

Semisimplicial spaces

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This is an exposition of homotopical results on the geometric realisation of semi-simplicial spaces. We then use these to derive basic foundational results about classifying spaces of topological categories, possibly without units. The topics considered include: fibrancy conditions on topological categories; the effect on classifying spaces of freely adjoining units; approximate notions of units; Quillen's Theorems A and B for nonunital topological categories; the effect on classifying spaces of changing the topology on the space of objects; the group-completion theorem.

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1. Semisimplicial spaces	2101
2. Results on the homotopy type of the geometric realisation	2110
3. (Nonunital) topological categories	2121
4. Quillen's Theorems A and B and bisemisimplicial resolutions	2126
5. Base changing spaces of objects	2133
6. The group-completion theorem	2134
7. Products of simplicial spaces	2143
References	2148

Semisimplicial spaces play an important role in the theory of moduli spaces of manifolds, beginning with their use by Galatius and Randal-Williams in [7; 28], and continuing in [8; 10; 11]. In those papers, a number of key properties of semisimplicial spaces are used. While such results are mostly well known to experts, a consistent exposition seems to be missing. The first goal of the present note is to give such an exposition; we hope that it helps to make the basic technology of those papers more understandable to the nonexpert. Results which are repeatedly used in [7; 8; 10; 11] are stated in this

paper as [Theorem 2.2](#), [Theorem 2.12](#) and [Lemma 2.4](#). (One piece of semisimplicial technology from those papers is not explained here, namely [\[8, Theorem 6.2\]](#) and its elaboration [\[9\]](#), which has been abstracted in Botvinnik and Perlmutter [\[2, Theorem 6.4\]](#). But these are explained in full detail in the cited papers.)

The second goal of this note is to establish basic foundational results about classifying spaces of topological categories, possibly without units (we define these in [Section 3](#)). The topics we will consider are: fibrancy conditions on topological categories; the effect on classifying spaces of freely adjoining units to a nonunital topological category; approximate notions of units; Quillen’s Theorems A and B; the effect on classifying spaces of changing the topology on the space of objects of a topological category. In order to prove Quillen’s Theorems A and B in this setting, in [Section 4](#) we describe a bisemisimplicial resolution of a semisimplicial map induced by a functor between topological categories. We use our version of Quillen’s Theorem B (whose formulation is a mild generalisation, due to Blumberg and Mandell [\[1\]](#), of the usual one) in a crucial way in our paper [\[5\]](#); clarifying the details of this foundational result has been our main motivation for writing this note.

The third goal of this note is to give a proof of the group-completion theorem, which plays a crucial role in Galatius, Tillmann, Madsen and Weiss [\[12\]](#) and Galatius and Randal-Williams [\[7; 8\]](#). The formulation of this theorem which is most convenient for geometric applications is due to McDuff and Segal [\[22\]](#), but their paper elides many details. A detailed exposition of McDuff and Segal’s proof has been given by Miller and Palmer [\[23\]](#), which in combination with Randal-Williams [\[27\]](#) proves a stronger result than the classical formulation. There are several other proofs of the group-completion theorem, due to Jardine [\[17; 14\]](#), Moerdijk [\[24\]](#) and Pitsch and Scherer [\[25\]](#). Our proof avoids the point-set topological subtleties of [\[22\]](#), and the model categorical subtleties of [\[17; 14; 24; 25\]](#); we think it is as elementary as possible.

Finally, we give an elementary proof that for two simplicial spaces there is a natural weak equivalence $\|(X \times Y)_{\bullet}\| \simeq \|X_{\bullet}\| \times \|Y_{\bullet}\|$ (this can be extracted from Segal’s paper [\[31\]](#)). This fact has been implicitly used at some places in the literature.

We have attempted to make this note as self-contained as possible, and a large portion can be read with relatively little background knowledge. We assume that the reader is familiar with the language of homotopy theory and with the definition of a simplicial object and the basic examples, though we repeat the definitions. Some key results on simplicial sets, namely Lemmas [1.7](#) and [1.11](#), are used without proof, but in both cases

there are easily accessible references. For the results of [Section 2](#), we use a fairly simple but powerful local-to-global principle for highly connected maps — see tom Dieck [\[3, Theorem 6.7.9\]](#) — and either Mather’s first cube theorem [\[20\]](#) or the Dold–Thom criterion for quasifibrations [\[4\]](#). In two proofs (of [Theorem 2.16](#) and [Lemma 6.8](#)) we use spectral sequences. [Section 7](#) is almost elementary.

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1 Semisimplicial spaces

1.1 (Semi)simplicial objects

For $n \in \mathbb{N}_0$, let us write $[n]$ for the linearly ordered set $\{0 < 1 < \cdots < n\}$. Let Δ denote the category with objects the linearly ordered sets $[n]$ with $n \in \mathbb{N}_0$, and with morphisms $[n] \rightarrow [m]$ the monotone functions, with composition given by composition of functions. Let $\Delta_{\text{inj}} \subset \Delta$ denote the subcategory containing all objects, but only the *injective* monotone maps.

Definition 1.1 A *simplicial object* in a category \mathcal{C} is a functor $\Delta^{\text{op}} \rightarrow \mathcal{C}$. A *semi-simplicial object* in \mathcal{C} is a functor $\Delta_{\text{inj}}^{\text{op}} \rightarrow \mathcal{C}$. We denote such a (semi)simplicial object by X_{\bullet} , and write $X_p = X_{\bullet}([p])$.

A morphism of (semi)simplicial objects is a natural transformation of functors. In this way the simplicial objects in a category \mathcal{C} form a category $s\mathcal{C}$, and the semisimplicial objects form a category $ss\mathcal{C}$. There is a functor $F: s\mathcal{C} \rightarrow ss\mathcal{C}$, defined by restricting functors along $\Delta_{\text{inj}}^{\text{op}} \subset \Delta^{\text{op}}$.

While the description of (semi)simplicial objects given above is convenient for certain manipulations, it is often convenient to also have a more hands-on description. The datum of a semisimplicial object in \mathcal{C} is equivalent to giving a collection of objects $X_p \in \text{Ob}(\mathcal{C})$ for $p \geq 0$, together with morphisms $d_i: X_p \rightarrow X_{p-1}$ for $0 \leq i \leq p$, called *face maps*, which satisfy

$$d_i d_j = d_{j-1} d_i \quad \text{if } i < j.$$

The morphism d_i is associated to the unique injective monotone map $[p-1] \rightarrow [p]$ which does not hit i : any monotone injective map can be written as a composition of such maps, uniquely up to the identity above.

Similarly, a simplicial object in \mathcal{C} is given by objects $X_p \in \text{Ob}(\mathcal{C})$, together with face maps $d_i: X_p \rightarrow X_{p-1}$ for $0 \leq i \leq p$ and degeneracy maps $s_i: X_p \rightarrow X_{p+1}$ for $0 \leq i \leq p$, which satisfy the *simplicial identities*

$$\begin{aligned} d_i d_j &= d_{j-1} d_i && \text{if } i < j, \\ s_i s_j &= s_{j+1} s_j && \text{if } i \leq j, \\ d_i s_j &= s_{j-1} d_i && \text{if } i < j, \\ d_j s_j &= d_{j+1} s_j = \text{Id}, \\ d_i s_j &= s_j d_{i-1} && \text{if } i > j + 1. \end{aligned}$$

In this paper, we usually think of simplicial objects as semisimplicial objects which are equipped with additional structure, namely the degeneracy maps.

Example 1.2 The *simplicial p -simplex* Δ_{\bullet}^p is the simplicial set $\Delta_q^p := \Delta([q], [p])$. For $p = 0$, one obtains $\Delta_q^0 = *$, which is a terminal object in the category \mathbf{sSet} .

The *semisimplicial p -simplex* ∇_{\bullet}^p is the semisimplicial set $\nabla_q^p := \Delta_{\text{inj}}([q], [p])$. It only has simplices in degrees $\leq p$. Note that ∇_q^0 is a point when $q = 0$ and empty when $q > 0$.

Definition 1.3 An *augmented semisimplicial object* in \mathcal{C} is a triple $(X_{\bullet}, X_{-1}, \epsilon_{\bullet})$, with $X_{-1} \in \text{Ob}(\mathcal{C})$, $X_{\bullet} \in \text{Ob}(\text{ss}\mathcal{C})$ a semisimplicial object and morphisms $\epsilon_p: X_p \rightarrow X_{-1}$ such that $\epsilon_p \circ d_i = \epsilon_{p-1}$ for all $p \geq 1$ and all $0 \leq i \leq p$.

Equivalently, it is a semisimplicial object in the over-category \mathcal{C}/X_{-1} (see [Section 3.3](#) for a reminder of this notion).

Bi(semi)simplicial objects As the (semi)simplicial objects in \mathcal{C} form a category in their own right, we may consider (semi)simplicial objects in this category. By adjunction, this leads to the following definition:

Definition 1.4 A *bisimplicial object* in \mathcal{C} is a functor $X_{\bullet, \bullet}: (\Delta \times \Delta)^{\text{op}} \rightarrow \mathcal{C}$ and a *bisemisimplicial object* in \mathcal{C} is a functor $X_{\bullet, \bullet}: (\Delta_{\text{inj}} \times \Delta_{\text{inj}})^{\text{op}} \rightarrow \mathcal{C}$. In either case we write $X_{p,q} = X_{\bullet, \bullet}([p], [q])$.

One can think of a bisimplicial object in \mathcal{C} as a simplicial object in $\mathbf{s}\mathcal{C}$ in two ways, namely as

$$[p] \mapsto ([q] \mapsto X_{p,q}) \quad \text{and} \quad [q] \mapsto ([p] \mapsto X_{p,q}),$$

and similarly for bisemisimplicial objects. The *diagonal* simplicial object δX_\bullet is the composition of $X_{\bullet,\bullet}$ with the diagonal functor $\Delta \rightarrow \Delta \times \Delta$, and similarly for bisemisimplicial objects. Hence $\delta X_p = X_{p,p}$.

If the category \mathcal{C} has finite products, we can form the *exterior* product $X_\bullet \otimes Y_\bullet$ of two simplicial objects $X_\bullet, Y_\bullet \in s\mathcal{C}$; it is

$$(X_\bullet \otimes Y_\bullet)([p], [q]) := X([p]) \times Y([q]).$$

The *interior* product of two simplicial objects is then $X_\bullet \times Y_\bullet := \delta(X_\bullet \otimes Y_\bullet)$; concretely,

$$(X_\bullet \times Y_\bullet)([p]) := X([p]) \times Y([p]).$$

Parallel notions can be defined for semisimplicial objects, but are not very useful.

Freely adding degeneracies If the category \mathcal{C} has finite coproducts, then the forgetful functor $F: s\mathcal{C} \rightarrow \text{ss}\mathcal{C}$ has a left adjoint E , which has the following explicit description. For a semisimplicial object $X_\bullet \in \text{ss}\mathcal{C}$, define

$$EX_p := \coprod_{\alpha: [p] \twoheadrightarrow [q]} X_q.$$

Let $\beta: [r] \rightarrow [p]$ be a morphism in Δ . For a surjection $\alpha: [p] \rightarrow [q]$, we factor $\alpha \circ \beta: [r] \rightarrow [q]$ as $[r] \xrightarrow{\alpha'} [s] \xrightarrow{\beta'} [q]$, and define $\beta: EX_p \rightarrow EX_r$ on the summand indexed by α as the map $\beta'^*: X_q \rightarrow X_s \subset EX_r$.

From this adjunction, we obtain the counit $c: EFY_\bullet \rightarrow Y_\bullet$ for each $Y_\bullet \in s\mathcal{C}$, and the unit $u: X_\bullet \rightarrow FEX_\bullet$ for each $X_\bullet \in \text{ss}\mathcal{C}$. Concretely, the counit is the map

$$EFY_p = \coprod_{\alpha: [p] \twoheadrightarrow [q]} Y_q \rightarrow Y_p$$

which on the summand indexed by α is given by α^* . Similarly, the unit is the map

$$X_p \rightarrow FEX_p = \coprod_{\alpha: [p] \twoheadrightarrow [q]} X_q$$

which sends X_p identically to the component indexed by $\text{Id}: [p] \rightarrow [p]$. Further details may be found in [6, page 166].

1.2 Semisimplicial spaces and their geometric realisation

Convention 1.5 Throughout this paper, we work in the category of compactly generated spaces as defined in [33] (the difference to the category considered by Steenrod in [32] is that we do not require the Hausdorff condition). All products of spaces

are understood to be taken in the category of compactly generated spaces. One key advantage of compactly generated spaces is that taking quotients commutes with taking products in full generality, by [33, Propositions 2.1 and 2.20]. Slightly abusing notation, we shall denote this category by \mathbf{Top} and call its objects topological spaces.

We think of the category \mathbf{Set} of sets as a full subcategory of \mathbf{Top} , namely that of spaces with the discrete topology. A similar convention applies to (semi)simplicial sets.

Recall that the *standard p -simplex* is the space

$$\Delta^p = \left\{ (t_0, t_1, \dots, t_p) \in \mathbb{R}^{p+1} \mid \sum_{i=0}^p t_i = 1 \text{ and } t_i \geq 0 \text{ for each } i \right\}.$$

To a morphism $\varphi: [p] \rightarrow [q]$ in Δ there is an associated continuous map $\varphi_*: \Delta^p \rightarrow \Delta^q$ given by $\varphi_*(t_0, t_1, \dots, t_p) = (s_0, s_1, \dots, s_q)$ where $s_j = \sum_{i \in \varphi^{-1}(j)} t_i$. In particular, let $d^i: \Delta^{p-1} \rightarrow \Delta^p$ be given by $(t_0, t_1, \dots, t_p) \mapsto (t_0, t_1, \dots, t_{i-1}, 0, t_i, \dots, t_p)$, and $s^i: \Delta^p \rightarrow \Delta^{p-1}$ be given by $(t_0, t_1, \dots, t_p) \mapsto (t_0, t_1, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_p)$.

The *geometric realisation* of a semisimplicial space X_\bullet is the quotient space

$$\|X_\bullet\| = \left(\coprod_p X_p \times \Delta^p \right) / \sim$$

by the equivalence relation $(x, \varphi_* t) \sim (\varphi^* x, t)$, where φ is a morphism in Δ_{inj} . This equivalence relation is generated by the requirement that $(x, d^i t) \sim (d_i x, t)$. The n -*skeleton* $\|X_\bullet\|^{(n)}$ of $\|X_\bullet\|$ is the image of $\coprod_{p=0}^n X_p \times \Delta^p$ under the quotient map. The natural map

$$\text{colim}_{n \rightarrow \infty} \|X_\bullet\|^{(n)} \rightarrow \|X_\bullet\|$$

is a homeomorphism.

Example 1.6 The geometric realisation of the semisimplicial p -simplex ∇_\bullet^p is the topological p -simplex, $\|\nabla_\bullet^p\| \cong \Delta^p$.

The (*thin*) *geometric realisation* of a simplicial space X_\bullet is the quotient space

$$|X_\bullet| = \left(\coprod_p X_p \times \Delta^p \right) / \approx,$$

with the equivalence relation $(x, \varphi_* t) \approx (\varphi^* x, t)$, where φ is a morphism in Δ . In addition to imposing the relation \sim above, the relation \approx imposes $(x, s^i t) \approx (s_i x, t)$. The *fat geometric realisation* of a simplicial space X_\bullet is by definition $\|X_\bullet\| := \|F(X_\bullet)\|$,

and it has a canonical map to $|X_\bullet|$. Skeleta of $|X_\bullet|$ are defined as above, and $|X_\bullet|$ is again the colimit of its skeleta.

Lemma 1.7 *For each simplicial set Y_\bullet , the quotient map $\|Y_\bullet\| \rightarrow |Y_\bullet|$ is a homotopy equivalence.*

The proof can be found in [29, Proposition 2.1]. It also follows from Proposition A.1(iv) of [31] as simplicial sets are “good”. The following lemma allows us to compare the geometric realisation of a semisimplicial set with the geometric realisation of the simplicial set obtained by freely adding degeneracies. Later, in Lemma 2.6, we will prove the analogue for semisimplicial spaces.

Lemma 1.8 *For each semisimplicial set X_\bullet , the map $\|X_\bullet\| \rightarrow \|EX_\bullet\|$ is a homotopy equivalence.*

Proof We will show that the composition

$$\|X_\bullet\| \rightarrow \|EX_\bullet\| \rightarrow |EX_\bullet|$$

is a homeomorphism, whence the claim follows from Lemma 1.7. For any simplicial set Y_\bullet , each point in $|Y_\bullet|^{(n)} \setminus |Y_\bullet|^{(n-1)}$ may be uniquely represented by a $(\sigma; t_0, \dots, t_n)$ with $\sigma \in Y_n$ a nondegenerate simplex. As the nondegenerate simplices of EX_n are precisely given by $X_n \subset EX_n$, we may describe $|EX_\bullet|^{(n)}$ as the pushout

$$\begin{array}{ccc} X_n \times \partial \Delta^n & \longrightarrow & X_n \times \Delta^n \\ \downarrow & & \downarrow \\ |EX_\bullet|^{(n-1)} & \longrightarrow & |EX_\bullet|^{(n)} \end{array}$$

Now $\|X_\bullet\|^{(n)}$ is obtained from $\|X_\bullet\|^{(n-1)}$ by precisely the same pushout description, which proves by induction that $\|X_\bullet\|^{(n)} \rightarrow |EX_\bullet|^{(n)}$ is a homeomorphism. Taking colimits over n gives the required result. □

If $X_{\bullet,\bullet}$ is a bisemisimplicial space, we define its geometric realisation as the quotient space

$$\|X_{\bullet,\bullet}\| := \coprod_{p,q} X_{p,q} \times \Delta^p \times \Delta^q / \sim$$

by the equivalence relation $((\varphi \times \psi)^*x, t, s) \sim (x, \varphi_*t, \psi_*s)$ for morphisms $\varphi \times \psi$ in $\Delta_{\text{inj}} \times \Delta_{\text{inj}}$. There are homeomorphisms

$$(1.9) \quad \|X_{\bullet, \bullet}\| \cong \|[p] \mapsto \|[q] \mapsto X_{p,q}\| \cong \|[q] \mapsto \|[p] \mapsto X_{p,q}\|$$

and

$$(1.10) \quad \|X_{\bullet} \otimes Y_{\bullet}\| \cong \|X_{\bullet}\| \times \|Y_{\bullet}\|,$$

which use that we are working in the category of compactly generated spaces.

