

Twisted differential generalized cohomology theories and their Atiyah–Hirzebruch spectral sequence

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We construct the Atiyah–Hirzebruch spectral sequence (AHSS) for twisted differential generalized cohomology theories. This generalizes to the twisted setting the authors’ corresponding earlier construction for differential cohomology theories, as well as to the differential setting the AHSS for twisted generalized cohomology theories, including that of twisted K –theory by Rosenberg and by Atiyah and Segal. In describing twisted differential spectra we build on the work of Bunke and Nikolaus, but we find it useful for our purposes to take an approach that highlights direct analogies with classical bundles and that is at the same time amenable for calculations. We will, in particular, establish that twisted differential spectra are bundles of spectra equipped with a flat connection. Our prominent case will be twisted differential K –theory, for which we work out the differentials in detail. This involves differential refinements of primary and secondary cohomology operations the authors developed in earlier papers. We illustrate our constructions and computational tools with examples.

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1 Introduction

There has been a lot of research activity constructing and using twisted cohomology theories. The main example has been twisted K –theory $K^*(X; h)$ of a space X , where the twist H is a cohomology class $h \in H^3(X, \mathbb{Z})$ (see Donovan and Karoubi [20], Rosenberg [59], Bouwknegt, Carey, Mathai, Murray and Stevenson [12], Tu, Xu and Laurent-Gengoux [70], Atiyah and Segal [8; 9], Freed, Hopkins and Teleman [26], Kupers [41], Ando, Blumberg and Gepner [2], Antieau, Gepner and Gómez [4], Karoubi [39] and Gomi [27]). Periodic de Rham cohomology may be twisted by any odd-degree cohomology class; see Teleman [68], Bunke, Schick and Spitzweck [18] and Mathai and Wu [52]. Twisted de Rham cohomology has roots that go back to Rohm and Witten [57], and has attracted attention more recently — see eg Mathai and Wu [52] and Sati [60; 61]. TMF can be twisted by a degree-four class (see Ando, Blumberg and Gepner [2]), and Morava K –theory and E –theory admit twists by cohomology classes

for every prime p and any chromatic level n (see Sati and Westerland [63]). More exotic spectra can be twisted in an even more unexpected way, including iterated algebraic K -theory of the connective K -theory spectrum; see Lind, Sati and Westerland [43].

Twisted generalized cohomology theories will admit an isomorphism with twisted periodic de Rham cohomology under the generalized twisted Chern character (or Chern–Dold character). For degree-three twisted K -theory, the twisted Chern character appears from various points of view in Atiyah and Segal [9], Karoubi [38], Mathai and Stevenson [50; 51], Tu and Xu [69], Bressler, Gorokhovsky, Nest and Tsygan [14], Karoubi [39], Gomi and Terashima [28] and Han and Mathai [34]. Chern characters can be considered for higher twisted theories, such as Morava K -theory and E -theory (see Sati and Westerland [63]) and iterated algebraic K -theory (see Lind, Sati and Westerland [43]). Differential refinements for a twisted generalized cohomology theory are considered by Bunke and Nikolaus [16].

On the other hand, differential cohomology has become an active area of research (eg Freed [25], Hopkins and Singer [36], Simons and Sullivan [67] and Schreiber [65]; for a more complete list of references for the untwisted case, see Grady and Sati [32]). Here there is also a Chern character map, as part of the data for a differential cohomology theory, that lands in a periodic form of de Rham cohomology, where the periodicity arises from the coefficients of the underlying generalized cohomology theory (see Bunke, Nikolaus and Völkl [17] and Upmeyer [71]). These differential generalized cohomology theories can in turn be twisted, the foundations of which are given by Bunke and Nikolaus [16], building on Bunke, Nikolaus and Völkl [17] and Schreiber [65].

Carey, Mickelsson and Wang [19] gave a construction of twisted differential K -theory that satisfies the square diagram and short exact sequences. Kahle and Valentino [37] gave a corresponding list of properties which can be generalized to any twisted differential cohomology theory. Indeed, a characterization of twisted differential cohomology including twisted differential K -theory is given in [16] (except the push-forward axiom). The model combines twisted cohomology groups and twisted differential forms in a homotopy theoretic way, analogous to (and as abstract as) the construction of Hopkins and Singer [36] in the untwisted case. Having a more concrete geometric description in mind, Park [56] provided a model for twisted differential K -theory in the case that the underlying topological twists represent torsion classes in degree-three integral cohomology. Recently, Gorokhovsky and Lott [29] provided a model of twisted differential K -theory using Hilbert bundles. In this paper, we seek an approach which builds on certain aspects of the above works and highlights both homotopic and

geometric aspects, and which we hope would be theoretically elegant and conceptually appealing, yet at the same time being amenable to computations and applications.

Among the various spectral sequences one could potentially construct, the Atiyah–Hirzebruch spectral sequence (AHSS) would perhaps be the most central for twisted theories, because of the way the theory is built by gluing data on the underlying space. As we will explicitly see, this continues to hold for differential twisted theories with extra data again assembled from patches on the manifold. Hence the Čech filtration of the manifold by good open covers is the most natural in this setting, as it captures the data of differential forms, as highlighted classically by Bott and Tu [11]. In the topological (and untwisted case) there is an equivalence between filtrations on the underlying topological space and filtrations on the spectrum of the generalized cohomology theory; see Maunder [53]. While we expect this to hold at the differential level (see Grady and Sati [32]) and also perhaps here at the twisted differential level, it will become clear that filtrations on the manifold are central to the construction and hence are a priori preferred.

The AHSS for the simplest nontrivial twisted cohomology theory, namely twisted de Rham cohomology, is studied in Atiyah and Segal [9], and more generally for any odd-degree twist in Mathai and Wu [52] and Li, Liu and Wang [42]. The AHSS for more involved theories has also been studied. The AHSS in the parametrized setting is described generally but briefly in May and Sigurdsson [54, Remark 20.4.2 and Proposition 22.1.5]. The twisted K -theory via the AHSS was first discussed in Rosenberg [58; 59] from the operator algebra point of view, and then briefly in Bouwknegt, Carey, Mathai, Murray and Stevenson [12] and in Atiyah and Segal [8] and more extensively in Atiyah and Segal [9] from the topological point of view. For $h \in H^3(X; \mathbb{Z})$, the twisted complex K -theory spectral sequence has initial terms with

$$E_3^{p,q} = E_2^{p,q} = \begin{cases} H^p(X; \mathbb{Z}) & \text{for } q \text{ even,} \\ 0 & \text{for } q \text{ odd,} \end{cases}$$

and the first nontrivial differential is the deformation of the cup product with twisting class, ie¹

$$d_3 = \text{Sq}_{\mathbb{Z}}^3 + (-) \cup h: H^p(X; \mathbb{Z}) = E_3^{p,q} \rightarrow E_3^{p+3, q-2} = H^{p+3}(X; \mathbb{Z}).$$

¹Throughout the paper we will use τ and $\hat{\tau}$ to denote the integral cohomology class and its differential refinement, respectively, for general cohomology theories. For K -theory, we will use h , \hat{h} , and H to denote, respectively, the integral cohomology class, its differential refinement, and the corresponding differential form representative.

In the case of twisted cohomology and twisted K -theory the higher differentials turn out to be modified by Massey products involving the rationalization of the twisting class h ; see Atiyah and Segal [9]. For twisted Morava K -theory and twisted Morava E -theory the differentials are discussed by Sati and Westerland [63]. All the above theories admit differential refinements and the AHSS for the corresponding untwisted forms of the theories is described in detail by the authors in [32]. In this paper we construct the AHSS for twisted differential generalized cohomology theories. Then we focus on twisted differential K -theory, whose untwisted version we considered, among other untwisted theories, in [32].

What we are constructing here is the AHSS for twisted differential cohomology theories, which we will denote by $\widehat{\text{AHSS}}_{\hat{\tau}}$, where $\hat{\tau}$ is a representative of a differential cohomology class, ie a higher bundle with connection. The E_2 -page of the spectral sequence looks considerably different than its topological counterpart and we feel a word of warning is in order. The spectral sequence is a half-plane spectral sequence, but the cohomology groups which appear in the first and fourth quadrants take different forms. On the x -axis, the entries are not homotopy-invariant cohomology groups, but are rather geometrically refined cohomology groups of some kind—often twisted closed differential forms. For a twisted differential theory $\widehat{R}(-; \hat{\tau})$ refining an underlying topological twisted theory $R(-; \tau)$, the E_2 -page takes the form

$$E_2^{p,q} = \begin{cases} H^p(M; R_{U(1)}^{-q-1}(*)) & \text{for } q < 0, \\ H^p(M; R^{-q}(*)) & \text{for } q > 0, \\ H^p(M; L(\widehat{R}(-; \hat{\tau}))) & \text{for } q = 0, \end{cases}$$

and (provided M is a compact smooth manifold) we have the convergence

$$E_2^{p,q} \Rightarrow \widehat{R}^0(M; \hat{\tau}).$$

Here $L(\widehat{R}(-; \hat{\tau}))$ is the sheafification of the presheaf given by evaluating the twisted differential theory $\widehat{R}(-; \hat{\tau})$ (which is not homotopy invariant!) on open sets of M . In the case when $\widehat{R}(-; \hat{\tau}) = \widehat{K}(-; \hat{\tau})$ is twisted differential K -theory, we show in Section 3.3 that $L(\widehat{K}(-; \hat{\tau})) \cong \Omega_{d_{H\text{-cl}}}^{\text{even}}(-)$, where on the right we have the sheaf of twisted-closed differential forms of even degree.

When the twist $\hat{\tau}$ is zero in differential cohomology, then we recover the AHSS for differential cohomology constructed in [32], which we denote by $\widehat{\text{AHSS}}$. On the other hand, if we forget the differential refinement and strip the theory to its underlying topological theory, then we recover the AHSS for twisted spectra, which

we denote by $AHSS_\tau$, a prominent example of which is that of twisted K -theory; see Rosenberg [58; 59] and Atiyah and Segal [9]. Of course when we take both a trivial twist and no differential refinement, then we restrict to the original case of Atiyah and Hirzebruch [7]. Summarizing, we will have a correspondence diagram of transformations of the corresponding spectral sequences,

$$\begin{array}{ccc} \widehat{AHSS}_{\widehat{\tau}} & \xrightarrow{I_I} & AHSS_\tau \\ \left. \begin{array}{c} \widehat{\tau}=0 \\ \downarrow \end{array} \right\} & & \left. \begin{array}{c} \tau=0 \\ \downarrow \end{array} \right\} \\ \widehat{AHSS} & \xrightarrow{I} & AHSS \end{array}$$

The picture that we have had in mind for the structure of the corresponding differentials, as cohomology operations, is schematically the following:

$$\begin{array}{ccc} \widehat{d}_{\widehat{\tau}} = \widehat{\text{primary}} + \widehat{\text{secondary}} + \dots & \xrightarrow{I_I} & d_\tau = \text{primary} + \text{secondary} + \dots \\ \left. \begin{array}{c} \widehat{\tau}=0 \\ \downarrow \end{array} \right\} & & \left. \begin{array}{c} \tau=0 \\ \downarrow \end{array} \right\} \\ \widehat{d} = \widehat{\text{primary}} & \xrightarrow{I} & d = \text{primary} \end{array}$$

The upper-left corner generalizes the corresponding forms in previous works. In [31], we identified \widehat{d} with a differential refinement of a primary cohomology operation. The differential secondary refinement that will appear in $\widehat{d}_{\widehat{\tau}}$ essentially consists of the differential Massey products we constructed in [30]. The differential d_τ is that of a twisted cohomology theory, as for instance for twisted K -theory, where these secondary operations are shown to be Massey products (rationally); see Atiyah and Segal [9]. The bare differential d is the Atiyah–Hirzebruch differential which, for the case of K -theory, is given by the integral Steenrod square $Sq_{\mathbb{Z}}^3$. The long-term goal that we had in previous projects was to arrive at the overall picture that we have assembled in this paper. In this sense, the current paper can be viewed a culmination of the series [30; 31; 32].

Twistings of a suitably multiplicative (that is, A_∞) cohomology theory R^* are governed by a space $BGL_1(R)$; see May and Sigurdsson [54] and Ando, Blumberg, Gepner, Hopkins and Rezk [3; 2]. Since twists usually considered arise from cohomology classes, there are maps from a source Eilenberg–Mac Lane space $K(G, m + 1)$ for appropriate abelian group G and degree m , depending on the theory under consideration. We have the inclusions

$$K(G, m + 1) \rightarrow BGL_1(R) \hookrightarrow \text{Pic}(R),$$

where $\text{Pic}(R)$ is the ∞ -groupoid of invertible R -module spectra, whose connected component containing the identity is equivalent to $\text{BGL}_1(R)$. For integral twists, ie for $G = \mathbb{Z}$, the natural replacement of the Eilenberg–Mac Lane spaces are going to be classifying stacks $\mathbf{B}^m U(1)_\nabla$. We also use the refinement by Bunke and Nikolaus [16] of the space $\text{Pic}(R)$. We do so by highlighting the analogies between classical line bundles and $\widehat{K}(X; \widehat{h})$ -theory. We have the following table (see remark (i) on page 2929): There

line bundles	twisted spectra
$\mathbf{B} \mathbb{Z}/2$	$\text{Pic}_{\mathcal{R}}^{\text{top}}$
$\mathbf{B} \mathbb{R}^\times$	$\text{Pic}_{\mathcal{R}}^{\text{dR}}$
Ω_{cl}^1	$\text{Pic}_{\mathcal{R}}^{\text{form}}$
$\mathbf{B} \mathbb{Z}/2_\nabla$	$\text{Tw}_{\widehat{\mathcal{R}}}$

Table 1

are classically two viewpoints on real line bundles: as bundles with fiber \mathbb{R} (which is one-dimensional as an \mathbb{R} -module) and also as locally free invertible \mathcal{O}_M -modules, ie sheaves which are invertible over the ring of smooth functions. Similarly, there are two viewpoints on twisted spectra: as bundles of spectra \mathcal{R} (which is rank one as an \mathcal{R} -module), and also as invertible \mathcal{R} -module spectra. The point we highlight is that there is a direct analogy between each one of the two descriptions of the pair. Even more, we can show (see remark (ii) on page 2929) that the sheaf cohomology with coefficients in a line bundle \mathcal{L} over M is isomorphic to the twisted $H\mathbb{R}$ cohomology with twist $\eta: M \rightarrow \mathbf{B} \mathbb{Z}/2$ classifying \mathcal{L} . We will be using stacks, in our context in the sense described by Fiorenza, Sati, Schreiber and Stasheff [24; 65; 62; 22]. In general, twisted topological spectra will satisfy descent over the base space, hence are stacks [16]. For the case of twisted topological K -theory this is also shown by Antieau and Williams [5]. Differential theories similarly satisfy descent over the base manifold M [16].

The paper is organized as follows. In Section 2 we provide our slightly modified take on twisted differential spectra for the purpose of constructing their AHSS. In particular, in Section 2.1 we start recalling the construction of [16] with slight modifications to suit our purposes. This leads us in Section 2.2 to describe a canonical bundle of spectra over the Picard ∞ -groupoid which comes from the ∞ -Grothendieck construction. In Theorem 8 we present several different, but equivalent, ways to think about twisted differential cohomology theories. We use this in Section 2.3 to provide analogies between these objects and the notion of a line bundle equipped with flat connection,

leading to the identification of a twisted differential theory as a bundle of spectra equipped with flat connection, whose meaning we explain. This is captured in Table 2, continuing the analogies in Table 1.

line bundle	bundle of spectra
$\begin{array}{ccc} \mathbb{R} & \longrightarrow & E \\ \downarrow & & \downarrow \pi \\ * & \hookrightarrow & M \end{array}$	$\begin{array}{ccc} H(\mathbb{R}[u, u^{-1}]) \cong_{\text{ch}} K \wedge H\mathbb{R} & \longrightarrow & E \\ \downarrow & & \downarrow \pi \\ * & \hookrightarrow & M \end{array}$
set of global sections $\Gamma(M, E)$ $\lim\{\prod C^\infty(U_\alpha, \mathbb{R}) \xrightarrow{g_{\alpha\beta}} \prod C^\infty(U_{\alpha\beta}, \mathbb{R})\}$	space of global sections $\Gamma(M, E)$ $\text{holim}\{\prod \Omega^*(U_\alpha) \xrightarrow{e^{dA_{\alpha\beta}}} \prod \Omega^*(U_{\alpha\beta}) \xrightarrow{\text{CS}} \dots\}$
local ring of smooth real functions $\Omega^* \otimes \mathbb{R}[0] \simeq C^\infty(-, \mathbb{R})$	local DGA of smooth functions $\Omega^* \otimes \mathbb{R}[u, u^{-1}] \simeq \{\Omega^{\text{even}} \rightarrow \Omega^{\text{odd}} \rightarrow \dots\}$
flat connection $\nabla: \Gamma(M, E) \rightarrow \Gamma(M, E \otimes T^*M)$, $\nabla(\sigma \cdot s) = d\sigma s + \sigma \nabla s$ with $\nabla^2 = 0$	flat connection $d_H: \Gamma(M, E) \rightarrow \Gamma(M, E \wedge \wedge^3(T^*M))$, $d_H(\omega \wedge \alpha) = d_H(\omega) \wedge \alpha + \omega \wedge d\alpha$ with $d_H^2 = 0$
parallel section $s: M \rightarrow E, \quad \nabla s = 0$	parallel section $d_H\text{-parallel: } d_H(\omega) = 0$
discrete transition functions $g_{\alpha\beta} \in \check{H}^1(M; \mathbb{R}^\times)$ written in local trivializations: $s_\alpha: U_\alpha \rightarrow E, \quad \nabla(s_\alpha) = 0$	topological twist $\eta_{\alpha\beta\gamma\delta} \in \check{H}^3(M; \mathbb{Z})$ written in local trivializations: $e^{B_\alpha} \wedge \omega_\alpha, \quad d_H(e^{B_\alpha} \wedge \omega_\alpha) = 0$
representation action of \mathbb{R}^\times on \mathbb{R} as multiplication by units	representation action of $\text{BU}(1)$ on K via line bundles

Table 2: Analogies between twisted differential spectra and line bundles with connections.

The general construction of $\widehat{\text{AHSS}}_{\hat{\tau}}$, the AHSS for any twisted differential cohomology theory, is given in Section 2.4. The differentials are then identified on general grounds and the corresponding properties are discussed in Section 2.5. Having provided a general discussion, we start focusing on twisted differential K -theory in Section 3. Indeed, in Section 3.1 we give a detailed description of the twists in differential K -theory from various perspectives. In order to arrive at explicit formulas, we describe the Chern character in twisted differential K -theory in Section 3.2. Then in the following two sections we characterize the differentials more explicitly. Unlike the classical AHSS, in the differential setting, as we also noticed for the untwisted case [32], we find two kinds

of differentials: the low-degree differentials are characterized in Section 3.3, while the higher flat ones are described in Section 3.4. We end with explicit and detailed examples, illustrating the machinery, in Section 3.5. We calculate twisted differential K -theory of the 3-sphere (or the group $SU(2)$), $\widehat{K}^*(S^3; \widehat{h})$, in two ways: using the Mayer–Vietoris sequence of Proposition 5(iii) and using the $\widehat{AHSS}_{\widehat{h}}$ of Theorem 18, as adapted to K -theory with differential twist \widehat{h} in previous sections. While we present both approaches as being useful, the highlight is that the latter approach is much more powerful, at least in this case.

2 Twisted differential spectra and their AHSS

2.1 Review of the general construction of differential twisted spectra

In this section we start by recounting some of the main constructions in [16] and then provide a description of local triviality as well as general properties of twisted differential spectra. The point of view taken in [16] uses primarily Picard ∞ -groupoids to define the differential twists of a differential function spectrum. Topologically, given an E_∞ (or even an A_∞) ring spectrum \mathcal{R} , one can obtain this ∞ -groupoid as a nonconnected delooping of its E_∞ space of units. More precisely, if $\text{Pic}_{\mathcal{R}}$ denotes the infinity groupoid of invertible \mathcal{R} -module spectra (with respect to the smash product $\wedge_{\mathcal{R}}$), then we have an equivalence²

$$\text{BGL}_1(\mathcal{R}) \times \pi_0(\text{Pic}_{\mathcal{R}}) \simeq \text{Pic}_{\mathcal{R}},$$

where $\pi_0(\text{Pic}_{\mathcal{R}})$ is the group of isomorphism classes of invertible \mathcal{R} -module spectra (see [3; 2; 63]).

In order to discuss the differential refinement of twisted cohomology, we need to retain some geometric data which cannot be encoded in spaces. Indeed, twisted differential cohomology theories crucially use a sort of twisted de Rham theorem to connect information about differential forms on a smooth manifold with some cohomology theory evaluated on the manifold. This mixture of geometry and topology is effectively captured by smooth stacks, with the homotopy theory being captured by the simplicial direction of the stack and the geometry being captured by the local information encoded in the site. In general, we would like to consider not only ∞ -groupoid-valued smooth

²Note that this equivalence is *not* an equivalence of E_∞ -spaces.

stacks, but also stacks valued in the stable ∞ -category of spectra; see [47; 46] for more comprehensive accounts of these constructions.³

Purely topological theories can be regarded as *constant* smooth theories. For example, let \mathcal{R} be an “ordinary” ring spectrum. Then we can consider the assignment which associates to each manifold M the spectrum \mathcal{R} . This assignment defines a prestack, ie a functor

$$\text{const}(\mathcal{R}): \text{Mf} \rightarrow \text{Sp}$$

on the category of smooth manifolds with values in spectra. We can equip the category Mf with the Grothendieck topology of good open covers, turning it into a site. The stackification of $\text{const}(\mathcal{R})$ with respect to this topology will be denoted by $\underline{\mathcal{R}}$. It is obviously locally constant, ie constant on some open neighborhood of each point of M .

