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Boundaries of Baumslag–Solitar groups

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A $\mathcal{Z}$–structure on a group G was introduced by Bestvina in order to extend the notion of a group boundary beyond the realm of CAT(0) and hyperbolic groups. A refinement of this notion, introduced by Farrell and Lafont, includes a G–equivariance requirement, and is known as an $\mathcal{EZ}$–structure. The general questions of which groups admit $\mathcal{Z}$– or $\mathcal{EZ}$–structures remain open. Here we show that all Baumslag–Solitar groups admit $\mathcal{EZ}$–structures and all generalized Baumslag–Solitar groups admit $\mathcal{Z}$–structures.

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1 Introduction

In [3], Bestvina introduced the concept of a $\mathcal{Z}$–structure on a group G to provide an axiomatic treatment of group boundaries. Roughly speaking, the definition requires G to act geometrically (properly, cocompactly, by isometries) on a "nice" space X and for that space to admit a nice compactification $\overline{X}$ (a $\mathcal{Z}$–compactification). In addition, it is required that translates of compact subsets of X get small in $\overline{X}$ — a property called the *nullity condition*. Adding visual boundaries to CAT(0) spaces and Gromov boundaries to appropriately chosen Rips complexes provide the model examples. Bestvina posed the still-open question of whether or not every group that admits a finite $K(G,1)$ complex admits a $\mathcal{Z}$–structure.

In [3], the Baumslag–Solitar group $\mathrm{BS}(1,2)$ was put forward as a nonhyperbolic, non-CAT(0) group that, nevertheless, admits a $\mathcal{Z}$–structure. The Baumslag–Solitar groups $\mathrm{BS}(1,n)$ behave similarly, but from the beginning, the status of the general Baumslag–Solitar groups $\mathrm{BS}(m,n)$ was unclear. In this paper we resolve that issue in a strong way:

Theorem 1.1 *Every generalized Baumslag–Solitar group admits a $\mathcal{Z}$–structure.*

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A generalized Baumslag–Solitar group is the fundamental group of a graph of groups with vertex and edge groups $\mathbb{Z}$. By applying work of Whyte [17] and a boundary-swapping trick (see [3] and Guilbault and Moran [10]), it will suffice to show that the actual Baumslag–Solitar groups $\mathrm{BS}(m,n)$ admit $\mathcal{Z}$–structures. For those groups, we will prove the following stronger theorem:

Theorem 1.2 ($\mathcal{EZ}$–structures on Baumslag–Solitar groups) *All Baumslag–Solitar groups,* $\mathrm{BS}(m,n)$*, admit* $\mathcal{EZ}$*–structures.*

Here $\mathcal{EZ}$ stands for "equivariant $\mathcal{Z}$–structure", a $\mathcal{Z}$–structure in which the group action extends to the boundary. Torsion-free groups (which includes all groups studied in this paper) that admit $\mathcal{EZ}$–structures are known to satisfy the Novikov conjecture; see Farrell and Lafont [8]. The Novikov conjecture for Baumslag–Solitar groups originally follows from Béguin, Bettaieb and Valette [2]. It can also be obtained from Matsnev [12] combined with Yu [18] and the generalized Baumslag–Solitar case comes from these results plus the quasi-isometry classification from [17]. Our theorem, combined with [8], gives a new proof of the Novikov conjecture for Baumslag–Solitar groups. This is one reason to aim for the stronger condition.

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2 Background

2.1 Visual boundaries of CAT(0) spaces

In this section, we review the definition of CAT(0) spaces and the visual boundary as we will use these as a starting point for $\mathcal{EZ}$–structures on $\mathrm{BS}(m,n)$. For a more thorough treatment of CAT(0) spaces, see [5].

Definition 2.1 A geodesic metric space (X,d) is a *CAT*(0) *space* if all of its geodesic triangles are no fatter than their corresponding Euclidean comparison triangles. That is, if $\Delta(p,q,r)$ is any geodesic triangle in X and $\overline{\Delta}(\overline{p},\overline{q},\overline{r})$ is its comparison triangle in $\mathbb{E}^2$, then for any $x,y\in\Delta$ and their comparison points $\overline{x},\overline{y}\in\overline{\Delta}$, we have $d(x,y)\le d_{\mathbb{E}}(\overline{x},\overline{y})$.

Example 1 Basic examples of CAT(0) spaces include:

- $\mathbb{R}^n$ equipped with the Euclidean metric is a CAT(0) space as all geodesic triangles are already Euclidean and hence no fatter than their comparison triangles.

- A tree, T, is a CAT(0) space since all geodesic triangles are degenerate and thus have no thickness associated to them.
- If X and Y are CAT(0) spaces, then $X \times Y$ with the ℓ^2 metric is CAT(0). So, for example, $\mathbb{R} \times T$ is a CAT(0) space — a fact that will play a significant role in this paper.

A group G that acts properly, cocompactly and by isometries (also known as a *geometric* group action) on a proper CAT(0) space is called a *CAT*(0) *group*.

Definition 2.2 The *boundary* of a proper CAT(0) space X, denoted by ∂X, is the set of equivalence classes of rays, where two rays are equivalent if and only if they are asymptotic. We say that two geodesic rays α, α': $[0,\infty) \to X$ are *asymptotic* if there is some constant k such that $d(\alpha(t), \alpha'(t)) \le k$ for every $t \ge 0$.

If we fix a basepoint $x_0 \in X$, each equivalence class of rays in X contains exactly one representative emanating from x_0. So when x_0 is chosen, we can view ∂X as the set of all rays in X based at x_0. We may endow $\overline{X} = X \cup \partial X$ with the *cone topology*, described below, under which ∂X is a closed subspace of $\overline{X}$ and $\overline{X}$ is compact (provided X is proper). Equipped with the topology induced by the cone topology on $\overline{X}$, the boundary is called the *visual boundary* of X; we will denote it by $\partial_\infty X$.

The cone topology on $\overline{X}$, denoted by $\mathcal{T}(x_0)$ for $x_0 \in X$, is generated by the basis $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_\infty$ where $\mathcal{B}_0$ consists of all open balls $B(x,r) \subset X$ and $\mathcal{B}_\infty$ is the collection of all sets of the form

$$U(c,r,\epsilon) = \{x \in \overline{X} \mid d(x, c(0)) > r \text{ and } d(p_r(x), c(r)) < \epsilon\},$$

where c: $[0,\infty) \to X$ is any geodesic ray based at x_0, $r > 0$, $\epsilon > 0$ and p_r is the natural projection of $\overline{X}$ onto $\overline{B}(c(0), r)$.

Example 2 Boundaries of the simple examples given above are:

- $\partial_\infty \mathbb{R}^n \simeq S^{n-1}$.
- $\partial_\infty T$ is compact and 0–dimensional. If each vertex has degree ≥ 3, it is a Cantor set C. (In order for T to be proper, assume all vertices have finite degree.)
- If X and Y are CAT(0) spaces and $X \times Y$ is given the ℓ^2 metric, then $\partial_\infty(X \times Y) \simeq \partial X_\infty * \partial_\infty Y$, the (spherical) join of the two boundaries [5, Section 8.11]. For example, $\partial_\infty(\mathbb{R} \times T)$ is homeomorphic to $S^0 * \partial_\infty T$, the suspension of a 0–dimensional set (usually a Cantor set).

When G is a CAT(0) group acting geometrically on a proper CAT(0) space X, we call $\partial_\infty X$ a CAT(0) boundary for G. For example, since $\mathbb{Z}^n$ acts geometrically on $\mathbb{R}^n$, it is a CAT(0) group and S^{n-1} is a CAT(0) boundary. The free group on two generators, F_2, acts geometrically on a four-valent tree, so a CAT(0) boundary for F_2 is the Cantor set.

