathfrak{s})$ is the homology of the kernel $\mathrm{CF}^n(Y,\mathfrak{s})$ of the multiplication map

$$U^n\colon \mathrm{CF}^+(Y,\mathfrak{s}) \to \mathrm{CF}^+(Y,\mathfrak{s}),$$

where $n \in \mathbb{Z}_+$. The two-handle cobordism from $S^3_N K$ or $S^3_{-N} K$, respectively, to S^3 induces a map on the truncated Floer chain complex,

$$v^n_s\colon \mathrm{CF}^n(S^3_N K, \mathfrak{s}_s) \to \mathrm{CF}^n(S^3), \quad v^{-n}_s\colon \mathrm{CF}^n(S^3) \to \mathrm{CF}^n(S^3_{-N} K, \mathfrak{s}_s),$$

and on the truncated Floer homology,

$$v^n_{s,*}\colon \mathrm{HF}^n(S^3_N K, \mathfrak{s}_s) \to \mathrm{HF}^n(S^3), \quad v^{-n}_{s,*}\colon \mathrm{HF}^n(S^3) \to \mathrm{HF}^n(S^3_{-N} K, \mathfrak{s}_s),$$

where $\mathfrak{s}_s$ denotes the restriction to $S^3_N(K)$ or $S^3_{-N}(K)$, respectively, of a Spin^c structure $\mathfrak{t}$ on the corresponding 2–handle cobordism such that

$$\langle c_1(\mathfrak{t}), [\hat{F}]\rangle + N = 2s, \quad \langle c_1(\mathfrak{t}), [\hat{F}]\rangle - N = 2s,$$

where $\hat{F}$ is a capped-off Seifert surface for K. These cobordism maps on $\widehat{\mathrm{HF}}$ and HF^+ play a key role in defining the concordance invariants ν and ν^+.

We extend the large integer surgery formula of Ozsváth and Szabó to truncated Heegaard Floer homology:

Proposition 3.1 (large negative integer surgery formula for HF^n) *Consider the subquotient complex*

$$\mathrm{CFK}^{-n}(S^3, K, m) = C\{0 \le \min(i, j-m) \le n-1\}$$

of $\mathrm{CFK}^+(S^3, K, m)$ *generated by* $[\boldsymbol{x}, i, j]$ *with* $0 \le \min(i, j-m) \le n-1$*. For each* $m \in \mathbb{Z}$*, there is an integer* $N(m)$ *such that for all* $p \ge N(m)$*, the map* Φ *of Ozsváth and Szabó induces isomorphisms in the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{CFK}^{-n}(S^3, K, m) & \longrightarrow & \mathrm{CFK}^+(S^3, K, m) & \xrightarrow{U^n} & \mathrm{CFK}^+(S^3, K, m) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Phi(n)} & & \downarrow{\scriptstyle \Phi^+} & & \downarrow{\scriptstyle \Phi^+} & & \\ 0 & \longrightarrow & \mathrm{CF}^n(S^3_{-p}K, [m]) & \longrightarrow & \mathrm{CF}^+(S^3_{-p}K, [m]) & \xrightarrow{U^n} & \mathrm{CF}^+(S^3_{-p}K, [m]) & \longrightarrow & 0 \end{array}$$

Proof The map Φ^+ is an isomorphism of chain complexes by Theorem 4.1 of [11]. By the five lemma, so is $\Phi(n)$. □

Proposition 3.2 (large positive integer surgery formula for HF^n) *Consider the subquotient complex*

$$\mathrm{CFK}^{n}(S^3, K, m) = C\{0 \le \max(i, j-m) \le n-1\}$$

of $\mathrm{CFK}^+(S^3, K, m)$ *generated by* $[\boldsymbol{x}, i, j]$ *with* $0 \le \max(i, j-s) \le n-1$*. For each* $m \in \mathbb{Z}$*, there is an integer* $N(m)$ *such that for all* $p \ge N(m)$*, the map* Ψ *of Ozsváth and Szabó induces isomorphisms in the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{CF}^n(S^3_pK, [m]) & \longrightarrow & \mathrm{CF}^+(S^3_pK, [m]) & \xrightarrow{U^n} & \mathrm{CF}^+(S^3_pK, [m]) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Psi(n)} & & \downarrow{\scriptstyle \Psi^+} & & \downarrow{\scriptstyle \Psi^+} & & \\ 0 & \longrightarrow & \mathrm{CFK}^n(S^3, K, m) & \longrightarrow & {}^b\mathrm{CFK}^+(S^3, K, m) & \xrightarrow{U^n} & {}^b\mathrm{CFK}^+(S^3, K, m) & \longrightarrow & 0 \end{array}$$

Proof The map Ψ^+ is an isomorphism of chain complexes by Theorem 4.4 of [11]. By the five lemma, so is $\Psi(n)$. □

Notation We sometimes use the notation, for $n > 0$,

$$A^n_m = C\{0 \le \max(i, j-m) \le n-1\}, \quad A^+_m = C\{0 \le \max(i, j-m)\},$$
$$A^{-n}_m = C\{0 \le \min(i, j-m) \le n-1\}, \quad A^{+'}_m = C\{0 \le \min(i, j-m)\}$$

and

$$B^n = C\{0 \le i \le n-1\},$$
$$B^+ = C\{0 \le i\}.$$

The cobordism maps on truncated Heegaard Floer groups lead us to define concordance invariants ν_n.