For a fixed manifold M , we can restrict the above functor $\text{const}(\mathcal{R})$ to the overcategory Mf/M (ie the ∞ -category on arrows $N \rightarrow M$), equipped with the restricted coverage. After stackification, this gives a locally constant sheaf of spectra $\underline{\mathcal{R}}$ on M . More generally, we can do this for any locally constant sheaf of spectra on Mf . Then we can consider the monoidal ∞ -category of *locally constant* sheaves of $\underline{\mathcal{R}}$ -module spectra over M and consider the space of invertible objects.⁴ In [16], this ∞ -groupoid is denoted by $\text{Pic}_{\underline{\mathcal{R}}}(M)$. In order to simplify notation and to highlight the fact that this object is related to the twists of the underlying theory \mathcal{R} , we will denote this ∞ -groupoid by $\text{Pic}_{\mathcal{R}}^{\text{top}}(M)$. This is further motivated by the following properties [16, Section 3]:

- (1) **Descent** The assignment $M \mapsto \text{Pic}_{\mathcal{R}}^{\text{top}}(M)$ defines a smooth stack on the site of all manifolds, topologized with good open covers.
- (2) **Correspondence with topological twist** For a manifold M , we have equivalences $\text{Pic}_{\mathcal{R}}^{\text{top}}(M) \simeq \text{Map}(M, \text{Pic}_{\mathcal{R}}) \simeq \underline{\text{Pic}}_{\mathcal{R}}(M)$ with $\text{Pic}_{\mathcal{R}}$ the usual Picard space of invertible \mathcal{R} -module spectra, the second mapping ∞ -groupoid the nerve of the mapping space in CGWH-spaces and $\underline{\text{Pic}}_{\mathcal{R}}$ the constant stack associated to the nerve of $\text{Pic}_{\mathcal{R}}$. All three of the resulting spaces model the ∞ -groupoid of invertible \mathcal{R} -modules parametrized over M .

³The reader who is not interested in the technical details can regard this rigorous development as a sort of black box, bearing in mind that in the language of ∞ -categories no commutative diagram is strict: commutativity is only up to higher homotopy coherence. The familiar concepts at the 1-categorical level generally hold in the ∞ -context, but only up to homotopy coherence.

⁴Here the monoidal structure is given by the R -module smash product $\wedge_{\mathcal{R}}$. Clearly we are omitting the details of monoidal ∞ -categories and the symmetric monoidal smash product. It would be far too lengthy to include these details here and we encourage the reader to consult [46] for more information.

- (3) **Twisted cohomology theory** Suppose that M be a manifold with \mathcal{R} -twist $\mathcal{R}_\tau \in \Pi_1 \text{Pic}_{\mathcal{R}}^{\text{top}}(M)$, where Π_1 denotes the fundamental groupoid. Then the classical τ -twisted cohomology of M is computed via⁵

$$R(M; \tau) := \pi_0(\Gamma(M, \mathcal{R}_\tau)),$$

where Γ is the global sections functor $\Gamma: \text{Sh}_\infty(M; \text{Sp}) \rightarrow \text{Sp}$.

Remark (determinantal twist versus dimension shift) The third property does not include a degree of the theory. This is because shifts of the spectrum τ are also \mathcal{R} -twists. Thus the various degrees of $R(-; \tau)$ can be obtained by considering the shifted \mathcal{R} -twists. However, because we are dealing with differential cohomology and are interested in geometry and applications, we find it more practical to separate the two notions. Thus, if we have a combination of twists of the form τ and n , where n is the twist given by the dimension shift $\Sigma^n \mathcal{R}$ we will write

$$R^n(M; \tau) := \pi_0(\Gamma(M; \mathcal{R}_{\tau+n}))$$

for the cohomology of the product $\mathcal{R}_\tau \wedge_{\mathcal{R}} \Sigma^n \mathcal{R} \simeq \Sigma^n \mathcal{R}_\tau$.

The discussion has not yet taken into account differential data. It is merely the manifestation of the underlying topological theory, embedded as a geometrically discrete object in smooth stacks. To describe the differential theory, we first need to discuss the differential refinement of an E_∞ -ring spectrum \mathcal{R} . This is discussed extensively in [16] and we review only the essential pieces discussed there, adapted to our purposes.

Definition 1 Let RingSp denote the $(\infty, 1)$ -category of E_∞ -ring spectra, $\mathbb{R}\text{-AlgSp}$ denote the $(\infty, 1)$ -category of commutative $H\mathbb{R}$ -algebra spectra, and $\mathbb{R}\text{-CDGA}$ be the ordinary 1-category of commutative differential graded algebras. We define the *category of differential ring spectra* as the $(\infty, 1)$ -pullback

$$\begin{array}{ccc} \text{diff}(\text{RingSp}) & \longrightarrow & \mathbb{R}\text{-CDGA} \\ \downarrow & & \downarrow H\iota \\ \text{RingSp} & \xrightarrow{\wedge^{H\mathbb{R}}} & \mathbb{R}\text{-AlgSp} \end{array}$$

where the functor $\iota: \mathbb{R}\text{-CDGA} \rightarrow (\mathbb{R}\text{-CDGA})_\infty$ localizes at weak equivalences to obtain an $(\infty, 1)$ -category, $H: (\mathbb{R}\text{-CDGA})_\infty \rightarrow \mathbb{R}\text{-AlgSp}$ is the Eilenberg–Mac Lane

⁵Generally, we will denote a spectrum in a script font (eg \mathcal{R}), while the underlying theory it represents will be denoted in the standard math and talic font (eg R).

equivalence and $\wedge H\mathbb{R}$ takes the smash product with the real Eilenberg–Mac Lane spectrum (see eg [66] for details).

Remark (i) The objects of the $(\infty, 1)$ –category $\text{diff}(\mathcal{R}\text{ingSp})$ can be identified with triples (\mathcal{R}, c, A) , where \mathcal{R} is an E_∞ –ring spectrum, A is a CDGA and c is an equivalence of ring spectra

$$c: \mathcal{R} \wedge H\mathbb{R} \xrightarrow{\cong} HA.$$

(ii) It was shown in [66] that the Eilenberg–Mac Lane functor is essentially surjective onto $H\mathbb{Z}$ –module spectra, from which we can deduce that every ring spectrum admits a differential refinement since $H\mathbb{R}$ –module spectra can be regarded as $H\mathbb{Z}$ –module spectra.

Definition 2 Let A be a CDGA. We define the *de Rham complex with values in A* to be the sheaf of chain complexes

$$\Omega^*(-; A) := \Omega^* \otimes_{\mathbb{R}} \underline{A},$$

where \underline{A} is the locally constant sheaf of CDGAs associated to A .

To define the smooth stack of twists, Bunke and Nikolaus [16] define a certain stack which contains all the twisted de Rham complexes corresponding to the theory. We now recall this stack. Let \mathcal{L} be a sheaf of $\Omega^*(-; A)$ –modules on M (ie on the restricted site $\mathcal{M}f/M$). Then:

- (1) \mathcal{L} is called *invertible* if there is a \mathcal{K} such that $\mathcal{L} \otimes_{\Omega^*(-; A)} \mathcal{K}$ is isomorphic to $\Omega^*(-; A)$.
- (2) \mathcal{L} is called *K –flat* if the functor

$$\mathcal{L} \otimes_{\Omega^*(-; A)} -: \Omega^*(-; A)\text{-Mod} \rightarrow \Omega^*(-; A)\text{-Mod}$$

preserves objectwise quasi-isomorphisms.

- (3) \mathcal{L} is called *weakly locally constant* if it is quasi-isomorphic to a locally constant sheaf of modules.

Denote by Pic^{form} the stack which assigns, to each smooth manifold M , the Picard groupoid $\text{Pic}^{\text{form}}(M)$ of all invertible, K –flat, weakly locally constant sheaves of $\Omega^*(-; A)$ –modules. The passage from the topological twists to differential twists is

then obtained via the $(\infty, 1)$ -pullback in smooth stacks

$$(2-1) \quad \begin{array}{ccc} \mathrm{Tw}_{\widehat{\mathcal{R}}} & \longrightarrow & \mathrm{Pic}^{\mathrm{form}} \\ \downarrow & & \downarrow H \\ \mathrm{Pic}^{\mathrm{top}} & \xrightarrow{\wedge H\mathbb{R}} & \mathrm{Pic}^{\mathrm{dR}} \end{array}$$

where $\mathrm{Pic}^{\mathrm{top}}$, $\mathrm{Pic}^{\mathrm{form}}$, and $\mathrm{Pic}^{\mathrm{dR}}$ have the more detailed respective notations $\mathrm{Pic}_{\underline{\mathcal{R}}}^{\mathrm{loc}}$, $\mathrm{Pic}_{\Omega^*(-;A)}^{\mathrm{wloc,fl}}$, and $\mathrm{Pic}_{H(\Omega^*(-;A))}^{\mathrm{loc}}$ in [16].

A bit of explanation is in order. First, the condition of weakly locally constant for the sheaf of $\Omega^*(-; A)$ -modules \mathcal{L} gives rise to a twisted de Rham theorem. Indeed, for each fixed point $x \in M$, there is an open set $x \in U \subset M$ such that $\mathcal{L}(U)$ is quasi-isomorphic to a constant complex on U . This locally constant complex represents a cohomology theory with local coefficients on M and these local quasi-isomorphisms induce a quasi-isomorphism of sheaves of complexes on M . At the level of cohomology, the induced map can be regarded as a twisted de Rham isomorphism. Next, the condition of K -flatness ensures that the tensor product is equivalent to the derived tensor product and therefore preserves invertible objects. This way, we have a well-defined Picard groupoid.

Remark An object in the $(\infty, 1)$ -category $\mathrm{Tw}_{\widehat{\mathcal{R}}}$ is given by a triple $(\mathcal{R}_\tau, t, \mathcal{L})$, with \mathcal{R}_τ a topological \mathcal{R} -twist, \mathcal{L} an invertible, K -flat, weakly locally constant module over $\Omega^*(-; A)$ and t a zigzag of weak equivalences (or a single equivalence in the localization at quasi-isomorphisms)

$$t: H(\mathcal{L}) \xrightarrow{\cong} \mathcal{R}_\tau \wedge H\mathbb{R}.$$

The equivalence t exhibits a twisted de Rham theorem for the rationalization of the topological twist \mathcal{R}_τ .

Definition 3 Such a triple $(\mathcal{R}_\tau, t, \mathcal{L})$ is called a *differential refinement of a topological twist* \mathcal{R}_τ .

One of the hallmark features of twisted cohomology is that of local triviality. Indeed, if we are to think of twisted cohomology theories as a bundle of theories, parametrized over some base space (in the sense of [54]), then one expects that if we restrict to a contractible open set on the parametrizing space, the theories should all be isomorphic to the “typical fiber”, given by the underlying theory which is being twisted. This happens for differential cohomology theories, just as it does for topological theories.

Proposition 4 (local triviality of the twist) *Let $\widehat{\mathcal{R}}_{\hat{\tau}} := (\mathcal{R}_{\tau}, t, \mathcal{L})$ be a sheaf of spectra refining the twisted theory \mathcal{R}_{τ} on a smooth manifold M . Suppose the underlying topological twist comes from a map $\tau: M \rightarrow \text{BGL}_1(\mathcal{R}) \hookrightarrow \text{Pic}_{\mathcal{R}}^{\text{top}}$. Then, around each point $x \in M$, there is an open set U such that, when restricted via the change-of-base functor*

$$i^*: \text{Tw}_{\widehat{\mathcal{R}}}(M) \rightarrow \text{Tw}_{\widehat{\mathcal{R}}}(U)$$

induced by the inclusion $i: U \rightarrow M$, we have an equivalence in $\text{Tw}_{\widehat{\mathcal{R}}}(U)$

$$\widehat{\mathcal{R}}_{\hat{\tau}} \simeq \widehat{\mathcal{R}},$$

where $\widehat{\mathcal{R}} = (\mathcal{R}, c, A)$ is a differential refinement of the untwisted theory \mathcal{R} .

Proof Let $x: * \rightarrow U \subset M$ be a point in M . Since M is a manifold, without loss of generality, it suffices to prove the claim with U contractible. In this case, the restriction of \mathcal{R}_{τ} to U trivializes and we have an equivalence $\mathcal{R}_{\tau} \simeq \mathcal{R}$ in $\text{Sh}_{\infty}(U; \text{Sp})$. As part of the data for the differential refinement of \mathcal{R} , we have an equivalence in $\text{Sh}_{\infty}(\text{Mf}, \text{Sp})$

$$H(\Omega^*(-; A)) \simeq \mathcal{R} \wedge H\mathbb{R},$$

and hence an equivalence on the restricted site $\text{Sh}_{\infty}(U; \text{Sp})$. Combining, we have equivalences in $\text{Sh}_{\infty}(U; \text{Sp})$,

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{\cong} & \mathcal{R}_{\tau} \\ \wedge H\mathbb{R} \downarrow & & \downarrow \wedge H\mathbb{R} \\ \mathcal{R} \wedge H\mathbb{R} & \xrightarrow{\cong} & \mathcal{R}_{\tau} \wedge H\mathbb{R} \\ \cong \downarrow c & & t \downarrow \cong \\ H(\Omega^*(-; A)) & \xrightarrow{\cong} & H(\mathcal{L}) \end{array}$$

where the bottom map in the diagram is defined by composing the evident three equivalences. By the definition of localization and the properties of H , this implies that $\Omega^*(-; A)$ and \mathcal{L} are quasi-isomorphic. Putting this data together, we see that we have defined an equivalence $\widehat{\mathcal{R}}_{\hat{\tau}} \xrightarrow{\cong} \widehat{\mathcal{R}}$ in the ∞ -groupoid $\text{Tw}_{\widehat{\mathcal{R}}}(U)$. □

We now summarize some of the above properties of twisted differential cohomology.

Proposition 5 (properties of twisted differential cohomology) *Let $\widehat{R}^*(-; \hat{\tau})$ be a twisted counterpart of a differential cohomology theory \widehat{R}^* on a smooth manifold M . Then $\widehat{R}^*(-; \hat{\tau})$ satisfies the following:*

- (i) **Additivity** For a disjoint union $f: N = \coprod_{\alpha} N_{\alpha} \rightarrow M$ of maps $f_{\alpha}: N_{\alpha} \rightarrow M$, we have a decomposition

$$\widehat{R}^*(N; \widehat{\tau}) \cong \bigoplus_{\alpha} \widehat{R}^*(N_{\alpha}; \widehat{\tau}).$$

- (ii) **Local triviality** If $\widehat{\tau}$ has underlying topological twist τ coming from a map $\tau: M \rightarrow \text{BGL}_1(\mathbb{R})$, then for every point $x \in M$, there is an open neighborhood $x \in U \xrightarrow{i} M$ such that the restriction of $\widehat{R}^*(-; \widehat{\tau})$ to U trivializes. That is, we have a natural isomorphism of functors

$$i^* \widehat{R}^*(-; \widehat{\tau}) \cong \widehat{R}^*(-),$$

on the overcategory \mathcal{Mf}/U .

- (iii) **Mayer–Vietoris** Let $\{U, V\}$ be an open cover of M . Then we have a long exact sequence,

$$\begin{array}{ccccccc} \dots & \longrightarrow & \mathcal{R}_{\mathbb{R}/\mathbb{Z}}^{-2}(U; \widehat{\tau}) \oplus \mathcal{R}_{\mathbb{R}/\mathbb{Z}}^{-2}(V; \widehat{\tau}) & \longrightarrow & \mathcal{R}_{\mathbb{R}/\mathbb{Z}}^{-2}(U \cap V; \widehat{\tau}) & \longrightarrow & \dots \\ & & \searrow & & \searrow & & \\ & \widehat{\mathcal{R}}^0(M; \widehat{\tau}) & \longrightarrow & \widehat{\mathcal{R}}^0(U; \widehat{\tau}) \oplus \widehat{\mathcal{R}}^0(V; \widehat{\tau}) & \longrightarrow & \widehat{\mathcal{R}}^0(U \cap V; \widehat{\tau}) & \longrightarrow \dots \\ & & \searrow & & \searrow & & \\ & \mathcal{R}^1(M; \tau) & \longrightarrow & \mathcal{R}^1(U; \tau) \oplus \mathcal{R}^1(V; \tau) & \longrightarrow & \dots & \end{array}$$

Proof The first follows from the fact that the global sections on a coproduct is isomorphic to the product of global sections and that taking connected components commutes with products. The second property arises from Proposition 4. The third property is stated in [16, Proposition 5.2], but we give a detailed proof here. With respect to the coverage on \mathcal{Mf}/M via good open covers, the sheaf of spectra $\widehat{\mathcal{R}}_{\widehat{\tau}}$ satisfies descent. Consider the pushout diagram

$$U \cap V \rightrightarrows U \amalg V \longrightarrow U \cup V = M.$$

Since $\widehat{\mathcal{R}}_{\widehat{\tau}}$ satisfies descent on M , the pullbacks to $U \amalg V$ and $U \cap V$ also satisfy descent and (by restriction) yield an exact sequence

$$\widehat{\mathcal{R}}_{\widehat{\tau}}|_{U \cap V} \rightarrow \widehat{\mathcal{R}}_{\widehat{\tau}}|_U \times \widehat{\mathcal{R}}_{\widehat{\tau}}|_V \rightarrow \widehat{\mathcal{R}}_{\widehat{\tau}}|_{U \cup V}$$

of sheaves of spectra. After taking global sections, we get the iterated fiber/cofiber sequence

$$\dots \rightarrow \Sigma^{-1} \widehat{\mathcal{R}}_{\widehat{\tau}}(M) \rightarrow \widehat{\mathcal{R}}_{\widehat{\tau}}(U \cap V) \rightarrow \widehat{\mathcal{R}}_{\widehat{\tau}}(U) \times \widehat{\mathcal{R}}_{\widehat{\tau}}(V) \rightarrow \widehat{\mathcal{R}}_{\widehat{\tau}}(M) \rightarrow \dots$$

Passing to connected components, one identifies the long exact sequence as stated. \square

The local triviality property will be useful throughout and the Mayer–Vietoris property will be explicitly utilized in the examples in Section 3.5.

Now we have seen that a map $\hat{\tau}: M \rightarrow \text{Tw}_{\hat{\mathcal{R}}}$ specifies a twist $\widehat{\mathcal{R}}_{\hat{\tau}}$ for the differential ring spectrum $\widehat{\mathcal{R}}$ and we know how to define the twisted cohomology abstractly via global sections of this sheaf of spectra. In practice, however, it is useful to understand how the sheaf $\widehat{\mathcal{R}}_{\hat{\tau}}$ is built out of local data, just as it is useful to understand the transition functions of a vector bundle. Happily, descent allows us to understand how a map to the stack of twists specifies local gluing data via pullback by a universal bundle over the twists, although we next need to develop a bit of machinery to make this precise.

2.2 The canonical bundle associated to the Picard ∞ –groupoid

In this subsection, we describe a canonical bundle of spectra over the Picard ∞ –groupoid which comes from the ∞ –Grothendieck construction. While rather abstract at first glance, we will see that this perspective has many conceptual and computational advantages. The general framework for this construction was set up by Jacob Lurie in [46]. Here we are simply unpackaging and adapting the general construction to our context.

In [46, Theorem 3.2.0.1], a general Grothendieck construction for ∞ –categories is described for each simplicial set S , whose associated simplicial category is equivalent to a fixed simplicial category \mathcal{C}^{op} . The construction gives a Quillen equivalence (depending on a choice of equivalence $\phi: S \rightarrow \mathcal{C}^{\text{op}}$)

$$\text{St}_{\phi}^{+}: (\text{sSet}^{+})_{/S} \xleftarrow{\cong} (\text{sSet}^{+})^{\mathcal{C}} : \text{Un}_{\phi}^{+},$$

whose right-hand side is a presentation of the ∞ –category of functors $\text{Fun}(S, \text{Cat}_{\infty})$.⁶ The fibrant objects on the left-hand side are precisely Cartesian fibrations $X \rightarrow S$. Thus the construction associates to each functor $\bar{p}: \mathcal{C} \rightarrow \text{Cat}_{\infty}$ ⁷ a Cartesian fibration $p: X \rightarrow S$ whose fiber at $\phi^{-1}(c)$ can be identified with the ∞ –category $\bar{p}(c)$.⁸ Now, in all ∞ –categories, ie in Cat_{∞} , there are the stable ones

$$i: \text{Stab}_{\infty} \hookrightarrow \text{Cat}_{\infty},$$

⁶The functors St_{ϕ}^{+} and Un_{ϕ}^{+} are called the straightening and unstraightening functors, respectively. However, we will not need anything explicit about such an interpretation here.

⁷ Cat_{∞} is the ∞ –category of all ∞ –categories.

⁸The analogy to keep in mind here is the tautological line bundle over $\mathbb{C}P^n$, each point of which is represented by a line in \mathbb{C}^{n+1} , and the associated bundle is then defined by assigning to each one of these points a copy of that line as the fiber.

and this inclusion functor admits a right adjoint, Stab , which stabilizes an ∞ -category. In particular, we can apply this functor to the codomain fibration $\mathcal{C}^{\Delta[1]} \rightarrow \mathcal{C}$, which gives the *tangent* ∞ -category $p: T\mathcal{C} \rightarrow \mathcal{C}$. This map is a Cartesian fibration, and the associated functor under the Grothendieck construction sends each $c \in \mathcal{C}^{\text{op}}$ to the stabilization of the slice $\text{Stab}(\mathcal{C}/_c)$ (see [45, Section 1.1]). In particular, if we take $\mathcal{C} = \infty\mathcal{G}\text{rpd}$ and fix an ∞ -groupoid $X \in \infty\mathcal{G}\text{rpd}$, again by the ∞ -Grothendieck construction, we get an equivalence

$$\text{Stab}(\infty\mathcal{G}\text{rpd}/_X) \simeq \text{Stab}(\text{Fun}(X, \infty\mathcal{G}\text{rpd})).$$

By the discussion above, the stabilization on the left is precisely the fiber of the tangent ∞ -category $T(\infty\mathcal{G}\text{rpd}) \rightarrow \infty\mathcal{G}\text{rpd}$ at X . From the stable Giraud theorem (see [47, Remark 1.2]), one sees that the right-hand side is equivalent to $\text{Fun}(X, \mathcal{S}\text{p})$.

Now bringing all this down to earth, what we have is that a functor $\bar{p}: X \rightarrow \mathcal{S}\text{p}$ is equivalently an object in the tangent ∞ -category at X . Such an object can be expressed as a “map”⁹ $p: E \rightarrow X$, which is to be thought of as a bundle of spectra over X . Notice also that one can do usual bundle constructions in the tangent ∞ -category. In particular, given a map $f: X \rightarrow Y$ of ∞ -groupoids, the change-of-base functor assigns to each object $E \rightarrow Y$ in $T(\infty\mathcal{G}\text{rpd})_Y$ an object $f^*(E) \rightarrow X$ in $T(\infty\mathcal{G}\text{rpd})_X$. This construction gives a pullback diagram in the global tangent ∞ -category $T(\infty\mathcal{G}\text{rpd})$ given by

$$\begin{array}{ccc} f^*(E) & \longrightarrow & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where the ∞ -groupoids X and Y are embedded via the “global sections functor” $s: \infty\mathcal{G}\text{rpd} \rightarrow T(\infty\mathcal{G}\text{rpd})$, which assigns each infinity groupoid X to the constant functor $X: \Delta[1] \rightarrow \infty\mathcal{G}\text{rpd}$ and then to the corresponding object in the stabilization.