The following lemma, which is reminiscent of the Lebesgue covering lemma, will be useful in proving our main theorem:

Lemma 2.3 *Let (X, d) be a proper CAT(0) space and let $\mathcal{U}$ be an open cover of $\overline{X}$. Then there exists a $\delta > 0$ such that $U\left(z, \frac{1}{\delta}, \delta\right)$ lies in an element of $\mathcal{U}$ for every $z \in \partial_\infty X$.*

Proof Since $\partial_\infty X$ is compact, there is a finite subcollection $\{U_1, U_2, \dots, U_k\}$ of $\mathcal{U}$ that covers $\partial_\infty X$. For each $i \in \{1, 2, \dots, k\}$, define a function $\eta_i \colon \partial_\infty X \to [0, \infty)$ by $\eta_i(z) = \sup\left\{\epsilon \mid U\left(z, \frac{1}{\epsilon}, \epsilon\right) \subseteq U_i\right\}$. Note that η_i is continuous and $\eta_i(z) > 0$ if and only if $z \in U_i$. Thus, $\eta\colon \partial_\infty X \to [0, \infty)$ defined by $\eta(z) = \max\{\eta_i(z)\}_{i=1}^k$ is continuous and strictly positive. Let δ' be the minimum value of η and set $\delta = \frac{1}{2}\delta'$. $\square$

2.2 $\mathcal{Z}$–structures

Boundaries of CAT(0) groups have proven to be useful objects that can help us gain more information about the groups themselves. This led Bestvina to generalize the notion of group boundaries by defining "$\mathcal{Z}$–boundaries" for groups, a topic that we explore now. For more on $\mathcal{Z}$–structures, see [3; 10].

Definition 2.4 A closed subset A of a space X is a *$\mathcal{Z}$–set* if there exists a homotopy $H\colon X \times [0, 1] \to X$ such that $H_0 = \mathrm{id}_X$ and $H_t(X) \subset X - A$ for every $t > 0$.

Example 3 The prototypical $\mathcal{Z}$–set is the boundary of a manifold, or any closed subset of that boundary.

A *$\mathcal{Z}$–compactification* of a space X is a compactification $\overline{X}$ such that $\overline{X} - X$ is a $\mathcal{Z}$–set in $\overline{X}$.

Example 4 The addition of the visual boundary to a proper CAT(0) space X gives a $\mathcal{Z}$–compactification $\overline{X}$ of X. A simple way to see the visual boundary as a $\mathcal{Z}$–set in $\overline{X}$ is to imagine the homotopy that "reels" points of the boundary in along the geodesic rays.

Definition 2.5 A *$\mathcal{Z}$–structure* on a group G is a pair of spaces $(\overline{X}, Z)$ satisfying the four conditions

(1) $\overline{X}$ is a compact absolute retract (AR),

(2) Z is a $\mathcal{Z}$–set in $\overline{X}$,

(3) $X = \overline{X} - Z$ is a proper metric space on which G acts geometrically, and

(4) $\overline{X}$ satisfies the following *nullity condition* with respect to the G–action on X: for every compact $C \subseteq X$ and any open cover $\mathcal{U}$ of $\overline{X}$, all but finitely many G–translates of C lie in an element of $\mathcal{U}$.

When this definition is satisfied, Z is called a *$\mathcal{Z}$–boundary* for G. If only conditions (1)–(3) are satisfied, the result is called a *weak $\mathcal{Z}$–structure*. If, in addition to (1)–(4) above, the G–action on X extends to $\overline{X}$, the result is called an *$\mathcal{E}\mathcal{Z}$–structure* (equivariant $\mathcal{Z}$–structure).

Example 5 The following are the most common examples of $(\mathcal{E})\mathcal{Z}$–structures:

(1) If G acts geometrically on a proper CAT(0) space X, then $\overline{X} = X \cup \partial_\infty X$, with the cone topology, gives an $\mathcal{E}\mathcal{Z}$–structure for G.

(2) In [4] it is shown that if G is a torsion-free hyperbolic group, $P_\rho(G)$ is an appropriately chosen Rips complex and ∂G is the Gromov boundary, then $\overline{P}_\rho(G) = P_\rho(G) \cup \partial G$ (appropriately topologized) gives an $\mathcal{E}\mathcal{Z}$–structure for G. Results in [13] allow for the inclusion of hyperbolic groups with torsion.

(3) Osajda and Przytycki [14] have shown that systolic groups admit $\mathcal{E}\mathcal{Z}$–structures.

Other classes of groups that admit $(\mathcal{E})\mathcal{Z}$–structures have been addressed by Dahmani [6] (relatively hyperbolic groups), Martin [11] (nonpositively curved complexes of groups), Tirel [16] (free and direct products) and Pietsch [15] (semidirect products with $\mathbb{Z}$ and 3–manifold groups).

There is a small overlap between our main theorem and the above results. For example, the Baumslag–Solitar groups $\mathrm{BS}(m,m)$ are CAT(0). But none of the others are CAT(0) or hyperbolic (a topic discussed in Section 3). Since generalized Baumslag–Solitar groups are, by definition, fundamental groups of graphs of groups, one might hope to obtain $(\mathcal{E})\mathcal{Z}$–structures by applying [11]. That strategy fails since the corresponding actions on their Bass–Serre trees are not acylindrical. In short, almost all of the examples covered by Theorem 1.2 appear to be new.

A few comments are in order regarding the definition of $\mathcal{Z}$–structure. First, Bestvina's original definition did not explicitly require actions by isometries, but only by covering transformations. As we point out at the end of Section 3.3, there is no loss of generality in requiring actions by isometries. Bestvina also required $\overline{X}$ to be finite-dimensional and the action to be free. Dranishnikov relaxed both of these conditions in [7], and [10] shows that nothing is lost in doing so.

We close this section with a few observations about $\mathcal{Z}$–structures. The first makes the nullity condition more intuitive, the second is useful for verifying the nullity condition, and the third can (and will) be used to obtain $\mathcal{Z}$–structures for a broad class of groups without checking each group individually.

Every $\mathcal{Z}$–compactification $\overline{X}$ of a proper metric space (X,d) is metrizable (see [10]), but in general there is no canonical choice of metric for $\overline{X}$; moreover, whichever metric $\overline{d}$ one chooses will be quite different from d. Nevertheless, any such choice can be used to give the following intuitive meaning to the nullity condition. The proof is straightforward general topology.

Lemma 2.6 *Let $(\overline{X},Z)$ be a weak $\mathcal{Z}$–structure as described in Definition 2.5, and let $\overline{d}$ be a metric for $\overline{X}$. Then $(\overline{X},Z)$ satisfies the nullity condition* (and hence is a $\mathcal{Z}$–structure) *if and only if*:

> *For any compact set $C \subseteq X$ and $\epsilon > 0$, all but finitely many G–translates of C have $\overline{d}$–diameter less than ϵ.*

The next lemma allows us to verify the nullity condition without checking every compact subset C of X.