The singular simplicial set The *singular simplicial set* of a topological space X is the simplicial set with p -simplices $\text{Sing}_p X := \text{Top}(\Delta^p, X)$, the set of continuous maps from the standard p -simplex to X , where $\varphi: [p] \rightarrow [q]$ acts via $\text{Top}(\varphi_*, X)$. The evaluation maps

$$\text{Top}(\Delta^p, X) \times \Delta^p \rightarrow X, \quad (\sigma, t) \mapsto \sigma(t),$$

assemble to a map $|\text{Sing}_{\bullet} X| \rightarrow X$.

Lemma 1.11 *The maps*

$$\|\text{Sing}_{\bullet} X\| \xrightarrow{\sim} |\text{Sing}_{\bullet} X| \xrightarrow{\sim} X.$$

are weak homotopy equivalences.

Proof The first map is a weak homotopy equivalence by [Lemma 1.7](#). The second map is shown to be a weak equivalence in eg [\[21, Theorem 16.6\]](#) or [\[6, Theorem 4.5.30\]](#). \square

1.3 Extra degeneracies and semisimplicial (null)homotopies

If $(Y_{\bullet}, Y_{-1}, \epsilon)$ is an augmented semisimplicial space, then there is an induced map $\|\epsilon_{\bullet}\|: \|Y_{\bullet}\| \rightarrow Y_{-1}$. There is a standard technique for easily showing that such maps are homotopy equivalences, which goes under the name of “having an extra degeneracy”.

Lemma 1.12 *Let $(Y_{\bullet}, Y_{-1}, \epsilon)$ be an augmented semisimplicial space, and suppose there are maps $h_{p+1}: Y_p \rightarrow Y_{p+1}$ for $p \geq -1$ such that*

$$\begin{aligned} d_{p+1}h_{p+1} &= \text{Id}_{Y_p}, \\ d_i h_{p+1} &= h_p d_i \quad \text{for } 0 \leq i < p+1, \\ \epsilon_0 h_0 &= \text{Id}_{Y_{-1}}; \end{aligned}$$

then $\|\epsilon_{\bullet}\|: \|Y_{\bullet}\| \rightarrow Y_{-1}$ is a homotopy equivalence.

Dually, if there are maps $g_{p+1}: Y_p \rightarrow Y_{p+1}$ for $p \geq -1$ such that

$$\begin{aligned} d_0 g_{p+1} &= \text{Id}_{Y_p}, \\ d_i g_{p+1} &= g_p d_{i-1} \quad \text{for } 0 < i \leq p+1, \\ \epsilon_0 g_0 &= \text{Id}_{Y_{-1}}, \end{aligned}$$

then the same conclusion holds.

In the first case the conditions on the maps h_{p+1} are formally identical to the conditions relating face maps d_i to degeneracy maps s_i , except that h_{p+1} behaves like a hypothetical degeneracy map s_{p+1} , whereas in the definition of a simplicial object there are only degeneracy maps $s_0, s_1, \dots, s_p: Y_p \rightarrow Y_{p+1}$. For this reason such a collection of maps h_{p+1} is often called an *extra degeneracy*. (Similarly, g_{p+1} behaves like a hypothetical degeneracy map $s_{-1}: Y_p \rightarrow Y_{p+1}$.)

Proof Let us just consider the first case. We have $h_0: Y_{-1} \rightarrow Y_0 \subset \|Y_\bullet\|$ and $\|\epsilon_\bullet\| \circ h_0 = \text{Id}_{Y_{-1}}$. The maps $[0, 1] \times Y_p \times \Delta^p \rightarrow Y_{p+1} \times \Delta^{p+1} \rightarrow \|Y_\bullet\|$, defined by

$$(s; x; t_0, \dots, t_p) \mapsto (h_{p+1}(x); (1-s)t_0, \dots, (1-s)t_p, s),$$

respect the equivalence relation used in the definition of the geometric realisation. Since taking products and taking quotients commutes in compactly generated spaces, this yields a homotopy $H: [0, 1] \times \|Y_\bullet\| \rightarrow \|Y_\bullet\|$, and one verifies that $H(0, -) = \text{Id}_{\|Y_\bullet\|}$ and that $H(1, -) = h_0 \circ \|\epsilon_\bullet\|$. \square

Any semisimplicial space Y_\bullet is augmented over a point $*$ in a unique way. The data of an extra degeneracy in this case gives in particular a point $y_0: * \rightarrow Y_0$, and the homotopy in the proof gives a contraction of $\|Y_\bullet\|$ to the point $\{y_0\} \subset Y_0 \subset \|Y_\bullet\|$. This can be generalised to maps of semisimplicial spaces, as follows.

Lemma 1.13 *Let $f_\bullet: X_\bullet \rightarrow Y_\bullet$ be a map of semisimplicial spaces and $y_0 \in Y_0$. A semisimplicial nullhomotopy from f_\bullet to y_0 is a collection of continuous maps $h_{p+1}: X_p \rightarrow Y_{p+1}$ such that*

$$\begin{aligned} d_{p+1} h_{p+1} &= f_p, \\ d_i h_{p+1} &= h_p d_i \quad \text{for } 0 \leq i \leq p \text{ and } p \geq 1, \\ d_0 h_1 &\equiv y_0. \end{aligned}$$

Such a semisimplicial nullhomotopy induces a homotopy from $\|f_\bullet\|$ to the constant map $\|X_\bullet\| \rightarrow \{y_0\} \subset Y_0 \subset \|Y_\bullet\|$.

Proof Use the same formula as in the proof of [Lemma 1.12](#) to obtain a homotopy $H: [0, 1] \times \|X_\bullet\| \rightarrow \|Y_\bullet\|$ with $H(0, -) = \|f_\bullet\|$ and $H(1, -)$ the constant map with value y_0 . \square

Example 1.14 The fat geometric realisation of the simplicial n -simplex Δ_\bullet^n is contractible (it is not homeomorphic to Δ^n). Recall that $\Delta_p^n = \Delta([p], [n])$ and let $h_{p+1}: \Delta([p], [n]) \rightarrow \Delta([p+1], [n])$ be the map that sends $\eta: [p] \rightarrow [n]$ to the map $\eta': [p+1] \rightarrow [n]$ which is defined by $\eta'(i) = \eta(i)$ for $i \leq p$ and $\eta'(p+1) := n$. This is a simplicial nullhomotopy from $\text{Id}_{\Delta_\bullet^n}$ to the vertex $n \in \Delta_0^n$, and hence the claim follows from [Lemma 1.13](#).

More generally, we have the notion of a semisimplicial homotopy between semisimplicial maps.

Lemma 1.15 Let $f_\bullet, g_\bullet: X_\bullet \rightarrow Y_\bullet$ be maps of semisimplicial spaces. A semisimplicial homotopy from f_\bullet to g_\bullet is a collection of continuous maps $h_{p+1,i}: X_p \rightarrow Y_{p+1}$ for $i = 0, 1, \dots, p$ such that

$$\begin{aligned} d_i h_{p+1,i} &= d_i h_{p+1,i-1} & \text{for } 0 < i \leq p, \\ d_i h_{p+1,j} &= h_{p,j-1} d_i & \text{for } 0 \leq i < j, \\ d_i h_{p+1,j} &= h_{p,j} d_{i-1} & \text{for } j+1 < i \leq p, \\ d_0 h_{p+1,0} &= f_p, \\ d_{p+1} h_{p+1,p} &= g_p. \end{aligned}$$

Such a semisimplicial homotopy induces a homotopy from $\|f_\bullet\|$ to $\|g_\bullet\|$.

Proof Consider the maps

$$\psi_{p+1,i}: \Delta^{p+1} \rightarrow \Delta^1 \times \Delta^p, \quad \sum_{j=0}^{p+1} t_j e_j \mapsto \sum_{j=0}^i t_j (0, e_j) + \sum_{j=i}^p t_{j+1} (1, e_j),$$

for $i = 0, 1, \dots, p$, giving the standard decomposition of the prism into simplices. The maps

$$\psi_{p+1,i}(\Delta^{p+1}) \times X_p \rightarrow \Delta^{p+1} \times Y_{p+1} \subset \|Y_\bullet\|, \quad (\psi_{p+1,i}(t), x) \mapsto (t, h_{p+1,i}(x)),$$

glue to maps $\phi_p: [0, 1] \times \Delta^p \times X_p \rightarrow \|Y_\bullet\|$ (using the first set of identities) which in turn glue to a map $\phi: [0, 1] \times \|X_\bullet\| \rightarrow \|Y_\bullet\|$ (using the second and third set of identities). This gives the required homotopy (using the fourth and fifth set of identities). \square

1.4 Spectral sequences

The space $\|X_\bullet\|$ is filtered by its skeleta $\|X_\bullet\|^{(n)}$, where $\|X_\bullet\|^{(0)} = X_0$ and

$$(1.16) \quad \|X_\bullet\|^{(n)} = \|X_\bullet\|^{(n-1)} \cup_{X_n \times \partial \Delta^n} X_n \times \Delta^n.$$

This filtration has the property that each map $K \rightarrow \|X_\bullet\|$ from a compact Hausdorff space K factors through some finite stage; see eg [15, Proposition A.1] for a related argument, or [33, Lemma 3.6] for a general argument.

Recall that a *local coefficient system* on a space Y is a functor \mathcal{L} from the fundamental groupoid $\Pi_1(Y)$ to the category of R -modules for a commutative ring R . If Y is semilocally simply connected then we may also consider a local coefficient system on Y to be a bundle $\mathcal{L} \rightarrow Y$ of R -modules.

For any system of local coefficients \mathcal{L} on $\|X_\bullet\|$, the skeletal filtration yields a spectral sequence

$$E_{p,q}^1 = H_{p+q}(\|X_\bullet\|^{(q)}, \|X_\bullet\|^{(q-1)}; \mathcal{L}) \Rightarrow H_{p+q}(\|X_\bullet\|; \mathcal{L}),$$

which is strongly convergent as each map from a simplex to $\|X_\bullet\|$ lands in some finite skeleton. Let $\mathcal{L}|_{X_q \times \Delta^q}$ be the pullback of \mathcal{L} along $X_q \times \Delta^q \rightarrow \|X_\bullet\|$, and \mathcal{L}_q be the restriction of $\mathcal{L}|_{X_q \times \Delta^q}$ to $X_q \cong X_q \times b_q$, where $b_q \in \Delta^q$ is the barycentre. The natural map

$$H_{p+q}(X_q \times \Delta^q, X_q \times \partial \Delta^q; \mathcal{L}|_{X_q \times \Delta^q}) \rightarrow H_{p+q}(\|X_\bullet\|^{(q)}, \|X_\bullet\|^{(q-1)}; \mathcal{L})$$

is an isomorphism, using the description (1.16) and excision. The contraction of Δ^q to $b_q \in \Delta^q$ determines an isomorphism $\mathcal{L}|_{X_q \times \Delta^q} \cong \pi_1^* \mathcal{L}_q$, and the Künneth map

$$H_p(X_q; \mathcal{L}_q) \cong H_p(X_q; \mathcal{L}_q) \otimes H_q(\Delta^q, \partial \Delta^q; \mathbb{Z}) \rightarrow H_{p+q}(X_q \times \Delta^q, X_q \times \partial \Delta^q; \pi_1^* \mathcal{L}_q)$$

is an isomorphism (as the homology of $(\Delta^q, \partial \Delta^q)$ is free). Thus we obtain the description

$$E_{p,q}^1 \cong H_p(X_q; \mathcal{L}_q) \Rightarrow H_{p+q}(\|X_\bullet\|; \mathcal{L})$$

for this spectral sequence. Associated to each face map $d_i: X_q \rightarrow X_{q-1}$ there is a unique homotopy class of path in Δ^q from $d^i(b_{q-1})$ to b_q , monodromy along which

gives a preferred map of local coefficient systems $\phi_i: \mathcal{L}_q \rightarrow \mathcal{L}_{q-1}$ covering d_i . One may show (see [30, Section 5]) that the d^1 -differential is

$$d^1 = \sum_{i=0}^q (-1)^i (d_i, \phi_i)_*: H_p(X_q; \mathcal{L}_q) \rightarrow H_p(X_{q-1}; \mathcal{L}_{q-1}),$$

the alternating sum of the maps induced on homology by the face maps.

More generally, if $(X_\bullet, X_{-1}, \epsilon)$ is an augmented semisimplicial space then (replacing X_{-1} by the mapping cylinder of $\|\epsilon_\bullet\|: \|X_\bullet\| \rightarrow X_{-1}$ and) setting $F_{-1} = (X_{-1}, X_{-1})$ and $F_q = (X_{-1}, \|X_\bullet\|^{(q)})$ for $q \geq 0$ gives a filtration of pairs, and hence for each local coefficient system \mathcal{L} on X_{-1} a spectral sequence with $E_{p,q}^1 \cong H_p(X_q; \mathcal{L}_q)$ for $p \geq 0$ and $q \geq -1$, which converges to $H_{p+q+1}(X_{-1}, \|X_\bullet\|; \mathcal{L})$.

2 Results on the homotopy type of the geometric realisation

In this section we shall collect results which allow one to deduce homotopical statements about geometric realisation of a map $f_\bullet: X_\bullet \rightarrow Y_\bullet$ of semisimplicial spaces from homotopical statements about the maps $f_p: X_p \rightarrow Y_p$. One says that a semisimplicial map f_\bullet has a certain property *levelwise* if each map f_p has that property. As a basic technical tool for gluing together k -connected maps, we will take Theorem 6.7.9 of tom Dieck's book [3].

Lemma 2.1 *For $m \geq n$ the inclusion $\|X_\bullet\|^{(n)} \rightarrow \|X_\bullet\|^{(m)}$ is n -connected, and the inclusion $\|X_\bullet\|^{(n)} \rightarrow \|X_\bullet\|$ is n -connected.*

Proof For the first claim, it is enough to prove that the inclusion $\|X_\bullet\|^{(n)} \rightarrow \|X_\bullet\|^{(n+1)}$ is n -connected. To see this, let $b \in \Delta^{n+1}$ be the barycentre and consider the covering of $\|X_\bullet\|^{(n+1)}$ by the open sets

$$\begin{aligned} U_0^X &= \|X_\bullet\|^{(n+1)} \setminus (X_{n+1} \times \{b\}) \simeq \|X_\bullet\|^{(n)}, \\ U_1^X &= X_{n+1} \times \text{int}(\Delta^{n+1}) \simeq X_{n+1}, \end{aligned}$$

with intersection $U_0^X \cap U_1^X \simeq X_{n+1} \times \partial\Delta^{n+1}$. Applying [3, Theorem 6.7.9] to the map

$$(U_0^X, U_0^X, U_1^X \cap U_0^X) \rightarrow (\|X_\bullet\|^{(n+1)}, U_0^X, U_1^X)$$

shows that $\|X_\bullet\|^{(n)} \xrightarrow{\simeq} U_0^X \rightarrow \|X_\bullet\|^{(n+1)}$ is n -connected, as required. The second claim follows from the first one and the fact that a map from a compact Hausdorff space to $\|X_\bullet\|$ factors through a skeleton. \square

Theorem 2.2 Let $f_{\bullet}: X_{\bullet} \rightarrow Y_{\bullet}$ be a map of semisimplicial spaces which is a level-wise weak homotopy equivalence. Then $\|f_{\bullet}\|: \|X_{\bullet}\| \rightarrow \|Y_{\bullet}\|$ is a weak homotopy equivalence.

Proof By Lemma 2.1, it is enough to show that $\|f_{\bullet}\|: \|X_{\bullet}\|^{(n)} \rightarrow \|Y_{\bullet}\|^{(n)}$ is a weak equivalence for each n , and this may be shown by induction on n . The case $n = 0$ is trivial. For the induction step, consider the open sets $U_0^X, U_1^X \subset \|X_{\bullet}\|^{(n+1)}$ from the proof of Lemma 2.1 and the analogous $U_0^Y, U_1^Y \subset \|Y_{\bullet}\|^{(n+1)}$. By induction hypothesis, the restriction of $\|f_{\bullet}\|$ to $U_0^X \rightarrow U_0^Y$ is a weak equivalence, and so is the restriction $U_1^X \rightarrow U_1^Y$ and $U_0^X \cap U_1^X \rightarrow U_0^Y \cap U_1^Y$. The inductive step then follows using [3, Theorem 6.7.9]. \square

Remark 2.3 Theorem 2.2 is false in general for the thin geometric realisation of simplicial spaces. This is the main reason why—even for simplicial spaces—it is often preferable to consider the fat geometric realisation. A concrete counterexample was given by Lawson in response to a question on MathOverflow [19].

Theorem 2.2 has the following useful generalisation:

Lemma 2.4 Let $f_{\bullet}: X_{\bullet} \rightarrow Y_{\bullet}$ be a map of semisimplicial spaces. If $f_p: X_p \rightarrow Y_p$ is $(k-p)$ -connected for all p , then $\|f_{\bullet}\|$ is k -connected.

Proof By Lemma 2.1 it is enough to show that $\|f_{\bullet}\|^{(n)}: \|X_{\bullet}\|^{(n)} \rightarrow \|Y_{\bullet}\|^{(n)}$ is k -connected for each n . The case $n = 0$ is trivial. For the induction step, we may as well suppose that $X_i = Y_i = \emptyset$ for $i > n$ and that $\|f_{\bullet}\|^{(n-1)}$ is k -connected. We factorise f_{\bullet} as

$$(2.5) \quad X_{\bullet} \xrightarrow{j_{\bullet}} W_{\bullet} \xrightarrow{g_{\bullet}} Z_{\bullet} \xrightarrow{h_{\bullet}} Y_{\bullet}$$

as follows. The semisimplicial space W_{\bullet} has $W_i = Y_i$ for $i < n$, $W_n = X_n$ and $W_i = \emptyset$ for $i > n$. The face maps $W_n \rightarrow W_{n-1}$ are the compositions $f_{n-1} \circ d_i = d_i \circ f_n$, and the other face maps are the same as those for X_{\bullet} . The map j_n is the identity, and $j_i = f_i$ for $i < n$.

Then factorise f_n as

$$f_n: X_n \xrightarrow{g_n} Z_n \xrightarrow{h_n} Y_n,$$

where h_n is a weak homotopy equivalence, and Z_n is obtained from X_n by attaching cells of dimension at least $k - n + 1$. For $i < n$ let $Z_i = Y_i$, and for $i > n$ let $Z_i = \emptyset$.

The map g_i is the identity for $i < n$, and $h_i: Z_i \rightarrow Y_i$ is the identity as well. This yields the factorisation (2.5).