In our case, we take X to be the Picard ∞ -groupoid $\text{Pic}_{\mathcal{R}}^{\text{top}}$ (see the description just before the remark on page 2908) and define the functor

$$\bar{p}: \text{Pic}_{\mathcal{R}}^{\text{top}} \rightarrow \mathcal{S}\text{p}$$

as the functor which assigns to each module spectrum \mathcal{R}_{τ} over \mathcal{R} the spectrum \mathcal{R}_{τ} . For tautological reasons this assignment defines an ∞ -functor and we denote the associated

⁹The main point here is that E and X live in different categories. However, we can get an actual map if were to embed X into its tangent ∞ -category as the terminal object. For ordinary ∞ -groupoids, this sends X to the corresponding spectrum $\Sigma^{\infty} X_+$.

bundle of spectra by $\pi: \lambda_{\mathcal{R}}^{\text{top}} \rightarrow \text{Pic}_{\mathcal{R}}^{\text{top}}$. Since a map $\tau: * \rightarrow \text{Pic}_{\mathcal{R}}^{\text{top}}$ is nothing but a choice of object of $\text{Pic}_{\mathcal{R}}^{\text{top}}$, the construction immediately identifies the fiber with \mathcal{R}_{τ} . That is, we have the following diagram in the tangent ∞ -category:

$$\begin{array}{ccc} \mathcal{R}_{\tau} & \longrightarrow & \lambda_{\mathcal{R}}^{\text{top}} \\ \downarrow & & \downarrow \\ * & \xrightarrow{\tau} & \text{Pic}_{\mathcal{R}}^{\text{top}} \end{array}$$

We can promote the previous discussion to the context of smooth stacks in various ways, but perhaps the most transparent way is to consider the “parametrized” version of the above construction. More precisely, for each smooth manifold M , consider the functor

$$(2-2) \quad \bar{p}_M: \text{Tw}_{\widehat{\mathcal{R}}}(M) \rightarrow \mathcal{S}p$$

which assigns to each twist $\widehat{\mathcal{R}}_{\tau}$ the spectra of global sections $\Gamma(M, \widehat{\mathcal{R}}_{\tau})$ on M . To this functor, the above construction gives an associated bundle of spectra $p_M: \lambda_{\widehat{\mathcal{R}}}(M) \rightarrow \text{Tw}_{\widehat{\mathcal{R}}}(M)$ and the fiber at a point $\tau: * \rightarrow \text{Tw}_{\widehat{\mathcal{R}}}(M)$ consists of the global sections of the twist $\widehat{\mathcal{R}}_{\tau}$ on M .

Definition 6 (canonical bundle of sheaves of spectra) We define the *canonical bundle of sheaves of spectra* over the stack of twists $\text{Tw}_{\widehat{\mathcal{R}}}$ as the presheaf with values in $T(\infty\mathcal{G}rpd)$ which associates to each smooth manifold M the object

$$p_M: \lambda_{\widehat{\mathcal{R}}}(M) \rightarrow \text{Tw}_{\widehat{\mathcal{R}}}(M)$$

associated to the tautological functor \bar{p}_M given in (2-2) which assigns to each object $\widehat{\mathcal{R}}_{\tau}$ the sheaf of spectra $\widehat{\mathcal{R}}_{\tau}$ on M . We denote this object by $\lambda_{\widehat{\mathcal{R}}} \rightarrow \text{Tw}_{\widehat{\mathcal{R}}}$.

Because the stabilization of an ∞ -category is built as a limit of ∞ -categories, we have a canonical equivalence

$$\mathcal{P}Sh_{\infty}(\mathcal{M}f, T(\infty\mathcal{G}rpd)) \simeq T(\mathcal{P}Sh_{\infty}(\mathcal{M}f)).$$

By manipulating adjoints and using the fact that the stabilization functor preserves limits, we see that this equivalence holds even at the level of sheaves. Since the stack of twists Tw satisfies descent, this implies that (under the Grothendieck construction) the bundle p_M associated to the functor $\bar{p}_M: \text{Tw}(M) \rightarrow \mathcal{S}p$ must satisfy descent on M . Thus, we have the following.

Proposition 7 The presheaf $M \mapsto (p_M: \lambda_{\widehat{\mathcal{R}}}(M) \rightarrow \text{Tw}_{\widehat{\mathcal{R}}}(M))$ satisfies descent.

We now would like to use the canonical bundle over the stack of twists to understand how twisted differential cohomology theories are pieced together from local data. With the powerful machinery developed by Lurie, this is now fairly systematic to construct.

Theorem 8 (local-to-global construction of twisted differential cohomology) *Let M be a smooth manifold and let $\hat{\tau}: M \rightarrow \text{Tw}_{\hat{\mathcal{R}}}$ be a twist for a differential refinement $\hat{\mathcal{R}}$ of a ring spectrum \mathcal{R} such that the underlying topological twist τ lands in the identity component of $\text{Pic}_{\mathcal{R}}^{\text{top}}$. Then the following sets are in bijective correspondence:*

- (i) *Equivalence classes of differential twists $\hat{\mathcal{R}}_{\hat{\tau}}$ for a refinement $\hat{\mathcal{R}}$.*
- (ii) *Equivalence classes of pullback bundles $E \rightarrow M$ by classifying maps $\hat{\tau}: M \rightarrow \text{Tw}_{\hat{\mathcal{R}}}$,*

$$\begin{array}{ccc} E & \longrightarrow & \lambda_{\hat{\mathcal{R}}} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\hat{\tau}} & \text{Tw}_{\hat{\mathcal{R}}} \end{array}$$

- (iii) *Equivalence classes of colimits of the form*

$$\left\{ \begin{array}{c} E \\ \downarrow \\ M \end{array} \right\} \simeq \text{colim} \left\{ \begin{array}{ccc} \cdots \rightrightarrows \coprod_{\alpha\beta} \hat{\mathcal{R}} \times U_{\alpha\beta} \rightrightarrows \coprod_{\alpha} \hat{\mathcal{R}} \times U_{\alpha} \\ \downarrow \\ \cdots \rightrightarrows \coprod_{\alpha\beta} U_{\alpha\beta} \rightrightarrows \coprod_{\alpha} U_{\alpha} \end{array} \right\},$$

where the simplicial maps in the top row are completely determined by a commutative diagram

$$(2-3) \quad \begin{array}{ccc} \begin{array}{c} \vdots \\ \uparrow\uparrow\uparrow \\ \Delta[1] \end{array} & \longrightarrow & \begin{array}{c} \prod_{\alpha\beta} \text{Tw}_{\hat{\mathcal{R}}}(U_{\alpha\beta}) \\ \uparrow\uparrow\uparrow \\ \prod_{\alpha} \text{Tw}_{\hat{\mathcal{R}}}(U_{\alpha}) \end{array} \\ \begin{array}{c} \uparrow\uparrow \\ \Delta[0] \end{array} & \longrightarrow & \prod_{\alpha} \text{Tw}_{\hat{\mathcal{R}}}(U_{\alpha}) \end{array}$$

with $\Delta[0] \rightarrow \prod_{\alpha} \text{Tw}_{\hat{\mathcal{R}}}(U_{\alpha})$ picking out the trivial twist $\hat{\mathcal{R}}$ on each U_{α} .

Proof The bijection between (i) and (ii) is a simple consequence of the Yoneda lemma, $\text{Map}(M, \text{Tw}_{\hat{\mathcal{R}}}) \simeq \text{Tw}_{\hat{\mathcal{R}}}(M)$, along with the fact that the objects of $\text{Tw}_{\hat{\mathcal{R}}}(M)$ are by

definition precisely the twists $\widehat{\mathcal{R}}_{\widehat{\tau}}$ of $\widehat{\mathcal{R}}$. By definition, the global sections of E fit into a diagram¹⁰

$$\begin{array}{ccc} \Gamma(M; E) := \text{Map}_M(M, E) & \longrightarrow & \lambda_{\widehat{\mathcal{R}}}(M) \\ \downarrow & & \downarrow \\ * & \xrightarrow{\widehat{\tau}} & \text{Tw}_{\widehat{\mathcal{R}}}(M) \end{array}$$

and by the ∞ -Grothendieck construction, the fiber of $\lambda_{\widehat{\mathcal{R}}}(M) \rightarrow \text{Tw}_{\widehat{\mathcal{R}}}(M)$ is $\widehat{\mathcal{R}}_{\widehat{\tau}}(M) \simeq \Gamma(M; E)$.

For the bijection between (ii) and (iii) recall that, by Proposition 4, for each $x \in M$, there is a neighborhood $x \in U \subset M$ such that each differential twist $\widehat{\mathcal{R}}_{\widehat{\tau}}$ on U is equivalent to the trivial twist $\widehat{\mathcal{R}}$. Thus for each such U we can choose a representative of the class of $\widehat{\tau}$ so that the corresponding element in $\text{Tw}_{\widehat{\mathcal{R}}}(U)$ is $\widehat{\mathcal{R}}$. Now let $\{U_\alpha\}$ be a good open covering of such open sets. Consider the augmented colimiting diagram in $\text{Sh}_\infty(\text{Mf})$

$$\cdots \rightrightarrows \coprod_{\alpha\beta} U_{\alpha\beta} \rightrightarrows \coprod_{\alpha} U_{\alpha} \longrightarrow M \xrightarrow{\widehat{\tau}} \text{Tw}_{\widehat{\mathcal{R}}}.$$

By the preceding discussion this gives rise to a colimiting diagram in the tangent topos $T(\text{Sh}_\infty(\text{Mf}))$ via the global sections functor. Taking iterated pullbacks by the universal bundle $\lambda_{\widehat{\mathcal{R}}} \rightarrow \text{Tw}_{\widehat{\mathcal{R}}}$, we get a commutative diagram,

$$(2-4) \quad \begin{array}{ccccccc} \cdots \rightrightarrows \coprod_{\alpha\beta} \widehat{\mathcal{R}} \times U_{\alpha\beta} & \rightrightarrows & \coprod_{\alpha} \widehat{\mathcal{R}} \times U_{\alpha} & \longrightarrow & E & \longrightarrow & \lambda_{\widehat{\mathcal{R}}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots \rightrightarrows \coprod_{\alpha\beta} U_{\alpha\beta} & \rightrightarrows & \coprod_{\alpha} U_{\alpha} & \longrightarrow & M & \xrightarrow{\widehat{\tau}} & \text{Tw}_{\widehat{\mathcal{R}}} \end{array}$$

The tangent topos itself is always an ∞ -topos (see [47, Section 6.1.1]). By the pasting lemma for pullbacks, all the square diagrams are Cartesian. By the descent axiom for a topos, the top diagram must be colimiting, since the bottom diagram is colimiting. Finally, using descent for the stack $\text{Tw}_{\widehat{\mathcal{R}}}$, we have an equivalence of ∞ -groupoids

$$\lim \left\{ \cdots \left\{ \rightrightarrows \prod_{\alpha} \text{Tw}_{\widehat{\mathcal{R}}}(U_{\alpha\beta}) \left\{ \rightrightarrows \prod_{\alpha} \text{Tw}_{\widehat{\mathcal{R}}}(U_{\alpha}) \right\} \right\} \simeq \text{Tw}_{\widehat{\mathcal{R}}}(M). \right.$$

The limit on the left can be calculated by the local formula for the homotopy limit. This gives a bijective correspondence between the classes of $\widehat{\tau}$ and the commutative diagrams of the form (2-3), taken up to equivalence. □

¹⁰In the diagram, $\text{Map}_M(M, E)$ denotes the mapping spectrum in the ∞ -slice over M .

2.3 Twisted differential theories as flat bundles of spectra

Theorem 8 gives us several different, but equivalent, ways to think about twisted differential cohomology theories. When taken altogether, one can find a long list of analogies between these objects and the notion of a line bundle equipped with flat connection. Ultimately, we would like to say that a twisted differential theory is a bundle of spectra equipped with flat connection. However, the notion of connection on a bundle of spectra has not yet been defined, nor is it completely clear how natural such a definition would be. For this reason, we will first motivate the notion via a list of analogies. We will then define the connections rigorously and show that indeed twisted differential theories come equipped with a canonical connection as part of the data.

(I) Local triviality of line bundles Consider a locally trivial, real line bundle over a smooth manifold $p: \eta \rightarrow M$. The local triviality means that if we fix a good open cover $\{U_\alpha\}$ on M and we are given the choice of transition functions $g_{\alpha\beta}$ on intersections $U_\alpha \cap U_\beta$, then we can piece together the total space of the bundle via the local trivializations. Moreover, a choice of Čech cocycle $g_{\alpha\beta}$ on intersections can be described in the language of smooth stacks as a commutative diagram

$$\begin{array}{ccc}
 \begin{array}{c} \vdots \\ \uparrow \\ \uparrow \\ \uparrow \\ \Delta[1] \end{array} & \xrightarrow{g_{\alpha\beta}} & \begin{array}{c} \vdots \\ \uparrow \\ \uparrow \\ \uparrow \\ \prod_{\alpha\beta} \mathbf{B}\mathbb{R}^\times(U_{\alpha\beta}) \end{array} \\
 \uparrow & & \uparrow \\
 \Delta[0] & \longrightarrow & \prod_{\alpha} \mathbf{B}\mathbb{R}^\times(U_{\alpha})
 \end{array}$$

with the vertical maps on the right given by restriction and the maps on the left by the face inclusions. By descent, this data is equivalently the data of a map (see [24] for similar constructions) $g: M \rightarrow \mathbf{B}\mathbb{R}^\times$. We can construct a universal line bundle over $\mathbf{B}\mathbb{R}^\times$ via an action of \mathbb{R}^\times on \mathbb{R} . The quotient by this action leads to a stack $\mathbb{R} // \mathbb{R}^\times$ and we have a canonical map $\mathbb{R} // \mathbb{R}^\times \rightarrow \mathbf{B}\mathbb{R}^\times$ which projects out \mathbb{R} . The statement of local triviality is then translated into a statement about descent by considering the diagram

$$\begin{array}{ccccccc}
 \coprod_{\alpha\beta} \mathbb{R} \times U_{\alpha\beta} & \rightrightarrows & \coprod_{\alpha} \mathbb{R} \times U_{\alpha} & \longrightarrow & E & \longrightarrow & \mathbb{R} // \mathbb{R}^\times \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \coprod_{\alpha\beta} U_{\alpha\beta} & \rightrightarrows & \coprod_{\alpha} U_{\alpha} & \longrightarrow & M & \xrightarrow{g} & \mathbf{B}\mathbb{R}^\times
 \end{array}
 \tag{2-5}$$

If $E \rightarrow M$ is the pullback of the universal bundle map by g , and if all the other squares are Cartesian, then descent says that the top diagram is colimiting. Conversely, if all squares but possibly the first are Cartesian and the top diagram is colimiting, then the first square is Cartesian. This means that the map g gives precisely the data necessary to construct the total space of the bundle E over M .

(I') Local triviality of bundles of spectra Comparing the diagram (2-5) with the diagram (2-4), we see a close analogy between the concept of a locally trivial line bundle and that of bundle of spectra $E \rightarrow M$ classified by a twist $\hat{\tau}$. In the case of a twisted differential theory, the cocycles representing the transition functions are replaced by maps to the stack of twists. These maps encode how to piece together the total space of the bundle via automorphisms of the underlying theory. Given that we have local triviality available, almost all the constructions one can do for vector bundles go through the same way for bundles of spectra, where the basic operations are now operations on spectra, rather than on vector spaces.

Definition 9 (smash product of bundles of spectra) Let $E \rightarrow M$ and $F \rightarrow M$ be two bundles of spectra over a smooth manifold M with fibers \mathcal{R} and \mathcal{K} , respectively. We define the *smash product* of E and F as the bundle $E \wedge F \rightarrow M$ with fiber $\mathcal{R} \wedge \mathcal{K}$, given by the colimit

$$\begin{array}{ccccccc} \cdots & \rightrightarrows & \coprod_{\alpha\beta} \mathcal{R} \wedge \mathcal{K} \times U_{\alpha\beta} & \rightrightarrows & \coprod_{\alpha} \mathcal{R} \wedge \mathcal{K} \times U_{\alpha} & \longrightarrow & E \wedge F \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \rightrightarrows & \coprod_{\alpha\beta} U_{\alpha\beta} & \rightrightarrows & \coprod_{\alpha} U_{\alpha} & \longrightarrow & M \end{array}$$

where the maps are induced by the corresponding maps for \mathcal{R} and \mathcal{K} via the smash product operation

$$\wedge: \text{Sh}_{\infty}(\text{Mf}; \text{Sp}) \times \text{Sh}_{\infty}(\text{Mf}; \text{Sp}) \rightarrow \text{Sh}_{\infty}(\text{Mf}; \text{Sp}).$$

(II) Flat connections on a line bundle Now we would like to add flat connections to the picture. Let ∇ be a flat connection on the line bundle $L \rightarrow M$. A connection is an operator on the sheaf of local sections

$$\nabla: \Gamma(-; L) \rightarrow \Gamma(-; L \otimes T^*M)$$

which satisfies the Leibniz rule with respect to the module action of $C^{\infty}(-; \mathbb{R})$ on $\Gamma(-; L)$. If the connection is flat, then ∇ gives rise to a complex

$$\mathcal{L}_L := (0 \rightarrow \Gamma(-; L) \xrightarrow{\nabla} \Gamma(-; L \otimes T^*M) \xrightarrow{\nabla} \Gamma(-; L \otimes \wedge^2(TM)) \rightarrow \dots).$$

With the discussion right after Definition 2 in mind, we have the following.

Lemma 10 *The complex \mathcal{L}_L is an invertible module over the de Rham complex.*

Proof To see this, we consider the de Rham complex as an unbounded sheaf of chain complexes concentrated in nonpositive degrees. Then the tensor product $\mathcal{L}_L \otimes \Omega^*$ is computed as the complex with degree $-n$ given by

$$\bigoplus_{p+q=-n} \Gamma(-; L \otimes \wedge^p(TM)) \otimes \Gamma(-; \wedge^q(TM)).$$

The wedge product of forms then gives a map

$$\mu: \bigoplus_{p+q=n} \Gamma(-; L \otimes \wedge^p(TM)) \otimes \Gamma(-; \wedge^q(TM)) \rightarrow \Gamma(-; L \otimes \wedge^n(TM)).$$

To get a map at the level of complexes, we need to check the commutativity with the differentials. But this follows immediately from the Leibniz rule:

$$\begin{aligned} \mu((\nabla \otimes \text{id} \pm \text{id} \otimes d)(s \otimes \omega \otimes v)) &= \nabla(s \otimes \omega) \wedge v \pm (s \otimes \omega) \wedge dv \\ &= \nabla(s \otimes \omega \wedge v) \\ &= \nabla(\mu(s \otimes \omega \otimes v)). \end{aligned}$$

Thus we have a module action of Ω^* on the complex \mathcal{L}_L . This complex is invertible with inverse given by the complex \mathcal{L}_L^{-1} , defined to be degreewise identical to \mathcal{L}_L but equipped with the differential $-\nabla$. □

(II') Flat connections on a bundle of spectra Continuing with the analogy for twisted differential theories, we see that we can think of the K -flat, invertible module \mathcal{L} (see the description right after Definition 2) as the sheaf of local sections of the bundle of spectra $p: E \wedge H(\Omega^*) \rightarrow M$, where $E \rightarrow M$ is the underlying bundle of spectra corresponding to the topological twist. Being a complex, \mathcal{L} comes equipped with a differential, which we denote by ∇ . We will continue to make a distinction between a bundle of spectra E_τ and the corresponding sheaf of spectra \mathcal{R}_τ given by its local sections.

Before defining connections on bundles of spectra, we will need some set up. Let $\wedge^\bullet(T^*M)$ denote the exterior power bundle, viewed as a graded vector bundle over M . The sheaf of sections of this graded bundle gives the graded algebra of differential forms on M . Equipping this graded algebra with the exterior derivative gives a DGA, and an application of the Eilenberg–Mac Lane functor gives a corresponding sheaf of spectra on M . To simplify notation and highlight our analogy with bundles, we will denote the corresponding bundle of spectra by the same symbol $\wedge^\bullet(T^*M)$.

Now given a bundle of spectra $E \rightarrow M$, the sheaf of sections of the smash product $E \wedge \bigwedge^\bullet(T^*M)$ admits the structure of a sheaf of $H\mathbb{R}$ -module spectrum, inherited from $\bigwedge^\bullet(T^*M)$. By Shipley’s theorem [66], there is corresponding sheaf of DGAs \mathcal{L} , which is unique up to equivalence such that

$$H(\mathcal{L}) \simeq \Gamma(-; E \wedge \bigwedge^\bullet(T^*M)),$$

so that the sections of $\Gamma(-; E \wedge \bigwedge^\bullet(T^*M))$ admit the structure of a sheaf of DGAs. This structure is not unique, but is unique up to a contractible choice. From the definition of $\bigwedge^\bullet(T^*M)$, we also see that \mathcal{L} can be chosen so that it admits the structure of a module over the de Rham DGA Ω^* . This now allows us to use the familiar structure of DGAs to construct connections.

Definition 11 (connection on a bundle of spectra) Fix a sheaf of DGAs modeling the sections of the bundle $E \wedge \bigwedge^\bullet(T^*M)$. A *connection* ∇ on a bundle of spectra $E \rightarrow M$ is a morphism of sheaves of DGAs

$$\nabla: \Gamma(-; E \wedge \bigwedge^\bullet(T^*M)) \rightarrow \Gamma(-; E \wedge \bigwedge^\bullet(T^*M)[1]),$$

satisfying the Leibniz rule

$$\nabla(\omega \cdot s) = d(\omega) \cdot s + (-1)^{\deg \omega} \omega \cdot \nabla(s),$$

where \cdot denotes the module action by differential forms. Here $\bigwedge^\bullet(T^*M)[1]$ denotes the shift of the corresponding bundle of spectra.

The following proposition shows that every differential twist gives a flat connection on the bundle of spectra corresponding to its underlying topological twist.

