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On Ruan's cohomological crepant resolution conjecture for the complexified Bianchi orbifolds

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We give formulae for the Chen–Ruan orbifold cohomology for the orbifolds given by a Bianchi group acting on complex hyperbolic 3–space.

The Bianchi groups are the arithmetic groups $\mathrm{PSL}_2(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers in an imaginary quadratic number field. The underlying real orbifolds which help us in our study, given by the action of a Bianchi group on real hyperbolic 3–space (which is a model for its classifying space for proper actions), have applications in physics.

We then prove that, for any such orbifold, its Chen–Ruan orbifold cohomology ring is isomorphic to the usual cohomology ring of any crepant resolution of its coarse moduli space. By vanishing of the quantum corrections, we show that this result fits in with Ruan's cohomological crepant resolution conjecture.

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1 Introduction

Recently, motivated by string theory in theoretical physics, a stringy topology of orbifolds has been introduced in mathematics; see Adem, Leida and Ruan [1]. Its essential innovations consist of Chen–Ruan orbifold cohomology [11; 10] and orbifold K–theory. They are of interest as topological quantum field theories; see González, Lupercio, Segovia and Uribe [20]. Ruan's cohomological crepant resolution conjecture [41] associates Chen–Ruan orbifold cohomology with the ordinary cohomology of a resolution of the singularities of the coarse moduli space of the given orbifold. We place ourselves where the conjecture is still open: in three complex dimensions and outside the global quotient case. There, we are going to calculate the Chen–Ruan cohomology of an infinite family of orbifolds. And we prove, in Section 6, that this cohomology is isomorphic as a ring to the cohomology of their crepant resolution of singularities.

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Denoting by $\mathbb{Q}(\sqrt{-m})$, with m a square-free positive integer, an imaginary quadratic number field, and by $\mathcal{O}_{-m}$ its ring of integers, the *Bianchi groups* are the projective special linear groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$; throughout the paper, we denote them by Γ. The Bianchi groups may be considered as a key to the study of a larger class of groups, the *Kleinian* groups, which date back to work of Henri Poincaré [31]. In fact, each noncocompact arithmetic Kleinian group is commensurable with some Bianchi group; see Maclachlan and Reid [26]. A wealth of information on the Bianchi groups can be found in the monographs of Fine [15], Elstrodt, Grunewald and Mennicke [13] and Maclachlan and Reid [26]. These groups act in a natural way on real hyperbolic three-space $\mathcal{H}^3_{\mathbb{R}}$, which is isomorphic to the symmetric space associated to them. This yields orbifolds $[\mathcal{H}^3_{\mathbb{R}}/\Gamma]$ that are studied in cosmology; see Aurich, Steiner and Then [2].

The orbifold structure obtained in this way is determined by a fundamental domain and its stabilizers and identifications. The computation of this information has been implemented for all Bianchi groups; see Rahm [32].

In order to obtain complex orbifolds, we consider complex hyperbolic three-space $\mathcal{H}^3_{\mathbb{C}}$. Then $\mathcal{H}^3_{\mathbb{R}}$ is naturally embedded into $\mathcal{H}^3_{\mathbb{C}}$ as the fixed-point set of the complex conjugation (Construction 6). The action of a Bianchi group Γ on $\mathcal{H}^3_{\mathbb{R}}$ extends to an action on $\mathcal{H}^3_{\mathbb{C}}$ in a natural way. Since this action is properly discontinuous (Lemma 7), we obtain a complex orbifold $[\mathcal{H}^3_{\mathbb{C}}/\Gamma]$, which we call a *complexified Bianchi orbifold*.

The vector space structure of Chen–Ruan orbifold cohomology

Let Γ be a discrete group acting *properly discontinuously*, hence with finite stabilizers, by biholomorphisms on a complex manifold Y. For any element $g \in \Gamma$, denote by $C_\Gamma(g)$ the centralizer of g in Γ. Denote by Y^g the subspace of Y consisting of the fixed points of g.

Definition 1 [11] Let $T \subset \Gamma$ be a set of representatives of the conjugacy classes of elements of finite order in Γ. Then the Chen–Ruan orbifold cohomology vector space of $[Y/\Gamma]$ is

$$\mathrm{H}^*_{\mathrm{CR}}([Y/\Gamma]) := \bigoplus_{g \in T} \mathrm{H}^*(Y^g/C_\Gamma(g); \mathbb{Q}).$$

The grading on this vector space is reviewed in equation (1) below.

This definition is slightly different from the original one in [11], but it is equivalent to it. We can verify this fact using arguments analogous to those used by Fantechi and

Göttsche [14] in the case of a finite group Γ acting on Y. The additional argument needed when considering some element g in Γ of infinite order is the following: As the action of Γ on Y is properly discontinuous, g does not admit any fixed point in Y. Thus, $\mathrm{H}^*(Y^g/C_\Gamma(g);\mathbb{Q}) = \mathrm{H}^*(\varnothing;\mathbb{Q}) = 0$. For another proof, see [1].

Statement of the results

We first compute the Chen–Ruan orbifold cohomology for the complex orbifolds $[\mathcal{H}^3_\mathbb{C}/\Gamma]$ in the following way. In order to describe the vector space structure of $\mathrm{H}^*_{\mathrm{CR}}([\mathcal{H}^3_\mathbb{C}/\Gamma])$, we reduce our considerations on complex orbifolds to the easier case of real orbifolds. This is achieved by using Theorem 17 (Section 4), which says that there is a Γ–equivariant homotopy equivalence between $\mathcal{H}^3_\mathbb{C}$ and $\mathcal{H}^3_\mathbb{R}$.

As a result of Theorems 20 and 21, we can express the vector space structure of the Chen–Ruan orbifold cohomology in terms of the numbers of conjugacy classes of finite subgroups and the cohomology of the quotient space. Actually, Theorems 20 and 21 hold true for (finite-index subgroups in) Bianchi groups with units $\{\pm 1\}$. These latter groups are those of the groups $\mathrm{PSL}_2(\mathcal{O})$ where $\mathcal{O}$ is a ring of integers in an imaginary quadratic number field such that it admits as the only units $\{\pm 1\}$. The remaining cases are the Gaussian and Eisenstein integers, and we treat them separately in Sections 8.3 and 8.4, respectively.

More precisely, as a corollary to Theorems 20 and 21, which we are going to prove in Section 5, and using Theorem 17, we obtain:

Corollary 2 *Let Γ be a finite-index subgroup in a Bianchi group with units $\{\pm 1\}$. Denote by λ_{2n} the number of conjugacy classes of cyclic subgroups of order n in Γ. Denote by λ^*_{2n} the number of conjugacy classes of those of them which are contained in a dihedral subgroup of order $2n$ in Γ. Then*

$$\mathrm{H}^d_{\mathrm{CR}}([\mathcal{H}^3_\mathbb{C}/\Gamma]) \cong \mathrm{H}^d(\mathcal{H}^3_\mathbb{R}/\Gamma;\mathbb{Q}) \oplus \begin{cases} \mathbb{Q}^{\lambda_4+2\lambda_6-\lambda^*_6} & \text{if } d=2,\\ \mathbb{Q}^{\lambda_4-\lambda^*_4+2\lambda_6-\lambda^*_6} & \text{if } d=3,\\ 0 & \text{otherwise.}\end{cases}$$

Together with the example computations for the Gaussian and Eisensteinian cases (Sections 8.3 and 8.4), we obtain $\mathrm{H}^d_{\mathrm{CR}}([\mathcal{H}^3_\mathbb{C}/\Gamma])$ for all Bianchi groups Γ.

The (co)homology of the quotient space $\mathcal{H}^3_\mathbb{R}/\Gamma$ has been computed numerically for a large scope of Bianchi groups; see Vogtmann [45], Scheutzow [42] and Rahm [35].

Also, Krämer [24] has determined number-theoretic formulae for the numbers λ_{2n} and λ^*_{2n} of conjugacy classes of finite subgroups in the Bianchi groups.

Using the previous description of $\mathrm{H}^*_{\mathrm{CR}}([\mathcal{H}^3_{\mathbb{C}}/\Gamma])$ and Theorem 11, we can compute the Chen–Ruan cup product as follows. By degree reasons, the Chen–Ruan cup product $\alpha_g \cup_{\mathrm{CR}} \beta_h$ between cohomology classes of two twisted sectors is zero. On the other hand, if $\alpha_g \in \mathrm{H}^*((\mathcal{H}^3_{\mathbb{C}})^g/C_\Gamma(g))$ and $\beta \in \mathrm{H}^*(\mathcal{H}^3_{\mathbb{C}}/\Gamma)$, then $\alpha_g \cup_{\mathrm{CR}} \beta = \alpha_g \cup \iota_g^*\beta \in \mathrm{H}^*((\mathcal{H}^3_{\mathbb{C}})^g/C_\Gamma(g))$, where $\iota_g\colon (\mathcal{H}^3_{\mathbb{C}})^g/C_\Gamma(g) \to \mathcal{H}^3_{\mathbb{C}}/\Gamma$ is the natural map induced by the inclusion $(\mathcal{H}^3_{\mathbb{C}})^g \subset \mathcal{H}^3_{\mathbb{C}}$ (notice that, in this case, the obstruction bundle has fibre dimension zero by Theorem 11).

Let us consider now, for any complexified Bianchi orbifold $[\mathcal{H}^3_{\mathbb{C}}/\Gamma]$, its coarse moduli space $\mathcal{H}^3_{\mathbb{C}}/\Gamma$. It is a quasiprojective variety (see Baily and Borel [3]) with Gorenstein singularities (Lemma 10). Therefore, it admits a crepant resolution (see eg Roan [40] or Chen and Tseng [9]). Then, we prove the following result:

Theorem 3 *Let Γ be a Bianchi group and let $[\mathcal{H}^3_{\mathbb{C}}/\Gamma]$ be the corresponding orbifold, with coarse moduli space $\mathcal{H}^3_{\mathbb{C}}/\Gamma$. Let $f\colon Y \to \mathcal{H}^3_{\mathbb{C}}/\Gamma$ be a crepant resolution of $\mathcal{H}^3_{\mathbb{C}}/\Gamma$. Then there is an isomorphism as graded $\mathbb{Q}$–algebras between the Chen–Ruan cohomology ring of $[\mathcal{H}^3_{\mathbb{C}}/\Gamma]$ and the singular cohomology ring of Y,*

$$(H^*_{\mathrm{CR}}([\mathcal{H}^3_{\mathbb{C}}/\Gamma]), \cup_{\mathrm{CR}}) \cong (H^*(Y), \cup).$$

The proof of this theorem, which we shall give in Section 6, uses the McKay correspondence and our computations of the Chen–Ruan orbifold cohomology of the complexified Bianchi orbifolds. In Section 7, we compare this result with Ruan's cohomological crepant resolution conjecture (see Ruan [41] and Coates and Ruan [12]). Even though $\mathcal{H}^3_{\mathbb{C}}/\Gamma$ and Y are not projective varieties — hence Ruan's conjecture does not apply directly — our results confirm the validity of this conjecture.

Finally, in Section 8, we illustrate our study with the computation of some explicit examples.

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Notation

We use the following symbols for the finite subgroups in $\mathrm{PSL}_2(\mathcal{O})$: for cyclic groups of order n, we write $\mathbb{Z}/n$; for the Klein four group $\mathbb{Z}/2\times\mathbb{Z}/2$, we write $\mathcal{D}_2$; for the dihedral group with six elements, we write $\mathcal{D}_3$; and for the alternating group on four symbols, we write $\mathcal{A}_4$.

2 The orbifold cohomology product

In order to equip the orbifold cohomology vector space with a product structure, called the Chen–Ruan product, we use the complex orbifold structure on $[Y/\Gamma]$.

Let Y be a complex manifold of dimension D with a properly discontinuous action of a discrete group Γ by biholomorphisms. For any $g\in\Gamma$ and $y\in Y^g$, we consider the eigenvalues $\lambda_1,\dots,\lambda_D$ of the action of g on the tangent space T_yY. As the action of g on T_yY is complex linear, its eigenvalues are roots of unity.

Definition 4 Write $\lambda_j=e^{2\pi i r_j}$, where r_j is a rational number in the interval $[0,1)$. The *degree shifting number* of g in y is the rational number $\mathrm{shift}(g,y):=\sum_{j=1}^{D} r_j$.

The degree shifting number agrees with the original definition by Chen and Ruan (see [14]). It is also called the Fermionic shift number in [46]. The degree shifting number of an element g is constant on a connected component of its fixed-point set Y^g. For the groups under our consideration, Y^g is connected, so we can omit the argument y. Details for this and the explicit value of the degree shifting number are given in Lemma 9. Then, we can define the graded vector space structure of the Chen–Ruan orbifold cohomology as

$$\mathrm{H}^d_{\mathrm{CR}}([Y/\Gamma]) := \bigoplus_{g\in T} \mathrm{H}^{d-2\,\mathrm{shift}(g)}(Y^g/C_\Gamma(g);\mathbb{Q}). \tag{1}$$

Denote by g and h two elements of finite order in Γ, and by $Y^{g,h}$ their common fixed-point set. Chen and Ruan construct a certain vector bundle on $Y^{g,h}$, the *obstruction bundle*. We denote by $c(g,h)$ its top Chern class. In our cases, $Y^{g,h}$ is a connected manifold. In the general case, the fibre dimension of the obstruction bundle can vary between the connected components of $Y^{g,h}$, and $c(g,h)$ is the cohomology class restricting to the top Chern class of the obstruction bundle on each connected component. The obstruction bundle is at the heart of the construction [11] of the Chen–Ruan orbifold cohomology product. In [14], this product, when applied to a cohomology class associated to Y^g and one associated to Y^h, is described as a pushforward of the cup product of these classes restricted to $Y^{g,h}$ and multiplied by $c(g,h)$.

The following statement is made for global quotient orbifolds, but it is a local property, so we can apply it in our case.

Lemma 5 (Fantechi–Göttsche) *Let $Y^{g,h}$ be connected. Then the obstruction bundle on it is a vector bundle of fibre dimension*

$$\mathrm{shift}(g)+\mathrm{shift}(h)-\mathrm{shift}(gh)-\mathrm{codim}_{\mathbb{C}}(Y^{g,h}\subset Y^{gh}).$$

In [14], a proof is given in the more general setting that $Y^{g,h}$ need not be connected. Examples where the product structure is worked out in the nonglobal quotient case, are for instance given in [11, Example 5.3], [30] and [6].

2.1 Groups of hyperbolic motions

A class of examples with complex structures admitting the grading (1) is given by the discrete subgroups Γ of the orientation-preserving isometry group $\mathrm{PSL}_2(\mathbb{C})$ of real hyperbolic 3–space $\mathcal{H}^3_{\mathbb{R}}$. The Kleinian model of $\mathcal{H}^3_{\mathbb{R}}$ gives a natural identification of the orientation-preserving isometries of $\mathcal{H}^3_{\mathbb{R}}$ with matrices in $\mathrm{PSO}(1,3)$. By the subgroup inclusion $\mathrm{PSO}(1,3)\hookrightarrow \mathrm{PSU}(1,3)$, these matrices specify isometries of the complex hyperbolic space $\mathcal{H}^3_{\mathbb{C}}$. The details are as follows:

Construction 6 Given an orbifold $[\mathcal{H}^3_{\mathbb{R}}/\Gamma]$, we presently construct the complexified orbifold $[\mathcal{H}^3_{\mathbb{C}}/\Gamma]$. Recall the Kleinian model for $\mathcal{H}^3_{\mathbb{R}}$ described in [13]: For this, we take a basis $\{f_0, f_1, f_2, f_3\}$ for $\mathbb{R}^4$, and rewrite $\mathbb{R}^4$ as $\widetilde{E}_1 := \mathbb{R}f_0\oplus\mathbb{R}f_1\oplus\mathbb{R}f_2\oplus\mathbb{R}f_3$. Then, we define the quadratic form q_1 by

$$q_1(y_0f_0+y_1f_1+y_2f_2+y_3f_3)=y_0^2-y_1^2-y_2^2-y_3^2.$$

We consider the real projective 3–space $\mathbb{P}\widetilde{E}_1 = (\widetilde{E}_1 \setminus \{0\})/\mathbb{R}^*$, where $\mathbb{R}^*$ stands for the multiplicative group $\mathbb{R} \setminus \{0\}$. The set underlying the Kleinian model is then

$$\mathbb{K} := \{[y_0 : y_1 : y_2 : y_3] \in \mathbb{P}\widetilde{E}_1 \mid q_1(y_0, y_1, y_2, y_3) > 0\}.$$

Once $\mathbb{K}$ is equipped with the hyperbolic metric, its group of orientation-preserving isometries is $\mathrm{PSO}_4(q_1, \mathbb{R}) =: \mathrm{PSO}(1,3)$. The isomorphism of $\mathbb{K}$ to the upper-halfspace model of $\mathcal{H}^3_{\mathbb{R}}$ yields an isomorphism between the groups of orientation-preserving isometries, $\mathrm{PSO}(1,3) \cong \mathrm{PSL}_2(\mathbb{C})$. This is how we include Γ into $\mathrm{PSO}(1,3)$.

Now we consider complex Euclidean 4–space $\widetilde{E}_1 \otimes_{\mathbb{R}} \mathbb{C} := \mathbb{C} f_0 \oplus \mathbb{C} f_1 \oplus \mathbb{C} f_2 \oplus \mathbb{C} f_3$, and complex projective 3–space $\mathbb{P}(\widetilde{E}_1 \otimes_{\mathbb{R}} \mathbb{C}) = (\widetilde{E}_1 \otimes_{\mathbb{R}} \mathbb{C} \setminus \{0\})/\mathbb{C}^*$, and obtain a model

$$\mathbb{K}_{\mathbb{C}} := \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{P}(\widetilde{E}_1 \otimes_{\mathbb{R}} \mathbb{C}) \mid q_1(|z_0|, |z_1|, |z_2|, |z_3|) > 0\}$$

for complex hyperbolic 3–space $\mathcal{H}^3_{\mathbb{C}}$, where $|z|$ denotes the modulus of the complex number z. The latter model admits $\mathrm{PSU}(1,3)$ as its group of orientation-preserving isometries, with a natural inclusion of $\mathrm{PSO}(1,3)$.

This is how we obtain our action of Γ on $\mathcal{H}^3_{\mathbb{C}}$. In the remainder of this section, we show some properties of this action that will be used later on.

Lemma 7 *The action of Γ on $\mathcal{H}^3_{\mathbb{C}}$ just defined is properly discontinuous.*

Proof This fact should be well known and can be proved using the existence of Dirichlet fundamental domains for the Γ–action on $\mathcal{H}^3_{\mathbb{C}}$ [19, Section 9.3]. We include here for completeness a self-contained proof, which relies on the fact that the Γ–action on $\mathcal{H}^3_{\mathbb{R}}$ is properly discontinuous [13, Chapter 2, Theorem 1.2, page 34, and Chapter 7, Theorem 1.1, page 311].

