msp

Algebraic & Geometric Topology 19 (2019) 2763–2794

$C^{1,0}$ foliation theory

William H Kazez
Rachel Roberts

Transverse 1–dimensional foliations play an important role in the study of codimension-one foliations. In Geom. Topol. Monogr. 19 (2015) 21–72, the authors introduced the notion of flow box decomposition of a 3–manifold M. This is a combinatorial decomposition of M that reflects both the structure of a given codimension-one foliation and that of a given transverse dimension-one foliation, and that is amenable to inductive strategies.

In this paper, flow box decompositions are used to extend some classical foliation results to foliations that are not C^2. Enhancements of well-known results of Calegari on smoothing leaves, Dippolito on Denjoy blowup of leaves, and Tischler on approximations by fibrations are obtained. The methods developed are not intrinsically 3–dimensional techniques, and should generalize to prove corresponding results for codimension-one foliations in n–dimensional manifolds.

57M50

1 Introduction

Smoothness plays an important role in the theory of codimension-one foliations of 3–manifolds. Reeb constructed the first C^∞ foliation on S^3 as the union of two foliated solid tori, or Reeb components [27]. This work of Reeb, together with work of Alexander [1] and Wallace [30], led to the proofs by Lickorish [21] and Novikov and Zieschang [24] that any closed 3–manifold has a C^∞ codimension-one foliation. On the other hand, Haefliger [17] showed that no foliation of S^3 can be analytic. This was greatly improved by Novikov [24] to show that any C^2 foliation of S^3 must have Reeb components, and these never exist in analytic foliations.

The qualitative nature of foliation theory and its impact on the ambient 3–manifold was considerably advanced by Thurston's introduction of the norm on the homology of a 3–manifold, and in particular the minimizing properties of leaves of taut, transversely oriented, C^2 foliations [28].

Published: 20 October 2019 DOI: 10.2140/agt.2019.19.2763

A foliation is *taut* if closed smooth transversals to the foliation pass through every point of the manifold. This is also known as *everywhere taut* to distinguish it from the more familiar notion of *smoothly taut*, in which closed smooth transversals are only required to intersect every leaf of the foliation. For a discussion of these and other notions of tautness, and why they are different for $C^{k,0}$ foliations, the same for $C^{k,1}$ foliations, and interchangeable up to C^0 approximation and isotopy of foliations, see Colin, Kazez and Roberts [6].

Foliations as a tool for understanding problems in 3–dimensional topology came to the fore as a result of Gabai's constructions of both C^∞ and often less smooth, but finite depth, taut foliations [14; 15; 16]. The success of Gabai's applications of foliation theory led to many constructions of taut codimension-one foliations. Often these foliations are constructed using Denjoy blowup techniques that yield foliations that are only $C^{\infty,0}$; that is, leaves are smoothly immersed, but transversely, their tangent plane fields vary only continuously.

The impetus for our work starts with the Eliashberg–Thurston approximation theorem [13]. They showed that a taut, cooriented codimension-one C^2 foliation of a 3–manifold can be C^0 approximated by a pair of symplectically fillable contact structures. This allows nontrivial Heegaard Floer invariants to be assigned to any manifold that supports a taut foliation; see Ozsváth and Szabó [25]. This is, consequently, one of the pillars of the conjectural relationship between L–spaces, taut foliations and left orderability of the fundamental group. For details, see for example Ozsváth and Szabó [26] and Boyer, Gordon and Watson [3].

In [19; 20] we extended the Eliashberg–Thurston approximation theorem to the class of all $C^{1,0}$, cooriented taut foliations, thereby extending its reach to manifolds carrying the new constructions of foliations mentioned above. Similar results can be found in Bowden [2]. In doing so, we found that many of the standard tools for working with foliations had either not been developed for foliations with lesser smoothness than originally intended, or had not been developed with an eye towards C^0 approximation theory in which it is often necessary to produce a new foliation while only moving the tangent planes of the original foliation slightly.

This paper includes enhancements to well-known results of Calegari [4] on smoothing leaves, Dippolito [12] on Denjoy blowup of leaves, and Tischler [29] on approximations by fibrations. It is possible that some of our results can be obtained by "reading between the lines" of the original source. However, as is well known, subtleties, sometimes fatal, arise when smooth objects are replaced by objects that are merely continuous. (See for example [6].) The original theorems are foundational results in foliation theory, and proofs of these theorems, in the generality in which they are used, do not exist in the literature. An advantage to the decomposition approach we use is that each of these results can be proved directly with a single inductive strategy.

The methods developed in this paper are not intrinsically 3–dimensional techniques, and we expect they can be adapted to prove corresponding results for codimension-one foliations in n–dimensional manifolds.

Basic definitions (codimension-one foliation, flow, $(\mathcal{F}, \Phi)$–compatible, C^0 close and C^0 small) are given in Section 2. In Section 3 we recall the definition of flow box decomposition, define regular neighborhood structure, and prove a sequence of useful local smoothing results. The main result of Section 4 is a proof that any $C^{1,0}$ foliation is isotopic to a C^0 close $C^{\infty,0}$ foliation. In Section 5 we prove that any $C^{1,0}$ measured foliation is isotopic to a C^0 close smooth measured foliation. Basic facts from [19; 20] about holonomy neighborhoods are recalled in Section 6. We give Dippolito's definition [12] of Denjoy blowup in Section 7 and prove that particularly nice, C^0 close, Denjoy blowups of a $C^{1,0}$ codimension-one foliation always exist.

Throughout this paper, unless stated otherwise, M will denote a 3–manifold that is either smooth or smooth with corners. When $\partial M \neq \varnothing$, it is often useful to think of M as a sutured manifold, not necessarily orientable, in the sense of [14]. Recall that any topological 3–manifold admits a smooth structure, unique up to diffeomorphism; see Moise [22; 23].

The restriction of a map to a closed subset A is said to be *smooth* on A if its restriction to some open neighborhood of A is smooth.

Acknowledgements We thank the referee for many helpful suggestions, and we thank Greg Knese for showing us how convolution can be used to smooth foliations.

This work was partially supported by grants from the Simons Foundation (#244855 to Kazez, #317884 to Roberts) and from the National Science Foundation (DMS-1612036 to Kazez, DMS-1612475 to Roberts).

2 Codimension-one foliations and transverse dimension-one foliations

We begin by defining foliations in 3–manifolds with empty boundary. Near the end of this section, we extend these definitions to 3–manifolds with nonempty boundary that are smooth or smooth with corners, namely manifolds locally modeled by open sets in $[0,\infty)^3$.

Definition 2.1 Let M be a smooth 3–manifold with empty boundary. Let k and l be nonnegative integers or infinity with $l \leq k$. Both C^k and $C^{k,l}$ *codimension-one foliations* $\mathcal{F}$ are decompositions of M into a disjoint union of C^k immersed connected surfaces, called the *leaves* of $\mathcal{F}$, together with a collection of charts U_i covering M, with $\phi_i\colon \mathbb{R}^2 \times \mathbb{R} \to U_i$ a homeomorphism, such that the preimage of each component of a leaf intersected with U_i is a horizontal plane.

The foliation $\mathcal{F}$ is C^k if the charts (U_i, ϕ_i) can be chosen so that each ϕ_i is a C^k diffeomorphism.

The foliation $\mathcal{F}$ is $C^{k,l}$ if for all i and j,

(1) the derivatives $\partial_x^a \partial_y^b \partial_z^c$, taken in any order, on the domain of each ϕ_i and each transition function $\phi_j^{-1}\phi_i$ are continuous for all $a+b \leq k$ and $c \leq l$, and
(2) if $l \geq 1$, then ϕ_i is a C^1 diffeomorphism.

Remark 2.2 The smoothness conditions on both the charts and the transition functions are to ensure that the smooth structure on the leaves is compatible with the smooth structure on M.

In particular, $T\mathcal{F}$ exists and is continuous if and only if $\mathcal{F}$ is $C^{1,0}$. Also notice that $C^{k,l}$ foliations are C^l, but not conversely.

Two $C^{k,0}$ foliations $\mathcal{F}$ and $\mathcal{G}$ of M are called $C^{k,0}$ *equivalent* if there is a self-homeomorphism of M that maps the leaves of $\mathcal{F}$ to the leaves $\mathcal{G}$, and is C^k when restricted to any leaf of $\mathcal{F}$.

We use the terms *transverse*, *transversal* and *transversely* in the smooth sense; that is, they refer to smooth objects intersecting so that the associated tangent spaces intersect minimally.

Given a codimension-one foliation $\mathcal{F}$, it is useful to fix a one-dimensional foliation Φ transverse or topologically transverse to $\mathcal{F}$. Such a Φ always exists and can be realized as the union of curves $\phi_p(t)$ of continuous local flows ϕ. When $\mathcal{F}$ is transversely oriented and M closed, Φ can be realized as the union of curves $\phi_p(t)$ of a global flow $\phi\colon M \times \mathbb{R} \to M$. When $\mathcal{F}$ is C^0, this is proved in [18, Theorems 1.1.2 and 1.3.2]. When $\mathcal{F}$ is $C^{1,0}$, Φ can be chosen to be smooth; in fact, in this case, it is elementary to see that Φ exists and consists of the integral curves of a smooth line field transverse to $T\mathcal{F}$. See, for example, Lemma 5.1.1 of [5].

Conventions Unless otherwise stated, throughout the rest of this paper, we will use foliation to mean a codimension-one transversely oriented foliation of a 3–manifold M. Such an M will be assumed to be compact and oriented. Since the foliations studied will be $C^{1,0}$ we can assume without loss of generality that a smooth transverse dimension-one foliation to the foliation is chosen. To simplify the exposition, we will abuse language, and refer to a one-dimensional foliation transverse to a codimension-one foliation as a flow, even though the domain of the trajectories may not be $\mathbb{R}$.

When a foliation $\mathcal{F}$ is understood, a submanifold of positive codimension in M is called *horizontal* if each component is a submanifold of a leaf of $\mathcal{F}$. When both a foliation $\mathcal{F}$ and a transverse flow Φ in M are understood, a submanifold of positive codimension in M is called *vertical* if and only if it can be expressed as a union of subsegments of the flow Φ. A codimension-0 submanifold X of M is called $(\mathcal{F},\Phi)$*–compatible* if its boundary is piecewise horizontal and vertical, and hence $\mathcal{F}$ and Φ restrict naturally to foliation and flow on X. If X is $(\mathcal{F},\Phi)$–compatible, let $\partial_v X$ denote its vertical boundary, and let $\partial_h X$ denote its horizontal boundary.

Definition 2.3 Suppose X is an $(\mathcal{F},\Phi)$–compatible submanifold of M, where possibly $X = M$. An isotopy of X which maps each flow segment of $\Phi|_X$ to itself is called a *flow-compatible*, or Φ*–compatible, isotopy*. Note that a flow-compatible isotopy of X fixes $\partial_h X$ pointwise.

By allowing the foliation atlas to include boundary charts, Definition 2.1 naturally extends to the case that M has nonempty boundary that is either smooth or smooth with corners. Smooth boundary components must either be a leaf of $\mathcal{F}$, and hence horizontal, or transverse to $\mathcal{F}$. A boundary component with corners must decompose along its corners into smooth subsurfaces, where if two subsurfaces share a corner, one is horizontal and one is transverse to $\mathcal{F}$. Such an M is a sutured manifold, in the sense of [14]. Thus, if $\partial M \neq \varnothing$ and we double $(M,\mathcal{F})$ along $\partial_v M$, $D\mathcal{F}$ is a foliation of DM with all components of $\partial(DM)$ leaves of $\mathcal{F}$.