Proposition 12 Every differential twist $\widehat{\mathcal{R}}_\tau = (\mathcal{R}_\tau, t, \mathcal{L})$ defines a flat connection on the bundle of spectra $E_\tau \rightarrow M$, classified by the topological twist τ .

Proof By definition of the differential twist, we have an equivalence

$$t: H(\mathcal{L}) \simeq \mathcal{R}_\tau \wedge H\underline{\mathbb{R}}.$$

By the Poincaré lemma, we have an equivalence of sheaves of spectra $H\underline{\mathbb{R}} \simeq H(\Omega^*)$ so that $\mathcal{R}_\tau \wedge H\underline{\mathbb{R}} \simeq \mathcal{R}_\tau \wedge H(\Omega^*)$. Composing equivalences, we see that \mathcal{L} indeed gives the structure of a sheaf of DGAs to the sections of the bundle $E \wedge \bigwedge^\bullet(T^*M) \rightarrow M$. Denote the differential on \mathcal{L} by ∇ . Then the shift $\mathcal{L}[1]$ represents the sections of the shift $E \wedge \bigwedge^\bullet(T^*M)[1]$, and since $\nabla^2 = 0$, we see that indeed ∇ defines a flat connection on the bundle. □

(III) Reducing the structure group of a flat line bundle Finally, to complete the analogy, we need to discuss how the Riemann–Hilbert correspondence enters the picture. For the line bundle $p: L \rightarrow M$, if we take local parallel sections, we get a preferred choice of local trivialization of the bundle. To show this, fix a good open cover $\{U_\alpha\}$ of M and let s_α be local nonvanishing sections of E satisfying $\nabla(s_\alpha) = 0$. Let $\phi_\alpha(x, v) = (x, s_\alpha(x))$ be the corresponding trivialization on each U_α . The transition functions $g_{\alpha\beta}$ written in these local trivializations are then constant, since $\nabla(s_\alpha) = 0$. At the level of smooth stacks, this means that the map $g: M \rightarrow \mathbf{B}\mathbb{R}^\times$ can be chosen to factorize through the stack $\mathbf{B}\underline{\mathbb{R}}^\times$, where $\underline{\mathbb{R}}^\times$ denotes the constant sheaf on \mathbb{R}^\times , viewed as a set via the discrete topology. In these local coordinates, the flat connection trivializes. This story can be summarized succinctly in the category of smooth stacks as follows.

Lemma 13 *We have a homotopy commutative diagram*

$$\begin{array}{ccc}
 M & \longrightarrow & \mathbf{bLine}_\nabla \\
 \downarrow & \swarrow \text{RH} & \\
 \mathbf{B}\underline{\mathbb{R}}^\times & &
 \end{array}$$

where the stack \mathbf{bLine}_∇ is the stack of flat line bundles, given locally by the Dold–Kan image of the sheaf of complexes

$$(\dots \rightarrow 0 \rightarrow C^\infty(-; \mathbb{R}^\times) \xrightarrow{d} \Omega_{\text{cl}}^1 \rightarrow 0 \rightarrow \dots).$$

The diagonal map RH is an equivalence, given by the Riemann–Hilbert correspondence, which associates every flat bundle to its corresponding local system.

More generally, we can use the affine structure for flat connections to add globally defined closed 1–forms to the flat connection and the resulting connection is trivially compatible with the discrete structure. We can also ask for further compatibility on M as follows. Consider the inclusion $\mathbb{Z}/2 \hookrightarrow \mathbb{R}^\times$ as the units of \mathbb{Z} . If we ask for the flat connections to have monodromy representation factoring through $\mathbb{Z}/2$, then we are led to a homotopy commutative diagram,

$$\begin{array}{ccc}
 M & \longrightarrow & \mathbf{bLine}_\nabla \\
 \downarrow & & \downarrow \text{RH} \\
 \mathbf{B}\mathbb{Z}/2 & \longrightarrow & \mathbf{B}\underline{\mathbb{R}}^\times
 \end{array}$$

If we further ask that the chosen connection is globally defined and compatible with the $\mathbb{Z}/2$ -structure, then we are asking for a homotopy commutative diagram,

$$\begin{CD} M @>>> \Omega_{\text{cl}}^1 \\ @VVV @VVV \\ \mathbf{B}\mathbb{Z}/2 @>>> \mathbf{B}\underline{\mathbb{R}}^\times \simeq \text{bLine}_\nabla \end{CD}$$

Then, by the universal property of the homotopy pullback, we have an induced map¹¹

$$M \rightarrow \mathbf{B}(\mathbb{Z}/2)_\nabla := \mathbf{B}\mathbb{Z}/2 \times_{\mathbf{B}\underline{\mathbb{R}}^\times}^h \Omega_{\text{cl}}^1.$$

Remark The stack $\mathbf{B}(\mathbb{Z}/2)_\nabla$ is thus the moduli stack of line bundles equipped with globally defined flat connection, with monodromy factoring through $\mathbb{Z}/2$. Equivalently, it is the moduli stack of those (globally) flat bundles whose transition functions (when written in the trivializations provided by parallel sections) take values in $\mathbb{Z}/2$.

(III') Reducing the structure group of a flat bundle of spectra The analogue of the Riemann–Hilbert correspondence for twisted differential theories is essentially the twisted de Rham theorem, which gives an equivalence of stacks,

$$\text{Pic}_{\mathcal{R}}^{\text{dR}} \simeq \text{Pic}_{\mathcal{R} \wedge H\mathbb{R}}^{\text{top}}.$$

Geometrically, we begin by solving the equation $\nabla(s) = 0$ locally on M , where ∇ is the differential on the K -flat invertible module \mathcal{L} . In general, we need to know that such solutions exist locally, and for this, we need to better understand the structure of \mathcal{L} .

Proposition 14 (structure of invertible modules over \mathcal{R}) *If \mathcal{L} is an invertible module over \mathcal{R} , then \mathcal{L} is locally isomorphic to a product of (possibly shifted) finitely many copies of \mathcal{R} .*

Proof Let \mathcal{K} invert \mathcal{L} , ie be such that $\mathcal{L} \otimes_{\mathcal{R}} \mathcal{K} \cong \mathcal{R}$ is an isomorphism of sheaves of DG \mathcal{R} -modules. Then there is a point $x \in U \subset M$ such that, in that neighborhood, $\mathcal{L}(U) \otimes_{\mathcal{R}(U)} \mathcal{K}(U) \cong \mathcal{R}(U)$. Degreewise this reduces to a collection of isomorphisms

$$\phi_n: \bigoplus_{p+q=n} \mathcal{L}(U)_p \otimes_{\mathcal{R}(U)_0} \mathcal{K}(U)_q / \sim \xrightarrow{\cong} \mathcal{R}_n(U),$$

where \sim denotes the submodule generated by the relation $(rl, k) = (l, rk)$, with $|l| + |r| + |k| = n$. Let $1 \in \mathcal{R}(U)_0$ denote the identity. Then, for $n = 0$, there is a finite

¹¹The superscript h is for “homotopy”, and this should not be confused with the class of twists h .

sum $\sum_{i,p} l_{p,i} \otimes k_{-p,i}$ (with the $k_{p,i}$ and $l_{p,i}$ independent in \mathcal{L}_p and \mathcal{K}_p , respectively) that is mapped as $\phi_0(\sum_{i,p} l_{p,i} \otimes k_{-p,i}) = 1$. Since $d(1) = 0$ and ϕ is a chain map, we must have

$$0 = d(1) = d\phi_0\left(\sum_{i,p} l_{p,i} \otimes k_{-p,i}\right) = \phi_{-1}\left(d\left(\sum_{i,p} l_{p,i} \otimes k_{-p,i}\right)\right).$$

Since ϕ is an isomorphism, this implies that

$$0 = d\left(\sum_{i,p} l_{p,i} \otimes k_{-p,i}\right) = \sum_{p,i} dl_{p,i} \otimes k_{p,i} \pm l_{p,i} \otimes dk_{-p,i}.$$

Since the $k_{-p,i}$ and $l_{p,i}$ were chosen to be independent, this forces $dl_{p,i} = 0$ and $dk_{-p,i} = 0$ for each p and i . Define the complex $\mathcal{L}'(U)$ as the subcomplex of $\mathcal{L}(U)$ which is freely generated by $l_{p,i}$ as a graded module over $\mathcal{R}(U)$. Its underlying chain complex can be identified with the complex

$$\bigoplus_{p,i} \mathcal{R}(U)[p] \cong \bigoplus_p \langle l_{p,i} \rangle,$$

where $\langle l_{p,i} \rangle$ is the freely generated $\mathcal{R}(U)$ -module on generators $l_{p,i}$ (as i varies) shifted up by p . The differential d on this complex is inherited from $\mathcal{L}(U)$ and, therefore, satisfies $d^2 = 0$. Now the tensor product of the inclusion $i: \mathcal{L}'(U) \hookrightarrow \mathcal{L}(U)$ (which is a degreewise monomorphism) with $\mathcal{K}(U)$ is also a degreewise epimorphism. Indeed, the element $\sum_{i,p} l_{p,i} \otimes k_{-p,i}$ generates the tensor product as an $\mathcal{R}(U)$ -module. By invertibility, we must have a commutative diagram

$$\begin{array}{ccc} \mathcal{L}'(U) & \xrightarrow{i} & \mathcal{L}(U) \\ \downarrow \cong & & \downarrow \cong \\ \mathcal{L}'(U) \otimes_{\mathcal{R}(U)} \mathcal{K}(U) \otimes_{\mathcal{R}(U)} \mathcal{L}(U) & \longrightarrow & \mathcal{L}(U) \otimes_{\mathcal{R}(U)} \mathcal{K}(U) \otimes_{\mathcal{R}(U)} \mathcal{L}(U) \end{array}$$

with the bottom map a degreewise epimorphism via right exactness of the tensor product. Thus i is an isomorphism. □

Now focusing on the special case where $\mathcal{R}(U) = \Omega^*(U; A)$, we have the following.

Proposition 15 *Every invertible module \mathcal{L} over $\Omega^*(-; A)$ is locally isomorphic to a shift $\Omega^*(-; A)[p]$.*

Proof We need to show that, in this case, we can choose the $l_{p,i}$ in the proof of Proposition 14 to have a single generator in each degree (ie the indexing set for i

contains one element). To this end, recall that (locally) the ring $C^\infty(-; \mathbb{R})$ is freely generated by a single nonvanishing section. By Proposition 14, \mathcal{L} and \mathcal{K} locally look like a finite product of shifts of $\Omega^*(-; A)$. After a careful consideration of the grading for \mathcal{L} , \mathcal{K} and $\Omega^*(-; A)$, we find that there is a neighborhood $x \in U \subset M$ and free $C^\infty(U; \mathbb{R})$ -submodules $\mathcal{L}'(U)_p \cong \bigoplus_i C^\infty(U; \mathbb{R})$ and $\mathcal{K}'(U)_{-p} \cong \bigoplus_j C^\infty(U; \mathbb{R})$ of $\mathcal{L}(U)_p$ and $\mathcal{K}(U)_{-p}$, respectively, such that

$$\bigoplus_p \mathcal{L}'(U)_p \otimes_{C^\infty(U; \mathbb{R})} \mathcal{K}'(U)_{-p} \cong C^\infty(U; \mathbb{R});$$

ie the only elements which can get mapped to smooth functions in $\Omega^*(-; A)$ must come from the given submodules. Using basic properties of the tensor product, along with the fact that the isomorphism is that of $C^\infty(U; \mathbb{R})$ -modules, implies that the indexing sets for i , j and p must contain a single element. □

Remark Proposition 15 also implies that the condition of “weakly locally constant” can be dropped from the definition of the stack $\text{Pic}_{\mathcal{R}}^{\text{form}}$ in Section 2.1, as $\Omega^*(-; A)$ and its shifts are weakly locally constant by de Rham’s theorem.

As a consequence of Proposition 15, we can immediately solve our differential equation.

Corollary 16 *The equation $\nabla(s_\alpha^p) = 0$ admits a solution which is provided by the isomorphism in Proposition 15. Moreover, s_α gives rise to a local isomorphism via the module action*

$$s_\alpha \wedge (-): \Omega_{U_\alpha}^*(-; A)[p] \xrightarrow{\cong} \mathcal{L}_{U_\alpha}.$$

Proof In the proof of Proposition 15, take $s_\alpha^p = l_p$. □

It is natural to ask about the resulting structure globally. For that, if we compare the isomorphisms s_α on intersections, we get an automorphism

$$\begin{array}{ccc} \Omega_{U_{\alpha\beta}}^*(-; A) & \xrightarrow{g_{\alpha\beta}} & \Omega_{U_{\alpha\beta}}^*(-; A) \\ & \searrow s_\alpha \wedge & \nearrow s_\beta^{-1} \wedge \\ & \mathcal{L}_{U_{\alpha\beta}} & \end{array}$$

The inverse map $s_\beta^{-1} \wedge (-)$ takes a section of the form $s_\beta \wedge \omega$ to its coefficient ω . Thus, the automorphism $g_{\alpha\beta}$ is represented as a wedge product with a form. Moreover, since both s_α and s_β^{-1} are chain maps, this form is closed:

$$dg_{\alpha\beta} = d(s_\beta^{-1} s_\alpha) = s_\beta^{-1} (\nabla(s_\alpha)) = 0.$$

By the Poincaré lemma, it is also exact. This results in a degree-one automorphism $s_{\alpha\beta}$, represented by a wedge product with a form

$$d(s_{\alpha\beta} \wedge \omega) \pm s_{\alpha\beta} \wedge d\omega = g_{\alpha\beta} \wedge \omega.$$

Continuing to higher intersections, we get a whole hierarchy of automorphisms on various intersections which are compatible in a prescribed way with the automorphisms one step down. The compatibility is precisely that of an element in the total complex of the Čech–de Rham double complex with values in \mathcal{L} . More explicitly, on the n -fold intersecting patch $U_{\alpha_0 \dots \alpha_n}$, the sections $s_{\alpha_0 \dots \alpha_n}$ is closed under the total differential of the double complex, ie

$$\nabla(s_{\alpha_0 \dots \alpha_n}) = \pm \delta s_{\alpha_0 \dots \alpha_{n-1}}.$$

We now use these sections to consider lifting through the diagram (2-1).

Theorem 17 (twisted differential spectra from local data) *Let $\widehat{\mathcal{R}} = (\mathcal{R}, \text{ch}, A)$ be a differential cohomology theory and let \mathcal{L} be a K -flat, invertible module over $\Omega^*(-; A)$ with $s = (s_\alpha, s_{\alpha\beta}, \dots)$ the corresponding cocycle with values in the Čech total complex of \mathcal{L} . Assume that the isomorphism s_α identifies \mathcal{L} with the unshifted complex $\Omega^*(-; A)$ locally. Then s determines a commutative diagram,*

$$\begin{array}{ccc} \begin{array}{c} \vdots \\ \uparrow \uparrow \uparrow \\ \Delta[1] \end{array} & \xrightarrow{\eta_{\alpha\beta}} & \begin{array}{c} \vdots \\ \uparrow \uparrow \uparrow \\ \prod_{\alpha\beta} \text{BGL}_1(\mathcal{R} \wedge H\mathbb{R}) \end{array} \\ \begin{array}{c} \uparrow \uparrow \\ \Delta[0] \end{array} & \longrightarrow & \begin{array}{c} \uparrow \uparrow \\ \prod_{\alpha} \text{BGL}_1(\mathcal{R} \wedge H\mathbb{R}) \end{array} \end{array}$$

Moreover, if the η 's further lift through the canonical map $\text{BGL}_1(\mathcal{R}) \rightarrow \text{BGL}_1(\mathcal{R} \wedge H\mathbb{R})$, then s determines a commutative diagram,

$$(2-6) \quad \begin{array}{ccc} \begin{array}{c} \vdots \\ \uparrow \uparrow \uparrow \\ \Delta[1] \end{array} & \xrightarrow{h_{\alpha\beta}} & \begin{array}{c} \vdots \\ \uparrow \uparrow \uparrow \\ \prod_{\alpha\beta} \text{Tw}_{\widehat{\mathcal{R}}}(U_{\alpha\beta}) \end{array} \\ \begin{array}{c} \uparrow \uparrow \\ \Delta[0] \end{array} & \xrightarrow{h_\alpha} & \begin{array}{c} \uparrow \uparrow \\ \prod_{\alpha} \text{Tw}_{\widehat{\mathcal{R}}}(U_\alpha) \end{array} \end{array}$$

and hence a twisted differential cohomology theory $\widehat{\mathcal{R}}(-; \widehat{\tau})$, by Theorem 8.

Proof For each α , the map s_α is an isomorphism of $\Omega^*(-; A)$ -modules

$$s_\alpha: \Omega_{U_\alpha}^*(-; A) \xrightarrow{\cong} \mathcal{L}_{U_\alpha}$$

over the patch U_α . Precomposing this map with the canonical inclusion

$$A \hookrightarrow \Omega^*(-; A)$$

yields a morphism a_α of $\Omega^*(-; A)$ modules and therefore determines an edge in $\prod_\alpha \text{Pic}_{\mathbb{R}}^{\text{dR}}(U_\alpha)$. Consider the diagram determined by the transition forms $g_{\alpha\beta}$

$$\begin{array}{ccccc} \mathcal{R} \wedge H\mathbb{R} & \xrightarrow{\text{ch}} & H(\Omega_{U_{\alpha\beta}}^*(-; A)) & \xrightarrow{s_\alpha} & H(\mathcal{L}_{U_{\alpha\beta}}) \\ \downarrow \eta_{\alpha\beta} & & \downarrow g_{\alpha\beta} & & \parallel \\ \mathcal{R} \wedge H\mathbb{R} & \xrightarrow{\text{ch}} & H(\Omega_{U_{\alpha\beta}}^*(-; A)) & \xrightarrow{s_\beta} & H(\mathcal{L}_{U_{\alpha\beta}}) \end{array}$$

←————— curved arrow —————→

with the bottom curved arrow giving a homotopy inverse for the Chern character map and the composition determining the map $\eta_{\alpha\beta}$. A choice of differential form $s_{\alpha\beta}$ in $\Omega^*(U_{\alpha\beta}; A)$ satisfying $ds_{\alpha\beta} = g_{\alpha\beta}$ determines a homotopy that fills this diagram. Indeed, such a form gives a degree-one automorphism

$$(2-7) \quad s_{\alpha\beta} \wedge (-) \in Z(\text{hom}_{\text{Ch}}(\Omega_{U_\alpha}^*(-; A), \Omega_{U_\alpha}^*(-; A)))_1,$$

via the module action, where Z is the functor $\text{Ch} \rightarrow \text{Ch}^+$ which truncates an unbounded complex and puts cycles in degree zero. Since $ds_{\alpha\beta} = g_{\alpha\beta}$, one easily checks that the image of the automorphism $s_{\alpha\beta} \wedge (-)$ under the differential on the complex (2-7) is indeed the automorphism of degree zero given by $g_{\alpha\beta} \wedge (-)$. Applying the Dold–Kan functor to the above positively graded chain complex gives an edge in the automorphism space $\text{Aut}(\Omega_{U_\alpha}^*(-; A), \Omega_{U_\alpha}^*(-; A))$. Then pre- and postcomposing with ch and ch^{-1} give rise to an edge in $\text{Aut}(\mathcal{R} \wedge H\mathbb{R}, \mathcal{R} \wedge H\mathbb{R})$ with boundary $(\text{id}, \eta_{\alpha\beta})$.

The data described above is manifestly that of a map

$$s_{\alpha\beta}: \Delta[1] \times \Delta[1] \rightarrow \prod_{\alpha\beta} \text{Pic}_{\mathbb{R}}^{\text{dR}}(U_{\alpha\beta})$$

which connects the restrictions of the maps a_α and a_β on intersections. Continuing the process, one identifies the higher forms $s_{\alpha\beta\gamma}$, etc, with higher-degree automorphisms and gets compatibility with the lower-degree automorphisms analogously. Now,

recalling the diagram (2-1), the resulting structure is precisely a commutative diagram

$$(2-8) \quad \begin{array}{ccc} \begin{array}{c} \vdots \\ \uparrow \uparrow \uparrow \\ \Delta[1] \times \Delta[1] \end{array} & \xrightarrow{a_{\alpha\beta}} & \begin{array}{c} \vdots \\ \uparrow \uparrow \uparrow \\ \prod_{\alpha\beta} \widehat{\text{Pic}}_{\mathcal{R}}^{\text{dR}}(U_{\alpha\beta}) \end{array} \\ \uparrow \uparrow & & \uparrow \uparrow \\ \Delta[1] \times \Delta[0] & \xrightarrow{a_{\alpha}} & \prod_{\alpha} \widehat{\text{Pic}}_{\mathcal{R}}^{\text{dR}}(U_{\alpha}) \end{array}$$

such that

- the evaluation at one of the two endpoints of the 1-simplex $\Delta[1]$ factors through the composite map $\text{BGL}_1(\mathcal{R} \wedge H\mathbb{R}) \hookrightarrow \text{Pic}_{\mathcal{R} \wedge H\mathbb{R}}^{\text{top}} \rightarrow \widehat{\text{Pic}}_{\mathcal{R}}^{\text{dR}}$, hence lifting to a commutative diagram,

$$(2-9) \quad \begin{array}{ccc} \begin{array}{c} \vdots \\ \uparrow \uparrow \uparrow \\ \Delta[1] \end{array} & \xrightarrow{\eta_{\alpha\beta}} & \begin{array}{c} \vdots \\ \uparrow \uparrow \uparrow \\ \prod_{\alpha\beta} \text{BGL}_1(\mathcal{R} \wedge H\mathbb{R}) \end{array} \\ \uparrow \uparrow & & \uparrow \uparrow \\ \Delta[0] & \longrightarrow & \prod_{\alpha} \text{BGL}_1(\mathcal{R} \wedge H\mathbb{R}) \end{array}$$

- and the evaluation at the other endpoint factors through the map $\widehat{\text{Pic}}_{\mathcal{R}}^{\text{form}} \rightarrow \widehat{\text{Pic}}_{\mathcal{R}}^{\text{dR}}$, lifting to a commutative diagram,

$$(2-10) \quad \begin{array}{ccc} \begin{array}{c} \vdots \\ \uparrow \uparrow \uparrow \\ \Delta[1] \end{array} & \xrightarrow{s_{\alpha\beta}} & \begin{array}{c} \vdots \\ \uparrow \uparrow \uparrow \\ \prod_{\alpha\beta} \widehat{\text{Pic}}_{\mathcal{R}}^{\text{form}}(U_{\alpha\beta}) \end{array} \\ \uparrow \uparrow & & \uparrow \uparrow \\ \Delta[0] & \xrightarrow{s_{\alpha}} & \prod_{\alpha} \widehat{\text{Pic}}_{\mathcal{R}}^{\text{form}}(U_{\alpha}) \end{array}$$

If the η 's factor further through the rationalization map $\text{BGL}_1(\mathcal{R}) \rightarrow \text{BGL}_1(\mathcal{R} \wedge H\mathbb{R})$, then by definition of the twisting stack $\text{Tw}_{\widehat{\mathcal{R}}}$ (ie since it is a pullback), the diagrams (2-9) and (2-10) indeed furnish the commutative diagram (2-6). □

Remark (i) Now completing the analogy, we see the stacks $\text{Pic}_{\mathcal{R}}^{\text{top}}$, $\text{Pic}_{\mathcal{R}}^{\text{dR}} \simeq \text{Pic}_{\mathcal{R} \wedge H\mathbb{R}}^{\text{top}}$ and $\text{Pic}_{\mathcal{R}}^{\text{form}}$ (see the diagram (2-1)) can be thought of as the analogues of the stacks $\mathbf{B}\mathbb{Z}/2$, $\mathbf{B}\mathbb{R}^{\times} \simeq \text{bLine}_{\nabla}$ and Ω_{cl}^1 , respectively (see also Table 1). The stack of differential twists $\text{Tw}_{\widehat{\mathcal{R}}}$ is then analogous to $\mathbf{B}\mathbb{Z}/2_{\nabla}$. This comparison indicates that we should think of the twisting stack $\text{Tw}_{\widehat{\mathcal{R}}}$ as the moduli stack of bundles of spectra $E \rightarrow M$, equipped with a globally defined flat connection encoded by the differential on the invertible, K -flat, locally constant module \mathcal{L} , whose corresponding local coefficient system is given by the underlying topological twisted theory $\mathcal{R}(-; \tau)$.

