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Coproducts in brane topology

SHUN WAKATSUKI

We extend the loop product and the loop coproduct to the mapping space from the k–dimensional sphere, or more generally from any k–manifold, to a k–connected space with finite-dimensional rational homotopy group for $k \geq 1$. The key to extending the loop coproduct is the fact that the embedding $M \to M^{S^{k-1}}$ is of "finite codimension" in the sense of Gorenstein spaces. Moreover, we prove the associativity, commutativity and Frobenius compatibility of them.

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1 Introduction

Chas and Sullivan [1] introduced the loop product on the homology $H_*(LM)$ of the free loop space $LM = \mathrm{Map}(S^1, M)$ of a manifold. Cohen and Godin [2] extended this product to other string operations, including the loop coproduct.

Generalizing these constructions, Félix and Thomas [6] defined the loop product and coproduct in the case M is a Gorenstein space. A Gorenstein space is a generalization of a manifold in the point of view of Poincaré duality, including the classifying space of a connected Lie group and the Borel construction of a connected oriented closed manifold and a connected Lie group. But these operations tend to be trivial in many cases. Let $\mathbb{K}$ be a field of characteristic zero. For example, Tamanoi showed that the loop coproduct is trivial for a manifold with the Euler characteristic zero in [10, Corollary 3.2], and that the composition of the loop coproduct followed by the loop product is trivial for any manifold in [10, Theorem A]. Similarly, Félix and Thomas [6, Theorem 14] proved that the loop product over $\mathbb{K}$ is trivial for the classifying space of a connected Lie group. A space with the nontrivial composition of loop coproduct and product is not found.

On the other hand, Sullivan and Voronov generalized the loop product to the sphere space $L^k M = \mathrm{Map}(S^k, M)$ for $k \geq 1$. This product is called the *brane product*. See Cohen, Hess and Voronov [3, Part I, Chapter 5].

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In this article, we will generalize the loop coproduct to sphere spaces, to construct nontrivial and interesting operations. We call this coproduct the *brane coproduct*.

Here, we review briefly the construction of the loop product and the brane product. For simplicity, we assume M is a connected oriented closed manifold of dimension m. The loop product is constructed as a mixture of the Pontrjagin product $H_*(\Omega M \times \Omega M) \to H_*(\Omega M)$ defined by the composition of based loops and the intersection product $H_*(M \times M) \to H_{*-m}(M)$. More precisely, we use the diagram

$$\begin{array}{ccccc} LM \times LM & \xleftarrow{\text{incl}} & LM \times_M LM & \xrightarrow{\text{comp}} & LM \\ {\scriptstyle \mathrm{ev}_1 \times \mathrm{ev}_1}\downarrow & & \downarrow & & \\ M \times M & \xleftarrow{\quad \Delta \quad} & M & & \end{array} \tag{1.1}$$

Here, the square is a pullback diagram by the diagonal map Δ and the evaluation map ev_1 at 1, identifying S^1 with the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$, and comp is the map defined by the composition of loops. Since the diagonal map $\Delta\colon M \to M \times M$ is an embedding of finite codimension, we have the shriek map $\Delta^!\colon H_*(M \times M) \to H_{*-m}(M)$, which is called the intersection product. Using the pullback diagram, we can "lift" $\Delta^!$ to $\mathrm{incl}^!\colon H_*(LM \times LM) \to H_{*-m}(LM \times_M LM)$. Then, we define the loop product to be the composition $\mathrm{comp}_* \circ \mathrm{incl}^!\colon H_*(LM \times LM) \to H_{*-m}(LM)$.

The brane product can be defined by a similar way. Let k be a positive integer. We use the diagram

$$\begin{array}{ccccc} L^kM \times L^kM & \xleftarrow{\text{incl}} & L^kM \times_M L^kM & \xrightarrow{\text{comp}} & L^kM \\ \downarrow & & \downarrow & & \\ M \times M & \xleftarrow{\quad \Delta \quad} & M & & \end{array}$$

Since the base map of the pullback diagram is the diagonal map Δ, which is the same as that for the loop product, we can use the same method to define the shriek map $\mathrm{incl}^!\colon H_*(L^kM \times L^kM) \to H_{*-m}(L^kM \times_M L^kM)$. Hence we define the brane product to be the composition $\mathrm{comp}_* \circ \mathrm{incl}^!\colon H_*(L^kM \times L^kM) \to H_{*-m}(L^kM)$.

Next, we review the loop coproduct. Using the diagram

$$\begin{array}{ccccc} LM & \xleftarrow{\text{comp}} & LM \times_M LM & \xrightarrow{\text{incl}} & LM \times LM \\ {\scriptstyle \mathrm{ev}_1 \times \mathrm{ev}_{-1}}\downarrow & & \downarrow & & \\ M \times M & \xleftarrow{\quad \Delta \quad} & M & & \end{array} \tag{1.2}$$

we define the loop coproduct to be the composition $\mathrm{incl}_* \circ \mathrm{comp}^!\colon H_*(LM) \to H_{*-m}(LM \times LM)$.

But the brane coproduct cannot be defined in this way. To construct the brane coproduct, we have to use the diagram

$$\begin{array}{ccccc} L^kM & \xleftarrow{\text{comp}} & L^kM \times_M L^kM & \xrightarrow{\text{incl}} & L^kM \times L^kM \\ {\scriptstyle\text{res}}\downarrow & & \downarrow & & \\ L^{k-1}M & \xleftarrow{\quad c \quad} & M & & \end{array}$$

Here, $c\colon M \to L^{k-1}M$ is the embedding by constant maps and $\mathrm{res}\colon L^kM \to L^{k-1}M$ is the restriction map to S^{k-1}, which is embedded to S^k as the equator. In the usual sense, the base map c is not an embedding of finite codimension. But, using the algebraic method of Félix and Thomas [6], we can consider this map as an embedding of codimension $\overline{m} = \dim \Omega^{k-1}M$, which is defined as a finite integer when the iterated loop space $\Omega^{k-1}M$ is a $\mathbb{K}$–Gorenstein space. Hence, under this assumption, we have the shriek map $c^!\colon H_*(L^{k-1}M) \to H_{*-\overline{m}}(M)$ and the lift $\mathrm{comp}^!\colon H_*(L^kM) \to H_{*-\overline{m}}(L^kM \times_M L^kM)$. This enables us to define the brane coproduct to be the composition $\mathrm{incl}_* \circ \mathrm{comp}^!\colon H_*(L^kM) \to H_{*-\overline{m}}(L^kM \times L^kM)$.

Note that, if $\bigoplus_n \pi_n(M) \otimes \mathbb{K}$ is of finite dimension, then $\Omega^{k-1}M$ is a $\mathbb{K}$–Gorenstein space by a result of Félix, Halperin and Thomas [4, Proposition 3.4]; see also our Proposition 2.2. The converse also holds when $k \geq 2$ by [4, Proposition 1.7].

More generally, using connected sums, we define the product and coproduct for mapping spaces from manifolds. Let S and T be manifolds of dimension k. Let M be a k–connected $\mathbb{K}$–Gorenstein space of finite type. Denote $m = \dim M$. Then we define the (S,T)*–brane product*

$$\mu_{ST}\colon H_*(M^S \times M^T) \to H_{*-m}(M^{S\#T})$$

using the diagram

$$\begin{array}{ccccc} M^S \times M^T & \xleftarrow{\text{incl}} & M^S \times_M M^T & \xrightarrow{\text{comp}} & M^{S\#T} \\ \downarrow & & \downarrow & & \\ M \times M & \xleftarrow{\quad \Delta \quad} & M & & \end{array} \tag{1.3}$$

Assume that M is k–connected and $\Omega^{k-1}M$ is a Gorenstein space (or, equivalently, $\bigoplus_n \pi_n(M) \otimes \mathbb{K}$ is of finite dimension). Denote $\overline{m} = \dim \Omega^{k-1}M$. Then we define the (S,T)–*brane coproduct*

$$\delta_{ST}\colon H_*(M^{S\#T}) \to H_{*-\overline{m}}(M^S \times M^T)$$

using the diagram

$$\text{(1.4)}\qquad \begin{array}{ccccc} M^{S\#T} & \xleftarrow{\text{comp}} & M^S \times_M M^T & \xrightarrow{\text{incl}} & M^S \times M^T \\ \downarrow & & \downarrow & & \\ L^{k-1}M & \xleftarrow{c} & M & & \end{array}$$

Note that, if we take $S = T = S^k$, then μ_{ST} and δ_{ST} are the brane product and coproduct, respectively.

Next, we study some fundamental properties of the brane product and coproduct. For the loop product and coproduct on Gorenstein spaces, Naito [9] showed their associativity and the Frobenius compatibility. In this article, we generalize them to the case of the brane product and coproduct. Moreover, we show the commutativity of the brane product and coproduct, which was not known even for the case of the loop product and coproduct on Gorenstein spaces.