Lemma 2.7 *Let X be a proper metric space admitting a proper cocompact action by G and let $(\overline{X},\overline{d})$ be a $\mathcal{Z}$–compactification of X. If C is a compact subset of X with the property that $GC = X$ and the nullity condition is satisfied for C, then the nullity condition is satisfied for all compact subsets of X.*

Proof Choose $\epsilon > 0$ and let $K \subseteq X$ be an arbitrary compact set. By properness and the hypothesis, there are finitely many translates of C that cover K, that is, $K \subseteq g_1C \cup g_2C \cup \cdots \cup g_nC$ for $g_i \in G$. Since C satisfies the nullity condition, all but finitely many G translates of C have $\overline{d}$–diameter less than $\frac{\epsilon}{n}$. If we consider any translate gK, then $gK \subseteq gg_1C \cup gg_2C \cup \cdots \cup gg_nC$. Only finitely many gg_iC for $g \in G$ have diameter greater than $\frac{\epsilon}{n}$ and thus only finitely many gK have diameter greater than $n\frac{\epsilon}{n} = \epsilon$. □

The following useful fact is often referred to as the "boundary-swapping trick":

Proposition 2.8 [3; 10] *Suppose G and H are quasi-isometric groups that act geometrically on proper metric ARs X and Y, respectively, and Y can be compactified to a $\mathcal{Z}$–structure $(\overline{Y}, Z)$ for H; then X can be compactified by addition of the same boundary to obtain a $\mathcal{Z}$–structure $(\overline{X}, Z)$ for G.*

3 $\mathcal{Z}$–structures on generalized Baumslag–Solitar groups

A *Baumslag–Solitar group* $\mathrm{BS}(m,n)$ is a two-generator, one-relator group admitting a presentation of the form

$$\mathrm{BS}(m,n) = \langle s, t \mid ts^m t^{-1} = s^n \rangle.$$

Without loss of generality, we may assume that $0 < |m| \le n$. These groups are HNN extensions of $\mathbb{Z}$ with infinite cyclic associated subgroups, and the standard presentation 2–complex $K_{m,n}$ is a $K(\pi,1)$ space. If we begin with the canonical graph of groups representation of $\mathrm{BS}(m,n)$ with one vertex and one edge, the corresponding Bass–Serre tree is the directed tree $T(|m|,n)$ with $|m|$ incoming and n outgoing edges at each vertex, and the universal cover of $K_{m,n}$ is homeomorphic to $\mathbb{R} \times T(|m|,n)$. Gersten [9] has shown that, provided $|m| \ne n$, the Dehn function of $\mathrm{BS}(m,n)$ is not bounded by a polynomial. By contrast, Dehn functions of hyperbolic and CAT(0) groups are bounded by linear and quadratic functions, respectively. So most Baumslag–Solitar groups are neither hyperbolic nor CAT(0). As such, this collection of groups contains some of the simplest candidates for $\mathcal{Z}$–structures not covered by the motivating examples.

3.1 Generalized Baumslag–Solitar groups

A *generalized Baumslag–Solitar group* is the fundamental group G of a finite graph of groups with all vertex and edge groups $\mathbb{Z}$. In [17], Whyte classified generalized Baumslag–Solitar groups, up to quasi-isometry.

Theorem 3.1 [17] *If Γ is a graph of $\mathbb{Z}$s and $G = \pi_1\Gamma$, then exactly one of the following is true:*

(1) *G contains a subgroup of finite index of the form $\mathbb{Z} \times \mathbb{F}_n$.*

(2) *$G = \mathrm{BS}(1,n)$ for some $n > 1$.*

(3) *G is quasi-isometric to $\mathrm{BS}(2,3)$.*

As with the ordinary Baumslag–Solitar groups, each generalized Baumslag–Solitar group G acts properly and cocompactly on $\mathbb{R} \times T$, where T is the Bass–Serre tree of its graph of groups representation. If G is of the first type mentioned in Theorem 3.1, it is quasi-isometric to the CAT(0) group $\mathbb{Z} \times F_n$, so, by the boundary-swapping trick (Proposition 2.8), G admits a $\mathcal{Z}$–structure. By another application of Theorem 3.1 and the boundary-swapping trick, we can then obtain $\mathcal{Z}$–structures for all generalized Baumslag–Solitar groups provided we can obtain them for ordinary Baumslag–Solitar groups. That is where we turn our attention to now.

3.2 A "standard" action of $\mathrm{BS}(m,n)$ on $\mathbb{R} \times T(|m|,n)$

As noted above, $\mathrm{BS}(m,n)$ acts properly, freely and cocompactly on $\mathbb{R} \times T(|m|,n)$. In Example 2, we observed that this space admits a $\mathcal{Z}$–compactification by addition of the suspension of $\partial_\infty T(|m|,n)$. That is accomplished by giving $\mathbb{R} \times T(|m|,n)$ its natural CAT(0) metric and adding the visual boundary. This gives us a weak $\mathcal{Z}$–structure for $\mathrm{BS}(m,n)$, but since the action of $\mathrm{BS}(m,n)$ on this CAT(0) space is not by isometries, the nullity condition does not follow. In fact, if we subdivide $\mathbb{R} \times T(|m|,n)$ into rectangular principal domains for $\mathrm{BS}(m,n)$ in the traditional manner (see Figure 1) and if $|m| \neq n$, these rectangles grow exponentially as they are translated along the positive t–axis. More importantly (for our purposes), translates of the fundamental domain remain large in the compactification (details to follow). Arranging the nullity condition will require significantly more work.

Although this "standard" action of $\mathrm{BS}(m,n)$ on $\mathbb{R} \times T(|m|,n)$ with its CAT(0) metric and corresponding visual boundary does not give the desired $(\mathcal{E})\mathcal{Z}$–structure, the picture it provides is useful; therefore, we supply some additional details.

For the moment it is convenient to assume that $m > 0$. Choose a preferred vertex v_0 of $T(m,n)$ and place the Cayley graph Γ of $\mathrm{BS}(m,n)$ in $\mathbb{R} \times T(m,n)$ so that $\boldsymbol{v}_0 = (0, v_0)$ corresponds to $1 \in \mathrm{BS}(m,n)$, and the positively oriented edge-ray $\tau^+ \subseteq \Gamma$ whose edges are each labeled by an outward-pointing t and the negatively oriented edge-ray τ^- whose edges are each labeled by an inward-pointing t both lie in $\{0\} \times T(m,n)$. In other words, the line $\tau \equiv \tau^- \cup \tau^+ \subseteq \Gamma$, corresponding to the subgroup $\langle t \rangle$, is a subset of $\{0\} \times T(m,n)$. Subdivide $\mathbb{R} \times \{v_0\}$ into edges of length $\frac{1}{n}$, each oriented in the positive $\mathbb{R}$–direction and labeled by the generator s. Thus we have identified this line with the subgroup $\langle s \rangle$. Let $R_0 \subseteq \mathbb{R} \times \tau$ be the 1×1 rectangle with lower left-hand vertex at 1 and boundary labeled by the defining relator of $\mathrm{BS}(m,n)$. Tile

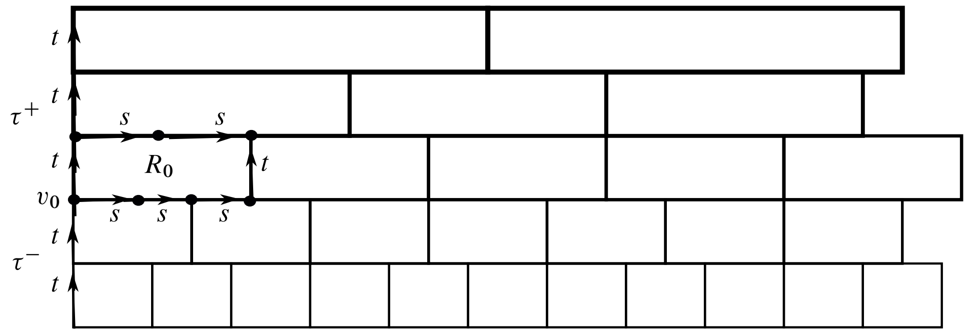

Figure 1: Tiling of $\mathrm{BS}(2,3)$

the plane $\mathbb{R} \times \tau$ with rectangular fundamental domains, each of whose boundaries is labeled by the relator as shown in Figure 1, keeping in mind that this plane represents only a small portion of the Cayley complex.