We restrict attention to flows Φ that meet ∂M in a similarly constrained way. A flow is required to be either everywhere transverse or everywhere tangent to a smooth component of ∂M. And if (S,γ) is a boundary component with annular sutures $A(\gamma)$, a flow is required to be transverse to $R(\gamma)$ and tangent to $A(\gamma)$. Thus the flow is transverse to $\partial_h M$, and $\partial_v M$ is vertical. In particular, if Φ is a flow transverse to $\mathcal{F}$, it is possible to double Φ along $\partial_v M$ so that M is a $(D\mathcal{F}, D\Phi)$–compatible submanifold of DM.

In the context of foliations, the terms C^0 close and C^0 small both refer to distances between tangent planes. More formally, suppose that a metric, d, has been chosen on the set of continuous sections of the Grassmann bundle of 2–planes in TM^3. Given a section, typically the tangent bundle of a foliation, $T\mathcal{F}$, we say that another section, $T\mathcal{G}$, can be found C^0 *close* to $T\mathcal{F}$, if for all $\epsilon > 0$ a $\mathcal{G}$ exists such that $d(T\mathcal{F}, T\mathcal{G}) < \epsilon$. For brevity, this is stated as $\mathcal{G}$ can be found C^0 *close* to $\mathcal{F}$. An isotopy $\mathcal{F}_t$ of $\mathcal{F}$ is called C^0 *small* if it can be chosen so that at all times $\mathcal{F}_t$ is C^0 close to $\mathcal{F}$. An isotopy can be found C^0 *close to the identity* if given any $\epsilon > 0$ an isotopy can be found that keeps every section within ϵ of its starting position.

Throughout the paper, I will be used to denote the closed interval $[0,1]$.

3 Flow boxes and local approximations

3.1 Flow boxes

Flow box decompositions were introduced and shown to exist in [19]. In the definition given below, an extra condition, (5), is added that is particularly useful for inductive arguments.

We note that our use of the term *flow box* differs from that standardly used in the theory of flows on manifolds, and reflects the fact that we are interested primarily in codimension-one foliations, with an emphasis on the usefulness of a fixed smooth transverse one-dimensional foliation. Flow boxes, in our sense, are combinatorial versions of biregular foliation charts.

Definition 3.1 [19] Let $\mathcal{F}$ be either a C^k or $C^{k,l}$ foliation, and let Φ be a smooth transverse flow. A *flow box*, F, is an $(\mathcal{F}, \Phi)$–compatible closed chart, possibly with corners. That is, it is a submanifold diffeomorphic to $D \times I$, where D is either a closed C^k disk or polygon (a closed disk with at least three corners), Φ intersects F in the arcs $\{(x,y)\} \times I$, and each component of $D \times \partial I$ is embedded in a leaf of $\mathcal{F}$.

Notice that the components of $\mathcal{F} \cap F$ give a family of C^k graphs over D. In the case that D is a polygon, it is often useful to view the disk D as a 2–cell with ∂D the cell complex obtained by letting the vertices correspond exactly to the corners of D. Similarly, it is useful to view the flow box F as a 3–cell possessing the product cell complex structure of $D \times I$. Then $\partial_h F$ is a union of two (horizontal) 2–cells and $\partial_v F$ is a union of c (vertical) 2–cells, where c is the number of corners of D. In the case that D has no corners, we abuse language slightly and consider $\partial_h F$ to be a union of two (horizontal) 2–cells and $\partial_v F$ to be a single vertical face, where the face is the entire vertical annulus $\partial D \times I$.

Definition 3.2 Suppose V is a compact, $(\mathcal{F}, \Phi)$–compatible, codimension-0 submanifold of M. A *flow box decomposition of M rel V*, or simply *flow box decomposition of M* if $V = \varnothing$, is a decomposition of $M \setminus \operatorname{int}(V)$ as a finite union $M = V \cup \left(\bigcup_{i=1}^{n} F_i\right)$ where

1. each F_i is a flow box,
2. $V \cap F_i$ is a union, possibly empty, of horizontal subsurfaces and vertical 2–cells of F_i, together possibly with some 0– and 1–cells,
3. the interiors of F_i and F_j are disjoint if $i \neq j$,
4. if $i \neq j$ and $F_i \cap F_j$ is nonempty, it must be homeomorphic to a point, an interval or a disk that is wholly contained either in $\partial_h F_i \cap \partial_h F_j$ or in a single face in each of $\partial_v F_i$ and $\partial_v F_j$, and
5. if Δ is a vertical 2–cell of F_n and the interior of Δ intersects a vertical 2–cell Δ' of some F_i with $i < n$, then $\Delta \subset \Delta'$.

Most of the results proved in this paper use flow box decompositions relative to an empty codimension-0 submanifold. The general definition is particularly useful for approximating foliations by contact structures, as described in [19; 20], and it appears in support of that work in Corollary 7.5.

Proposition 3.3 *Suppose $\mathcal{F}$ is either a C^k or a $C^{k,l}$ foliation of a compact manifold M and let Φ be a smooth flow transverse to $\mathcal{F}$. Suppose V is a compact $(\mathcal{F}, \Phi)$–compatible, codimension-0 submanifold of M. Then M has a flow box decomposition rel V. Moreover, any flow box decomposition of V can be extended to a flow box decomposition of M.*

Proof Conditions (1)–(4) follow from Proposition 4.4 of [19]. Thus it is enough to show that a flow box decomposition satisfying (1)–(4) can be inductively subdivided so that (5) is satisfied.

To do this, consider the union, X, of all vertical 2–cells contained in an F_i with $i < n$ that intersect the interior of some vertical 2–cell of F_n. Split F_n along a finite collection of leaves of $\mathcal{F} \cap F_n$ that contain $(\partial_h X) \cap F_n$, and let F_n^j be the resulting components. Redefine the polygonal structure on each F_n^j by decreeing that, in addition to the original vertical edges, every component of $\partial_v X \cap F_n^j$ is also a vertical edge. Replacing F_n by the F_n^j completes the inductive step of the construction. □

A flow box decomposition is called V*–transitive*, or in the case that $V = \varnothing$, *transitive*, if there exists an indexing set $i = 1, \dots, n$ and V_i such that $V_0 = V$, $V_i = V_{i-1} \cup F_i$, and, for $i = 1, \dots, n$,

(6) $V_{i-1} \cap F_i$ contains a vertical 2–cell of F_i.

Condition (6) is used in [19; 20], where flow boxes are needed to laterally propagate an approximating contact structure from V to the rest of M.

Proposition 3.4 [19, Proposition 4.4] *If M is compact and each point in M can be reached from V by a path in a leaf of $\mathcal{F}$, then there is a transitive flow box decomposition of M rel V.* □

If $\mathcal{B} = \mathcal{B}(\mathcal{F}, \Phi)$ is a flow box decomposition of M rel V, an isotopy of M is $\mathcal{B}$*–compatible* if it is Φ–compatible and, in addition, maps each cell of each flow box of $\mathcal{B}$ to itself setwise.

By condition (5), the set of vertical faces of the flow boxes F_i is partially ordered by set containment: if Δ_i and Δ_j are vertical faces of F_i and F_j, respectively, and their interiors have nonempty intersection, and $i < j$, then $\Delta_j \subseteq \Delta_i$. Call a vertical face *maximal* if it is maximal with respect to this partial ordering, namely if it is not properly contained in any vertical face.

Let $\sigma_1, \dots, \sigma_m$ be a listing of the maximal faces. It will sometimes be helpful to consider a regular neighborhood of $\bigcup_j \sigma_j$ of the following sort:

Definition 3.5 Let $F_1, \dots, F_n$ be a listing of the flow boxes of a flow box decomposition $\mathcal{B} = \mathcal{B}(\mathcal{F}, \Phi)$. A *regular neighborhood structure* $\mathcal{N}_\mathcal{B} = \mathcal{N}_\mathcal{B}(\mathcal{F}, \Phi)$ for $\mathcal{B}$ is a tuple of the form

$$(N, N_v, N(\sigma_1), \dots, N(\sigma_m)),$$

where

(1) $\sigma_1, \dots, \sigma_m$ is a listing of the maximal faces of $\mathcal{B}$,

(2) each $N(\sigma_j)$ is a flow box that properly contains σ_j,

(3) N_v is an $(\mathcal{F}, \Phi)$–compatible regular neighborhood of the union of the vertical 1–cells of the maximal faces σ_j,

(4) N_v decomposes as a finite union of flow boxes $B_p = D_p \times I$, where $B_p \cap B_q \subset \partial_h B_p \cap \partial_h B_q$ and $B_p \cap \left(\bigcup_i (\partial_v F_i)^{(1)}\right) \subset (\{0\} \times I)$ for each p,

(5) if $j \neq k$, then $N(\sigma_j) \cap N(\sigma_k)$ is contained in the interior of N_v,

(6) $N := N_v \cup \bigcup_j N(\sigma_j)$, and

(7) $\bigcup_i \partial_v F_i$ is a deformation retract of N.

Figure 1 illustrates a horizontal cross-section of a regular neighborhood structure in a neighborhood of a single flow box. Note that a $\mathcal{B}$–compatible isotopy takes a regular neighborhood structure $\mathcal{N}_{\mathcal{B}}(\mathcal{F}, \Phi)$ to a regular neighborhood structure $\mathcal{N}_{\mathcal{B}}(\mathcal{F}', \Phi)$.

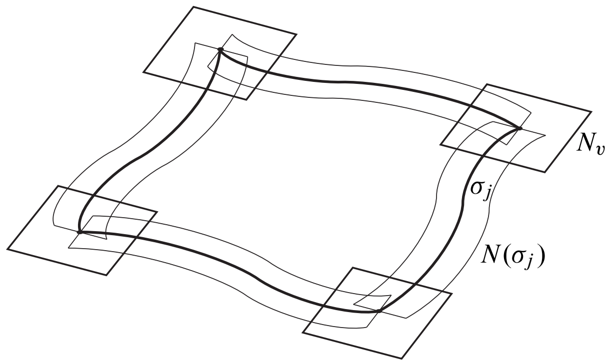

Figure 1: Horizontal cross-section of a flow box.

A standard method of proof is to work inductively with a cell complex, working first with 0–cells, and then extending over the 1–cells, followed by the 2–cells, and finally the 3–cells. Since a foliation is determined by its restriction to the 2–skeleton, the focus of most constructions is on the 1–dimensional foliations of the vertical faces. When smoothness is a priority, it is often useful to work instead with regular neighborhoods of the cells. Regular neighborhood structures provide a vocabulary for this approach in the context of flow box decompositions, namely establish a property first on N_v, then on the union $\bigcup_j N(\sigma_j)$, and finally extend this property over the 3–cells complementary to $N \cup \bigcup_i \partial_h F_i$.

Definition 3.6 A flow box decomposition $\mathcal{B}$ is *smooth-sided* if the interior of every vertical face of every flow box F_i of $\mathcal{B}$ is a smooth surface. The flow box decomposition is called *smooth* if it is smooth-sided and every horizontal face has a neighborhood in the leaf it is contained in that is smoothly embedded.

Lemma 3.7 *Let M be compact. If $\mathcal{F}$ is $C^{1,0}$ and Φ is a smooth transverse flow in M, then there exists a smooth-sided flow box decomposition of M. If $\mathcal{F}$ is $C^{\infty,0}$ and Φ is a smooth transverse flow in M, then there exists a smooth flow box decomposition of M.*

Proof The proof of Proposition 4.4 of [19] starts by choosing initial flow boxes and these may be taken to be smooth-sided. The rest of the construction involves transversality of vertical intersections and splitting along leaves, and both of these operations work with smooth vertical faces. □

Lemma 3.8 *If $\mathcal{F}$ is $C^{1,0}$, Φ is a smooth transverse flow in M and $\mathcal{B}$ is a smooth-sided flow box decomposition, then there exists a flow-compatible isotopy that takes $\mathcal{F}$ to a C^0 close $C^{1,0}$ foliation and takes $\mathcal{B}$ to a smooth flow box decomposition.*

Proof Let U be the union of small neighborhoods, in leaves of $\mathcal{F}$, of each of the horizontal faces of all $F_i \in \mathcal{B}$. Then U is a C^1 embedded surface. This may be isotoped, while preserving flow lines of Φ and moving the tangent planes of $\mathcal{F}$ by no more than ϵ, to a smoothly embedded surface. Applying this isotopy to $\mathcal{B}$ produces the desired smooth flow box decomposition. □

Remark 3.9 Suppose $\mathcal{B}$ is a smooth $(\mathcal{F}, \Phi)$–flow box decomposition, where $\mathcal{F}$ is a $C^{k,0}$ foliation for some $k \geq 1$, and Φ is a smooth flow transverse to Φ. The interior of each flow box describes a smooth chart for M in which the flow restricts to the union of vertical segments $\{x\} \times I$ and the leaves of $\mathcal{F}$ restrict to a C^0 family of C^k graphs. After fixing a point x_0, an index t may be chosen so that the leaf containing (x_0, t) is given by the graph $z = f_t(x)$.