Theorem 1.5 *Let M be a k–connected space such that $\Omega^{k-1}M$ is a Gorenstein space. Then the above product and coproduct satisfy the following properties*:

(1) *The product is associative and commutative.*

(2) *The coproduct is associative and commutative.*

(3) *The product and coproduct satisfy the Frobenius compatibility.*

In particular, if we take $S = T = S^k$, the shifted homology $\mathbb{H}_*(L^kM) = H_{*+m}(L^kM)$ is a nonunital and noncounital Frobenius algebra, where m is the dimension of M as a Gorenstein space.

Note that M is a Gorenstein space by the assumption $\dim \pi_*(M) \otimes \mathbb{K} < \infty$ (see Proposition 2.2). The associativity of the product holds even if we assume that M is a Gorenstein space instead of assuming $\dim \pi_*(M) \otimes \mathbb{K} < \infty$. But we need the assumption to prove the commutativity of the *product*.

A nonunital and noncounital Frobenius algebra corresponds to a "positive boundary" TQFT, in the sense that TQFT operations are defined only when each component of the cobordism surfaces has a *positive* number of incoming and outgoing boundary components. See a paper of Cohen and Godin [2] for details.

See Section 7 for the precise statement and the proof of the associativity, the commutativity and the Frobenius compatibility. It is interesting that the proof of the commutativity of the loop coproduct (ie $k=1$) is easier than that of the brane coproduct with $k \geq 2$. In fact, we prove the commutativity of the loop coproduct using the explicit description of the loop coproduct constructed in another article of the author [11]. On the other hand, we prove the commutativity of the brane coproduct with $k \geq 2$ directly from the definition.

Moreover, we compute an example of the brane product and coproduct. Here, we consider the shifted homology $\mathbb{H}_*(L^k M) = H_{*+m}(L^k M)$. We also have the shifts of the brane product and coproduct on $\mathbb{H}_*(L^k M)$ with the sign determined by the Koszul sign convention.

Theorem 1.6 *The shifted homology $\mathbb{H}_*(L^2 S^{2n+1})$ for $n \geq 1$ equipped with the brane product μ is isomorphic to the exterior algebra $\wedge(y,z)$ with $|y| = -2n-1$ and $|z| = 2n-1$. The brane coproduct δ is described as follows*:

$$\begin{aligned}
\delta(1) &= 1 \otimes yz - y \otimes z + z \otimes y + yz \otimes 1, \\
\delta(y) &= y \otimes yz + yz \otimes y, \\
\delta(z) &= z \otimes yz + yz \otimes z, \\
\delta(yz) &= -yz \otimes yz.
\end{aligned}$$

Note that both the brane product and coproduct are nontrivial. Moreover, $(\delta \otimes 1) \circ \delta \neq 0$, in contrast with the case of the loop coproduct, in which the similar composition is always trivial [10, Theorem A].

On the other hand, the brane coproduct is trivial in some cases.

Theorem 1.7 *If the minimal Sullivan model $(\wedge V, d)$ of M is pure and satisfies* $\dim V^{\text{even}} > 0$, *then the brane coproduct on $H_*(L^2 M)$ is trivial.*

See Definition 6.4 for the definition of a pure Sullivan algebra.

Remark 1.8 If we fix embeddings of disks $D^k \hookrightarrow S$ and $D^k \hookrightarrow T$ instead of assuming S and T are manifolds, we can define the product and coproduct using "connected sums" defined by these embedded disks. Moreover, if we have two disjoint embeddings $i, j\colon D^k \hookrightarrow S$ to the same space S, we can define the "connected sum" along i and j, and hence we can define the product and coproduct using this. We call these the (S,i,j)–brane product and coproduct, and give definitions in Section 4.

Section 2 contains brief background material on string topology on Gorenstein spaces. We define the (S,T)–brane product and coproduct in Section 3 and the (S,i,j)–brane product and coproduct in Section 4. Here, we defer the proof of Corollary 3.2 to Section 5. In Section 6, we compute examples and prove Theorems 1.6 and 1.7. Section 7 is devoted to the proof of Theorem 1.5, where we defer the determination of some signs to Sections 8 and 9.

2 Construction by Félix and Thomas

In this section, we recall the construction of the loop product and coproduct by Félix and Thomas [6]. Since the cochain models are good for fibrations, the duals of the loop product and coproduct are defined at first, and then we define the loop product and coproduct as the duals of them. Moreover we focus on the case when the characteristic of the coefficient $\mathbb{K}$ is zero. So we make full use of rational homotopy theory. For the basic definitions and theorems on homological algebra and rational homotopy theory, we refer the reader to [5].

Definition 2.1 [4] Let $m \in \mathbb{Z}$ be an integer.

(1) An augmented dga (differential graded algebra) (A,d) is called a *($\mathbb{K}$-)Gorenstein algebra of dimension m* if

$$\dim \operatorname{Ext}_A^l(\mathbb{K}, A) = \begin{cases} 1 & \text{if } l = m, \\ 0 & \text{otherwise,} \end{cases}$$

where the field $\mathbb{K}$ and the dga (A,d) are (A,d)–modules via the augmentation map and the identity map, respectively.

(2) A path-connected topological space M is called a ($\mathbb{K}$–)*Gorenstein space of dimension m* if the singular cochain algebra $C^*(M)$ of M is a Gorenstein algebra of dimension m.

Here, $\operatorname{Ext}_A(M,N)$ is defined using a semifree resolution of (M,d) over (A,d) for a dga (A,d) and (A,d)–modules (M,d) and (N,d). $\operatorname{Tor}_A(M,N)$ is defined similarly. See [6, Section 1] for details of semifree resolutions.

An important example of a Gorenstein space is given by the following proposition:

Proposition 2.2 [4, Proposition 3.4] *A 1–connected topological space M is a $\mathbb{K}$–Gorenstein space if $\pi_*(M) \otimes \mathbb{K}$ is finite-dimensional. Similarly, a Sullivan algebra $(\wedge V, d)$ is a Gorenstein algebra if V is finite-dimensional.*

Note that this proposition is stated only for $\mathbb{Q}$–Gorenstein spaces in [4], but the proof can be applied for any $\mathbb{K}$ and Sullivan algebras.

Let M be a 1–connected $\mathbb{K}$–Gorenstein space of dimension m whose cohomology $H^*(M)$ is of finite type. As a preparation to define the loop product and coproduct, Félix and Thomas proved the following theorem:

Theorem 2.3 [6, Theorem 12] *The diagonal map $\Delta\colon M \to M^2$ makes $C^*(M)$ into a $C^*(M^2)$–module. We have an isomorphism*

$$\operatorname{Ext}^*_{C^*(M^2)}(C^*(M), C^*(M^2)) \cong H^{*-m}(M).$$

By Theorem 2.3, we have $\operatorname{Ext}^m_{C^*(M^2)}(C^*(M), C^*(M^2)) \cong H^0(M) \cong \mathbb{K}$, hence the generator

$$\Delta_! \in \operatorname{Ext}^m_{C^*(M^2)}(C^*(M), C^*(M^2))$$

is well defined up to multiplication by a nonzero scalar. We call this element the *shriek map* for Δ.

Using the map $\Delta_!$, we can define the duals of the loop product and coproduct. Then, using the diagram (1.1), we define the dual of the loop product to be the composition

$$\mathrm{incl}_! \circ \mathrm{comp}^*\colon H^*(LM) \xrightarrow{\mathrm{comp}^*} H^*(LM \times_M LM) \xrightarrow{\mathrm{incl}_!} H^{*+m}(LM \times LM).$$

Here, the map $\mathrm{incl}_!$ is defined by the composition

$$H^*(LM \times_M LM) \underset{\cong}{\xleftarrow{\mathrm{EM}}} \operatorname{Tor}^*_{C^*(M^2)}(C^*(M), C^*(LM \times LM)) \\ \xrightarrow{\operatorname{Tor}_{\mathrm{id}}(\Delta_!,\mathrm{id})} \operatorname{Tor}^{*+m}_{C^*(M^2)}(C^*(M^2), C^*(LM \times LM)) \xrightarrow[\cong]{} H^{*+m}(LM \times LM),$$

where the map EM is the Eilenberg–Moore map, which is an isomorphism (see [5, Theorem 7.5] for details). Similarly, using the diagram (1.2), we define the dual of the loop coproduct to be the composition

$$\mathrm{comp}_! \circ \mathrm{incl}^*\colon H^*(LM \times LM) \xrightarrow{\mathrm{incl}^*} H^*(LM \times_M LM) \xrightarrow{\mathrm{comp}_!} H^*(LM).$$

Here, the map $\mathrm{comp}_!$ is defined by the composition

$$H^*(LM \times_M LM) \underset{\cong}{\xleftarrow{\mathrm{EM}}} \operatorname{Tor}^*_{C^*(M^2)}(C^*(M), C^*(LM)) \\ \xrightarrow{\operatorname{Tor}_{\mathrm{id}}(\Delta_!,\mathrm{id})} \operatorname{Tor}^{*+m}_{C^*(M^2)}(C^*(M^2), C^*(LM)) \xrightarrow[\cong]{} H^{*+m}(LM).$$

3 Definition of the (S, T)–brane coproduct

Let $\mathbb{K}$ be a field of characteristic zero, S and T manifolds of dimension k, and M a k–connected Gorenstein space of finite type. As in the construction by Félix and Thomas, which we reviewed in Section 2, we construct the duals

$$\mu_{ST}^{\vee}\colon H^*(M^{S\#T}) \to H^{*+\dim M}(M^S \times M^T),$$
$$\delta_{ST}^{\vee}\colon H^*(M^S \times M^T) \to H^{*+\dim \Omega^{k-1} M}(M^{S\#T})$$

of the (S, T)–brane product and the (S, T)–brane coproduct.