Definition 3.3 For $n > 0$, define

$$\nu_n(K) = \min\{s \in \mathbb{Z} \mid v_s^n\colon \mathrm{CF}^n(S^3_N(K), s) \to \mathrm{CF}^n(S^3) \text{ induces a surjection on homology}\},$$

and for $n < 0$, define

$$\nu_n(K) = \max\{s \in \mathbb{Z} \mid v_s^n\colon \mathrm{CF}^{-n}(S^3) \to \mathrm{CF}^{-n}(S^3_{-N}(K), s) \text{ induces an injection on homology}\},$$

where N is sufficiently large that the Ozsváth–Szabó large integer surgery formula of [11] holds. For $n = 0$, we define $\nu_0(K) = \tau(K)$.

Remark 3.4 For $n = \pm 1$, these invariants are already known as $\nu_1(K) = \nu(K)$ and $\nu_{-1}(K) = \nu'(K)$.

Proposition 3.5 *$\nu_n(K)$ is a concordance invariant.*

Proof Suppose K_1 is concordant to K_2. Then $S^3_N(K_1)$ is homology cobordant to $S^3_N(K_2)$. This implies that there exists a (smooth, connected, oriented) cobordism W from $S^3_N(K_1)$ to $S^3_N(K_2)$ with $H_i(W, \mathbb{Q}) = 0$ for $i = 1, 2$.

The map

$$\mathrm{HF}^n(S^3_N(K_1), s) \to \mathrm{HF}^n(S^3)$$

induced by the cobordism obtained by adding a two-handle along K_1 factors through $\mathrm{HF}^n(S^3_N(K_2), s)$. So, if it is surjective, then the map

$$\mathrm{HF}^n(S^3_N(K_2), s) \to \mathrm{HF}^n(S^3)$$

is also surjective. So we get that $\nu_n(K_1) \geq \nu_n(K_2)$. The same argument with K_1 and K_2 switched shows the inequality $\nu_n(K_2) \geq \nu_n(K_1)$. Therefore, $\nu_n(K_1) = \nu_n(K_2)$.

For negative n, that $\nu_n(K)$ is a concordance invariant follows from a similar argument to the above. □

Proposition 3.6 (mirroring property) $\nu_n(-K) = -\nu_{-n}(K)$.

Proof Recall the symmetry of CFK^∞ under mirroring [11, Section 3.5],

$$\mathrm{CFK}^\infty(-K) \simeq \mathrm{CFK}^\infty(K)^*,$$

where $\mathrm{CFK}^\infty(K)^*$ is the dual complex $\mathrm{Hom}_{\mathbb{F}[U,U^{-1}]}(\mathrm{CFK}^\infty(K), \mathbb{F}[U,U^{-1}])$. Letting $C = \mathrm{CFK}^\infty(S^3, K)$ and $n > 0$, the following conditions are equivalent:

- $v_{-s,*}^{-n}(K)\colon \mathrm{HF}^n(S^3) \to \mathrm{HF}^n(S^3_{-N}(K), \mathfrak{s}_{-s})$ is injective.
- $v_{-s}^{-n}(K)\colon C\{0 \le i \le n-1\} \to C\{0 \le \min(i, j+s) \le n-1\}$ is injective on H_*.
- $U^{n-1} v_s^n(-K)\colon C\{-(n-1) \le \max(i, j-s) \le 0\} \to C\{-(n-1) \le i \le 0\}$ is surjective on H_*.
- $v_s^n(-K)\colon C\{0 \le \max(i, j-s) \le n-1\} \to C\{0 \le i \le n-1\}$ is surjective on H_*.
- $v_{s,*}^n(-K)\colon \mathrm{HF}^n(S^3_N(-K), \mathfrak{s}_s) \to \mathrm{HF}^n(S^3)$ is surjective.

Here U^{n-1} is a degree-shifting isomorphism on $\mathrm{CFK}^\infty(K)$. Therefore,

$$\begin{aligned}\nu_n(-K) &= \min(s \in \mathbb{Z} \mid v_{s,*}^n(-K) \text{ is surjective})\\ &= -\max(-s \in \mathbb{Z} \mid v_{-s,*}^{-n}(K) \text{ is injective}) = -\nu_{-n}(K).\end{aligned}$$ □

Proposition 3.7 (monotonicity) $\nu_n(K) \le \nu_{n+1}(K)$.

Proof It is known that $\nu_{-1}(K) \le \tau(K) \le \nu_1(K)$, so we focus on the two separate cases where $n > 0$ and $n < 0$.

For $n > 0$, consider the commutative diagram

$$\begin{array}{ccc}\mathrm{HF}^{n+1}(S^3_N K, s) & \xrightarrow{v_{s,*}^{n+1}} & \mathrm{HF}^{n+1}(S^3)\\ \big\downarrow \cdot U & & \big\downarrow \cdot U\\ \mathrm{HF}^{n}(S^3_N K, s) & \xrightarrow{v_{s,*}^{n}} & \mathrm{HF}^{n}(S^3)\end{array}$$

where the vertical maps are given by multiplication by U. The vertical map on the right is surjective. Thus, if $v_{s,*}^{n+1}$ is surjective, then so is $v_{s,*}^n$.