For each edge-ray $\rho \subseteq T(m,n)$ emanating from v_0, we refer to the half-plane $\mathbb{R} \times \rho$ as a *sheet* of $\mathbb{R} \times T(m,n)$. If all edges on ρ are positively oriented, call $\mathbb{R} \times \rho$ a *positive sheet*; if all edges are negatively oriented, call $\mathbb{R} \times \rho$ a *negative sheet*; and if ρ contains both orientations, call $\mathbb{R} \times \rho$ a *mixed sheet*. Call $\mathbb{R} \times \tau^+$ the *preferred positive sheet and* $\mathbb{R} \times \tau^-$ the *preferred negative sheet*. (Note: although the oriented tree $\{0\} \times T(m,n)$ plays a useful role, most of its edges are not contained in Γ.)

Notice that each sheet is a convex subset of $\mathbb{R} \times T(m,n)$ isometric to a Euclidean half-plane. Up to horizontal translation, all positive sheets inherit a tiling identical to that of $\mathbb{R} \times \tau^+$ and all negative sheets inherit a tiling identical (up to translation) to $\mathbb{R} \times \tau^-$. So, in positive sheets the widths of the fundamental domains increase (exponentially) as one gets further from v_0 in the $T(m,n)$–direction, while in the negative sheets the widths decrease. In mixed sheets, widths do not change in a monotone manner — sometimes they increase and sometimes they decrease — but the resulting tiling is always finer than that of an appropriately placed positive sheet. In other words, the tiles in a generic sheet always fit inside those of a correspondingly subdivided positive sheet. Finally, note also that for $m < 0$, the tiling of $\mathbb{R} \times T(|m|,n)$ is the same, but with the s edges at odd integer heights oriented in the negative $\mathbb{R}$–direction.

3.3 An adjusted action of $\mathrm{BS}(m,n)$ on $\mathbb{R} \times T(|m|,n)$

Under the above setup, the nullity condition fails badly. For example, translates of R_0 by powers of t limit out on the entire quarter circle bounding the right-hand quadrant of $\mathbb{R} \times \tau^+$ in the visual compactification of the CAT(0) space $\mathbb{R} \times T(|m|, n)$. Instead of changing the space or its compactification, we will remedy this problem by changing the action. Some of the resulting calculations are lengthy, but the idea is simple. Define $f\colon \mathbb{R} \times T(|m|, n) \to \mathbb{R} \times T(|m|, n)$ by

$$f(x, y) = (\operatorname{sgn}(x) \log\log(|x| + e), y).$$

Our new action is via conjugation by this homeomorphism. More specifically, for each $g \in \mathrm{BS}(m, n)$, viewed as a self-homeomorphism of $\mathbb{R} \times T(|m|, n)$ under the original $\mathrm{BS}(m, n)$ action, define $\overline{g}\colon \mathbb{R} \times T(|m|, n) \to \mathbb{R} \times T(|m|, n)$ by $\overline{g} = f \circ g \circ f^{-1}$. Here $f^{-1}\colon \mathbb{R} \times T(|m|, n) \to \mathbb{R} \times T(|m|, n)$ can be specified by

$$f^{-1}(x, y) = (\operatorname{sgn}(x)(\exp\exp|x| - e), y).$$

For simplicity, we refer to this as the $\overline{\mathrm{BS}(m, n)}$–action on $\mathbb{R} \times T(|m|, n)$. Our goal then is to show that *with this action*, the visual compactification of $\mathbb{R} \times T(|m|, n)$ satisfies the definition of $\mathcal{Z}$–structure. After that task is completed, we will show that this action also extends to the visual boundary, thereby completing the proof of Theorem 1.2.

Before proceeding with the calculations, note that the $\overline{\mathrm{BS}(m, n)}$–action on the CAT(0) space $\mathbb{R} \times T(|m|, n)$ is still not by isometries — as noted earlier, that would be impossible since $\mathrm{BS}(m, n)$ is not CAT(0) when $|m| \neq n$. To obtain the isometry requirement implicit in Definition 2.5 we can apply the following proposition. It reveals that the isometry requirement is mostly just a technicality.

Proposition 3.2 [1] *Suppose G acts properly and cocompactly on a locally compact space X. Then there is a topologically equivalent proper metric for X under which the action is by isometries.*

3.4 Nullity condition for the $\overline{\mathrm{BS}(m, n)}$–action on $\mathbb{R} \times T(|m|, n)$

Recall the 1×1 rectangle $R_0 \subseteq \mathbb{R} \times \tau$ defined earlier. Under the standard action of $\mathrm{BS}(m, n)$ on $\mathbb{R} \times T(|m|, n)$ acts as our preferred fundamental domain. Translates of R_0 by elements of $\mathrm{BS}(m, n)$ produce a "tiling" of $\mathbb{R} \times T(|m|, n)$, part of which is pictured in Figure 1. The most notable trait of this tiling is that, while the heights of all rectangles in the tiling are 1 (measured along the $T(|m|, n)$–coordinate), the widths of rectangles in the positive sheets grow exponentially with the $T(|m|, n)$–coordinate whenever $|m| \neq n$. For example, a generic tile in a positive sheet with lower edge at height b will have width $(n/|m|)^b$. Widths of tiles in generic sheets are

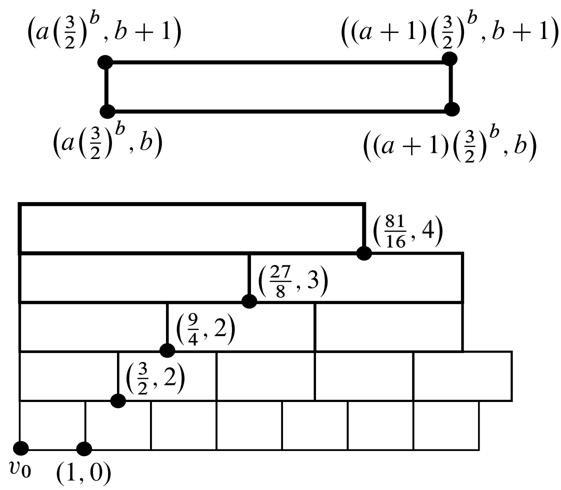

Figure 2: Coordinates of $\mathrm{BS}(2,3)$

bounded above by this number. Under the $\overline{\mathrm{BS}(m,n)}$–action on $\mathbb{R}\times T(|m|,n)$, the role of R_0 is played by the compressed rectangle $\bar{R}_0 = f(R_0)$, and every $\overline{\mathrm{BS}(m,n)}$–tile has its width compressed by the $\log\log$ function. Most importantly, for the sake of calculations, a generic $\mathrm{BS}(2,3)$–tile in the preferred positive sheet will have the coordinates shown in Figure 2. For a generic $\mathrm{BS}(m,n)$–tile, simply replace 2 and 3 by m and n, respectively.

For a CAT(0) space X, the reason $\partial_\infty X$ is called the "visual boundary" is because, in a flat geometry, the size of a set $A\subseteq X$ viewed within $\overline{X}$ is related to the angle of vision it subtends for a viewer stationed at a fixed origin. For that reason (with more precision to be provided shortly), the following lemma and its corollary are key. To keep calculations as simple as possible, we begin by analyzing the preferred positive sheet of $\mathbb{R}\times T(|m|,n)$.