We now give several elementary and frequently used smoothing operations that will be used in a neighborhood of a surface. To streamline statements, let S be a surface, possibly with boundary, and let $S \times I \subset M$. A *strictly horizontal foliation of* $S \times I$ is the foliation with leaves $S \times \{t\}, t \in I$. An *almost horizontal foliation* of $S \times I$ is a foliation transverse to the I–fibers which contains $S \times \partial I$ as leaves. A product submanifold, $S \times I$ of M, is called an $(\mathcal{F}, \Phi)$*–compatible product* if the restriction of $\mathcal{F}$ to $S \times I$ is almost horizontal, and the I–fibers $\{x\} \times I$ are flow segments of Φ.

Notation 3.10 If S is a proper subsurface of a leaf of a given foliation, $N(S)$ will denote a regular neighborhood of S in its leaf.

3.2 Approximating $C^{1,0}$ by $C^{\infty,0}$

We begin by showing that any $C^{1,0}$ almost horizontal product foliation can be approximated by $C^{\infty,0}$ almost horizontal product foliations.

Proposition 3.11 (leafwise smoothing a product foliation) *Let S be a compact smooth surface, possibly with boundary, and let $\mathcal{F}$ be a $C^{1,0}$ almost horizontal product foliation on $S \times I$. Then $\mathcal{F}$ can be C^0 deformed to a C^0 close, $C^{\infty,0}$, almost horizontal product foliation $\mathcal{S}$ on $S \times I$. If $\mathcal{F}$ is $C^{\infty,0}$, on some compact $(\mathcal{F}, \Phi)$–compatible submanifold, then we may choose the deformation to fix this submanifold pointwise. In addition, if some finite number of leaves of $\mathcal{F}$ are $C^{\infty,0}$ embedded, then the deformation can be chosen to fix these leaves pointwise, and if $L'_1, \dots, L'_n$ are subsurfaces of leaves $L_1, \dots, L_n$ of $\mathcal{F}$ so that regular neighborhoods $N(L'_i)$ of L'_i in L_i are smoothly embedded in M, then the deformation can be chosen to fix each L'_i pointwise.*

Proof Pick a metric on the bundle of tangent 2–planes to $S \times I$. Fix a point x_0 in S, and denote the leaf of $\mathcal{F}$ that contains (x_0, t) by P_t. Given $s < t$ and $x \in S$, let $[s,t]_x$ denote the subinterval of $\{x\} \times I$ with boundary points in $P_s \cup P_t$. Each P_t is the graph of a C^1 function $f_t \colon S \to I$. It is enough to deform the continuously varying family f_t to a smoothly varying family whose graphs foliate $S \times I$.

Let $\epsilon > 0$. Choose a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$ with the property that for all $x \in S$, for each i, the tangent planes to leaves of $\mathcal{F}$ at each point of $[t_{i-1}, t_i]_x$ are all within ϵ of each other. Perform a C^0 small isotopy so that $\bigcup_i P_{t_i}$ is smoothly embedded.

If some finite number of leaves of P are smoothly embedded, the partition can be chosen so that these leaves appear as P_{t_i}. In the relative case, each L_i can be smoothed by a C^0 small isotopy relative to L'_i, and the partition can be chosen so these leaves also appear as P_{t_i}.

For $i = 1, \dots, n$ let $\ell_i \colon [t_{i-1}, t_i] \to [0,1]$ be a smooth bijection that vanishes to infinite order at the endpoints. For $t \in [t_{i-1}, t_i]$ define

$$g_t = (1 - \ell_i(t)) f_{t_{i-1}} + \ell_i(t) f_{t_i}.$$

Then g_t is a continuous family of smooth functions whose graphs give a $S^{\infty,0}$ foliation $\mathcal{S}$ of $S \times I$. Since g_t is a linear combination of $f_{t_{i-1}}$ and f_{t_i}, it is easily checked using local coordinates on S that the normal vector to the graph of g_t is a linear combination of normals to $f_{t_{i-1}}$ and f_{t_i}. It follows that tangent planes to g_t are C^0 close to the tangent planes to f_t.

Since f_t and g_t are graphs, there is an I–fiber-preserving deformation of $\mathcal{F}$ to $\mathcal{S}$. □

3.3 Approximating $C^{\infty,0}$ by C^∞

Next we show that any $C^{\infty,0}$ almost horizontal product foliation can be approximated by C^∞ almost horizontal product foliations. The proof of the following theorem is due to Greg Knese, who showed us how convolution can be used to smooth foliations.

Theorem 3.12 (smoothing a $C^{\infty,0}$ product foliation) *Let S be a compact smooth surface, and let $\mathcal{F}$ be a $C^{\infty,0}$ almost horizontal product foliation on $S\times I$. Then $\mathcal{F}$ can be C^0 approximated by a C^∞, almost horizontal product foliation $\mathcal{S}$ on $S\times I$. If some finite number of leaves of $\mathcal{F}$ are chosen, then the approximation can be chosen to fix these leaves pointwise, and if $L_1',\dots,L_n'$ are subsurfaces of leaves $L_1,\dots,L_n$ of $\mathcal{F}$ such that regular neighborhoods $N(L_i')$ of L_i' in L_i are smoothly embedded in M, then the approximation can be chosen to fix each L_i' pointwise. If $\mathcal{F}$ is C^∞ on some compact $(\mathcal{F},\Phi)$–compatible submanifold, then we may choose the approximation to fix this submanifold pointwise.*

Proof An almost horizontal $C^{\infty,0}$ product foliation $\mathcal{F}$ on $S\times I$ is the homeomorphic image of the product foliation on $S\times I$ by a map of the form $(p,t)\mapsto(p,F(p,t))$.

Since $\mathcal{F}$ is $C^{\infty,0}$,

(a) $\partial^{i+j}F/\partial x^i\partial y^j$ exists and is jointly continuous in p and t for all i, j and smooth local coordinates x, y on S.

Let $f_t(p)=F(p,t)$. A function F satisfying (a) defines an almost horizontal product foliation if

(b) the graphs G_t of f_t are pairwise disjoint and $\bigcup_{t\in I}G_t=S\times I$.

It is convenient to consider the homeomorphism that is the inverse of $(p,t)\mapsto(p,F(p,t))$. This is denoted by $(p,t)\mapsto(p,T(p,t))$. Thus $T(p,z)$ is the unique t such that $F(p,t)=z$; hence the graph $G_{T(p,z)}$ passes through (p,z).

Note that T must be continuous. For, if $(p_n,z_n)\to(p,z)$ and $|T(p_n,z_n)-T(p,z)|>\epsilon$, then there is a subsequence with $T(p_n,z_n)\to t\neq T(p,z)$. But then

$$0=z_n-F(p_n,T(p_n,z_n))\to z-F(p,t)$$

and hence $t=T(p,z)$.

The first goal is to find a family of functions $F^s\colon S\times I\to I$ for $0<s<\frac{1}{2}$ such that each F^s is smooth, satisfies (b) and has the following limiting property, (c). To specify this, define $F^0(p,t) = F(p,t)$ and let $T^s(p,t)$ be defined analogously to $T(p,t)$ so that $F^s(p, T^s(p,t)) = t$ for $0 \le s \le \frac{1}{2}$. To force the tangent planes of the almost horizontal foliation $\mathcal{F}^s$ defined by F^s to limit on the tangent planes of $\mathcal{F}$ as $s \to 0$ we require

(c) $\partial F^s(p, T^s(p,t))/\partial x \to \partial F^0(p, T^0(p,t))/\partial x$ uniformly as $s \to 0$, where x is a smooth local coordinate for S.

We may assume that $F(p,0) = 0$ and $F(p,1) = 1$ for all $p \in S$, so that condition (b) is equivalent to $t \mapsto F(p,t)$ being a monotone increasing function for all p.

Next construct an extension of F to $S \times [-1,2] \to [-1,2]$ so that

(1) property (a) holds on $S \times [-1,2]$,

(2) $t \mapsto F(p,t)$ is increasing on $\left[-\frac{1}{2}, \frac{3}{2}\right]$, and

(3) $F = 0$ on $S \times \{-1, 2\}$.

This can be done by first defining $F(p,t) = 2 - F(p, 2-t)$ when $t > 1$ and $F(p,t) = -F(p,-t)$ when $t < 0$, and then multiplying by a smooth bump function for $\left[-\frac{1}{2}, \frac{3}{2}\right]$ so that (3) is satisfied. Choose this bump function so that the extension of F is "odd" about both t equal to 0 and 1; namely, for $-\frac{1}{2} \le t \le \frac{1}{2}$, $F(p,-t) = -F(p,t)$ and $F(p, 1-t) - 1 = 1 - F(p, 1+t)$.

Let ϕ: $\mathbb{R} \to [0,\infty)$ be smooth and even with the support of ϕ contained in $[-1,1]$ and $\int_{\mathbb{R}} \phi(t)\,dt = 1$.

Define $\phi^s(t) = \frac{1}{s}\phi\left(\frac{t}{s}\right)$ for $s > 0$. Note that the support of ϕ^s is contained in $[-s,s]$ and $\int_{\mathbb{R}} \phi^s(t)\,dt = 1$.

Define, for $0 < s < \frac{1}{2}$,

$$\begin{aligned} F^s(p,t) = (F(p,\cdot\,) * \phi^s)(t) &= \int_{[t-s,t+s]} F(p,u)\phi^s(t-u)\,du \\ &= \int_{[-s,s]} F(p,t-u)\phi^s(u)\,du. \end{aligned}$$

The map $t \mapsto F^s(p,t)$ is increasing on $[0,1]$ since if $t_1 > t_2$ then, for $s < \frac{1}{2}$,

$$F^s(p,t_1) - F^s(p,t_2) = \int_{[-s,s]} (F(p,t_1-u) - F(p,t_2-u))\phi^s(u)\,du$$

and the integrand is positive since $t \mapsto F(p,t)$ is increasing on $\left[-\frac{1}{2}, \frac{3}{2}\right]$.

Next,

$$F^s(p,0) = \int_{[-s,s]} F(p,-u)\phi^s(u)\,du = 0$$

since $F(p,\cdot)$ is odd across $t=0$ and ϕ^s is even. Similarly, $F^s(p,1)=1$ for all $p \in S$. Now,

$$\begin{aligned} |F^s(p,t) - F(p,t)| &\le \int_{[-s,s]} |F(p,t-u) - F(p,t)|\phi^s(u)\,du \\ &\le \sup_{|u-v|\le s} |F(p,u) - F(p,v)| \to 0. \end{aligned}$$

The limit of 0 is taken as s approaches 0 and exists by uniform continuity of F. Similarly, if x is a smooth local coordinate on S, then $\partial F^s/\partial x$ approaches $\partial F/\partial x$ uniformly as s tends to 0.

Claim $\lim_{s\to 0} T^s(p,z) = T^0(p,z)$.