For $n < 0$, consider the commutative diagram

$$\begin{array}{ccc}\mathrm{HF}^{-n}(S^3) & \xrightarrow{v_{s,*}^{n}} & \mathrm{HF}^{-n}(S^3_{-N} K, s)\\ \big\downarrow i'_a & & \big\downarrow i'_b\\ \mathrm{HF}^{-(n-1)}(S^3) & \xrightarrow{v_{s,*}^{n-1}} & \mathrm{HF}^{-(n-1)}(S^3_{-N} K, s)\end{array}$$

where the vertical maps are induced by inclusion of chain groups. In particular, the left map i'_a is injective on homology. Therefore, if $v^{n-1}_{s,*}(K)$ is injective, then so is $v^n_{s,*}(K)$. We conclude that $\nu_{n-1}(K) \le \nu_n(K)$. □

Proposition 3.8 (boundedness) $\nu^{+'}(K) \le \nu_n(K) \le \nu^+(K)$ *for all* n*.*

Proof It is known that $\nu(K) \le \nu^+(K)$ from [7]. For $n \ge 1$, consider the commutative diagram

$$\begin{array}{ccc} H_*(A_k^-) & \xrightarrow{j_A} & H_*(A_k^n) \\ \downarrow{\scriptstyle v^-_{k,*}} & & \downarrow{\scriptstyle v^n_{k,*}} \\ H_*(B^-) & \xrightarrow{j_B} & H_*(B^n) \end{array}$$

The map j_B is surjective, so if $v^-_{k,*}$ is surjective, then so is $v^n_{k,*}$.

For $n \le -1$, consider the commutative diagram

$$\begin{array}{ccc} H_*(B^n) & \xrightarrow{i_B} & H_*(B^+) \\ \downarrow{\scriptstyle v^n_{k,*}} & & \downarrow{\scriptstyle v^{+'}_{k,*}} \\ H_*(A_k^n) & \xrightarrow{i_A} & H_*(A_k^{+'}) \end{array}$$

The map i_B is injective, so if $v^{+'}_{k,*}$ is injective, then so is $v^n_{k,*}$. □

Proposition 3.9 (stabilization) *For sufficiently large positive* n*,* $\nu_n(K) = \nu^+(K)$ *and* $\nu_{-n}(K) = \nu^{+'}(K)$*.*

Proof Let $C_1 = \mathrm{CF}^-(S^3_N K, s)$ and $C_2 = \mathrm{CF}^-(S^3)$. There is a canonical degree-shifting isomorphism

$$\mathrm{CF}^n(Y, \mathfrak{s}) \cong \mathrm{CF}^-(Y, \mathfrak{s}) \otimes_{\mathbb{F}[U]} \frac{\mathbb{F}[U]}{U^n}.$$

Moreover, the map

$$v^-_{n,s}\colon C_1 \otimes_{\mathbb{F}[U]} \frac{\mathbb{F}[U]}{U^n} \to C_2 \otimes_{\mathbb{F}[U]} \frac{\mathbb{F}[U]}{U^n}$$

is the same as the map $v^n_s\colon \mathrm{CF}^n(S^3_N K, s) \to \mathrm{CF}^n(S^3)$. We show that if $v^-_{s,*}$ is not surjective, then neither is $v^n_{s,*}$ for sufficiently large n. By the universal coefficient theorem,

$$\begin{array}{ccccccc} 0 \to H_*(C_1) \otimes \dfrac{\mathbb{F}[U]}{U^n} & \xrightarrow{i_1} & H_*\left(C_1 \otimes \dfrac{\mathbb{F}[U]}{U^n}\right) & \longrightarrow & \operatorname{Tor}\left(H_*C_1, \dfrac{\mathbb{F}[U]}{U^n}\right) \to 0 \\ \Big\downarrow v^-_{s,*}\otimes \mathrm{id} & & \Big\downarrow v^n_{s,*} & & \Big\downarrow \operatorname{Tor}(v^-_s) \\ 0 \to H_*(C_2) \otimes \dfrac{\mathbb{F}[U]}{U^n} & \xrightarrow{i_2} & H_*\left(C_2 \otimes \dfrac{\mathbb{F}[U]}{U^n}\right) & \longrightarrow & \operatorname{Tor}\left(H_*C_2, \dfrac{\mathbb{F}[U]}{U^n}\right) \to 0 \end{array}$$

where all tensor products are taken over $\mathbb{F}[U]$.

We note the following facts:

- For a rational homology 3–sphere Y, $\mathrm{HF}^-(Y,\mathfrak{s})/\{U\text{–torsion}\} = \mathfrak{T}^- = \mathbb{F}[X]$. So $H_*(C_1) = \mathfrak{T}^- \oplus \left(\bigoplus \mathbb{F}[U]/U^{m_i}\right)$.
- $\mathfrak{T}^- \otimes \mathbb{F}[U]/U^n = \mathbb{F}[U]/U^n$ and $\mathbb{F}[U]/U^{m_i} \otimes \mathbb{F}[U]/U^n = \mathbb{F}[U]/U^{m_i}$. So $H_*(C_1) \otimes \mathbb{F}[U]/U^n = \mathbb{F}[U]/U^n \oplus \left(\bigoplus \mathbb{F}[U]/U^{m_i}\right)$.
- $\operatorname{Tor}(\mathbb{F}[U]/U^m, \mathbb{F}[U]/U^n) = \mathbb{F}[U]/U^m$ if $m < n$.
- $H_*(C_2 \otimes \mathbb{F}[U]/U^n) = \mathbb{F}[U]/U^n$.

Assume n is sufficiently large that $m_i < n$ for all m_i. So the above Tor groups are $n-1$ torsion.