Let $\{\gamma_n\}_{n\geqslant 1}$ be a sequence of elements of Γ and let $x \in \mathcal{H}^3_{\mathbb{C}}$ be a point such that $\{\gamma_n \cdot x\}_{n\geqslant 1}$ is infinite. We show that $\{\gamma_n \cdot x\}_{n\geqslant 1}$ has no accumulation point in $\mathcal{H}^3_{\mathbb{C}}$. To this aim, assume by contradiction that $x_\infty \in \mathcal{H}^3_{\mathbb{C}}$ is an accumulation point for $\{\gamma_n \cdot x\}_{n\geqslant 1}$. Let $p\colon \mathcal{H}^3_{\mathbb{C}} \to \mathcal{H}^3_{\mathbb{R}}$ be the projection defined in the proof of Theorem 17 and consider $p(x_\infty)$ and $\{p(\gamma_n \cdot x)\}_{n\geqslant 1}$. Notice that $p(\gamma_n \cdot x) = \gamma_n \cdot p(x)$ (since p is Γ–equivariant) and $\{\gamma_n \cdot p(x)\}_{n\geqslant 1}$ is infinite (because Γ acts properly discontinuously on $\mathcal{H}^3_{\mathbb{R}}$). It follows that $p(x_\infty)$ is an accumulation point for $\{\gamma_n \cdot p(x)\}_{n\geqslant 1}$, whence a contradiction. □

Lemma 8 *For any $g \in \Gamma$, the natural map $(\mathcal{H}^3_{\mathbb{C}})^g / C_\Gamma(g) \to \mathcal{H}^3_{\mathbb{C}} / \Gamma$ induced by the inclusion $(\mathcal{H}^3_{\mathbb{C}})^g \subset \mathcal{H}^3_{\mathbb{C}}$ is proper.*

Proof The proof is given in two steps; in the first one we show that the map has finite fibres. Since this fact holds true in general for any discrete group Γ acting properly discontinuously by biholomorphisms on a complex manifold M, we prove it in this generality. Let us denote by $f\colon M^g / C_\Gamma(g) \to M/\Gamma$ the natural map induced by the inclusion $M^g \subset M$. For any $x \in M^g$, let $[x] \in M^g / C_\Gamma(g)$ be its equivalence class. Then

$$f^{-1}(f([x])) = \{y \in M^g \mid y \in \Gamma \cdot x\} / C_\Gamma(g),$$

where $\Gamma \cdot x$ denotes the orbit of x. Notice that for any $h \in \Gamma$, if $h \cdot x \in M^g$, then $g \in \mathrm{Stab}(h \cdot x) = h \mathrm{Stab}(x) h^{-1}$, and so there exists a unique $g_h \in \mathrm{Stab}(x)$ such that $h g_h h^{-1} = g$ and $g_h = h^{-1} g h$. Here, for any element y, $\mathrm{Stab}(y)$ denotes its stabilizer. Furthermore, if $h_1, h_2 \in \Gamma$ are such that $h_1^{-1} g h_1 = h_2^{-1} g h_2$, then $g = h_2 h_1^{-1} g h_1 h_2^{-1} = (h_2 h_1^{-1}) g (h_2 h_1^{-1})^{-1}$. Therefore, $h_2 h_1^{-1} \in C_\Gamma(g)$ and hence $h_2 \in C_\Gamma(g) \cdot h_1$. This implies that, if we define

$$\Gamma_{x,g} := \{h \in \Gamma \mid h \cdot x \in M^g\},$$

then the map $f_{x,g}\colon \Gamma_{x,g} \to \mathrm{Stab}(x)$, $h \mapsto g_h = h^{-1} g h$, descends to an injective map $\Gamma_{x,g} / C_\Gamma(g) \to \mathrm{Stab}(x)$. The claim now follows from the fact that $\mathrm{Stab}(x)$ is finite and $\Gamma_{x,g} / C_\Gamma(g)$ is bijective to $f^{-1}(f([x]))$.

In the second step of the proof, $M = \mathcal{H}^3_{\mathbb{C}}$ and $f\colon (\mathcal{H}^3_{\mathbb{C}})^g / C_\Gamma(g) \to \mathcal{H}^3_{\mathbb{C}} / \Gamma$. Let $d\colon \mathcal{H}^3_{\mathbb{C}} \times \mathcal{H}^3_{\mathbb{C}} \to \mathbb{R}$ be the distance function induced by the Bergman metric, that is, the positive definite Hermitian form $\sum_{\alpha,\beta}^3 (\partial^2 \log \mathcal{K} / \partial z_\alpha \partial \overline{z}_\beta) \mathrm{d}z_\alpha \mathrm{d}\overline{z}_\beta$ on $\mathcal{H}^3_{\mathbb{C}}$, where $\mathcal{K}$ is the Bergman kernel of $\mathcal{H}^3_{\mathbb{C}}$ (see [29, page 145]). By restriction d induces a distance function on $(\mathcal{H}^3_{\mathbb{C}})^g$. Moreover, defining for any $[x], [y] \in \mathcal{H}^3_{\mathbb{C}} / \Gamma$ (respectively $[x], [y] \in (\mathcal{H}^3_{\mathbb{C}})^g / C_\Gamma(g)$),

$$\widetilde{d}([x],[y]) := \mathrm{Inf}\{d(\xi, \eta) \mid \xi \in \Gamma \cdot x,\ \eta \in \Gamma \cdot y\},$$

we have a distance function on $\mathcal{H}^3_{\mathbb{C}} / \Gamma$ (respectively on $(\mathcal{H}^3_{\mathbb{C}})^g / C_\Gamma(g)$, where $\widetilde{d}$ is defined accordingly). By elementary topology, for metric spaces, a subspace K is compact if and only if any infinite subset $Z \subset K$ has an accumulation point in K. So, let $K \subset \mathcal{H}^3_{\mathbb{C}} / \Gamma$ be a compact subspace. To show that $f^{-1}(K)$ is compact, let $Z \subset f^{-1}(K)$ be an infinite subset. Since f has finite fibres, $f(Z)$ is infinite, so it has an accumulation point, say $[x_0] \in K$. Notice that $f^{-1}([x_0]) \neq \varnothing$, since $\mathrm{Im}(f)$

is closed. To see this, let $[x] \notin \mathrm{Im}(f)$. Then $\Gamma \cdot x \cap (\mathcal{H}^3_{\mathbb{C}})^g = \varnothing$, in other words, for any $y \in \Gamma \cdot x$, $g \notin \mathrm{Stab}(y)$. Since the action is properly discontinuous, any $y \in \Gamma \cdot x$ has a neighbourhood U such that $\gamma \cdot U \cap U \neq \varnothing$, if and only if $\gamma \in \mathrm{Stab}(y)$, for any $\gamma \in \Gamma$. In particular, the stabilizer of any point in U is contained in $\mathrm{Stab}(y)$, and hence $\Gamma \cdot U \cap (\mathcal{H}^3_{\mathbb{C}})^g = \varnothing$. So, $\Gamma \cdot U$ gives an open neighbourhood of $[x]$ which has empty intersection with $\mathrm{Im}(f)$. To finish the proof of the lemma, we observe that, if $[x_0] \in K$ is an accumulation point for $f(Z)$ and $f^{-1}([x_0]) \neq \varnothing$, then there exists $[y_0] \in f^{-1}([x_0]) \subset f^{-1}(K)$ which is an accumulation point for Z, since f has finite fibres. □

As we will see in Section 4 (Remark 18), if $g \in \mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{PSO}(1,3)$ is different from ± 1 and $(\mathcal{H}^3_{\mathbb{C}})^g \neq \varnothing$, then $(\mathcal{H}^3_{\mathbb{C}})^g \cap \mathcal{H}^3_{\mathbb{R}} \neq \varnothing$. Therefore, g is an elliptic element of $\mathrm{PSL}_2(\mathbb{C})$ [13, Proposition 1.4, page 34]; in particular, it has exactly two fixed points on $\partial\mathcal{H}^3_{\mathbb{R}} \cong \mathbb{P}^1_{\mathbb{C}}$, and the geodesic line in $\mathcal{H}^3_{\mathbb{R}}$ joining these two points is contained in $(\mathcal{H}^3_{\mathbb{C}})^g$. Moreover, g acts as a rotation around this geodesic line. For this reason, we call any such element g a *nontrivial rotation* of $\mathcal{H}^3_{\mathbb{C}}$.

Lemma 9 *The degree shifting number of any nontrivial rotation of $\mathcal{H}^3_{\mathbb{C}}$ on its fixed-point set is* 1.

Proof For any rotation $\widehat{\theta}$ of angle θ around a geodesic line in $\mathcal{H}^3_{\mathbb{R}}$, there is a basis for the construction of the Kleinian model such that the matrix of $\widehat{\theta}$ takes the shape [13, Proposition 1.13, page 40]

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \in \mathrm{PSO}(1,3).$$

This matrix, considered as an element of $\mathrm{PSU}(1,3)$, performs a rotation of angle θ around the "complexified geodesic line" with respect to the inclusion $\mathcal{H}^3_{\mathbb{R}} \hookrightarrow \mathcal{H}^3_{\mathbb{C}}$. The fixed points of this rotation are exactly the points x lying on this complexified geodesic line, and the action on their tangent space $\mathrm{T}_x\mathcal{H}^3_{\mathbb{C}} \cong \mathbb{C}^3$ is again a rotation of angle θ. Hence, we can choose a basis of this tangent space such that this rotation is expressed by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{-i\theta} \end{pmatrix} \in \mathrm{SL}_3(\mathbb{C}).$$

Therefore, the degree shifting number of the rotation $\widehat{\theta}$ at x is 1.

Now let $x \in (\mathcal{H}^3_{\mathbb{C}})^g \setminus \mathcal{H}^3_{\mathbb{R}}$. From Remark 18 it follows that x and $p(x) \in (\mathcal{H}^3_{\mathbb{R}})^g$ belongs to the same connected component of $(\mathcal{H}^3_{\mathbb{C}})^g$, where $p\colon \mathcal{H}^3_{\mathbb{C}} \to \mathcal{H}^3_{\mathbb{R}}$ is the projection defined in the proof of Theorem 17. Therefore, $\operatorname{shift}(g,x) = \operatorname{shift}(g,p(x)) = 1$. □

Lemma 10 *Let Γ be a Bianchi group acting on $\mathcal{H}^3_{\mathbb{C}}$ as in Construction 6. Then, for any point $x \in \mathcal{H}^3_{\mathbb{C}}$, the stabilizer $\operatorname{Stab}_\Gamma(x)$ of x in Γ is a finite group isomorphic to one of the following groups: the cyclic group of order $1, 2$ or 3; the dihedral group $\mathcal{D}_2$ of order 4; the dihedral group $\mathcal{D}_3$ of order 6; the alternating group $\mathcal{A}_4$ of order 12.*

Furthermore, the map $\operatorname{Stab}_\Gamma(x) \to \mathrm{GL}_3(\mathbb{C})$ given by $\gamma \mapsto T_x\gamma$, where $T_x\gamma$ is the differential of γ at x, is an injective group homomorphism whose image is contained in $\mathrm{SL}_3(\mathbb{C})$ and it is conjugate to one of the following subgroups G of $\mathrm{SL}_3(\mathbb{C})$:

(1) *If $\operatorname{Stab}_\Gamma(x)$ is cyclic of order $n = 1, 2$ or 3, then*

$$G = \left\langle \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle,$$

where $\omega \in \mathbb{C}^*$ is a primitive n^{th} root of 1.

(2) *If $\operatorname{Stab}_\Gamma(x) \cong \mathcal{D}_2$, then*

$$G = \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle.$$

(3) *If $\operatorname{Stab}_\Gamma(x) \cong \mathcal{D}_3$, then*

$$G = \left\langle \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle,$$

where $\omega \in \mathbb{C}^*$ is a primitive third root of 1.

(4) *If $\operatorname{Stab}_\Gamma(x) \cong \mathcal{A}_4$, then*

$$G = \left\langle \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle.$$

Proof Since the action of Γ on $\mathcal{H}^3_{\mathbb{C}}$ is properly discontinuous (Lemma 7), $\operatorname{Stab}_\Gamma(x)$ is finite. The first part of the lemma follows now from the classification of the finite subgroups of Γ (see Lemma 13).

From the proof of Lemma 9 we deduce that, if $\gamma \in \mathrm{Stab}_\Gamma(x) \setminus \{1\}$, then $T_x\gamma$ is different from the identity and $\det(T_x\gamma) = 1$, hence we obtain an injective group homomorphism $\mathrm{Stab}_\Gamma(x) \to \mathrm{SL}_3(\mathbb{C})$. The description of the images of these morphisms follows from elementary representation theory, as we briefly explain.

The case (1) is clear. In case (2), G is generated by two matrices $A, B \in \mathrm{SL}_3(\mathbb{C})$ such that $A^2 = B^2 = I_3$ and $A \cdot B = B \cdot A$. From Schur's lemma it follows that A and B are simultaneously diagonalisable, hence there exists a basis of $\mathbb{C}^3$ such that A and B are diagonal of the given form.

In case (3), G is generated by two matrices, A and B, such that $A^3 = B^2 = (A \cdot B)^2 = I_3$. Let $\{u, v, w\}$ be a basis of $\mathbb{C}^3$ such that $Au = \omega u$, $Av = \omega^2 v$ and $Aw = w$, where $\omega \in \mathbb{C}^*$, $\omega^3 = 1$ and $\omega \neq 1$. From the relation $A \cdot B = B \cdot A^2$, we deduce that $Bw = \pm w$, $Bu = av$ and $Bv = bu$ for some $a, b \in \mathbb{C}^*$. Since $B^2 = I_3$, it follows that $ab = 1$. Hence, in the basis $\left\{\frac{1}{a}u, v, w\right\}$ of $\mathbb{C}^3$, the matrices A and B have the desired form.

Finally, in case (4), we use the fact that $\mathcal{A}_4$ has four irreducible representations (see eg [43, Theorem 7, page 19]), three of dimension one that are induced by the representations of $\mathcal{A}_4/H \cong \mathbb{Z}/3\mathbb{Z}$, where H is the normal subgroup of $\mathcal{A}_4$ consisting of the permutations of order two. The remaining irreducible representation of $\mathcal{A}_4$ is of dimension three. Therefore, up to conjugation, there is only one injective group homomorphism $\mathcal{A}_4 \to \mathrm{SL}_3(\mathbb{C})$. The result follows from the fact that the three given matrices generate a subgroup of $\mathrm{SL}_3(\mathbb{C})$ isomorphic to $\mathcal{A}_4$. □

Theorem 11 *Let Γ be a group generated by translations and rotations of $\mathcal{H}^3_{\mathbb{C}}$. Then all obstruction bundles of the orbifold $[\mathcal{H}^3_{\mathbb{C}}/\Gamma]$ are of fibre dimension zero, except in the case corresponding to two elements $b, c \in \Gamma \setminus \{1\}$ with $c = b^k$ and $bc \neq 1$. In this last case, the obstruction bundle is of fibre dimension 1 and it is trivial.*

Proof Nontrivial obstruction bundles can only appear for two elements of Γ with common fixed points. The translations of $\mathcal{H}^3_{\mathbb{C}}$ have their fixed points on the boundary and not in $\mathcal{H}^3_{\mathbb{C}}$. So let b and c be nontrivial hyperbolic rotations around distinct axes intersecting in the point $x \in \mathcal{H}^3_{\mathbb{C}} =: Y$. Then bc is again a hyperbolic rotation around a third distinct axis passing through x. Obviously, these rotation axes constitute the fixed-point sets Y^b, Y^c and Y^{bc}. Hence, the only fixed point of the group generated by b and c is x. Now, Lemma 5 yields the following fibre dimension for the obstruction bundle on $Y^{b,c}$:

$$\mathrm{shift}(b) + \mathrm{shift}(c) - \mathrm{shift}(bc) - \mathrm{codim}_{\mathbb{C}}(Y^{b,c} \subset Y^{bc}).$$

After computing degree shifting numbers using Lemma 9, we see that this fibre dimension is zero.

Now let b and c be nontrivial hyperbolic rotations around the same axis $Y^b = Y^c$. Then $c = b^k$ and either $bc = 1$ or $bc \neq 1$. As before we conclude that the fibre dimension of the obstruction bundle is 0 in the first case, and 1 in the second. However, if $bc \neq 1$, then $Y^{b,c} = Y^{bc}$, which is a noncompact Riemann surface contained in $\mathcal{H}^3_{\mathbb{C}}$, hence the obstruction bundle is trivial in this case [17, Theorem 30.3, page 229].

Finally, if $b = 1$ or $c = 1$, the claim follows from Lemmas 5 and 9 as before. $\square$

3 The centralizers of finite cyclic subgroups in the Bianchi groups

In this section, as well as in Theorems 20 and 21, we will reduce all our considerations to the action on real hyperbolic 3–space $\mathcal{H}^3_{\mathbb{R}}$. For the latter action, there are Poincaré's formulae [31] on the upper-halfspace model, which extend the Möbius transformations from the hyperbolic plane. Let Γ be a finite-index subgroup in a Bianchi group $\mathrm{PSL}_2(\mathcal{O}_{-m})$. In 1892, Luigi Bianchi [5] computed fundamental domains for some of the full Bianchi groups. Such a fundamental domain has the shape of a hyperbolic polyhedron (up to a missing vertex at certain cusps, which represent the ideal classes of $\mathcal{O}_{-m}$), so we will call it the *Bianchi fundamental polyhedron*. We use the Bianchi fundamental polyhedron to induce a Γ–equivariant cell structure on $\mathcal{H}^3_{\mathbb{R}}$, namely we start with this polyhedron as a 3–cell, record its polyhedral facets, edges and vertices, and tessellate out $\mathcal{H}^3_{\mathbb{R}}$ with their Γ–images.

It is well known [23] (see also [13, Proposition 1.13, page 40]) that any element of Γ fixing a point inside real hyperbolic 3–space $\mathcal{H}^3_{\mathbb{R}}$ acts as a rotation of finite order. And the rotation axis does not pass through the interior of the Bianchi fundamental polyhedron, because the interior of the latter contains only one point on each Γ–orbit. Therefore, we can easily refine our Γ–equivariant cell structure so that the stabilizer in Γ of any cell σ fixes σ pointwise: we just have to subdivide the facets and edges of the Bianchi fundamental polyhedron by their symmetries (and then again spread out the subdivided cell structure on $\mathcal{H}^3_{\mathbb{R}}$ using the Γ–action). This has been implemented in practice [36], and we shall denote $\mathcal{H}^3_{\mathbb{R}}$ with this refined cell structure by Z.

Definition 12 Let ℓ be a prime number. The *ℓ–torsion subcomplex* is the subcomplex of Z consisting of all the cells which have stabilizers in Γ containing elements of order ℓ.

For ℓ being one of the two occurring primes, 2 and 3, the orbit space of this subcomplex is a finite graph, because the cells of dimension greater than 1 are trivially stabilized in the refined cellular complex. We reduce this subcomplex with the following procedure, motivated in [37].

Condition A In the ℓ–torsion subcomplex, let σ be a cell of dimension $n-1$, lying in the boundary of precisely two n–cells τ_1 and τ_2, the latter cells representing two different orbits. Assume further that no higher-dimensional cells of the ℓ–torsion subcomplex touch σ, and that the n–cell stabilizers admit an isomorphism $\Gamma_{\tau_1} \cong \Gamma_{\tau_2}$.