Otherwise, there exist $s_n \to 0$ and $(p_n, z_n) \in S \times I$ such that

$$(*) \qquad |T^{s_n}(p_n,z_n) - T^0(p_n,z_n)| > \epsilon.$$

By passing to a subsequence we may assume that

$$(p_n,z_n) \to (p,z) \in S \times I, \quad T^{s_n}(p_n,z_n) \to t \ne T^0(p,z),$$
$$|F^{s_n} - F| < \frac{1}{n} \quad \text{at all points in } S \times I.$$

Evaluating the last inequality at $(p_n, T^{s_n}(p_n,z_n))$, and using $F^{s_n}(p_n, T^{s_n}(p_n,z_n)) = z_n$, gives

$$|z_n - F(p_n, T^{s_n}(p_n,z_n))| < \frac{1}{n}.$$

Letting n tend to ∞ implies $z - F(p,t) = 0$, thus $T^0(p,z) = t$. So, by continuity of T^0, $T^0(p_n,z_n) \to T^0(p,z) = t$, and this contradicts $(*)$.

Hence, (c) is satisfied.

Finally, $F^s(p,t)$ is smooth; indeed,

$$\frac{\partial^{i+j+k} F^s}{\partial x^i \partial y^j \partial t^k} = \frac{\partial^{i+j} F}{\partial x^i \partial y^j} * \frac{\partial^k \phi^s}{\partial t^k} = \int \frac{\partial^{i+j} F}{\partial x^i \partial y^j}(p,u) \frac{\partial^k \phi^s}{\partial t^k}(t-u)\,du$$

exists and is continuous.

To ensure that the family of graphs of F^s for a fixed $s > 0$ form a C^∞ foliation on $S \times I$, we show that $(p,t) \mapsto (p, F^s(p,t))$ is a diffeomorphism. This map is smooth, injective, and the image is a union of smooth graphs, so it is enough, by the inverse function theorem, to show that the derivative of $F^s(p,t)$ with respect to t is greater than 0.

By the previous formula, with $i=j=0$ and $k=1$,

$$\frac{\partial F^s}{\partial t}=\int F(p,u)\frac{\partial \phi^s}{\partial t}(t-u)\,du.$$

By choice, ϕ^s is an even positive function, thus $\partial\phi^s(t-u)/\partial t$ is an odd function about $u=t$ that is negative on $u\in(t-s,t)$ and positive on $u\in(t,t+s)$. Since $F(p,u)$ is an increasing function of u, it follows that $\partial F^s/\partial t>0$.

This completes the proof of the first portion of the proposition. Now consider the case that $\mathcal{F}$ has a finite number of subsurfaces L_i' of leaves L_i that are smoothly embedded. An open neighborhood $N(L_i')$ in L_i is smoothly embedded. Hence we may smooth each leaf L_i as necessary, keeping L_i' fixed. Thus we reduce to the case that $\mathcal{F}$ has a finite number of leaves L_i that are smoothly embedded.

Begin by noting that if $\mathcal{F}$ is smooth in a neighborhood of $S\times\partial I$, then we may assume that $\mathcal{F}$ is horizontal; that is, F is constant in p in this neighborhood. Hence, the foliation given by F^s is horizontal for s small enough and when in a small enough neighborhood of $S\times\partial I$. We may therefore proceed as follows. First, replace $\mathcal{F}$ with a C^0 close $C^{\infty,0}$ foliation obtained by replacing each leaf L_i with a small smoothly embedded closed I–bundle $N_i=L_i\times I_i$ of parallel copies of L_i. (See, for example, the proof of Lemma 7.4.) For each i, let $N_i'=L_i\times J_i$, where J_i is a closed interval contained in the interior of I_i. Now proceed as before on the complement of the union of these N_i'. □

3.4 Smoothing with constraints

Definition 3.13 Let $\mathcal{F}$ be an almost horizontal foliation on $S\times I$. If $\beta\subset S$ is an arc with $\beta(0)=*$ and $\beta(1)=x$, let $\rho_{\mathcal{F}}(\beta)$ denote the homeomorphism from $\{x\}\times I\to\{*\}\times I$ given by lifting β to leaves of $\mathcal{F}$. More precisely, given such an arc β, let β_t be the path in a leaf of $\mathcal{F}$ that ends at (x,t) and projects to β, and define $\rho_{\mathcal{F}}(\beta)(x,t)=\beta_t(0)$.

The next two results are concerned with extending the smoothing of product foliations of a surface crossed with I. Proposition 3.14 shows how such a foliation that has been smoothed above the neighborhoods of two vertices can be smoothed above a neighborhood of an edge connecting the two vertices. Proposition 3.15 shows that the smoothing can be extended from the 1–skeleton of the surface to the 2–skeleton.

Proposition 3.14 (smoothing above a 1–cell) *Let $D = I \times I$, with core curve $\alpha = \{\frac{1}{2}\} \times I$, oriented as above. Let Φ denote the 1–dimensional foliation by I–fibers $\{x\} \times I$ for $x \in D$. Suppose $\mathcal{F}$ is a $C^{1,0}$ almost horizontal foliation on $D \times I$ that is $C^{\infty,0}$ on the neighborhood $N_v = I \times (J_0 \cup J_1) \times I$ of $I \times \partial I \times I$, where J_0 and J_1 are nondegenerate closed intervals containing 0 and 1, respectively. Then $\mathcal{F}$ can be C^0 deformed to a C^0 close, $C^{\infty,0}$, almost horizontal foliation $\mathcal{G}$ on $D \times I$ such that $\mathcal{G}$ agrees with $\mathcal{F}$ on N_v and $\rho_{\mathcal{G}}(\alpha) = \rho_{\mathcal{F}}(\alpha)$. If $L'_1, \dots, L'_n$ are smoothly embedded leaves of $\mathcal{F}$, then the deformation can be chosen to fix these leaves pointwise. Moreover, if $L'_1, \dots, L'_n$ are subsurfaces of leaves $L_1, \dots, L_n$ of $\mathcal{F}$ such that regular neighborhoods $N(L'_i)$ of L'_i in L_i are smoothly embedded in $D \times I$, then the deformation can be chosen to fix each L'_i pointwise.*

Moreover, if $\mathcal{F}$ is C^∞ on N_v and $\rho_{\mathcal{F}}$ is smooth, then $\mathcal{G}$ can be chosen to be C^∞.

Proof Pick a metric on the bundle of tangent 2–planes to $D \times I$. Denote the leaf of $\mathcal{F}$ that contains $\left(\frac{1}{2}, 0, t\right)$ by F_t. Each F_t is the graph of a C^1 function $f_t\colon D \to I$. Fix a smooth monotonic bump function $\ell\colon I \to I$ that vanishes on J_1 and is identically 1 on J_0.

By Proposition 3.11, there is a $C^{\infty,0}$, almost horizontal, foliation $\mathcal{S}$ of $D \times I$ that is C^0 close to $\mathcal{F}$ and agrees, in the sense that the decompositions as unions of surfaces agree, with $\mathcal{F}$ on N_v. Denote the leaf of $\mathcal{S}$ that contains $\left(\frac{1}{2}, 0, t\right)$ by S_t. Each S_t is the graph of a C^∞ function $s_t\colon D \to I$. If f_t describes the graph of the leaf L'_i for some i, then choose $\mathcal{S}$ so that $s_t = f_t$ along L'_i.

It may be helpful, even though not necessary, to recall that by the construction of $\mathcal{S}$ found in the proof of Proposition 3.11, we may assume that there is a partition $0 = t_0 < t_1 < \dots < t_m = 1$ such that $S_{t_i} = F_{t_i}$ for each i, and each leaf S_t for $t \in [t_i, t_{i+1}]$ has tangent plane field C^0 close to the tangent plane field of each of F_{t_i} and $F_{t_{i+1}}$.

Define a homeomorphism $h\colon I \to I$ by

$$\left(\left(\tfrac{1}{2}, 0\right), h(t)\right) = \rho_{\mathcal{S}}(\alpha) \circ \rho_{\mathcal{F}}^{-1}(\alpha)\left(\left(\tfrac{1}{2}, 0\right), t\right).$$

The point of the definition of $h(t)$ is that a leaf of $\mathcal{S}$ that contains $\left(\left(\frac{1}{2}, 0\right), h(t)\right)$ will intersect the leaf of $\mathcal{F}$ that contains $\left(\left(\frac{1}{2}, 0\right), t\right)$ at the point $\left(\left(\frac{1}{2}, 1\right), s_{h(t)}\left(\frac{1}{2}, 1\right)\right) = \left(\left(\frac{1}{2}, 1\right), f_t\left(\frac{1}{2}, 1\right)\right)$.

Since $\mathcal{F}$ and $\mathcal{S}$ agree on $I \times (J_0 \cup J_1)$, $s_{h(t)} = f_t$ on $I \times J_1$. Since $\mathcal{F}$ and $\mathcal{S}$ are C^0 close, h is C^0 close to the identity map and, for each $t \in I$, the graphs s_t and $s_{h(t)}$ are C^1 close on the rest of D.

We obtain a $C^{\infty,0}$ foliation $\mathcal{G}$ approximating $\mathcal{F}$ and satisfying the holonomy constraint $\rho_{\mathcal{G}}(\alpha) = \rho_{\mathcal{F}}(\alpha)$ as follows. Define

$$\begin{aligned} g_t(x,y) &= \ell(y)s_t(x,y) + (1-\ell(y))s_{h(t)}(x,y) \\ &= \ell(y)(s_t(x,y) - s_{h(t)}(x,y)) + s_{h(t)}(x,y). \end{aligned}$$

Then, for a fixed t,

$$\partial_x g_t = \ell \cdot (\partial_x s_t - \partial_x s_{h(t)}) + \partial_x s_{h(t)}$$

and

$$\partial_y g_t = \partial_y \ell (s_t - s_{h(t)}) + \ell \cdot (\partial_y s_t - \partial_y s_{h(t)}) + \partial_y s_{h(t)}.$$

By choosing $\mathcal{S}$ sufficiently close to $\mathcal{F}$, we guarantee that the $C^{\infty,0}$ foliation $\mathcal{G}$ with leaves the graphs of the functions $g_t\colon D \to I$ is C^0 close to $\mathcal{F}$. Since $\mathcal{S} = \mathcal{F}$ on N_v, the choice of ℓ implies $\mathcal{G} = \mathcal{F}$ on N_v. This foliation $\mathcal{G}$ is obtained by a C^0 small deformation of $\mathcal{F}$. □

Proposition 3.15 (smoothing above a 2–cell) *Let $D \times I$ be a C^∞ flow box and let A be a closed smooth regular collar neighborhood of ∂D in D. Let Φ denote the 1–dimensional foliation by I–fibers $\{x\} \times I$ for $x \in D$. Suppose $\mathcal{F}$ is a $C^{1,0}$ almost horizontal foliation on $D \times I$ that is $C^{\infty,0}$ on $A \times I$. Then $\mathcal{F}$ can be C^0 deformed to a C^0 close, $C^{\infty,0}$, almost horizontal foliation $\mathcal{G}$ on $D \times I$ such that $\mathcal{G}$ agrees with $\mathcal{F}$ on $A \times I$.*

Moreover, if $\mathcal{F}$ is C^∞ on $A \times I$, then $\mathcal{G}$ can be chosen to be C^∞.

Proof This follows as an immediate corollary to Propositions 3.11 and 3.12. □

4 Any $C^{1,0}$ foliation is a limit of $C^{\infty,0}$ foliations

The next theorem adds C^0 approximation to a theorem of Calegari [4]. It is applied and cited as Theorem 2.10 in [20]. Calegari shows that a foliation with continuous leaves can be isotoped to a foliation with smooth leaves. Instead, we start with a $C^{1,0}$ foliation so that the leaves have continuously varying tangent planes, and then isotope the foliation to have smooth leaves while controlling the amount the tangent planes move.