If v^-_s is not surjective, we can further choose n sufficiently large that the image of $v^-_{s,*} \otimes \mathrm{id}$ is $n-1$ U–torsion. By commutativity of the diagram, the image of $v^n_{s,*} \circ i_1$ is $n-1$ $U-$torsion.

Suppose $\xi \in H_*(C_1 \otimes \mathbb{F}[U]/U^n)$ is such that $v^n_{s,*}(\xi)$ is an element of order n. Then, since the short exact sequence in the universal coefficient theorem splits, $\xi = \alpha + \beta$, where $\alpha \in H_*(C_1) \otimes \mathbb{F}[U]/U^n$ and $\beta \in \operatorname{Tor}(H_*C_1, \mathbb{F}[U]/U^n)$. But

$$U^{n-1} \cdot v^n_{s,*}(\alpha + \beta) = v^n_{s,*}(U^{n-1}\alpha) + v^n_{s,*}(U^{n-1}\beta) = 0.$$

Since $H_*(C_2 \otimes \mathbb{F}[U]/U^n) = \mathbb{F}[U]/U^n$, the invariant $v^n_{s,*}$ is not surjective. Therefore, for sufficiently large n, $\nu_n(K) = \nu^+(K)$.

Finally, by the mirroring property, $\nu_n(K) = \nu^{+'}(K)$ for sufficiently large negative integers n. □

The fact that $\nu_n(K)$ are not concordance homomorphisms from $\mathcal{C}$ to $\mathbb{Z}$ can easily be seen. Note $\nu_n(K)$ is not additive under connected sum of knots. For $n = 1$, just consider two knots with $\varepsilon(K)=\varepsilon(K')=-1$. Then

$$\nu(K)=\tau(K)+1 \quad \text{and} \quad \nu(K')=\tau(K')+1,$$

but

$$\nu(K\,\#\,K')=\tau(K\,\#\,K')+1=\tau(K)+\tau(K')+1<\nu(K)+\nu(K').$$

4 Computations

Knot Floer homology groups can be easily computed for certain special families of knots. For example, homologically thin knots are knots with $\widehat{\mathrm{HFK}}(K)$ supported in a single δ–grading, where $\delta=A-M$. If the homology is supported on the diagonal $\delta=-\frac{1}{2}\sigma(K)$, where $\sigma(K)$ denotes the knot signature, then we say the knot is σ–thin. The class of σ–thin knots contains as a proper subset all quasialternating knots, and in particular all alternating knots. The following theorem shows that $\nu_n(K)$ of thin knots only depends on $\tau(K)$:

Proposition 4.1 *Let K be a homologically thin knot with $\tau(K)=\tau$.*

(i) *If $\tau=0$, $\nu_n(K)=0$ for all n.*

(ii) *If $\tau>0$,*

$$\nu_n(K)=\begin{cases}0 & \text{for } n\le -\frac{1}{2}(\tau+1),\\ \tau+2n+1 & \text{for } -\frac{1}{2}\tau\le n\le -1,\\ \tau & \text{for } n\ge 0.\end{cases}$$

(iii) *If $\tau<0$,*

$$\nu_n(K)=\begin{cases}\tau & \text{for } n\le 0,\\ \tau+2n-1 & \text{for } 1\le n\le -\frac{1}{2}\tau,\\ 0 & \text{for } n\ge \frac{1}{2}(-\tau+1).\end{cases}$$

Proof In [15, Theorem 4] Petkova constructs model complexes for $\mathrm{CFK}^\infty(K)$ of homologically thin knots. She shows the model chain complex contains a direct summand (called the "staircase") isomorphic to

$$\begin{cases}\mathrm{CFK}^\infty(T_{2,2\tau+1}) & \text{if } \tau(K)>0,\\ \mathrm{CFK}^\infty(T_{2,2\tau-1}) & \text{if } \tau(K)\le 0.\end{cases}$$

The "staircase" summand supports $H_*(\mathrm{CFK}^\infty(K))$; that is,

$$H_*(\mathrm{CFK}^\infty(K))=H_*(\mathrm{CFK}^\infty(T_{2,2\tau-1})).$$

The maps induced on homology by v^n_s (or $v^-_{n,s}$) will thus only depend on the "staircase" summand and not the acyclic summands. Thus, $\nu_n(T_{2,2\tau\pm1}) = \nu_n(K)$.

Without loss of generality, assume $\tau(K) > 0$. The chain complex $\mathrm{CFK}^-(T_{2,2\tau+1})$ is generated over $\mathbb{F}[U]$ by generators $\{z_p\}_{p=1}^{2\tau+1}$ with U–filtration levels i and Alexander filtration levels j specified (for all $1 \le p \le 2\tau+1$) by

$$j(z_p) = \begin{cases} \tau - \frac{1}{2}(p-1) & \text{if } p \text{ odd,} \\ \tau - \frac{1}{2}(p-2) & \text{if } p \text{ even,} \end{cases} \qquad i(z_p) = \begin{cases} \frac{1}{2}(p-1) & \text{if } p \text{ odd,} \\ \frac{1}{2}p & \text{if } p \text{ even,} \end{cases}$$

and differential

$$\partial z_p = \begin{cases} 0 & \text{if } p \text{ odd,} \\ z_{p-1} + z_{p+1} & \text{if } p \text{ even.} \end{cases}$$

The above complex with generators $\{z_p\}_{p=1}^{2\tau+1}$ and given differential maps forms the generating "staircase" complex C_τ, and $\mathrm{CFK}^\infty(T_{2,2\tau+1}) = C_\tau \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}]$. The U–action lowers i and j by one.