Where this condition is fulfilled in the ℓ–torsion subcomplex, we merge the cells τ_1 and τ_2 along σ, and do so for their entire orbits, if and only if they meet the following additional condition. We never merge two cells the interior of which contains two points on the same orbit.

Condition B The inclusion $\Gamma_{\tau_1} \subset \Gamma_\sigma$ induces an isomorphism on group homology with $\mathbb{Z}/\ell$–coefficients under the trivial action.

A reduced ℓ–torsion subcomplex is a Γ–complex obtained by orbitwise merging two n–cells of the ℓ–torsion subcomplex satisfying Conditions A and B

We use the following classification of Felix Klein [23]:

Lemma 13 (Klein) *The finite subgroups in* $\mathrm{PSL}_2(\mathcal{O})$ *are exclusively of isomorphism types the cyclic groups of orders* 1, 2 *and* 3, *the dihedral groups* $\mathcal{D}_2$ *and* $\mathcal{D}_3$ (isomorphic to the Klein four group $\mathbb{Z}/2 \times \mathbb{Z}/2$ and the symmetric group on three symbols, respectively) *and the alternating group* $\mathcal{A}_4$.

Now we investigate the associated normalizer groups. Straightforward verification using the multiplication tables of the implied finite groups yields the following:

Lemma 14 *Let G be a finite subgroup of* $\mathrm{PSL}_2(\mathcal{O}_{-m})$. *Then the type of the normalizer of any subgroup of type $\mathbb{Z}/\ell$ in G is given as follows for $\ell = 2$ and $\ell = 3$, where* *we print only cases with existing subgroup of type $\mathbb{Z}/\ell$:*

isomorphism type of G	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathcal{D}_2$	$\mathcal{D}_3$	$\mathcal{A}_4$
normalizer of $\mathbb{Z}/2$		$\mathbb{Z}/2$		$\mathcal{D}_2$	$\mathbb{Z}/2$	$\mathcal{D}_2$
normalizer of $\mathbb{Z}/3$			$\mathbb{Z}/3$		$\mathcal{D}_3$	$\mathbb{Z}/3$

Lemma 15 *Let $v \in \mathcal{H}^3_{\mathbb{R}}$ be a vertex with stabilizer in Γ of type $\mathcal{D}_2$ or $\mathcal{A}_4$. Let γ in Γ be a rotation of order 2 around an edge e adjacent to v. Then the centralizer $C_\Gamma(\gamma)$ reflects $\mathcal{H}^\gamma$ — which is the geodesic line through e — onto itself at v.*

Proof Denote by Γ_v the stabilizer of the vertex v. In the case that Γ_v is of type $\mathcal{D}_2$, which is abelian, it admits two order-2 elements β and $\beta\cdot\gamma$ centralizing γ and turning the geodesic line through e onto itself so that the image of e touches v from the side opposite to e (illustration: $\bullet\overset{e}{\text{—}}\underset{v}{\bullet}\overset{\beta e}{\text{—}}\bullet$). In the case that Γ_v is of type $\mathcal{A}_4$, it contains a normal subgroup of type $\mathcal{D}_2$ that admits again two such elements. □

Any edge of a maximally reduced torsion subcomplex is obtained by merging a chain of edges on the intersection of one geodesic line with some strict fundamental domain for Γ in $\mathcal{H}$. We call this chain the *chain of edges associated to α*. It is well defined up to translation along the rotation axis of α.

Lemma 16 *Let α be any 2–torsion element in Γ. Then the chain of edges associated to α is a fundamental domain for the action of the centralizer of α on the rotation axis of α.*

Proof We distinguish the following two cases of how $\langle\alpha\rangle \cong \mathbb{Z}/2$ is included into Γ.

First case Suppose that there is no subgroup of type $\mathcal{D}_2$ in Γ which contains $\langle\alpha\rangle$. Then the connected component to which the rotation axis of α passes in the quotient of the 2–torsion subcomplex is homeomorphic to a circle, with cell structure . We can write $\Gamma_e = \langle\alpha\rangle$ and $\Gamma_{e'} = \langle\gamma\alpha\gamma^{-1}\rangle$. One immediately checks that any fixed point $x \in \mathcal{H}$ of α induces the fixed point $\gamma\cdot x$ of $\gamma\alpha\gamma^{-1}$. As $\mathrm{PSL}_2(\mathbb{C})$ acts by isometries, the whole fixed-point set in $\mathcal{H}$ of α is hence identified by γ with the fixed-point set of $\gamma\alpha\gamma^{-1}$. This gives us the identification γ^{-1} from e' to an edge on the rotation axis of α, adjacent to e because of the first condition on γ. We repeat this step until we have attached an edge δe on the orbit of the first edge e, with $\delta \in \Gamma$. As δ is an isometry, the whole chain is translated by δ from the start at e to the start at δe. So the group $\langle\delta\rangle$ acts on the rotation axis with fundamental domain our chain of edges, and $\delta\alpha\delta^{-1}$ is again the rotation of order 2 around the axis of α. So, $\delta\alpha\delta^{-1} = \alpha$ and therefore $\langle\delta\rangle < C_\Gamma(\alpha)$.

Second case Suppose that there is a subgroup G of Γ of type $G \cong \mathcal{D}_2$ containing $\langle\alpha\rangle$. Then the rotation axis of α passes in the quotient of the 2–torsion subcomplex to an edge on a connected component of homeomorphism type , or (see [4]). If there is no further inclusion $G < G' < \Gamma$ with $G' \cong \mathcal{A}_4$, let $G' := G$. Then the chain associated to α can be chosen so that one of its endpoints is stabilized by G'. The other endpoint of this chain must then lie on a different Γ–orbit, and admits as stabilizer a group H' containing $\langle\alpha\rangle$, of type $\mathcal{D}_2$ or $\mathcal{A}_4$. By Lemma 15, each G' and H' contains a reflection of the rotation axis of α, centralizing α. These two reflections must differ from one another because they do not fix the chain of edges. So their free product tessellates the rotation axis of α with images of the chain of edges associated to α. $\square$

4 A spine for the complexified Bianchi orbifolds

In this section, we prove the following theorem, which will be used to prove Theorem 3:

Theorem 17 *Let Γ be a Bianchi group. Then there is a Γ–equivariant homotopy equivalence between $\mathcal{H}^3_{\mathbb{C}}$ and $\mathcal{H}^3_{\mathbb{R}}$. In particular, $\mathcal{H}^3_{\mathbb{C}}/\Gamma$ is homotopy equivalent to $\mathcal{H}^3_{\mathbb{R}}/\Gamma$.*

Proof We consider the ball model for complex hyperbolic 3–space $\mathcal{H}^3_{\mathbb{C}}$ [19] (which is called the Klein model in [13]). This provides us with a complex structure such that $\mathcal{H}^3_{\mathbb{R}}$ is naturally embedded into $\mathcal{H}^3_{\mathbb{C}}$ as the fixed-point set of the complex conjugation. In the other direction, following [19], we define a projection as follows. For any point $z \in \mathcal{H}^3_{\mathbb{C}}$, there is a unique geodesic arc, with respect to the Bergman metric, $\alpha_{z,\overline{z}}$ from z to its complex conjugate $\overline{z}$ (see eg [19, Theorem 3.1.11]), and the intersection point $p(z) = \alpha_{z,\overline{z}} \cap \mathcal{H}^3_{\mathbb{R}}$ is equidistant to z and $\overline{z}$ [19, Section 3.3.6]. This defines a projection $p\colon \mathcal{H}^3_{\mathbb{C}} \to \mathcal{H}^3_{\mathbb{R}}$. Notice that p is $\mathrm{PSO}(1,3)$–equivariant and hence also Γ–equivariant.

Clearly, the restriction $p|_{\mathcal{H}^3_{\mathbb{R}}}$ is the identity. On the other hand, let

$$H\colon \mathcal{H}^3_{\mathbb{C}} \times [0,1] \to \mathcal{H}^3_{\mathbb{C}}, \qquad H(z,t) = \alpha_{z,\overline{z}}\big(t\rho(z,p(z))\big),$$

where ρ is the hyperbolic distance and we have parametrized the geodesic arc so that $\alpha_{z,\overline{z}}(0) = p(z)$ and $\alpha_{z,\overline{z}}\big(\rho(z,p(z))\big) = z$. Then H is an homotopy between p and the identity map of $\mathcal{H}^3_{\mathbb{C}}$. Furthermore, since $\mathrm{PSO}(1,3)$ is a group of isometries of $\mathcal{H}^3_{\mathbb{C}}$, it sends geodesics to geodesics and so, for any $M \in \mathrm{PSO}(1,3)$,

$$(2)\quad H(Mz,t) = \alpha_{Mz,\overline{Mz}}\big(t\rho(Mz, p(Mz))\big) = M\alpha_{z,\overline{z}}\big(t\rho(z, p(z))\big) = MH(z,t).$$

It follows that H is $\mathrm{PSO}(1,3)$–equivariant; in particular, it is Γ–equivariant. □

Remark 18 From (2) it follows that, if $g \in \mathrm{PSO}(1,3)$ fixes a point z with $z \neq \overline{z}$, then g fixes the geodesic arc $\alpha_{z,\overline{z}}$ pointwise. Indeed, from the fact that $g\cdot\overline{z} = \overline{g\cdot z} = \overline{z}$, we deduce that $g(\alpha_{z,\overline{z}}) = \alpha_{z,\overline{z}}$. Moreover, for every $z' \in \alpha_{z,\overline{z}}$, we see that $g\cdot z' = z'$, because otherwise we get a contradiction from the equalities

$$\rho(z,z') = \rho(g\cdot z, g\cdot z') = \rho(z, g\cdot z'),$$

where ρ is the hyperbolic distance.

Remark 19 From Lemma 10 it follows that the points $z \in \mathcal{H}^3_{\mathbb{C}}$ such that the stabilizer $\mathrm{Stab}_\Gamma(z) \subset \Gamma$ is not cyclic are isolated; hence, Theorem 17 implies that such points z belong to $\mathcal{H}^3_{\mathbb{R}}$.

5 Orbifold cohomology of real Bianchi orbifolds

Our main results on the vector space structure of the Chen–Ruan orbifold cohomology of Bianchi orbifolds are the below two theorems.

Theorem 20 *For any element γ of order 3 in a finite-index subgroup Γ in a Bianchi group with units $\{\pm 1\}$, the quotient space $\mathcal{H}^\gamma / C_\Gamma(\gamma)$ of the rotation axis modulo the centralizer of γ is homeomorphic to a circle.*

Proof As γ is a nontrivial torsion element, by [37, Lemma 22] the Γ–image of the chain of edges associated to γ contains the rotation axis $\mathcal{H}^\gamma$. Now we can observe two cases:

- First, assume that the rotation axis of γ does not contain any vertex of stabilizer type $\mathcal{D}_3$ (from [37], we know that this gives us a circle as a path component in the quotient of the 3–torsion subcomplex). Assume that there exists a reflection of $\mathcal{H}^\gamma$ onto itself by an element of Γ. Such a reflection would fix a point on $\mathcal{H}^\gamma$. Then the normalizer of $\langle\gamma\rangle$ in the stabilizer of this point would contain the reflection. This way, Lemma 14 yields that this stabilizer is of type $\mathcal{D}_3$, which we have excluded. Thus, there can be no reflection of $\mathcal{H}^\gamma$ onto itself by an element of Γ.

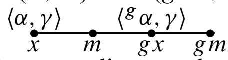
As Γ acts by isometries preserving a metric of nonpositive curvature (a CAT(0) metric), every element $g \in \Gamma$ sending an edge of the chain for γ to an edge on $\mathcal{H}^\gamma$ outside the fundamental domain can then only perform a translation on $\mathcal{H}^\gamma$. A translation along the rotation axis of γ commutes with γ, so $g \in C_\Gamma(\gamma)$. Hence the quotient space $\mathcal{H}^\gamma / C_\Gamma(\gamma)$ is homeomorphic to a circle.

•—• If $\mathcal{H}^\gamma$ contains a point with stabilizer in Γ of type $\mathcal{D}_3$, then there are exactly two Γ–orbits of such points. The elements of order 2 do not commute with the elements of order 3 in $\mathcal{D}_3$, so the centralizer of γ does not contain the former ones. Hence, $C_\Gamma(\gamma)$ does not contain any reflection of $\mathcal{H}^\gamma$ onto itself. Denote by α and β elements of order 2 of each of the stabilizers of the two endpoints of a chain of edges for γ. Then $\alpha\beta$ performs a translation on $\mathcal{H}^\gamma$ and hence commutes with γ. A fundamental domain for the action of $\langle\alpha\beta\rangle$ on $\mathcal{H}^\gamma$ is given by the chain of edges for γ united with its reflection through one of its endpoints. As no such reflection belongs to the centralizer of γ and the latter endpoint is the only one on its Γ–orbit in this fundamental domain, the quotient $\mathcal{H}^\gamma / C_\Gamma(\gamma)$ matches with the quotient $\mathcal{H}^\gamma / \langle\alpha\beta\rangle$, which is homeomorphic to a circle. □

Theorem 21 *Let γ be an element of order 2 in a Bianchi group Γ with units $\{\pm 1\}$. Then the homeomorphism type of the quotient space $\mathcal{H}^\gamma / C_\Gamma(\gamma)$ is*

- •—• *an edge without identifications if $\langle\gamma\rangle$ is contained in a subgroup of type $\mathcal{D}_2$ inside Γ, and*
- ○ *a circle otherwise.*

Proof By Lemma 16, the chain of edges for γ is a fundamental domain for $C_\Gamma(\gamma)$ on the rotation axis $\mathcal{H}^\gamma$ of γ. Again, we have two cases:

•—• If $\langle\gamma\rangle$ is contained in a subgroup of type $\mathcal{D}_2$ inside Γ, then any chain of edges for γ admits endpoints of stabilizer types $\mathcal{D}_2$ or $\mathcal{A}_4$, because we can merge any two adjacent edges on a 2–torsion axis with touching point of stabilizer type $\mathbb{Z}/2$ or $\mathcal{D}_3$. As $\mathcal{D}_2$ is an abelian group and the reflections in $\mathcal{A}_4$ are contained in the normal subgroup $\mathcal{D}_2$, the reflections in these endpoint stabilizers commute with γ, so the quotient space $\mathcal{H}^\gamma / C_\Gamma(\gamma)$ is represented by a chain of edges for γ. What remains to show is that there is no element of $C_\Gamma(\gamma)$ identifying the two endpoints of stabilizer type $\mathcal{D}_2$ (respectively $\mathcal{A}_4$). Assume that there is an element $g \in C_\Gamma(\gamma)$ carrying out this identification. Any one of the two endpoints — denote it by x — contains in its stabilizer a reflection α of the rotation axis of γ. The other endpoint is then $g\cdot x$ and contains in its stabilizer the conjugate ${}^g\alpha$ by g. Denote by m the point in the middle of $(x, g\cdot x)$, ie the point on $\mathcal{H}^\gamma$ with equal distance to x and to $g\cdot x$. As $\langle {}^g\alpha, \gamma\rangle$ is abelian, ${}^g\alpha$ is in $C_\Gamma(\gamma)$ and hence (x, m) and $(g\cdot x, m)$ are equivalent modulo $C_\Gamma(\gamma)$ via the element ${}^g\alpha g$ (illustration: 

). Then the chain of edges for γ does not reach from x to $g\cdot x$. This contradicts our hypotheses, so the homeomorphism type of $\mathcal{H}^\gamma/C_\Gamma(\gamma)$ is an edge without identifications.

- $\circ\!\!\bullet$ The other case is analogous to the first case of the proof of Theorem 20, the role of $\mathcal{D}_3$ being played by $\mathcal{D}_2$ and $\mathcal{A}_4$. □

Furthermore, the following easy-to-check statement will be useful for our orbifold cohomology computations.

Observation 22 *There is only one conjugacy class of elements of order 2 in $\mathcal{D}_3$ as well as in $\mathcal{A}_4$. In $\mathcal{D}_3$, there is also only one conjugacy class of elements of order 3, whilst in $\mathcal{A}_4$ there is an element γ such that γ and γ^2 represent the two conjugacy classes of elements of order 3.*

Proof In cycle type notation, we can explicitly establish the multiplication tables of $\mathcal{D}_3$ and $\mathcal{A}_4$, and compute the conjugacy classes. □

Corollary 23 *Let γ be an element of order 3 in a Bianchi group Γ with units $\{\pm 1\}$. Then γ is conjugate in Γ to its square γ^2 if and only if there exists a group $G \cong \mathcal{D}_3$ with $\langle\gamma\rangle \subsetneq G \subsetneq \Gamma$.*

Denote by $\lambda_{2\ell}$ the number of conjugacy classes of subgroups of type $\mathbb{Z}/\ell\mathbb{Z}$ in a finite-index subgroup Γ in a Bianchi group with units $\{\pm 1\}$. Denote by $\lambda_{2\ell}^*$ the number of conjugacy classes of those of them which are contained in a subgroup of type $\mathcal{D}_\ell$ in Γ. By Corollary 23, there are $2\lambda_6 - \lambda_6^*$ conjugacy classes of elements of order 3. As a result of Theorems 20 and 21, we have the isomorphism of vector spaces

$$\bigoplus_{\gamma\in T} \mathrm{H}^\bullet((\mathcal{H}_\mathbb{R})^\gamma/C_\Gamma(\gamma);\mathbb{Q}) \cong$$

$$\mathrm{H}^\bullet(\mathcal{H}_\mathbb{R}/\Gamma;\mathbb{Q}) \oplus \bigoplus^{\lambda_4^*} \mathrm{H}^\bullet(\bullet\!\!-\!\!\!-\!\!\bullet;\mathbb{Q}) \oplus \bigoplus^{\lambda_4-\lambda_4^*} \mathrm{H}^\bullet(\circ\!\!\bullet;\mathbb{Q}) \oplus \bigoplus^{2\lambda_6-\lambda_6^*} \mathrm{H}^\bullet(\circ\!\!\bullet;\mathbb{Q}),$$

where $T \subset \Gamma$ is a set of representatives of conjugacy classes of elements of finite order in Γ. The (co)homology of the quotient space $\mathcal{H}_{\mathbb{R}}/\Gamma$ has been computed numerically for a large scope of Bianchi groups [45; 42; 35]. Bounds for its Betti numbers have been given in [25]. Krämer [24] has determined number-theoretic formulae for the numbers $\lambda_{2\ell}$ and $\lambda^*_{2\ell}$ of conjugacy classes of finite subgroups in the full Bianchi groups. Krämer's formulae have been evaluated for hundreds of thousands of Bianchi groups [37], and these values are matching with the ones from the orbifold structure computations with [32] in the cases where the latter are available.