Theorem 4.1 *Suppose $\mathcal{F}$ is a $C^{1,0}$ foliation of a compact manifold M. Then there is a C^0 small isotopy of M taking $\mathcal{F}$ to a $C^{\infty,0}$ foliation $\mathcal{G}$ that is C^0 close to $\mathcal{F}$. If Φ is a smooth flow transverse to $\mathcal{F}$, the isotopy may be taken to be flow-compatible.*

Proof Let Φ be a smooth flow transverse to $\mathcal{F}$, and apply Lemma 3.7 to obtain a smooth-sided $(\mathcal{F},\Phi)$–flow box decomposition $\mathcal{B}'$ of M. Applying the isotopy of Lemma 3.8 to $\mathcal{B}'$ and $\mathcal{F}$, respectively, produces a smooth flow box decomposition $\mathcal{B}$ and C^0 close foliation $\mathcal{F}_1$. Let $\sigma_1,\dots,\sigma_n$ be a listing of the maximal vertical faces of $\mathcal{B}$, and choose a regular neighborhood structure $(N, N_v, N(\sigma_1),\dots,N(\sigma_n))$ for $\mathcal{B}$.

By Proposition 3.11 and Theorem 3.12, there is a C^0 small, $\mathcal{B}$–compatible isotopy of M that takes $\mathcal{F}_1$ to a C^0 close $C^{1,0}$ foliation $\mathcal{F}_2$ that is smooth on N_v.

By Proposition 3.14, there is a C^0 small, $\mathcal{B}$–compatible isotopy of M that takes $\mathcal{F}_2$ to a C^0 close $C^{1,0}$ foliation $\mathcal{F}_3$ that is smooth on N_v and $C^{\infty,0}$ on each $N(\sigma_i)$.

Finally, by Proposition 3.15, there is a C^0 small, $\mathcal{B}$–compatible isotopy of M that takes $\mathcal{F}_3$ to a C^0 close $C^{\infty,0}$ foliation $\mathcal{G}$. □

Corollary 4.2 *Any $C^{1,0}$ foliation is a limit of $C^{\infty,0}$ foliations.* □

5 Measured foliations

A *transverse measure* on a codimension-one foliation $\mathcal{F}$ is a *continuous, nondegenerate, invariant* measure, μ, on each arc transverse to $\mathcal{F}$. It is continuous in the sense that if τ is smoothly parametrized as $\tau=[0,x]$, then $\mu([0,x])$ is continuous in x. Nondegenerate means that μ is positive on every open interval. Invariant, in this context, means that the measure of a transverse arc is unchanged under isotopies of the arc that keep each point on the same leaf of $\mathcal{F}$.

Lemma 5.1 (smoothing a measure near a transversal) *Let $(\mathcal{F},\mu)$ be a $C^{\infty,0}$ measured foliation in M. Suppose τ is a smoothly embedded arc or closed curve which is everywhere transverse to $\mathcal{F}$. Then there is a C^0 small isotopy of M which is the identity outside some small regular neighborhood N of τ and takes the measured foliation $(\mathcal{F},\mu)$ to a C^0 close $C^{\infty,0}$ measured foliation $(\mathcal{F}',\mu')$ such that $\mathcal{F}'$ is smooth in a neighborhood of τ and the measure, μ', restricted to τ is smooth. If μ is smooth on a closed submanifold A of τ, then the isotopy can be chosen so that $\mu'=\mu$ on A. If, in addition, $\mathcal{F}$ is smooth in an $(\mathcal{F},\Phi)$–compatible regular neighborhood N_0 of A, then the isotopy can be chosen to be the identity on N_0.*

Proof It suffices to consider the case that τ is a smoothly embedded arc.

Regard τ as a smooth map $I\to M$. Then $\mu(\tau[0,t])$ is a homeomorphism, $h\colon I\to\mathbb{R}$, onto its image. Approximate h by a diffeomorphism, relative to endpoints, g. The goal is to make a small, continuous change of coordinates on τ so that μ is smooth in the new coordinates. In other words, we must choose a homeomorphism $f\colon I \to I$ so that $h \circ f$ is smooth. This is accomplished by defining $f = h^{-1}g$.

The next step is to use this reparametrization to describe a C^0 small isotopy of M which is the identity outside a small neighborhood of τ and takes the measure μ to a measure μ' that on τ satisfies $\mu'[0,t] = \mu(\tau \circ f[0,t])$ for each $t \in [0,1]$. In particular, μ' restricted to τ is smooth.

To accomplish this, let N_1 and N_2 be small smoothly embedded tubular neighborhoods of τ satisfying $\overline{N}_1 \subset \operatorname{int}(N_2)$. Choose these neighborhoods small enough that $\mathcal{F}$ meets each in a foliation by meridian disks. Parametrize these disks by their intersection with $\tau = [0,t]$ and so that N_1 is identified with the smooth family of meridian disks $D_t = D^2 \times \{t\}$.

Next isotope $\mathcal{F}$ in N_2 so that the disk $D_{f(t)}$ is taken to the disk D_t. Choose this isotopy to be C^0 small and Φ–compatible, taking $\mathcal{F}$ to a C^0 close $C^{\infty,0}$ foliation. If we choose N_1 small enough and f sufficiently close to the identity, we may choose these isotopies to be as close as desired to the identity.

Define μ' near τ so that it is invariant and agrees with μ away from τ. □

Corollary 5.2 (smoothing $\mathcal{F}$ near a transversal) *Let $\mathcal{F}$ be a $C^{\infty,0}$ foliation in M. Suppose τ is a smoothly embedded arc or closed curve which is everywhere transverse to $\mathcal{F}$. Then there is a C^0 small isotopy of M that is the identity outside some small regular neighborhood of τ and takes $\mathcal{F}$ to a C^0 close $C^{\infty,0}$ foliation $\mathcal{F}'$ such that $\mathcal{F}'$ is smooth in a neighborhood of τ. If $\mathcal{F}$ is smooth in an $(\mathcal{F},\Phi)$–compatible regular neighborhood N_0 of a closed subset of τ, then the isotopy can be chosen to be the identity on N_0.*

Proof Choose a small regular neighborhood N of τ so that $\mathcal{F}$ meets it in a product foliation by disks. Use distance along τ to define a smooth transverse measure on the restriction of $\mathcal{F}$ to N. The result now follows immediately from Lemma 5.1. □

The next lemma shows how the existence of a transverse measure allows the foliation to be smoothed near a compact portion of a leaf.

Lemma 5.3 (smoothing in the neighborhood of a compact subsurface of a leaf) *Let $(\mathcal{F},\mu)$ be a $C^{\infty,0}$ measured foliation in M. Suppose S is a compact subsurface of a leaf of $\mathcal{F}$. Then there is a C^0 small isotopy of M which is the identity outside some small regular neighborhood of S in M and takes the measured foliation $(\mathcal{F},\mu)$ to a*

$C^{\infty,0}$ measured foliation $(\mathcal{F}', \mu')$ such that $\mathcal{F}'$ is smooth in a neighborhood of S and the measure μ' restricted to this neighborhood is smooth.

Proof Let L be the leaf of $\mathcal{F}$ containing S. If $S = L$, let $N(S) = L$. Otherwise, let $N(S)$ be the closure of a regular neighborhood of S in L. Use the measure, μ, to give a continuous parametrization of the flow Φ in a neighborhood of $N(S)$. To avoid confusion, let Φ' denote this reparametrized restriction of Φ. Choose this parametrization so that $\Phi'(x,0) = x$ for all $x \in N(S)$ and $\mu(\Phi'(x,[s,t])) = t - s$ for $s < t$ sufficiently close to 0.

For some $\epsilon > 0$, $\Phi'\colon N(S) \times [-\epsilon, \epsilon] \to M$ is a topological embedding, and for each $t \in [-\epsilon, \epsilon]$, $\Phi'(N(S) \times \{t\})$ is a compact subsurface of a leaf of $\mathcal{F}$, necessarily isotopic to $N(S)$. Since $\mathcal{F}$ is $C^{\infty,0}$ and Φ' is smooth when restricted to a leaf, $\Phi'(N(S) \times [-\epsilon, \epsilon])$ is a smooth codimension-zero submanifold, possibly with corners.

Use Theorem 3.12 to C^0 isotope $\mathcal{F}$ in $\Phi'(N(S) \times [-\epsilon, \epsilon])$ so that it is a smooth foliation by surfaces isotopic to $N(S)$. The resulting measured foliation $(\mathcal{F}', \mu')$ and the measure μ' are necessarily smooth on the neighborhood $\Phi'(N(S) \times (-\epsilon, \epsilon))$ of $N(S)$. □

The next theorem is applied and cited as Theorem 8.10 in [20]:

Theorem 5.4 *Suppose $\mathcal{F}$ is a transversely orientable $C^{1,0}$ measured foliation in M. Then there is an isotopy of M taking $\mathcal{F}$ to a C^∞ measured foliation which is C^0 close to $\mathcal{F}$. If Φ is a smooth flow transverse to $\mathcal{F}$, the isotopy may be taken to be flow-compatible.*

Proof By Theorem 4.1 we may assume $\mathcal{F}$ is $C^{\infty,0}$. From Lemma 3.7 it follows that M has a smooth flow box decomposition, $M = F_1 \cup \dots \cup F_n$.

Using Lemma 5.3, we may assume, after a C^0 small isotopy, that $\mathcal{F}$ and μ are smooth in a small regular neighborhood N_h of $\bigcup_i \partial_h F_i$. The vertical 1–skeleton of $\mathcal{B}$ is a disjoint union of transversals to $\mathcal{F}$, and hence, by applying Lemma 5.1, we may assume, after a C^0 small isotopy, that $\mathcal{F}$ and μ are smooth on $N_h \cup N_v$, where N_v is a union of flow boxes that form an $(\mathcal{F}, \Phi)$–compatible regular neighborhood of the union of the vertical 1–cells of $\bigcup_i \partial^{(1)} F_i$.

Let $\sigma_1, \dots, \sigma_n$ be a listing of the maximal vertical faces of $\mathcal{B}$, and let

$$(N, N_v, N(\sigma_1), \dots, N(\sigma_n))$$

be a regular neighborhood structure for $\mathcal{B}$. Choose $N(N(\sigma_i))$ for $1 \le i \le n$ so that

$$\Big(N \cup \bigcup_i N(N(\sigma_i)), N_v, N(N(\sigma_1)), \dots, N(N(\sigma_n))\Big)$$

is also a regular neighborhood structure for $\mathcal{B}$.

Let τ_i^- and τ_i^+ denote the two vertical edges of σ_i. Let ρ_i denote the homeomorphism obtained by following leaves of $\mathcal{F}$ across σ_i. Since the measure μ is smooth on $\tau_i^\pm$ and ρ_i preserves μ, ρ_i is smooth. By Proposition 3.14, there is a C^0 small, $\mathcal{B}$–compatible isotopy of M that is the identity on $N_v \cup N_h$ and outside the union $\bigcup_i N(N(\sigma_i))$, and takes $\mathcal{F}$ to a foliation $\mathcal{G}$ that is smooth on N and C^0 close to $\mathcal{F}$.

Finally, we apply Proposition 3.15, to extend $\mathcal{G}$ above each σ_i to a smooth foliation that is C^0 close and isotopic, by a C^0 small isotopy, to $\mathcal{F}$. Since μ is defined on the vertical boundary of every flow box, it extends to all of $\mathcal{G}$. □

The next corollary now follows from Theorem 8.11 in [20], which uses Theorem 5.4 in its proof. Alternatively, it also follows from Theorem 5.4 together with Tischler's theorem [29], which states that any transversely oriented, measured C^2 foliation on a compact n–manifold can be C^∞ approximated by a smooth fibration over S^1.