Computation of $\nu^+(T_{2,2\tau+1})$ and $\nu_n(T_{2,2\tau+1})$ for $n > 0$ Since

$$C\{i < 0,\ j \ge \tau\} = 0,$$

the map v^+_τ is the same as the inclusion

$$C\{0 \le i \le n-1,\ j \le \tau + n - 1\} \to C\{0 \le i \le n-1\} = B^n.$$

Moreover, the generator with the highest Alexander grading in $C\{0 \le i \le n-1\}$ is $U^{n-1}z_1$, with

$$j(U^{n-1}z_1) = \tau + n - 1.$$

Thus, $C\{0 \le i \le n-1,\ j > \tau+n-1\} = 0$. That is, the inclusion v^+_τ is an isomorphism of chain complexes, so $\nu^+(K) = \tau(K)$. Therefore, $\nu_n(K) = \tau(K)$ for all $n \ge 0$.

Computation of $\nu^{+'}(T_{2,2\tau+1})$ The homology of B^+ is generated by $\{[U^{-i}z_1]\}$ for all $i \ge 0$. The subquotient complex $A^{+'}_0$ contains $U^{-i}C_\tau$ for all $i \ge 0$, and the homology of $U^{-i}C_\tau$ is generated by the class $[U^{-i}z_1]$. Therefore, $v^{+'}_{0,*}[U^{-i}z_1] \ne 0$ in $H_*(A^{+'}_0)$, and $v^{+'}_{0,*}$ is injective. So $\nu^{+'}(T_{2,2\tau+1}) \ge 0$. But since $\nu^{+'}(K) \le 0$ for any knot K, we conclude $\nu^{+'}(T_{2,2\tau+1}) = 0$.

Computation of $\nu_n(T_{2,2\tau+1})$ for $-\frac{1}{2}\tau \le n \le -1$ Consider the subquotient complex A^n_k where $k = \tau + 2n + 1$. For each $1 \le p \le 2\tau + 1$,

$$\min(i(z_p), j(z_p) - k) = \begin{cases} \min\left(\frac{1}{2}(p-1), -\frac{1}{2}(p-1) - 2n - 1\right) & \text{if } p \text{ is odd,} \\ \min\left(\frac{1}{2}p, -\frac{1}{2}(p-2) - 2n - 1\right) & \text{if } p \text{ is even} \end{cases}$$

$$= \begin{cases} \frac{1}{2}(p-1) & \text{if } p \text{ is odd and } p \le -2n-1, \\ -\frac{1}{2}(p-1)-2n-1 & \text{if } p \text{ is odd and } p > -2n, \\ \frac{1}{2}p & \text{if } p \text{ is even and } p \le -2n, \\ -\frac{1}{2}(p-2)-2n-1 & \text{if } p \text{ is even and } p > -2n. \end{cases}$$

Using these formulas, it is straightforward to check that A_k^n contains z_p for $1 \le p \le -2n-1$ but $z_{-2n} \notin A_k^n$. Therefore, $[z_1] \neq 0$ in A_k^n. Similarly, for $1 \le i \le -n-1$, A_k^n contains $U^{-i} z_p$ for $1 \le p \le -2(n+i)-1$ but $z_{-2(n+i)} \notin A_k^n$. Therefore, $[U^{-i} z_1] \neq 0$ in $H_*(A_k^n)$. Since $H_*(B^n)$ is generated by $[U^{-i} z_1]$ for $0 \le i \le -n-1$, v_k^n is injective on homology.

To check that $\nu_n(T_{2,2\tau+1}) = \tau+2n+1$, consider the subquotient complex $A^n_{\tau+2n+2}$. For each $1 \le p \le 2\tau+1$,

$$\min(i(z_p), j(z_p)-k) = \begin{cases} \min\left(\frac{1}{2}(p-1), -\frac{1}{2}(p-1)-2n-2\right) & \text{if } p \text{ is odd}, \\ \min\left(\frac{1}{2}p, -\frac{1}{2}(p-2)-2n-2\right) & \text{if } p \text{ is even} \end{cases}$$

$$= \begin{cases} \frac{1}{2}(p-1) & \text{if } p \text{ is odd and } p \le -2n-1, \\ -\frac{1}{2}(p-1)-2n-2 & \text{if } p \text{ is odd and } p > -2n, \\ \frac{1}{2}p & \text{if } p \text{ is even and } p \le -2n-2, \\ -\frac{1}{2}(p-2)-2n-2 & \text{if } p \text{ is even and } p \ge -2n. \end{cases}$$

Using the above, it is straightforward to check that $A^n_{\tau+2n+2}$ contains z_p for $1 \le p \le -4n-2$ but $z_{-4n-1} \notin A^n_{\tau+2n+2}$. Therefore, $[z_1] = 0$ in $H_*(A^n_{\tau+2n+2})$. Thus, $\nu_n(T_{2,2\tau+1}) = \tau+2n+1$.