The map h_\bullet is a levelwise weak equivalence, and so $\|h_\bullet\|$ is a weak equivalence by Theorem 2.2. Moreover, $\|W_\bullet\|^{(n-1)} = \|Z_\bullet\|^{(n-1)}$, and the pair $(Z_n \times \Delta^n, Z_n \times \partial\Delta^n)$ is obtained from the pair $(W_n \times \Delta^n, W_n \times \partial\Delta^n)$ by attaching cells of dimension at least $k + 1$, so $\|Z_\bullet\|^{(n)}$ is obtained from $\|W_\bullet\|^{(n)}$ by attaching cells of dimension at least $k + 1$; in particular, $\|g_\bullet\|: \|W_\bullet\|^{(n)} \rightarrow \|Z_\bullet\|^{(n)}$ is k -connected. By the inductive hypothesis, $\|j_\bullet\|^{(n-1)}: \|X_\bullet\|^{(n-1)} \rightarrow \|W_\bullet\|^{(n-1)}$ is k -connected and j_n is the identity. Using the notation introduced in the proof of Theorem 2.2, we get that $U_0^X \rightarrow U_0^W$ is k -connected, while $U_1^X \rightarrow U_1^W$ and $U_0^X \cap U_1^X \rightarrow U_0^W \cap U_1^W$ are weak equivalences. From [3, Theorem 6.7.9], it follows that $\|j_\bullet\|$ is k -connected. \square

Using this we can now prove the analogue of Lemma 1.8 for semisimplicial spaces, rather than semisimplicial sets.

Lemma 2.6 *For each semisimplicial space X_\bullet , the map $\|X_\bullet\| \rightarrow \|EX_\bullet\|$ is a weak homotopy equivalence.*

Proof Consider the bisemisimplicial set $\text{Sing}_p(X_q)$, with $\text{Sing}_p(EX_\bullet) = E(\text{Sing}_p X_\bullet)$ so giving a commutative square

$$\begin{array}{ccc} \|\text{Sing}_\bullet X_\bullet\| & \longrightarrow & \|E(\text{Sing}_\bullet X_\bullet)\| = \| \text{Sing}_\bullet (EX_\bullet) \| \\ \downarrow & & \downarrow \\ \|X_\bullet\| & \longrightarrow & \|EX_\bullet\| \end{array}$$

The vertical maps are weak equivalences by Lemma 1.11 and Theorem 2.2, and the top map is a weak equivalence by Lemma 1.8 and Theorem 2.2; hence the bottom map is a weak equivalence. \square

Definition 2.7 A commutative square

$$\begin{array}{ccc} X_1 & \xrightarrow{k_1} & Y_1 \\ \downarrow f & & \downarrow g \\ X_0 & \xrightarrow{k_0} & Y_0 \end{array}$$

is called *homotopy cartesian* if for each basepoint $x \in X_0$, the map $\text{hofib}_x(f) \rightarrow \text{hofib}_{k_0(x)}(g)$, induced by k_0 and k_1 , is a weak homotopy equivalence.

Remark 2.8 Equivalently, one can express this condition by saying that for all $y \in Y_1$, the induced map $\text{hofib}_y(k_1) \rightarrow \text{hofib}_{g(y)}(k_0)$ is a weak homotopy equivalence.

More symmetrically, one can express this condition by saying that the canonical map from X_1 to the *homotopy fibre product*

$$X_0 \times_{Y_0}^h Y_1 := \{(x_0, y_1, \gamma) \in X_0 \times Y_1 \times \text{map}([0, 1], Y_0) \mid \gamma(0) = k_0(x_0), \gamma(1) = g(y_1)\}$$

is a weak homotopy equivalence.

Let us record the 2-out-of-3 properties enjoyed by homotopy cartesian squares. If we have adjacent commutative squares

$$\begin{array}{ccccc} X_1 & \xrightarrow{k_1} & Y_1 & \xrightarrow{l_1} & Z_1 \\ \downarrow f & & \downarrow g & & \downarrow h \\ X_0 & \xrightarrow{k_0} & Y_0 & \xrightarrow{l_0} & Z_0 \end{array}$$

then

- (i) if the left and right squares are homotopy cartesian, the outer square is homotopy cartesian;
- (ii) if the right and outer squares are homotopy cartesian, the left square is homotopy cartesian;
- (iii) if the left and outer squares are homotopy cartesian and k_0 is 0-connected, the right square is homotopy cartesian.

Definition 2.9 A map $f_\bullet: X_\bullet \rightarrow Y_\bullet$ of semisimplicial spaces is called *homotopy cartesian* if for each $p \geq 1$ and each $0 \leq i \leq p$, the square

$$(2.10) \quad \begin{array}{ccc} X_p & \xrightarrow{d_i} & X_{p-1} \\ \downarrow f_p & & \downarrow f_{p-1} \\ Y_p & \xrightarrow{d_i} & Y_{p-1} \end{array}$$

is homotopy cartesian.

Such maps are occasionally known as *equifibred*. For each p there are $p + 1$ conditions to be checked. The next lemma shows that the number of conditions to be checked can be drastically reduced.

Lemma 2.11 *To prove that $f_{\bullet} \colon X_{\bullet} \rightarrow Y_{\bullet}$ is homotopy cartesian, it is enough to verify that (2.10) is homotopy cartesian for those (p, i) with $i = 0$ and for $(p, i) = (1, 1)$. Dually, it is enough to verify that (2.10) is homotopy cartesian for those (p, i) with $i = p$ and for $(p, i) = (1, 0)$.*

Proof We treat only the first case. Consider the commutative cube

$$\begin{array}{ccccc} & & X_k & \xrightarrow{d_0^{k-1}} & X_1 \\ & \swarrow d_k & \downarrow f_k & \swarrow d_1 & \downarrow f_1 \\ X_{k-1} & \xrightarrow{d_0^{k-1}} & X_0 & & \\ \downarrow f_{k-1} & & \downarrow d_0^{k-1} & & \downarrow f_0 \\ & & Y_k & \xrightarrow{d_0^{k-1}} & Y_1 \\ & \swarrow d_k & \downarrow d_0^{k-1} & \swarrow d_1 & \downarrow f_1 \\ Y_{k-1} & \xrightarrow{d_0^{k-1}} & Y_0 & & \end{array}$$

By hypothesis the front, back and right faces are homotopy cartesian, so the left face is too. But each structure map $X_p \rightarrow X_0$ can be written as the composition of maps of the form d_0 and $d_k \colon X_k \rightarrow X_{k-1}$. Therefore, for each $\eta \colon [0] \rightarrow [p]$, the square

$$\begin{array}{ccc} X_p & \xrightarrow{\eta^*} & X_0 \\ \downarrow f_p & & \downarrow f_0 \\ Y_p & \xrightarrow{\eta^*} & Y_0 \end{array}$$

is homotopy cartesian. The result then follows easily. □

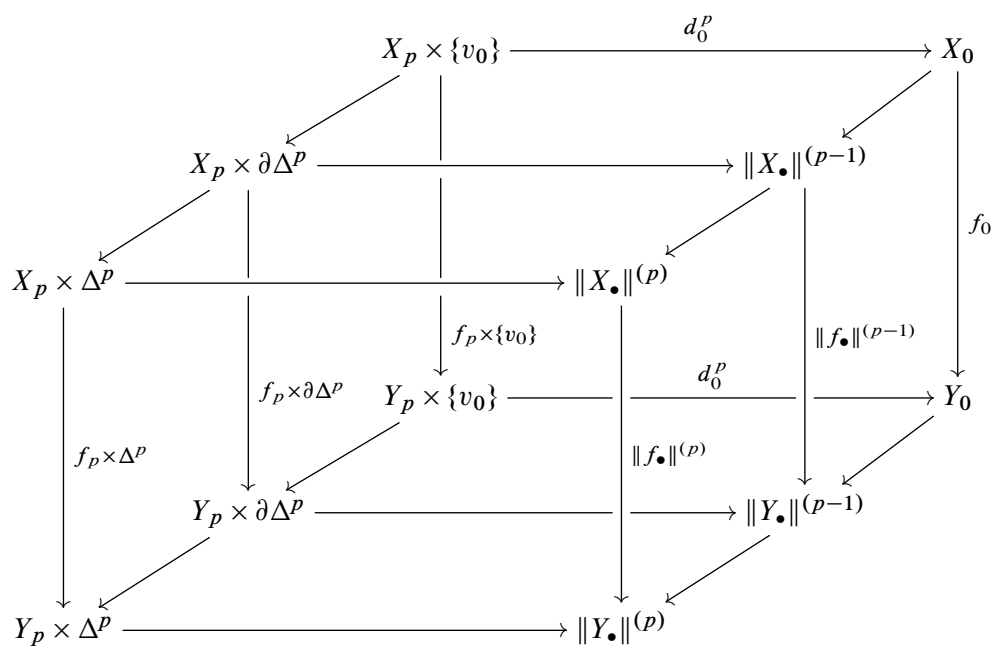
The following is due to Segal [31, Proposition 1.6]:

Theorem 2.12 *Let $f \colon X_{\bullet} \rightarrow Y_{\bullet}$ be a homotopy cartesian map of semisimplicial spaces. Then the square*

$$\begin{array}{ccc} X_0 & \longrightarrow & \|X_{\bullet}\| \\ \downarrow f_0 & & \downarrow \|f_{\bullet}\| \\ Y_0 & \longrightarrow & \|Y_{\bullet}\| \end{array}$$

is also homotopy cartesian.

First proof We prove the result by induction on skeleta. There are commutative cubes



Consider first the back cube. If $p = 1$ then the front face is homotopy cartesian by hypothesis. If $p > 1$ then the right-hand face is homotopy cartesian by inductive assumption, the left-hand face is homotopy cartesian, the back face is homotopy cartesian by hypothesis, and $Y_p \times \{v_0\} \rightarrow Y_p \times \partial \Delta^p$ is 0-connected; thus by the 2-out-of-3 property of homotopy cartesian squares the front face of the back cube is homotopy cartesian.

Consider now the front cube. The left-hand face is homotopy cartesian and by the above the back face is too. The top and bottom faces are homotopy cocartesian, so this cube satisfies the hypotheses of Mather’s first cube theorem [20]. Thus the right-hand face of the front cube is homotopy cartesian, and hence the right-hand face of the outer cube is also homotopy cartesian, as required. \square

Second proof First consider the case where each f_p is a fibration. In this case, the lemma follows from the fact that the geometric realisation $\|f_\bullet\|: \|X_\bullet\| \rightarrow \|Y_\bullet\|$ is a quasifibration, which in turn follows from applying the Dold–Thom criterion [4, Satz 2.2, Hilfssatz 2.10 and Satz 2.12] (a convenient reference is [15, Lemma 4.K.3]).

In the general case, we factor f_p functorially as a composition $X_p \xrightarrow{h_p} Z_p \xrightarrow{g_p} Y_p$ with a weak equivalence h_p and a fibration g_p . Then Z_\bullet is a semisimplicial space,

and h_\bullet and g_\bullet are semisimplicial maps. In the diagram

$$\begin{array}{ccccc} X_p & \xrightarrow{h_p} & Z_p & \xrightarrow{g_p} & Y_p \\ \downarrow d_i & & \downarrow d_i & & \downarrow d_i \\ X_{p-1} & \xrightarrow{h_{p-1}} & Z_{p-1} & \xrightarrow{g_{p-1}} & Y_{p-1} \end{array}$$

the maps h_p and h_{p-1} are weak homotopy equivalences, and it follows that the right square is homotopy cartesian. The lower square in

$$\begin{array}{ccc} X_0 & \longrightarrow & \|X_\bullet\| \\ \downarrow h_0 & & \downarrow \|h_\bullet\| \\ Z_0 & \longrightarrow & \|Z_\bullet\| \\ \downarrow g_0 & & \downarrow \|g_\bullet\| \\ Y_0 & \longrightarrow & \|Y_\bullet\| \end{array}$$

is homotopy cartesian by the first part of the proof, and the upper square is homotopy cartesian as h_0 and $\|h_\bullet\|$ are both weak equivalences, by [Theorem 2.2](#). \square

Lemma 2.13 *Let $\epsilon: X_\bullet \rightarrow X_{-1}$ and $\epsilon: Y_\bullet \rightarrow Y_{-1}$ be augmented semisimplicial spaces and let $(f_\bullet, f): (X_\bullet, X_{-1}) \rightarrow (Y_\bullet, Y_{-1})$ be a map of augmented semisimplicial spaces. If for each $p \geq 0$ the square*

$$\begin{array}{ccc} X_p & \xrightarrow{f_p} & Y_p \\ \downarrow \epsilon_p & & \downarrow \epsilon_p \\ X_{-1} & \xrightarrow{f} & Y_{-1} \end{array}$$

is homotopy cartesian, then so is the square

$$\begin{array}{ccc} \|X_\bullet\| & \xrightarrow{\|f_\bullet\|} & \|Y_\bullet\| \\ \downarrow \| \epsilon_\bullet \| & & \downarrow \| \epsilon_\bullet \| \\ X_{-1} & \xrightarrow{f} & Y_{-1} \end{array}$$

Proof The diagram

$$\begin{array}{ccccc} X_p & \xrightarrow{d_i} & X_{p-1} & \xrightarrow{\epsilon_{p-1}} & X_{-1} \\ \downarrow f_p & & \downarrow f_{p-1} & & \downarrow f \\ Y_p & \xrightarrow{d_i} & Y_{p-1} & \xrightarrow{\epsilon_{p-1}} & Y_{-1} \end{array}$$

has right-hand and outer squares homotopy cartesian by hypothesis, so the left-hand square is also homotopy cartesian. Thus the map f_\bullet is homotopy cartesian and we can apply [Theorem 2.12](#), which shows that the left-hand square in

$$\begin{array}{ccccc} X_0 & \xrightarrow{\iota} & \|X_\bullet\| & \xrightarrow{\|\epsilon_\bullet\|} & X_{-1} \\ \downarrow f_0 & & \downarrow \|f_\bullet\| & & \downarrow f \\ Y_0 & \xrightarrow{\iota} & \|Y_\bullet\| & \xrightarrow{\|\epsilon_\bullet\|} & Y_{-1} \end{array}$$

is homotopy cartesian. As $\|\epsilon_\bullet\| \circ \iota = \epsilon_0$ the outer square is homotopy cartesian by hypothesis, and $\iota: Y_0 \rightarrow \|Y_\bullet\|$ is 0-connected, so the right-hand square is also homotopy cartesian, as required. \square

Lemma 2.14 *Let $\epsilon_\bullet: X_\bullet \rightarrow X_{-1}$ be an augmented semisimplicial space such that each $\epsilon_p: X_p \rightarrow X_{-1}$ is a quasifibration. Then for each $x \in X_{-1}$, the natural map*

$$\|\epsilon_\bullet^{-1}(x)\| \rightarrow \text{hofib}_x \|\epsilon_\bullet\|$$

is a weak homotopy equivalence.

Proof The diagrams

$$\begin{array}{ccc} \epsilon_p^{-1}(x) & \longrightarrow & X_p \\ \downarrow & & \downarrow \epsilon_p \\ \{x\} & \longrightarrow & X_{-1} \end{array}$$

form a map of augmented semisimplicial spaces, and by assumption this map is homotopy cartesian. The statement then follows from [Lemma 2.13](#). \square

Corollary 2.15 *Let X be a topological space and consider the constant semisimplicial space X_\bullet (that is, $X_p := X$ and all face maps are the identity). Then the inclusion $\iota: X = \|X_\bullet\|^{(0)} \rightarrow \|X_\bullet\|$ is a weak equivalence.*

Proof The identity map(s) $X_p \rightarrow X$ form an augmentation $\epsilon_\bullet: X_\bullet \rightarrow X$ and the composition $\|\epsilon_\bullet\| \circ \iota$ is the identity. The semisimplicial space $\epsilon_\bullet^{-1}(x)$ is the terminal semisimplicial space and hence has contractible geometric realisation. It then follows from [Lemma 2.14](#) that $\|\epsilon_\bullet\|$ is a weak homotopy equivalence, whence the claim follows. \square

The following result is due to Segal [31, Proposition 1.5], though we have generalised the formulation a little. It plays a key role in his theory of Γ -spaces, and is also a key ingredient in [7].

Theorem 2.16 *Let X_\bullet be a semisimplicial space and assume that:*

- (i) $X_0 \simeq *$.
- (ii) *The map $\kappa_p\colon X_p \rightarrow (X_1)^p$ given by $(\iota_1^*, \dots, \iota_p^*)$, where $\iota_j\colon [1] \rightarrow [p]$ is the map $0 \mapsto j-1$ and $1 \mapsto j$, is a weak homotopy equivalence.*

Then

- (iii) *If X_1 is k -connected, then $\|X_\bullet\|$ is $(k+1)$ -connected.*
- (iv) *If the squares*

$$\begin{array}{ccc} X_2 & \xrightarrow{d_1} & X_1 \\ \downarrow d_2 & & \downarrow d_1 \\ X_1 & \xrightarrow{d_1} & X_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} X_2 & \xrightarrow{d_1} & X_1 \\ \downarrow d_0 & & \downarrow d_0 \\ X_1 & \xrightarrow{d_1} & X_0 \end{array}$$

are homotopy cartesian and $X_1 \neq \varnothing$, then the tautological map

$$X_1 \rightarrow \tilde{\Omega}\|X_\bullet\|$$

(the target is the space of paths that begin and end in the contractible subspace $X_0 \subset \|X_\bullet\|$) is a weak homotopy equivalence.

If X_\bullet is simplicial, and not just semisimplicial, then the statement of this theorem and its proof can be simplified, which we shall explain in Remark 2.17 below.

Under assumption (ii) we can form the morphism

$$\mu\colon X_1 \times X_1 \xleftarrow[\simeq]{d_0 \times d_2} X_2 \xrightarrow{d_1} X_1$$

in the homotopy category, which makes X_1 into a nonunital homotopy associative H -space. Assumptions (i) and (ii) should be thought of as saying that X_\bullet is a model for the nerve of this H -space.