Lemma 3.3 *For each $\epsilon>0$, there exists $M_\epsilon>0$ such that if $g\bar{R}_0$ is a $\overline{\mathrm{BS}(m,n)}$–tile lying in the preferred positive sheet $\mathbb{R}\times\tau^+$ of $\mathbb{R}\times T(|m|,n)$ and outside the closed M_ϵ–ball of $\mathbb{R}\times T(|m|,n)$ centered at $\boldsymbol{v}_0$, and if $\boldsymbol{w}_1,\boldsymbol{w}_2\in g\bar{R}_0$, then the angle between segments $\overline{\boldsymbol{v}_0\boldsymbol{w}_1}$ and $\overline{\boldsymbol{v}_0\boldsymbol{w}_2}$ is less than ϵ.*

Proof First note that $\mathbb{R}\times\tau^+$ is a Euclidean half-plane, so angle refers to standard angle measure. Similarly, since $\mathbb{R}\times\tau^+$ is a convex subset of $\mathbb{R}\times T(|m|,n)$ with $\boldsymbol{v}_0$ corresponding to the origin, a closed M_ϵ–ball of $\mathbb{R}\times T(|m|,n)$ intersects $\mathbb{R}\times\tau^+$

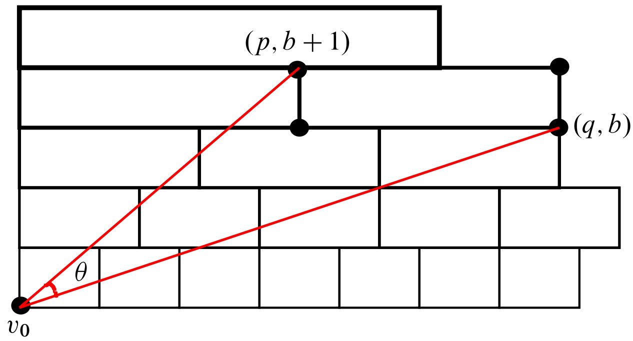

Figure 3: Angle measurement in preferred sheet of BS(2, 3)

precisely in the closed half-disk of the same radius. As such, Figure 3 accurately captures the situation.

Since our tiling is symmetric about the vertical axis, we may assume that $\bar{g}\bar{R}_0$ lies in the right-hand quadrant and has vertices with Euclidean coordinates

- $\left(\log\log(a(n/|m|)^b + e), b\right)$,
- $\left(\log\log((a+1)(n/|m|)^b + e), b\right)$,
- $\left(\log\log(a(n/|m|)^b + e), b+1\right)$,
- $\left(\log\log((a+1)(n/|m|)^b + e), b+1\right)$,

where all numbers in the formulas, except possibly m, are nonnegative.

For simplicity of notation, let

$$p = \log\log\left(a\left(\frac{n}{|m|}\right)^b + e\right) \quad \text{and} \quad q = \log\log\left((a+1)\left(\frac{n}{|m|}\right)^b + e\right).$$

Note that the angle between $\overline{\boldsymbol{v}_0\boldsymbol{w}_1}$ and $\overline{\boldsymbol{v}_0\boldsymbol{w}_2}$ is no larger than the angle between segments $\overline{\boldsymbol{v}_0, (p, b+1)}$ and $\overline{\boldsymbol{v}_0, (q, b)}$.

Representing that angle by θ, we have the formula

$$\theta = \tan^{-1}\left(\frac{b+1}{p}\right) - \tan^{-1}\left(\frac{b}{q}\right)$$

and by application of a few inverse tangent identities we obtain

$$\theta = \tan^{-1}\left(\left(\frac{b+1}{p} + \frac{-b}{q}\right) \Big/ \left(1 - \frac{-b(b+1)}{pq}\right)\right).$$

With some algebraic manipulation we may simplify this equation for θ as follows:

$$\theta = \tan^{-1}\left(\frac{b(q-p)}{pq+b^2+b} + \frac{q}{pq+b^2+b}\right).$$

Next we analyze this formula when $\sqrt{a^2+b^2} \to \infty$. Recall that p and q are both defined in terms of a and b. In particular, $p \to \infty$ and $q \to \infty$ as $\sqrt{a^2+b^2} \to \infty$. Thus, the second term in the above sum clearly gets small as $\sqrt{a^2+b^2} \to \infty$. So, to deduce that θ approaches 0 as $\sqrt{a^2+b^2} \to \infty$, we need only check that the first term in that sum goes to zero.

We direct our attention to proving that the term

$$\text{(\#)} \qquad \frac{b(q-p)}{pq+b^2+b}$$

approaches 0 as $\sqrt{a^2+b^2} \to \infty$.

Recall that

$$\begin{aligned} b(q-p) &= b\left(\log\log\left((a+1)\left(\frac{n}{|m|}\right)^b + e\right) - \log\log\left(a\left(\frac{n}{|m|}\right)^b + e\right)\right) \\ &= b\log\frac{\log((a+1)(n/|m|)^b+e)}{\log(a(n/|m|)^b+e)}. \end{aligned}$$

We split our analysis into two cases:

Case 1 $(a=0)$ Observe that

$$b\log\log\left(\left(\frac{n}{|m|}\right)^b + e\right) \sim b\log(b)$$

and since there is a b^2 term in the denominator, (#) approaches 0, as desired.

Case 2 $(a \geq 1)$ Without loss of generality, we can assume $a(n/|m|)^b > e$, in which case

$$\begin{aligned} \frac{\log((a+1)(n/|m|)^b+e)}{\log(a(n/|m|)^b+e)} &< \frac{\log((a+1)(n/|m|)^b)+1}{\log(a(n/|m|)^b)} \\ &= \frac{\log(a+1)+\log(n/|m|)^b+1}{\log a+\log(n/|m|)^b} \\ &< \frac{\log a+\log(n/|m|)^b+2}{\log a+\log(n/|m|)^b} \to 1 \quad \text{as } \sqrt{a^2+b^2} \to \infty, \end{aligned}$$

which implies that (#)$\to 0$, as desired. $\square$

Now suppose $\bar{g}\bar{R}_0$ is an arbitrary tile of $\mathbb{R}\times T(|m|,n)$. We may choose an edge ray ρ in $T(|m|,n)$ emanating from ν_0 so that $\bar{g}\bar{R}_0$ lies in the sheet $\mathbb{R}\times\rho$, which inherits the geometry of a Euclidean half-plane with $\boldsymbol{v}_0$ at the origin. For points $\boldsymbol{w}_1,\boldsymbol{w}_2\in\bar{g}\bar{R}_0$ we can measure the angle between segments $\overline{\boldsymbol{v}_0\boldsymbol{w}_1}$ and $\overline{\boldsymbol{v}_0\boldsymbol{w}_2}$ in this half-plane. That measure does not depend on the sheet chosen.