When we pass to the complexified orbifold $[\mathcal{H}^3_{\mathbb{C}}/\Gamma]$, the real line that is the rotation axis in $\mathcal{H}_{\mathbb{R}}$ of an element of finite order becomes a complex line. However, the centralizer still acts in the same way by reflections and translations. So, the interval $\bullet\!\!-\!\!\bullet$ as a quotient of the real line yields a stripe $\bullet\!\!-\!\!\bullet \times \mathbb{R}$ as a quotient of the complex line. And the circle $\bigcirc\!\!\bullet$ as a quotient of the real line yields a cylinder $\bigcirc\!\!\bullet \times \mathbb{R}$ as a quotient of the complex line. Therefore, using the degree shifting numbers of Lemma 9, we obtain the result of Corollary 2,

$$\mathrm{H}^d_{\mathrm{CR}}([\mathcal{H}^3_{\mathbb{C}}/\Gamma]) \cong \mathrm{H}^d(\mathcal{H}_{\mathbb{C}}/\Gamma; \mathbb{Q}) \oplus \begin{cases} \mathbb{Q}^{\lambda_4 + 2\lambda_6 - \lambda_6^*} & \text{if } d = 2, \\ \mathbb{Q}^{\lambda_4 - \lambda_4^* + 2\lambda_6 - \lambda_6^*} & \text{if } d = 3, \\ 0 & \text{otherwise.} \end{cases}$$

As the authors have calculated the Bredon homology $\mathrm{H}_0^{\mathfrak{Fin}}(\Gamma; R_{\mathbb{C}})$ of the Bianchi groups with coefficients in the complex representation ring functor $R_{\mathbb{C}}$ (see [38]), Mislin's following lemma allows us a verification of our computations (we calculate both sides of Mislin's isomorphism explicitly).

Lemma 24 (Mislin [28]) *Let Γ be an arbitrary group and write* $\mathrm{FC}(\Gamma)$ *for the set of conjugacy classes of elements of finite order in Γ. Then there is an isomorphism*

$$\mathrm{H}_0^{\mathfrak{Fin}}(\Gamma; R_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}[\mathrm{FC}(\Gamma)].$$

6 The cohomology ring isomorphism

In this section, we prove Theorem 3. To this aim, we first prove that there is a bijective correspondence between conjugacy classes of elements of finite order in $\Gamma \setminus \{1\}$ and exceptional prime divisors of the crepant resolution $f\colon Y \to \mathcal{H}^3_{\mathbb{C}}/\Gamma$. Here we follow and we use results from [22], therefore we interpret the aforementioned correspondence as a McKay correspondence for complexified Bianchi orbifolds. In Section 6.3, we use this correspondence to define a morphism of graded vector spaces $\Phi\colon \mathrm{H}^*_{\mathrm{CR}}([\mathcal{H}^3_{\mathbb{C}}/\Gamma]) \to \mathrm{H}^*(Y)$. Finally, using a Mayer–Vietoris argument, together with results from [22; 30], we show that Φ is an isomorphism and that it preserves the cup products.

Throughout this section, Γ is a Bianchi group and $[\mathcal{H}^3_{\mathbb{C}}/\Gamma]$ is the corresponding complexified Bianchi orbifold.

6.1 The singular locus of complexified Bianchi orbifolds and the existence of crepant resolutions

Let us recall that the singular points of $\mathcal{H}^3_{\mathbb{C}}/\Gamma$ are the image, under the projection $\mathcal{H}^3_{\mathbb{C}} \to \mathcal{H}^3_{\mathbb{C}}/\Gamma$, of the points with nontrivial stabilizer. Moreover, every element $\gamma \in \Gamma \setminus \{1\}$ such that $(\mathcal{H}^3_{\mathbb{C}})^\gamma \neq \varnothing$ is a nontrivial rotation of $\mathcal{H}^3_{\mathbb{C}}$ of order 2 or 3 (see the discussion before Lemma 9) and the fixed-point locus $(\mathcal{H}^3_{\mathbb{C}})^\gamma$ is a Riemann surface. More precisely, we get the following result:

Lemma 25 *Let $\Sigma \subset \mathcal{H}^3_{\mathbb{C}}/\Gamma$ be the singular locus. Then the following statements hold true*:

(1) *Σ is an analytic space of dimension 1 with finitely many singular points $x_1, \dots, x_s$.*

(2) *For any $\gamma \in \Gamma$, let $\iota_\gamma\colon (\mathcal{H}^3_{\mathbb{C}})^\gamma / C_\Gamma(\gamma) \to \mathcal{H}^3_{\mathbb{C}}/\Gamma$ be the morphism induced by the inclusion $(\mathcal{H}^3_{\mathbb{C}})^\gamma \hookrightarrow \mathcal{H}^3_{\mathbb{C}}$ and let $\Sigma_\gamma := \mathrm{Im}(\iota_\gamma)$ be the image of ι_γ. Then every irreducible component of Σ is equal to $\Sigma_\gamma \subset \Sigma$ for some $\gamma \in \Gamma$.*

(3) *For any $\gamma \in \Gamma$, the centralizer $C_\Gamma(\gamma)$ is a normal subgroup of $N_\Gamma(\langle\gamma\rangle)$, the normalizer of $\langle\gamma\rangle$ in Γ. Moreover, $N_\Gamma(\langle\gamma\rangle)/C_\Gamma(\gamma)$ acts on $(\mathcal{H}^3_{\mathbb{C}})^\gamma / C_\Gamma(\gamma)$ and $(\mathcal{H}^3_{\mathbb{C}})^\gamma / N_\Gamma(\langle\gamma\rangle)$ is the normalization of Σ_γ.*

(4) *Let $\gamma \in \Gamma$. If γ has order 2, or it has order 3 and it is not conjugate to γ^2 in Γ, then $C_\Gamma(\gamma) = N_\Gamma(\langle\gamma\rangle)$. If γ has order 3 and it is conjugate to γ^2 in Γ, then $C_\Gamma(\gamma)$ has index 2 in $N_\Gamma(\langle\gamma\rangle)$.*

Proof (1) As observed before, if $\gamma \in \Gamma \setminus \{1\}$ is such that $(\mathcal{H}^3_{\mathbb{C}})^\gamma \neq \varnothing$, then it is a nontrivial rotation. Therefore, $(\mathcal{H}^3_{\mathbb{C}})^\gamma$ is a Riemann surface and so Σ is an analytic space of dimension 1. The singular points of Σ are the image of the points $z \in \mathcal{H}^3_{\mathbb{C}}$ with stabilizer not cyclic. As observed in Remark 19, such points belong to $\mathcal{H}^3_{\mathbb{R}}$. Now, the fact that Σ has finitely many singular points follows from the existence of a fundamental domain for the action of Γ on $\mathcal{H}^3_{\mathbb{R}}$, which is bounded by finitely many geodesic surfaces [13, Theorem 1.1, page 311].

(2) This is a consequence of the fact that $(\mathcal{H}^3_{\mathbb{C}})^\gamma$ is irreducible, as it is isomorphic to the disk $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$, and the image of an irreducible analytic space is irreducible.

(3) Let $\eta \in C_\Gamma(\gamma)$, and let $\delta \in N_\Gamma(\langle\gamma\rangle)$. Then

$$\delta^{-1}\eta\delta\gamma = \delta^{-1}\eta\gamma^k\delta = \delta^{-1}\gamma^k\eta\delta = \gamma\delta^{-1}\eta\delta,$$

where $k \in \mathbb{N}$ is such that $\delta\gamma\delta^{-1} = \gamma^k$. Hence $\delta^{-1}\eta\delta \in C_\Gamma(\gamma)$ and so $C_\Gamma(\gamma)$ is a normal subgroup of $N_\Gamma(\langle\gamma\rangle)$.

The natural action of $N_\Gamma(\langle\gamma\rangle)$ on $(\mathcal{H}^3_{\mathbb{C}})^\gamma$ is properly discontinuous, hence every point has finite stabilizer. From this it follows that $(\mathcal{H}^3_{\mathbb{C}})^\gamma/N_\Gamma(\langle\gamma\rangle)$ is a normal analytic space. Furthermore, let $z, z' \in (\mathcal{H}^3_{\mathbb{C}})^\gamma$ be two points that are mapped to the same point $x \in \mathcal{H}^3_{\mathbb{C}}/\Gamma$, and suppose that x is a smooth point of Σ. Under these hypotheses, $\mathrm{Stab}_\Gamma(z) = \mathrm{Stab}_\Gamma(z') = \langle\gamma\rangle$ and so, if $g \in \Gamma$ is such that $g \cdot z = z'$, we know that $g\langle\gamma\rangle g^{-1} = \langle\gamma\rangle$, that is $g \in N_\Gamma(\langle\gamma\rangle)$. This implies that ι_γ induces a birational map between $(\mathcal{H}^3_{\mathbb{C}})^\gamma/N_\Gamma(\langle\gamma\rangle)$ and Σ_γ, hence (3) follows.

To prove (4), let us consider the action of $N_\Gamma(\langle\gamma\rangle)/C_\Gamma(\gamma)$ on $\langle\gamma\rangle \setminus \{1\}$ given by conjugation. If γ has order 2, or it has order 3 and it is not conjugate to γ^2, then this action is trivial, hence $C_\Gamma(\gamma) = N_\Gamma(\langle\gamma\rangle)$. In the remaining case, the orbit of γ has two elements, so the result follows. □

The existence of a crepant resolution of $\mathcal{H}^3_{\mathbb{C}}/\Gamma$ follows from [40] (see also [9]), since $\mathcal{H}^3_{\mathbb{C}}/\Gamma$ has Gorenstein singularities (Lemma 10). For later use, and to fix notation, we briefly review its construction. Under the notation of Lemma 25, let $x_1, \dots, x_s \in \Sigma$ be the singular points of Σ (the singular locus of $\mathcal{H}^3_{\mathbb{C}}/\Gamma$). By Lemma 10, there are disjoint open neighbourhoods $U_1, \dots, U_s \subset \mathcal{H}^3_{\mathbb{C}}/\Gamma$ of $x_1, \dots, x_s$, respectively, each of them isomorphic to the quotient of an open neighbourhood of the origin in $\mathbb{C}^3$ by a finite subgroup of $\mathrm{SL}_3(\mathbb{C})$. Therefore, for any $i = 1, \dots, s$, there exists a crepant resolution $f_i \colon V_i \to U_i$ of U_i.

Let $X := (\mathcal{H}^3_{\mathbb{C}}/\Gamma) \setminus \{x_1, \dots, x_s\}$ be the complement of $x_1, \dots, x_s$. It is an analytic space with *transverse singularities of type A*. That is, every singular point $x \in X$ has a neighbourhood isomorphic to a neighbourhood of a singular point of $\{(u, v, w) \in \mathbb{C}^3 \mid w^{n+1} = uv\} \times \mathbb{C}^{d-2}$ for some integer $n \geqslant 1$, where $d = \dim(X)$ is equal to 3 in our case. Notice that n is constant on each connected component C of the singular locus of X, hence we say that X has transverse singularities of type A_n on C.

Every analytic space with transverse singularities of type A admits a unique crepant resolution (see eg [30, Proposition 4.2]), up to canonical isomorphism. So let $f_0\colon V \to X$ be a crepant resolution of X. By uniqueness, the restriction of $f_0\colon V \to X$ to $U_i \setminus \{x_i\}$ is canonically isomorphic to the restriction of $f_i\colon V_i \to U_i$ to $U_i \setminus \{x_i\}$ for all $i = 1, \dots, s$. Therefore, $f_0, f_1, \dots, f_s$ can be glued together, yielding a crepant resolution $f\colon Y \to \mathcal{H}^3_{\mathbb{C}}/\Gamma$.

6.2 McKay correspondence for complexified Bianchi orbifolds

In this section, we prove that there is a natural one-to-one correspondence between conjugacy classes of elements of finite order of $\Gamma \setminus \{1\}$ and exceptional prime divisors of the crepant resolution $f\colon Y \to \mathcal{H}^3_{\mathbb{C}}/\Gamma$. Let us recall that the authors of [22] define a natural bijective correspondence between, on one side, conjugacy classes of junior (degree shifting number 1) elements of $G\setminus\{1\}$ (here, G is a finite subgroup of $\mathrm{SL}_n(\mathbb{C})$) and, on the other side, exceptional prime divisors of a minimal model of $\mathbb{C}^n/G$ (if $f\colon Y \to \mathbb{C}^n/G$ is a crepant resolution, then Y is a minimal model of $\mathbb{C}^n/G$). This result has been interpreted, and extended in several directions, using derived categories in [7; 9].

We will need some general facts about analytic spaces with transverse singularities of type A, which we briefly recall. Let M be a complex manifold with an effective and properly discontinuous action of a discrete group Γ, and let $X = M/\Gamma$ be the quotient space. For $\gamma \in \Gamma$, let $C \subset X$ be the image under the quotient map $M \to M/\Gamma$ of the fixed-point locus M^γ. Let us suppose that X has transverse singularities of type A_n on C. In particular, the stabilizer of any point $z \in M^\gamma$ is $\langle\gamma\rangle \cong \mathbb{Z}/(n+1)\mathbb{Z}$, so two points $z, z' \in M^\gamma$ are identified by the projection $M \to M/\Gamma$ if and only if they are in the same $N_\Gamma(\langle\gamma\rangle)$–orbit (see the proof of Lemma 25(3)), where $N_\Gamma(\langle\gamma\rangle)$ is the normalizer of $\langle\gamma\rangle$ in Γ.

Note 26 In this situation, $g\gamma g^{-1} = \gamma^{\pm 1}$ for any $g \in N_\Gamma(\langle\gamma\rangle)$.

Proof Let us consider the normal bundle of M^γ in M, $N_{M^\gamma/M}$. The group $\langle\gamma\rangle$ acts fibrewise on $N_{M^\gamma/M}$, so we have a splitting $N_{M^\gamma/M} = (N_{M^\gamma/M})^\chi \oplus (N_{M^\gamma/M})^{\chi^{-1}}$, where $(N_{M^\gamma/M})^{\chi^{\pm 1}}$ is the subbundle of $N_{M^\gamma/M}$ where $\langle\gamma\rangle$ acts as multiplication by the character $\chi^{\pm 1}$, and χ is a generator of the group of characters of $\langle\gamma\rangle$. Assume that $g\gamma g^{-1} = \gamma^k$, and let $z \in M^\gamma$, $z' := g \cdot z$. Then the tangent map of g at z, $T_z g$, induces an isomorphism

$$N_{M^\gamma/M}(z) \cong N_{M^\gamma/M}(z') \tag{3}$$

between the fibre of $N_{M^\gamma/M}$ at z and that at z'. Since $T_z g \circ T_z \gamma = T_z(g \circ \gamma) = T_{z'}\gamma^k \circ T_z g$, the isomorphism (3) yields an isomorphism between the following representations of $\langle\gamma\rangle$: $\langle\gamma\rangle \to \mathrm{GL}(N_{M^\gamma/M}(z))$, $\gamma \mapsto T_z\gamma$, and $\langle\gamma\rangle \to \mathrm{GL}(N_{M^\gamma/M}(z'))$, $\gamma \mapsto T_{z'}\gamma^k$. But the last representation is the direct sum of the irreducible representations of $\langle\gamma\rangle$ having characters χ^k and χ^{-k}, so $k \equiv \pm 1 \bmod n+1$. □

We say that X has *transverse singularities of type A_n on C and nontrivial monodromy* if γ is conjugate to γ^{-1} in Γ. Otherwise we say that X has *transverse singularities of type A_n on C and trivial monodromy*. We refer to [30, Section 3.1] for an equivalent definition of the monodromy. Notice also that in [30] the monodromy is referred to a suitable neighbourhood of $[M^\gamma/N_\Gamma(\langle\gamma\rangle)]$ in the orbifold $[M/\Gamma]$. However, by [30, Proposition 2.9], such an orbifold structure is determined uniquely by X.

Now let $\widetilde{U}$ be a neighbourhood of M^γ in M that is isomorphic to a neighbourhood of the 0–section of $N_{M^\gamma/M}$ (ie a tubular neighbourhood of M^γ in M). The natural action of $N_\Gamma(\langle\gamma\rangle)$ on $N_{M^\gamma/M}$ induces an action of $N_\Gamma(\langle\gamma\rangle)$ on $\widetilde{U}$, such that $\widetilde{U}/N_\Gamma(\langle\gamma\rangle)$ is an open neighbourhood of C in X. Moreover, if X has nontrivial monodromy on C, then $\widetilde{U}/C_\Gamma(\gamma)$ is an analytic space with transverse singularities of type A_n on $M^\gamma/C_\Gamma(\gamma)$ and trivial monodromy, and the natural map $\widetilde{U}/C_\Gamma(\gamma) \to \widetilde{U}/N_\Gamma(\langle\gamma\rangle)$ is a two-to-one topological covering (this is analogous to [30, Corollary 3.6]).

We summarize in the following proposition the previous considerations, in the case of complexified Bianchi orbifolds:

Proposition 27 *Let Γ be a Bianchi group, let Σ be the singular locus of $\mathcal{H}^3_{\mathbb{C}}/\Gamma$, and let $x_1,\dots,x_s$ be the singular points of Σ. Let $X := (\mathcal{H}^3_{\mathbb{C}}/\Gamma)\setminus\{x_1,\dots,x_s\}$. Then the following holds true:*

(1) *For every connected component C of $\Sigma\setminus\{x_1,\dots,x_s\}$, X has transverse singularities of type A_n on C, with $n \in \{1,2\}$.*

(2) *If C is contained in the image of $(\mathcal{H}^3_{\mathbb{C}})^\gamma$, with either $n=2$ and γ not conjugate to γ^2, or $n=1$, then X has trivial monodromy on C.*

(3) *If $n=2$ and γ is conjugate to γ^2, then X has nontrivial monodromy on C. Furthermore, using the same notation as before, $\widetilde{U}/C_\Gamma(\gamma)$ is an analytic space with transverse singularities of type A_2 and trivial monodromy; the map $\widetilde{U}/C_\Gamma(\gamma) \to \widetilde{U}/N_\Gamma(\langle\gamma\rangle)$ is a two-to-one topological covering.*

In the following proposition, we establish a McKay correspondence for complexified Bianchi orbifolds following [22]:

Proposition 28 *Let Γ be a Bianchi group, and let $f\colon Y \to \mathcal{H}^3_{\mathbb{C}}/\Gamma$ be a crepant resolution. Then there is a one-to-one correspondence between conjugacy classes of elements of finite order of $\Gamma\setminus\{1\}$ and exceptional prime divisors of f.*

Proof Let $\gamma \in \Gamma\setminus\{1\}$ be an element of finite order. Then γ is an elliptic element of $\mathrm{PSL}_2(\mathbb{C})$ [13, Definition 1.3, page 34] and so $(\mathcal{H}^3_{\mathbb{C}})^\gamma \neq \varnothing$ [13, Proposition 1.4, page 34]. By Lemma 9, the degree shifting number of γ is 1; in other words, in the notation of [22], γ is a junior element of $\mathrm{Stab}_\Gamma(z)$ for any $z \in (\mathcal{H}^3_{\mathbb{C}})^\gamma$.

As observed in Lemma 25, the image of $(\mathcal{H}^3_{\mathbb{C}})^\gamma$ in $\mathcal{H}^3_{\mathbb{C}}/\Gamma$ is an irreducible component Σ_γ of the singular locus Σ of $\mathcal{H}^3_{\mathbb{C}}/\Gamma$.

Now we distinguish two cases.