The assumption in (iv) can be expressed, by taking vertical homotopy fibres, as asking that for each $x \in X_1$ the maps $\mu(x, -), \mu(-, x)\colon X_1 \rightarrow X_1$ be weak homotopy equivalences. That is, it models the H -space X_1 being *grouplike*. In particular, μ induces an associative product $- \cdot -\colon \pi_0(X_1) \times \pi_0(X_1) \rightarrow \pi_0(X_1)$ for which

$[x] \cdot - , - \cdot [x] \colon \pi_0(X_1) \rightarrow \pi_0(X_1)$ are bijections for all $[x]$. As $X_1 \neq \varnothing$ we may choose an $[x] \in \pi_0(X_1)$ for which there is a unique $[e] \in \pi_0(X_1)$ such that $[x] \cdot [e] = [x]$. But then for any y we have

$$[x] \cdot [y] = ([x] \cdot [e]) \cdot [y] = [x] \cdot ([e] \cdot [y]),$$

so $[e] \cdot [y] = [y]$ for any $[y]$ as $[x] \cdot -$ is injective. But then

$$([y] \cdot [e]) \cdot [y] = [y] \cdot ([e] \cdot [y]) = [y] \cdot [y]$$

and so $[y] \cdot [e] = [y]$ as $- \cdot [y]$ is injective. Hence $[e]$ is an two-sided identity element for $- \cdot -$, making $(\pi_0(X_1), \cdot, [e])$ an associative unital monoid. As each $[y] \cdot -$ is a bijection, it is easy to see that this is in fact a group. One consequence is that the map $\mu(e, -) \colon X_1 \rightarrow X_1$ satisfies $\mu(e, \mu(e, -)) \simeq \mu(\mu(e, e), -) \simeq \mu(e, -)$, so is homotopy-idempotent, but it is also a weak equivalence, so is weakly homotopic to the identity.

Proof The first part is an immediate consequence of Lemma 2.4: the map $X_p \rightarrow *$ is $(k+2-p)$ -connected for each p , and hence $\|X_\bullet\| \rightarrow \|\ast_\bullet\|$ is $(k+2)$ -connected. But the geometric realisation of the terminal semisimplicial space is contractible, and so $\|X_\bullet\|$ is $(k+1)$ -connected.

For the second part, we use the *semisimplicial path space* PX_\bullet . This is the semisimplicial space $PX_p := X_{p+1}$, with face maps $d_i \colon PX_p \rightarrow PX_{p-1}$ given by those of X_\bullet having the same names. The maps $d_{p+1} \colon PX_p \rightarrow X_p$ define a simplicial map $PX_\bullet \rightarrow X_\bullet$ and we will prove that it is homotopy cartesian. To verify this, we use Lemma 2.11. The condition for $(p, i) = (1, 1)$ holds by hypothesis, so it remains to prove that the diagrams

$$\begin{array}{ccc} X_{p+1} & \xrightarrow{d_0} & X_p \\ \downarrow d_{p+1} & & \downarrow d_p \\ X_p & \xrightarrow{d_0} & X_{p-1} \end{array}$$

are homotopy cartesian. Under the weak equivalences κ_i (for $p-1 \leq i \leq p+1$), this diagram becomes

$$\begin{array}{ccc} X_1^{p+1} & \xrightarrow{\text{pr}_{\{2, \dots, p+1\}}} & X_1^p \\ \downarrow \text{pr}_{\{1, \dots, p\}} & & \downarrow \text{pr}_{\{1, \dots, p-1\}} \\ X_1^p & \xrightarrow{\text{pr}_{\{2, \dots, p\}}} & X_1^{p-1} \end{array}$$

which is obviously homotopy cartesian. Therefore

$$\begin{array}{ccc} X_1 & \longrightarrow & \|PX_\bullet\| \\ \downarrow d_1 & & \downarrow \\ X_0 & \longrightarrow & \|X_\bullet\| \end{array}$$

is homotopy cartesian, by [Theorem 2.12](#).

We will now show that $\|PX_\bullet\|$ is weakly contractible. Using the simplicial identities, one quickly checks that the maps $\epsilon_p = d_0^{p+1} \colon PX_p = X_{p+1} \rightarrow X_0$ form an augmentation $PX_\bullet \rightarrow PX_{-1} := X_0$. We shall show this is a weak equivalence by showing that $H_*(PX_{-1}, \|PX_\bullet\|; \mathbb{Z}) = 0$ and then showing that $\|PX_\bullet\|$ is simply connected; the claim then follows from Whitehead’s theorem.

To see that the homology of the pair $(PX_{-1}, \|PX_\bullet\|)$ vanishes, consider the morphism

$$g_{q+1} \colon X_q \xrightarrow{\kappa_q} X_1^q \xrightarrow{e \times \text{Id}} X_1^{q+1} \xleftarrow[\simeq]{\kappa_{q+1}} X_{q+1}$$

in the homotopy category, where $e \in X_1$ represents the identity element of $\pi_0(X_1)$ as discussed above. This satisfies the identities of [Lemma 1.12](#) up to weak homotopy. Thus in the spectral sequence

$$E_{p,q}^1 = H_p(PX_q; \mathbb{Z}) \Rightarrow H_{p+q}(PX_{-1}, \|PX_\bullet\|; \mathbb{Z})$$

the maps g_{q+1} give a chain contraction of $(E_{p,*}^1, d^1)$, as we have

$$\begin{aligned} (g_q)_* d^1 + d^1 (g_{q+1})_* &= \left(\sum_{i=0}^q (-1)^i (g_q)_* (d_i)_* \right) + \left(\sum_{j=0}^{q+1} (-1)^j (d_j)_* (g_{q+1})_* \right) \\ &= (d_0)_* (g_{q+1})_* + \sum_{i=0}^q (-1)^i ((g_q)_* (d_i)_* - (d_{i+1})_* (g_{q+1})_*) \\ &= \text{Id}. \end{aligned}$$

Thus $E_{*,*}^2 = 0$ and hence $H_*(PX_{-1}, \|PX_\bullet\|; \mathbb{Z}) = 0$, as claimed.

To show that $\|PX_\bullet\|$ is simply connected, let PX'_\bullet be obtained by collapsing down the 0–simplices of PX_\bullet to a point. Consider the map of homotopy cofibre sequences

$$\begin{array}{ccccc} X_1 & \longrightarrow & \|PX_\bullet\| & \longrightarrow & \|PX'_\bullet\| \\ \downarrow & & \downarrow & & \downarrow \\ \pi_0 X_1 & \longrightarrow & \|\pi_0 PX_\bullet\| & \longrightarrow & \|\pi_0 PX'_\bullet\| \end{array}$$

The map $X_1 \rightarrow \pi_0 X_1$ is 1-connected. The map $PX'_p \rightarrow \pi_0(PX'_p)$ is $(2-p)$ -connected, so $\|PX'_\bullet\| \rightarrow \|\pi_0 PX'_\bullet\|$ is 2-connected by [Lemma 2.4](#). The semisimplicial set $\pi_0 PX_\bullet$ is in bijection with $\pi_0(X_1)^{p+1}$ in degree p , and can be identified with $E_\bullet \pi_0(X_1)$ for the group $\pi_0(X_1)$, so $\|\pi_0 PX_\bullet\| \simeq *$. Now the map $X_1 \rightarrow \|PX_\bullet\|$ is nullhomotopic (it is homotopic to $x \mapsto \mu(x, e) \in X_1 \subset \|PX_\bullet\|$, which in turn is homotopic to $x \mapsto \pi_2(x, e) = e \in X_1 \subset \|PX_\bullet\|$), so the middle map is a retract of the right-hand map, so is also an isomorphism on fundamental groups. Thus $\|PX_\bullet\|$ is simply connected. \square

Remark 2.17 If X_\bullet is a simplicial space satisfying (i) and (ii) of [Theorem 2.16](#), then instead of the hypothesis of (iv) it is enough to just ask for the square

$$\begin{array}{ccc} X_2 & \xrightarrow{d_1} & X_1 \\ \downarrow d_2 & & \downarrow d_1 \\ X_1 & \xrightarrow{d_1} & X_0 \end{array}$$

to be homotopy cartesian for the same conclusion to hold. This is because the maps $h_{p+1} = s_{p+1}: PX_p = X_{p+1} \rightarrow PX_{p+1} = X_{p+2}$ form a system of extra degeneracies, so [Lemma 1.12](#) shows that the augmentation map $\|PX_\bullet\| \rightarrow X_0$ is a weak homotopy equivalence, and we have assumed that $X_0 \simeq *$.

3 (Nonunital) topological categories

Definition 3.1 A *nonunital topological category* \mathcal{C} consists of an object space $\mathcal{C}_0 = \text{Ob}(\mathcal{C})$, a morphism space $\mathcal{C}_1 = \text{Mor}(\mathcal{C})$ and three maps

$$s, t: \mathcal{C}_1 \rightarrow \mathcal{C}_0 \quad \text{and} \quad m: \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 := \{(f, g) \in \mathcal{C}_1 \times \mathcal{C}_1 \mid t(f) = s(g)\} \rightarrow \mathcal{C}_1,$$

such that

$$m(m(f, g), h) = m(f, m(g, h)), \quad t(m(f, g)) = t(g), \quad s(m(f, g)) = s(f)$$

for all f, g and h for which these expressions are defined.

One thinks of s as the map associating to a morphism its source, t as the map associating to a morphism its target, and m as the composition of morphisms, whence we write $g \circ f := m(f, g)$. We write $\mathcal{C}(b_0, b_1) := (s, t)^{-1}(b_0, b_1)$ for the space of morphisms from b_0 to b_1 . A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between nonunital topological categories is a pair

of continuous maps $F_i: \mathcal{C}_i \rightarrow \mathcal{D}_i$ for $i = 0, 1$ such that $sF_1 = F_0s$, $tF_1 = F_0t$ and $m \circ (F_1 \times F_1) = F_1 \circ m$. The set $\text{Fun}(\mathcal{C}, \mathcal{D})$ of functors is endowed with a topology as a subspace of $\text{map}(\mathcal{C}_0, \mathcal{D}_0) \times \text{map}(\mathcal{C}_1, \mathcal{D}_1)$.

Definition 3.2 A *unital topological category* is a nonunital topological category \mathcal{C} together with a map $u: \text{Ob}(\mathcal{C}) \rightarrow \text{Mor}(\mathcal{C})$ such that $t \circ u = s \circ u = \text{Id}$ and $m(f, u(t(f))) = m(u(s(f)), f) = f$ for all $f \in \text{Mor}(\mathcal{C})$.

We shall say, slightly informally, that \mathcal{C} has units if there is the structure of a unital topological category on it.

Definition 3.3 Let \mathcal{C} be a nonunital topological category. The *(semisimplicial) nerve* $N_\bullet \mathcal{C} = \mathcal{C}_\bullet$ of \mathcal{C} is the semisimplicial space whose space of p -simplices is the space $\text{Fun}([p], \mathcal{C})$. For a morphism $\alpha: [q] \rightarrow [p]$, the map $\alpha^*: N_p \mathcal{C} \rightarrow N_q \mathcal{C}$ is given by precomposition with α .

The *classifying space* BC of \mathcal{C} is by definition the geometric realisation of its nerve, $BC := \|\mathcal{C}_\bullet\|$. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ induces a semisimplicial map $F_\bullet: \mathcal{C}_\bullet \rightarrow \mathcal{D}_\bullet$ of semisimplicial spaces and hence a map $BF: BC \rightarrow BD$ of classifying spaces.

More explicitly, $N_0 \mathcal{C} = \mathcal{C}_0$, $N_1 \mathcal{C} = \mathcal{C}_1$, $d_1 = s$ and $d_0 = t: \mathcal{C}_1 \rightarrow \mathcal{C}_0$. For higher values of p , $N_p \mathcal{C}$ is the space $\mathcal{C}_p := \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \cdots \times_{\mathcal{C}_0} \mathcal{C}_1$ (with p factors) with face maps given by composition, and even more explicitly, the points of $N_p \mathcal{C}$ are the sequences

$$c_0 \xrightarrow{f_1} c_1 \rightarrow \cdots \rightarrow c_{p-1} \xrightarrow{f_p} c_p$$

of composable morphisms in \mathcal{C} , and the face maps are given by

$$d_i(f_1, \dots, f_p) := \begin{cases} (f_2, \dots, f_p) & \text{if } i = 0, \\ (f_1, \dots, f_{i+1} \circ f_i, f_p) & \text{if } 0 < i < p, \\ (f_1, \dots, f_{p-1}) & \text{if } i = p. \end{cases}$$

From this point of view, the data of a nonunital topological category is captured precisely by spaces of 0-, 1- and 2-simplices of \mathcal{C}_\bullet and the face maps between them: the source and target maps are given by $d_1: \mathcal{C}_1 \rightarrow \mathcal{C}_0$ and $d_0: \mathcal{C}_1 \rightarrow \mathcal{C}_0$, respectively, and composition of morphisms is given by $d_1: \mathcal{C}_2 \rightarrow \mathcal{C}_1$. For this reason we shall freely confuse the target and source maps with $d_0, d_1: \mathcal{C}_1 \rightarrow \mathcal{C}_0$.

Lemma 3.4 If $\eta: F \Rightarrow G: \mathcal{C} \rightarrow \mathcal{D}$, is a natural transformation, then there is an induced homotopy $BF \simeq BG: BC \rightarrow BD$, of maps on classifying spaces.

Proof We apply Lemma 1.15 with

$$h_{p+1,i}(f_1, f_2, \dots, f_p) = (F(f_1), F(f_2), \dots, F(f_i), \eta_{c_i}, G(f_{i+1}), \dots, G(f_p)),$$

where the hypotheses are immediately verified. \square

3.1 Fibrancy conditions

We shall only be able to make homotopical statements about the classifying spaces of (nonunital) topological categories when some of the structure maps involved are fibrations.

Remark 3.5 We wish to record a technical point about the meaning of the term “fibration” here and in later sections. While we have in mind the class of Serre fibrations, what will be used in the arguments is: Hurewicz fibrations are “fibrations”; “fibrations” are preserved under pullback; composition of “fibrations” are “fibrations”; “fibrations” are quasifibrations. For example, this allows one to take the class of Dold fibrations or, even more generally, Dold–Serre fibrations (ie maps which have the weak covering homotopy property with respect to discs).

Definition 3.6 A nonunital topological category \mathcal{C} is called *left fibrant* if the source map $d_1: \mathcal{C}_1 \rightarrow \mathcal{C}_0$ is a fibration. It is called *right fibrant* if the target map $d_0: \mathcal{C}_1 \rightarrow \mathcal{C}_0$ is a fibration.

Moreover, \mathcal{C} is called *fibrant* if $(d_0, d_1): \mathcal{C}_1 \rightarrow \mathcal{C}_0 \times \mathcal{C}_0$ is a fibration.

If \mathcal{C} is fibrant then it is both left and right fibrant, but the converse need not hold: consider the topological category with objects and morphisms a space X , and all structure maps the identity; this is always left and right fibrant, but is fibrant only if there are no nonconstant paths in X .

Lemma 3.7 If \mathcal{C} is left fibrant, then $d_p: \mathcal{C}_p \rightarrow \mathcal{C}_{p-1}$ is a fibration. If \mathcal{C} is right fibrant, then $d_0: \mathcal{C}_p \rightarrow \mathcal{C}_{p-1}$ is a fibration.

Proof This follows because the squares

$$\begin{array}{ccc} \mathcal{C}_p & \xrightarrow{d_0 \cdots d_0} & \mathcal{C}_1 \\ \downarrow d_p & & \downarrow d_1 \\ \mathcal{C}_{p-1} & \xrightarrow{d_0 \cdots d_0} & \mathcal{C}_0 \end{array} \quad \begin{array}{ccc} \mathcal{C}_p & \xrightarrow{d_2 \cdots d_p} & \mathcal{C}_1 \\ \downarrow d_0 & & \downarrow d_0 \\ \mathcal{C}_{p-1} & \xrightarrow{d_1 \cdots d_{p-1}} & \mathcal{C}_0 \end{array}$$

are cartesian. \square

3.2 The unitalisation

If \mathcal{C} has units, then the semisimplicial space $N_\bullet \mathcal{C}$ has the structure of a simplicial space [30]. Just as we can freely add degeneracies to a semisimplicial object to form a simplicial one, we can freely add units to a nonunital topological category to form a unital one.

Definition 3.8 The *unitalisation* of a nonunital topological category \mathcal{C} is the topological category \mathcal{C}^+ with object space $\text{Ob}(\mathcal{C}^+) = \text{Ob}(\mathcal{C})$ and morphism space $\text{Mor}(\mathcal{C}^+) = \text{Mor}(\mathcal{C}) \sqcup \text{Ob}(\mathcal{C})$. The source and target maps are extended by the identity on $\text{Ob}(\mathcal{C})$. The composition map m^+ for \mathcal{C}^+ is defined so that $c \in \text{Ob}(\mathcal{C}) \subset \text{Mor}(\mathcal{C}^+)$ behaves as the identity morphism at c .

The category \mathcal{C}^+ is never fibrant unless the object space $\text{Ob}(\mathcal{C})$ has no nonconstant paths. However, \mathcal{C}^+ is left (or right) fibrant if \mathcal{C} is left (or right) fibrant. This limits the use of the unitalisation. But unitalisation has one very pleasant property, which we learnt from M Krannich [18, Lemma 1.3.11]:

Proposition 3.9 Let \mathcal{C} be a nonunital topological category. Then the natural map $BC \rightarrow BC^+$ is a weak homotopy equivalence.

Proof There is an isomorphism $N_\bullet \mathcal{C}^+ \cong E(N_\bullet \mathcal{C})$ of simplicial spaces such that the inclusion $N_\bullet \mathcal{C} \rightarrow N_\bullet \mathcal{C}^+$ corresponds to the unit map $N_\bullet \mathcal{C} \rightarrow EN_\bullet \mathcal{C}$. Apply Lemma 2.6. □

3.3 Soft units

From the point of view of the homotopy theory of classifying spaces of (unital, discrete) categories, such as Quillen’s Theorems A and B, an important role is played by over-categories \mathcal{C}/c (and dually under-categories $c \backslash \mathcal{C}$).

Recall that for an object $c \in \text{Ob}(\mathcal{C})$, the *over-category* \mathcal{C}/c has objects the arrows $f: b \rightarrow c$, and morphisms $(g: a \rightarrow c) \rightarrow (f: b \rightarrow c)$ given by a morphism $h: a \rightarrow b$ such that $f \circ h = g$. This definition can be made equally well for nonunital topological categories, by topologising both objects and morphisms as subspaces of \mathcal{C}_1 . Dually (by reversing arrows), one defines the under-category $c \backslash \mathcal{C}$.

If \mathcal{C} is a *unital* topological category then \mathcal{C}/c has an object $\text{Id}_c: c \rightarrow c$ which is terminal: there is a natural transformation from $\text{Id}_{\mathcal{C}/c}$ to the constant functor to Id_c .

By Lemma 3.4 this gives a contraction of $B(\mathcal{C}/c)$. Similarly, $B(c \backslash \mathcal{C})$ is contractible if \mathcal{C} is unital.