Corollary 5.5 *A $C^{1,0}$, transversely oriented, measured foliation on a compact 3–manifold is C^0 close to a smooth fibration over S^1.* □

6 Holonomy neighborhoods

In this section, we recall some definitions and results used in [20], giving those proofs which, for clarity of exposition, were deferred to this paper.

Let γ be an *oriented* simple closed curve in a leaf L of $\mathcal{F}$, and let p be a point in γ. We are interested in the behavior of $\mathcal{F}$ in a neighborhood of γ. Let h_γ be a holonomy map for $\mathcal{F}$ along γ, and let σ and τ be small closed segments of the flow Φ which contain p in their interiors and satisfy $h_\gamma(\tau) = \sigma$. Choose τ small enough that $\sigma \cup \tau$ is a closed segment and not a loop. Notice that $\sigma \cap \tau$ is necessarily a closed segment containing p in its interior. There are three possibilities:

(1) $\sigma = \tau$,

(2) one of σ and τ is properly contained in the other, or

(3) $\sigma \cap \tau$ is properly contained in each of σ and τ.

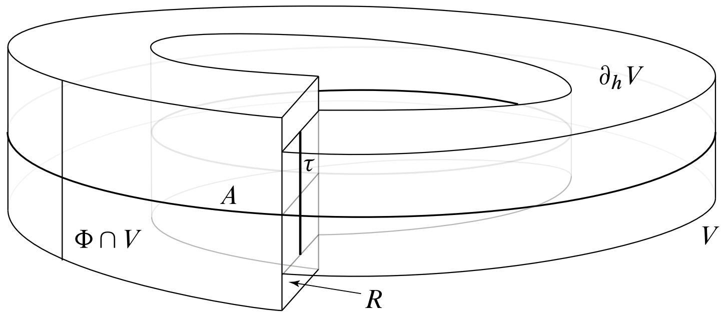

Figure 2: Holonomy neighborhood.

We will need to consider very carefully a regular neighborhood of γ which lies nicely with respect to both $\mathcal{F}$ and Φ. To this end, restrict attention to foliations $\mathcal{F}$ which are $C^{\infty,0}$ and transversely oriented, and transverse flows Φ which are smooth, and suppose that γ is smoothly embedded in L. Let A be the closure of a smooth regular neighborhood of γ in L; so A is a smoothly embedded annulus in L.

Lemma 6.1 [20, Lemma 3.1] *Suppose $\mathcal{F}$ is $C^{\infty,0}$ and transversely oriented, and Φ is smooth. If τ and A are chosen to be small enough, there is a compact submanifold V of M, smoothly embedded with corners, which satisfies the following:*

(1) *V is homeomorphic to a solid torus.*

(2) *∂V is piecewise vertical and horizontal; namely, ∂V decomposes as a union of subsurfaces $\partial_v V \cup \partial_h V$, where $\partial_v V$ is a union of flow segments of Φ and $\partial_h V$ is a union of two surfaces L_- and L_+, each of which is either a disk or an annulus, contained in leaves of $\mathcal{F}$.*

(3) *Each flow segment of $\Phi \cap V$ runs from L_- to L_+.*

(4) *τ is contained in a component of the flow segments of $\Phi \cap V$.*

(5) *A is a leaf of the foliation $\mathcal{F} \cap V$.*

Proof Cover a small open neighborhood of γ by finitely many smooth flow boxes. By passing to a smaller τ and A as necessary, we may suppose that A is covered by two flow boxes with union, V, satisfying the properties (1)–(5). □

Notation 6.2 Denote the neighborhood V of Lemma 6.1 by $V_\gamma(\tau, A)$. Let $R_\gamma(\tau, A)$ denote any smooth vertical rectangle embedded in $V_\gamma(\tau, A)$ such that the result of cutting $V_\gamma(\tau, A)$ open along $R_\gamma(\tau, A)$, and taking the metric closure, is diffeomorphic to a solid cube. Denote this cube by $Q_\gamma(\tau, A)$.

Notice that $R_\gamma(\tau, A)$ is uniquely determined if and only if $\tau \neq \sigma$. If γ is essential, then $Q_\gamma(\tau, A)$ can be viewed as an $(\widetilde{\mathcal{F}}, \widetilde{\Phi})$–flow box, where $(\widetilde{\mathcal{F}}, \widetilde{\Phi})$ is the lift of $(\mathcal{F}, \Phi)$ to the universal cover of M.

Definition 6.3 The neighborhood $V_\gamma(\tau, A)$ is called the *holonomy neighborhood determined by* (τ, A), and is called an *attracting neighborhood* if $h_\gamma(\tau)$ is contained in the interior of σ.

Fix a set of pairwise disjoint holonomy neighborhoods $V_{\gamma_1}(\tau_1, A_1), \dots, V_{\gamma_n}(\tau_n, A_n)$ for $\mathcal{F}$, and let V denote their union. Let $R_i = R_{\gamma_i}(\tau_i, A_i)$ for $1 \leq i \leq n$, and let R denote the union of the R_i. For each i, $1 \leq i \leq n$, fix a smooth open neighborhood N_{R_i} of R_i in M. Choose each N_{R_i} small enough that its closure, $\overline{N}_{R_i}$, is a closed regular neighborhood of R_i. Let N_R denote the union of the N_{R_i}.

Now, given V, R and N_R, we further constrain the set of foliations $\mathcal{F}$ to $C^{\infty,0}$ foliations which are smooth on N_R. The following lemma, applied and cited as Lemma 3.7 in [20], establishes that we can do this with no loss of generality.

Lemma 6.4 *Let $\mathcal{F}$ be a transversely oriented, $C^{\infty,0}$ foliation, and let Φ be a smooth flow transverse to $\mathcal{F}$. Let V denote the union of a set of pairwise disjoint holonomy neighborhoods for $\mathcal{F}$ and fix N_R as above. There is an isotopy of M taking $\mathcal{F}$ to a $C^{\infty,0}$ foliation which is both C^0 close to $\mathcal{F}$ and smooth on N_R. This isotopy may be taken to preserve V and be flow-compatible.*

Proof Apply Theorem 3.12 to an $(\mathcal{F}, \Phi)$–flow box that intersects V in a regular neighborhood of N_R, and contains N_R in its interior. □

Next we describe a preferred product parametrization on a closed set containing V. In this paper, we express S^1 as the quotient $S^1 = [-1, 1]/\sim$, where $\sim$ is the equivalence relation on $[-1, 1]$ which identifies -1 and 1.

Lemma 6.5 *Let $\mathcal{F}$ be a transversely oriented, $C^{\infty,0}$ foliation, and let Φ be a smooth flow transverse to $\mathcal{F}$. Let V denote the union of pairwise disjoint holonomy neighborhoods $V_i = V_{\gamma_i}(\tau_i, A_i)$ for $\mathcal{F}$ for $1 \leq i \leq n$, and fix N_R as above. Suppose $\mathcal{F}$ is smooth on N_R. Then for each i, $1 \leq i \leq n$, there is a pairwise disjoint collection of closed solid tori P_i such that P_i contains V_i and there is a diffeomorphism $P_i \to [-1, 1] \times S^1 \times [-1, 1]$ which satisfies the following:*

(1) *the flow segments $\Phi \cap P_i$ are identified with the segments $\{(x, y)\} \times [-1, 1]$,*

(2) *A_i is identified with $[-1, 1] \times S^1 \times \{0\}$,*

(3) *γ_i is identified with $\{0\} \times S^1 \times \{0\}$,*

(4) *R_i is identified with $[-1,1] \times \{1 \sim -1\} \times [-1,1]$, and*

(5) *the restriction of the diffeomorphism to N_{R_i} maps leaves of $\mathcal{F}$ to horizontal level surfaces $D_z = D \times \{z\}$, where $D = [-1,1] \times \big(\big(\big[\frac{1}{2},1\big] \cup \big[-1,-\frac{1}{2}\big]\big)/{\sim}\big)$.*

Proof Since $\mathcal{F}$ is smooth on N_R, there is a choice of smooth coordinates on an open neighborhood of N_R with respect to which the leaves of $\mathcal{F}$ are horizontal and the restrictions of the conditions (1)–(5) hold true. This parametrization extends to a smooth parametrization of a smooth solid torus neighborhood of V satisfying conditions (1)–(5). □

Definition 6.6 Fix V and N_R as above. Let P_i and $P_i \to [-1,1] \times S^1 \times [-1,1]$ be as given in Lemma 6.5. Abuse notation and use the diffeomorphism to identify P_i with $[-1,1] \times S^1 \times [-1,1]$. Let $\mathcal{P}_i$ be the product foliation of P_i with leaves $([-1,1] \times S^1) \times \{t\}$, and call such a foliated solid torus, $(P_i, \mathcal{P}_i)$, a *product neighborhood of* $(V_i; N_{R_i})$. Letting P denote the union of the P_i and $\mathcal{P}$ denote the union of the $\mathcal{P}_i$, call $(P, \mathcal{P})$ a *product neighborhood of* $(V; N_R)$.

Definition 6.7 Let $\mathcal{F}$ be a transversely oriented, $C^{\infty,0}$ foliation and V the union of pairwise disjoint, holonomy neighborhoods $V_{\gamma_i}(\tau_i, A_i)$ for $\mathcal{F}$ for $1 \le i \le k$. Let R denote the union of the $R_{\gamma_i}(\tau_i, A_i)$ for $1 \le i \le k$, and let N_R be an open regular neighborhood of R in V. Let $(P, \mathcal{P})$ be a product neighborhood of $(V; N_R)$. The foliation $\mathcal{F}$ is *strongly (V, P)–compatible* if

(1) $\mathcal{F} \cap N_R = \mathcal{P} \cap N_R$, and

(2) in the coordinates inherited from P, $\mathcal{F} \cap V$ is a product foliation $[-1,1] \times \mathcal{F}_0$, where $\mathcal{F}_0$ is a $C^{\infty,0}$ foliation of $V \cap (\{1\} \times S^1 \times [-1,1])$ (ie $\mathcal{F} \cap V$ is x–invariant).

The following lemma, applied and cited as Lemma 3.11 in [20], establishes that, up to a C^0 small perturbation of $\mathcal{F}$, we can always choose the product neighborhood $(P, \mathcal{P})$ so that $\mathcal{F}$ is strongly (V, P)–compatible.