The (S, T)–brane product is defined by a similar way to that of Félix and Thomas. Using the diagram (1.3), we define $\mu_{ST}^{\vee}$ to be the composition

$$\mathrm{incl}_! \circ \mathrm{comp}^*\colon H^*(M^{S\#T}) \xrightarrow{\mathrm{comp}^*} H^*(M^S \times_M M^T) \xrightarrow{\mathrm{incl}_!} H^{*+m}(M^S \times M^T).$$

Here, the map $\mathrm{incl}_!$ is defined by the composition

$$H^*(M^S \times_M M^T) \xleftarrow[\cong]{\mathrm{EM}} \mathrm{Tor}^*_{C^*(M^2)}(C^*(M), C^*(M^S \times M^T))$$
$$\xrightarrow{\mathrm{Tor}_{\mathrm{id}}(\Delta_!, \mathrm{id})} \mathrm{Tor}^{*+m}_{C^*(M^2)}(C^*(M^2), C^*(M^S \times M^T)) \xrightarrow[\cong]{} H^{*+m}(M^S \times M^T).$$

Next, we begin the definition of the (S, T)–brane coproduct. But Theorem 2.3 cannot be applied to this case since the base map of the pullback is $c\colon M \to L^{k-1}M$.

Instead of Theorem 2.3, we use the following theorem to define the (S, T)–brane coproduct. A graded algebra A is *connected* if $A^0 = \mathbb{K}$ and $A^i = 0$ for any $i < 0$. A dga (A, d) is *connected* if A is connected.

Theorem 3.1 *Let $(A \otimes B, d)$ be a dga such that A and B are connected commutative graded algebras, (A, d) is a sub-dga of finite type and $(A \otimes B, d)$ is semifree over (A, d). Let $\eta\colon (A \otimes B, d) \to (A, d)$ be a dga homomorphism. Assume that the following conditions hold*:

(a) *The restriction of η to A is the identity map of A.*

(b) *The dga $(B, \bar{d}) = \mathbb{K} \otimes_A (A \otimes B, d)$ is a Gorenstein algebra of dimension $\overline{m}$.*

(c) *For any $b \in B$, the element $db - \bar{d}b$ lies in $A^{\geq 2} \otimes B$.*

Then we have an isomorphism

$$\mathrm{Ext}^*_{A \otimes B}(A, A \otimes B) \cong H^{*-\overline{m}}(A).$$

This can be proved by a similar method to Theorem 2.3 [6, Theorem 12]. The proof is given in Section 9.

Applying to sphere spaces, we have the following corollary:

Corollary 3.2 *Let M be a $(k-1)$–connected (and 1–connected) space of finite type such that $\Omega^{k-1}M$ is a Gorenstein space of dimension $\overline{m}$. Then we have an isomorphism*

$$\mathrm{Ext}^{*}_{C^*(L^{k-1}M)}(C^*(M), C^*(L^{k-1}M)) \cong H^{*-\overline{m}}(M).$$

To prove the corollary, we need to construct models of sphere spaces satisfying the conditions of Theorem 3.1. This will be done in Section 5.

Note that, since $L^0M = M \times M$, this is a generalization of Theorem 2.3 (in the case that the characteristic of $\mathbb{K}$ is zero).

Assume that M is a k–connected space such that $\Omega^{k-1}M$ is a Gorenstein space. Then we have $\mathrm{Ext}^{\overline{m}}_{C^*(L^{k-1}M)}(C^*(M), C^*(L^{k-1}M)) \cong H^0(M) \cong \mathbb{K}$, hence the shriek map for $c\colon M \to L^{k-1}M$ is defined to be the generator

$$c_! \in \mathrm{Ext}^{\overline{m}}_{C^*(L^{k-1}M)}(C^*(M), C^*(L^{k-1}M)),$$

which is well defined up to multiplication by a nonzero scalar. Using $c_!$ with the diagram (1.4), we define the dual $\delta^{\vee}_{ST}$ of the (S,T)–brane coproduct to be the composition

$$\mathrm{comp}_! \circ \mathrm{incl}^*\colon H^*(M^S \times M^T) \xrightarrow{\mathrm{incl}^*} H^*(M^S \times_M M^T) \xrightarrow{\mathrm{comp}_!} H^*(M^{S\#T}).$$

Here, the map $\mathrm{comp}_!$ is defined by the composition

$$H^*(M^S \times_M M^T) \overset{\mathrm{EM}}{\underset{\cong}{\longleftarrow}} \mathrm{Tor}^{*}_{C^*(L^{k-1}M)}(C^*(M), C^*(M^{S\#T}))$$
$$\xrightarrow{\mathrm{Tor}_{\mathrm{id}}(c_!,\mathrm{id})} \mathrm{Tor}^{*+\overline{m}}_{C^*(L^{k-1}M)}(C^*(L^{k-1}M), C^*(M^{S\#T})) \xrightarrow[\cong]{} H^{*+\overline{m}}(M^{S\#T}).$$

Note that the Eilenberg–Moore isomorphism can be applied since $L^{k-1}M$ is 1–connected.

4 Definition of the (S,i,j)–brane product and coproduct

In this section, we give a definition of the (S,i,j)–brane product and coproduct. Let S be a topological space, and i and j embeddings $D^k \to S$. Fix a small k–disk $D \subset D^k$

and denote its interior by D° and its boundary by ∂D. Then we define three spaces $\#(S,i,j)$, $Q(S,i,j)$ and $\bigvee(S,i,j)$ as follows. The space $\#(S,i,j)$ is obtained from $S \setminus (i(D^\circ) \cup j(D^\circ))$ by gluing $i(\partial D)$ and $j(\partial D)$ by an orientation-reversing homeomorphism. We obtain $Q(S,i,j)$ by collapsing two disks $i(D)$ and $j(D)$ to two points, respectively. The space $\bigvee(S,i,j)$ is defined as the quotient space of $Q(S,i,j)$ identifying the two points. Then, since the quotient space D^k/D is homeomorphic to the disk D^k, we identify $Q(S,i,j)$ with S itself. By the above definitions, we have the maps $\#(S,i,j) \to \bigvee(S,i,j)$ and $S = Q(S,i,j) \to \bigvee(S,i,j)$. For a space M, these maps induce the maps comp: $M^{\bigvee(S,i,j)} \to M^{\#(S,i,j)}$ and incl: $M^{\bigvee(S,i,j)} \to M^S$. Moreover, we have diagrams

$$\begin{array}{ccccc} M^S & \xleftarrow{\text{incl}} & M^{\bigvee(S,i,j)} & \xrightarrow{\text{comp}} & M^{\#(S,i,j)} \\ \downarrow & & \downarrow & & \\ M \times M & \xleftarrow{\Delta} & M & & \end{array}$$

and

$$\begin{array}{ccccc} M^{\#(S,i,j)} & \xleftarrow{\text{comp}} & M^{\bigvee(S,i,j)} & \xrightarrow{\text{incl}} & M^S \\ \downarrow & & \downarrow & & \\ L^{k-1}M & \xleftarrow{c} & M & & \end{array}$$

in which the squares are pullback diagrams. If M is a k–connected space such that $\Omega^{k-1}M$ is a Gorenstein space, we define the (S,i,j)–brane product and coproduct by a similar method to Section 3, using these diagrams instead of the diagrams (1.3) and (1.4). Note that this generalizes the (S,T)–brane product and coproduct defined in Section 3.

5 Construction of models and proof of Corollary 3.2

In this section, we give a proof of Corollary 3.2, constructing a Sullivan model of the map $c\colon M \to L^{k-1}M$ satisfying the assumptions of Theorem 3.1.

First, we construct models algebraically. Let $(\wedge V, d)$ be a Sullivan algebra. For an integer $l \in \mathbb{Z}$, let $s^l V$ be a graded module defined by $(s^l V)^n = V^{n+l}$, and $s^l v$ denotes the element in $s^l V$ corresponding to the element $v \in V$.