If \mathcal{C} is a *nonunital* topological category then $B(\mathcal{C}/c)$ need not be contractible; for example, it can be empty. Instead, we axiomatise this property as follows:

Definition 3.10 A nonunital topological category \mathcal{C} has *soft left units* if for each $c \in \mathcal{C}_0$ we have $B(\mathcal{C}/c) \simeq *$. It has *soft right units* if for each $c \in \mathcal{C}_0$ we have $B(c \backslash \mathcal{C}) \simeq *$.

We will describe a convenient property, more general than having units, which implies that a nonunital topological category has soft left or right units. This property arises naturally for nonunital topological categories such as cobordism categories.

Lemma 3.11 Let \mathcal{C} be a nonunital topological category and let $f \in \mathcal{C}(c, c')$ be a morphism in \mathcal{C} . Then the induced functor $f_*: \mathcal{C}/c \rightarrow \mathcal{C}/c'$ given by postcomposition with f induces a nullhomotopic map on classifying spaces.

Therefore, if an object $c \in \mathcal{C}_0$ is either the source or target of a morphism f which induces a weak equivalence on over-categories, it follows that $B(\mathcal{C}/c) \simeq *$. The analogous statement holds for under-categories.

Proof We consider the case of the over-categories. There are maps $h_p: N_p(\mathcal{C}/c) \rightarrow N_{p+1}(\mathcal{C}/c')$, given by sending a p -tuple of composable morphisms $c_0 \rightarrow c_1 \rightarrow \cdots \rightarrow c_p \rightarrow c$ in \mathcal{C}/c to the $(p+1)$ -tuple of composable morphisms $c_0 \rightarrow c_1 \rightarrow \cdots \rightarrow c_p \rightarrow c \xrightarrow{f} c'$ in \mathcal{C}/c' . These form a semisimplicial nullhomotopy from $N_\bullet f_*$ to the constant map to $(c \xrightarrow{f} c')$. Then apply Lemma 1.13. \square

This observation may be applied to many nonunital topological categories arising in practice, because while they do not have units they do have many morphisms composition with which induce weak equivalences on morphism spaces, as follows.

Definition 3.12 Let \mathcal{C} be a topological category. We say that \mathcal{C} has *weak left units* if for each object $b \in \mathcal{C}_0$, there is a morphism $u: b \rightarrow b'$ in \mathcal{C} such that the map

$$\mathcal{C}(-, b) := d_0^{-1}(b) \xrightarrow{u \circ -} \mathcal{C}(-, b')$$

is a weak homotopy equivalence. Dually, \mathcal{C} has *weak right units* if for each object $b \in \mathcal{C}_0$, there is a morphism $u: b' \rightarrow b$ in \mathcal{C} such that

$$\mathcal{C}(b, -) := d_1^{-1}(b) \xrightarrow{- \circ u} \mathcal{C}(b', -)$$

is a weak homotopy equivalence.

Remark 3.13 If \mathcal{C} is left fibrant, then $u \circ - \colon \mathcal{C}(-, b) \rightarrow \mathcal{C}(-, b')$ is a weak equivalence if and only if $u \circ - \colon \mathcal{C}(a, b) \rightarrow \mathcal{C}(a, b')$ is a weak equivalence for each $a \in \mathcal{C}_0$.

Lemma 3.14 If \mathcal{C} has weak left units and is right fibrant, then it has soft left units. Dually, if \mathcal{C} has weak right units and is left fibrant, then it has soft right units.

Proof We only treat the first case. Let $u \in \mathcal{C}(c, c')$ be a weak left unit. The squares

$$\begin{array}{ccccc} N_p(\mathcal{C}/c) & \xrightarrow{N_p(u_*)} & N_p(\mathcal{C}/c') & \longrightarrow & N_p(\mathcal{C}) \\ \downarrow d_0^p & & \downarrow d_0^p & & \downarrow d_0^p \\ N_0(\mathcal{C}/c) & \xrightarrow{N_0(u_*)} & N_0(\mathcal{C}/c') & \longrightarrow & N_0(\mathcal{C}) \end{array}$$

are both cartesian. By [Lemma 3.7](#) the right-hand vertical map is a fibration, and so all the vertical maps are fibrations and hence both squares are homotopy cartesian. We now consider the left-hand square: Since the bottom horizontal map is a weak equivalence by assumption, it follows that the upper horizontal one is as well. Therefore, the functor $u_* \colon \mathcal{C}/c \rightarrow \mathcal{C}/c'$ induces a levelwise equivalence on nerves. But the map $Bu_* \colon B(\mathcal{C}/c) \rightarrow B(\mathcal{C}/c')$ is also nullhomotopic by [Lemma 3.11](#), so $B(\mathcal{C}/c) \simeq *$. \square

4 Quillen’s Theorems A and B and bisemisimplicial resolutions

Let $F \colon \mathcal{C} \rightarrow \mathcal{D}$ be a functor of discrete and unital categories. Quillen’s Theorem A [\[26\]](#) is a classical and well-known criterion to show that $BF \colon B\mathcal{C} \rightarrow B\mathcal{D}$ is a weak equivalence. Similarly, Quillen’s Theorem B [\[26\]](#) is a device to identify the homotopy fibre of BF . In this section, we prove generalisations of Quillen’s theorems for topological and nonunital categories. Those are stated as Theorems [4.7](#), [4.8](#) and [4.9](#) below, but before we can state them precisely, we need to introduce a construction that is used in the proofs.

Definition 4.1 Let $F \colon \mathcal{C} \rightarrow \mathcal{D}$ be a continuous functor between nonunital topological categories. Let $(F/\mathcal{D})_{p,q}$ be the space of all pairs in $N_p\mathcal{C} \times N_{q+1}\mathcal{D}$ of the form $(a_0 \rightarrow \cdots \rightarrow a_p, F(a_p) \rightarrow b_0 \rightarrow \cdots \rightarrow b_q)$ (of course, the unnamed arrows are part of the data). The $(F/\mathcal{D})_{p,q}$ form, in an evident way, a bisemisimplicial space. It has augmentation maps

$$\epsilon_{p,q} \colon (F/\mathcal{D})_{p,q} \rightarrow \mathcal{C}_p, \quad (a_0 \rightarrow \cdots \rightarrow a_p, F(a_p) \rightarrow b_0 \rightarrow \cdots \rightarrow b_q) \mapsto (a_0 \rightarrow \cdots \rightarrow a_p),$$

and

$\eta_{p,q} \colon (F/\mathcal{D})_{p,q} \rightarrow \mathcal{D}_q, \quad (a_0 \rightarrow \cdots \rightarrow a_p, F(a_p) \rightarrow b_0 \rightarrow \cdots \rightarrow b_q) \mapsto (b_0 \rightarrow \cdots \rightarrow b_q).$

Dually, let $(\mathcal{D}/F)_{p,q}$ be the space of all pairs in $N_p\mathcal{C} \times N_{q+1}\mathcal{D}$ of the form

$(a_0 \rightarrow \cdots \rightarrow a_p, b_0 \rightarrow \cdots \rightarrow b_q \rightarrow F(a_0)).$

The $(\mathcal{D}/F)_{p,q}$ form, in an obvious way, a bisemisimplicial space. It has augmentation maps

$\xi_{p,q} \colon (\mathcal{D}/F)_{p,q} \rightarrow \mathcal{C}_p, \quad (a_0 \rightarrow \cdots \rightarrow a_p, b_0 \rightarrow \cdots \rightarrow b_q \rightarrow F(a_0)) \mapsto (a_0 \rightarrow \cdots \rightarrow a_p),$

and

$\zeta_{p,q} \colon (\mathcal{D}/F)_{p,q} \rightarrow \mathcal{D}_q, \quad (a_0 \rightarrow \cdots \rightarrow a_p, b_0 \rightarrow \cdots \rightarrow b_q \rightarrow F(a_0)) \mapsto (b_0 \rightarrow \cdots \rightarrow b_q).$

For the rest of this section we shall makes statements about both constructions, but only prove them in the first case: the second is dual.

Lemma 4.2 *The diagrams*

$$\begin{array}{ccc} & \|(F/\mathcal{D})_{\bullet,\bullet}\| & \\ \|\epsilon_{\bullet,\bullet}\| \swarrow & & \searrow \|\eta_{\bullet,\bullet}\| \\ \|\mathcal{C}_{\bullet}\| & \xrightarrow{\|F_{\bullet}\|} & \|\mathcal{D}_{\bullet}\| \end{array}$$

and

$$\begin{array}{ccc} & \|(\mathcal{D}/F)_{\bullet,\bullet}\| & \\ \|\xi_{\bullet,\bullet}\| \swarrow & & \searrow \|\zeta_{\bullet,\bullet}\| \\ \|\mathcal{C}_{\bullet}\| & \xrightarrow{\|F_{\bullet}\|} & \|\mathcal{D}_{\bullet}\| \end{array}$$

are (naturally) homotopy commutative.

Proof For $p, q \geq 0$, we define a map

$H_{p,q} \colon I \times (F/\mathcal{D})_{p,q} \times \Delta^p \times \Delta^q \rightarrow \|\mathcal{D}_{\bullet}\|$

by sending $(t; a_0 \rightarrow \cdots \rightarrow a_p, F(a_p) \rightarrow b_0 \rightarrow \cdots \rightarrow b_q; r, s)$ to

$(F(a_0) \rightarrow \cdots \rightarrow F(a_p) \rightarrow b_0 \rightarrow \cdots \rightarrow b_q; tr, (1-t)s) \in \mathcal{D}_{p+q+1} \times \Delta^{p+q+1}.$

This respects the simplicial relations and hence descends to a map $H \colon I \times \|(F/\mathcal{D})_{\bullet,\bullet}\| \rightarrow \|\mathcal{D}_{\bullet}\|$ (we have used that taking products preserves quotient maps in the category

of compactly generated spaces). This satisfies $H(0, -) = \|\eta_{\bullet, \bullet}\|$ and $H(1, -) = \|F_{\bullet}\| \circ \|\epsilon_{\bullet, \bullet}\|$. \square

Lemma 4.3 *If \mathcal{D} is unital, then*

$$\|\epsilon_{\bullet, \bullet}\|: \|(F/\mathcal{D})_{\bullet, \bullet}\| \rightarrow \|\mathcal{C}_{\bullet}\| \quad \text{and} \quad \|\xi_{\bullet, \bullet}\|: \|(\mathcal{D}/F)_{\bullet, \bullet}\| \rightarrow \|\mathcal{C}_{\bullet}\|$$

are weak homotopy equivalences.

Proof By Theorem 2.2, it is enough to prove that $\|(F/\mathcal{D})_{p, \bullet}\| \rightarrow \mathcal{C}_p$ is a weak homotopy equivalence for all p . We show that the augmented semisimplicial space $\epsilon_{p, \bullet}: (F/\mathcal{D})_{p, \bullet} \rightarrow \mathcal{C}_p$ has an extra degeneracy of the second type described in Lemma 1.12. Define $g_0: \mathcal{C}_p \rightarrow (F/\mathcal{D})_{p, 0}$ by

$$(a_0 \rightarrow \cdots \rightarrow a_p) \mapsto (a_0 \rightarrow \cdots \rightarrow a_p, F(a_p) \xrightarrow{\text{Id}} F(a_p))$$

and $g_{q+1}: (F/\mathcal{D})_{p, q} \rightarrow (F/\mathcal{D})_{p, q+1}$ by

$$(a_0 \rightarrow \cdots \rightarrow a_p, F(a_p) \rightarrow b_0 \rightarrow \cdots \rightarrow a_q) \mapsto (a_0 \rightarrow \cdots \rightarrow a_p, F(a_p) \xrightarrow{\text{Id}} F(a_p) \rightarrow b_0 \rightarrow \cdots \rightarrow a_q).$$

These satisfy the conditions in Lemma 1.12, showing that $\|(F/\mathcal{D})_{p, \bullet}\| \rightarrow \mathcal{C}_p$ is a homotopy equivalence. \square

For nonunital categories, the conclusion of Lemma 4.3 does not hold without further hypotheses. If we do not have units then, rather than the explicit homotopy coming from an extra degeneracy used in the proof of the last lemma, note that for $a = (a_0 \rightarrow \cdots \rightarrow a_p) \in \mathcal{C}_p$, we have

$$\epsilon_{p, \bullet}^{-1}(a) = N_{\bullet}(F(a_p)/\mathcal{D})$$

and

$$\xi_{p, \bullet}^{-1}(a) = N_{\bullet}(\mathcal{D}/F(a_0)),$$

the semisimplicial nerves of over- and under-categories. We have axiomatised the contractibility of these as soft left or right units, and we will show that under appropriate fibrancy conditions this is enough to get the conclusion of Lemma 4.3.

Lemma 4.4 *If \mathcal{D} is left fibrant, then the augmentation map $\epsilon_{p, q}: (F/\mathcal{D})_{p, q} \rightarrow \mathcal{C}_p$ is a fibration. If \mathcal{D} is right fibrant, then the augmentation map $\xi_{p, q}: (\mathcal{D}/F)_{p, q} \rightarrow \mathcal{C}_p$ is a fibration.*

Proof Observe that both squares

$$\begin{array}{ccccc}
 (F/\mathcal{D})_{p,q} & \xrightarrow{d_0^p} & (F/\mathcal{D})_{0,q} & \xrightarrow{\gamma} & \mathcal{D}_{q+1} \\
 \downarrow \epsilon_{p,q} & & \downarrow \epsilon_{0,q} & & \downarrow d_1 \cdots d_{q+1} \\
 \mathcal{C}_p & \xrightarrow{d_0^p} & \mathcal{C}_0 & \xrightarrow{F_0} & \mathcal{D}_0
 \end{array}$$

with $\gamma(a_0, F(a_0) \rightarrow b_0 \rightarrow \cdots \rightarrow b_q) := (F(a_0) \rightarrow b_0 \rightarrow \cdots \rightarrow b_q)$ are cartesian, and use [Lemma 3.7](#). \square

Corollary 4.5 *If \mathcal{D} is left fibrant and has soft right units, then $\|\epsilon_{\bullet,\bullet}\|: \|(F/\mathcal{D})_{\bullet,\bullet}\| \rightarrow BC$ is a weak equivalence. Dually, if \mathcal{D} is right fibrant and has soft left units, then $\|\xi_{\bullet,\bullet}\|: \|(\mathcal{D}/F)_{\bullet,\bullet}\| \rightarrow BC$ is a weak equivalence.*

Proof By [Lemma 4.4](#) the maps $\epsilon_{p,q}: (F/\mathcal{D})_{p,q} \rightarrow \mathcal{C}_p$ are fibrations, so [Lemma 2.14](#) applies to $\epsilon_{p,\bullet}: (F/\mathcal{D})_{p,\bullet} \rightarrow \mathcal{C}_p$ and so for each $a = (a_0 \rightarrow \cdots \rightarrow a_p) \in \mathcal{C}_p$ the map

$$B(F(a_p)/\mathcal{D}) = \|\epsilon_{p,\bullet}^{-1}(a)\| \rightarrow \text{hofib}_a \|\epsilon_{p,\bullet}\|$$

is a weak equivalence. But, as \mathcal{D} has soft right units, the source of this map is contractible, and hence $\|(F/\mathcal{D})_{p,\bullet}\| \rightarrow \mathcal{C}_p$ is a weak equivalence. The claim then follows by geometrically realising in the p -direction and using [Theorem 2.2](#). \square

To make use of these resolutions, we shall also need to know that the maps $\eta_{p,q}$ and $\zeta_{p,q}$ are fibrations, and the final result of this section is a criterion for this to hold.

Lemma 4.6 (i) *If $\eta_{p,0}$ is a fibration, then so is $\eta_{p,q}$ for all $q \geq 0$.*

(ii) *If $\eta_{0,0}$ is a fibration and \mathcal{C} is right fibrant, then $\eta_{p,0}$ is a fibration for all $p \geq 0$.*

(iii) *If $F_0: \mathcal{C}_0 \rightarrow \mathcal{D}_0$ is a fibration and \mathcal{D} is right fibrant, then $\eta_{0,0}$ is a fibration.*

Dually:

(iv) *If $\zeta_{p,0}$ is a fibration, then so is $\zeta_{p,q}$ for all $q \geq 0$.*

(v) *If $\zeta_{0,0}$ is a fibration and \mathcal{C} is left fibrant, then $\zeta_{p,0}$ is a fibration for all $p \geq 0$.*

(vi) *If $F_0: \mathcal{C}_0 \rightarrow \mathcal{D}_0$ is a fibration and \mathcal{D} is left fibrant, then $\zeta_{0,0}$ is a fibration.*

Proof The square

$$\begin{array}{ccc} (F/\mathcal{D})_{p,q} & \xrightarrow{d_1 \cdots d_q} & (F/\mathcal{D})_{p,0} \\ \downarrow \eta_{p,q} & & \downarrow \eta_{p,0} \\ \mathcal{D}_q & \xrightarrow{d_1 \cdots d_q} & \mathcal{D}_0 \end{array}$$

is cartesian, which proves (i). For part (ii), use that

$$\begin{array}{ccc} (F/\mathcal{D})_{p,0} & \xrightarrow{\epsilon_{p,0}} & \mathcal{C}_p \\ \downarrow d_0 \cdots d_0 & & \downarrow d_0 \cdots d_0 \\ (F/\mathcal{D})_{0,0} & \xrightarrow{\epsilon_{0,0}} & \mathcal{C}_0 \end{array}$$

is cartesian, [Lemma 3.7](#), and that $\eta_{p,0} = \eta_{0,0} \circ (d_0)^p$. For part (iii), let $\gamma: (F/\mathcal{D})_{0,0} \rightarrow \mathcal{D}_1$ be given by $\gamma(a, F(a) \rightarrow b) = (F(a) \rightarrow b)$. The diagram

$$\begin{array}{ccc} (F/\mathcal{D})_{0,0} & \xrightarrow{\gamma} & \mathcal{D}_1 \\ \downarrow \epsilon_{0,0} & & \downarrow d_1 \\ \mathcal{C}_0 & \xrightarrow{F_0} & \mathcal{D}_0 \end{array}$$

is cartesian, so γ is a fibration, hence so is $d_0 \circ \gamma = \eta_{0,0}$. □

We can now state and prove our version of Quillen's Theorems A and B for nonunital topological categories.

Theorem 4.7 (Quillen's Theorem A) *Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a continuous functor. Assume that*

- (i) $B(F/b)$ is contractible for each $b \in \mathcal{D}_0$,
- (ii) $\|\epsilon_{\bullet,\bullet}\|: \|(F/\mathcal{D})_{\bullet,\bullet}\| \rightarrow BC$ is a weak equivalence,
- (iii) $\eta_{p,0}: (F/\mathcal{D})_{p,q} \rightarrow N_q \mathcal{D}$ is a fibration for each $p \geq 0$.