Computation of $\nu_n(T_{2,2\tau+1})$ for $n \le -\frac{1}{2}(\tau+1)$ Consider $A_0^{n'}$, where

$$n' = \begin{cases} -\frac{1}{2}(\tau+1) & \text{if } \tau \text{ is odd}, \\ -\frac{1}{2}\tau - 1 & \text{if } \tau \text{ is even}. \end{cases}$$

For each $1 \le p \le 2\tau+1$,

$$\min(i(z_p), j(z_p)-0) = \begin{cases} \min\left(\frac{1}{2}(p-1), \tau-\frac{1}{2}(p-1)\right) & \text{if } p \text{ is odd}, \\ \min\left(\frac{1}{2}p, \tau-\frac{1}{2}(p-2)\right) & \text{if } p \text{ is even} \end{cases}$$

$$= \begin{cases} \frac{1}{2}(p-1) & \text{if } p \text{ is odd and } p \le \tau+1, \\ \tau-\frac{1}{2}(p-1) & \text{if } p \text{ is odd and } p > \tau+1, \\ \frac{1}{2}p & \text{if } p \text{ is even and } p \le \tau+1, \\ \tau-\frac{1}{2}(p-2) & \text{if } p \text{ is even and } p > \tau+1. \end{cases}$$

These computations show that $v_0^{n'}$ is injective on homology:

- If τ is odd, $A_0^{n'}$ contains z_p for $1 \le p \le -2n-1$ but $z_{-2n} \notin A_0^{n'}$. Similarly, for $1 \le i \le -n-1$, $A_0^{n'}$ contains $U^{-i} z_p$ for $1 \le p \le -2(n+i)-1$ but

$z_{-2(n+i)} \notin A_0^{n'}$. Therefore, $[U^{-i} z_1] \neq 0$ in $H_*(A_0^{n'})$ for $0 \leq i \leq -n-1$. So $v_0^{n'}$ is injective on homology.

- If τ is even, $A_0^{n'}$ contains all z_p for $1 \leq p \leq 2\tau+1$. Furthermore, for $1 \leq i \leq -n-1$, $A_0^{n'}$ contains $U^{-i} z_p$ for $1 \leq p \leq -2(n+i)-1$ but $z_{-2(n+i)} \notin A_0^{n'}$. Therefore, $[U^{-i} z_1] \neq 0$ in $H_*(A_0^{n'})$ for $0 \leq i \leq -n-1$. So $v_0^{n'}$ is injective on homology.

Since $\nu^{+'}(T_{2,2\tau+1}) = 0$ is a lower bound on $\nu_n(T_{2,2\tau+1})$, we conclude that

$$\nu_n(T_{2,2\tau+1}) = 0$$

for all $n \leq -\frac{1}{2}(\tau+1)$. □

We have the following result for strongly quasipositive knots (see [4] for background on strongly quasipositive knots):

Proposition 4.2 *If K is strongly quasipositive, then $\nu_n(K) = \tau(K) = g_4(K) = g(K)$ for all positive n.*

Proof Theorem 1.2 of [4] states that K is strongly quasipositive if and only if $\tau(K) = g_4(K) = g(K)$. The result immediately follows since $\tau(K) \leq \nu_n(K) \leq \nu^+(K) \leq g_4(K)$ for positive n. See also [7, Proposition 3]. □

Example 4.3 Figure 1 (top-left) shows the knot Floer chain complex CFK^∞ of the $(2,9)$–torus knot. The computation of $\nu_{-2}(T_{2,9})$ is shown in Figure 1 (top-right and bottom-left). We have

$$\nu_n(T_{2,9}) = \begin{cases} 4 & \text{for all } n \geq 0, \\ 3 & \text{for } n = -1, \\ 1 & \text{for } n = -2, \\ 0 & \text{for all } n \leq -3. \end{cases}$$

The computation of $\nu_n(K)$ for thin knots shows that the sequence ν_n can increase by more than one at a time, in contrast to the local h–invariants defined by Rasmussen, which jump by at most one [16, Proposition 7.6].

In fact, the gap between $\nu_n(K)$ and $\nu_{n+1}(K)$ can be arbitrarily big. For example, a straightforward (partial) computation of $\nu_n(T_{p,p+1})$ using $\mathrm{CFK}^\infty(T_{p,p+1})$ shows that for $p > 3$,