Define two derivations $s^{(k-1)}$ and $\bar{d}^{(k-1)}$ on the graded algebra $\wedge V \otimes \wedge s^{k-1} V$ by

$$s^{(k-1)}(v) = s^{k-1}v, \qquad s^{(k-1)}(s^{k-1}v) = 0,$$
$$\bar{d}^{(k-1)}(v) = dv, \qquad \bar{d}^{(k-1)}(s^{k-1}v) = (-1)^{k-1}s^{(k-1)}dv.$$

Then it is easy to see that $\bar{d}^{(k-1)} \circ \bar{d}^{(k-1)} = 0$ and hence $(\wedge V \otimes \wedge s^{k-1} V, \bar{d}^{(k-1)})$ is a dga.

Similarly, define derivations $s^{(k)}$ and $d^{(k)}$ on the graded algebra $\wedge V \otimes \wedge s^{k-1} V \otimes \wedge s^k V$ by

$$s^{(k)}(v) = s^k v, \quad s^{(k)}(s^{k-1} v) = s^{(k)}(s^k v) = 0,$$
$$d^{(k)}(v) = dv, \quad d^{(k)}(s^{k-1} v) = \bar{d}^{(k-1)}(s^{k-1} v), \quad d^{(k)}(s^k v) = s^{k-1} v + (-1)^k s^{(k)} dv.$$

Then it is easy to see that $d^{(k)} \circ d^{(k)} = 0$ and hence $(\wedge V \otimes \wedge s^{k-1} V \otimes \wedge s^k V, d^{(k)})$ is a dga.

The tensor product $(\wedge V, d) \otimes_{\wedge V \otimes \wedge s^{k-1} V} (\wedge V \otimes \wedge s^{k-1} V \otimes \wedge s^k V, d^{(k)})$ is canonically isomorphic to $(\wedge V \otimes \wedge s^k V, \bar{d}^{(k)})$, where $(\wedge V, d)$ is a $(\wedge V \otimes \wedge s^{k-1} V, \bar{d}^{(k-1)})$–module via the dga homomorphism $\phi\colon (\wedge V \otimes \wedge s^{k-1} V, \bar{d}^{(k-1)}) \to (\wedge V, d)$ defined by $\phi(v) = v$ and $\phi(s^{k-1} v) = 0$.

It is clear that, if $V^{\leq k-1} = 0$, the dga $(\wedge V \otimes \wedge s^{k-1} V, \bar{d}^{(k-1)})$ is a Sullivan algebra and, if $V^{\leq k} = 0$, the dga $(\wedge V \otimes \wedge s^{k-1} V \otimes \wedge s^k V, d^{(k)})$ is a relative Sullivan algebra over $(\wedge V \otimes \wedge s^{k-1} V, \bar{d}^{(k-1)})$.

Define a dga homomorphism

$$\tilde{\varepsilon}\colon (\wedge V \otimes \wedge s^{k-1} V \otimes \wedge s^k V, d^{(k)}) \to (\wedge V, d)$$

by $\tilde{\varepsilon}(v) = v$ and $\tilde{\varepsilon}(s^{k-1} v) = \tilde{\varepsilon}(s^k v) = 0$. Then the linear part

$$Q(\tilde{\varepsilon})\colon (V \oplus s^{k-1} V \oplus s^k V, d_0^{(k)}) \to (V, d_0)$$

is a quasi-isomorphism, and hence $\tilde{\varepsilon}$ is a quasi-isomorphism [5, Proposition 14.13].

Define a relative Sullivan algebra $\mathcal{M}_{\mathrm{P}} = (\wedge V^{\otimes 2} \otimes \wedge s V, d)$ over $(\wedge V, d)^{\otimes 2}$ by the formula

$$d(sv) = 1 \otimes v - v \otimes 1 - \sum_{i=1}^{\infty} \frac{(sd)^i}{i!}(v \otimes 1)$$

inductively (see [5, Section 15(c)] or [11, Appendix A] for details).

For simplicity, we write $\mathcal{M}_{S^k} = (\wedge V \otimes \wedge s^k V, \bar{d}^{(k)})$ for $k \geq 1$ and $\mathcal{M}_{S^0} = (\wedge V, d)^{\otimes 2}$, and $\mathcal{M}_{D^k} = (\wedge V \otimes \wedge s^{k-1} V \otimes \wedge s^k V, d^{(k)})$ for $k \geq 2$ and $\mathcal{M}_{D^1} = \mathcal{M}_{\mathrm{P}}$.

Let $A^*(-)$ be the functor of the algebra of polynomial differential forms. Note that, for a space X, $A^*(X)$ is a *commutative* dga which is naturally quasi-isomorphic to the singular cochain algebra $C^*(X)$ as differential graded algebras. See [5, Section 10] for details.

Using these algebras, we have the following proposition:

Proposition 5.1 *Let $k \geq 2$ be an integer, M a $(k-1)$–connected space of finite type and $f\colon (\wedge V, d) \to A^*(M)$ its Sullivan model such that $V^{\leq k-1} = 0$ and V is of finite type. Then, for any l with $0 \leq l \leq k-1$, there are quasi-isomorphisms $f_l\colon \mathcal{M}_{S^l} \xrightarrow{\simeq} A^*(L^l M)$ and $g_l\colon \mathcal{M}_{D^l} \to A^*(D^l M)$ such that the diagrams*

$$
\begin{array}{ccccc}
\mathcal{M}_{S^l} & \xrightarrow{\phi} & (\wedge V, d) & \hookrightarrow & \mathcal{M}_{S^l} \\
{\scriptstyle f_l}\downarrow{\scriptstyle \simeq} & & {\scriptstyle f}\downarrow{\scriptstyle \simeq} & & {\scriptstyle f_l}\downarrow{\scriptstyle \simeq} \\
A^*(L^l M) & \xrightarrow{c^*} & A^*(M) & \xrightarrow{\mathrm{ev}^*} & A^*(L^l M)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{M}_{S^{l-1}} & \hookrightarrow & \mathcal{M}_{D^l} \\
{\scriptstyle f_{l-1}}\downarrow{\scriptstyle \simeq} & & {\scriptstyle g_l}\downarrow{\scriptstyle \simeq} \\
A^*(L^{l-1} M) & \xrightarrow{\mathrm{res}^*} & A^*(D^l M)
\end{array}
$$

commute strictly, where $D^l M = \mathrm{Map}(D^l, M)$. In particular, the dga homomorphism $\phi\colon \mathcal{M}_{S^{k-1}} \to (\wedge V, d)$ is a Sullivan representative of the map $c\colon M \to L^{k-1} M$ with strict commutativity $c^* \circ f_l = f \circ \phi$

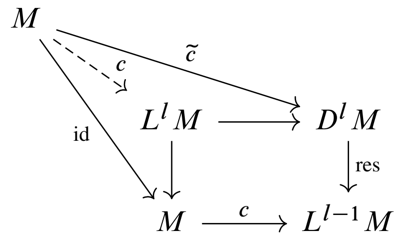
Proof We prove the proposition by induction on l. The case $l = 0$ is well known, since c is the diagonal map and ϕ is the multiplication map.

Let l be an integer with $1 \leq l \leq k-1$ and assume that we already have f_{l-1} satisfying $c^* \circ f_l = f \circ \phi$. Let $\tilde{c}\colon M \to D^l M$ be the embedding by constant maps, and res: $D^l M \to L^{l-1} M$ the restriction map to the boundary. Since $\mathrm{res} \circ \tilde{c} = c$, the outer square in the diagram

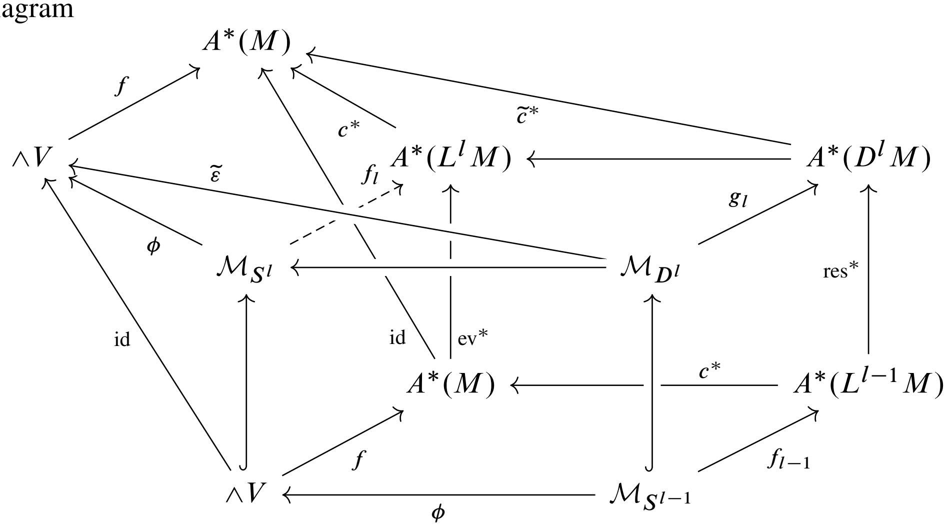
$$
\begin{array}{ccccc}
\mathcal{M}_{S^{l-1}} & \underset{\simeq}{\xrightarrow{f_{l-1}}} & A^*(L^{l-1} M) & \xrightarrow{\mathrm{res}^*} & A^*(D^l M) \\
\Big\downarrow & & {\scriptstyle g_l} \ \dashrightarrow & & {\scriptstyle \simeq}\Big\downarrow{\scriptstyle \tilde{c}^*} \\
\mathcal{M}_{D^l} & \underset{\simeq}{\xrightarrow{\tilde{\varepsilon}}} & \wedge V & \underset{\simeq}{\xrightarrow{f}} & A^*(M)
\end{array}
$$