Then $BF: BC \rightarrow B\mathcal{D}$ is a weak homotopy equivalence.

Conditions (ii) and (iii) are satisfied if either

- (iv) \mathcal{C} is right fibrant, \mathcal{D} is left fibrant and has soft right units and $\eta_{0,0}$ is a fibration, or
- (v) \mathcal{C} is right fibrant, \mathcal{D} has units and $\eta_{0,0}$ is a fibration.

There is a dual version, with a parallel proof:

Theorem 4.8 (Quillen's Theorem A, dual version) *Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a continuous functor. Assume that*

- (i) $B(b/F)$ is contractible for each $b \in \mathcal{D}_0$,
- (ii) $\|\xi_{\bullet, \bullet}\|: \|(\mathcal{D}/F)_{\bullet, \bullet}\| \rightarrow BC$ is a weak equivalence,
- (iii) $\zeta_{p,q}: (\mathcal{D}/F)_{p,q} \rightarrow N_q \mathcal{D}$ is a fibration for each $p, q \geq 0$.

Then $BF: BC \rightarrow BD$ is a weak homotopy equivalence.

Conditions (ii) and (iii) are satisfied if either

- (iv) \mathcal{C} is left fibrant, \mathcal{D} is right fibrant and has soft left units and $\zeta_{0,0}$ is a fibration, or
- (v) \mathcal{C} is left fibrant, \mathcal{D} has units and $\zeta_{0,0}$ is a fibration.

In the case of discrete (unital) categories, this is a classical result of Quillen [26]. A version for (unital) simplicial categories was proven by Waldhausen [34, Section 4].

Proof of Theorem 4.7 That conditions (iv) or (v) imply conditions (ii) and (iii) follows from Lemmas 4.3 and 4.6 and Corollary 4.5.

By Lemma 4.2, it is enough to prove that $\|\eta_{\bullet, \bullet}\|: \|(F/\mathcal{D})_{\bullet, \bullet}\| \rightarrow \|\mathcal{D}_{\bullet}\|$ is a weak equivalence. Since each $\eta_{p,0}$ is a fibration, it follows by Lemma 4.6 that $\eta_{p,q}$ is a fibration for all $p, q \geq 0$, so by Lemma 2.14 for each $b = (b_0 \rightarrow \cdots \rightarrow b_q) \in \mathcal{D}_q$ the natural map

$$B(F/b) = \|\eta_{\bullet, q}^{-1}(b)\| \rightarrow \text{hofib}_b \|\eta_{\bullet, q}\|$$

is a weak equivalence. The source is contractible by assumption, so $\|\eta_{\bullet, q}\|$ is a weak equivalence. \square

Quillen's Theorem B [26] gives a criterion for identifying the homotopy fibre of a functor between ordinary categories. We now state and prove a version of this for nonunital topological categories; in fact we give a mild generalisation, due to Blumberg and Mandell [1, Theorem 4.5]. In this case we only state one version; it has a dual version which we leave to the reader.

Theorem 4.9 (Quillen's Theorem B) *Let*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{J} & \mathcal{C} \\ \downarrow G & & \downarrow F \\ \mathcal{B} & \xrightarrow{H} & \mathcal{D} \end{array}$$

be a commuting square of nonunital topological categories. Assume that:

- (i) \mathcal{B} and \mathcal{D} are left fibrant and have soft right units.
- (ii) \mathcal{A} and \mathcal{C} are right fibrant, and the maps

$$\eta_{0,0}: (G/\mathcal{B})_{0,0} \rightarrow \mathcal{B}_0 \quad \text{and} \quad \eta_{0,0}: (F/\mathcal{D})_{0,0} \rightarrow \mathcal{D}_0$$

are fibrations.

- (iii) For each morphism $u: d \rightarrow d'$ in \mathcal{D} , the functor $u_*: F/d \rightarrow F/d'$ induced by composition with u induces a weak equivalence on classifying spaces.
- (iv) For each object $b \in \mathcal{B}_0$, the functor $G/b \rightarrow F/H(b)$ induced by J and H is a weak equivalence.

Then the square

$$\begin{array}{ccc} B\mathcal{A} & \xrightarrow{BJ} & B\mathcal{C} \\ \downarrow BG & & \downarrow BF \\ B\mathcal{B} & \xrightarrow{BH} & B\mathcal{D} \end{array}$$

is homotopy cartesian.

Proof Using the resolutions of the functors F and G , by assumption (i) and [Corollary 4.5](#) it is enough to show that the square

$$\begin{array}{ccc} \|(G/\mathcal{B})_{\bullet,\bullet}\| & \longrightarrow & \|(F/\mathcal{D})_{\bullet,\bullet}\| \\ \downarrow \|\eta_{\bullet,\bullet}^G\| & & \downarrow \|\eta_{\bullet,\bullet}^F\| \\ B\mathcal{B} & \xrightarrow{BH} & B\mathcal{D} \end{array}$$

is homotopy cartesian. Arguing as in the proof of [Theorem 4.7](#), which requires assumption (ii), we see that the maps

$$\|\eta_{\bullet,q}^F\|: \|(F/\mathcal{D})_{\bullet,q}\| \rightarrow \mathcal{D}_q, \quad \|\eta_{\bullet,q}^G\|: \|(G/\mathcal{B})_{\bullet,q}\| \rightarrow \mathcal{B}_q$$

are quasifibrations. In the commutative square

(4.10)

$$\begin{array}{ccc} \|(F/\mathcal{D})_{\bullet,q}\| & \xrightarrow{d_i} & \|(F/\mathcal{D})_{\bullet,q-1}\| \\ \downarrow \|\eta_{\bullet,q}^F\| & & \downarrow \|\eta_{\bullet,q-1}^F\| \\ \mathcal{D}_q & \xrightarrow{d_i} & \mathcal{D}_{q-1} \end{array}$$

the fibre over $x = (d_0 \xrightarrow{u_1} \dots \xrightarrow{u_q} d_q) \in \mathcal{D}_q$ is $B(F/d_0)$, and the induced map on fibres is either the identity (if $i > 0$) or it is the fibre transport map $(u_1)_*: B(F/d_0) \rightarrow B(F/d_1)$, which is a weak equivalence by assumption (iii). Therefore, by [Theorem 2.12](#),

the squares

$$(4.11) \quad \begin{array}{ccccc} B(F/d) & \longrightarrow & \|(F/\mathcal{D})_{\bullet,0}\| & \longrightarrow & \|(F/\mathcal{D})_{\bullet,\bullet}\| \\ \downarrow & & \downarrow & & \downarrow \|\eta_{\bullet,\bullet}^F\| \\ \{d\} & \longrightarrow & \mathcal{D}_0 & \longrightarrow & B\mathcal{D} \end{array}$$

are both homotopy cartesian. For each morphism $f: b \rightarrow b'$ in \mathcal{B} , the induced map $f_*: B(G/b) \rightarrow B(G/b')$ is a weak equivalence, since it fits into a commutative diagram

$$\begin{array}{ccc} B(G/b) & \xrightarrow{\simeq} & B(F/Hb) \\ \downarrow f_* & & \simeq \downarrow H(f)_* \\ B(G/b') & \xrightarrow{\simeq} & B(F/Hb') \end{array}$$

in which all other maps are weak equivalences by assumptions (iii) and (iv). Therefore, in the analogue of the diagram (4.11) for the functor G both squares are also homotopy cartesian. For $b \in \mathcal{B}_0$ the composition

$$B(G/b) \xrightarrow{\simeq} \mathrm{hofib}_b \|\eta_{\bullet,\bullet}^G\| \xrightarrow{BJ} \mathrm{hofib}_{Hb} \|\eta_{\bullet,\bullet}^F\|$$

is equal to the composition

$$B(G/b) \xrightarrow{\simeq} BF/(Hb) \xrightarrow{\simeq} \mathrm{hofib}_{Hb} \|\eta_{\bullet,\bullet}^F\|.$$

Therefore, $BJ: \mathrm{hofib}_b \|\eta_{\bullet,\bullet}^G\| \rightarrow \mathrm{hofib}_{Hb} \|\eta_{\bullet,\bullet}^F\|$ is a weak equivalence for each $b \in \mathcal{B}_0$; since the inclusion $\iota: \mathcal{B}_0 \rightarrow B\mathcal{B}$ is 0-connected, this finishes the proof. \square

5 Base changing spaces of objects

For a nonunital topological category \mathcal{C} and a continuous map $f: X \rightarrow \mathcal{C}_0$, we may form a new nonunital topological category \mathcal{C}^X as follows. We let \mathcal{C}_0^X be X , and $F_0: \mathcal{C}_0^X \rightarrow \mathcal{C}_0$ be f . Then we define \mathcal{C}_1^X as the pullback

$$(5.1) \quad \begin{array}{ccc} \mathcal{C}_1^X & \xrightarrow{F_1} & \mathcal{C}_1 \\ \downarrow & & \downarrow s \times t \\ \mathcal{C}_0^X \times_{\mathcal{C}_0^X} \mathcal{C}_0^X & \xrightarrow{F_0 \times F_0} & \mathcal{C}_0 \times \mathcal{C}_0 \end{array}$$

The left-hand maps define $s, t: \mathcal{C}_1^X \rightarrow \mathcal{C}_0^X$, and the universal property of the pullback provides a map $c: \mathcal{C}_1^X \times_{\mathcal{C}_0^X} \mathcal{C}_1^X \rightarrow \mathcal{C}_1^X$; this defines a nonunital topological category, and the F_i define a continuous functor $F: \mathcal{C}^X \rightarrow \mathcal{C}$. (If \mathcal{C} has units, then \mathcal{C}^X does too.)

Theorem 5.2 *If \mathcal{C} is fibrant and has weak right (or left) units, and f is 0-connected, then $BF: BC^X \rightarrow BC$ is a weak equivalence.*

Proof We consider the resolution $(F/\mathcal{C})_{\bullet,\bullet}$ of the functor F . As \mathcal{C} is left fibrant and has weak right (say) units, it has soft right units by [Lemma 3.14](#), and so [Corollary 4.5](#) applies and shows that $\|\epsilon_{\bullet,\bullet}\|: \|(F/\mathcal{C})_{\bullet,\bullet}\| \rightarrow BC^X$ is a weak equivalence. It remains to show that $\|\eta_{\bullet,\bullet}\|: \|(F/\mathcal{C})_{\bullet,\bullet}\| \rightarrow BC$ is a weak equivalence.

The space $(F/\mathcal{C})_{0,0}$ fits into a cartesian square

$$\begin{array}{ccc} (F/\mathcal{C})_{0,0} & \longrightarrow & \mathcal{C}_1 \\ \downarrow \epsilon_{0,0} \times \eta_{0,0} & & \downarrow s \times t \\ \mathcal{C}_0^X \times \mathcal{C}_0 & \xrightarrow{F_0 \times \text{id}} & \mathcal{C}_0 \times \mathcal{C}_0 \end{array}$$

and as \mathcal{C} is fibrant the right-hand vertical map is a fibration, and so $\eta_{0,0}$ is a fibration too. Furthermore, as \mathcal{C} is fibrant, [\(5.1\)](#) shows that \mathcal{C}^X is too. Hence, by applying (ii) then (i) of [Lemma 4.6](#), each $\eta_{p,q}$ is a fibration. Hence, by [Lemma 2.14](#), for each $b = (b_0 \rightarrow \cdots \rightarrow b_q) \in \mathcal{C}_q$ the map

$$B(\mathcal{C}^X/b_0) = \|\eta_{\bullet,q}^{-1}(b)\| \rightarrow \text{hofib}_b \|\eta_{\bullet,q}\|$$

is a weak equivalence, so it is enough to show that the over-categories $B(\mathcal{C}^X/b_0)$ are contractible for some object $b_0 \in \mathcal{C}_0$ in each path component. As $f: X \rightarrow \mathcal{C}_1$ is 0-connected, we may suppose that $b_0 = F(x_0)$, but in this case $\mathcal{C}^X/F(x_0) = \mathcal{C}^X/x_0$, by [\(5.1\)](#), so it is enough to show that \mathcal{C}^X has soft right units. As \mathcal{C} has weak right units, so does \mathcal{C}^X (by [Remark 3.13](#) and because both categories are fibrant), so by [Lemma 3.14](#), \mathcal{C}^X has soft right units, as required. \square

A typical application of this result is to take $X = \mathcal{C}_0^\delta$ to be the set of objects of \mathcal{C} with the discrete topology, and $f: \mathcal{C}_0^\delta \rightarrow \mathcal{C}_0$ to be the identity function (which is 0-connected). This yields a category \mathcal{C}^δ with discrete space of objects but the same space of maps between any two objects, which has a homotopy equivalent classifying space under the conditions given above.

6 The group-completion theorem

We shall take care to formulate and prove the group-completion theorem, and the main technical result underlying it, for homology with local coefficients. We therefore make the following definitions:

Definition 6.1 Let \mathcal{L} be a local coefficient system of R -modules on a space X .

- (i) The *monodromy* of \mathcal{L} at $x \in X$ is the homomorphism

$$\mu_x: \pi_1(X, x) \rightarrow \text{Aut}_{R\text{-Mod}}(\mathcal{L}(x))$$

induced from \mathcal{L} .

- (ii) \mathcal{L} is called *constant* if all monodromy homomorphisms are trivial.
 (iii) \mathcal{L} is called *abelian* if the images of all monodromy homomorphisms are abelian groups.

Assumptions 6.2 In the sequel, let \mathcal{A} be either

- (i) the class of constant local coefficient systems of R -modules,
 (ii) the class of abelian local coefficient systems of R -modules, or
 (iii) the class of all local coefficient systems of R -modules.

We say a map $f: X \rightarrow Y$ is an \mathcal{A} -*equivalence* if for every local coefficient system \mathcal{L} on Y in the class \mathcal{A} , the map

$$f_*: H_*(X; f^* \mathcal{L}) \rightarrow H_*(Y; \mathcal{L})$$

is an isomorphism.

In case (iii) with $R = \mathbb{Z}$, the \mathcal{A} -equivalences are precisely the acyclic maps; see eg [16].

Definition 6.3 A commutative square of spaces

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow g & & \downarrow f \\ Z & \xrightarrow{h} & Y \end{array}$$

is called \mathcal{A} -*cartesian* if the induced map $\text{hofib}_z(g) \rightarrow \text{hofib}_{h(z)}(f)$ is an \mathcal{A} -equivalence for all $z \in Z$.

Remark 6.4 Unlike for homotopy cartesian diagrams, the symmetry explained in Remark 2.8 does not generally hold for \mathcal{A} -cartesian diagrams (though it does in case (iii)). A counterexample in case (i) is $R = \mathbb{Z}$ when $W = Z = Y = *$ and $X = BG$ is the classifying space of an infinite acyclic group.

The following homological analogue of Theorem 2.12 is the technical heart of the “group-completion theorem” and is due to McDuff and Segal [22]. The notion of an \mathcal{A} -cartesian map $f_\bullet: X_\bullet \rightarrow Y_\bullet$ is defined in analogy to Definition 2.9.

Theorem 6.5 *If $f_{\bullet}: X_{\bullet} \rightarrow Y_{\bullet}$ is an \mathcal{A} -cartesian map of semisimplicial spaces, then the diagram*

$$(6.6) \quad \begin{array}{ccc} X_0 & \longrightarrow & \|X_{\bullet}\| \\ \downarrow f_0 & & \downarrow \|f_{\bullet}\| \\ Y_0 & \longrightarrow & \|Y_{\bullet}\| \end{array}$$

is \mathcal{A} -cartesian.

The presentation of McDuff and Segal omits many details, to say the least. A more detailed exposition of the proof, with some imprecisions fixed, can be found in [23]. These proofs involve some fairly complicated point-set topology. There are proofs of an analogous result in the context of bisimplicial sets by Jardine [17; 14], Moerdijk [24] and Pitsch and Scherer [25]. These proofs use heavy machinery from simplicial homotopy theory (either model structures on the category of bisimplicial sets, or (unpublished) results for manipulating homotopy colimits). The proof we shall give is essentially that of McDuff and Segal, but our argument replaces the point-set topology considerations with simplicial arguments.

6.1 Proof of Theorem 6.5

The main portion of the proof of Theorem 6.5 will be to prove the following version for simplicial spaces; the last step is the generalisation to semisimplicial spaces. We shall say that a map $f_{\bullet}: X_{\bullet} \rightarrow Y_{\bullet}$ of simplicial spaces is \mathcal{A} -cartesian if the underlying map of semisimplicial spaces has this property.

Proposition 6.7 *Let $f_{\bullet}: X_{\bullet} \rightarrow Y_{\bullet}$ be an \mathcal{A} -cartesian map of simplicial spaces. Then the diagram (6.6) is \mathcal{A} -cartesian.*

The proof will be sequence of lemmas, each of which extends the class of base spaces Y_{\bullet} for which the conclusion of Proposition 6.7 holds. To this end, let us say that a simplicial space Y_{\bullet} is *basic* if for every \mathcal{A} -cartesian map of simplicial spaces $f_{\bullet}: X_{\bullet} \rightarrow Y_{\bullet}$ the diagram (6.6) is \mathcal{A} -cartesian. Given this definition, the statement of Proposition 6.7 is that every simplicial space is basic.

Lemma 6.8 *If Y_{\bullet} is a simplicial set with contractible geometric realisation then it is basic.*

Proof The proof only uses the semisimplicial structure. Let $y \in Y_0$ be a basepoint. Since $\|Y_\bullet\|$ is contractible, the natural map $\eta\colon \operatorname{hofib}_y(\|f_\bullet\|) \rightarrow \|X_\bullet\|$ is a weak equivalence. Hence any coefficient system \mathcal{L}' on $\operatorname{hofib}_y(\|f_\bullet\|)$ is of the form $\eta^*\mathcal{L}$ for a coefficient system on $\|X_\bullet\|$, and if \mathcal{L}' lies in the class \mathcal{A} , then so does \mathcal{L} . Therefore, we have to prove that for each point $y \in Y_0$, the inclusion map $j\colon f_0^{-1}(y) \rightarrow \|X_\bullet\|$ induces an isomorphism $H_*(f_0^{-1}(y); j^*\mathcal{L}) \rightarrow H_*(\|X_\bullet\|; \mathcal{L})$.