$$\nu_{-1}(T_{p,p+1}) - \nu_{-2}(T_{p,p+1}) = p.$$

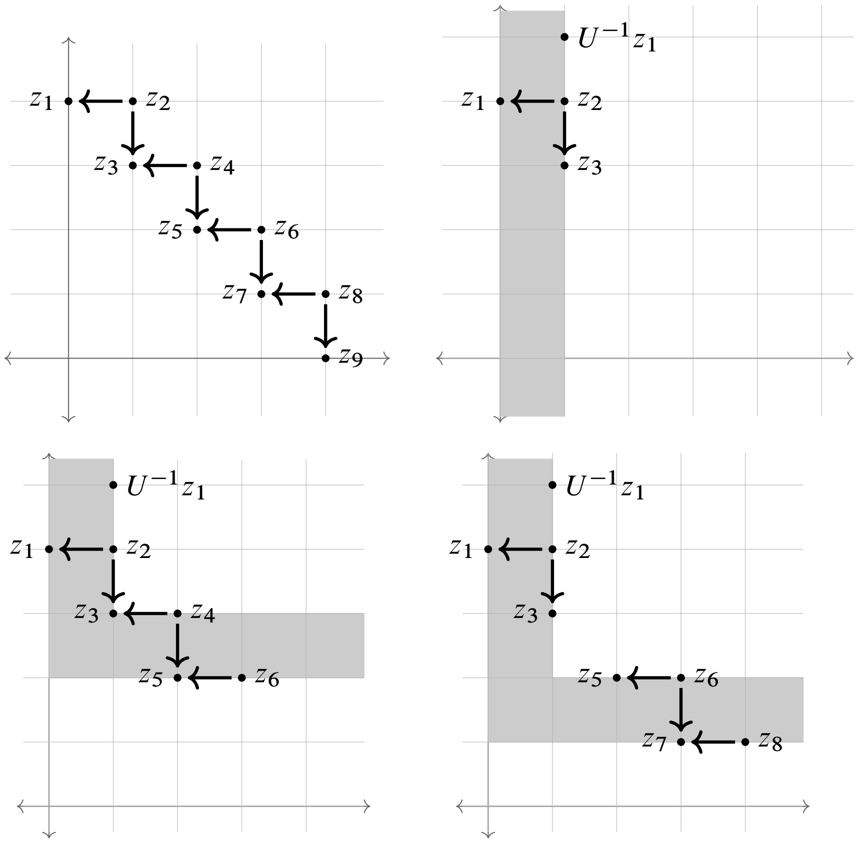

Figure 1: Top-left: Generating complex for CFK^∞ of the (2, 9)-torus knot $T_{2,9}$. Top-right: The classes $[U^{-1}z_1]$ and $[z_1]=[z_3]$ generate $\mathrm{HF}^2(S^3)$. Bottom-left: The classes $[z_1]$ and $[z_3]$ vanish in $\mathrm{HF}^2(S^3_{-N}K,[2])$. Bottom-right: The classes $[z_1]=[z_3]$ and $[U^{-1}z_1]$ survive in $\mathrm{HF}^2(S^3_{-N}K,[1])$.

Theorem 4.4 *Let $T_{p,p+1}$ denote the $(p,p+1)$–torus knot for $p>3$. Let $\tau=\tau(T_{p,p+1})=\frac{1}{2}(p-1)p$. Then*

$$\nu_n(T_{p,p+1})=\begin{cases}\tau & \text{for } n\geq 0,\\ \tau-1 & \text{for } n=-1,\\ \tau-1-p & \text{for } n=-2.\end{cases}$$

Thus, $\nu_{-1}(T_{p,p+1})-\nu_{-2}(T_{p,p+1})=p$.

Proof In [1], Allen shows that the staircase model chain complex for $\mathrm{CFK}^\infty(T_{p,p+1})$ takes the form

$$[1,\ p-1,\ 2,\ p-2,\ \ldots,\ j,\ p-j,\ \ldots,\ p-1,\ 1],$$

where the indices alternate between the widths of the horizontal and vertical steps. From this staircase description, there exists a (i, j)–filtered basis for $\mathrm{CFK}^{\infty}(T_{p,p+1})$ consisting of generators $\{b_l\}_{l=0}^{2(p-1)}$ lying in (i, j)–filtrations

$$b_{2m}: \quad \Bigl(\sum_{k=1}^{m} k,\ \tfrac{1}{2}(p-1)p - \sum_{k=1}^{m}(p-k)\Bigr),$$

$$b_{2m+1}: \quad \Bigl(\sum_{k=1}^{m+1} k,\ \tfrac{1}{2}(p-1)p - \sum_{k=1}^{m}(p-k)\Bigr),$$

and differential

$$\partial b_{2m} = 0, \quad \partial b_{2m+1} = b_{2m} + b_{2m+2}.$$

The same argument for showing that $\nu^+(T_{2,2\tau+1}) = \tau(T_{2,2\tau+1})$ in Proposition 4.1 holds for the knots $T_{p,p+1}$. Moreover, in the terminology of [5], the basis $\{b_l\}_{l=0}^{2(p-1)}$ satisfies:

- b_0 is the vertically distinguished element of a vertically simplified basis.
- b_0 has a unique incoming horizontal arrow (from b_1) (and no outgoing horizontal arrows).

We immediately conclude that $\varepsilon(T_{p,p+1}) = 1$ and $\nu_{-1}(T_{p,p+1}) = \tau - 1$.

To show $\nu_{-2}(T_{p,p+1}) = \tau - p - 1$, we observe:

- $A^{-2}_{\tau-p-1}$ contains the generators b_0, b_1 and b_2, but $b_3 \notin A^{-2}_{\tau-p-1}$. Therefore, $[b_0] \neq 0$ in $H_*(A^{-2}_{\tau-p-1})$. Moreover, $[U^{-1}b_0] \neq 0$ in $H_*(A^{-2}_{\tau-p-1})$. Thus, $v^{-2}_{\tau-p-1}$ is injective on homology.
- $A^{-2}_{\tau-p}$ contains the generators b_0, b_1, b_2 and b_3, but $b_4 \notin A^{-2}_{\tau-p}$. Therefore, $[b_0] = 0$ in $H_*(A^{-2}_{\tau-p})$. □

We show the concordance invariants $\{\nu_n(K)\}$ contain more concordance information than the collection $\{\tau, \nu, \nu', \nu^+, \nu^{+'}\}$:

Proposition 4.5 *There exist knots K and K' with equal τ, ν, ν', ν^+ and $\nu^{+'}$ invariants, but $\nu_n(K) \neq \nu_n(K')$ for some $n \in \mathbb{Z}$.*

Proof The torus knot $T_{4,5}$ and the torus knot $T_{2,13}$ share the following invariants in common:

$$\nu^{+'}(T_{4,5}) = 0 = \nu^{+'}(T_{2,13}),$$

$$\nu'(T_{4,5}) = 5 = \nu'(T_{2,13}),$$

$$\tau(T_{4,5}) = \nu(T_{4,5}) = \nu^+(T_{4,5}) = 6 = \tau(T_{2,13}) = \nu(T_{2,13}) = \nu^+(T_{2,13}).$$