is commutative by the induction hypothesis. Here, $\tilde{c}^*$ is a surjective quasi-isomorphism, since the map $\tilde{c}$ is a homotopy equivalence and has a retraction, namely the evaluation map at the basepoint. Hence, by the lifting property of a relative Sullivan algebra with respect to a surjective quasi-isomorphism, there is a dga homomorphism $g_l\colon \mathcal{M}_{D^l} \to A^*(D^l M)$ which makes *both* of the triangles in the above diagram commute strictly. Note that, when $l = 1$, this diagram is constructed in [8, Section 4.5], without the *strict* commutativity of the lower right triangle.

Here the map $c\colon M \to L^l M$ is given by the pullback diagram



Applying the functor $A^*(-)$ to the diagram and considering its model, we have the diagram



where the faces are strictly commutative and the square in the front face is a pushout diagram. By the universality of the pushout, we have the dga homomorphism $f_l\colon \mathcal{M}_{S^l} \to A^*(L^l M)$, which makes the diagram commutative. In particular, it satisfies $f \circ \phi = c^* \circ f_l$. Note that f_l is a quasi-isomorphism by the Eilenberg–Moore theorem [5, Section 15(c)]. This completes the induction. □

Proof of Corollary 3.2 In the case $k = 1$, apply Theorem 3.1 to the multiplication map $(\wedge V, d)^{\otimes 2} \to (\wedge V, d)$. (Note that this case is a result of Félix and Thomas [6].)

In the case $k \geq 2$, using Proposition 5.1, apply Theorem 3.1 to the map ϕ. □

6 Computation of examples

In this section, we will compute the brane product and coproduct for some examples, which proves Theorems 1.6 and 1.7.

In [9], the duals of the loop product and coproduct are described in terms of Sullivan models using the torsion functor description of [7]. By a similar method, we can describe the brane product and coproduct as follows:

Theorem 6.1 *Let M be a k–connected $\mathbb{K}$–Gorenstein space of finite type and $(\wedge V, d)$ its Sullivan model such that $V^{\le k} = 0$ and V is of finite type. Then the dual of the brane product on $H^*(L^k M)$ is induced by the composition*

$$\begin{aligned}\mathcal{M}_{S^k} &\xrightarrow{\cong} \wedge V \otimes_{\mathcal{M}_{S^{k-1}}} \mathcal{M}_{D^k} \xleftarrow[\simeq]{\tilde{\varepsilon}\otimes\mathrm{id}} \mathcal{M}_{D^k} \otimes_{\mathcal{M}_{S^{k-1}}} \mathcal{M}_{D^k} \\ &\xrightarrow{(\phi\otimes\mathrm{id})\otimes_{\phi}(\phi\otimes\mathrm{id})} \mathcal{M}_{S^k} \otimes_{\wedge V} \mathcal{M}_{S^k} \xrightarrow{\cong} \wedge V \otimes_{\wedge V^{\otimes 2}} {\mathcal{M}_{S^k}}^{\otimes 2} \\ &\xleftarrow[\simeq]{\bar{\varepsilon}\otimes\mathrm{id}} \mathcal{M}_{\mathrm{P}} \otimes_{\wedge V^{\otimes 2}} {\mathcal{M}_{S^k}}^{\otimes 2} \xrightarrow{\delta_!\otimes\mathrm{id}} \wedge V^{\otimes 2} \otimes_{\wedge V^{\otimes 2}} {\mathcal{M}_{S^k}}^{\otimes 2} \xrightarrow{\cong} {\mathcal{M}_{S^k}}^{\otimes 2},\end{aligned}$$

where $\delta_!$ is a representative of $\Delta_!$. (See Section 5 for the definitions of the other maps.)

Assume that $\Omega^{k-1} M$ is a Gorenstein space. Then the dual of the brane coproduct is induced by the composition

$$\begin{aligned}{\mathcal{M}_{S^k}}^{\otimes 2} &\xrightarrow{\cong} \wedge V^{\otimes 2} \otimes_{{\mathcal{M}_{S^{k-1}}}^{\otimes 2}} {\mathcal{M}_{D^k}}^{\otimes 2} \xrightarrow{\mu\otimes_{\mu'}\eta} \wedge V \otimes_{\mathcal{M}_{S^{k-1}}} (\mathcal{M}_{D^k} \otimes_{\mathcal{M}_{S^{k-1}}} \mathcal{M}_{D^k}) \\ &\xleftarrow[\simeq]{\tilde{\varepsilon}\otimes\mathrm{id}} \mathcal{M}_{D^k} \otimes_{\mathcal{M}_{S^{k-1}}} (\mathcal{M}_{D^k} \otimes_{\mathcal{M}_{S^{k-1}}} \mathcal{M}_{D^k}) \\ &\xrightarrow{\gamma_!\otimes\mathrm{id}} \mathcal{M}_{S^{k-1}} \otimes_{\mathcal{M}_{S^{k-1}}} (\mathcal{M}_{D^k} \otimes_{\mathcal{M}_{S^{k-1}}} \mathcal{M}_{D^k}) \\ &\xrightarrow{\cong} \mathcal{M}_{D^k} \otimes_{\mathcal{M}_{S^{k-1}}} \mathcal{M}_{D^k} \xrightarrow[\simeq]{\tilde{\varepsilon}\otimes\mathrm{id}} \wedge V \otimes_{\mathcal{M}_{S^{k-1}}} \mathcal{M}_{D^k} \xrightarrow{\cong} \mathcal{M}_{S^k},\end{aligned}$$

where $\gamma_!$ is a representative of $c_!$, the maps μ and μ' are the multiplication maps, and η is the quotient map.

Proof We omit the detail of the proof for the brane product, since it is the same as that for the usual loop product. Here we give a detailed proof of the construction of the model for the brane coproduct.

Here we use two pullback diagrams

$$\begin{array}{ccc} L^k M \times_M L^k M & \longrightarrow & L^k M \\ \big\downarrow & & \big\downarrow \mathrm{ev} \\ L^k M & \xrightarrow{\quad \mathrm{ev} \quad} & M \end{array} \qquad \begin{array}{ccc} M \times_{L^{k-1} M} L^k M & \longrightarrow & L^k M \\ \big\downarrow & & \big\downarrow \mathrm{res} \\ M & \xrightarrow{\quad c \quad} & L^{k-1} M \end{array}$$

The spaces $L^k M \times_M L^k M$ and $M \times_{L^{k-1} M} L^k M$ are obviously homeomorphic and hence we identify them outside of this proof, but we distinguish them in this proof in order to specify the pullback diagrams. By a similar method to the proof of Proposition 5.1, we have dga homomorphisms $h_k\colon \mathcal{M}_{D^k} \otimes_{\mathcal{M}_{S^{k-1}}} \mathcal{M}_{D^k} \to A^*(L^k M)$ and $i_k\colon \wedge V \otimes_{\mathcal{M}_{S^{k-1}}} (\mathcal{M}_{D^k} \otimes_{\mathcal{M}_{S^{k-1}}} \mathcal{M}_{D^k}) \to A^*(M \times_{L^{k-1}M} L^k M)$ such that the diagrams

$$\begin{array}{ccc}
\mathcal{M}_{S^{k-1}} & \longrightarrow & \mathcal{M}_{D^k} \otimes_{\mathcal{M}_{S^{k-1}}} \mathcal{M}_{D^k} \\
\simeq \big\downarrow f_{k-1} & & \simeq \big\downarrow h_k \\
A^*(L^{k-1}M) & \xrightarrow{\ \mathrm{res}^*\ } & A^*(L^k M)
\end{array} \tag{6.2}$$

$$\begin{array}{ccc}
\mathcal{M}_{S^k} \otimes_{\wedge V} \mathcal{M}_{S^k} & \xrightarrow[\cong]{\qquad} & \wedge V \otimes_{\mathcal{M}_{S^{k-1}}} (\mathcal{M}_{D^k} \otimes_{\mathcal{M}_{S^{k-1}}} \mathcal{M}_{D^k}) \\
\big\downarrow f_k \otimes f_k & & \\
A^*(L^k M) \otimes_{A^*(M)} A^*(L^k M) & & \simeq \big\downarrow i_k \\
\big\downarrow & & \\
A^*(L^k M \times_M L^k M) & \xrightarrow[\cong]{\qquad} & A^*(M \times_{L^{k-1}M} L^k M)
\end{array} \tag{6.3}$$

commute strictly, where the horizontal maps in the second diagram are the canonical isomorphisms.