Corollary 3.4 *For each $\epsilon>0$, there exists $N_\epsilon>0$ such that if $\bar{g}\bar{R}_0$ is a $\overline{\mathrm{BS}(m,n)}$–tile of $\mathbb{R}\times T(|m|,n)$ lying outside the closed N_ϵ–ball of $\mathbb{R}\times T(|m|,n)$ centered at $\boldsymbol{v}_0$, and if $\boldsymbol{w}_1,\boldsymbol{w}_2\in\bar{g}\bar{R}_0$, then the angle between segments $\overline{\boldsymbol{v}_0\boldsymbol{w}_1}$ and $\overline{\boldsymbol{v}_0\boldsymbol{w}_2}$ is less than ϵ.*

Proof Let $\epsilon>0$ be fixed, and apply Lemma 3.3 to obtain $M_{\epsilon/2}$ so large that if $\bar{g}\bar{R}_0$ is a $\overline{\mathrm{BS}(m,n)}$–tile in the preferred positive sheet $\mathbb{R}\times\tau^+$ and lying outside the closed $M_{\epsilon/2}$–ball of $\mathbb{R}\times T(|m|,n)$ centered at $\boldsymbol{v}_0$, and if $\boldsymbol{w}_1,\boldsymbol{w}_2\in\bar{g}\bar{R}_0$, then the angle between $\overline{\boldsymbol{v}_0\boldsymbol{w}_1}$ and $\overline{\boldsymbol{v}_0\boldsymbol{w}_2}$ is less than $\frac{\epsilon}{2}$. Then let $N_\epsilon=M_{\epsilon/2}+R$, where $R>0$ is chosen so large that every $\overline{\mathrm{BS}(m,n)}$–tile that intersects $B(\boldsymbol{v}_0,M_{\epsilon/2})$ is contained in $B(\boldsymbol{v}_0,M_{\epsilon/2}+R)$.

Now let $\bar{g}\bar{R}_0$ be an arbitrary $\overline{\mathrm{BS}(m,n)}$–tile and $\mathbb{R}\times\rho$ a sheet of $T(|m|,n)$ that contains $\bar{g}\bar{R}_0$.

Case 1 ($\mathbb{R}\times\rho$ is a positive sheet) In the case of the standard tiling of $\mathbb{R}\times T(|m|,n)$ (by exponentially growing rectangles) we observed that the standard tiling of $\mathbb{R}\times\rho$ is identical up to horizontal translation to that of $\mathbb{R}\times\tau^+$. So if the standardly tiled template of $\mathbb{R}\times\tau^+$ were superimposed on $\mathbb{R}\times\rho$, each tile of $\mathbb{R}\times\rho$ would be contained in a pair of side-by-side tiles of $\mathbb{R}\times\tau^+$. This remains true after conjugating the action by f. Therefore the tile $\bar{g}\bar{R}_0$ fits within a pair of side-by-side $\overline{\mathrm{BS}(m,n)}$–tiles of $\mathbb{R}\times\tau^+$ superimposed upon $\mathbb{R}\times\rho$. So by the triangle inequality for angle measure and the choice of N_ϵ, the angle between $\overline{\boldsymbol{v}_0\boldsymbol{w}_1}$ and $\overline{\boldsymbol{v}_0\boldsymbol{w}_2}$ is less than ϵ provided $\bar{g}\bar{R}_0$ lies outside the closed N_ϵ–ball.

Case 2 ($\mathbb{R}\times\rho$ is arbitrary) As noted previously, the standard tiling of an arbitrary sheet of $\mathbb{R}\times T(|m|,n)$ refines the standard tiling of an appropriately chosen positive sheet. The same then is true for the $\overline{\mathrm{BS}(m,n)}$–tiling. Hence, the general case can be deduced from Case 1. □

Theorem 3.5 *The $\overline{\mathrm{BS}(m,n)}$–action on $\mathbb{R}\times T(|m|,n)$, together with the visual compactification $\overline{\mathbb{R}\times T(|m|,n)}$ of $\mathbb{R}\times T(|m|,n)$ with the ℓ^2 metric, is a $\mathcal{Z}$–structure for* $\mathrm{BS}(m,n)$.

Proof We need only verify the nullity condition of Definition 2.5. Toward that end let $\mathcal{U}$ be an open cover of $\overline{\mathbb{R} \times T(|m|, n)}$, and apply Lemma 2.3 to obtain a $\delta > 0$ with the property that every basic open subset of $\overline{\mathbb{R} \times T(|m|, n)}$ of the form $U(z, 1/\delta, \delta)$, with $z \in \partial_\infty(\mathbb{R} \times T(|m|, n))$, is contained in some element of $\mathcal{U}$. By Lemma 2.7 and properness of the action, it then suffices to find $N > 0$ such that every $\overline{\mathrm{BS}(m,n)}$–translate $\bar{g}\bar{R}_0$ of $\bar{R}_0$ which lies outside $\overline{B(\boldsymbol{v}_0, N)}$ is contained in $U(z, 1/\delta, \delta)$ for some $z \in \partial_\infty(\mathbb{R} \times T(|m|, n))$.

Suppose $\bar{g}\bar{R}_0$ lies outside $\overline{B(\boldsymbol{v}_0, N)}$, where N is yet to be specified. Choose a sheet $\mathbb{R} \times \rho$ containing $\bar{g}\bar{R}_0$ and a point $\boldsymbol{w}_0 \in \bar{g}\bar{R}_0$. The Euclidean ray $\overrightarrow{\boldsymbol{v}_0\boldsymbol{w}_0}$ in $\mathbb{R} \times \rho$ is an element of $\partial_\infty\mathbb{R} \times T(|m|, n))$; call it z. Its projection onto the $(1/\delta)$–sphere of $\mathbb{R} \times T(|m|, n)$ is the point $z(1/\delta)$ where the ray $\overrightarrow{\boldsymbol{v}_0\boldsymbol{w}}$ intersects the semicircle of radius $1/\delta$ in $\mathbb{R} \times \rho$. For any other point $\boldsymbol{w} \in \bar{g}\bar{R}_0$, let $p_{1/\delta}(\boldsymbol{w})$ denote the projection onto the $(1/\delta)$–sphere. By the law of cosines, the distance between $p_{1/\delta}(\boldsymbol{w})$ and $z(1/\delta)$ is $\sqrt{(2/\delta^2)(1 - \cos\theta)}$, where θ is the angle between the segments $\overline{\boldsymbol{w}_0\boldsymbol{v}_0}$ and $\overline{\boldsymbol{v}_0\boldsymbol{w}}$. Since δ is constant, this distance can be made arbitrarily small (in particular $< \delta$), by forcing θ to be small. By Corollary 3.4, this can be arranged by making N sufficiently large. Lastly, one should be sure to choose $N > 1/\delta$. □

Corollary 3.6 *Every generalized Baumslag–Solitar group admits a $\mathcal{Z}$–structure.*

Proof This argument was provided in Section 3.1. □

Remark Our choice of "compressing function" for the $\mathbb{R}$–coordinate, essentially $x \xrightarrow{f_1} \log\log(x + e)$, is somewhat arbitrary. (In an earlier draft we used $x \mapsto \sqrt{\log(x+1)}$, which also worked.) A key property is that $\lim_{x\to\infty} f_1(\exp x)/x = 0$, or equivalently $\lim_{x\to\infty} f_1(x)/\log x = 0$; this ensures that translates of compacta become small when pushed vertically by powers of t. If another function is chosen, care must be taken to maintain control over compacta pushed in other directions (see Figure 3 and the corresponding calculations).

Another issue affected by the choice of compressing function is the ability, or lack thereof, to extend the action to the visual boundary (a topic to be discussed in the next section). Some flexibility still exists, but a compressing function that leads to a $\mathcal{Z}$–structure may not yield an $\mathcal{EZ}$–structure; moreover, among those that give $\mathcal{EZ}$–structures, the induced boundary actions can vary.

Our choice of compressing function was made because it works: it leads to $\mathcal{EZ}$–structures for Baumslag–Solitar groups, and the resulting calculations are reasonably clean. For geometrically similar groups, an analogous "compressing trick" might be useful, and in those cases the ability to vary the compressing function might become important. An example of this phenomenon can be found in [15].