Case 1 (γ has order 2, or it has order 3 and is not conjugate to γ^2 in Γ) Then there are open subsets $U_1,\dots,U_r \subset \mathcal{H}^3_{\mathbb{C}}/\Gamma$ of the form $U_i = \widetilde{U}_i/G_i$, where $\widetilde{U}_i \subset \mathcal{H}^3_{\mathbb{C}}$ is an open neighbourhood of a point $z \in (\mathcal{H}^3_{\mathbb{C}})^\gamma$, $G_i = \mathrm{Stab}_\Gamma(z)$ for $i = 1,\dots,r$, and such that $\Sigma_\gamma \subset \bigcup_{i=1}^r U_i$. Notice that, in this case, the conjugacy class of $\gamma \in G_i$ consists only of γ, so by [22, Corollary 1.5], γ corresponds to an exceptional prime divisor $E_{\gamma,i}$ of the restriction $f|_{f^{-1}(U_i)}\colon f^{-1}(U_i) \to U_i$ for any $i = 1,\dots,r$ (as observed before, γ is a junior element of G_i). Moreover, by the definition of the $E_{\gamma,i}$ (see [22]), it follows that on $f^{-1}(U_i \cap U_j)$, the divisors $E_{\gamma,i}$ and $E_{\gamma,j}$ coincide, so they glue together to form an exceptional prime divisor $E_\gamma \subset Y$ of f.

Case 2 (γ has order 3 and is conjugate to γ^2 in Γ) Now let $C \subset \Sigma_\gamma$ be the complement in Σ_γ of the singular points of Σ. By Proposition 27, there is an open subset $\widetilde{U} \subset \mathcal{H}^3_{\mathbb{C}}$ with an action of $N_\Gamma(\langle\gamma\rangle)$ such that $\widetilde{U}/N_\Gamma(\langle\gamma\rangle) \subset \mathcal{H}^3_{\mathbb{C}}/\Gamma$ is an open neighbourhood of C and $\widetilde{U}/C_\Gamma(\gamma) \to \widetilde{U}/N_\Gamma(\langle\gamma\rangle)$ is a two-to-one topological covering. Furthermore, $\widetilde{U}/C_\Gamma(\gamma)$ is an analytic space with transverse singularities of type A_2 and trivial monodromy. Let $\widetilde{V} \to \widetilde{U}/C_\Gamma(\gamma)$ be a crepant resolution; then, by the uniqueness of the crepant resolution for spaces with transverse singularities of type A, there is a morphism

$$\widetilde{V} \to f^{-1}(\widetilde{U}/N_\Gamma(\langle\gamma\rangle)) \tag{4}$$

such that the following diagram commutes:

$$\begin{tikzcd}
\widetilde{V} \arrow[r] \arrow[d] & f^{-1}(\widetilde{U}/N_\Gamma(\langle\gamma\rangle)) \arrow[d, "f_|"] \\
\widetilde{U}/C_\Gamma(\gamma) \arrow[r] & \widetilde{U}/N_\Gamma(\langle\gamma\rangle)
\end{tikzcd}$$

From Case 1, γ corresponds to an exceptional prime divisor F_γ of $\widetilde{V} \to \widetilde{U}/C_\Gamma(\gamma)$. Let us denote by $E_{\gamma,0} \subset Y$ the image of F_γ under the morphism (4).

In order to extend $E_{\gamma,0}$ over the whole Σ_γ, let $W \subset \mathcal{H}^3_{\mathbb{C}}/\Gamma$ be a (possibly disconnected) neighbourhood of $\Sigma_\gamma \setminus C$ such that each connected component is of the form $\widetilde{W}/G$, where $\widetilde{W} \subset \mathcal{H}^3_{\mathbb{C}}$ is an open neighbourhood of a point $z \in (\mathcal{H}^3_{\mathbb{C}})^\gamma$ and $G = \mathrm{Stab}_\Gamma(z)$. By [22, Corollary 1.5], (the conjugacy class of) γ corresponds to an exceptional prime divisor $E'_\gamma \subset f^{-1}(W)$. By construction, $E_{\gamma,0}$ and E'_γ glue together to form an exceptional prime divisor E_γ of $f\colon Y \to \mathcal{H}^3_{\mathbb{C}}/\Gamma$.

Notice that if we apply the same procedure starting from $\gamma^2 = \gamma^{-1}$, we obtain the same divisor E_γ. This concludes the proof of the proposition. $\square$

6.3 The linear map

Let Γ be a Bianchi group and let $f\colon Y \to \mathcal{H}^3_{\mathbb{C}}/\Gamma$ be a crepant resolution of $\mathcal{H}^3_{\mathbb{C}}/\Gamma$. In this section, we define a linear map

$$\Phi\colon \mathrm{H}^*_{\mathrm{CR}}([\mathcal{H}^3_{\mathbb{C}}/\Gamma], \mathbb{Q}) \to \mathrm{H}^*(Y, \mathbb{Q}).$$

To this aim, let us fix the following presentation of the Chen–Ruan orbifold cohomology of $[\mathcal{H}^3_{\mathbb{C}}/\Gamma]$ (see Definition 1):

$$\mathrm{H}^*_{\mathrm{CR}}([\mathcal{H}^3_{\mathbb{C}}/\Gamma], \mathbb{Q}) = \bigoplus_{\gamma \in T} \mathrm{H}^{*-2\mathrm{shift}(\gamma)}((\mathcal{H}^3_{\mathbb{C}})^\gamma / C_\Gamma(\gamma), \mathbb{Q}), \tag{5}$$

where $T \subset \Gamma$ is a set of representatives of the conjugacy classes of elements of finite order of Γ. Then Φ is defined as the sum of linear maps

$$\Phi_\gamma\colon \mathrm{H}^{*-2\mathrm{shift}(\gamma)}((\mathcal{H}^3_{\mathbb{C}})^\gamma / C_\Gamma(\gamma), \mathbb{Q}) \to \mathrm{H}^*(Y, \mathbb{Q}) \quad \text{for } \gamma \in T. \tag{6}$$

If $\gamma = 1$, then we define $\Phi_1 := f^*\colon \mathrm{H}^*(\mathcal{H}^3_{\mathbb{C}}/\Gamma, \mathbb{Q}) \to \mathrm{H}^*(Y, \mathbb{Q})$. Now let $\gamma \in T \setminus \{1\}$ and consider the commutative diagram

$$\begin{tikzcd} \widetilde{E}_\gamma \arrow[r, "\tilde{\jmath}_\gamma"] \arrow[d, "\pi"'] & E_\gamma \arrow[d, "f|_{E_\gamma}"] \\ (\mathcal{H}^3_{\mathbb{C}})^\gamma / C_\Gamma(\gamma) \arrow[r, "\iota_\gamma"] & \mathcal{H}^3_{\mathbb{C}}/\Gamma \end{tikzcd}$$

where ι_γ is the morphism induced by the inclusion $(\mathcal{H}^3_{\mathbb{C}})^\gamma \hookrightarrow \mathcal{H}^3_{\mathbb{C}}$ and $E_\gamma \subset Y$ is the exceptional prime divisor corresponding to the class of γ by Proposition 28. Let

$\jmath_\gamma\colon \widetilde{E}_\gamma \to Y$ be the composition of $\tilde{\jmath}_\gamma\colon \widetilde{E}_\gamma \to E_\gamma$ followed by the inclusion $E_\gamma \hookrightarrow Y$. Notice that $\jmath_\gamma$ is proper since $\imath_\gamma$ is so (Lemma 8). Then we define

$$\Phi_\gamma(\alpha) := (\jmath_\gamma)_*(\pi^*(\alpha)) \quad \text{for all } \alpha \in \mathrm{H}^{*-2\,\mathrm{shift}(\gamma)}((\mathcal{H}^3_{\mathbb{C}})^\gamma / C_\Gamma(\gamma), \mathbb{Q}), \tag{7}$$

where for all $\beta \in \mathrm{H}^*(\widetilde{E}_\gamma, \mathbb{Q})$, $(\jmath_\gamma)_*(\beta) \in \mathrm{H}^{*+2}(Y, \mathbb{Q})$ is the cohomology class that corresponds via Poincaré duality (see [27, Chapter XIV]) to the following element of $\mathrm{H}^{4-*}_c(Y,\mathbb{Q})^\vee$ (the dual space of the cohomology of Y with compact support):

$$\omega \in \mathrm{H}^{4-*}_c(Y,\mathbb{Q}) \mapsto \int_{\widetilde{E}_\gamma} \beta \cup \jmath_\gamma^*(\omega). \tag{8}$$

Remark 29 In (8), $\widetilde{E}_\gamma$ is a complex analytic space of real dimension 4 (it is a divisor of Y). If it is singular, by the integral $\int_{\widetilde{E}_\gamma} \beta \cup \jmath_\gamma^*(\omega)$ we mean the integral of the pullback of $\beta \cup \jmath_\gamma^*(\omega)$ on a resolution of the singularities of $\widetilde{E}_\gamma$ (which is a complex manifold and hence it has a natural orientation). Notice that this does not depend on the particular resolution of $\widetilde{E}_\gamma$. If $\rho'\colon \widetilde{E}'_\gamma \to \widetilde{E}_\gamma$ and $\rho''\colon \widetilde{E}''_\gamma \to \widetilde{E}_\gamma$ are two resolutions of $\widetilde{E}_\gamma$, then there exists a third resolution $\widetilde{E}'''_\gamma$ with two morphisms $\rho_1\colon \widetilde{E}'''_\gamma \to \widetilde{E}'_\gamma$, $\rho_2\colon \widetilde{E}'''_\gamma \to \widetilde{E}''_\gamma$, such that $\rho' \circ \rho_1 = \rho'' \circ \rho_2$. One can take, for example, $\widetilde{E}'''_\gamma$ to be a resolution of the Cartesian product $\widetilde{E}'_\gamma \times_{\rho', \widetilde{E}_\gamma, \rho''} \widetilde{E}''_\gamma$. In particular, $\widetilde{E}'''_\gamma$ differs from $\widetilde{E}'_\gamma$ (respectively $\widetilde{E}''_\gamma$) by a closed analytic subspace of (complex) codimension $\geqslant 1$, which has measure zero and so the integral in (8) does not depend on the resolution of $\widetilde{E}_\gamma$.

Let us first notice that Φ is degree-preserving, since any $\gamma \in T \setminus \{1\}$ has $\mathrm{shift}(\gamma) = 1$ and $(\jmath_\gamma)_*\colon \mathrm{H}^*(\widetilde{E}_\gamma,\mathbb{Q}) \to \mathrm{H}^{*+2}(Y,\mathbb{Q})$ increases the degrees by two (the real codimension of $\widetilde{E}_\gamma$ in Y).

In the proof of Theorem 3, we will use a compatibility property of Φ with respect to open embeddings, as follows. Let $U \subset \mathcal{H}^3_{\mathbb{C}}/\Gamma$ be an open subset, and let $\widetilde{U} \subset \mathcal{H}^3_{\mathbb{C}}$ be the preimage of U with respect to the quotient map $\mathcal{H}^3_{\mathbb{C}} \to \mathcal{H}^3_{\mathbb{C}}/\Gamma$. Then the action of Γ on $\mathcal{H}^3_{\mathbb{C}}$ restricts to an action on $\widetilde{U}$, in such a way that $[\widetilde{U}/\Gamma]$ is an open suborbifold of $[\mathcal{H}^3_{\mathbb{C}}/\Gamma]$. The same definition of Φ gives a linear map

$$\Phi^U\colon \mathrm{H}^*_{\mathrm{CR}}([\widetilde{U}/\Gamma],\mathbb{Q}) \to \mathrm{H}^*(f^{-1}(U),\mathbb{Q}).$$

Lemma 30 *Under the previous notation, let $i\colon [\widetilde{U}/\Gamma] \hookrightarrow [\mathcal{H}^3_{\mathbb{C}}/\Gamma]$ and $j\colon f^{-1}(U) \hookrightarrow Y$ be the open inclusions. Then*

$$\Phi^U \circ i^* = j^* \circ \Phi.$$

Proof It suffices to prove that $\Phi_\gamma^U \circ i^* = j^* \circ \Phi_\gamma$ for any $\gamma \in T$ of finite order, where Φ_γ^U and Φ_γ are defined as in (7). If $\gamma = 1$ the claim follows by the functoriality property of the pullback. So, let us assume that $\gamma \neq 1$ and consider the commutative diagram

$$\begin{tikzcd}
\widetilde{E}_\gamma^U \arrow[r, "\jmath_\gamma^U"] \arrow[d, "\pi_|"'] \arrow[dr, "\tilde{\imath}"] & f^{-1}(U) \arrow[dr, "j"] & \\
\widetilde{U}^\gamma/C_\Gamma(\gamma) \arrow[dr, "i"'] & \widetilde{E}_\gamma \arrow[r, "\jmath_\gamma"] \arrow[d, "\pi"] & Y \\
& (\mathcal{H}_\mathbb{C}^3)^\gamma/C_\Gamma(\gamma) &
\end{tikzcd}$$

Here, by abuse of notation, we have denoted with $i\colon \widetilde{U}^\gamma/C_\Gamma(\gamma) \to (\mathcal{H}_\mathbb{C}^3)^\gamma/C_\Gamma(\gamma)$ the map induced by the inclusion $i\colon [\widetilde{U}/\Gamma] \hookrightarrow [\mathcal{H}_\mathbb{C}^3/\Gamma]$; π, $\widetilde{E}_\gamma$ and $\jmath_\gamma$ are defined as in the definition of Φ; $\widetilde{E}_\gamma^U := \pi^{-1}(\widetilde{U}^\gamma/C_\Gamma(\gamma))$; $\pi_|$ is the restriction of π; $\tilde{\imath}$ is the open inclusion; and $\jmath_\gamma^U$ is the restriction of $\jmath_\gamma$. The result follows if we prove that $(\jmath_\gamma^U)_* \circ \tilde{\imath}^* = j^* \circ (\jmath_\gamma)_*$. So, let $\beta \in \mathrm{H}^*(\widetilde{E}_\gamma)$. Then, for all $\delta \in \mathrm{H}_c^*(f^{-1}(U))$, we get

$$\int_{f^{-1}(U)} [(\jmath_\gamma^U)_* \circ \tilde{\imath}^*](\beta) \cup \delta = \int_{\widetilde{E}_\gamma^U} \tilde{\imath}^*(\beta) \cup (\jmath_\gamma^U)^*(\delta).$$

On the other hand, there exists $\widetilde{\delta} \in \mathrm{H}_c^*(Y)$ such that $\delta = j^*\widetilde{\delta}$ (this follows from the excision property [27, pages 320, 362 and 363]). Therefore,

$$\begin{aligned}
\int_{\widetilde{E}_\gamma^U} \tilde{\imath}^*(\beta) \cup (\jmath_\gamma^U)^*(\delta) &= \int_{\widetilde{E}_\gamma^U} \tilde{\imath}^*(\beta) \cup (\jmath_\gamma^U)^*(j^*\widetilde{\delta}) \\
&= \int_{\widetilde{E}_\gamma^U} \tilde{\imath}^*(\beta) \cup \tilde{\imath}^*(\jmath_\gamma^*(\widetilde{\delta})) \\
&= \int_{\widetilde{E}_\gamma^U} \tilde{\imath}^*[\beta \cup \jmath_\gamma^*(\widetilde{\delta})] \\
&= \int_{\widetilde{E}_\gamma} \beta \cup \jmath_\gamma^*(\widetilde{\delta}) \\
&= \int_Y (\jmath_\gamma)_*(\beta) \cup \widetilde{\delta} \\
&= \int_{f^{-1}(U)} [j^* \circ (\jmath_\gamma)_*](\beta) \cup \delta.
\end{aligned}$$

Using Poincaré duality, we conclude that $[(\jmath_\gamma^U)_* \circ \tilde{\imath}^*](\beta) = [j^* \circ (\jmath_\gamma)_*](\beta)$. $\square$

6.4 Proof of Theorem 3

In this section, we prove that the map Φ defined in the previous section is an isomorphism of graded $\mathbb{Q}$–algebras. Our approach has been inspired by [9]. As a stepping stone, we prove the following result:

Proposition 31 *The linear map $\Phi\colon \mathrm{H}^*_{\mathrm{CR}}([\mathcal{H}^3_{\mathbb{C}}/\Gamma],\mathbb{Q}) \to \mathrm{H}^*(Y,\mathbb{Q})$ defined in the previous section is an isomorphism of vector spaces.*

Proof We use the Mayer–Vietoris exact sequence for Chen–Ruan orbifold cohomology. We will define an appropriate open covering of the orbifold $[\mathcal{H}^3_{\mathbb{C}}/\Gamma]$. This induces an open covering of the inertia orbifold. Since the Chen–Ruan orbifold cohomology is the usual cohomology of the inertia orbifold, we have a Mayer–Vietoris long exact sequence.

The open covering is defined as follows. As before, let $x_1,\dots,x_s \in \mathcal{H}^3_{\mathbb{C}}/\Gamma$ be the singular points of the singular locus Σ of $\mathcal{H}^3_{\mathbb{C}}/\Gamma$, and let $X := (\mathcal{H}^3_{\mathbb{C}}/\Gamma)\setminus\{x_1,\dots,x_s\}$. Then there is a unique open suborbifold $\mathcal{X} \subset [\mathcal{H}^3_{\mathbb{C}}/\Gamma]$ having X as coarse moduli space. Notice that X is an analytic space with transverse singularities of type A. Now let, for any $i=1,\dots,s$, $W_i \subset \mathcal{H}^3_{\mathbb{C}}/\Gamma$ be an open neighbourhood of x_i isomorphic to $\widetilde{W}_i/G_i$, where $\widetilde{W}_i$ is an open subset of $\mathcal{H}^3_{\mathbb{C}}$ isomorphic to an open ball, and G_i is the stabilizer of a point $z_i \in \widetilde{W}_i$ that maps onto x_i under the quotient map $\mathcal{H}^3_{\mathbb{C}} \to \mathcal{H}^3_{\mathbb{C}}/\Gamma$. Without loss of generality, we suppose that $W_1,\dots,W_s$ are pairwise disjoint. Then $\mathcal{W} := \bigsqcup_{i=1}^s [\widetilde{W}_i/G_i]$ is an open suborbifold of $[\mathcal{H}^3_{\mathbb{C}}/\Gamma]$. Let us denote with $W := \bigsqcup_{i=1}^s W_i$ the coarse moduli space of $\mathcal{W}$.