Using the above maps, we obtain the diagram

$$\begin{array}{ccccc}
H^*(L^kM \times L^kM) & \xleftarrow[\cong]{\mathrm{EM}} & \mathrm{Tor}_{\mathbb{K}}(A^*(L^kM), A^*(L^kM)) & \xleftarrow[\cong]{\mathrm{Tor}_{\mathrm{id}}(f_k, f_k)} & H^*(\mathcal{M}_{S^k} \otimes \mathcal{M}_{S^k}) \\
\big\downarrow \mathrm{incl}^* & & \big\downarrow & & \big\downarrow \\
H^*(L^kM \times_M L^kM) & \xleftarrow[\cong]{\mathrm{EM}} & \mathrm{Tor}_{A^*(M)}(A^*(L^kM), A^*(L^kM)) & \xleftarrow[\cong]{\mathrm{Tor}_f(f_k, f_k)} & H^*(\mathcal{M}_{S^k} \otimes_{\wedge V} \mathcal{M}_{S^k}) \\
\big\downarrow \cong & & & & \big\downarrow \cong \\
H^*(M \times_{L^{k-1}M} L^kM) & \xleftarrow[\cong]{\mathrm{EM}} & \mathrm{Tor}_{A^*(L^{k-1}M)}(A^*(M), A^*(L^kM)) & \xleftarrow[\cong]{\mathrm{Tor}_{f_{k-1}}(f, h_k)} & H^*(\wedge V \otimes_{\mathcal{M}_{S^{k-1}}} \overline{\mathcal{M}}_{S^k}) \\
 & & & & \big\uparrow \cong \\
\big\downarrow \mathrm{comp}_! & & \big\downarrow \mathrm{Tor}_{\mathrm{id}}(c_!, \mathrm{id}) & & H^*(\mathcal{M}_{D^k} \otimes_{\mathcal{M}_{S^{k-1}}} \overline{\mathcal{M}}_{S^k}) \\
 & & & & \big\downarrow H^*(\gamma_! \otimes \mathrm{id}) \\
H^*(L^kM) & \xleftarrow[\cong]{\qquad} & \mathrm{Tor}_{A^*(L^{k-1}M)}(A^*(L^{k-1}M), A^*(L^kM)) & \xleftarrow[\cong]{\mathrm{Tor}_{f_{k-1}}(f_{k-1}, h_k)} & H^*(\mathcal{M}_{S^{k-1}} \otimes_{\mathcal{M}_{S^{k-1}}} \overline{\mathcal{M}}_{S^k}) \\
 & & & & \big\downarrow \cong \\
 & & & & H^*(\mathcal{M}_{S^k})
\end{array}$$

where $\overline{\mathcal{M}}_{S^k} = \mathcal{M}_{D^k} \otimes_{\mathcal{M}_{S^{k-1}}} \mathcal{M}_{D^k}$. The composition of the vertical maps in the left column is the definition of the brane coproduct and the one in the right column is the description in the statement of the theorem. The horizontal maps in the right squares are defined by the strict commutativity of the diagrams in Proposition 5.1 and (6.2). The commutativity of the central square follows from (6.3) and that of the other squares are obvious from the definitions. The commutativity of this diagram proves the theorem. □

As a preparation of computation, recall the definition of a pure Sullivan algebra.

Definition 6.4 (see [5, Section 32]) A Sullivan algebra $(\wedge V, d)$ with $\dim V < \infty$ is called *pure* if $d(V^{\mathrm{even}}) = 0$ and $d(V^{\mathrm{odd}}) \subset \wedge V^{\mathrm{even}}$.

For a pure Sullivan algebra, we have an explicit construction of the shriek map $\delta_!$ and $\gamma_!$. For $\delta_!$, see [9]. For $\gamma_!$, we have the following proposition:

Proposition 6.5 *Let $(\wedge V, d)$ be a pure minimal Sullivan algebra. Take bases $V^{\mathrm{even}} = \mathbb{K}\{x_1, \dots x_p\}$ and $V^{\mathrm{odd}} = \mathbb{K}\{y_1, \dots y_q\}$. Define a $(\wedge V \otimes \wedge sV, d)$–linear map*

$$\gamma_!\colon (\wedge V \otimes \wedge sV \otimes \wedge s^2 V, d) \to (\wedge V \otimes \wedge sV, d)$$

by $\gamma_!(s^2 y_1 \cdots s^2 y_q) = sx_1 \cdots sx_p$ and $\gamma_!(s^2 y_{j_1} \cdots s^2 y_{j_l}) = 0$ for $l < q$. Then $\gamma_!$ defines a nontrivial element in $\mathrm{Ext}_{\wedge V \otimes \wedge sV}(\wedge V, \wedge V \otimes \wedge sV)$

Proof By a straightforward calculation, $\gamma_!$ is a cocycle in

$$\mathrm{Hom}_{\wedge V \otimes \wedge sV}(\wedge V \otimes \wedge sV \otimes \wedge s^2 V, \wedge V \otimes \wedge sV).$$

In order to prove the nontriviality, we define an ideal

$$I = (x_1, \dots, x_p, y_1, \dots, y_q, sy_1, \dots, sy_q) \subset \wedge V \otimes \wedge sV.$$

By purity and minimality, we have $d(I) \subset I$. Using this ideal, we have the evaluation map of the form

$$\begin{aligned}\mathrm{Ext}_{\wedge V \otimes \wedge sV}(\wedge V, \wedge V \otimes \wedge sV) \otimes \mathrm{Tor}_{\wedge V \otimes \wedge sV}(\wedge V, \wedge V \otimes \wedge sV/I) \\ \xrightarrow{\mathrm{ev}} \mathrm{Tor}_{\wedge V \otimes \wedge sV}(\wedge V \otimes \wedge sV, \wedge V \otimes \wedge sV/I) \xrightarrow{\cong} \wedge sV^{\mathrm{even}}.\end{aligned}$$

By this map, the element $[\gamma_!] \otimes [s^2 y_1 \cdots s^2 y_q \otimes 1]$ is mapped to the element $sx_1 \cdots sx_p$, which is obviously nontrivial. Hence $[\gamma_!]$ is also nontrivial. □

Now, we give proofs of Theorems 1.6 and 1.7.

Proof of Theorem 1.6 Using the descriptions in Theorem 6.1, we compute the brane product and coproduct for $M = S^{2n+1}$ and $k = 2$. In this case, we can take $(\wedge V, d) = (\wedge x, 0)$ with $|x| = 2n+1$, and have $\mathcal{M}_{S^1} = (\wedge(x, sx), 0)$ and $\mathcal{M}_{D^2} = (\wedge(x, sx, s^2x), d)$, where $dx = dsx = 0$ and $ds^2x = sx$. The computation is straightforward except for the shriek maps $\delta_!$ and $\gamma_!$. The map $\delta_!$ is the linear map $\mathcal{M}_{\mathrm{P}} \to (\wedge x, 0)^{\otimes 2}$ over $(\wedge x, 0)^{\otimes 2}$ determined by $\delta_!(1) = 1 \otimes x - x \otimes 1$ and $\delta_!((sx)^l) = 0$ for $l \geq 1$. By Proposition 6.5, the map $\gamma_!$ is the linear map $\mathcal{M}_{D^k} \to \mathcal{M}_{S^{k-1}}$ over $\mathcal{M}_{S^{k-1}}$ determined by $\gamma_!(s^2x) = 1$ and $\gamma_!(1) = 0$.

Then the dual of the brane product $\mu^\vee$ is a linear map

$$\mu^\vee \colon \wedge(x, s^2x) \to \wedge(x, s^2x) \otimes \wedge(x, s^2x)$$

of degree $m = 2n+1$ over $\wedge(x) \otimes \wedge(x)$, which is characterized by

$$\mu^\vee(1) = 1 \otimes x - x \otimes 1, \quad \mu^\vee(s^2x) = (1 \otimes x - x \otimes 1)(s^2x \otimes 1 + 1 \otimes s^2x).$$

Similarly, the dual of the brane coproduct $\delta^\vee$ is a linear map

$$\delta^\vee \colon \wedge(x, s^2x) \otimes \wedge(x, s^2x) \to \wedge(x, s^2x).$$

of degree $\overline{m} = 1 - 2n$ over $\wedge(x) \otimes \wedge(x)$, which is characterized by

$$\delta^\vee(1) = 0, \quad \delta^\vee(s^2x \otimes 1) = -1, \quad \delta^\vee(1 \otimes s^2x) = 1, \quad \delta^\vee(s^2x \otimes s^2x) = -s^2x.$$

Dualizing these results, we get the brane product and coproduct on the homology, which proves Theorem 1.6. □

Proof of Theorem 1.7 By Proposition 6.5, we have that $\mathrm{Im}(\gamma_! \otimes \mathrm{id})$ is contained in the ideal $(sx_1, \dots sx_p)$, which is mapped to zero by the map $\tilde{\varepsilon} \otimes \mathrm{id}$. □

7 Proof of the associativity, the commutativity and the Frobenius compatibility

In this section, we give a precise statement and the proof of Theorem 1.5.