4 $\mathcal{EZ}$–structures on Baumslag–Solitar groups

We complete the proof of Theorem 1.2 by showing that the $\overline{\mathrm{BS}(m,n)}$–action on $\mathbb{R}\times T(|m|,n)$ extends to the visual compactification $\overline{\mathbb{R}\times T(|m|,n)}$. Since this action is not by isometries and, more specifically, this action does not send rays to rays, this observation is not immediate.

Note that, since $T(|m|,n)$ is a Bass–Serre tree for $\mathrm{BS}(m,n)$, there is a natural action by isometries of $\mathrm{BS}(m,n)$ on $T(|m|,n)$. As such, this action extends to the visual compactification of $T(|m|,n)$ (which is just its end-point compactification) in the obvious way. As noted previously, $\partial_\infty(\mathbb{R}\times T(|m|,n))$ is the suspension $S^0 * \partial_\infty T(|m|,n)$, which we may parametrize as the quotient space $[0,\pi]\times\partial_\infty T(|m|,n)/\sim$. Here the equivalence relation identifies the sets $\{0\}\times\partial_\infty T(|m|,n)$ and $\{\pi\}\times\partial_\infty T(|m|,n)$ to the right- and left-hand *suspension points*, which we denote by $\boldsymbol{R}$ and $\boldsymbol{L}$. Each edge path ray ρ in $T(|m|,n)$ emanating from v_0 uniquely determines both a point of $\partial_\infty T(|m|,n)$ and a sheet $\mathbb{R}\times\rho\subseteq\mathbb{R}\times T(|m|,n)$. The great semicircle C_ρ of rays in $\mathbb{R}\times\rho$ based at $\boldsymbol{v}_0$ (parametrized by the angles they make with the positive x–axis), trace out the set $[0,\pi]\times\{\rho\}\subseteq S^0 * \partial_\infty T(|m|,n)$.

Given a homeomorphism $h\colon \partial_\infty T(|m|,n)\to\partial_\infty T(|m|,n)$, the *suspension of* h is the homeomorphism of $S^0 * \partial_\infty T(|m|,n)$ which fixes $\boldsymbol{R}$ and $\boldsymbol{L}$ and takes each great semicircle C_ρ to $C_{h(\rho)}$ in a parameter-preserving manner. The *reflected suspension of* h switches $\boldsymbol{R}$ and $\boldsymbol{L}$ and takes the point on C_ρ with parameter θ to the point on $C_{h(\rho)}$ with parameter $\pi-\theta$. We will complete the proof of Theorem 1.2 for cases $m>0$ by proving the following proposition.

Proposition 4.1 *For $m>0$, the suspension of the $\mathrm{BS}(m,n)$–action on $\partial_\infty T(m,n)$ extends the $\overline{\mathrm{BS}(m,n)}$–action on $\mathbb{R}\times T(m,n)$.*

Remark Cases where $m<0$ require the use of reflected suspensions; we will handle those cases after completing Proposition 4.1.

The proof of Proposition 4.1 requires some additional terminology and notation. Thus far we have understood the space $\mathbb{R}\times T(|m|,n)$ as a union of sheets, each with a common origin $\boldsymbol{v}_0 = (0, v_0)$ and a common "edge", $\mathbb{R} \times \{v_0\}$. As such, each sheet has a natural system of Euclidean *local coordinates*, where a point $(x, y) \in \mathbb{R} \times \rho$ is represented by the pair of real numbers (x, d), where d is the distance along ρ from v_0 to y.

Since the actions of $\mathrm{BS}(m,n)$ on $\mathbb{R} \times T(|m|, n)$ (standard and conjugated) do not send sheets to sheets, it is useful to expand our perspective slightly. If σ is an arbitrary edge path ray in $T(|m|, n)$ emanating from a vertex v, then $\mathbb{R} \times \sigma$ is again convex and isometric to a Euclidean half-plane. Call $\mathbb{R} \times \sigma$ a *generalized sheet* and attach to it the obvious system of Euclidean local coordinates, where $\boldsymbol{v} = (0, v)$ plays the role of the origin. Note that

- if v_0 lies on σ, then $\mathbb{R} \times \sigma$ contains the sheet $\mathbb{R} \times \sigma'$ where $\sigma' \subseteq \sigma$ is the subray beginning at v_0; and
- if $v_0 \notin \sigma$, there is an edge path ray σ' emanating from v_0 and containing σ as a subray, in which case the sheet $\mathbb{R} \times \sigma'$ contains $\mathbb{R} \times \sigma$.

In each of the above cases, the edges of half-planes $\mathbb{R} \times \sigma$ and $\mathbb{R} \times \sigma'$ cobound a Euclidean strip in the larger of the two sets. As a result, a ray in $\mathbb{R} \times \sigma$ emanating from an arbitrary edge point (x, v) at an angle θ with $[x, \infty) \times \{v\}$ is asymptotic in $\mathbb{R} \times T(|m|, n)$ to the ray in $\mathbb{R} \times \sigma'$ emanating from $\boldsymbol{v}_0$ and forming the same angle with $[0, \infty) \times v_0$. As such, both rays represent the same element of $S^0 * \partial_\infty T(m, n)$, the point on the semicircle $C_{\sigma'}$ with parameter θ.

Proof of Proposition 4.1 In this proof we allow s and t to represent the isometries generating the action of $\mathrm{BS}(m,n)$ on the Bass–Serre tree $T(m,n)$ as well as the extensions of those isometries to the visual compactification of $T(m,n)$. We use the same symbols to denote the homeomorphisms generating the standard $\mathrm{BS}(m,n)$–action on $\mathbb{R} \times T(m,n)$, as described in Section 3.2.[1] It will be useful to have formulaic representations of these functions.

As an isometry of $T(m,n)$, s fixes v_0, but permutes the collection of rays emanating from that vertex. As a self-homeomorphism of $\mathbb{R} \times T(m,n)$, the action of s on the $\mathbb{R}$–coordinate is translation by $\frac{1}{n}$. So, if $\mathbb{R} \times \rho$ is an arbitrary sheet and ρ' is the image of ρ under s in the Bass–Serre tree, then, as a homeomorphism of $\mathbb{R} \times T(m,n)$,

[1] This notation is reasonable since the isometries $s, t\colon T(m,n) \to T(m,n)$ are precisely the $T(m,n)$–coordinate functions of the corresponding self-homeomorphisms of $\mathbb{R} \times T(m,n)$.

s takes points of $\mathbb{R} \times \rho$ with local coordinates (x, d) to points of $\mathbb{R} \times \rho'$ with local coordinates $\bigl(x + \frac{1}{n}, d\bigr)$.

As an isometry of $T(m, n)$, t sends v_0 to a vertex v_1, one unit away; and as a self-homeomorphism of $\mathbb{R} \times T(m, n)$, the action of t on the $\mathbb{R}$–coordinate is multiplication by $\frac{n}{m}$. So, if $\mathbb{R} \times \rho$ is an arbitrary sheet and ρ' is the image of ρ under t in the Bass–Serre tree, t takes points of $\mathbb{R} \times \rho$ with local coordinates (x, d) to points of $\mathbb{R} \times \rho'$ with local coordinates $\bigl(\frac{n}{m}x, d\bigr)$.