The spectral sequence of the semisimplicial space X_\bullet with coefficients in \mathcal{L} discussed in Section 1.4 takes the form

$$E^1_{p,q} = H_q(X_p; \mathcal{L}_p) \Rightarrow H_{p+q}(\|X_\bullet\|; \mathcal{L}).$$

Since Y_p is discrete, we can write the E^1 -term as

$$H_q(X_p; \mathcal{L}_p) = \bigoplus_{s \in Y_p} H_q(f^{-1}(s); \mathcal{L}_p|_{f^{-1}(s)}).$$

To simplify notation, we write $H_q(f^{-1}(s); \mathcal{L}_p) := H_q(f^{-1}(s); \mathcal{L}_p|_{f^{-1}(s)})$. Because the map f_\bullet is \mathcal{A} -cartesian, the map $H_q(f^{-1}(s); \mathcal{L}_p) \rightarrow H_q(f^{-1}(d_i s); \mathcal{L}_{p-1})$ induced by the face map d_i is an isomorphism. Hence $s \mapsto H_q(f^{-1}(s); \mathcal{L}_p)$ is a locally constant coefficient system $H_q(f; \mathcal{L})$ on the simplicial set Y_\bullet . Hence $E^2_{p,q} = H_p(\|Y_\bullet\|; H_q(f; \mathcal{L}))$. Because $\|Y_\bullet\|$ is contractible, it follows that $E^2_{p,q} = 0$ for $p > 0$. If $y \in Y_0$ is a basepoint, the induced map $\Delta^0_\bullet \rightarrow Y_\bullet$ of simplicial sets gives a comparison diagram

$$\begin{array}{ccc} f^{-1}(s) & \longrightarrow & X_\bullet \\ \downarrow & & \downarrow f_\bullet \\ \Delta^0_\bullet & \xrightarrow{y} & Y_\bullet \end{array}$$

It induces an isomorphism on the E^2 -term of the spectral sequence, and therefore $f^{-1}(y) \rightarrow \|X_\bullet\|$ induces an isomorphism in homology with coefficients in \mathcal{L} , as claimed. □

The next step is a discretisation argument. For a simplicial space Y_\bullet , we consider the bisimplicial set $(p, q) \mapsto \operatorname{Sing}_q Y_p$ and the associated diagonal simplicial set $\delta Y_p := \operatorname{Sing}_p Y_p$. By Theorem 7.1, Lemma 1.11 and Theorem 2.2, the maps

$$\|\delta Y_\bullet\| \xrightarrow{D} \|\operatorname{Sing}_\bullet Y_\bullet\| \rightarrow \|Y_\bullet\|$$

are weak equivalences.

Lemma 6.9 *If Y_\bullet is a simplicial space such that δY_\bullet is basic, then Y_\bullet is basic.*

Proof The proof uses the simplicial structure in an essential way. As in the second proof of [Theorem 2.12](#) we may assume that $f_p: X_p \rightarrow Y_p$ is a fibration for each p . Let $Y_{p,q} := \text{Sing}_q Y_p$, giving a bisimplicial set $Y_{\bullet,\bullet}$, and define a bisimplicial space $X_{\bullet,\bullet}$ and a map $f_{\bullet,\bullet}: X_{\bullet,\bullet} \rightarrow Y_{\bullet,\bullet}$ as follows. Let $X_{p,q} := \coprod_{\sigma \in Y_{p,q}} \text{Lift}(\sigma, f_p)$, where $\text{Lift}(\sigma, f_p)$ is the space of all maps $h: \Delta^q \rightarrow X_p$ with $f_p \circ h = \sigma$, equipped with the compact–open topology. The simplicial structure in the p direction is given by $h \mapsto d_i \circ h$ and in the q –direction by $h \mapsto h \circ d^j$ (similarly for the degeneracy maps). The evident maps $f_{p,q}: X_{p,q} \rightarrow Y_{p,q}$ are the components of a bisimplicial map. Because f_p is a fibration, the map $f_{p,q}^{-1}(\sigma) \rightarrow f_{p,q-1}^{-1}(d_i \sigma)$ is a weak equivalence for each q and i . Hence the simplicial map $f_{p,\bullet}: X_{p,\bullet} \rightarrow Y_{p,\bullet}$ is homotopy cartesian.

Analogous to the evaluation map $\|Y_{p,\bullet}\| \rightarrow Y_p$, let $u_p: \|X_{p,\bullet}\| \rightarrow X_p$ be the map which sends $(h, t) \in X_{p,q} \times \Delta^q$ to $h(t) \in X_p$. These are the components of a map of simplicial spaces, and the diagram

$$\begin{array}{ccccc} X_{p,0} & \longrightarrow & \|X_{p,\bullet}\| & \xrightarrow{u_p} & X_p \\ \downarrow f_{p,0} & & \downarrow \|f_{p,\bullet}\| & & \downarrow f_p \\ Y_{p,0} & \longrightarrow & \|Y_{p,\bullet}\| & \longrightarrow & Y_p \end{array}$$

commutes. As $f_{p,\bullet}$ is homotopy cartesian, it follows from [Theorem 2.12](#) that the left-hand square is homotopy cartesian. The space $Y_{p,0}$ is Y_p with the discrete topology, and $f_{p,0}^{-1}(y) = f_p^{-1}(y)$. Therefore, the outer rectangle is homotopy cartesian. Moreover, $Y_{p,0} \rightarrow \|Y_{p,\bullet}\|$ is 0–connected, so it follows that the right-hand square is homotopy cartesian as well. The bottom right-hand map is a weak equivalence by [Lemma 1.11](#), so the map u_p is also a weak equivalence.

So far, we set the stage for the following diagonal argument. Consider the commutative square

$$\begin{array}{ccccccc} X_{0,0} & \longrightarrow & \|\delta X_\bullet\| & \xrightarrow{\sim} & \|X_{\bullet,\bullet}\| & \xrightarrow[\|u_\bullet\|]{\sim} & \|X_\bullet\| \\ \downarrow f_{0,0} & & \downarrow \delta f_\bullet & & \downarrow \|f_{\bullet,\bullet}\| & & \downarrow \|f_\bullet\| \\ Y_{0,0} & \longrightarrow & \|\delta Y_\bullet\| & \xrightarrow{\sim} & \|Y_{\bullet,\bullet}\| & \xrightarrow{\sim} & \|Y_\bullet\| \end{array}$$

where the weak equivalences in the middle come from [Theorem 7.1](#).

Since f_p is a fibration, and the original map f_\bullet was \mathcal{A} –cartesian, it follows that $f_{\bullet,\bullet}$ is \mathcal{A} –cartesian (in the obvious sense: we require that the diagrams in [Definition 2.9](#) to

be \mathcal{A} -cartesian in both simplicial directions), and hence that δf_\bullet is \mathcal{A} -cartesian. By the hypothesis of the lemma, δY_\bullet is basic and so the left square is \mathcal{A} -cartesian. Since the other horizontal maps are weak equivalences, it follows that the outer rectangle is \mathcal{A} -cartesian, which concludes the proof. \square

The next step is to show that the property of being basic descends along homotopy cartesian maps.

Lemma 6.10 *Let $h_\bullet: Z_\bullet \rightarrow Y_\bullet$ be a homotopy cartesian map of simplicial spaces and assume that h_0 is 0-connected. If Z_\bullet is basic then Y_\bullet is basic.*

Proof The proof only uses the semisimplicial structure. Let $f_\bullet: X_\bullet \rightarrow Y_\bullet$ be a \mathcal{A} -cartesian map of simplicial spaces. As in the second proof of [Theorem 2.12](#) we may assume that each f_p is a fibration. We form the levelwise pullback

$$(6.11) \quad \begin{array}{ccc} W_p & \xrightarrow{k_p} & X_p \\ \downarrow g_p & & \downarrow f_p \\ Z_p & \xrightarrow{h_p} & Y_p \end{array}$$

and this diagram is homotopy cartesian, because f_p is a fibration. The map g_\bullet is \mathcal{A} -cartesian. To see this, let $z \in Z_p$ be a point and consider the commutative diagram

$$\begin{array}{ccc} \mathrm{hofib}_z(g_p) & \longrightarrow & \mathrm{hofib}_{d_i z}(g_{p-1}) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{hofib}_{h_p(z)}(f_p) & \longrightarrow & \mathrm{hofib}_{d_i h_p(z)}(f_{p-1}) \end{array}$$

and use that \mathcal{A} -equivalences satisfy the 2-out-of-3 property. A similar argument (using also [Remark 2.8](#)) shows that k_\bullet is homotopy cartesian. The square

$$\begin{array}{ccc} \|W_\bullet\| & \xrightarrow{\|k_\bullet\|} & \|X_\bullet\| \\ \downarrow \|g_\bullet\| & & \downarrow \|f_\bullet\| \\ \|Z_\bullet\| & \xrightarrow{\|h_\bullet\|} & \|Y_\bullet\| \end{array}$$

is homotopy cartesian. This follows by applying [Theorem 2.12](#) to both h_\bullet and k_\bullet , using that (6.11) is homotopy cartesian for $p = 0$ and using [Remark 2.8](#). Since (6.11)

for $p = 0$ is homotopy cartesian, comparing homotopy fibres gives a commutative square

$$\begin{array}{ccc} \mathrm{hofib}_z(g_0) & \xrightarrow{\cong} & \mathrm{hofib}_{h_0(z)}(f_0) \\ \downarrow & & \downarrow \\ \mathrm{hofib}_{\iota(z)}(\|g_\bullet\|) & \xrightarrow{\cong} & \mathrm{hofib}_{\iota(z)}(\|f_\bullet\|) \end{array}$$

in which the horizontal maps are weak equivalences. Since $g_\bullet \colon W_\bullet \rightarrow Z_\bullet$ is \mathcal{A} -cartesian, and by assumption Z_\bullet is basic, it follows that the left vertical map is an \mathcal{A} -equivalence. Therefore, the right vertical map is also an \mathcal{A} -equivalence. This holds for any $z \in Z_0$, but the map h_0 is 0-connected, which finishes the proof. \square

The next lemma provides an appropriate resolution of a simplicial *set* by a contractible simplicial space.

Lemma 6.12 *Let Y_\bullet be a 0-connected simplicial set. Then there is a simplicial space QY_\bullet with $\|QY_\bullet\| \simeq *$ and a homotopy cartesian morphism $f_\bullet \colon QY_\bullet \rightarrow Y_\bullet$ such that f_0 is 0-connected.*

The same statement is true for semisimplicial sets, with the same proof.

Proof Fix a vertex $y \in Y_0$. For each simplex $\sigma \in Y_p$, we let $\chi_\sigma \colon \Delta^p \rightarrow \|Y_\bullet\|$ denote its characteristic map. Furthermore, we view $\Delta^p \subset \Delta^{p+1}$ as the last face, ie the face opposite to e_{p+1} . We let

$$QY_p := \coprod_{\sigma \in Y_p} \{(\sigma, h) \mid h \colon \Delta^{p+1} \rightarrow \|Y_\bullet\|, h|_{\Delta^p} = \chi_\sigma, h(e_{p+1}) = y\},$$

topologised as a subspace of $Y_p \times \|Y_\bullet\|^{\Delta^p}$. Define $d_i \colon QY_p \rightarrow QY_{p-1}$ by $d_i(\sigma, h) := (d_i\sigma, h \circ d^i)$ (and the degeneracy maps in an analogous way) and $f_p \colon QY_p \rightarrow Y_p$ by $f_p(\sigma, h) := \sigma$. Then $f_\bullet \colon QY_\bullet \rightarrow Y_\bullet$ is a map of simplicial spaces.

This should be viewed as an analogue of the path fibration, and we now verify that indeed it has the characteristic properties of that construction. The maps $d_i \colon f_p^{-1}(\sigma) \rightarrow f_{p-1}^{-1}(\sigma)$ are homotopy equivalences, so f_\bullet is homotopy cartesian. It remains to show that $\|QY_\bullet\| \simeq *$.

First observe that the fibre $f_0^{-1}(y)$ is the based loop space $\Omega_y \|Y_\bullet\|$. Let $P_y \|Y_\bullet\|$ denote the path space, that is, the space of all paths in $\|Y_\bullet\|$ with endpoint y . The map

$$g \colon \|QY_\bullet\| \rightarrow P_y \|Y_\bullet\|, \quad (\sigma, h, t) \mapsto (s \mapsto h((1-s)t, s)),$$

makes the diagram

$$\begin{array}{ccccc}
 \Omega_y \|Y_\bullet\| & \longrightarrow & \|QY_\bullet\| & \xrightarrow{g} & P_y \|Y_\bullet\| \\
 \downarrow & & \downarrow \|f_\bullet\| & & \downarrow \text{ev}_0 \\
 \{y\} & \longrightarrow & \|Y_\bullet\| & \xlongequal{\quad} & \|Y_\bullet\|
 \end{array}$$

commute, by inspection. Since f_\bullet is homotopy cartesian, the left-hand square is homotopy cartesian. The outer rectangle is also homotopy cartesian, as it is cartesian and ev_0 is a fibration. Thus the map between vertical homotopy fibres over y of the right-hand square is an equivalence; this holds for all y , so the right-hand square is homotopy cartesian, and hence g is a weak equivalence. Thus $\|QY_\bullet\| \simeq *$, as desired. \square

The deduction of [Proposition 6.7](#) is fairly easy.

Proof of Proposition 6.7 We have to show that every simplicial space Y_\bullet is basic. It is no loss of generality to assume that $\|Y_\bullet\|$ is 0-connected. Using the construction from [Lemma 6.12](#), we consider the simplicial set $\delta(Q(\delta Y))_\bullet$. This is contractible (by [Lemma 6.12](#) and [Theorem 7.1](#)) so, by [Lemma 6.8](#), $\delta(Q(\delta Y))_\bullet$ is basic. By [Lemma 6.9](#), it follows that $Q(\delta Y)_\bullet$ is basic. As the map $f_\bullet: Q(\delta Y)_\bullet \rightarrow \delta Y_\bullet$ provided by [Lemma 6.12](#) is homotopy cartesian and f_0 is 0-connected, it follows from [Lemma 6.10](#) that δY_\bullet is basic. Finally, using [Lemma 6.9](#) again, it follows that Y_\bullet is basic. \square

Proof of Theorem 6.5 By [Proposition 6.7](#), every simplicial space is basic. We will make use of the functor $E: \text{ssTop} \rightarrow \text{sTop}$ which freely adds degeneracies. Let $f_\bullet: X_\bullet \rightarrow Y_\bullet$ be an \mathcal{A} -cartesian map of semisimplicial spaces, giving a map $Ef_\bullet: EX_\bullet \rightarrow EY_\bullet$ of simplicial spaces. It follows from the description of the simplices and face maps of EY_\bullet that Ef_\bullet is also \mathcal{A} -cartesian. Consider the commutative diagram

$$\begin{array}{ccccc}
 X_0 & \longrightarrow & \|X_\bullet\| & \longrightarrow & \|EX_\bullet\| \\
 \downarrow f_0 & & \downarrow \|f_\bullet\| & & \downarrow \|Ef_\bullet\| \\
 Y_0 & \longrightarrow & \|Y_\bullet\| & \longrightarrow & \|EY_\bullet\|
 \end{array}$$

The simplicial space EY_\bullet is basic by [Proposition 6.7](#), so as $EX_0 = X_0$ and $EY_0 = Y_0$ we have that the outer rectangle is \mathcal{A} -cartesian. As the two rightmost horizontal maps are weak equivalences, by [Lemma 2.6](#) it follows that the left-hand square is \mathcal{A} -cartesian, as claimed. \square

6.2 Group-completion

Let us describe the application of [Theorem 6.5](#) to group-completion. Let M be a (topological) monoid acting on the left on a space X and on the right on a space Y . One may form the *two-sided bar construction* $B_\bullet(Y, M, X)$, the semisimplicial space having p -simplices $Y \times M^p \times X$, with face maps

$$\begin{aligned} d_0(y, m_1, \dots, m_p, x) &= (y \cdot m_1, m_2, \dots, m_p, x), \\ d_i(y, m_1, \dots, m_p, x) &= (y, m_1, \dots, m_{i-1}, m_i \cdot m_{i+1}, m_{i+2}, \dots, m_p, x) \quad \text{for } 0 < i < p, \\ d_p(y, m_1, \dots, m_p, x) &= (y, m_1, m_2, \dots, m_p \cdot x). \end{aligned}$$

Now let $Y = *$ and suppose that M acts on X by \mathcal{A} -equivalences. Then the projection map $B_\bullet(*, M, X) \rightarrow B_\bullet(*, M, *)$ is \mathcal{A} -cartesian, and so by [Theorem 6.5](#) the square

$$(6.13) \quad \begin{array}{ccc} X & \longrightarrow & \|B_\bullet(*, M, X)\| \\ \downarrow f_0 & & \downarrow \|f_\bullet\| \\ \{*\} & \longrightarrow & \|B_\bullet(*, M, *)\| = BM \end{array}$$

is \mathcal{A} -cartesian.

We apply this as follows. Suppose that the set of path components of M is countable and let $m_1, m_2, m_3, \dots \in M$ be a sequence of points with infinitely many lying in each path component. We may form the homotopy colimit

$$M_\infty = \operatorname{hocolim}(M \xrightarrow{-m_1} M \xrightarrow{-m_2} M \xrightarrow{-m_3} \dots)$$

over right multiplication in the monoid M by the m_i ; this has a residual left M -action. If the monoid M is homotopy commutative, then $H_*(M; \mathbb{Z})$ has the structure of a commutative ring, and we can identify

$$H_*(M_\infty; \mathbb{Z}) \cong \operatorname{colim}(H_*(M; \mathbb{Z}) \xrightarrow{(-m_1)_*} H_*(M; \mathbb{Z}) \xrightarrow{(-m_2)_*} H_*(M; \mathbb{Z}) \xrightarrow{(-m_3)_*} \dots)$$

with the localisation $H_*(M; \mathbb{Z})[\pi_0(M)^{-1}]$ of the ring $H_*(M)$ at the multiplicative subset $\pi_0(M) \subset H_0(M; \mathbb{Z})$. In particular, the map $m \cdot -: M_\infty \rightarrow M_\infty$ given by left multiplication by m induces an isomorphism on homology. We may thus apply the above observation to the left action of M on M_∞ . Now $B_\bullet(*, M, M)$ has an extra degeneracy (as in [Lemma 1.12](#)), so $\|B_\bullet(*, M, M)\| \simeq *$ and hence

$$\|B_\bullet(*, M, M_\infty)\| \simeq \operatorname{hocolim}(\|B_\bullet(*, M, M)\| \xrightarrow{-m_1} \|B_\bullet(*, M, M)\| \rightarrow \dots) \simeq *.$$

The homology-cartesian square (6.13) therefore gives a map

$$(6.14) \quad M_\infty \rightarrow \text{hofib}_*(\|B_\bullet(*, M, M_\infty)\| \rightarrow \|B_\bullet(*, M, *)\|) \simeq \Omega BM$$

which is an integral homology equivalence; in particular,

$$H_*(M; \mathbb{Z})[\pi_0(M)^{-1}] \cong H_*(\Omega BM; \mathbb{Z}).$$

Remark 6.15 In fact, the argument of [27] shows that in the situation above the monoid M acts on M_∞ by abelian homology equivalences, and so the map (6.14) is an abelian homology equivalence, but the fundamental group of the target is abelian, so it follows that (6.14) is in fact an acyclic map.