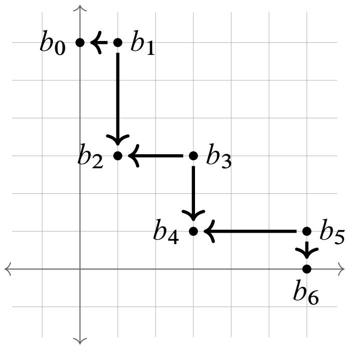

Figure 2: Generating complex for CFK^∞ of the left-handed $(4,5)$–torus knot $T_{4,5}$. $\mathrm{CFK}^\infty(T_{4,5})$ is generated over $\mathbb{F}[U, U^{-1}]$ by the above chain complex. The arrows, representing terms in the differential, are drawn to scale, with lengths of arrows ranging between one and three.

However, the invariants $\nu_n(T_{4,5})$ are different from $\nu_n(T_{2,13})$:

$$\nu_n(T_{4,5}) = \begin{cases} 6 & \text{for } n \geq 0, \\ 5 & \text{for } n = -1, \\ 1 & \text{for } n = -2, \\ 0 & \text{for } n \leq -3, \end{cases} \qquad \nu_n(T_{2,13}) = \begin{cases} 6 & \text{for } n \geq 0, \\ 5 & \text{for } n = -1, \\ 3 & \text{for } n = -2, \\ 1 & \text{for } n = -3, \\ 0 & \text{for } n \leq -4, \end{cases}$$

where $\nu_n(T_{4,5})$ is computed from the knot Floer chain complex $\mathrm{CFK}^\infty(T_{4,5})$ as shown in Figure 2. □

5 Further directions

One question is the effectiveness of $\nu_n(K)$ when compared to other concordance invariants such as $\Upsilon_K(t)$, coming from t–modified knot Floer homology [10], or V_k, coming from surgery formulas of Ozsváth and Szabó [13].

The invariants $\nu_n(K)$ do not define concordance homomorphisms $\mathcal{C} \to \mathbb{Z}$, where $\mathcal{C}$ is the concordance group of knots. This implies that they do not necessarily vanish on knots of finite concordance order. The existence of p–torsion, with $p \neq 2$, in the concordance group $\mathcal{C}$ is an open question. A related conjecture, based on a question of Gordon [3], as phrased in [8] is:

Conjecture 5.1 (Gordon) A knot has order two in $\mathcal{C}$ if and only if $K = -K$ is negative amphichiral.

Recently, Hendricks and Manolescu defined involutive Heegaard Floer concordance invariants $\overline{V}_0$ and $\underline{V}_0$, which detects the nonsliceness of the figure eight knot. The nonsliceness of 4_1 was previously known through classical methods, but this is the first method of detection coming from the Heegaard Floer package. By additivity of τ and the behavior of ε under connected sums, $\tau(K)$ and $\nu(K)$ vanish for all knots K of finite concordance order. This leaves open the cases $\nu_n(K)$ for $n > 1$ and $n < -1$. We pose the question:

Question 5.2 Does there exist a knot K of finite concordance order such that $\nu_n(K) \neq 0$ for some n?

Another question is how the invariants $\nu_n(K)$ behave under connected sum. It is known that $\nu^+(K)$ is subadditive by [2]. That is,

$$\nu^+(K \# L) \leq \nu^+(K) + \nu^+(L).$$

Using mirroring relations and subadditivity of $\nu^+(K)$ shows that $\nu^{+'}(K)$ is superadditive:

Lemma 5.3 *For any two knots K and L,*

$$\nu^{+'}(K \# L) \geq \nu^{+'}(K) + \nu^{+'}(L).$$

Proof By subadditivity of ν^+ and the mirroring relations,

$$\begin{aligned} \nu^+(-K \# -L) &\leq \nu^+(-K) + \nu^+(-L), \\ -\nu^{+'}(K \# L) &\leq -\nu^{+'}(K) + -\nu^{+'}(L), \\ \nu^{+'}(K \# L) &\geq \nu^{+'}(K) + \nu^{+'}(L). \end{aligned}$$

□

As pointed out to the author by Jen Hom, it can also be seen by additivity of τ and the behavior of ε under connected sum that $\nu(K)$ is subadditive. A similar argument shows that $\nu'(K)$ is superadditive. This leads us to ask the following two questions:

Question 5.4 Is $\nu_n(K \# K') \leq \nu_n(K) + \nu_n(K')$ for all positive integers $n \in \mathbb{Z}_+$?

Question 5.5 Is $\nu_n(K \# K') \geq \nu_n(K) + \nu_n(K')$ for all negative integers $n \in \mathbb{Z}_-$?

The next question was posed by Zhongtao Wu:

Question 5.6 (Wu) If $\nu_n(K) = \nu_n(K')$ for all $n \in \mathbb{Z}$, then is $\nu^+(K \# -K') = \nu^+(-K \# K') = 0$?

The condition that $\nu^+(K \# -K') = \nu^+(-K \# K') = 0$ implies that

$$\mathrm{CFK}^\infty(K \# -K') \simeq \mathrm{CFK}^\infty(U) \oplus A,$$

where A is an acyclic complex [6].

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Department of Mathematics, Columbia University
New York, NY, United States

`ltruong@math.columbia.edu`

Received: 21 January 2018 Revised: 10 November 2018