Lemma 6.8 *Let $\mathcal{F}$ be a transversely oriented, $C^{\infty,0}$ foliation and let Φ be a smooth flow transverse to $\mathcal{F}$. Let V denote the union of a set of pairwise disjoint holonomy neighborhoods for $\mathcal{F}$ and fix N_R as above. There is a C^0 small, flow-compatible* *isotopy of M that takes $\mathcal{F}$ to a $C^{\infty,0}$ foliation that is C^0 close to $\mathcal{F}$ and strongly (V,P)–compatible for some choice of product neighborhood $(P,\mathcal{P})$ of $(V;N_R)$. This isotopy may be taken to preserve V.*

Proof We may assume $\mathcal{F}\cap N_R=\mathcal{P}\cap N_R$. Let $N(N_R)$ be an $(\mathcal{F},\Phi)$–flow box neighborhood that contains N_R in its interior. After applying an isotopy as described in Theorem 3.12, we may assume that $\mathcal{F}$ is smooth on $V\setminus N(N_R)$. Thus, we may choose the smooth coordinates (x,y,z) on $V\setminus N(N_R)$ so that with respect to these coordinates, $\mathcal{F}$ is x–invariant. Finally, proceeding as in the proof of Proposition 3.14, perform a C^0 small isotopy of $\mathcal{F}$ so that the leaves of its restriction to $N(N_R)\setminus N_R$ are obtained by smoothed linear extensions. Thus, the smooth coordinates $\mathcal{P}=[-1,1]\times S^1\times[-1,1]$ for P can be chosen to restrict to the preferred coordinates on N_R and $V\setminus N(N_R)$, and consequently these smoothed linear extensions are $C^{\infty,0}$ and x–invariant. □

7 Denjoy blowup

In [10; 7; 8; 9; 11], Denjoy gave examples of C^1 foliations on T^2 with exceptional minimal sets. In [12], Dippolito generalized Denjoy's method to $C^{\infty,0}$ codimension-one foliations of n–manifolds. This generalized construction is commonly referred to as *Denjoy blowup*, and is defined precisely as follows:

Definition 7.1 Let L be a countable (finite or countably infinite) union of leaves of a $C^{k,0}$ foliation $\mathcal{F}$ of M with $k\geq 1$, and let Φ be a smooth flow transverse to $\mathcal{F}$. A $C^{k,0}$ foliation, $\mathcal{F}'$, is a *Denjoy blowup of $\mathcal{F}$ along L* if there is an open subset $U\subset M$ and a continuous collapsing map $\pi\colon M\to M$ satisfying the following properties:

(1) $\mathcal{F}'$ is transverse to Φ,

(2) there is an injective map $j\colon L\times I\to M$ such that $j|_{L\times(0,1)}$ is a C^k immersion and $j(L\times(0,1))=U$,

(3) for each $p\in L$, $j(\{p\}\times I)$ is contained in a flow line of Φ,

(4) $j(L\times\{0\})$ and $j(L\times\{1\})$ are leaves of $\mathcal{F}'$,

(5) $\pi^{-1}(p)$ is a point if $p\notin L$ and equals $j(\{p\}\times I)$ if $p\in L$,

(6) π is Φ–compatible and maps leaves of $\mathcal{F}'$ to leaves of $\mathcal{F}$,

(7) π is C^k when restricted to any leaf of $\mathcal{F}'$, and

(8) there is a Φ–compatible C^k homotopy $\pi_t\colon M\to M$ such that π_t is an isotopy for $t\in[0,1)$ and $\pi=\pi_1$.

If $\mathcal{L}$ is a $C^{k,0}$ almost horizontal foliation of $L \times I$ and the pullback of the Denjoy blowup $\mathcal{F}'$ to $L \times I$ is C^k equivalent to $\mathcal{L}$, then $\mathcal{F}'$ is a Denjoy blowup of $\mathcal{F}$ along L *by* $\mathcal{L}$.

Lemma 7.2 *Suppose $\mathcal{F}$ and $\mathcal{G}$ are $C^{1,0}$ foliations of M transverse to a common smooth flow Φ. If $\mathcal{F}$ and $\mathcal{G}$ are Φ–compatible isotopic, then a Denjoy blowup of $\mathcal{G}$ is a Denjoy blowup of $\mathcal{F}$.*

Proof Denjoy blowup is defined only up to Φ–compatible isotopy, and so varying a foliation by a Φ–compatible isotopy does not change its Denjoy blowup. □

The following result extends Dippolito's generalization of Denjoy's construction [12, Theorem 7] in two ways. First, it allows for foliations which are not $C^{\infty,0}$. Second, it shows that the resulting foliation, $\mathcal{F}'$, can be constructed arbitrarily C^0 close to $\mathcal{F}$.

Theorem 7.3 (Denjoy blowup) *Let $\mathcal{F}$ be a $C^{1,0}$ foliation in a compact 3–manifold M. Suppose that $\mathcal{F}$ is transverse to a smooth flow Φ. Let L be a countable collection of leaves of $\mathcal{F}$ and let $\mathcal{L}$ be a $C^{1,0}$ almost horizontal foliation of $L \times I$. Then there exists a $C^{\infty,0}$ $\mathcal{F}'$ arbitrarily C^0 close to $\mathcal{F}$ that is a Denjoy blowup of $\mathcal{F}$ along L by $\mathcal{L}$.*

Moreover, if $\mathcal{F}$ is $C^{\infty,0}$ and $\mathcal{B}$ is a $(\mathcal{F}, \Phi)$–flow box decomposition of M, with flow boxes $F_1, \dots, F_n$, and L is disjoint from $\bigcup_j \partial_h F_i$, then the Denjoy blowup can be chosen to be strongly $\mathcal{B}$–compatible in the following sense: the restriction of $\mathcal{F}'$ to each F_i is the Denjoy blowup of the restriction of $\mathcal{F}$ to F_i.

By Theorem 4.1, a $C^{k,0}$ foliation on a compact 3–manifold can be isotoped by a C^0 small Φ–compatible isotopy to a C^0 close $C^{\infty,0}$ foliation. Thus, by Lemma 7.2, it is enough to prove the theorem with the assumption that $\mathcal{F}$ is $C^{\infty,0}$ and $\mathcal{L}$ is $C^{k,0}$. While it may be possible to conclude C^0 proximity of $\mathcal{F}'$ from Dippolito's original proof, the method of flow box decompositions gives a direct and elementary proof.

We first describe the Denjoy blowup of a strictly horizontal (and therefore smooth) foliation $\mathcal{F}$ of a single flow box.

Lemma 7.4 *Let $\mathcal{F}$ be a strictly horizontal foliation of a C^∞–flow box $F = D \times I$. Let L be a countable union of leaves of $\mathcal{F}$, and let $\mathcal{L}$ be a $C^{k,0}$ almost horizontal foliation of $L \times I$ for some $k \geq 1$. Then there exists a $C^{\infty,0}$ strictly horizontal foliation $\mathcal{F}'$ that is a Denjoy blowup of $\mathcal{F}$ along L by $\mathcal{L}$, with associated collapsing map π that is C^0 close to the identity.*

Moreover, given finitely many leaves $D \times \{t_j\}$ of $\mathcal{F}$ that are disjoint from L, $\mathcal{F}'$ can be chosen so that the restriction of π to each $D \times \{t_j\}$ is the identity map.

Proof In this case, Φ is a flow along the vertical segments $\{x\} \times I$.

Let $D_t = D \times \{t\}$, and let the components of L be the leaves D_{z_i} for some set of points $z_i \in (0,1)$ for $i \in \mathcal{A}$.

We begin by describing the Denjoy blowup of I along the points z_i. Let w_i denote a summable sequence of positive numbers, with sum $w = \sum_i w_i$. Cut I at each z_i and insert an interval J_i of length w_i. The result is a new interval of length $1 + w$. The left inverse of this operation is a Cantor function; denote this Cantor function by $c\colon [0, 1+w] \to [0,1]$. Let $p\colon [0,1] \to [0,1]$ denote the function obtained by composing the function c with the linear scaling $s\colon [0,1] \to [0, 1+w]$; so $p = c \circ s$. These functions are illustrated in Figure 3.

Let $[z_i^-, z_i^+] = s^{-1}(J_i)$ and notice that $z_i^+ - z_i^- = w_i/(1+w)$. Set $C = I \setminus \bigsqcup_i (z_i^-, z_i^+)$ and let Λ be the strictly horizontal lamination with leaves D_t for $t \in C$. Let $U_i = D \times (z_i^-, z_i^+)$, and $U = \bigsqcup_i U_i$, the open set $F \setminus \Lambda$.

Let $\pi = \mathrm{id} \times p\colon F \to F$, where id is the identity map on D. In particular, π takes each flow segment $(x \times I) \cap U_i$ to the point (x, z_i).

Fix $i \in \mathcal{A}$, and let $f_i\colon I \to [z_i^-, z_i^+]$ be the affine diffeomorphism. Define $j_i = \mathrm{id} \times f_i\colon (D_{z_i} \times I) \to U_i$, and define $j = \bigcup j_i\colon L \times I \to U$ to be the map that restricts to j_i on $D_{z_i} \times I$.

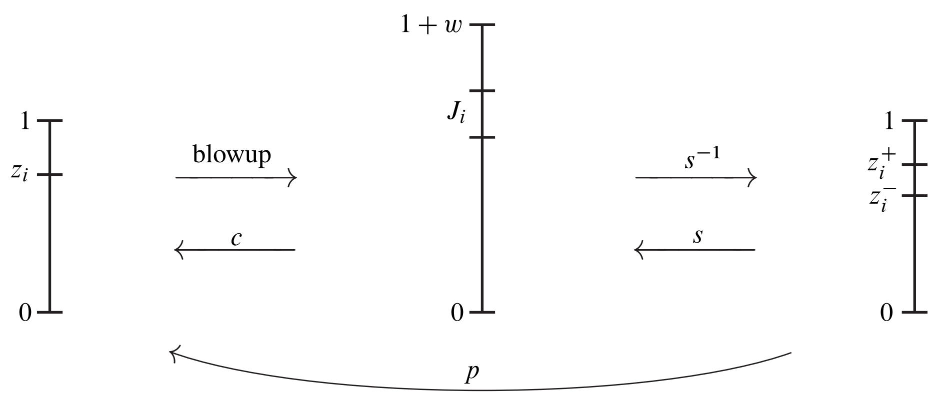

Figure 3

Let $\mathcal{F}'$ denote the foliation obtained by taking the union of Λ with $j(\mathcal{L})$. Now fix $\epsilon > 0$. Since $f_i' = w_i/(1+w) < w_i$ and $\mathcal{F}$ is strictly horizontal, the w_i can be chosen so that $\mathcal{F}'$ is a Denjoy blowup of $\mathcal{F}$ along L by $\mathcal{L}$, with associated collapsing map π that is C^0 close to the identity.

Properties (1)–(7) of Definition 7.1 then follow immediately. The homotopy π_t of property (8) is given by the straight line, Φ–compatible homotopy from the identity map to π. By Theorem 4.1, we may isotope the resulting $C^{1,0}$ Denjoy blowup to a C^0 close $C^{\infty,0}$ Denjoy blowup.

Finally, if $D \times \{t_j\}$ is a listing of finitely many leaves of $\mathcal{F}$ that are disjoint from L, cut F open along each $D \times \{t_j\}$, and perform Denjoy blowup, as just described, on each resulting flow box. □

Hence, Theorem 7.3 holds for a strictly horizontal (and therefore smooth) foliation $\mathcal{F}$ of a single flow box.

Proof of Theorem 7.3 By Theorem 4.1, $\mathcal{F}$ is Φ–compatible isotopic to a C^0 close $C^{\infty,0}$ foliation. By Lemma 7.2, therefore, we may restrict attention to the case that $\mathcal{F}$ is $C^{\infty,0}$.

Let $\mathcal{B}$ be a smooth $(\mathcal{F}, \Phi)$–flow box decomposition of M. Let $F_1, \dots, F_n$ be a listing of the flow boxes of $\mathcal{B}$. Choose $\mathcal{B}$ so that $\bigcup_i \partial_h F_i$ is disjoint from L.

Let σ_i for $1 \le i \le n$ be a listing of the maximal vertical faces of $\mathcal{B}$, and let

$$(N, N_v, N(\sigma_1), \dots, N(\sigma_n))$$

be a regular neighborhood structure for $\mathcal{B}$. By Corollary 5.2 and Lemma 7.2, it suffices to further restrict attention to the case that $\mathcal{F}$ is $C^{\infty,0}$, and smooth when restricted to N_v.

We will describe a C^0 close Denjoy splitting $\mathcal{F}'$ by considering first N_v, then the union $\bigcup_i N(\sigma_i)$, and finally the flow box interiors forming the complement of N.

Recall that N_v is a union of flow boxes, $B_j = D_j \times I$, satisfying conditions (3) and (4) of Definition 3.5. Rechoose the B_j, if necessary, so that $D_j \times (0,1)$ has empty intersection with $\bigcup_i \partial_h F_i$. This can be achieved by cutting each B_j open along any horizontal level that has nonempty intersection with $\bigcup_i \partial_h F_i$.

Let $B = D \times I$ be a B_j that has nonempty intersection with L. Since the restriction of $\mathcal{F}$ to B is smooth, there is a smooth parametrization (x, z) of B such that the restriction of $\mathcal{F}$ to B is strictly horizontal and the restriction of Φ to B has flow lines the vertical line segments $x \times I$. By Lemma 7.4, therefore, there is a Denjoy blowup $\mathcal{F}'_B$ of the restriction of $\mathcal{F}$ to B, and hence functions $\pi_B\colon B \to B$ and $j_B\colon L_B \times I \to B$ satisfying the conditions of Definition 7.1.