Consider the open covering $\{\mathcal{X},\mathcal{W}\}$ of $[\mathcal{H}^3_{\mathbb{C}}/\Gamma]$, the open covering $\{f^{-1}(X), f^{-1}(W)\}$ of Y and the corresponding long exact cohomology sequences of Mayer–Vietoris. By Lemma 30, Φ induces a morphism between long exact sequences as follows:

$$\begin{tikzcd}
\mathrm{H}^{k-1}_{\mathrm{CR}}(\mathcal{X}\cap\mathcal{W}) \arrow[r, "\Phi^{X\cap W}"] \arrow[d] & \mathrm{H}^{k-1}(f^{-1}(X)\cap f^{-1}(W)) \arrow[d] \\
\mathrm{H}^{k}_{\mathrm{CR}}([\mathcal{H}^3_{\mathbb{C}}/\Gamma]) \arrow[r, "\Phi"] \arrow[d] & \mathrm{H}^{k}(Y) \arrow[d] \\
\mathrm{H}^{k}_{\mathrm{CR}}(\mathcal{X})\oplus\mathrm{H}^{k}_{\mathrm{CR}}(\mathcal{W}) \arrow[r, "\Phi^X\oplus\Phi^W"] \arrow[d] & \mathrm{H}^{k}(f^{-1}(X))\oplus\mathrm{H}^{k}(f^{-1}(W)) \arrow[d] \\
\mathrm{H}^{k}_{\mathrm{CR}}(\mathcal{X}\cap\mathcal{W}) \arrow[r, "\Phi^{X\cap W}"] & \mathrm{H}^{k}(f^{-1}(X)\cap f^{-1}(W))
\end{tikzcd}$$

The map Φ^W is an isomorphism by [22]. Therefore, the Proposition follows from the five lemma if Φ^X and $\Phi^{X\cap W}$ are isomorphisms. To see that they are isomorphisms, recall that X and $X\cap W$ are analytic spaces with transverse singularities of type A. Therefore, Φ^X and $\Phi^{X\cap W}$ are isomorphisms if the monodromy is trivial [30, Propositions 4.8 and 4.9]. On the other hand, if the monodromy is not trivial, then there is an unramified double covering $\widetilde{\mathcal{X}}\to\mathcal{X}$ such that $\widetilde{\mathcal{X}}$ has transverse singularities of type A and trivial monodromy (Proposition 27). Let $\widetilde{f^{-1}(X)}\to\widetilde{X}$ be the crepant resolution of $\widetilde{X}$ (the coarse moduli space of $\widetilde{\mathcal{X}}$). Then, there is a natural map $\widetilde{f^{-1}(X)}\to f^{-1}(X)$, which is an unramified double covering. Since $\mathrm{H}^*_{\mathrm{CR}}(\mathcal{X})\cong\mathrm{H}^*_{\mathrm{CR}}(\widetilde{\mathcal{X}})^{\mathbb{Z}/2\mathbb{Z}}$ [30, Proposition 3.13] and $\mathrm{H}^*(f^{-1}(X))\cong\mathrm{H}^*(\widetilde{f^{-1}(X)})^{\mathbb{Z}/2\mathbb{Z}}$, we conclude that Φ^X is an isomorphism. The same proof works for $\Phi^{X\cap W}$. □

The proof of Theorem 3 is now completed when combining Proposition 31 with the following statement.

Proposition 32 $\Phi\colon(\mathrm{H}^*_{\mathrm{CR}}([\mathcal{H}^3_{\mathbb{C}}/\Gamma],\mathbb{Q}),\cup_{\mathrm{CR}})\to(\mathrm{H}^*(Y,\mathbb{Q}),\cup)$ *is a ring homomorphism.*

Proof Notice that on the nontwisted sector, Φ preserves the cup products because f^* is a ring homomorphism. So let α_g and β_h be cohomology classes of the twisted sectors $(\mathcal{H}^3_{\mathbb{C}})^g/C_\Gamma(g)$, $(\mathcal{H}^3_{\mathbb{C}})^h/C_\Gamma(h)$. Since $\mathrm{shift}(g)=\mathrm{shift}(h)=1$, the Chen–Ruan degrees $\deg(\alpha_g)$ and $\deg(\beta_h)$ are $\geqslant 2$, hence $\deg(\alpha_g\cup_{\mathrm{CR}}\beta_h)\geqslant 4$. By Theorem 17, we conclude that $\mathrm{H}^d_{\mathrm{CR}}([\mathcal{H}^3_{\mathbb{C}}/\Gamma])=0$ if $d\geqslant 4$; so $\alpha_g\cup_{\mathrm{CR}}\beta_h=0$. On the other hand, since Φ is grading-preserving, $\deg(\Phi(\alpha_g)\cup\Phi(\beta_h))\geqslant 4$, so also $\Phi(\alpha_g)\cup\Phi(\beta_h)=0$ because $\mathrm{H}^d(Y)\cong\mathrm{H}^d_{\mathrm{CR}}([\mathcal{H}^3_{\mathbb{C}}/\Gamma])$ for any d. Finally, let $\alpha_g\in\mathrm{H}^*((\mathcal{H}^3_{\mathbb{C}})^g/C_\Gamma(g))$ and $\beta\in\mathrm{H}^*(\mathcal{H}^3_{\mathbb{C}}/\Gamma)$. Then, $\alpha_g\cup_{\mathrm{CR}}\beta=\alpha_g\cup\iota_g^*\beta\in\mathrm{H}^*((\mathcal{H}^3_{\mathbb{C}})^g/C_\Gamma(g))$, so $\Phi(\alpha_g\cup_{\mathrm{CR}}\beta)=(\jmath_g)_*\pi^*(\alpha_g\cup\iota_g^*\beta)$. On the other hand,

$$\begin{aligned}\Phi(\alpha_g)\cup\Phi(\beta)&=(\jmath_g)_*\pi^*(\alpha_g)\cup f^*(\beta)\\&=(\jmath_g)_*\big(\pi^*(\alpha_g)\cup\jmath_g^*(f^*(\beta))\big)\\&=(\jmath_g)_*\big(\pi^*(\alpha_g)\cup\pi^*(\iota_g^*(\beta))\big)\\&=(\jmath_g)_*\pi^*(\alpha_g\cup\iota_g^*(\beta))\\&=\Phi(\alpha_g\cup_{\mathrm{CR}}\beta).\end{aligned}$$

where the second equality is due to the projection formula and the third because $f\circ\jmath_g=\iota_g\circ\pi$. □

7 Cohomological crepant resolution conjecture for Bianchi orbifolds

In this section, we compare the results obtained so far with the cohomological crepant resolution conjecture of Ruan. We begin by briefly reviewing the statement of this conjecture, referring to [41; 12] for further details.

Let $\mathcal{X}$ be a complex orbifold, and let X be its coarse moduli space. We assume that X is a complex projective variety which has a crepant resolution $f\colon Y \to X$. The corrected quantum cohomology ring of $f\colon Y \to X$ is a ring structure on the vector space $\mathrm{H}^*(Y,\mathbb{C}) = \bigoplus_{d\geqslant 0} \mathrm{H}^d(Y,\mathbb{C})$, which is a deformation of the standard cohomology ring of Y. Its definition depends on the choice of a basis of $\ker(f_*\colon \mathrm{H}_2(Y,\mathbb{Q}) \to \mathrm{H}_2(X,\mathbb{Q}))$ consisting of homology classes of effective curves $\beta_1,\dots,\beta_n$. One defines the 3–point function

$$(\alpha_1,\alpha_2,\alpha_3)(q_1,\dots,q_n) = \sum_{(k_1,\dots,k_n)\in\mathbb{N}^n} \langle\alpha_1,\alpha_2,\alpha_3\rangle^Y_\beta q_1^{k_1}\cdots q_n^{k_n}, \tag{9}$$

where $\beta = k_1\beta_1 + \cdots + k_n\beta_n \in \mathrm{H}_2(Y,\mathbb{Z})$, and $\langle\alpha_1,\alpha_2,\alpha_3\rangle^Y_\beta$ is the Gromov–Witten invariant of Y, of genus 0, of homology class β, with respect to the cohomology classes $\alpha_1,\alpha_2,\alpha_3 \in \mathrm{H}^*(Y,\mathbb{C})$. Recall that a compact complex curve $D \subset Y$ of homology class β is called an *exceptional curve* for f. To simplify the discussion, we assume that the 3–point function (9) converges in a neighbourhood of the origin $(q_1,\dots,q_n) = (0,\dots,0)$ (see [12] for the general case); then, for any $(q_1,\dots,q_n)$ in this neighbourhood, we define a product $\star_f$ on the cohomology of Y as follows: given cohomology classes α_1 and α_2, $\alpha_1 \star_f \alpha_2$ is the cohomology class which satisfies the equation

$$(\alpha_1 \star_f \alpha_2, \alpha_3) = (\alpha_1,\alpha_2,\alpha_3)(q_1,\dots,q_n) \quad \text{for all } \alpha_3 \in \mathrm{H}^*(Y,\mathbb{C}),$$

where the pairing $(\,,)$ to the left-hand side is the Poincaré pairing of Y. The product $\star_f$ satisfies the usual properties of the cup product, eg it is associative and graded-commutative, and 1 is its neutral element. The family of rings $(\mathrm{H}^*(Y,\mathbb{C}),\star_f)$, as $(q_1,\dots,q_n)$ varies, is part of the (small) quantum cohomology of Y. Assigning to $q_1,\dots,q_n$ specific values, we obtain the so-called corrected quantum cohomology ring of $f\colon Y \to X$ [41; 12]. Notice that if $(q_1,\dots,q_n) = (0,\dots,0)$, then $\star_f$ coincides with the usual cup product, as it follows from the fact that $(\alpha_1,\alpha_2,\alpha_3)(0,\dots,0) = \int_Y \alpha_1\cup\alpha_2\cup\alpha_3$. So, the corrected quantum cohomology ring of $f\colon Y \to X$ is regarded as a deformation of the usual cohomology ring of Y.

Ruan's cohomological crepant resolution conjecture predicts that there is an analytic continuation of (9) to a region containing a point $(\overline{q}_1, \dots, \overline{q}_n)$ such that, for $(q_1, \dots, q_n) = (\overline{q}_1, \dots, \overline{q}_n)$, the ring $(\mathrm{H}^*(Y, \mathbb{C}), \star_f)$ is isomorphic to the Chen–Ruan orbifold cohomology ring $(\mathrm{H}^*_{\mathrm{CR}}(\mathcal{X}), \cup_{\mathrm{CR}})$ of $\mathcal{X}$.

In the case of a Bianchi orbifold $[\mathcal{H}^3_{\mathbb{C}}/\Gamma]$, the coarse moduli space $\mathcal{H}^3_{\mathbb{C}}/\Gamma$ is not a projective variety [13], and so, for every crepant resolution $f\colon Y \to \mathcal{H}^3_{\mathbb{C}}/\Gamma$, Y is not a projective variety. Hence the Gromov–Witten invariants of Y are, in general, not well defined. However, we will see that $[\mathcal{H}^3_{\mathbb{C}}/\Gamma]$ has a Kähler structure, and that one does not expect nonzero quantum corrections coming from exceptional curves for f. This is motivated by a deformation-theoretic argument about the complex structure of Y (let us recall that the Gromov–Witten invariants are invariant under deformations of the complex structure). More precisely, we conjecture that, for any homology class $\beta \in \mathrm{H}_2(Y, \mathbb{Z})$ of a connected exceptional curve for f, there is an open subset $\mathcal{U} \subset Y$ containing all the connected curves $D \subset Y$ of homology class β, and a deformation of the complex structure of $\mathcal{U}$ that does not contain any compact complex curve.

In this article we prove the latter conjecture in one special case, namely $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-5})$, while the general case should be feasible with similar arguments. Hence, in accordance with Ruan's conjecture, there should be a ring isomorphism $(\mathrm{H}^*_{\mathrm{CR}}([\mathcal{H}^3_{\mathbb{C}}/\Gamma]), \cup_{\mathrm{CR}}) \cong (\mathrm{H}^*(Y), \cup)$. This is confirmed by our Theorem 3.

Proposition 33 *Let $[\mathcal{H}^3_{\mathbb{C}}/\Gamma]$ be a Bianchi orbifold. Then the Bergman metric on $\mathcal{H}^3_{\mathbb{C}}$ descends to a Kähler (orbifold) metric on $[\mathcal{H}^3_{\mathbb{C}}/\Gamma]$.*

Proof Let

$$\mathrm{d}s^2 = \sum g_{\alpha\overline{\beta}}\, \mathrm{d}z_\alpha\, \mathrm{d}\overline{z}_\beta$$

be the Bergman metric on $\mathcal{H}^3_{\mathbb{C}}$. By [29, Theorem 8.4, page 144], $\mathrm{d}s^2$ is invariant under the action of Γ, hence it induces a Kähler metric on the orbifold $[\mathcal{H}^3_{\mathbb{C}}/\Gamma]$. □

Now let $f\colon Y \to \mathcal{H}^3_{\mathbb{C}}/\Gamma$ be a crepant resolution. Let $D \subset Y$ be an exceptional, compact, complex and connected curve, that is, $f_*([D]) = 0$, where $[D]$ is the fundamental class of D. Since $[\mathcal{H}^3_{\mathbb{C}}/\Gamma]$ is Kähler, $f(D)$ is a point, so D is contained in the exceptional divisor of f. In particular, for any homology class $\beta \in \ker(f_*\colon \mathrm{H}_2(Y, \mathbb{Q}) \to \mathrm{H}_2(\mathcal{H}^3_{\mathbb{C}}/\Gamma, \mathbb{Q}))$ and for any stable map $\mu\colon C \to Y$ such that $\mu_*([C]) = \beta$, the image of μ is contained in the exceptional divisor of f. Hence it suffices to consider the problem locally in a neighbourhood of the exceptional divisor.

From the results of Sections 3 and 5, we see that the singular locus of $\mathcal{H}^3_{\mathbb{C}}/\Gamma$ is the union of several irreducible components, each of which is isomorphic either to $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ or to $\Delta^* = \Delta \setminus \{0\}$. Furthermore, the generic point of each irreducible component of the singular locus is a transverse singularity of type A_n of $\mathcal{H}^3_{\mathbb{C}}/\Gamma$, with $n = 1$ or 2.

Let us now consider the special case where $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-5})$ (see Section 8.2). In this case we show that the quantum corrections to the cohomology ring of Y coming from exceptional curves vanish. The singular locus of $X = \mathcal{H}^3_{\mathbb{C}}/\Gamma$ has two connected components, $X_{(2)} \cong \Delta^*$, whose points are transverse singularities of type A_2, and $X_{(1)}$, which is the union of three irreducible components, $X'_{(1)}, X''_{(1)}, X'''_{(1)} \cong \Delta$, which meet in two points P and Q such that the complement $X_{(1)} \setminus \{P, Q\}$ is a locus of transverse singularities of type A_1 (see Figure 2 and Section 8.2). The exceptional divisor of $f\colon Y \to X$ has two connected components: $E_{(2)}$, which is mapped to $X_{(2)}$ by f, and $E_{(1)}$ such that $f(E_{(1)}) = X_{(1)}$. Furthermore, $E_{(1)}$ has three irreducible components, $E'_{(1)}$, $E''_{(1)}$ and $E'''_{(1)}$, which are mapped by f to $X'_{(1)}$, $X''_{(1)}$ and $X'''_{(1)}$, respectively.

Let us consider first the A_2–singularities $X_{(2)}$. Notice that from the presentation of the Chen–Ruan cohomology (Section 8.2), it follows that $\mathcal{H}^3_{\mathbb{C}}/\Gamma$ has trivial monodromy on $X_{(2)}$. Hence there is an open neighbourhood U of $X_{(2)}$, such that $U \cong \widetilde{U}/(\mathbb{Z}/3\mathbb{Z})$, where $\widetilde{U}$ is a complex manifold with an action of $\mathbb{Z}/3\mathbb{Z}$, such that the fixed-point locus $\widetilde{U}^{\mathbb{Z}/3\mathbb{Z}}$ is a smooth submanifold of $\widetilde{U}$ isomorphic to $X_{(2)}$. Furthermore, up to deformation, we can assume that $\widetilde{U}$ is an open neighbourhood of the zero section of the normal bundle $N_{\widetilde{U}^{\mathbb{Z}/3\mathbb{Z}}}|_{\widetilde{U}}$ of $\widetilde{U}^{\mathbb{Z}/3\mathbb{Z}}$ in $\widetilde{U}$. This can be achieved using the *deformation to the normal cone* of the embedding $\widetilde{U}^{\mathbb{Z}/3\mathbb{Z}} \subset \widetilde{U}$ [18, Chapter 5]. The vector bundle map $N_{\widetilde{U}^{\mathbb{Z}/3\mathbb{Z}}}|_{\widetilde{U}} \to \widetilde{U}^{\mathbb{Z}/3\mathbb{Z}} \cong X_{(2)}$ induces a morphism $\widetilde{U}/(\mathbb{Z}/3\mathbb{Z}) \to X_{(2)}$ that equips $U \cong \widetilde{U}/(\mathbb{Z}/3\mathbb{Z})$ with the structure of a fibration over $X_{(2)}$, with fibres all isomorphic to the surface singularity of type A_2. The important fact is that this fibration is trivial. To see this, let us recall that the action of $\mathbb{Z}/3\mathbb{Z}$ on the fibres of $N_{\widetilde{U}^{\mathbb{Z}/3\mathbb{Z}}}|_{\widetilde{U}}$ induces a splitting, $N_{\widetilde{U}^{\mathbb{Z}/3\mathbb{Z}}}|_{\widetilde{U}} = \mathbb{L} \oplus \mathbb{M}$, where $\mathbb{L}$ and $\mathbb{M}$ are the eigenbundles corresponding to the irreducible characters of the representation of $\mathbb{Z}/3\mathbb{Z}$ on the fibres of $N_{\widetilde{U}^{\mathbb{Z}/3\mathbb{Z}}}|_{\widetilde{U}}$. In our case, $\mathbb{L}$ and $\mathbb{M}$ are trivial line bundles on $X_{(2)}$ (see [17, Theorem 30.3, page 229]), therefore the fibration $N_{\widetilde{U}^{\mathbb{Z}/3\mathbb{Z}}}|_{\widetilde{U}}/(\mathbb{Z}/3\mathbb{Z}) \to X_{(2)}$ is trivial, that is, it is isomorphic to the projection to the first factor of $X_{(2)} \times \{(u, v, w) \in \mathbb{C}^3 \mid uv = w^3\}$. Now, using the theory of deformations of rational double points (see [8; 44]), we deform the family $N_{\widetilde{U}^{\mathbb{Z}/3\mathbb{Z}}}|_{\widetilde{U}}/(\mathbb{Z}/3\mathbb{Z}) \to X_{(2)}$ to a family of affine smooth surfaces. Finally, consider the neighbourhood $\mathcal{U} := f^{-1}(U)$ of $E_{(2)}$. Taking a simultaneous resolution of the previous deformation of $N_{\widetilde{U}^{\mathbb{Z}/3\mathbb{Z}}}|_{\widetilde{U}}/(\mathbb{Z}/3\mathbb{Z}) \to X_{(2)}$, we obtain a deformation of $\mathcal{U}$ to a manifold that does not contain compact complex curves.