(ii) The theory of bundles of spectra can be viewed as a generalization of the theory of ordinary line bundles, so that Table 2 is not just an analogy. Indeed, the sheaf of sections of a line bundle can be viewed as a sheaf of DGAs, concentrated in degree zero. Application of the Eilenberg–Mac Lane functor gives a sheaf of ring spectra, which in turn gives rise to a trivial bundle over any fixed manifold M . The fiber of this bundle is then easily computed as the Eilenberg–Mac Lane spectrum $H\mathbb{R}$.

(iii) In [33], we found that the stack $\mathbf{B}(\mathbb{Z}/2)_{\nabla}$ gives the twists for smooth Deligne cohomology, so that the constructions here indeed represent a generalization to other generalized cohomology theories.

2.4 The general construction of the spectral sequence $\widehat{\text{AHSS}}_{\widehat{\tau}}$

The general machinery established in [32] works in the twisted case as well with some modifications. More precisely, let $\widehat{\mathcal{R}} = (\mathcal{R}, c, A)$ be a differential ring spectrum. If we are given a smooth manifold M , equipped with a map

$$\widehat{\tau}: M \rightarrow \text{Tw}_{\widehat{\mathcal{R}}}$$

to the stack of twists, then $\widehat{\mathcal{R}}_{\widehat{\tau}} = (\mathcal{R}_{\tau}, \text{ch}_{\tau}, \mathcal{L})$ is a twisted differential spectrum which lives over the manifold M ; ie it is an object in $\text{Sh}_{\infty}(M; \text{Sp})$. Fix a good open cover $\{U_{\alpha}\}$ of the smooth manifold M . By the second property in Proposition 5, the restrictions of $\widehat{\mathcal{R}}_{\widehat{\tau}}$ to the various contractible intersections can be identified with the untwisted differential theory $\widehat{\mathcal{R}}$.

This locally trivializing phenomenon is really the key observation in constructing the differential twisted AHSS and is similar to the familiar analogue in spaces. Indeed, the topological twisted AHSS has E_2 -page which looks like cohomology with coefficients in the underlying *untwisted* theory, and again this amounts to the fact that the twisted \mathcal{R} -theory of a contractible space reduces to untwisted \mathcal{R} -theory.

We now sketch the filtration in the twisted context. Essentially, this is the same filtration as considered in [32], but where each level of the filtration is equipped with a map to the stack of twists $\text{Tw}_{\widehat{\mathcal{R}}}$.

Remark (the filtration for differential twisted spectra) Let us consider a smooth manifold M and the Čech nerve of a cover $\{U_\alpha\}$ of M . The restriction of a twist $\widehat{\tau}: M \rightarrow \text{Tw}_{\widehat{\mathcal{R}}}$ to various intersections of the cover gives a simplicial diagram in the slice $\text{Sh}_\infty(\mathcal{M}f)/\text{Tw}_{\widehat{\mathcal{R}}}$. That is, we have the simplicial diagram

$$\begin{array}{ccccc} \cdots & \rightrightarrows & \coprod_{\alpha\beta\gamma} U_{\alpha\beta\gamma} & \rightrightarrows & \coprod_{\alpha\beta} U_{\alpha\beta} & \rightrightarrows & \coprod_{\alpha} U_{\alpha} \\ & & \searrow & & \downarrow & & \swarrow \\ & & & & \text{Tw}_{\widehat{\mathcal{R}}} & & \end{array}$$

where the maps to $\text{Tw}_{\widehat{\mathcal{R}}}$ are given by the restriction of the twisted spectrum $\widehat{\mathcal{R}}_{\widehat{\tau}}$ to various intersections. Let F_p denote the skeletal filtration on this simplicial object. In this case, the successive quotients F_p/F_{p-1} take the form

$$F_p/F_{p-1} \simeq (\widehat{\tau}|_{U_{\alpha_0 \dots \alpha_p}} : \Sigma_{\text{Tw}_{\widehat{\mathcal{R}}}}^p (\bigvee_{\alpha_0 \dots \alpha_p} (U_{\alpha_0 \dots \alpha_p})_+) \rightarrow \text{Tw}_{\widehat{\mathcal{R}}}).$$

Using the properties of $\widehat{\mathcal{R}}_{\widehat{\tau}} = (\mathcal{R}_{\widehat{\tau}}, t, \mathcal{L})$, we see that the connected components of the global sections of $\widehat{\mathcal{R}}_{\widehat{\tau}}$ are given by

$$\widehat{R}^0(\Sigma_{\text{Tw}_{\widehat{\mathcal{R}}}}^p (\bigvee_{\alpha_0 \dots \alpha_p} (U_{\alpha_0 \dots \alpha_p})_+); \widehat{\tau}) \cong \bigoplus_{\alpha_0 \dots \alpha_p} \widehat{R}^{-p}(U_{\alpha_0 \dots \alpha_p}; \widehat{\tau}) \cong \bigoplus_{\alpha_0 \dots \alpha_p} \widehat{R}^{-p}(U_{\alpha_0 \dots \alpha_p}).$$

Here the last isomorphism uses local triviality (Proposition 4) since $U_{\alpha_0 \dots \alpha_p}$ is contractible as a space.

The same arguments used in the ordinary AHSS hold for this filtration and we have the following.

Theorem 18 (AHSS for twisted differential spectra) *Let M be a compact manifold and let $\widehat{\mathcal{R}}_{\widehat{\tau}}: M \rightarrow \text{Tw}_{\widehat{\mathcal{R}}}$ be a twist for a differential ring spectrum $\widehat{\mathcal{R}}$. Then:*

- (i) *There is a half-plane spectral sequence $\widehat{\text{AHSS}}_{\widehat{\tau}}$ with E_2 -page given by*

$$E_2^{p,q} = \begin{cases} H^p(M; R_{U(1)}^{-q-1}(*)) & \text{for } q < 0, \\ H^p(M; R^{-q}(*)) & \text{for } q > 0, \\ H^p(M; L(\widehat{R}^0(-; \widehat{\tau}))) & \text{for } q = 0, \end{cases}$$

where the group $H^p(M; L(\widehat{R}^0(-; \widehat{\tau})))$ is the degree-zero Čech cohomology with coefficients in the sheafification of the presheaf $\widehat{R}^0(-; \widehat{\tau})$.

(ii) For $p = 0$, the group $H^0(M; L(\widehat{R}^0(-; \widehat{\tau})))$ is just the evaluation of the sheaf $L(\widehat{R}^0(-; \widehat{\tau}))$ on M and can be computed as the kernel of the map

$$(2-11) \quad \prod_{\alpha} \widehat{R}^0(U_{\alpha}; \widehat{\tau}) \xrightarrow{r_{\alpha\beta} - r_{\beta\alpha}} \prod_{\alpha\beta} \widehat{R}^0(U_{\alpha\beta}; \widehat{\tau}),$$

where $\{U_{\alpha}\}$ is a good open cover of M , the map $r_{\alpha\beta}$ is the restriction induced by the inclusion $U_{\alpha\beta} \hookrightarrow U_{\alpha}$, and $r_{\beta\alpha}$ is induced by $U_{\alpha\beta} \hookrightarrow U_{\beta}$.

(iii) This spectral sequence converges to the graded group $E_{\infty}^{p,q}$ associated to the filtration F_p described in the remark on page 2930.

Convergence of spectral sequences is studied in general in [10; 55]. In our case, convergence is understood in the same sense as in the untwisted differential case [32], similarly to how in the classical topological case convergence in twisted K -theory [58; 59; 9] has been understood in the same way as for untwisted K -theory [7]. In the differential case, convergence is guaranteed by virtue of the fact that M is compact and, therefore, admits a finite good open cover. This is analogous to the classical topological case, where the underlying space is assumed to be a CW complex of finite dimension. Note, however, that in both the topological and the differential case this can be extended to certain infinite dimensional spaces and manifolds, respectively, by taking appropriate direct limits. However, we will not consider this extension explicitly here.

Remark The $E^{0,0}$ -entry involves the twisted theory as coefficients. This might seem to disagree with the local triviality condition for the twisted theory (ie one might expect coefficients in the untwisted theory to appear). However, note that moving from the E_1 -page to the E_2 -page, we have to find the kernel of the map (2-11) as explained in [32]. The isomorphisms which identify the theory locally with the untwisted theory do not commute with the usual restriction maps and so we cannot identify the result with its analogue in the untwisted case. However, we can compute $E_2^{0,0}$ alternatively as the kernel

$$E_2^{0,0} \cong \ker \left\{ \prod_{\alpha} \widehat{R}^0(U_{\alpha}) \xrightarrow{r_{\alpha\beta} - \phi_{\alpha\beta}} \prod_{\alpha\beta} \widehat{R}^0(U_{\alpha\beta}) \right\},$$

where the maps $\phi_{\alpha\beta}$ are the transition maps induced from the local trivializations

$$\begin{array}{ccc} \prod_{\alpha\beta} \widehat{R}^0(U_{\alpha\beta}) & \xrightarrow{\phi_{\alpha\beta}} & \prod_{\alpha\beta} \widehat{R}^0(U_{\alpha\beta}) \\ s_{\alpha} \downarrow \cong & & \cong \uparrow s_{\beta}^{-1} \\ \prod_{\alpha\beta} \widehat{R}^0(U_{\alpha\beta}, \widehat{\tau}) & \xlongequal{\quad} & \prod_{\alpha\beta} \widehat{R}^0(U_{\alpha\beta}, \widehat{\tau}) \end{array}$$

2.5 Properties of the differentials

In this section, we would like to describe some of the properties which the differentials in the $\widehat{\text{AHSS}}_{\hat{\tau}}$ for twisted differential cohomology theories enjoy. Some of these properties involve the module structure of a twisted differential theory over the untwisted theory. We have not yet described the module structure. To this end, we note that for a twisted differential theory $\widehat{\mathcal{R}}_{\tau} = (\mathcal{R}_{\tau}, t, \mathcal{L})$, both \mathcal{R}_{τ} and \mathcal{L} are modules over \mathcal{R} and $\Omega^*(-; A)$, respectively. The map t is an equivalence in $\text{Pic}_{\mathcal{R}}^{\text{form}}$ and so is, in particular, a map of module spectra. In the same way one defines a differential refinement of a ring spectrum (see [15; 71]), one sees that the twisted theory $\widehat{\mathcal{R}}_{\tau}$ admits module structure over $\widehat{\mathcal{R}}$. At the level of the underlying theory, we have maps

$$(2-12) \quad \mu: \widehat{R}^n(M) \times \widehat{R}^n(M; \hat{\tau}) \rightarrow \widehat{R}^{n+m}(M; \hat{\tau}),$$

and so at least it makes sense to talk about the behavior of the differentials with respect to the module structure.

Some of the classical properties of the differentials in the spectral sequence for twisted K -theory are discussed in [8; 9; 40]. Many of the analogous properties also hold for twisted Morava K -theory [63]. We will first review these properties and then discuss the differential refinement. Let \mathcal{R}_{τ} be a twist for a ring spectrum \mathcal{R} . In general, we have:

- (i) **Linearity** Each differential d_i is an $R^*(*)$ -module map.
- (ii) **Normalization** The twisted differential with a zero twist reduces to the untwisted differential (which may be zero).
- (iii) **Naturality** Let \mathcal{R}_{τ} be a twist of \mathcal{R} on X . If $f: Y \rightarrow X$ is continuous then $f^*\mathcal{R}_{\tau}$ is a twist of \mathcal{R} on Y and f induces a map of spectral sequences from the twisted AHSS for \mathcal{R}_{τ} to the twisted AHSS for $f^*\mathcal{R}_{\tau}$. On E_2 -terms it is induced by f^* in cohomology, and on E_{∞} -terms it is the associated graded map induced by f^* in the twisted \mathcal{R} -theory \mathcal{R}_{τ} .
- (iv) **Module** The AHSS_{τ} for \mathcal{R}_{τ} is a spectral sequence of modules for the untwisted AHSS. Specifically, $d_i(ab) = d_i^u(a)b + (-1)^{|a|}ad_i(b)$ where a comes from the untwisted spectral sequence (with untwisted differentials d_i^u).

All of these properties lift to the twisted differential case with the natural modifications needed in order to make the statement sensible. We will spell this out in detail.

Proposition 19 (properties of the differentials in $\widehat{\text{AHSS}}_{\hat{\tau}}$) *Let $\widehat{\mathcal{R}}$ be a differential ring spectrum refining a ring spectrum \mathcal{R} . Let $\widehat{\mathcal{R}}_{\hat{\tau}}$ be a differential twist. Then we have the following properties:*

- (i) **Linearity** Each differential d_i is an $\widehat{R}^*(*) \simeq R(*)$ -module map.
- (ii) **Normalization** The twisted differential with a zero twist reduces to the untwisted differential (which may be zero).
- (iii) **Naturality** If $f: M \rightarrow N$ is smooth map between compact manifolds, f^* induces a map of spectral sequences from the $\widehat{\text{AHSS}}_{\widehat{\mathcal{R}}_{\widehat{\tau}}}$ for $(N, \widehat{\mathcal{R}}_{\widehat{\tau}})$ to the $\widehat{\text{AHSS}}_{f^*\widehat{\mathcal{R}}_{\widehat{\tau}}}$ for $(M, f^*\widehat{\mathcal{R}}_{\widehat{\tau}})$. On E_2 -terms it is induced by f^* in Čech cohomology, and on E_∞ -terms it is the associated graded map induced by f^* in the twisted theory $\widehat{\mathcal{R}}_{\widehat{\tau}}$.
- (iv) **Module** The $\widehat{\text{AHSS}}_{\widehat{\mathcal{R}}_{\widehat{\tau}}}$ for $(N, \widehat{\mathcal{R}}_{\widehat{\tau}})$ is a spectral sequence of modules for the untwisted $\widehat{\text{AHSS}}$. Specifically, $d_i(ab) = d_i^u(a)b + (-1)^{|a|}ad_i(b)$ where a comes from the untwisted spectral sequence (with differentials d_i^u).

Proof To prove (i), note that the map μ in (2-12) gives $\widehat{R}(M; \widehat{\tau})$ the structure of a module over $\widehat{R}^*(*)$ for each M by precomposing with the map induced by the terminal map $M \rightarrow *$. Clearly this structure is natural with respect to maps of pairs $f: (M, f^*\widehat{\tau}) \rightarrow (N, \widehat{\tau})$, with τ a twist on M . Therefore, the maps in the exact couple defining the $\widehat{\text{AHSS}}_{\widehat{\mathcal{R}}_{\widehat{\tau}}}$ commute with the module action. Since the differentials are defined via these maps, the claim follows.

Property (ii) follows immediately by noting that the zero twist $\widehat{\tau}: M \rightarrow * \rightarrow \text{Tw}_{\widehat{\mathcal{R}}}$ picks out the triple (\mathcal{R}, c, A) , where the ring spectrum \mathcal{R} and the algebra A are thought of as modules over themselves.

Property (iii) also follows directly, as a smooth map $f: M \rightarrow N$ induces a morphism of stacks $f^*: \text{Tw}_{\widehat{\mathcal{R}}}(N) \rightarrow \text{Tw}_{\widehat{\mathcal{R}}}(M)$. This map sends the sheaf of spectra $\widehat{\mathcal{R}}_{\tau} = (\mathcal{R}_{\tau}, t, \mathcal{L})$ on N to the sheaf of spectra $f^*\widehat{\mathcal{R}}_{\tau} = (f^*\mathcal{R}_{\tau}, f^*(t), f^*(\mathcal{L}))$ on M via the change-of-base functor

$$f^*: \text{Sh}_{\infty}(M; \mathcal{S}p) \rightarrow \text{Sh}_{\infty}(N; \mathcal{S}p).$$

Moreover, the filtration F_p leads to a pullback filtration f^*F_p which is the simplicial diagram formed from the nerve of the cover $\{f^{-1}(U_{\alpha})\}$. This induces the desired morphism of spectral sequences. Considering the relevant diagram that gives the Čech differential on the E^1 -page [32] and comparing with the corresponding diagram for the cover $\{f^{-1}(U_{\alpha})\}$ one immediately sees that f gives rise to the induced map in Čech cohomology on the E_2 -page.

The proof for property (iv) follows verbatim as in the proof for ring spectrum presented in the differential untwisted case [32], by simply replacing the product map with the module map. □

3 The AHSS in twisted differential K -theory

By the discussion in the introduction, and via the general construction in [54], given a cohomology theory \mathcal{E} (represented by an E_∞ -ring spectrum), the action by automorphisms $\mathrm{GL}_1(\mathcal{E})$ on \mathcal{E} gives rise to a bundle of spectra over the quotient, which is classified by a map into the delooping $\mathrm{BGL}_1(\mathcal{E})$. This space classifies the twists of the theory. For K -theory, by [4] (see also [40]), there is essentially a unique equivalence class of maps of Picard ∞ -groupoids $K(\mathbb{Z}, 2) \rightarrow \mathrm{GL}_1(\mathcal{K})$, which gives the most interesting part of the twists. Delooping the embedding gives a map $\mathrm{B}^2\mathrm{U}(1) \hookrightarrow \mathrm{BGL}_1(\mathcal{K})$ and we consider only the twists arising as maps to $\mathrm{B}^2\mathrm{U}(1)$, ie degree-three cohomology classes.

3.1 Twisting differential K -theory by gerbes

In this section, we give a comprehensive account of the twists in differential K -theory and discuss the situation from various angles, unifying certain perspectives taken up in [16; 19; 37; 56]. Our end goal in this section will be to show that differential K -theory can indeed be twisted by gerbes, equipped with connection. This is implicit in [16], but here we rephrase this result in a way that makes contact with the moduli stack of gerbes $\mathbf{B}^2\mathrm{U}(1)_\nabla$ (see [65; 24; 62]).¹²

We begin by recounting the topological case. Let \mathcal{K} denote the complex K -theory spectrum. The delooping of the units $\mathrm{BGL}_1(\mathcal{K})$ split into three factors. The factor which has attracted the most attention is the Eilenberg–Mac Lane space $K(\mathbb{Z}, 3) \hookrightarrow \mathrm{BGL}_1(\mathcal{K})$. Now, recalling that $\mathrm{Pic}_{\mathbb{R}} \cong \mathrm{BGL}_1(\mathcal{R}) \times \pi_0(\mathrm{Pic}_{\mathcal{R}})$, the inclusion at the identity component of $\mathrm{Pic}_{\mathcal{R}}$ then gives rise to a canonical map,

$$(3-1) \quad \mathrm{B}^2\mathrm{U}(1) \simeq K(\mathbb{Z}, 3) \hookrightarrow \mathrm{Pic}_{\mathcal{K}}.$$

The left-hand side can be identified with the moduli space of bundles with fiber $\mathrm{PU}(\mathcal{H})$, the projective unitary group acting on an infinite-dimensional, separable Hilbert space \mathcal{H} . Indeed, a map $h: M \rightarrow \mathrm{B}^2\mathrm{U}(1)$ classifies a fibration $\mathrm{BU}(1) \hookrightarrow P \twoheadrightarrow M$. The fiber $\mathrm{BU}(1)$ is a classifying space of complex line bundles and has the homotopy type of $\mathrm{PU}(\mathcal{H})$. Alternatively, we can think of $\mathrm{B}^2\mathrm{U}(1)$ as the classifying space of *topological* gerbes. Now $\mathrm{PU}(\mathcal{H})$ acts on the space of Fredholm operators $\mathcal{F}\mathrm{r}$, which is well known

¹²While this might seem like a further undesirable abstraction, it will be more of a computational and applicable type of abstraction. For example, it will make it relatively easy to write down local cocycle data for the twist.

to be a classifying space for K -theory (see [6]). Given a map $h: M \rightarrow B^2U(1)$, this action leads to an associated bundle of spectra $E_h \rightarrow M$, which we can regard as a bundle of spectra with fiber \mathcal{K} . Taking the homotopy classes of sections of this bundle gives the twisted K -theory $K(M; h)$.