There is also a group-completion theorem for categories, rather than monoids: it can also be deduced immediately from Theorem 6.5; we refer the reader to [12, Section 7] for a formulation.

7 Products of simplicial spaces

Let $X_{\bullet, \bullet}$ be a bisimplicial space and let $\delta(X_{\bullet, \bullet})$ be the diagonal simplicial space. To define the *diagonal map* $D: \|\delta(X_{\bullet, \bullet})\| \rightarrow \|X_{\bullet, \bullet}\|$, take the diagonal map $d: \Delta^p \rightarrow \Delta^p \times \Delta^p$ and

$$(\text{Id}_{X_{p,p}} \times d): X_{p,p} \times \Delta^p \rightarrow X_{p,p} \times \Delta^p \times \Delta^p.$$

This respects the equivalence relations used for the definition of the fat geometric realisation and so induces a map D as indicated.

Theorem 7.1 *The diagonal map D is a weak equivalence.*

This is false if one considers bisemisimplicial spaces instead: if Y_\bullet is an arbitrary semisimplicial space and $X_{\bullet, \bullet} = \nabla_\bullet^0 \otimes Y_\bullet$, then $\|X_{\bullet, \bullet}\| = \|Y_\bullet\|$ and $\|\delta(X)_{\bullet}\| = Y_0$. Let us note an application of Theorem 7.1.

Theorem 7.2 *Let X_\bullet and Y_\bullet be simplicial spaces. Then the map*

$$\|(X \times Y)_\bullet\| \rightarrow \|X_\bullet\| \times \|Y_\bullet\|,$$

induced by the two projection maps $(X \times Y)_\bullet \rightarrow X_\bullet$ and $(X \times Y)_\bullet \rightarrow Y_\bullet$, is a weak homotopy equivalence.

Proof The diagram

$$\begin{array}{ccc} \|(X \times Y)_\bullet\| & \longrightarrow & \|X_\bullet\| \times \|Y_\bullet\| \\ \parallel & & \cong \uparrow \\ \|\delta(X \otimes Y)_\bullet\| & \xrightarrow{\simeq} & \|X_\bullet\| \otimes \|Y_\bullet\| \end{array}$$

commutes, and the indicated homeomorphism and weak equivalence are true by [Theorem 7.1](#) and [\(1.10\)](#). □

[Theorem 7.2](#) is important when one applies Segal’s theory of Γ –spaces to deloop spaces which arise as geometric realisations of simplicial spaces. This will be done in [\[5\]](#) and has been done at various places in the literature.

One could derive [Theorem 7.1](#) from the classical result [\[13, Theorem I.3.7\]](#) that for a bisimplicial set one has a homeomorphism $|\delta(X_\bullet, \bullet)| \cong |X_\bullet, \bullet|$ and [\[31, Proposition A.1\]](#). However, it seems to be easier to give an argument from scratch. The main bulk of work for the proof of [Theorem 7.1](#) is the proof for bisimplicial *sets*, and the proof of that case resembles in some sense the proof of the classical Eilenberg–Zilber theorem in singular homology, using the method of acyclic models. The first step is to prove that the “models” are contractible.

Lemma 7.3 *Let $\Delta_{\bullet, \bullet}^{n, m} := \Delta_\bullet^n \otimes \Delta_\bullet^m$ be the “bisimplicial (n, m) –simplex”. The spaces $\|\Delta_{\bullet, \bullet}^{n, m}\|$ and $\|\delta(\Delta_{\bullet, \bullet}^{n, m})_\bullet\|$ are contractible. In particular, [Theorem 7.1](#) is true when $X_{\bullet, \bullet} = \Delta_{\bullet, \bullet}^{n, m}$.*

Proof By [\(1.10\)](#) and [Example 1.14](#), we have

$$\|\Delta_{\bullet, \bullet}^{n, m}\| \cong \|\Delta_\bullet^n\| \times \|\Delta_\bullet^m\| \simeq *.$$

To prove that $\|\delta(\Delta_{\bullet, \bullet}^{n, m})_\bullet\| \simeq *$, consider the ordered set $[n]$ as a (unital) category. Then Δ_\bullet^n is the nerve of $[n]$. Moreover, $\delta(\Delta_{\bullet, \bullet}^{n, m})_\bullet$ is the nerve of the category $[n] \times [m]$. This category has a terminal object, namely (n, m) , so a natural transformation from the identity functor to a constant functor. It follows from [Lemma 3.4](#) that $\|\delta(\Delta[n, m])_\bullet\|$ is contractible. □

It is in this step that the degeneracies are used. The analogous claim for bisemisimplicial sets is false. The role of $\Delta_{\bullet, \bullet}^{n, m}$ is then taken by $\nabla_{\bullet, \bullet}^{n, m} := \nabla_\bullet^n \otimes \nabla_\bullet^m$. While $\|\nabla_{\bullet, \bullet}^{n, m}\|$ is contractible, $\|\delta(\nabla_{\bullet, \bullet}^{n, m})_\bullet\|$ usually is not. This may be seen by calculating the Euler number of these finite complexes.

The identity $\text{Id}_{[n]}$ defines an element $\iota_n \in \Delta_n^n$ and its characteristic map $\hat{\iota}_n: \Delta^n \rightarrow \|\Delta_\bullet^n\|^{(n)} \subset \|\Delta_\bullet^n\|$. The restriction to the topological boundary $\partial\Delta^n$ goes into the $(n-1)$ -skeleton $\|\Delta_\bullet^n\|^{(n-1)}$ and is denoted by $\partial\hat{\iota}_n$. In a similar vein, the tautological element $\iota_{n,m} = (\iota_n, \iota_m) \in \Delta_{n,m}^{n,m}$ induces a map $\hat{\iota}_{n,m}: \Delta^n \times \Delta^m \rightarrow \|\Delta_{\bullet,\bullet}^{n,m}\|^{n+m}$ with boundary $\partial\hat{\iota}_{n,m}: (\Delta^n \times \partial\Delta^m \cup \partial\Delta^n \times \Delta^m) =: \partial(\Delta^n \times \Delta^m) \rightarrow \|\Delta_{\bullet,\bullet}^{n,m}\|^{n+m-1}$. Moreover, composition with the diagonal map $d: \Delta^n \rightarrow \Delta^n \times \Delta^n$ (whose restriction to $\partial\Delta^n$ goes into $\partial(\Delta^n \times \Delta^n)$) defines a map $\hat{\iota}_{n,n} \circ d: \Delta^n \rightarrow \|\delta(\Delta^{n,n})_\bullet\|^{(n)}$, with boundary map $\partial(\hat{\iota}_{n,n} \circ d): \partial\Delta^n \rightarrow \|\delta(\Delta^{n,n})_\bullet\|^{(n-1)}$.

Note that $X_{\bullet,\bullet} \mapsto \|\delta(X)_\bullet\|$ and $X_{\bullet,\bullet} \mapsto \|X_{\bullet,\bullet}\|$ are functors from the category of bisimplicial sets to **Top** and the diagonal map D is a natural transformation. Moreover, both $\|\delta(X)_\bullet\|$ and $\|X_{\bullet,\bullet}\|$ are naturally filtered spaces; their 0-skeleta are equal, that is,

$$\|\delta(X)_\bullet\|^{(0)} = \|X_{\bullet,\bullet}\|^{(0)} = X_{0,0};$$

and D restricts to the identity between the 0-skeleta.

Lemma 7.4 (i) *There is a natural map $F: \|X_{\bullet,\bullet}\| \rightarrow \|\delta(X)_\bullet\|$ which is the identity on the 0-skeleton.*

(ii) *The map $D \circ F: \|X_{\bullet,\bullet}\| \rightarrow \|X_{\bullet,\bullet}\|$ is naturally homotopic to the identity.*

(iii) *The map $F \circ D: \|\delta(X)_\bullet\| \rightarrow \|\delta(X)_\bullet\|$ is naturally homotopic to the identity.*

In particular, D is a homotopy equivalence for each bisimplicial set.

One can add the statements that the maps F and D are unique up to natural homotopy among those natural maps which are the identity on the 0-skeleton. These statements will not enter the proof of [Theorem 7.1](#) and so we do not prove them, but the method of proof can easily be adapted.

Proof We shall construct the map F and the homotopies inductively on skeleta. More precisely, we shall construct natural maps

$$F_n = F_n^X: \|X_{\bullet,\bullet}\|^{(n)} \rightarrow \|\delta(X)_\bullet\|$$

and natural homotopies

$$h_n: F \circ D_n \leadsto \text{Id}, \quad k_n: D \circ F_n \leadsto \text{Id}.$$

We begin with the construction of F_n . The map F_0 is the identity, and we assume that F_0, \dots, F_{n-1} are already constructed. Let $p + q = n$, and we first construct a suitable

map $\mu_{p,q}: \|\Delta_{\bullet,\bullet}^{p,q}\|^{(n)} \rightarrow \|\delta(\Delta^{p,q})_{\bullet}\|$. The inclusion map $\|\Delta_{\bullet,\bullet}^{p,q}\|^{(n-1)} \rightarrow \|\Delta_{\bullet,\bullet}^{p,q}\|^{(n)}$ is a cellular inclusion. By [Lemma 7.3](#), the space $\|\delta(\Delta^{p,q})_{\bullet}\|$ is contractible. Hence there exists a solution $\mu_{p,q}$ to the extension problem

$$\begin{array}{ccc} \|\Delta_{\bullet,\bullet}^{p,q}\|^{(n-1)} & \xrightarrow{F_{n-1}^{\Delta^{p,q}}} & \|\delta(\Delta^{p,q})_{\bullet}\| \\ \downarrow & \nearrow \mu_{p,q} & \\ \|\Delta_{\bullet,\bullet}^{p,q}\|^{(n)} & & \end{array}$$

Now we construct F_n^X for a bisimplicial set X . Observe that

$$X_{p,q} = \text{bisSet}(\Delta_{\bullet,\bullet}^{p,q}, X_{\bullet,\bullet}),$$

the set of morphisms of bisimplicial sets (this is an instance of the Yoneda lemma). For each $s \in X_{p,q}$, we have the characteristic map $\hat{s}: \Delta^p \times \Delta^q \rightarrow \|X_{\bullet,\bullet}\|^{p+q}$, and if we view s as a map of bisimplicial sets, we obtain $\|s\|: \|\Delta_{\bullet,\bullet}^{p,q}\| \rightarrow \|X_{\bullet,\bullet}\|$. The relation between these two maps is that $\|s\| \circ \hat{l}_{p,q} = \hat{s}$. The following diagram is a pushout diagram:

$$\begin{array}{ccc} \coprod_{\substack{p+q=n \\ s \in X_{p,q}}} \partial \Delta^{p,q} & \xrightarrow{\varphi} & \|X_{\bullet,\bullet}\|^{(n-1)} \\ \downarrow \text{inc} & & \downarrow \\ \coprod_{\substack{p+q=n \\ s \in X_{p,q}}} \Delta^{p,q} & \xrightarrow{\phi} & \|X_{\bullet,\bullet}\|^{(n)} \end{array}$$

where the map ϕ is

$$\phi = \coprod_{\substack{p+q=n \\ s \in X_{p,q}}} \hat{s} = \coprod_{\substack{p+q=n \\ s \in X_{p,q}}} \|s\|^{(n)} \circ \hat{l}_{p,q}$$

and similarly

$$\varphi = \coprod_{\substack{p+q=n \\ s \in X_{p,q}}} \|s\|^{(n-1)} \circ \partial \hat{l}_{p,q}.$$

We claim that the two maps

$$F_{n-1}^{X_{\bullet,\bullet}} \circ \varphi, \quad \coprod_{\substack{p+q=n \\ s \in X_{p,q}}} \|\delta(s)_{\bullet}\| \circ \mu_{p,q} \circ \hat{l}_{p,q} \circ \text{inc} : \coprod_{\substack{p+q=n \\ s \in X_{p,q}}} \|\Delta_{\bullet,\bullet}^{p,q}\|^{(n-1)} \rightarrow \|\delta(X)_{\bullet}\|$$

are equal, and so they induce a map from the pushout, ie $F_n^X: \|X_{\bullet, \bullet}\|^{(n)} \rightarrow \|\delta(X)_{\bullet}\|$, which finishes the inductive construction of F_n (it is obvious that F_n becomes a natural map). To verify this claim, it is enough to check that for each $s \in X_{p,q}$, the diagram

$$\begin{array}{ccc} \|\Delta_{\bullet, \bullet}^{p,q}\|^{(n-1)} & \xrightarrow{\|s\|^{(n-1)}} & \|X_{\bullet, \bullet}\|^{(n-1)} \\ \downarrow & & \searrow F_{n-1}^X \\ \|\Delta_{\bullet, \bullet}^{p,q}\|^{(n)} & \xrightarrow{\mu_{p,q}} & \|\delta(\Delta^{p,q})_{\bullet}\| \xrightarrow{\|\delta(s)\|} \|\delta(X)_{\bullet}\| \end{array}$$

commutes. But this is clear because F_{n-1} is a natural transformation:

$$F_{n-1}^X \circ \|s\|^{(n-1)} = \|\delta(s)_{\bullet}\| \circ F_{n-1}^{\Delta^{p,q}},$$

and we constructed $\mu_{p,q}$ so that $\mu_{p,q} \circ \text{inc} = F_{n-1}^{\Delta^{p,q}}$. This finishes the construction of F .

Now we turn to the construction of natural homotopies $h_n: I \times \|\delta(X)_{\bullet}\|^{(n)} \rightarrow \|\delta(X)_{\bullet}\|$ from $F \circ D_n$ to the “identity” (ie inclusion map). We can take h_0 to be the constant homotopy. Assume that h_0, \dots, h_{n-1} are already constructed. As before, we first construct a certain map $\lambda_n: I \times \|\delta(\Delta^{n,n})_{\bullet}\|^{(n)} \rightarrow \|\delta(\Delta^{n,n})_{\bullet}\|$. The inclusion map

$$I \times \|\delta(\Delta^{n,n})_{\bullet}\|^{(n-1)} \cup \{0, 1\} \times \|\delta(\Delta^{n,n})_{\bullet}\|^{(n)} \rightarrow I \times \|\delta(\Delta^{n,n})_{\bullet}\|^{(n)}$$

is a cellular inclusion. We define a map

$$I \times \|\delta(\Delta^{n,n})_{\bullet}\|^{(n-1)} \cup \{0, 1\} \times \|\delta(\Delta^{n,n})_{\bullet}\|^{(n)} \rightarrow \|\delta(\Delta^{n,n})_{\bullet}\|$$

by taking the homotopy $h_{n-1}^{\Delta^{n,n}}$ on the first part, $F \circ D_n$ on $\{0\} \times \|\delta(\Delta^{n,n})_{\bullet}\|^{(n)}$ and the “identity” on $\{1\} \times \|\delta(\Delta^{n,n})_{\bullet}\|^{(n)}$. Those fit together by assumption and so define a continuous map. It can be extended to a map

$$\lambda_n: I \times \|\delta(\Delta^{n,n})_{\bullet}\|^{(n)} \rightarrow \|\delta(\Delta^{n,n})_{\bullet}\|,$$

because the target space is contractible by [Lemma 7.3](#). There is a pushout diagram

$$\begin{array}{ccc} \coprod_{s \in X_{n,n}} I \times \partial \Delta^n & \xrightarrow{\varphi} & I \times \|\delta(X)_{\bullet}\|^{(n-1)} \\ \downarrow \text{inc} & & \downarrow \\ \coprod_{s \in X_{n,n}} I \times \Delta^n & \xrightarrow{\phi} & I \times \|\delta(X)_{\bullet}\|^{(n)} \end{array}$$

whose horizontal maps are given by

$$\varphi = \text{Id}_I \times \left(\coprod_{s \in X_{n,n}} \|\delta(s)\|^{(n-1)} \circ \partial \hat{\iota}_n \right)$$

and

$$\phi = \text{Id}_I \times \left(\coprod_{s \in X_{n,n}} \|\delta(s)\|^{(n)} \circ \hat{\iota}_n \right).$$

Let $\psi: \coprod_{s \in X_{n,n}} I \times \Delta^n \rightarrow \|\delta(X)_\bullet\|$ be the map

$$\coprod_{s \in X_{n,n}} \|s\| \circ \lambda_n \circ (\text{Id}_I \times \hat{\iota}_n).$$

Then $\psi \circ \text{inc} = h_{n-1}^X \circ \varphi$ by construction, and so these maps together induce a map h_n^X from the pushout $I \times \|\delta(X)_\bullet\|^{(n)}$ to $\|\delta(X)_\bullet\|$ which extends h_{n-1}^X and is natural.

The construction of the homotopies k_n is very similar and left to the reader. \square

Proof of Theorem 7.1 Consider the trisimplicial set $(p, q, r) \mapsto \text{Sing}_r X_{p,q}$. The following diagram commutes:

$$\begin{array}{ccc} \|p \mapsto X_{p,p}\| & \xrightarrow{D} & \|(p, q) \mapsto X_{p,q}\| \\ \uparrow & & \uparrow \\ \|p \mapsto \|r \mapsto \text{Sing}_r X_{p,p}\|\| & \xrightarrow{D} & \|(p, q) \mapsto \|r \mapsto \text{Sing}_r X_{p,q}\|\| \\ \downarrow \cong & & \downarrow \cong \\ \|r \mapsto \|p \mapsto \text{Sing}_r X_{p,p}\|\| & \xrightarrow{D} & \|r \mapsto \|(p, q) \mapsto \text{Sing}_r X_{p,q}\|\| \end{array}$$

The upper vertical maps are weak equivalences, by Lemma 1.11 and Theorem 2.2. The lower vertical maps are the homeomorphisms from (1.9). The bottom horizontal map is a weak equivalence by Lemma 7.4 and Theorem 2.2. Hence so is the upper horizontal map, which proves the claim. \square

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