First, we give a precise statement of Theorem 1.5. For simplicity, we omit the statement for the (S, i, j)–brane product and coproduct, which is almost the same as that for the (S, T)–brane product and coproduct. Let M be a k–connected $\mathbb{K}$–Gorenstein space of finite type such that $\Omega^{k-1}M$ is also a Gorenstein space. Denote $m = \dim M$. Then the precise statement of (1) is that the diagrams

$$\begin{array}{ccc} H^*(M^{S\#T\#U}) & \xrightarrow{\mu^\vee_{S\#T,U}} & H^*(M^{S\#T}\times M^U) \\ \downarrow{\scriptstyle \mu^\vee_{S,T\#U}} & & \downarrow{\scriptstyle \mu^\vee_{S,T\sqcup U}} \\ H^*(M^S\times M^{T\#U}) & \xrightarrow{\mu^\vee_{S\sqcup T,U}} & H^*(M^S\times M^T\times M^U) \end{array} \tag{7.1}$$

and

$$\begin{array}{ccc} H^*(M^{T\#S}) & \xrightarrow{\mu^\vee_{T,S}} & H^*(M^T\times M^S) \\ \downarrow{\scriptstyle \tau^*_\#} & & \downarrow{\scriptstyle \tau^*_\times} \\ H^*(M^{S\#T}) & \xrightarrow{\mu^\vee_{S,T}} & H^*(M^S\times M^T) \end{array} \tag{7.2}$$

commute by the sign $(-1)^m$. Here, $\tau_\times$ and $\tau_\#$ are defined as the transposition of S and T. Note that the associativity of the product holds even if the assumption that $\Omega^{k-1}M$ is a Gorenstein space is dropped.

Denote $\overline{m}=\dim\Omega^{k-1}M$. Then (2) states that the diagrams

$$\begin{array}{ccc} H^*(M^S\times M^T\times M^U) & \xrightarrow{\delta^\vee_{S\sqcup T,U}} & H^*(M^S\times M^{T\#U}) \\ \downarrow{\scriptstyle \delta^\vee_{S,T\sqcup U}} & & \downarrow{\scriptstyle \delta^\vee_{S,T\#U}} \\ H^*(M^{S\#T}\times M^U) & \xrightarrow{\delta^\vee_{S\#T,U}} & H^*(M^{S\#T\#U}) \end{array} \tag{7.3}$$

and

$$\begin{array}{ccc} H^*(M^{T\times S}) & \xrightarrow{\delta^\vee_{T,S}} & H^*(M^T\# M^S) \\ \downarrow{\scriptstyle \tau^*_\#} & & \downarrow{\scriptstyle \tau^*_\times} \\ H^*(M^{S\times T}) & \xrightarrow{\delta^\vee_{S,T}} & H^*(M^S\# M^T) \end{array} \tag{7.4}$$

commute by the sign $(-1)^{\overline{m}}$. Similarly, (3) states that the diagram

$$\begin{array}{ccc} H^*(M^S\times M^{T\#U}) & \xrightarrow{\delta^\vee_{S,T\#U}} & H^*(M^{S\#T\#U}) \\ \downarrow{\scriptstyle \mu^\vee_{S\#T,U}} & & \downarrow{\scriptstyle \mu^\vee_{S\sqcup T,U}} \\ H^*(M^S\times M^T\times M^U) & \xrightarrow{\delta^\vee_{S,T\sqcup U}} & H^*(M^{S\#T}\times M^U) \end{array} \tag{7.5}$$

commutes by the sign $(-1)^{m\overline{m}}$.

Before proving Theorem 1.5, we give a notation g_α for a shriek map.

Definition 7.6 Consider a pullback diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{f} & B \end{array}$$

of spaces, where q is a fibration. Let α be an element of $\mathrm{Ext}^m_{C^*(B)}(C^*(A), C^*(B))$. Assume that the Eilenberg–Moore map

$$\mathrm{EM}\colon \mathrm{Tor}^*_{C^*(B)}(C^*(A), C^*(Y)) \xrightarrow{\cong} H^*(X)$$

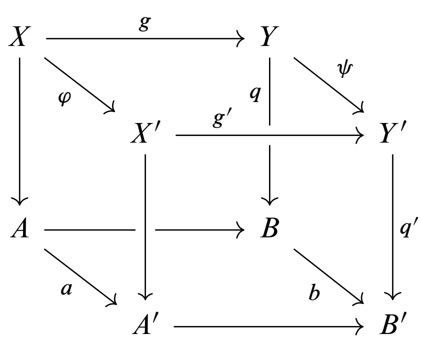
is an isomorphism (eg B is 1–connected and the cohomology of the fiber is of finite type). Then we define g_α to be the composition

$$g_\alpha\colon H^*(X) \xleftarrow{\cong} \mathrm{Tor}^*_{C^*(B)}(C^*(A), C^*(Y)) \xrightarrow{\mathrm{Tor}(\alpha,\mathrm{id})} \mathrm{Tor}^{*+m}_{C^*(B)}(C^*(B), C^*(Y)) \xrightarrow{\cong} H^{*+m}(Y).$$

Using this notation, we can write the shriek map $\mathrm{incl}_!$ as $\mathrm{incl}_{\Delta_!}$ for the diagram (1.3), and the shriek map $\mathrm{comp}_!$ as $\mathrm{comp}_{c_!}$ for the diagram (1.4).

Now we have the following two propositions as preparation of the proof of Theorem 1.5:

Proposition 7.7 *Consider a diagram*



where q and q' are fibrations and the front and back squares are pullback diagrams. Let $\alpha \in \mathrm{Ext}^m_{C^*(B)}(C^*(A), C^*(B))$ and $\alpha' \in \mathrm{Ext}^m_{C^*(B')}(C^*(A'), C^*(B'))$. Assume that the elements α and α' are mapped to the same element in $\mathrm{Ext}^m_{C^*(B')}(C^*(A'), C^*(B))$ by the morphisms induced by a and b, and that the Eilenberg–Moore maps of two *pullback diagrams are isomorphisms. Then the following diagram commutes*:

$$\begin{array}{ccc} H^*(X') & \xrightarrow{g'_{\alpha'}} & H^{*+m}(Y') \\ \big\downarrow{\scriptstyle \varphi^*} & & \big\downarrow{\scriptstyle \psi^*} \\ H^*(X) & \xrightarrow{g_\alpha} & H^{*+m}(Y) \end{array}$$

Proposition 7.8 *Consider a diagram*

$$\begin{array}{ccccc} X & \xrightarrow{\tilde{f}} & Y & \xrightarrow{\tilde{g}} & Z \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle r} \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

where the two squares are pullback diagrams. Let α *be an element of the module* $\mathrm{Ext}^m_{C^*(B)}(C^*(A), C^*(B))$ *and* β *an element of* $\mathrm{Ext}^n_{C^*(C)}(C^*(B), C^*(C))$. *Assume that the Eilenberg–Moore maps are isomorphisms for two pullback diagrams. Then we have*

$$(\tilde{g} \circ \tilde{f})_{\beta \circ (g_* \alpha)} = \tilde{g}_\beta \circ \tilde{f}_\alpha,$$

where $g_*\colon \mathrm{Ext}^m_{C^*(B)}(C^*(A), C^*(B)) \to \mathrm{Ext}^m_{C^*(C)}(C^*(A), C^*(B))$ *is the morphism induced by the map* $g\colon B \to C$.

These propositions can be proved by straightforward arguments.

Proof of Theorem 1.5 First, we give a proof for (3). Note that the associativity in (1) and (2) can be proved similarly.