This is the well-known description for twisted topological K -theory established by Atiyah and Segal in [8]. In order to make contact with the stacky differential spectra provided by Bunke and Nikolaus, it is useful to take a different perspective. The transition from the above approach to the approach of [16] is achieved by considering the bundle $E_h \rightarrow M$ as a sheaf of spectra on M via its local sections. For an object $(i: N \rightarrow M) \in \mathcal{Mf}/M$, the value of this sheaf at N is given by the function spectrum $\mathcal{K}_h(N) := \Gamma(N, i^*(E_h))$. One can show that the assignment leads to a functor $\mathcal{Mf}/M \rightarrow \mathcal{Sp}$ and satisfies descent.¹³ Since every point on a manifold M admits a geodesically convex neighborhood, this immediately implies that this sheaf of spectra is locally constant and is equivalent to the untwisted K -theory spectrum (regarded as a geometrically discrete sheaf of spectra) locally. As we have seen in Sections 2.1 and 2.2, the actual equivalences which identify the spectrum with untwisted K -theory locally can be regarded as a local trivialization of the bundle $E_h \rightarrow M$. In this context, descent can be summarized by the diagram in $T(\infty\mathcal{G}rpd)$

$$\begin{array}{ccccccc}
 \cdots \coprod_{\alpha\beta\gamma} \mathcal{K} \times (U_\alpha \cap U_\beta \cap U_\gamma) & \rightrightarrows & \coprod_{\alpha\beta} \mathcal{K} \times (U_\alpha \cap U_\beta) & \rightrightarrows & \coprod_{\alpha} \mathcal{K} \times U_\alpha & \rightarrow & E_h \rightarrow \mathcal{K} // \text{PU}(\mathcal{H}) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \cdots \coprod_{\alpha\beta\gamma} U_\alpha \cap U_\beta \cap U_\gamma & \rightrightarrows & \coprod_{\alpha\beta} U_\alpha \cap U_\beta & \rightrightarrows & U_\alpha & \longrightarrow & M \xrightarrow{h} K(\mathbb{Z}, 3)
 \end{array}
 \tag{3-2}$$

where each square is Cartesian and the top and bottom simplicial diagrams are colimiting. This gives us all the data needed to construct the twisted theory in terms of local data. The simplicial maps in the top diagram are determined by the twist $h: M \rightarrow K(\mathbb{Z}, 3)$ which, more explicitly, is determined by a Čech 3-cocycle on M via descent. What we have described so far is the relationship between the bundle of spectra $E_h \rightarrow M$ associated to a classifying map $h: M \rightarrow K(\mathbb{Z}, 3)$ and the locally constant sheaf \mathcal{K}_h given by its sheaf of sections.

¹³Descent follows from the definition of the coverage via *good* open covers, along with homotopy invariance for \mathcal{K} .

The next step is to go differential. Crucially, doing so involves a sort of twisted de Rham theorem which relates twisted K -theory to the complex of periodic forms $(\Omega^*[u, u^{-1}](M), d_H)$ equipped with the differential $d_H = d + H$, where H is a closed 3-form and the differential acts by

$$\omega_{2k}u^k + \omega_{2k-2}u^{k-1} \mapsto (d\omega_{2k} + H \wedge \omega_{2k-2})u^k.$$

To understand this interaction, it is important to consider the untwisted case first. K -theory is an example of a *differentially simple* spectrum (see [16]). For such untwisted spectra, there is a rather canonical differential refinement, which in this case is given by taking $A = \pi_*(K) \otimes \mathbb{R} \cong \mathbb{R}[u, u^{-1}]$. The usual Chern character map gives an isomorphism of rings

$$\text{ch}: (\mathcal{K} \wedge H\mathbb{R})^*(M) \xrightarrow{\cong} H^*(\Omega^*[u, u^{-1}](M)),$$

where $\Omega^*[u, u^{-1}](M)$ is the periodic de Rham complex, equipped with the usual differential. This map refines to a morphism of sheaves of spectra and gives rise to the triple $\widehat{\mathcal{K}} = (\mathcal{K}, \text{ch}, \mathbb{R}[u, u^{-1}])$, which is a differential spectrum representing differential K -theory.

Returning to the twisted case, we begin with a closed 3-form H , defined on a smooth manifold M and consider the twisted periodic complex $(\Omega^*[u, u^{-1}], d_H)$ (as a sheaf of DGAs on M). This complex defines a K -flat, invertible module over the untwisted complex $\Omega^*[u, u^{-1}]$ (see [16]), with inverse given by $(\Omega^*[u, u^{-1}], d_{-H})$. From the discussion in Section 2.3, we see that the differential d_H should be thought of as a connection on the bundle of spectra $E_h \wedge \bigwedge^\bullet(T^*M) \rightarrow M$ and the local parallel sections of this connection should give rise to a system of local trivializations of the bundle. We have the following well-known fact which will be essential for a detailed description of the trivialization.

Lemma 20 *Let H be a closed 3-form on a finite-dimensional smooth manifold M and let $\{U_\alpha\}$ be a good open cover of M . Let B_α be a local potential (ie trivialization) for H . Then the local sections given by the formal exponentials*

$$e^{-B_\alpha} = 1 - B_\alpha + \frac{1}{2!}B_\alpha^2 - \dots$$

are twisted closed over U_α . Moreover, they define isomorphisms of modules via the module action

$$e^{-B_\alpha} \wedge (-): (\Omega^*_{U_\alpha}[u, u^{-1}], d) \rightarrow (\Omega^*_{U_\alpha}[u, u^{-1}], d_H).$$

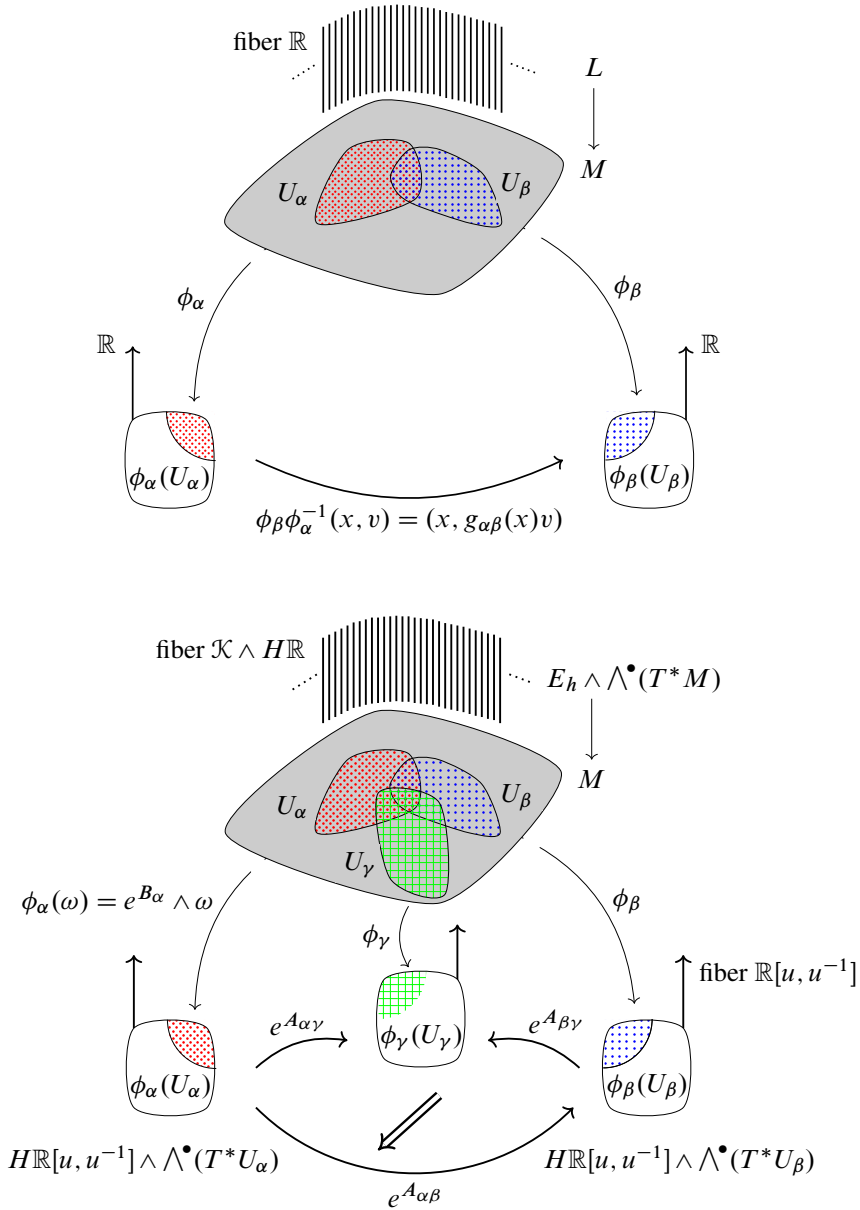


Figure 1: Flat line bundle (top) versus twisted differential cohomology theory (bottom).

Proof Applying the twisted differential to our section gives

$$(d + H)(e^{-B_\alpha}) = d(e^{-B_\alpha}) + H \wedge e^{-B_\alpha} = -dB_\alpha e^{B_\alpha} + H \wedge e^{-B_\alpha} = 0,$$

so that e^{-B_α} is indeed twisted closed. Now the map $e^{-B_\alpha} \wedge (-)$ defines a map of modules since, for each smooth map $f: N \rightarrow U_\alpha$ the pullback form $f^*e^{-B_\alpha}$ satisfies

$$\begin{aligned} (d + f^*H)(f^*e^{-B_\alpha} \wedge \omega) &= df^*(e^{-B_\alpha}) \wedge \omega + f^*(H \wedge e^{-B_\alpha}) \wedge \omega \\ &= f^*(-dB_\alpha \wedge e^{-B_\alpha}) \wedge \omega + f^*(e^{-B_\alpha}) \wedge d\omega + f^*(H \wedge e^{-B_\alpha}) \wedge \omega \\ &= f^*(e^{-B_\alpha}) \wedge d\omega. \end{aligned}$$

Consequently, $f^*(e^{-B_\alpha}) \wedge (-)$ defines a chain map on N and, as a result, $e^{-B_\alpha} \wedge (-)$ defines a morphism of sheaves of chain complexes. That this map commutes with the module action is obvious from the definition. Moreover, it is easy to check that this map admits an inverse given by $e^{B_\alpha} \wedge (-)$. □

Thus, we see that the forms e^{-B_α} play the role of the s_α in the general discussion from Section 2.3. We depict this in Figure 1.

Taking \mathcal{R} to be \mathcal{K} in Theorem 17, we get an induced diagram,

$$\begin{array}{ccc} \begin{array}{c} \vdots \\ \uparrow \uparrow \uparrow \\ \Delta[1] \end{array} & \xrightarrow{\eta_{\alpha\beta}} & \begin{array}{c} \vdots \\ \uparrow \uparrow \uparrow \\ \prod_{\alpha\beta} \text{BGL}_1(\mathcal{K} \wedge H\mathbb{R}) \end{array} \\ \begin{array}{c} \uparrow \uparrow \\ \Delta[0] \end{array} & \longrightarrow & \begin{array}{c} \uparrow \uparrow \\ \prod_{\alpha} \text{BGL}_1(\mathcal{K} \wedge H\mathbb{R}) \end{array} \end{array}$$

To see exactly what the maps in this diagram look like, let us choose 1-forms $A_{\alpha\beta}$, defined on intersections of the fixed open cover $\{U_\alpha\}$, which satisfy

$$dA_{\alpha\beta} = B_\alpha - B_\beta,$$

and smooth functions satisfying $df_{\alpha\beta\gamma} = A_{\alpha\beta} + A_{\beta\gamma} - A_{\alpha\gamma}$. A choice of such data defines a representative for the closed 3-form H in the total complex of the Čech-de Rham double complex. Then we have an induced commutative diagram,

$$\begin{array}{ccc} (\Omega_{U_{\alpha\beta}}^*[u, u^{-1}], d) & \xrightarrow{\exp(-B_\alpha)} & (\Omega_{U_{\alpha\beta}}^*[u, u^{-1}], d_H) \\ \exp(dA_{\alpha\beta}) \downarrow & & \swarrow \exp(B_\beta) \\ (\Omega_{U_{\alpha\beta}}^*[u, u^{-1}], d) & & \end{array}$$

and a similar (3–dimensional) diagram for the triple intersections. Now the equivalence $\text{ch}: \mathcal{K} \wedge H\mathbb{R} \rightarrow \mathbb{R}[u, u^{-1}]$ has a pleasant geometric interpretation arising from the fact that K –theory represents isomorphism classes of vector bundles. Indeed, if we take the model for K –theory whose zero space is that of Fredholm operators (see [6]), then a map

$$f: M \rightarrow \mathcal{F}r$$

defines a virtual vector bundle $(E \ominus F) \rightarrow M$ whose fiber at $x \in M$ is given by the virtual difference $\ker(f(x)) - \text{coker}(f(x))$. If we fix curvature 2–forms \mathcal{F} and \mathcal{G} for the bundles E and F , respectively, then the Chern character form

$$\text{ch}(E \ominus F) = \text{tr}\left(\exp\left(\frac{i}{2\pi}\mathcal{F} - \frac{i}{2\pi}\mathcal{G}\right)\right)$$

constitutes a geometric representative for $\text{ch}: \mathcal{K} \wedge H\mathbb{R} \rightarrow H(\Omega^*[u, u^{-1}])$, defined at the level of sheaves of spectra (see [17] for a discussion on the cycle map). In order to spell out the data guaranteed by Theorem 17 in the case of K –theory, let us consider the task of finding a homotopy commutative diagram

$$(3-3) \quad \begin{array}{ccccccc} \mathcal{K} & \longrightarrow & \mathcal{K} \wedge H\mathbb{R} & \xrightarrow{\text{ch}} & H(\Omega_{U_{\alpha\beta}}^*[u, u^{-1}]) & \xrightarrow{e^{-B_\alpha}} & H(\Omega_{U_{\alpha\beta}}^*[u, u^{-1}], d_H) \\ \downarrow \otimes \mathcal{L}_{\alpha\beta} & & \downarrow \otimes \mathcal{L}_{\alpha\beta} \wedge \text{id} & & \downarrow e^{dA_{\alpha\beta}} & & \parallel \\ \mathcal{K} & \longleftarrow & \mathcal{K} \wedge H\mathbb{R} & \xrightarrow{\text{ch}} & H(\Omega_{U_{\alpha\beta}}^*[u, u^{-1}]) & \xrightarrow{e^{-B_\beta}} & H(\Omega_{U_{\alpha\beta}}^*[u, u^{-1}], d_H) \end{array}$$

where $\mathcal{L}_{\alpha\beta}: U_{\alpha\beta} \rightarrow PU(\mathcal{H})$, giving an action on sections $\Gamma(U_{\alpha\beta}; \mathcal{K})$ via¹⁴

$$(E_\alpha \ominus F_\alpha) \mapsto (\mathcal{L}_{\alpha\beta} \otimes E_\beta \ominus \mathcal{L}_{\alpha\beta} \otimes F_\beta),$$

with $E_\alpha \ominus F_\alpha$ and $\mathcal{L}_{\alpha\beta} \otimes E_\beta \ominus \mathcal{L}_{\alpha\beta} \otimes F_\beta$ being virtual difference bundles on double intersections. After choosing connections on each bundle and, using the fact that the Chern character defines a ring homomorphism, we get the following transformation at the level of forms:

$$\begin{aligned} \text{ch}(\mathcal{L}_{\alpha\beta} \otimes E_\beta \ominus \mathcal{L}_{\alpha\beta} \otimes F_\beta) &= \text{ch}(\mathcal{L}_{\alpha\beta}) \wedge \text{ch}(E_\beta) - \text{ch}(\mathcal{L}_{\alpha\beta}) \wedge \text{ch}(F_\beta) \\ &= \text{ch}(\mathcal{L}_{\alpha\beta}) \wedge (\text{ch}(E_\beta) - \text{ch}(F_\beta)). \end{aligned}$$

Here we have $\text{ch}(\mathcal{L}_{\alpha\beta}) = \exp((i/2\pi)f_{\alpha\beta})$ and $dA_{\alpha\beta} = (i/2\pi)f_{\alpha\beta}$ a closed 2–form on $U_{\alpha\beta}$, with $f_{\alpha\beta}$ giving the curvature of $\mathcal{L}_{\alpha\beta}$. Thus, if H represents the curvature of

¹⁴Note that although one can describe the action this way, twisted K –theory classes cannot generally be represented by twisted vector bundles, unless the Dixmier–Douady class corresponding to the full gerbe associated to the $\mathcal{L}_{\alpha\beta}$ is torsion.

a gerbe then we can choose cocycle data

$$\widehat{H} = (g_{\alpha\beta\gamma}, A_{\alpha\beta}, B_\alpha)$$

such that $A_{\alpha\beta}$ defines the connection of a line bundle on intersections, whose transition functions $g_{\alpha\beta\gamma}$ on an intersecting patch $U_\alpha \cap U_\beta \cap U_\gamma$ satisfy the Čech cocycle condition on fourfold intersections. In this case, we are able to construct such a homotopy commutative diagram (3-3) and the forms $A_{\alpha\beta}$ and $g_{\alpha\beta\gamma}$ define the homotopies and higher homotopies, respectively, filling the diagram. We now summarize the above discussion.

Proposition 21 (twisted differential K -theory from gerbe data) *Let $\widehat{h}: M \rightarrow B^2U(1)_\nabla$ be a gerbe on M , with corresponding cocycle data given by the triple $(g_{\alpha\beta\gamma}, A_{\alpha\beta}, B_\alpha)$ satisfying the usual gerbe compatibility. Then \widehat{h} determines a twisted differential K -theory spectrum $\widehat{\mathcal{K}}_{\widehat{h}}$, unique up to a contractible choice.*

3.2 The differential K -theory twisted Chern character

In this section we will consider the effect of rationalization, making contact with information encoded in the differential form part of twisted differential K -theory. To begin with, tensoring with the rationals in the underlying topological K -theory gives the isomorphisms

$$\begin{aligned} (K^0 \otimes \mathbb{Q})(X) &\cong \mathbb{Q} \times \prod_{k>0} H^{2k}(X; \pi_{2k}(\text{BU}) \otimes \mathbb{Q}) \\ &= H^{\text{even}}(X; \mathbb{Q}), \\ (K^1 \otimes \mathbb{Q})(X) &\cong H^{\text{odd}}(X; \mathbb{Q}). \end{aligned}$$

The rationalized Chern character is $\text{ch}_\mathbb{Q}: K^*(X; \mathbb{Q}) \rightarrow H^*(-; \mathbb{Q})$, so that the Chern character can be viewed as the composite map (see [1])

$$K^*(X) \rightarrow K^*(X) \otimes \mathbb{Q} \cong K^*(X; \mathbb{Q}) \xrightarrow{\text{ch}_\mathbb{Q}} H^*(X; \mathbb{Q}),$$

where the isomorphism follows from the fact that \mathbb{Q} is torsion-free. A key ingredient in the identification of the differentials in the spectral sequence for twisted K -theory is the twisted Chern character map

$$\text{ch}_H: K(-; h) \rightarrow H^{\text{even}}(-; H),$$

which is a natural transformation from h -twisted K -theory to the H -twisted de Rham cohomology of a space, with H a differential form representative for the rationalization of h . In order to identify the differentials in the AHSS, we will need a differential

refinement of this map. To this end, let us recall (see [18; 17]) that 2–periodic rational cohomology admits a differential refinement via the homotopy pullback

$$(3-4) \quad \begin{array}{ccc} \widehat{H}\mathbb{Q}[u, u^{-1}] & \longrightarrow & H(\tau_{\geq 0}\Omega^* \otimes \mathbb{R}[u, u^{-1}]) \\ \downarrow & & \downarrow \\ H\mathbb{Q}[u, u^{-1}] & \xrightarrow{i} & H\mathbb{R}[u, u^{-1}] \end{array}$$

where $\tau_{\geq 0}$ denotes the truncation functor which discards all information in negative degrees. This leads to a differential cohomology diamond diagram [67] and, in particular, to an exact sequence

$$0 \rightarrow H^{\text{odd}}(M; \mathbb{R}/\mathbb{Q}) \rightarrow \widehat{H}^{\text{even}}(M; \mathbb{Q}) \rightarrow \Omega_{\text{cl}}^{\text{even}}(M; \mathbb{R}).$$

If we apply the AHSS to this differential spectrum, then we get the following.

Lemma 22 (AHSS for refinement of 2–periodic rational cohomology) *The E_2 –page for refined 2–periodic rational cohomology $\widehat{H}\mathbb{Q}[u, u^{-1}]$ looks as follows:*

$$(3-5) \quad \begin{array}{c} 1 \\ 0 \\ -1 \\ -2 \\ -3 \\ -4 \end{array} \quad \begin{array}{ccc} \bigoplus_{k \in \mathbb{N}} \Omega_{\text{cl}}^{2k}(M) \oplus \mathbb{Q} & \xrightarrow{d_2} & H^2(M; \mathbb{R}/\mathbb{Q}) \\ & \searrow & \downarrow \\ H^1(M; \mathbb{R}/\mathbb{Q}) & \xrightarrow{\quad} & H^3(M; \mathbb{R}/\mathbb{Q}) \\ & \searrow & \downarrow \\ & 0 & \rightarrow 0 \\ & & \searrow \\ & & H^4(M; \mathbb{R}/\mathbb{Q}) \end{array}$$

In particular, the form of the spectral sequence is exactly the same as for \widehat{K} , but with \mathbb{Q} ’s appearing as the quotients instead of \mathbb{Z} ’s. In this case, all the odd differentials vanish, while the effect of the even ones, which were calculated in [32], was shown there to extract the class of the singular cocycle¹⁵

$$\omega \mapsto \left(c \mapsto \oint_c \omega_{2k} \pmod{\mathbb{Q}} \right),$$

¹⁵In [32], the calculation was performed for ordinary differential cohomology, but the same calculation applies to \mathbb{Q} –coefficients.

where $\omega = \omega_0 + \omega_2 + \dots + \omega_{2k} + \dots$. There is a differential refinement of the ordinary Chern character map to

$$\widehat{\text{ch}}: \widehat{K}(-) \rightarrow \widehat{H}^{\text{even}}(-; \mathbb{Q}),$$

where $\widehat{H}^{\text{even}}(-; \mathbb{Q})$ is the cohomology theory represented by the differential spectrum (3-4) (see [32; 15] for details). The induced map at the level of spectral sequences for *untwisted* K -theory does not yield much information since the only nonvanishing differentials are the even ones, which say something about the values that the Chern character can take on cycles. With rational coefficients, this simply says that the Chern character should take rational values, which is already the case.