Repeat this process for each B_j that has nonempty intersection with L. And let $\mathcal{F}'_{B_j} = \mathcal{F}$ on the remaining B_j. Thus, we get a $C^{\infty,0}$ Denjoy blowup $\mathcal{F}'_{N_v}$ of the restriction of $\mathcal{F}$ to N_v that is strongly compatible with $\mathcal{B}$, together with functions $\pi_{N_v}\colon N_v \to N_v$ and $j_{N_v}\colon L_{N_v} \times I \to N_v$ satisfying the conditions of Definition 7.1. The foliation $\mathcal{F}'_{N_v}$ can be chosen to be C^0 close to the restriction of $\mathcal{F}$ to N_v.

Next, let σ be any maximal vertical face of $\mathcal{B}$ and let $\tau_\pm$ be the vertical edges of σ. Let $N(\tau_-)$ and $N(\tau_+)$ denote the components of $N_v \cap N(\sigma)$ that contain τ_- and τ_+, respectively. The Denjoy blowup, $\mathcal{F}'_{N_v}$, is defined on N_v, and hence on each $N(\tau_\pm)$. Let L_ℓ be a listing of the components of $L \cap N(\sigma)$. Set $a_\ell = L_\ell \cap N(\tau_-)$, and set $b_\ell = L_\ell \cap N(\tau_+)$. Writing $N(\sigma) = D \times I$, where $D = I \times I$, $\sigma = \{0\} \times I \times I$ and $\alpha = \{0\} \times I \subset D$. Orient α so that $\rho_{\mathcal{F}}(\alpha)(a_\ell) = b_\ell$. On each $[a_\ell^-, a_\ell^+]$, define $\rho(z) = j_{N_v} \circ \rho_{\mathcal{L}}(\alpha) \circ j_{N_v}^{-1}(z)$. On the complement of $\bigcup_\ell [a_\ell^-, a_\ell^+]$, where π^{-1} is single-valued, define $\rho(z) = \pi^{-1} \circ \rho_{\mathcal{F}}(\alpha) \circ \pi(z)$.

By Proposition 3.14, the foliation $\mathcal{F}'_{N_v}$ defined on N_v extends to a $C^{\infty,0}$ Denjoy blowup of $\mathcal{F}$ on $N_v \cup N(\sigma)$ that is $\mathcal{B}$–compatible, C^0 close to $\mathcal{F}$ on $N(\sigma_i)$ and satisfies $\rho_{\mathcal{F}'}(\alpha) = \rho$. Repeating this construction for each maximal face σ_i extends the definition of the C^0 close Denjoy blowup of $\mathcal{F}$ to N.

Finally, the Denjoy blowup $\mathcal{F}'_N$ defined on N extends to a strongly $\mathcal{B}$–compatible Denjoy blowup $\mathcal{F}'$, C^0 close to $\mathcal{F}$, on each flow box F_m of $\mathcal{B}$ by Proposition 3.15. Moreover, since at each step we are extending over a flow box F_m, the resulting packet, $j_{F_m}\colon (L \times I) \cap F_m \to F_m$, of inserted leaves will be $C^{k,0}$ equivalent to $\mathcal{L}$ on F_m. Similarly, the extension of the collapsing map $\pi\colon N \to N$ to F_m is uniquely determined by the properties that it maps leaves of $\mathcal{F}'$ to leaves of $\mathcal{F}$ and maps each I–fiber to itself.

Since the resulting foliation $\mathcal{F}'$ satisfies the conditions of Definition 7.1 on N and on each flow box of $\mathcal{B}$, it satisfies these conditions on M. □

The following corollary is cited as Theorem 5.2 in [20].

Corollary 7.5 *Let $\mathcal{F}$ be a transversely oriented, $C^{k,0}$ foliation with $k \ge 1$ that is transverse to a smooth flow Φ. Let L be a countable collection of leaves of $\mathcal{F}$, and let*

$\mathcal{F}_1$ be a $C^{k,0}$ foliation of $L \times I$ transverse to the I–coordinate that contains $L \times \partial I$ as leaves. Then there exists a $C^{k,0}$ $\mathcal{F}'$ arbitrarily C^0 close to $\mathcal{F}$ that is a Denjoy blowup of $\mathcal{F}$ along L and such that the pullback of $\mathcal{F}'$ to $L \times I$ is equivalent to $\mathcal{F}_1$.

Moreover, if V is the union of a set of pairwise disjoint holonomy neighborhoods for $\mathcal{F}$, $(W, \mathcal{P})$ is a product neighborhood of V, and $\mathcal{F}$ is strongly (V, W)–compatible, then $\mathcal{F}'$ can be chosen to be strongly (V, W)–compatible.

Proof The first paragraph of the corollary is stated as it is used in [20], and it follows directly from Theorem 7.3.

For the second paragraph, it suffices to consider the case that V consists of a single holonomy neighborhood.

Using Notation 6.2, V can be cut open along R into a cube Q. Parametrize $Q = I^3$ so that

$$N_R = I \times \left(\left(\tfrac{3}{4}, 1\right] \cup \left[0, \tfrac{1}{4}\right)\right) \times I,$$

and decompose Q along $I \times \left\{\tfrac{1}{2}\right\} \times I$ into two flow boxes. Thus V is realized as a union of two flow boxes, and, by Proposition 3.3, this flow box decomposition of V extends to a flow box decomposition $\mathcal{B}$ of M. Moreover, this extension, $\mathcal{B}$, can be chosen so that each vertical face of V, except for the proper subface of R, is maximal. Let σ_1 denote the maximal face R.

Choose a regular neighborhood structure $(N, N_v, N(\sigma_1), \dots, N(\sigma_m))$ for $\mathcal{B}$ such that the following two properties are satisfied:

(1) The decomposition of N_v into flow boxes $B_p = D_p \times I$ satisfies: each vertical edge of V appears as $0 \times I \subset D_p \times I$ for some p.

(2) $N(\sigma_1) \cap V = N_R$.

It now follows that if $\mathcal{F}$ is strongly (V, W)–compatible, then we can apply the construction of Theorem 7.3 so that

(1) $\mathcal{F}'$ is strictly horizontal in the flow boxes B_p of N_v that contain vertical edges of R,

(2) $\mathcal{F}'$ is strictly horizontal in $N(\sigma_1)$, and

(3) in the coordinates inherited from W, the fixed product neighborhood of V, $\mathcal{F}' \cap V$, is x–invariant.

Hence, if $\mathcal{F}$ is strongly (V, W)–compatible, then a C^0 close, $C^{k,0}$, Denjoy blowup $\mathcal{F}'$ can be chosen to be strongly (V, W)–compatible. □

References

[1] **J W Alexander**, *A lemma on systems of knotted curves*, Proc. Nat. Acad. Sci. U.S.A. 9 (1923) 93–95

[2] **J Bowden**, *Approximating C^0–foliations by contact structures*, Geom. Funct. Anal. 26 (2016) 1255–1296 MR

[3] **S Boyer**, **C M Gordon**, **L Watson**, *On L–spaces and left-orderable fundamental groups*, Math. Ann. 356 (2013) 1213–1245 MR

[4] **D Calegari**, *Leafwise smoothing laminations*, Algebr. Geom. Topol. 1 (2001) 579–585 MR

[5] **A Candel**, **L Conlon**, *Foliations, I*, Graduate Studies in Mathematics 23, Amer. Math. Soc., Providence, RI (2000) MR

[6] **V Colin**, **W H Kazez**, **R Roberts**, *Taut foliations*, Comm. Anal. Geom. (2019) 357–375

[7] **A Denjoy**, *Sur les caractéristiques à la surface du tore*, C. R. Acad. Sci. Paris 194 (1932) 830–833

[8] **A Denjoy**, *Sur les caractéristiques du tore*, C. R. Acad. Sci. Paris 194 (1932) 2014–2016

[9] **A Denjoy**, *Sur les caractéristiques du tore*, C. R. Acad. Sci. Paris 195 (1932) 934–936

[10] **A Denjoy**, *Sur les courbes définies par les équations différentielles à la surface du tore*, J. Math. Pures Appl. 11 (1932) 333–375

[11] **A Denjoy**, *Les trajectoires à la surface du tore*, C. R. Acad. Sci. Paris 223 (1946) 5–8 MR

[12] **P R Dippolito**, *Codimension one foliations of closed manifolds*, Ann. of Math. 107 (1978) 403–453 MR

[13] **Y M Eliashberg**, **W P Thurston**, *Confoliations*, University Lecture Series 13, Amer. Math. Soc., Providence, RI (1998) MR

[14] **D Gabai**, *Foliations and the topology of 3–manifolds*, J. Differential Geom. 18 (1983) 445–503 MR

[15] **D Gabai**, *Foliations and the topology of 3–manifolds, II*, J. Differential Geom. 26 (1987) 461–478 MR

[16] **D Gabai**, *Foliations and the topology of 3–manifolds, III*, J. Differential Geom. 26 (1987) 479–536 MR

[17] **A Haefliger**, *Variétés feuilletées*, Ann. Scuola Norm. Sup. Pisa 16 (1962) 367–397 MR

[18] **G Hector**, **U Hirsch**, *Introduction to the geometry of foliations, B: Foliations of codimension one*, Aspects of Mathematics E3, Friedr., Braunschweig (1983) MR

[19] **W H Kazez**, **R Roberts**, *Approximating $C^{1,0}$–foliations*, from "Interactions between low-dimensional topology and mapping class groups" (R I Baykur, J Etnyre, U Hamenstädt, editors), Geom. Topol. Monogr. 19, Geom. Topol. Publ., Coventry (2015) 21–72 MR

[20] **W H Kazez**, **R Roberts**, *C^0 approximations of foliations*, Geom. Topol. 21 (2017) 3601–3657 MR

[21] **W B R Lickorish**, *A foliation for 3–manifolds*, Ann. of Math. 82 (1965) 414–420 MR

[22] **E E Moise**, *Affine structures in 3–manifolds, V: The triangulation theorem and Hauptvermutung*, Ann. of Math. 56 (1952) 96–114 MR

[23] **E E Moise**, *Geometric topology in dimensions 2 and 3*, Graduate Texts in Mathematics 47, Springer (1977) MR

[24] **S P Novikov**, *The topology of foliations*, Trudy Moskov. Mat. Obšč. 14 (1965) 248–278 MR In Russian; translated in Trans. Mosc. Math. Soc. 14 (1965) 268–304

[25] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and genus bounds*, Geom. Topol. 8 (2004) 311–334 MR

[26] **P Ozsváth**, **Z Szabó**, *On Heegaard diagrams and holomorphic disks*, from "European Congress of Mathematics" (A Laptev, editor), Eur. Math. Soc., Zürich (2005) 769–781 MR

[27] **G Reeb**, *Sur certaines propriétés topologiques des variétés feuilletées*, from "Sur les espaces fibrés et les variétés feuilletées", Actualités Sci. Ind. 1183, Hermann, Paris (1952) 91–158 MR

[28] **W P Thurston**, *A norm for the homology of 3–manifolds*, Mem. Amer. Math. Soc. 339, Amer. Math. Soc., Providence, RI (1986) 99–130 MR

[29] **D Tischler**, *On fibering certain foliated manifolds over S^1*, Topology 9 (1970) 153–154 MR

[30] **A H Wallace**, *Modifications and cobounding manifolds*, Canadian J. Math. 12 (1960) 503–528 MR

Department of Mathematics, University of Georgia
Athens, GA, United States

Department of Mathematics, Washington University
St Louis, MO, United States

will@math.uga.edu, roberts@math.wustl.edu

Received: 10 December 2016 Revised: 22 January 2019