Let us now consider the exceptional curves that are contained in $E_{(1)}$. Notice that each component $E'_{(1)}$, $E''_{(1)}$ and $E'''_{(1)}$ can be seen as the exceptional divisor of a crepant resolution of a transverse singularity of type A_1. Hence, from our description of the obstruction bundles (Theorem 11) and from [30, Theorem 7.6], it follows that the exceptional curves contained in one of these components do not contribute to the quantum corrected cohomology ring of Y. If $D \subset Y$ is a connected exceptional curve which is contained in more than one component of $E_{(1)}$, then $f(D)$ coincides with P or Q, the points where the components $X'_{(1)}$, $X''_{(1)}$ and $X'''_{(1)}$ meet together. Near P and Q, X is isomorphic to the singularity $\mathbb{C}^3/\mathcal{D}_2$ (see Section 8.2), where $\mathcal{D}_2 = \langle \xi, \eta \mid \xi^2 = \eta^2 = (\xi\eta)^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. We can realize the quotient $\mathbb{C}^3/\mathcal{D}_2$ as $(\mathbb{C}^3/\langle\xi\rangle)/\langle\eta\rangle$, and notice that $\mathbb{C}^3/\langle\xi\rangle \cong \{(u,v,w,z) \in \mathbb{C}^3 \times \mathbb{C} \mid uv = w^2\}$ with the action of $\langle\eta\rangle$ given by $\eta \cdot (u,v,w,z) \mapsto (u,v,-w,-z)$. The semiuniversal deformation of $\{(u,v,w) \in \mathbb{C}^3 \mid uv = w^2\}$ is $uv = w^2 + t$, where t is the deformation parameter. Notice that the action of $\langle\eta\rangle$ on $\mathbb{C}^3/\langle\xi\rangle$ extends to $\{(u,v,w,z) \in \mathbb{C}^3 \times \mathbb{C} \mid uv = w^2 + t\}$ for all t, as follows: $\eta \cdot (u,v,w,z) = (u,v,-w,-z)$. Hence $\{(u,v,w,z) \in \mathbb{C}^3 \times \mathbb{C} \mid uv = w^2 + t\}/\langle\eta\rangle$ for $t \in \mathbb{C}$ is a deformation of $\mathbb{C}^3/\mathcal{D}_2$. Notice that for $t \neq 0$, $\{(u,v,w,z) \in \mathbb{C}^3 \times \mathbb{C} \mid uv = w^2 + t\}/\langle\eta\rangle$ has transverse singularities of type A_1, and they can be smoothed by a deformation as follows. Taking the invariants of the $\langle\eta\rangle$–action, we see that

$$\{(u,v,w,z) \in \mathbb{C}^3 \times \mathbb{C} \mid uv = w^2 + t\}/\langle\eta\rangle \cong \{(u,v,\rho,\sigma,\tau) \in \mathbb{C}^5 \mid uv = \rho + t,\ \rho\sigma = \tau^2\},$$

where $\rho = w^2$, $\sigma = z^2$ and $\tau = wz$. Thus $\{(u,v,\rho,\sigma,\tau) \in \mathbb{C}^5 \mid uv = \rho + t,\ \rho\sigma = \tau^2 + s\}$ is a deformation of $\mathbb{C}^3/\mathcal{D}_2$, with deformation parameters t and s. For $t \neq 0$ and $s \neq 0$, the variety $\{(u,v,\rho,\sigma,\tau) \in \mathbb{C}^5 \mid uv = \rho + t,\ \rho\sigma = \tau^2 + s\}$ is an affine smooth variety. Thus, a simultaneous resolution of this family yields a deformation of a neighbourhood of $f^{-1}(P)$ (respectively $f^{-1}(Q)$), such that the generic member of the family is a smooth affine variety. Hence, it does not contain compact complex curves.

8 Orbifold cohomology computations for sample Bianchi orbifolds

We will carry out our computations in the upper-halfspace model

$$\{x + iy + rj \in \mathbb{C} \oplus \mathbb{R}j \mid r > 0\}$$

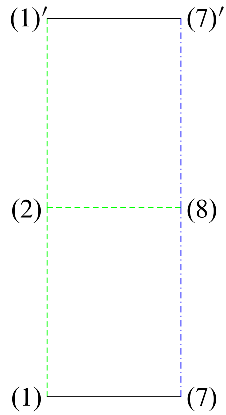


Figure 1: Fundamental domain in the case $m=2$.

for $\mathcal{H}^3_{\mathbb{R}}$ in six cases. Details on how to compute Chen–Ruan orbifold cohomology can be found in [30]. In the case $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\sqrt{-5}\,])$, we also compute the cohomology ring structure.

8.1 The case $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\sqrt{-2}\,])$

Let $\omega := \sqrt{-2}$. A fundamental domain for $\Gamma := \mathrm{PSL}_2(\mathbb{Z}[\omega])$ in real hyperbolic 3–space $\mathcal{H}$ has been found by Bianchi [5]. We can obtain it by taking the geodesic convex envelope of its lower boundary (half of which is depicted in Figure 1) and the vertex ∞, and then removing the vertex ∞, making it noncompact. The other half of the lower boundary consists of one isometric Γ–image of each of the depicted 2–cells (in fact, the depicted 2–cells are a fundamental domain for a Γ–equivariant retract of $\mathcal{H}$, which is described in [39]). The coordinates of the vertices of Figure 1 in the upper-halfspace model are $(1) = j$, $(1)' = \omega + j$, $(2) = \frac{1}{2}\omega + \sqrt{1/2}\,j$, $(7) = \frac{1}{2} + \sqrt{3/4}\,j$, $(7)' = \frac{1}{2} + \omega + \sqrt{3/4}\,j$, $(8) = \frac{1}{2} + \frac{1}{2}\omega + \frac{1}{2}j$.

The 2–torsion subcomplex (dashed) and the 3–torsion subcomplex (dotted) are indicated in the figure. The set of representatives of conjugacy classes can be chosen to be

$$T = \{\mathrm{Id}, \alpha, \gamma, \beta, \beta^2\},$$

with

$$\alpha = \pm\begin{pmatrix} 1 & \omega \\ \omega & -1 \end{pmatrix}, \quad \beta = \pm\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \gamma = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so α and γ are of order 2, and β is of order 3. Using Lemma 24 and with the help of our Bredon homology computations, we check the cardinality of T. The fixed-point sets are then the following subsets of complex hyperbolic space $\mathcal{H} := \mathcal{H}^3_{\mathbb{C}}$:

- $\mathcal{H}^{\mathrm{Id}} = \mathcal{H}$,
- $\mathcal{H}^{\alpha} =$ the complex geodesic line through (2) and (8),
- $\mathcal{H}^{\gamma} =$ the complex geodesic line through (1) and (2),
- $\mathcal{H}^{\beta} = \mathcal{H}^{\beta^2} =$ the complex geodesic line through (7) and (8).

The matrix

$$g = \pm \begin{pmatrix} 1 & -\omega \\ 0 & 1 \end{pmatrix}$$

performs a translation preserving the j–coordinate and sends the edge $(1)(7)$ onto the edge $(1)'(7)'$, so the orbit space $\mathcal{H}_{\mathbb{R}}/\,\Gamma$ is homotopy equivalent to a circle. Consider the real geodesic line $\mathcal{H}^{\gamma}_{\mathbb{R}}$ on the unit circle of real part zero. The edge $g^{-1}\cdot((2)(1)') = (g^{-1}(2))(1)$ lies on $\mathcal{H}^{\gamma}_{\mathbb{R}}$ and is not Γ–equivalent to the edge $(1)(2)$. Because of Lemma 15, the centralizer $C_\Gamma(\gamma)$ reflects the line $\mathcal{H}^{\gamma}_{\mathbb{R}}$ onto itself at (2), and again at $g^{-1}(2)$. Furthermore, none of the four elements of Γ sending (2) to $g^{-1}(2)$ belongs to $C_\Gamma(\gamma)$. Hence the quotient space $\mathcal{H}^{\gamma}_{\mathbb{R}}/C_\Gamma(\gamma)$ consists of a contractible segment of two adjacent edges. Thus

$$\mathrm{H}^{d-2}(\mathcal{H}^{\gamma}_{\mathbb{C}}/C_\Gamma(\gamma);\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } d = 2, \\ 0 & \text{else,} \end{cases}$$

is contributed to the orbifold cohomology.

Next, consider the real geodesic line $\mathcal{H}^{\beta}_{\mathbb{R}}$ on the circle of constant real coordinate $\frac{1}{2}$, of centre $\frac{1}{2}$ and radius $\sqrt{3/4}$. The edge $g^{-1}\cdot((8)(7)') = (g^{-1}(8))(7)$ lies on $\mathcal{H}^{\beta}_{\mathbb{R}}$ and is not Γ–equivalent to the edge $(7)(8)$. The centralizer of β contains the matrix

$$V := \pm \begin{pmatrix} 21-\omega & \\ \omega-1 & 1+\omega \end{pmatrix}$$

of infinite order, which sends the edge $(g^{-1}(8))(7)$ to $(8)z$ with $z = \frac{1}{2} + \frac{3}{5}\omega + \sqrt{3/100}\, j$. We conclude that the translation action of the group $\langle V \rangle$ on the line $\mathcal{H}^{\beta}_{\mathbb{R}}$ is transitive with quotient space represented by the circle $(g^{-1}(8))(7) \cup (7)(8)$, first and last vertex identified. Thus

$$\mathrm{H}^{d-2}(\mathcal{H}^{\beta}_{\mathbb{C}}/C_\Gamma(\beta);\mathbb{Q}) \cong \mathrm{H}^{d-2}(\mathcal{H}^{\beta^2}_{\mathbb{C}}/C_\Gamma(\beta^2);\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } d = 2, 3, \\ 0 & \text{else,} \end{cases}$$

is contributed to the orbifold cohomology.

Because of Lemma 15, the centralizer $C_\Gamma(\alpha)$ reflects the line $\mathcal{H}^\alpha_{\mathbb{R}}$ onto itself at (2), and again at (8). So, the quotient space $\mathcal{H}^\alpha_{\mathbb{R}}/C_\Gamma(\alpha)$ is represented by the single contractible edge $(2)(8)$. This yields that

$$\mathrm{H}^{d-2}(\mathcal{H}^\alpha_{\mathbb{C}}/C_\Gamma(\alpha);\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } d=2, \\ 0 & \text{else,} \end{cases}$$

is contributed to the orbifold cohomology.

Summing up over T, we obtain

$$\mathrm{H}^d_{\mathrm{CR}}\big([\mathcal{H}^3_{\mathbb{C}}/\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-2}\,])]\big) \cong \mathrm{H}^d(\mathcal{H}_{\mathbb{C}}/\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-2}\,]);\mathbb{Q}) \oplus \begin{cases} \mathbb{Q}^4 & \text{if } d=2, \\ \mathbb{Q}^2 & \text{if } d=3, \\ 0 & \text{otherwise.} \end{cases}$$

8.2 The case $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-5})$

Let $\mathcal{O}_{-5} = \mathbb{Z}[\sqrt{-5}]$, $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-5})$. In this case the singular locus of X has two connected components. One component is a transverse singularity of type A_2 (we write $\mathrm{t}A_2$). The other component, drawn in Figure 2, contains two singular points P and Q which are analytically isomorphic to the singularity at the origin of $\mathbb{C}^3/\mathcal{D}_2$, where

$$\mathcal{D}_2 = \langle \xi, \eta \mid \xi^2 = \eta^2 = (\xi\eta)^2 = 1 \rangle \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

is the Klein four group acting via the standard diagonal representation $\mathcal{D}_2 \to \mathrm{SL}_3(\mathbb{C})$,

$$\xi \mapsto \mathrm{diag}(-1,-1,1), \qquad \eta \mapsto \mathrm{diag}(-1,1,-1).$$

The points P and Q are joined by three curves of transverse singularities of type A_1 (we write $\mathrm{t}A_1$), which correspond in a neighbourhood of P (resp. Q) to the image in $\mathbb{C}^3/\mathcal{D}_2$ of the coordinate axes of $\mathbb{C}^3$.

From Corollary 2, we get the following presentation of the Chen–Ruan cohomology:

$$H^d_{\mathrm{CR}}([\mathcal{H}^3_{\mathbb{C}}/\mathrm{PSL}_2(\mathcal{O}_{-5})],\mathbb{Q}) \cong H^d(\mathcal{H}^3_{\mathbb{C}}/\mathrm{PSL}_2(\mathcal{O}_{-5}),\mathbb{Q}) \oplus \begin{cases} \mathbb{Q}^2 \oplus \mathbb{Q}^3 & \text{if } d=2, \\ \mathbb{Q}^2 \oplus \{0\} & \text{if } d=3, \end{cases}$$

where the first direct summand is the cohomology of the nontwisted sector. The second direct summand $\binom{\mathbb{Q}^2}{\mathbb{Q}^2}$ is the cohomology of the 3–torsion twisted sector $\mathcal{X}_{(2)}$ whose coarse moduli space is the connected component of the singular locus of X corresponding to the $\mathrm{t}A_2$–singularity. Notice that this locus is topologically isomorphic to $S^1 \times \mathbb{R} \cong \mathbb{C}^*$, $\lambda_6 = 1$ and $\lambda_6^* = 0$, where λ_{2n} and λ^*_{2n} are as defined in Corollary 2. Finally, the third direct summand $\binom{\mathbb{Q}^3}{\{0\}}$ is the cohomology of the 2–torsion twisted sector $\mathcal{X}_{(1)}$. This sector has three connected components, each one homeomorphic

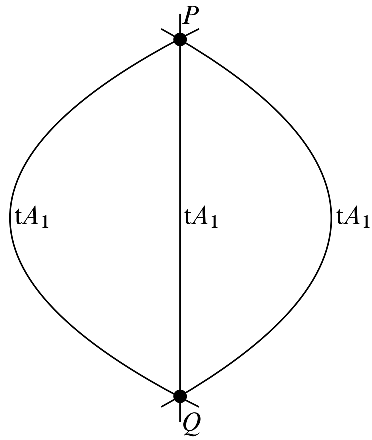


Figure 2: Two singular points P and Q which are analytically isomorphic to the singularity at the origin of $\mathbb{C}^3/\mathcal{D}_2$

to the strip $[0,1]\times\mathbb{R}$ and corresponding to the tA_1–singularities joining the points P and Q in Figure 2. In the coarse moduli space X, these components form the configuration in Figure 2. Here, we have $\lambda_4=\lambda_4^*=3$.

Now we study the Chen–Ruan cup product $\cup_{\mathrm{CR}}$, verifying first that the ordinary cup product on the nontwisted sector $H^*(\mathcal{H}^3_{\mathbb{C}}/\Gamma,\mathbb{Q})$ vanishes. From the explicit

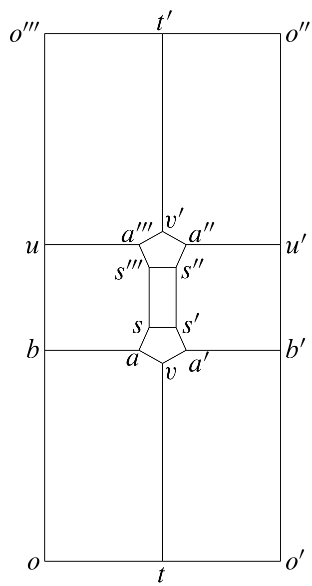


Figure 3: Fundamental domain for the Borel–Serre compactification in the case $m=5$.

description of the quotient space $\mathcal{H}^3_{\mathbb{R}}/\mathrm{PSL}_2(\mathcal{O}_{-5})$ in [39], we get the picture of the Borel–Serre compactification of $\mathcal{H}^3_{\mathbb{R}}/\mathrm{PSL}_2(\mathcal{O}_{-5})$ drawn in Figure 3. Here, we have expanded the singular cusp at $\frac{1}{2}(\sqrt{-5}+1)$ to a fundamental rectangle (s, s', s'', s''') for the action of the cusp stabilizer $\Gamma_{(\sqrt{-5}+1)/2}$ on the plane attached by the Borel–Serre bordification. In the same way, we expand the cusp at infinity to a fundamental rectangle $(\infty, \infty', \infty'', \infty''')$ for the action of the cusp stabilizer Γ_∞ on the plane attached there. This is not visible in our 2–dimensional diagram, but is located above the rectangle (o, o', o'', o'''), where o is of height 1 one above the cusp 0. The fundamental polyhedron for the Γ–action is then spanned by the rectangle $(\infty, \infty', \infty'', \infty''')$ and the polygons of Figure 3. The face identifications of the fundamental polyhedron are

$$(\infty, o, t, o', \infty') \sim (\infty''', o''', t', o'', \infty''), \tag{10}$$

$$(\infty, o, b, u, o''', \infty''') \sim (\infty', o', b', u', o'', \infty''), \tag{11}$$

$$(a''', s''', s'', a'', v') \sim (a, s, s', a', v), \tag{12}$$

$$(u, a''', s''', s, a, b) \sim (u', a'', s'', s', a', b'), \tag{13}$$

$$(o, t, v, a, b) \sim (o', t, v, a', b'), \tag{14}$$

$$(o''', t', v', a''', u) \sim (o'', t', v', a'', u'). \tag{15}$$

Here, we did not respect the orientation of the 2–cells, but have written them in the way in which their vertices are identified.

It is well known that the Borel–Serre compactification of $\mathcal{H}^3_{\mathbb{R}}/\mathrm{PSL}_2(\mathcal{O}_{-m})$ is homotopy equivalent to $\mathcal{H}^3_{\mathbb{R}}/\mathrm{PSL}_2(\mathcal{O}_{-m})$ itself, and it has been worked out in [34] how the boundary is attached in the compactification.

So we can describe the cohomology cocycles of $\mathcal{H}^3_{\mathbb{R}}/\mathrm{PSL}_2(\mathcal{O}_{-5})$ in terms of the above fundamental polyhedron and face identifications. By [19, Section 9.3], $\mathcal{H}^3_{\mathbb{C}}$ admits a fundamental polyhedron $P_{\mathbb{C}}$ for Γ with the interior of its top-dimensional facets (called *sides*) being open smooth submanifolds. This yields a Γ–equivariant cell structure on $\mathcal{H}^3_{\mathbb{C}}$. The natural map $\mathcal{H}^3_{\mathbb{R}} \hookrightarrow \mathcal{H}^3_{\mathbb{C}} \to \mathcal{H}^3_{\mathbb{R}}$ induces a map of the sides with respect to the fundamental polyhedron $P_{\mathbb{R}}$ for Γ on $\mathcal{H}^3_{\mathbb{C}}$,

$$\mathrm{sides}(P_{\mathbb{R}}) \hookrightarrow \mathrm{sides}(P_{\mathbb{C}}) \to \mathrm{sides}(P_{\mathbb{R}}),$$

which respects the side identifications (*side pairings*). All of the side pairings of $P_{\mathbb{C}}$ are detected this way, because they generate the group Γ (see [19, Section 9.3]), and so do already the side pairings of $P_{\mathbb{R}}$. Hence there are no additional identifications when complexifying the orbifold, and thus there are no additional cohomology cocycles on

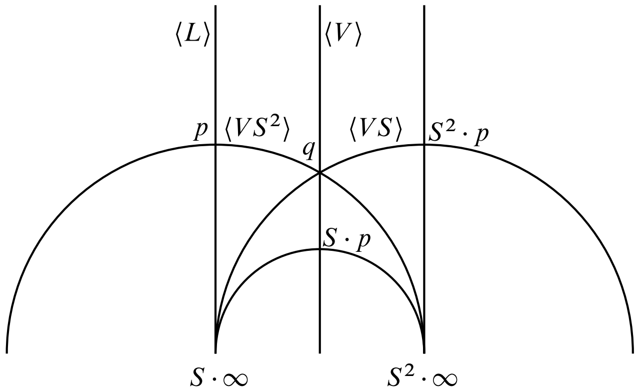


Figure 4: Geodesics fixed by certain finite-order elements of $\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-1}\,])$.