We now move to the twisted case. Here, we would like to get a differential refinement of the twisted Chern character and see what new conditions arise from the corresponding morphism of AHSSs. To this end, notice that if \widehat{h} is a twist of \widehat{K} , then under the map

$$r: \mathbf{B}^2U(1)_{\nabla} \cong \mathbf{B}^2(\mathbb{R}/\mathbb{Z})_{\nabla} \rightarrow \mathbf{B}^2(\mathbb{R}/\mathbb{Q})_{\nabla}$$

induced by the inclusion $r: \mathbb{Z} \hookrightarrow \mathbb{Q}$, we get a twist of $\widehat{H}^{\text{even}}(-; \mathbb{Q})$. Recall from Theorem 8 that the Čech cocycle data for \widehat{h} gave the correct gluing conditions for the differential spectrum associated with differential K -theory. More precisely, we have a colimiting diagram in $T(\text{Sh}_{\infty}(\text{Mf}))$

$$\begin{array}{ccccc} \cdots & \rightrightarrows & \coprod_{\alpha\beta} \widehat{\mathcal{K}} \times U_{\alpha\beta} & \rightrightarrows & \coprod_{\alpha} \widehat{\mathcal{K}} \times U_{\alpha} & \longrightarrow & \widehat{\mathcal{K}}_{\widehat{h}} \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \rightrightarrows & \coprod_{\alpha\beta} U_{\alpha\beta} & \rightrightarrows & \coprod_{\alpha} U_{\alpha} & \longrightarrow & M \end{array}$$

where the maps in the diagram are completely determined by the cocycle data for the twist $\widehat{h}: M \rightarrow \mathbf{B}^2U(1)_{\nabla}$. Applying the differential Chern character map locally to each restriction $\widehat{\mathcal{K}} \times U_{\alpha_0 \dots \alpha_n}$ and using the fact that $\widehat{H}^{\text{even}}(-; \mathbb{Q})$ is itself twisted by \widehat{h} , we get an induced morphism of bundles,

$$(3-6) \quad \begin{array}{ccc} \widehat{\mathcal{K}}_{\widehat{h}} & \xrightarrow{\widehat{\text{ch}}_{\widehat{h}}} & \widehat{H}\mathbb{Q}[u, u^{-1}]_{\widehat{h}} \\ & \searrow & \swarrow \\ & M & \end{array}$$

Definition 23 (twisted differential Chern character) We define the twisted differential Chern character $\widehat{\text{ch}}_{\widehat{h}}$ as the map in the diagram (3-6).

We would like to get some information about the induced map at the level of spectral sequences. However, we first need to identify what types of differentials arise for $\widehat{H}_{\mathbb{Q}}^{\text{even}}(-; \widehat{h})$. Since this theory refines $H_{\mathbb{Q}}^{\text{even}}(-; h)$, we expect the differentials to be related, and indeed they are. The E_2 -page for $\widehat{H}_{\mathbb{Q}}^{\text{even}}(-; \widehat{h})$ takes the same form as (3-5), with the only difference occurring at the level of the form part in bidegree $(0, 0)$. Here, we have differentials on the E_{2k} -page of the form

$$(3-7) \quad d_{2k}: \Omega_{d_{H\text{-cl}, \mathbb{Z}}}^{\text{even}}(M) \rightarrow H^{2k}(M; \mathbb{R}/\mathbb{Q}),$$

where $\Omega_{d_{H\text{-cl}, \mathbb{Z}}}^{\text{even}}(M)$ is the group of twisted-closed differential forms of even degree, whose degree-zero component is a constant integer-valued function, ie an element $\omega \in \Omega_{d_{H\text{-cl}, \mathbb{Z}}}^{\text{even}}(M)$ satisfies

$$(d + H)(\omega) = d\omega_0 + (d\omega_2 + H \wedge \omega_0) + \dots = 0,$$

and $\omega_0 \in \mathbb{Z}$ is a constant. In the twisted case, the differentials d_{2k} in the spectral sequence are modified from the untwisted case. We will explore this in the following sections.

3.3 Identification of the low-degree differentials

The formula for the first differential d_3 in the twisted AHSS for K -theory can be obtained by observing that the differential defines a cohomology operation and that the only cohomology operations (for spaces equipped with maps to $K(\mathbb{Z}, 3)$) which raise the degree by 3 are given by elements of

$$H^{k+3}(K(\mathbb{Z}, k) \times K(\mathbb{Z}, 3); \mathbb{Z}) \cong H^{p+3}(K(\mathbb{Z}, k); \mathbb{Z}) \oplus H^{p+3}(K(\mathbb{Z}, 3); \mathbb{Z}) \oplus \mathbb{Z},$$

where the third summand of the right-hand side is generated by the product of the generators for $H^k(K(\mathbb{Z}, k); \mathbb{Z})$ and $H^3(K(\mathbb{Z}, 3); \mathbb{Z})$. This implies the formula

$$d_3 = \text{Sq}_{\mathbb{Z}}^3 + (-) \cup \lambda h: H^p(X; \mathbb{Z}) = E_3^{p,q} \rightarrow E_3^{p+3,q-2} = H^{p+3}(X; \mathbb{Z}),$$

where λ is an integer which has yet to be determined. To compute this integer, it is sufficient to consider the spectral sequence on the sphere S^3 , where one computes $\lambda = -1$ (see [9]).

We would like to apply similar reasoning in the differential setting. To do this, however, we should take some care. The AHSS for differential cohomology theory has the additional datum of the level at which we are computing the theory. For example, in the case

of K -theory, we have a different spectral sequence for each of the differential spectra

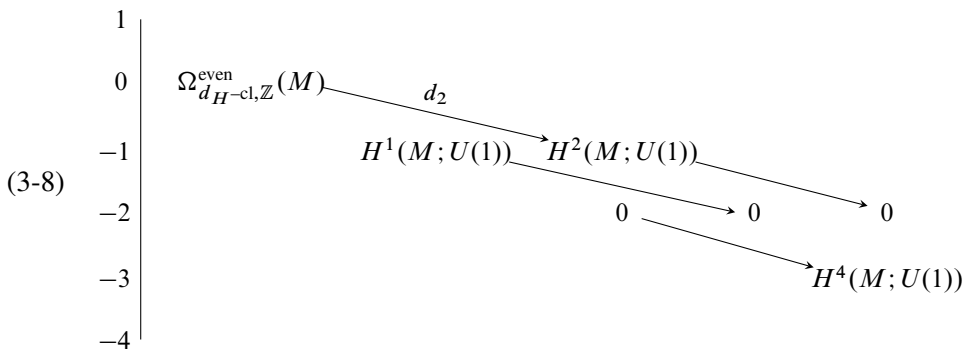
$$\widehat{\mathcal{K}}_{(n)} := (\Sigma^n K, \text{ch}, \mathbb{R}[u, u^{-1}][n]),$$

where we have shifted both the K -theory spectrum and the complex $\mathbb{R}[u, u^{-1}]$ by n . The degree-zero cohomology of this differential spectrum computes the degree n differential K -theory of the underlying manifold. This is not really too surprising since the spectral sequence is based off the Mayer-Vietoris sequence (as is the AHSS for ordinary K -theory). In [17], the Mayer-Vietoris sequence is considered and this sequence depends on the degree of the cohomology theory in question. Therefore, we will have to differentially refine one degree at a time. For illustration, we will consider the case $n = 0$ as this will compute the differential refinement of K^0 . Similar effects hold for K^1 , which we will consider in Section 3.5.

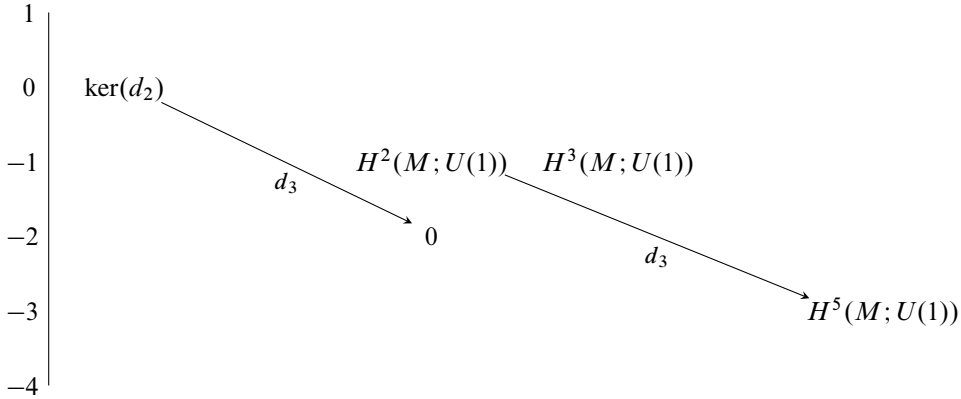
Suppose that we are given a twist $\widehat{h}: M \rightarrow \mathbf{B}^2U(1)_\nabla$ for the theory \widehat{K}^0 . Setting $\widehat{\mathcal{R}} = \widehat{\mathcal{K}}$ in Theorem 18, we observe that the $E_2^{0,0}$ -entry in the spectral sequence can be calculated as

$$H^0(M; \widehat{K}(-; \widehat{h})) = \ker \left\{ \prod_{\alpha} \mathbb{Z} \times \Omega_{>0}^{\text{even}}(U_{\alpha}) \xrightarrow{e^{dA_{\alpha\beta}} - \text{id}} \prod_{\alpha\beta} \mathbb{Z} \times \Omega_{>0}^{\text{even}}(U_{\alpha\beta}) \right\},$$

where the subscript > 0 indicates that we are taking forms with vanishing degree-zero term. Note that this group is isomorphic to the subgroup of twisted closed forms, whose elements are of the type $\omega = \omega_0 + \omega_2 + \dots$, with ω_0 an integer. If the curvature of \widehat{h} is nontrivial in de Rham cohomology, then ω_0 must vanish (otherwise we could trivialize H by the form ω_2/ω_0). In this case, the $E_2^{0,0}$ -entry is precisely the twisted closed forms and the full E_2 -page looks like



while the E_3 -page looks like



This pattern continues and we are led to the following.

Lemma 24 (types of differentials in $\widehat{AHSS}_{\hat{h}}$ for $\widehat{K}^0(-; \hat{h})$) *There are two types of differentials:*

(i) **Obstructions associated to curvature forms** *These are of the form*

$$(3-9) \quad d_{2k}^{curv}: S \subset \Omega_{d_{H-cl}, \mathbb{Z}}^{even}(M) \rightarrow H^{2k}(M; U(1)),$$

where $S = \bigcap \ker(d_{2j})$ for $k < j$, emanating from the entry $E_{2k}^{0,0}$.

(ii) **Obstructions associated to flat classes** *We also have differentials occurring on the odd pages of another type, namely*

$$(3-10) \quad d_{2k+1}^{fl}: H^p(M; U(1)) \rightarrow H^{p+2k+1}(M; U(1)),$$

emanating from the entries $E_{2k+1}^{p, -2q+1}$ with $q \geq 0$.

As the notation suggests, the differentials of the first type give obstructions for the curvature forms to lift to differential K -theory, while the differentials of the second type give rise to obstructions to lifting to flat classes (see eg [44; 35] for a description of such classes).

In [31], we classified all differential cohomology operations by computing the connected components of the mapping space $\text{Map}(\mathbf{B}^n U(1)_{\nabla}, \mathbf{B}^m U(1)_{\nabla})$. We showed that all such operations either arise as an integral multiple of a power of the Dixmier–Douady class (in the sense of [22; 23])

$$DD := \text{id}: \mathbf{B}^n U(1)_{\nabla} \rightarrow \mathbf{B}^n U(1)_{\nabla},$$

or arise from an operation of the form $\phi: H^n(-; \mathbb{Z}) \rightarrow H^m(-; U(1))$, via the natural map $I: \widehat{H}^n(-; \mathbb{Z}) \rightarrow H^n(-; \mathbb{Z})$, and the inclusion $j: H^m(-; U(1)) \rightarrow \widehat{H}^{m+1}(-; \mathbb{Z})$ of flat classes. In particular, we can get transformations of the second type by lifting integral cohomology operations through the Bockstein for the exponential sequence $\beta: H^m(-; U(1)) \rightarrow H^{m+1}(-; \mathbb{Z})$. In fact, such lifts are exactly the refinements of the torsion operations in integral cohomology. This allows us to compute the differentials of the second type (3-10).

Proposition 25 (flat first differential for twisted differential K -theory) *Suppose that $\widehat{h}: M \rightarrow \mathbf{B}^2U(1)_\nabla$ is a twist for differential K -theory. Then we have*

$$d_3^{\text{fl}} = \widehat{\text{Sq}}^3 + \widehat{h} \cup_{\text{DB}} (-),$$

where $\widehat{\text{Sq}}^3$ is the operation $\Gamma_2 \text{Sq}^2 \rho_2 \beta: H^{p-1}(-; U(1)) \rightarrow H^{p+2}(-; U(1))$ and \cup_{DB} is the Deligne–Beilinson cup product.¹⁶

Proof By the characterization in [31], the only operation which can raise the degree by 3 comes from integral multiples of powers of the Dixmier–Douady class or torsion operations in integral cohomology. Moreover, in [32], we showed that the differentials for $\widehat{\text{AHSS}}$ must refine the differentials in the underlying AHSS, in the sense that we have commutative diagrams, one for each p ,

$$\begin{CD} H^{p-1}(M; U(1)) @>\beta>> H^p(M; \mathbb{Z}) \\ @Vd_3^{\text{fl}}VV @VVd_3V \\ H^{p+2}(M; U(1)) @>\beta>> H^{p+3}(M; \mathbb{Z}) \end{CD}$$

This is because the differentials in the underlying theory are given by the cup product with the integral class h and $\beta(\widehat{h} \cup_{\text{DB}} \widehat{\alpha}) = h \cup \beta(\widehat{\alpha})$, for a flat class $\widehat{\alpha}$. In [32], we showed that $\widehat{\text{Sq}}^3$ refines the differential d_3 in untwisted K -theory. Thus, the only possibility is $d_3 = \widehat{\text{Sq}}^3 + \widehat{h} \cup_{\text{DB}} (-)$. □

We now turn our attention to the differentials d_p^{curv} , the first of the two types in Lemma 24. Consider the sheaf of spectra $\tau_{\geq p} \mathbb{Q}[u, u^{-1}] \otimes \Omega^*$, where $\tau_{\geq p} \mathbb{Q}[u, u^{-1}]$ is the truncated complex, with degrees $\leq p$ replaced by zeros. We have natural maps

$$i: \tau_{\geq p} \mathbb{Q}[u, u^{-1}] \hookrightarrow \tau_{\geq p} \mathbb{Q}[u, u^{-1}] \otimes \Omega^* \quad \text{and} \quad j: \Omega^* \hookrightarrow \tau_{\geq p} \mathbb{Q}[u, u^{-1}] \otimes \Omega^*.$$

¹⁶Note that we are considering cohomology with $U(1)$ coefficients as sitting inside differential cohomology by the inclusion j .

We denote the pullback of i by j in sheaves of spectra by $F_p \widehat{H}\mathbb{Q}[u, u^{-1}]$ and we have a canonical map

$$F_p \widehat{H}\mathbb{Q}[u, u^{-1}] \rightarrow \widehat{H}\mathbb{Q}[u, u^{-1}],$$

which is induced by the inclusion $\tau_{\geq p}\mathbb{Q}[u, u^{-1}] \hookrightarrow \mathbb{Q}[u, u^{-1}]$. Since a choice of twist $\widehat{h}: M \rightarrow \mathbf{B}^2(\mathbb{R}/\mathbb{Q})_{\nabla}$ raises the degree, we can consider the complex $\tau_{\geq p}\mathbb{Q}[u, u^{-1}] \otimes \Omega^*$ with differential $d + uH \wedge (-)$,¹⁷ and the bundle of $H(\tau_{\geq p}\mathbb{Q}[u, u^{-1}])$ -spectra $\mathcal{H} \rightarrow M$, induced by the topological twist $h: M \rightarrow K(\mathbb{Q}, 3)$. The local exponential map still gives rise to a twisted Chern character and thus we have an \widehat{h} -twisted theory $F_p \widehat{H}\mathbb{Q}[u, u^{-1}]_{\widehat{h}}$, which comes equipped with a canonical map

$$F_p \widehat{H}\mathbb{Q}[u, u^{-1}]_{\widehat{h}} \rightarrow \widehat{H}\mathbb{Q}[u, u^{-1}]_{\widehat{h}}$$

induced by the inclusion i . This map gives rise to a morphism of spectral sequences which on the E_2 -page determines a commutative diagram,

$$(3-11) \quad \begin{array}{ccc} F_p(\Omega_{d_{H-cl}, \mathbb{Q}}^{\text{even}})(M) & \hookrightarrow & \Omega_{d_{H-cl}, \mathbb{Q}}^{\text{even}}(M) \\ \downarrow d'_p & & \downarrow d_p \\ \ker(d'_{p-1}) & \longrightarrow & \ker(d_{p-1})/\text{im}(d_{p-1}) \end{array}$$

The spectra $F_p \widehat{H}\mathbb{Q}[u, u^{-1}]_{\widehat{h}}$ give rise to a filtration of the spectrum $\widehat{H}\mathbb{Q}[u, u^{-1}]_{\widehat{h}}$, and we also have an exact sequence of sheaves of spectra

$$(3-12) \quad F_{p+1} \widehat{H}\mathbb{Q}[u, u^{-1}]_{\widehat{h}} \rightarrow F_p \widehat{H}\mathbb{Q}[u, u^{-1}]_{\widehat{h}} \rightarrow F_p \widehat{H}\mathbb{Q}[u, u^{-1}]_{\widehat{h}} / F_{p+1} \widehat{H}\mathbb{Q}[u, u^{-1}]_{\widehat{h}}$$

induced by the inclusion $\tau_{\geq p+1}\mathbb{Q}[u, u^{-1}] \hookrightarrow \tau_{\geq p}\mathbb{Q}[u, u^{-1}]$. The rightmost map in (3-12) gives rise to a morphism of spectral sequences and, consequently, to a commutative diagram,¹⁸

$$\begin{array}{ccc} F_p(\Omega_{d_{H-cl}, \mathbb{Q}}^{\text{even}})(M) & \longrightarrow & \Omega_{cl}^p(M) \\ \downarrow d'_p & & \downarrow d''_p \\ \ker(d'_{p-1}) & \hookrightarrow & H^p(M; \mathbb{R}/\mathbb{Q}) \end{array}$$

¹⁷Note that we have chosen to work with chain complexes rather than in cochain complexes, so that differential forms are weighted with negative degrees.

¹⁸This does require a quick calculation, but essentially the point is that the subcomplexes $F_p(\Omega_{d_{H-cl}, \mathbb{Q}}^{\text{even}})(M)$ contain those twisted closed forms of the form $\omega = 0 + 0 + \dots + \omega_p + \omega_{p+2} + \dots$. Thus, at the level of forms, $F_p(\Omega_{d_{H-cl}, \mathbb{Q}}^{\text{even}})(M) / F_{p+1}(\Omega_{d_{H-cl}, \mathbb{Q}}^{\text{even}})(M) \cong \Omega_{cl}^p(M)$.

The differential d_p'' was identified in [32] as the map which takes the class of ω_p in singular cohomology and mods by \mathbb{Q} . Thus we identify the differential d_p' as the map

$$d_p'(0 + 0 + \dots + \omega_p + \omega_{p+2} + \dots) = [\omega_p] \pmod{\mathbb{Q}}.$$

Returning to the diagram (3-11), we see that when restricted to the forms in the p^{th} level of the filtration $F_p(\Omega_{d_{H\text{-cl},\mathbb{Q}}}^{\text{even}}(M))$, the differential d_p must give the class of the leading term, modulo \mathbb{Q} . This gives the following characterization of the differentials d_p^{curv} , at least when restricted to forms in the p^{th} level of the filtration on $(\Omega^*[u, u^{-1}], d_H)$.

Proposition 26 (characterization of the differentials d_p^{curv}) *Let p be an even integer. For the refinement of periodic rational cohomology, the differential d_p^{curv} take the form*

$$d_p: \Omega_{d_{H\text{-cl},\mathbb{Q}}}^{\text{even}}(M) \rightarrow H^p(M; \mathbb{R}/\mathbb{Q}),$$

where $\Omega_{d_{H\text{-cl},\mathbb{Q}}}^{\text{even}}(M)$ is the subgroup of twisted closed forms with degree-zero term ω_0 given by a constant function taking values in \mathbb{Q} . Moreover, the differential d_p maps a twisted closed form of the type $\omega = 0 + 0 + \dots + \omega_p + \omega_{p+2} + \dots$ to the class of the leading term ω_p , modulo \mathbb{Q} , ie

$$d_p(0 + 0 + \dots + \omega_p + \omega_{p+2} + \dots) = [\omega_p] \pmod{\mathbb{Q}}.$$

Remark Identifying the differentials d_p^{curv} in full generality seems to be complicated. The structure appears to be related to lifting the terms of the locally defined closed form $e^{-B\alpha} \wedge \omega$ to a flat bundle, using the gerbe data for the twist \hat{h} , but we are unsure of how to present the differential in such a way that the lifting condition is useful in practice (ie a checkable condition on the twisted forms such that they lift to curvatures of twisted differential K -theory). Moreover, we have only discussed the rational case here and the situation for twisted differential K -theory seems to be further complicated by the presence of torsion. Since we believe such a presentation exists and would be extremely useful in applications, we save this for future investigation.

3.4 The higher flat differentials

In [9], Atiyah and Segal considered the spectral sequence for the twisted de Rham complex $(\Omega^*[u, u^{-1}](M), d_H)$ on a smooth manifold M . This complex admits a filtration¹⁹ with the p^{th} level $F_p(\Omega^*[u, u^{-1}](M), d_H)$ given by the complex which in degree k has elements of the form

$$\omega = 0 + 0 + \dots + \omega_p + \omega_{p+2} + \dots.$$

¹⁹Note that this is the same filtration we considered in the previous section.

The spectral sequence with respect to this filtration has E_2 -page given by the ordinary de Rham cohomology groups

$$E_2^{p,q} = H_{\text{dR}}^{p+q}(M).$$

Atiyah and Segal showed that the differentials d_{2k+1} in the spectral sequence for the periodic complex of forms are given by the Massey product operation $\langle H, H, \dots, H, x \rangle$, where H is repeated k times.