Now consider the homeomorphisms $\overline{s} = f \circ s \circ f^{-1}$ and $\overline{t} = f \circ t \circ f^{-1}$ which generate the $\overline{B(m,n)}$–action on $\mathbb{R} \times T(|m|, n))$. Since the suspension of a composition is the composition of the suspensions, it is enough to verify the proposition for these two elements. Recall that f and f^{-1} are given by the formulas

$$(x, y) \xrightarrow{f} \bigl(\operatorname{sgn}(x) \log\log(|x| + e), y\bigr)$$

and

$$(x, y) \xrightarrow{f^{-1}} \bigl(\operatorname{sgn}(x)\bigl(\exp\exp|x| - e\bigr), y\bigr).$$

Let $\mathbb{R} \times \rho$ be an arbitrary sheet, and for $p, q \in \mathbb{Z}$ with $p \geq 0$, let $\vec{\boldsymbol{r}}_{p/q} = \{(qx, px) \mid x \in \mathbb{R}^+\}$, ie $\vec{\boldsymbol{r}}_{p/q}$ is the ray in $\mathbb{R} \times \rho$ with slope $\frac{p}{q}$. If ρ' is the image of ρ under s in the Bass–Serre tree, then $\overline{s}$ takes $\mathbb{R} \times \rho$ onto $\mathbb{R} \times \rho'$ and the image of $\vec{\boldsymbol{r}}_{p/q}$ is the set of points with local coordinates

$$\left\{\left(\delta_{q,x,n} \cdot \log\log\left(\exp\exp(|q|x) + \frac{1}{n}\right), px\right) \mid x \in \mathbb{R}^+\right\}, \tag{4-1}$$

where $\delta_{q,x,n} = \pm 1$ is a small variation on $\operatorname{sgn}(q)$. Specifically,

$$\delta_{q,x,n} = \operatorname{sgn}\left(\operatorname{sgn}(q) \log\log(x + e) + \frac{1}{n}\right),$$

which is identical to $\operatorname{sgn}(q)$ except when $\log\log(x + e) < \frac{1}{n}$ and $q < 0$. Most importantly, the image of $\vec{\boldsymbol{r}}_{p/q}$ under $\overline{s}$ is a topologically embedded (nongeodesic) ray in $\mathbb{R} \times \rho'$ which, in local coordinates, emanates from $\bigl(\log\log\bigl(e + \frac{1}{n}\bigr), 0\bigr)$ and is asymptotic to geodesic rays in $\mathbb{R} \times \rho'$ with slope $\frac{p}{q}$. That is easily seen by letting x approach infinity in formula (4-1). From this it can be seen that the restriction of $\overline{s}$ taking $\mathbb{R} \times \rho$ onto $\mathbb{R} \times \rho'$ extends to the visual boundaries of these half-planes by taking C_ρ onto $C_{\rho'}$ in a parameter-preserving manner. Since this is true for each sheet, it follows that the suspension of the homeomorphism $s\colon \partial_\infty T(m, n) \to \partial_\infty T(m, n)$ extends $\overline{s}\colon \mathbb{R} \times T(m, n)) \to \mathbb{R} \times T(m, n))$ over the visual boundary.

Next consider the homeomorphism $\overline{t}$. Again let $\vec{\boldsymbol{r}}_{p/q}$ be a ray (as described above) in an arbitrary sheet $\mathbb{R}\times\rho$ and let ρ' be the t–image of ρ under the action on $T(m,n)$. In local coordinates, the image of $\vec{\boldsymbol{r}}_{p/q}$ is the set of points in $\mathbb{R}\times\rho'$ with local coordinates

$$\left\{\left(\operatorname{sgn}(q)\log\log\left(\frac{n}{m}\exp\exp(|q|x)+\frac{m-n}{m}\cdot e\right), px\right)\;\middle|\; x\in\mathbb{R}^+\right\}.$$

Consider now the ratios of the coordinates of these points as x gets large, ie

$$\operatorname{sgn}(q)\cdot\lim_{x\to\infty}\frac{px}{\log\log\left(\frac{n}{m}\exp\exp(|q|x)+\frac{m-n}{m}\cdot e\right)}.$$

By another elementary but messy calculation, this limit is $\frac{p}{q}$. As such, the image of $\vec{\boldsymbol{r}}_{p/q}$ under $\overline{t}$ is a topologically embedded (nongeodesic) ray in $\mathbb{R}\times\rho'$ emanating (in local coordinates) from $(0,0)$ and asymptotic to rays in $\mathbb{R}\times\rho'$ with slope $\frac{p}{q}$. As before, the restriction of $\overline{t}$ taking $\mathbb{R}\times\rho$ onto $\mathbb{R}\times\rho'$ extends to the visual boundaries of these half-planes by taking C_ρ onto $C_{\rho'}$ in a parameter-preserving manner. And since this is true for all sheets, the suspension of $t\colon \partial_\infty T(m,n)\to\partial_\infty T(m,n)$ extends $\overline{t}\colon \mathbb{R}\times T(m,n)\to\mathbb{R}\times T(m,n)$ over the visual boundary. □

To complete Theorem 1.2, we need an analog of Proposition 4.1 for $m<0$. In those cases, we cannot simply suspend the $\mathrm{BS}(m,n)$–action on $\partial_\infty T(|m|,n)$ to get the appropriate extension of the $\overline{\mathrm{BS}(m,n)}$–action on $\mathbb{R}\times T(|m|,n)$. That is because the homeomorphisms t and $\overline{t}$ now flip the orientation of the $\mathbb{R}$–factor. More precisely, if $r\colon \mathbb{R}\times T(|m|,n)\to\mathbb{R}\times T(|m|,n)$ is the reflection homeomorphism taking (x,y) to $(-x,y)$, then t and $\overline{t}$ are the homeomorphisms $r\circ t'$ and $r\circ\overline{t'}$, where t' and $\overline{t'}$ are the homeomorphisms studied earlier in cases where $m>0$. Obviously, if $\overline{t'}$ extends to the visual boundary of $\mathbb{R}\times T(|m|,n)$, then $\overline{t}$ extends to the visual boundary of $\mathbb{R}\times T(|m|,n)$ via the reflected suspension of that same homeomorphism. By contrast, the homeomorphisms s and $\overline{s}$ are no different when $m<0$ than they are when $m>0$.

For $m<0$ define $\phi\colon \mathrm{BS}(m,n)\to\mathbb{Z}$ to be the quotient map obtained by modding out by the normal closure of the subgroup $\langle s\rangle$. Then, for an action of $\mathrm{BS}(m,n)$ on $\partial_\infty T(|m|,n)$, define the corresponding *t–reflected action* of $\mathrm{BS}(m,n)$ on the space $S^0*\partial_\infty T(|m|,n)$ as follows:

- if $\phi(g)$ is even, then $g\colon S^0*\partial_\infty T(|m|,n)\to S^0*\partial_\infty T(|m|,n)$ is the suspension of $g\colon \partial_\infty T(|m|,n)\to\partial_\infty T(|m|,n)$; and
- if $\phi(g)$ is odd, then $g\colon S^0*\partial_\infty T(|m|,n)\to S^0*\partial_\infty T(|m|,n)$ is the reflected suspension of $g\colon \partial_\infty T(|m|,n)\to\partial_\infty T(|m|,n)$.

The proof of the following is now essentially the same as Proposition 4.1.

Proposition 4.2 *For $m < 0$, the t–reflected suspension of the* $\mathrm{BS}(m,n)$*–action on* $\partial_\infty T(|m|,n)$ *extends the* $\overline{\mathrm{BS}(m,n)}$*–action on* $\mathbb{R} \times T(|m|,n)$.

Remark The argument by which $\mathcal{Z}$–structures for generalized Baumslag–Solitar groups were obtained from the existence of $\mathcal{Z}$–structures on ordinary Baumslag–Solitar groups does not extend to $\mathcal{EZ}$–structures. That is because equivariance can be lost when applying Proposition 2.8. We leave the issue of $\mathcal{EZ}$–structures for generalized Baumslag–Solitar groups for later.

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