$\mathcal{H}^3_{\mathbb{C}}/\mathrm{PSL}_2(\mathcal{O}_{-5})$. Generators for $H^1(\mathcal{H}^3_{\mathbb{R}}/\mathrm{PSL}_2(\mathcal{O}_{-5}),\mathbb{Q})$ are, with reference to the above numbering of the identifications, obtained from

$$(\infty,\infty''') \quad \text{under } (1) \qquad \text{and} \qquad (s,s''') \quad \text{under } (3).$$

Both (∞,∞') under (2) and (s,s') under (4) yield trivial cocycles because of the identifications (5) and (6).

For instance, using the arc method introduced in [21, Section 3.2], we can now check explicitly that the cup product of the two cocycles obtained from (∞,∞''') under (1) and (s,s''') under (3) vanishes. In addition to the two 1–dimensional cocycles, $\mathcal{H}^3_{\mathbb{R}}/\,\Gamma$ only admits a 0– and a 2–dimensional cocycle, and as there are no further identifications when complexifying, we arrive at the claimed vanishing of the ordinary cup product on the nontwisted sector $H^*(\mathcal{H}^3_{\mathbb{C}}/\,\Gamma,\mathbb{Q})$.

The cup product of two classes coming from the twisted sectors would be a class in dimension $\geqslant 4$, where the cohomology of the twisted sectors vanishes, and by the above calculation, so does the cohomology of the nontwisted sector.

Therefore, the Chen–Ruan cup product $\cup_{\mathrm{CR}}$ is trivial on $[\mathcal{H}^3_{\mathbb{C}}/\mathrm{PSL}_2(\mathcal{O}_{-5})]$.

8.3 The case $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\sqrt{-1}\,])$

Let $i := \sqrt{-1}$. A fundamental domain for the action of $\Gamma := \mathrm{PSL}_2(\mathbb{Z}[i])$ on real hyperbolic 3–space $\mathcal{H}$ has been found by Bianchi, and the stabilizers have been computed by Flöge [16], whose notation we are going to adopt. It is drawn in Figure 5.

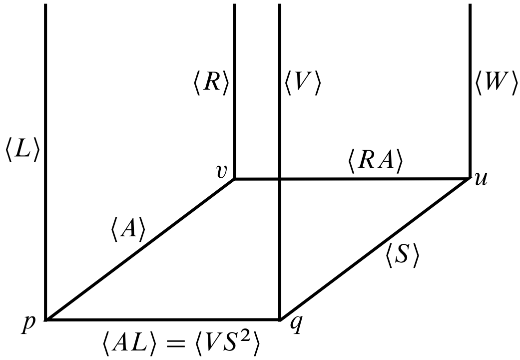


Figure 5: Half of a fundamental domain for the action of $\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-1}\,])$ on $\mathcal{H}$, open towards the cusp at ∞. The second half can be obtained as a copy of the open pyramid glued from below to its base square.

Here the vertex stabilizers are

$$\Gamma_p = \langle A, L\rangle \cong \mathcal{D}_2, \quad \Gamma_q = \langle V, S\rangle \cong \mathcal{D}_3, \quad \Gamma_u = \langle W, S\rangle \cong \mathcal{A}_4, \quad \Gamma_v = \langle R, A\rangle \cong \mathcal{D}_3,$$

where

$$A = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad L = \pm\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad S = \pm\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix},$$
$$R = \pm\begin{pmatrix} -i & 1 \\ 0 & i \end{pmatrix}, \quad V = \pm\begin{pmatrix} -i & -i \\ 0 & i \end{pmatrix} \quad \text{and} \quad W = \pm\begin{pmatrix} -i & 1-i \\ 0 & i \end{pmatrix},$$
$$1 = A^2 = L^2 = V^2 = R^2 = S^3 = W^2.$$

The matrices mentioned in Figure 5 (and their square when they are of order 3) constitute a system of representatives modulo Γ of the nontrivial elements of finite order. So we compute the respective quotients of their rotation axis by their centralizer, in order to obtain the CR orbifold cohomology. For the elements of order 3, namely RA and S, Theorem 20 and its proof pass unchanged, so $\mathcal{H}^{RA}/C_\Gamma(\langle RA\rangle) \cong \bigcirc\!\!\bullet$ and $\mathcal{H}^{S}/C_\Gamma(\langle S\rangle) \cong \bigcirc\!\!\bullet$.

For the elements of order 2, we study the quotient of their fixed geodesic by their centralizer through Figure 4. Further, we obtain another such figure useful for our purpose by making the following replacements in Figure 4: $q \mapsto v$, $S \mapsto (RA)^2$, $VS^2 \mapsto A$ and $V \mapsto R$. The symmetries obtained from combining complex conjugation with the rotation by L ensure that the relabelled figure is isometric to the printed one.

The points p, $S\cdot p$, $S^2\cdot p$, $(RA)^2\cdot p$ and $R\cdot p$ all have stabilizer type $\mathcal{D}_2$, because they are on the orbit of p, and hence the 2–torsion axes passing through them are mirrored by order-2 elements commuting with the rotation around the respective axis. We immediately conclude that $\mathcal{H}^L/C_\Gamma(\langle L\rangle)$ is represented by the half-open interval $[p,\infty)$. In the stabilizer of q, which is of type $\mathcal{D}_3$, apart from the trivial element, only the order-3 element and its square commute with each other. So there are no mirrorings at q in the centralizer of the rotations with axis passing through q. Hence, $\mathcal{H}^V/C_\Gamma(\langle V\rangle)\cong[S\cdot p,q,\infty)$ and $\mathcal{H}^{VS^2}/C_\Gamma(\langle VS^2\rangle)\cong[p,q,S^2\cdot\infty)$.

By the above described-replacements in Figure 4, we obtain analogously that

$$\mathcal{H}^R/C_\Gamma(\langle R\rangle)\cong[(RA)^2\cdot p,v,\infty)\quad\text{and}\quad\mathcal{H}^A/C_\Gamma(\langle A\rangle)\cong[p,v,RA\cdot\infty).$$

In the stabilizer of the point u, there are order-2 elements commuting with W, and therefore

$$\mathcal{H}^W/C_\Gamma(\langle W\rangle)\cong[u,\infty).$$

Summing up, and taking into account that $\mathcal{H}/\Gamma$ is contractible, we obtain the CR orbifold cohomology

$$\mathrm{H}^d_{\mathrm{CR}}\big([\mathcal{H}^3_{\mathbb{C}}/\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-1}\,])]\big)\cong\begin{cases}\mathbb{Q} & \text{if } d=0,\\ \mathbb{Q}^{10} & \text{if } d=2,\\ \mathbb{Q}^4 & \text{if } d=3,\\ 0 & \text{otherwise.}\end{cases}$$

8.4 The case $\Gamma=\mathrm{PSL}_2(\mathcal{O}_{-3})$

Let $\omega:=\frac{1}{2}(\sqrt{-3}-1)$. A fundamental domain for the action of $\Gamma:=\mathrm{PSL}_2(\mathbb{Z}[\omega])$ on real hyperbolic 3–space $\mathcal{H}$ has been found by Bianchi, and the stabilizers have been computed by Flöge [16], whose notation we are going to adopt. It is drawn in Figure 6. Here the vertex stabilizers are

$$\Gamma_u=\langle A,L\rangle\cong\mathcal{D}_3,\quad \Gamma_v=\langle K,S\rangle\cong\mathcal{A}_4,\quad \Gamma_w=\langle M,S\rangle\cong\mathcal{A}_4,$$

where

$$A=\pm\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad L=\pm\begin{pmatrix}-\omega^2&0\\0&\omega\end{pmatrix},\quad S=\pm\begin{pmatrix}0&-1\\1&1\end{pmatrix},\quad K=\pm\begin{pmatrix}\omega^2&-\omega\\0&\omega\end{pmatrix},$$
$$1=A^2=L^3=K^3=S^3=M^2.$$

As $\langle S\rangle\cong\mathbb{Z}/3$ and $\Gamma_v\cong\mathcal{A}_4\cong\Gamma_w$, the latter two vertex stabilizers neither reflect $\mathcal{H}^S$, nor do they contribute any element to $C_\Gamma(\langle S\rangle)$. That is why, though all cusps are on one Γ–orbit, the centralizer $C_\Gamma(\langle S\rangle)\cong\langle S\rangle$ leaves pointwise fixed $\mathcal{H}^S$, which is the geodesic line through (v,w) starting at a cusp s in the Γ_v–orbit of ∞ and ending at a

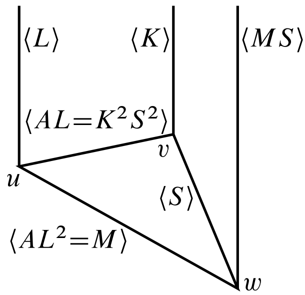


Figure 6: Half of a fundamental domain for the action of $\mathrm{PSL}_2(\mathcal{O}_{-3})$ on $\mathcal{H}$, open towards the cusp at ∞. The second half can be obtained as a copy of the open pyramid glued from below to its base square.

cusp e in the Γ_w–orbit of ∞. By the $\mathcal{A}_4$–symmetries in v and w, (s, v) is mapped to (∞, v) and (e, w) is mapped to (∞, w). Hence there can be no translations of $\mathcal{H}^S$ in Γ, and therefore $\mathcal{H}^S/C_\Gamma(\langle S\rangle) = \mathcal{H}^S$. The $\mathcal{A}_4$–symmetries force all 3–torsion axes passing through a representative of v or w to admit the same centralizer quotient. Hence also $\mathcal{H}^K/C_\Gamma(\langle K\rangle) = \mathcal{H}^K$ and $\mathcal{H}^{MS}/C_\Gamma(\langle MS\rangle) = \mathcal{H}^{MS}$ are open geodesic lines starting and ending at cusps.

In contrast, $\mathcal{H}^L$ is getting reflected onto itself by Γ_u. But the elements of order 2 in $\Gamma_u \cong \mathcal{D}_3$ do not commute with L, and hence $\mathcal{H}^L = \mathcal{H}^L/C_\Gamma(\langle L\rangle)$ is the geodesic line $(\infty, M\cdot\infty)$ with the vertex u on its middle.

Concerning the 2–torsion axes, $\mathcal{H}^M$ does not get reflected by $\Gamma_u \cong \mathcal{D}_3$. It gets reflected by order-2 elements in Γ_w and $\Gamma_{v'}$ commuting with M (see Figure 7); hence $\mathcal{H}^M/C_\Gamma(\langle M\rangle) \cong \bullet\!\!-\!\!\bullet$. By the $\mathcal{D}_3$–symmetry in u, the same happens for $\mathcal{H}^{AL}$: it gets reflected in v and w''' by centralizing elements and not in u; therefore $\mathcal{H}^{AL}/C_\Gamma(\langle AL\rangle) \cong \bullet\!\!-\!\!\bullet$.

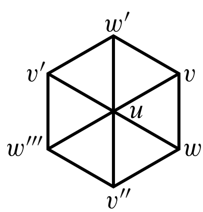


Figure 7: The 2–cells in $\mathcal{H}$ equidistant to the cusps at 0 and ∞, with no other $\mathrm{PSL}_2(\mathcal{O}_{-3})$–cusp being closer. The triangle (u, v, w) is the same one as in Figure 6, and the vertex u sits on the middle of the geodesic $(0, \infty)$.

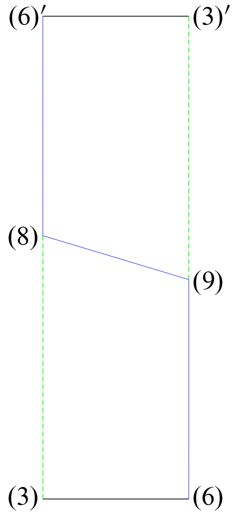


Figure 8: Fundamental domain in the case $m = 11$.

Summing up, and taking into account that $\mathcal{H}/\Gamma$ is contractible, we obtain the CR orbifold cohomology

$$\mathrm{H}^d_{\mathrm{CR}}\big(\big[\mathcal{H}^3_{\mathbb{C}}/\mathrm{PSL}_2\big(\mathbb{Z}\big[\tfrac{1}{2}(\sqrt{-3}-1)\big]\big)\big]\big) \cong \begin{cases} \mathbb{Q} & \text{if } d = 0, \\ \mathbb{Q}^{10} & \text{if } d = 2, \\ 0 & \text{otherwise.} \end{cases}$$

8.5 The case $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-11})$

Let $\mathcal{O}_{-11}$ be the ring of integers in $\mathbb{Q}(\sqrt{-11})$.

Then $\mathcal{O}_{-11} = \mathbb{Z}[\omega]$ with $\omega = \frac{1}{2}(-1+\sqrt{-11})$.

A fundamental domain for $\Gamma := \mathrm{PSL}_2(\mathcal{O}_{-11})$ in real hyperbolic 3–space $\mathcal{H}$ has been found by Bianchi [5]. Half of its lower boundary is given in Figure 8. The coordinates of the vertices of Figure 8 in the upper-halfspace model are $(3) = j$, $(3)' = 1+\omega+j$, $(6) = \frac{1}{2}+\sqrt{3/4}\,j$, $(6)' = \frac{1}{2}+\omega+\sqrt{3/4}\,j$, $(8) = \frac{3}{11}+\frac{3}{11}\omega+\sqrt{2/11}\,j$ and $(9) = \frac{8}{11}+\frac{5}{11}\omega+\sqrt{2/11}\,j$. The set of representatives of conjugacy classes can be chosen to be

$$T = \{\mathrm{Id}, \gamma, \beta, \beta^2\},$$

with $\beta = \pm\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ and $\gamma = \pm\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, so γ is of order 2, and β is of order 3. Using Lemma 24 and with the help of our Bredon homology computations, we check the cardinality of T. That we have one less conjugacy class of finite-order elements than in the case $\mathcal{O}_{-2}$ comes from the fact that, by Observation 22, there is only one conjugacy class of order-2 elements in $\mathcal{A}_4$.

The fixed-point sets are then the following subsets of complex hyperbolic space $\mathcal{H} := \mathcal{H}^3_{\mathbb{C}}$: $\mathcal{H}^{\mathrm{Id}} = \mathcal{H}$, $\mathcal{H}^{\gamma} =$ the complex geodesic line through (3) and (8), and $\mathcal{H}^{\beta} = \mathcal{H}^{\beta^2} =$ the complex geodesic line through (6) and (9).

The 2–torsion subcomplex is of homeomorphism type $\bullet\!\!-\!\!\bullet$ and the 3–torsion subcomplex is of homeomorphism type $\circ\!\!\bullet$. Therefore, we obtain

$$\mathrm{H}^d_{\mathrm{CR}}\big([\mathcal{H}^3_{\mathbb{C}}/\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-11}\,])]\big) \cong \mathrm{H}^d(\mathcal{H}_{\mathbb{C}}/\mathrm{PSL}_2(\mathcal{O}_{-11});\mathbb{Q}) \oplus \begin{cases} \mathbb{Q}^{1+2} & \text{if } d = 2, \\ \mathbb{Q}^2 & \text{if } d = 3, \\ 0 & \text{otherwise.} \end{cases}$$

8.6 The case $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-191})$

Let $\mathcal{O}_{-191}$ be the ring of integers in $\mathbb{Q}(\sqrt{-191}\,)$. Again, the set of representatives of conjugacy classes can be chosen to be

$$T = \{\mathrm{Id}, \gamma, \beta, \beta^2\},$$

with $\beta = \pm\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ and $\gamma = \pm\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, so γ is of order 2, and β is of order 3. Both the 2– and the 3–torsion subcomplexes are of homeomorphism type $\circ\!\!\bullet$. Then,

$$\mathrm{H}^d_{\mathrm{CR}}\big([\mathcal{H}^3_{\mathbb{C}}/\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-191}\,])]\big) \cong \mathrm{H}^d(\mathcal{H}_{\mathbb{C}}/\mathrm{PSL}_2(\mathcal{O}_{-191});\mathbb{Q}) \oplus \begin{cases} \mathbb{Q}^{1+2} & \text{if } d = 2, \\ \mathbb{Q}^{1+2} & \text{if } d = 3, \\ 0 & \text{otherwise.} \end{cases}$$

We conclude this section with the following explanation of why, in our fundamental domain diagrams, there occurs only one representative per torsion-stabilized edge:

Remark 34 Let e be a nontrivially stabilized edge in the fundamental domain for the refined cell complex. Then the fundamental domain for the 2–dimensional retract can be chosen so that it contains e as the only edge on its orbit.

Sketch of proof Observe that the inner dihedral angle $\frac{2\pi}{q}$ of the Bianchi fundamental polyhedron is $\frac{2\pi}{\ell}$ or $\frac{\pi}{\ell}$ at its edges admitting a rotation of order ℓ from the Bianchi group. We can verify this in the vertical half-plane, where the action of $\mathrm{PSL}_2(\mathbb{Z})$ is embedded into the action of the Bianchi group, for the generators of orders $\ell = 2$ and $\ell = 3$ of $\mathrm{PSL}_2(\mathbb{Z})$ which fix edges orthogonal to the vertical half-plane. These angles are transported to all edges stabilized by Bianchi group elements conjugate under $\mathrm{SL}_2(\mathbb{C})$ to these two rotations. Poincaré [31] partitions the edges of the Bianchi

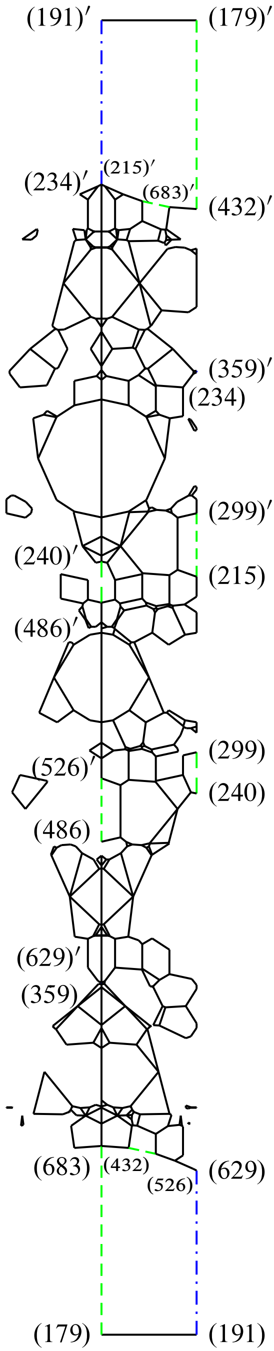


Figure 9: Fundamental domain in the case $m = 191$. The coordinates of the vertices can be displayed by [32].

fundamental polyhedron into cycles, consisting of the edges on the same orbit, of length $\frac{q}{\ell} = 1$ or 2. In the case of length 2, Poincaré's description implies that each of the two 2–cells separated by the first edge of the cycle is respectively on the same orbit as one of the 2–cells separated by the second edge of the cycle. As the fundamental domain for the 2–dimensional retract is strict with respect to the 2–cells, it can be chosen so that it contains e as the only edge on its orbit. □

Note that we can check our computations using the algorithm of [33, Section 5.3] for the computation of subgroups in the centralizers.

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