We highlight that the Massey products are related to solving the twisted closed equation $dx = H \wedge x$ in the following way. Let x_2 be a degree-two de Rham class and consider the Massey triple product $\langle H, H, x_2 \rangle$. A representative for the product is constructed by trivializing each of the wedge products $H \wedge H = 0$ and $H \wedge x_2$. Clearly, we can choose the zero form as a trivialization of the first product. Assuming that the second product vanishes in cohomology, then we can find x_4 such that $dx_4 = -H \wedge x_2$. In this case, we have the representative of the Massey triple product

$$\langle H, H, x_2 \rangle = H \wedge x_4,$$

which is a class in cohomology of degree seven. In general, there is some ambiguity in defining the Massey product coming from the choice of x_4 . This ambiguity is measured precisely by classes in $[H] \wedge H_{\text{dR}}^4(M)$. Thus, the product is well defined in the quotient

$$d_5[x_2] = -\langle H, H, x_2 \rangle \in H_{\text{dR}}^7(M)/[H] \wedge H_{\text{dR}}^4(M),$$

and this is precisely the quotient one finds on the E_5 -page of the spectral sequence. Thus, if d_5 vanishes on $[x_2]$, then there is an x_6 such that $H \wedge x_4 = dx_6$ and we can form the Massey product $H \wedge x_6$. Again the ambiguity is measured by the subgroup $[H] \wedge H_{\text{dR}}^6(M)$. If we iterate the process, we find that we are actually asking for a solution to the twisted equation $dx_{2k+2} = -H \wedge x_{2k}$.

We now want to find the relationship between the Massey products on the higher differentials in the spectral sequence for twisted differential K -theory. In [30] we established the basic theory of differential Massey products. Using these, we have the following.

Proposition 27 (the higher differentials in $\widehat{\text{AHSS}}_{\hat{h}}$ of $\widehat{K}^0(-; \hat{h})$) *Suppose that $\hat{h}: M \rightarrow \mathbf{B}^2U(1)_{\nabla}$ is a twist for differential K -theory, regarded as a twist for periodic rational cohomology via the differential Chern character map (see Section 3.2). Then the differentials d_{2k+1} can be identified with the differential Massey product operation*

$$d_{2k+1} = -\underbrace{\langle \hat{h}, \hat{h}, \dots, \hat{h}, - \rangle}_{k \text{ times}}.$$

Proof Consider the shifted twisted theory $H(\mathbb{R}[u, u^{-1}][1])_h$ (ie the shift of periodic real cohomology). By restriction, \hat{h} twists the underlying flat theory for $\widehat{H}(\mathbb{Q}[u, u^{-1}])$, which is equivalent to the shifted spectrum $H(\mathbb{R}/\mathbb{Q}[u, u^{-1}][1])$. The equivalence allows us to twist periodic \mathbb{R}/\mathbb{Q} -cohomology by \hat{h} . The mod \mathbb{Q} map thus gives a well-defined morphism

$$H(\mathbb{R}[u, u^{-1}][1])_h \rightarrow H(\mathbb{R}/\mathbb{Q}[u, u^{-1}][1])_{\hat{h}},$$

which induces a morphism of corresponding spectral sequences. At the level of spectral sequences, the shift of $H(\mathbb{R}[u, u^{-1}])_h$ manifests itself by simply shifting the entries on each page. By Atiyah and Segal [9, Proposition 6.1], the differentials for this spectral sequence are the Massey products

$$d_{2k+1}[x] = -\underbrace{\langle h, h, \dots, [x] \rangle}_{k \text{ times}}.$$

Since we have a morphism of spectral sequences, it follows that the differentials d'_{2k+1} on the $(2k+1)$ -page of the spectra sequence for $H(\mathbb{R}/\mathbb{Q}[u, u^{-1}][1])_{\hat{h}}$ refine the Massey product operation in that we have a commutative diagram

$$\begin{CD} H^p(M; \mathbb{R}) @>\text{mod } \mathbb{Q}>> H^p(M; \mathbb{R}/\mathbb{Q}) \\ @VV-\langle h, h, \dots, - \rangle V @VVd'_{2k+1} V \\ H^{p+2k+1}(M; \mathbb{R}) @>\text{mod } \mathbb{Q}>> H^{p+2k+1}(M; \mathbb{R}/\mathbb{Q}) \end{CD}$$

Using the formula for the Deligne–Beilinson cup product on flat classes (see [22] or expression (3-15)), it is easy to see that the differential Massey product operation $-\langle \hat{h}, \hat{h}, \dots, - \rangle$ indeed refines the underlying Massey product.

We claim that this is the only possible operation. Indeed, let

$$(3-13) \quad \phi: H^p(M; \mathbb{R}/\mathbb{Q}) \rightarrow H^{p+2k+1}(M; \mathbb{R}/\mathbb{Q})$$

be a natural operation on manifolds equipped with maps $M \rightarrow \mathbf{B}^2\mathbb{R}/\mathbb{Q}_\nabla$ refining the Massey product operation in cohomology with real coefficients. Then, from the diagram, the difference $\phi + \langle \hat{h}, \hat{h}, \dots, - \rangle$, when restricted to $H^p(M; \mathbb{R})$, must vanish. But the mod \mathbb{Q} map is surjective since the Bockstein

$$H^p(M; \mathbb{R}/\mathbb{Q}) \xrightarrow{\beta_{\mathbb{Q}}} H^{p+1}(M; \mathbb{Q}) \hookrightarrow H^{p+1}(M; \mathbb{R})$$

vanishes. Thus, $\phi = -\langle \hat{h}, \hat{h}, \dots, - \rangle$ on $H^p(M; \mathbb{R}/\mathbb{Q})$. □

We will now show that the corresponding Massey products in the topological and the differential cases are compatible under the composite map

$$(3-14) \quad \varphi: H_{\text{dR}}^{*-1}(-; \mathbb{R}) \xrightarrow{\text{mod } \mathbb{Q}} H^{*-1}(-; \mathbb{R}/\mathbb{Q}) \xrightarrow{j} \widehat{H}^*(-; \mathbb{Q}).$$

Lemma 28 (correspondence of Massey products) *Let $\widehat{h}: M \rightarrow \mathbf{B}^n U(1)_{\nabla}$ be a differential cohomology class with curvature H and let ω be a degree m closed differential form on M . If the Massey product is defined, then, for each representative*

$$x \in \underbrace{\langle H, H, \dots, H, \omega \rangle}_{k \text{ times}} \subset H_{\text{dR}}^{m+k(n-1)+1}(M),$$

the map (3-14) gives a corresponding representative

$$\varphi(x) \in \underbrace{\langle \widehat{h}, \widehat{h}, \dots, \widehat{h}, \varphi(\omega) \rangle}_{k \text{ times}} \subset \widehat{H}^{m+k(n-1)+2}(M; \mathbb{Q}).$$

Proof Recall that we have a formula for the Deligne–Beilinson cup product at the level of cochains with values in the Deligne complex with \mathbb{Q} -coefficients. At the level of local sections $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_0, \beta_1, \dots, \beta_m)$, this product takes the form (see [22; 32])

$$(3-15) \quad \alpha \cup_{\text{DB}} \beta = \begin{cases} \alpha_0 \beta_\ell & \text{if } \ell \neq m, \\ \alpha_p \wedge d\beta_m & \text{if } p \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

If ω is a globally defined closed form, then $\varphi([\omega])$ is the class of the cocycle in Deligne cohomology $\varphi(\omega) = (0, 0, \dots, \omega)$. On \widehat{h} the above formula gives

$$\varphi(\omega) \cup_{\text{DB}} \widehat{h} = \varphi(\omega \wedge H).$$

Since the DB cup product defines a graded commutative product at the level of cohomology, the above cocycle represents the class $\varphi([\omega]) \cup_{\text{DB}} \widehat{h}$. Since φ arose from a morphism of DGAs, it follows immediately that φ must send a defining system for the Massey product to a defining system for the differential Massey product. □

3.5 Examples

The basic properties listed above are enough to get started with computations. We illustrate with the example of the 3–sphere S^3 , which generalizes to differential twisted K -theory the corresponding twisted K -theory calculation in [48] (after MJ Hopkins)

and [12; 13; 21; 26; 49]. The twisted differential K -theory of the Lie group $SU(2)$ (and also $SU(3)$) was studied in [19] using index and group theoretic methods.

Let $h: S^3 \rightarrow K(\mathbb{Z}, 3)$ be a map representing an integer (which we also denote by h) in $H^3(S^3; \mathbb{Z}) \cong \mathbb{Z}$. We denote the corresponding twisted K -theory on S^3 by $K^*(S^3; h)$. Now the map h can be refined to a gerbe with connection $\hat{h}: S^3 \rightarrow \mathbf{B}^2U(1)_\nabla$, with curvature form H . Let $(\Omega^*[u, u^{-1}](S^3), d_H)$ denote the periodic, H -twisted de Rham complex on S^3 with differential $d + H \wedge (-)$. Thus, the triple $\hat{\mathcal{K}}_{\hat{h}} := (\mathcal{K}_h, \text{ch}, (\Omega^*[u, u^{-1}], d_H))$ gives the data of a differential refinement of an h -twisted K -theory spectrum. We denote the underlying theory by $\hat{K}^*(S^3; \hat{h})$.

Proposition 29 (twisted differential K^0 of the 3-sphere) *We have*

$$\hat{K}^0(S^3; \hat{h}) \cong \Omega_{\text{cl}}^2(S^3) = d\Omega^1(S^3).$$

Proof Consider the cover $\{U, V\}$, given by removing the north and south poles of S^3 , respectively. Then the Mayer-Vietoris sequence from Proposition 5(iii) gives an exact sequence

$$(3-16) \quad K_{\mathbb{R}/\mathbb{Z}}^{-2}(S^2; \hat{h}) \rightarrow \hat{K}^0(S^3; \hat{h}) \rightarrow \hat{K}^0(U; \hat{h}) \oplus \hat{K}^0(V; \hat{h}) \xrightarrow{f} \hat{K}^0(U \cap V; \hat{h}) \rightarrow K^1(S^3; h).$$

The twist trivializes on each local patch U and V . Moreover, since the restriction of h to $U \cap V \simeq S^2$ trivializes by dimension, the twisted differential K -theory of $U \cap V$ also reduces to the untwisted differential K -theory. Our job is now reduced to calculating the untwisted groups and understanding the maps in (3-16). We have

$$(3-17) \quad \hat{K}^0(U) \cong \mathbb{Z} \oplus \Omega_{\text{cl}}^{\text{even}}(U),$$

$$(3-18) \quad \hat{K}^0(V) \cong \mathbb{Z} \oplus \Omega_{\text{cl}}^{\text{even}}(V),$$

$$(3-19) \quad \hat{K}^0(U \cap V) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus d\Omega^{\text{odd}}(U \cap V),$$

where the first factor of \mathbb{Z} in (3-19) is generated by 1 and the second factor is generated by the canonical line bundle L on S^2 . Now the maps in the sequence depend on the twist because the identification of the twisted theory with the untwisted theory depends on a choice of trivialization, and that trivialization is not canonical (ie it depends on the U and V). The transition maps between the trivializations give the correction from the untwisted case.

Notice that the factor $\mathbb{Z} \oplus \mathbb{Z}$ in the last group (3-19) corresponds to the \mathbb{Z} -linear span of each of the basis elements $\{1, L\}$ in $K(S^2) \cong \mathbb{Z}[L]/(1-L)^2$. The copies of \mathbb{Z} in the

previous two equations (3-17) and (3-18) correspond to the \mathbb{Z} -linear span of the trivial bundle. Topologically, the twist h restricts on the intersection $U \cap V \simeq S^2$ to a cocycle representing L . Geometrically, the group structure on $H^3(S^3; \mathbb{Z}) \cong \mathbb{Z}$ corresponds to the group generated by L under tensor product of line bundles. The action of the transition functions for the local trivializations corresponds to multiplication by $L^{\otimes h}$, which, when written in the basis $\{1, L\}$, is $(h - 1)1 - hL$. Thus, if we restrict the map f in (3-16) to the copies of \mathbb{Z} , we get the map

$$\begin{pmatrix} 1 & h - 1 \\ 0 & -h \end{pmatrix}: \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}.$$

This map is coming from the topological part of twisted K -theory.

We still need to identify the geometric part. To this end, recall that we also have local trivializations for the twisted complex of forms, coming from the formal exponentials e^{B_U} and e^{B_V} , where B_U and B_V are local potentials (ie trivialization) for H . Using the transition data e^{dA} on intersections (ie A is a 1-form on $U \cap V$ such that $dA = B_U - B_V$), one sees that the map f in the sequence (3-16) can be expressed fully as the block matrix

$$f = \begin{pmatrix} 1 & h - 1 & 0 & 0 \\ 0 & -h & 0 & 0 \\ 0 & 0 & 1 & -e^{dA} \end{pmatrix}: \mathbb{Z} \oplus \mathbb{Z} \oplus \Omega_{\text{cl}}^{\text{even}}(U) \oplus \Omega_{\text{cl}}^{\text{even}}(V) \rightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus d\Omega^{\text{odd}}(U \cap V).$$

The upper-left block coming from the topological part is easily seen to be injective when $h \neq 0$. Thus, we need only understand the lower-right block $(1, -e^{dA})$. Let $\omega = \omega_0 + \omega_2$ and $v = v_0 + v_2$ be a pair of closed forms on U and V , respectively.²⁰ Expanding $e^{dA} = 1 + dA$ on $U \cap V$, we see that the map $(1, -e^{dA})$ is zero precisely if the equations

$$\omega_0 = v_0 \quad \text{and} \quad \omega_2 = v_2 - dA \wedge v_0$$

are satisfied. Notice that if H is not exact and $v_0 \neq 0$, then no such solutions exist. Indeed, since ω and v are closed, v_0 is constant and the second equation implies that $\omega_2/v_0 - v_2/v_0 = dA$. On the other hand, there are local potentials B_U and B_V for H on U and V which satisfy $B_U - B_V = dA$. Combining these equations, we have $\omega_2/v_0 - B_U = v_2/v_0 - B_V$ on $U \cap V$. By the sheaf condition for differential forms, there is a global form η such that $\eta_U = \omega_2/v_0 - B_U$ and $\eta_V = v_2/v_0 - B_V$. But this means $d\eta = H$. Thus, if H represents a nonzero cohomology class, then $v_0 = \omega_0 = 0$

²⁰Recall that U and V have dimension 3, so there are no higher-degree forms in ω and v .

and $\omega_2 = v_2$. By gluing, the pair (ω_2, v_2) are the restrictions of a global closed 2–form on S^3 . We now have an exact sequence

$$(3-20) \quad K_{\mathbb{R}/\mathbb{Z}}^{-2}(U; h) \oplus K_{\mathbb{R}/\mathbb{Z}}^{-2}(V; h) \xrightarrow{g} K_{\mathbb{R}/\mathbb{Z}}^{-2}(S^2; h) \rightarrow \widehat{K}^0(S^3; \widehat{h}) \rightarrow \Omega_{\text{cl}}^2(S^3) \rightarrow 0.$$

Turning our attention to the map g , we again observe that since the twist h trivializes on S^2 , the Bockstein sequence for K –theory with coefficients, along with periodicity, gives an identification $K_{\mathbb{R}/\mathbb{Z}}^{-2}(S^2; h) \cong \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z}$. As we have already observed, the map is induced by restrictions and the twist, which in this case gives the map

$$\begin{pmatrix} 1 & h-1 \\ 0 & -h \end{pmatrix}: \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z},$$

where the matrix is to be understood as the map of \mathbb{Z} –modules which sends a pair $(\theta, \varphi) \in \mathbb{R}/\mathbb{Z}$ to $(\theta(h-1)\varphi, -h\varphi) \in \mathbb{R}/\mathbb{Z}$. This map is surjective and hence the sequence (3-20) reduces to

$$(3-21) \quad 0 \rightarrow \widehat{K}^0(S^3; \widehat{h}) \rightarrow \Omega_{\text{cl}}^2(S^3) \rightarrow 0,$$

and we have an isomorphism $\widehat{K}^0(S^3; \widehat{h}) \cong \Omega_{\text{cl}}^2(S^3)$. □

Proposition 30 (twisted differential K^1 of the 3–sphere) *We have*

$$\widehat{K}^1(S^3; \widehat{h}) \cong \Omega_{d_H\text{-cl}}^{\text{odd}}(S^3) \oplus \mathbb{Z}/h,$$

where $d_H\text{-cl}$ indicates closed forms with respect to the twisted de Rham differential d_H and h is the integer multiple of the generator of $H^3(S^3; \mathbb{Z}) \cong \mathbb{Z}$ corresponding to the topological twist.

Proof As in the case for degree zero above, we consider the portion of the Mayer–Vietoris sequence

$$(3-22) \quad K_{\mathbb{R}/\mathbb{Z}}^{-1}(S^2; h) \rightarrow \widehat{K}^1(S^3; \widehat{h}) \rightarrow \widehat{K}^1(U; \widehat{h}) \oplus \widehat{K}^1(V; \widehat{h}) \xrightarrow{f} \widehat{K}^1(U \cap V; \widehat{h}) \rightarrow K^2(S^3; h).$$

By periodicity, this immediately reduces to

$$(3-23) \quad 0 \rightarrow \widehat{K}^1(S^3; \widehat{h}) \rightarrow \widehat{K}^1(U; \widehat{h}) \oplus \widehat{K}^1(V; \widehat{h}) \xrightarrow{f} \widehat{K}^1(U \cap V; \widehat{h}) \rightarrow 0.$$

As before, we arrive at the identifications

$$\begin{aligned} \widehat{K}^1(U) &\cong \mathbb{R}/\mathbb{Z} \oplus \Omega_{\text{cl}}^{\text{odd}}(U), \\ \widehat{K}^1(V) &\cong \mathbb{R}/\mathbb{Z} \oplus \Omega_{\text{cl}}^{\text{odd}}(V), \\ \widehat{K}^1(U \cap V) &\cong \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z} \oplus d\Omega^{\text{even}}(U \cap V), \end{aligned}$$

and we identify the map f with

$$f = \begin{pmatrix} 1 & h-1 & 0 & 0 \\ 0 & -h & 0 & 0 \\ 0 & 0 & 1 & -e^{dA} \end{pmatrix} : \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z} \oplus \Omega_{\text{cl}}^{\text{odd}}(U) \oplus \Omega_{\text{cl}}^{\text{odd}}(V) \rightarrow \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z} \oplus d\Omega^{\text{even}}(U \cap V).$$

In f , the upper-left block matrix

$$(3-24) \quad \begin{pmatrix} 1 & h-1 \\ 0 & -h \end{pmatrix} : \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z}$$

is again understood as the map of \mathbb{Z} -modules which sends a pair $(\theta, \varphi) \in \mathbb{R}/\mathbb{Z}$ to $(\theta(h-1)\varphi, -h\varphi) \in \mathbb{R}/\mathbb{Z}$. Unlike in the degree-zero case, however, now the kernel of the map (3-24) is nonzero since we are taking \mathbb{R}/\mathbb{Z} -coefficients. The requirement $-h\varphi \in \mathbb{Z}$ implies that (identifying $\mathbb{R}/\mathbb{Z} \cong U(1)$) φ is an h -root of unity. On the other hand, this combined with the second requirement $(1-h)\theta\varphi \in \mathbb{Z}$ implies that $\theta \in \mathbb{Z}$. Hence we identify the kernel with the subgroup isomorphic to the image of the map

$$\mathbb{Z}/h \xrightarrow{\times \frac{1}{h}} \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z},$$

sending $a \mapsto (1, a/h)$. Turning to the lower-left block matrix $(1, -e^{dA})$ in f , we consider a pair of odd forms $\omega = \omega_1 + \omega_3$ and $v = v_1 + v_3$ on U and V , respectively. Again, writing $e^{dA} = 1 + dA$, we identify the elements in the kernel as those forms satisfying the equations

$$\omega_1 = v_1 \quad \text{and} \quad \omega_3 = v_3 - dA \wedge v_1.$$

Again we see that globally defined closed 3-forms give solutions (setting $v_1 = 0$). However we also have other solutions, even when H is nontrivial in cohomology. The resulting forms are the twisted-closed odd forms on S^3 . □

Although the Mayer-Vietoris sequence provides a powerful tool for calculating twisted differential K -theory, the $\widehat{\text{AHSS}}_{\hat{h}}$ can simplify some of these calculations drastically. We illustrate the power of the spectral sequence in the following.

Example (the 3-sphere, revisited) Let $\hat{h}: S^3 \rightarrow \mathbf{B}^2U(1)_{\nabla}$ be a twist. Recall from Proposition 25 that we identified the differential on the E_3 -page as $d_3 = \widehat{\text{Sq}}_{\mathbb{Z}}^3 + \hat{h} \cup_{\text{DB}} (-)$. For the 3-sphere, the $U(1)$ -cohomology is calculated from the exponential sequence as

$$H^2(S^3; U(1)) \cong 0 \quad \text{and} \quad H^3(S^3; U(1)) \cong U(1).$$

Then for \widehat{K}^0 , we see that all relevant differentials must vanish and the spectral sequence collapses at the E_2 -page in the diagram (3-8). There is no extension problem in this case, and we have

$$\widehat{K}^0(S^3; \widehat{h}) \cong \Omega_{d_H\text{-cl}}^{\text{even}}(S^3) \cong \Omega_{\text{cl}}^2(S^3).$$

For \widehat{K}^1 , we need to calculate the kernel of the map

$$d_3: U(1) \rightarrow U(1) \cong H^3(S^3; U(1)).$$

For degree reasons, $\widehat{Sq}_{\mathbb{Z}}^3$ vanishes on $U(1)$ and we are reduced to finding the kernel of \widehat{h} . To this end, recall that the formula for the Deligne–Beilinson cup product sends an element in $U(1)$, written as $e^{2\pi i\theta}$, to $e^{2\pi ih\theta}$, where h is the integer representing the underlying topological twist. The kernel is thus easily identified with the h -roots of unity, which as an abelian group is isomorphic to $\mathbb{Z}/h\mathbb{Z}$. For degree reasons, there are no nontrivial differentials out of the term $(\Omega^*(S^3), d_H)_{\text{cl}}$. In this case, there is no extension problem and we recover the differential cohomology group, as in Proposition 30,

$$\widehat{K}^1(S^3; \widehat{h}) \cong \mathbb{Z}/h\mathbb{Z} \oplus \Omega_{d_H\text{-cl}}^{\text{odd}}(S^3).$$

Generalizations to higher chromatic levels and higher spheres are studied in [64], where twisted Morava K -theory of spheres is calculated. We plan to consider explicit refinements of those twisted theories elsewhere. We also plan to address many applications in a separate treatment, including the relation to classes in twisted Deligne cohomology [33], as well as implications for physics.

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