Consider the diagram

$$\begin{array}{ccccc} H^*(M^S \times M^{T\#U}) & \xrightarrow{\mathrm{incl}^*} & H^*(M^S \times_M M^{T\#U}) & \xrightarrow{\mathrm{comp}_{c!}} & H^*(M^{S\#T\#U}) \\ \big\downarrow{\scriptstyle \mathrm{comp}^*} & & \big\downarrow{\scriptstyle \mathrm{comp}^*} & & \big\downarrow{\scriptstyle \mathrm{comp}^*} \\ H^*(M^S \times M^T \times_M M^U) & \xrightarrow{\mathrm{incl}^*} & H^*(M^S \times_M M^T \times_M M^U) & \xrightarrow{\mathrm{comp}_{(c\times \mathrm{id})!}} & H^*(M^{S\#T} \times_M M^U) \\ \big\downarrow{\scriptstyle \mathrm{incl}_{\Delta!}} & & \big\downarrow{\scriptstyle \mathrm{incl}_{(\mathrm{id}\times\Delta)!}} & & \big\downarrow{\scriptstyle \mathrm{incl}_{(\mathrm{id}\times\Delta)!}} \\ H^*(M^S \times M^T \times M^U) & \xrightarrow{\mathrm{incl}^*} & H^*(M^S \times_M M^T \times M^U) & \xrightarrow{\mathrm{comp}_{(c\times \mathrm{id})!}} & H^*(M^{S\#T} \times M^U) \end{array}$$

Note that the boundary of the whole square is the same as the diagram (7.5). The upper left square is commutative by the functoriality of the cohomology and so are the upper right and lower left squares by Proposition 7.7. Next, we consider the lower right square. Applying Proposition 7.8 to the diagram

$$\begin{array}{ccccc} M^S \times_M M^T \times_M M^U & \xrightarrow{\text{comp}} & M^{S\#T} \times_M M^U & \xrightarrow{\text{incl}} & M^{S\#T} \times M^U \\ \downarrow & & \downarrow & & \downarrow \\ M \times M & \xrightarrow{c\times\mathrm{id}} & L^{k-1}M \times M & \xrightarrow{\mathrm{id}\times\Delta} & L^{k-1}M \times M^2 \end{array}$$

we have

$$\mathrm{incl}_{(\mathrm{id}\times\Delta)_!} \circ \mathrm{comp}_{(c\times\mathrm{id})_!} = (\mathrm{incl}\circ\mathrm{comp})_{(\mathrm{id}\times\Delta)_!\circ((\mathrm{id}\times\Delta)_*(c\times\mathrm{id})_!)}.$$

In order to compute the element

$$(\mathrm{id}\times\Delta)_!\circ((\mathrm{id}\times\Delta)_*(c\times\mathrm{id})_!) \in \mathrm{Ext}_{C^*(L^{k-1}M\times M^2)}(C^*(M\times M), C^*(L^{k-1}M\times M^2)),$$

we use the models constructed in Section 5. Let $\delta_! \in \mathrm{Hom}^m_{\wedge V^{\otimes 2}}(\mathcal{M}_{\mathrm{P}}, \wedge V^{\otimes 2})$ and $\gamma_! \in \mathrm{Hom}^{\overline{m}}_{\mathcal{M}_{S^{k-1}}}(\mathcal{M}_{D^k}, \mathcal{M}_{S^{k-1}})$ be representatives of the generators:

$$[\delta_!] = \Delta_! \in \mathrm{Ext}^m_{C^*(M^2)}(C^*(M), C^*(M^2)),$$
$$[\gamma_!] = c_! \in \mathrm{Ext}^{\overline{m}}_{C^*(L^{k-1}M)}(C^*(M), C^*(L^{k-1}M)).$$

Then, using the isomorphism

$$\mathrm{Ext}^{m+\overline{m}}_{C^*(M^2\times L^{k-1}M)}(C^*(M\times M), C^*(M^2\times L^{k-1}M)) \cong H^{m+\overline{m}}(\mathrm{Hom}_{\wedge V^{\otimes 2}\otimes\mathcal{M}_{S^{k-1}}}(\mathcal{M}_{\mathrm{P}}\otimes\mathcal{M}_{D^k}, \wedge V^{\otimes 2}\otimes\mathcal{M}_{S^{k-1}})),$$

we have a representation

$$(\mathrm{id}\times\Delta)_!\circ((\mathrm{id}\times\Delta)_*(c\times\mathrm{id})_!) = [\mathrm{id}\otimes\delta_!]\circ[\gamma_!\otimes\mathrm{id}] = [(-1)^{m\overline{m}}\gamma_!\otimes\delta_!]$$

as a chain map. Similarly, we compute the other composition to be

$$\mathrm{comp}_{(c\times\mathrm{id})_!}\circ\mathrm{incl}_{(\mathrm{id}\times\Delta)_!} = (\mathrm{comp}\circ\mathrm{incl})_{(c\times\mathrm{id})_!\circ((c\times\mathrm{id})_*)(\mathrm{id}\times\Delta)_!}$$

with

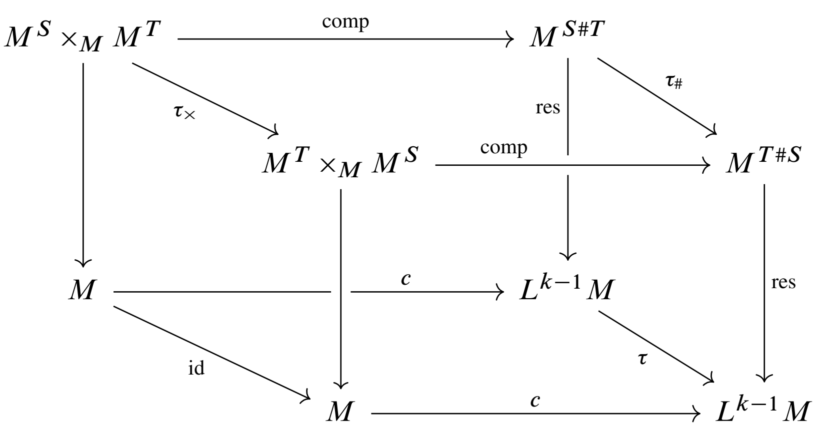
$$(c\times\mathrm{id})_!\circ((c\times\mathrm{id})_*)(\mathrm{id}\times\Delta)_! = [\gamma_!\otimes\delta_!].$$

This proves the commutativity by the sign $(-1)^{m\overline{m}}$ of the lower right square.

Next, we prove the commutativity of the coproduct in (2). This follows from the commutativity of the diagram

(7.9)
$$\begin{array}{ccccc} H^*(M^T \times M^S) & \xrightarrow{\mathrm{incl}^*} & H^*(M^T \times_M M^S) & \xrightarrow{\mathrm{comp}_!} & H^*(M^{T\#S}) \\ \downarrow{\scriptstyle \tau_\times^*} & & \downarrow{\scriptstyle \tau_\times^*} & & \downarrow{\scriptstyle \tau_\#^*} \\ H^*(M^S \times M^T) & \xrightarrow{\mathrm{incl}^*} & H^*(M^S \times_M M^T) & \xrightarrow{\mathrm{comp}_!} & H^*(M^{S\#T}) \end{array}$$

The commutativity of the left square is obvious. If one can apply Proposition 7.7 to the diagram

(7.10)


we obtain the commutativity of the right square of (7.9). In order to apply Proposition 7.7, it suffices to prove the equation

(7.11)
$$\mathrm{Ext}_{\tau^*}(\mathrm{id}, \tau^*)(c_!) = (-1)^{\overline{m}} c_!$$

in $\mathrm{Ext}_{C^*(L^{k-1}M)}(C^*(M), C^*(L^{k-1}M))$. Since

$$\mathrm{Ext}^{\overline{m}}_{C^*(L^{k-1}M)}(C^*(M), C^*(L^{k-1}M)) \cong \mathbb{K}$$

and $\mathrm{Ext}_{\tau^*}(\mathrm{id}, \tau^*) \circ \mathrm{Ext}_{\tau^*}(\mathrm{id}, \tau^*) = \mathrm{id}$, we have (7.11) up to sign. In Section 9, we will determine the sign to be $(-1)^{\overline{m}}$.

Similarly, in order to prove the commutativity of the product in (1), we need to prove the equation

(7.12)
$$\mathrm{Ext}_{\tau^*}(\mathrm{id}, \tau^*)(\Delta_!) = (-1)^m \Delta_!$$

in $\mathrm{Ext}_{C^*(M^2)}(C^*(M), C^*(M^2))$. As above, we have (7.12) up to sign. The sign is determined to be $(-1)^m$ in Section 8.

The same proofs can be applied for the (S, i, j)–brane product and coproduct. □

8 Proof of (7.12)

In this section, we will prove (7.12), determining the sign. Here, we need the explicit description of $\Delta_!$ in [11].

Let M be a 1–connected space with $\dim \pi_*(M) \otimes \mathbb{K} < \infty$. By [11, Theorem 1.6], we have a Sullivan model $(\wedge V, d)$ of M which is semipure, ie $d(I_V) \subset I_V$, where I_V is the ideal generated by V^{even}. Let $\varepsilon\colon (\wedge V, d) \to \mathbb{K}$ be the augmentation map and pr: $(\wedge V, d) \to (\wedge V/I_V, d)$ the quotient map. Take bases $V^{\mathrm{even}} = \mathbb{K}\{x_1, \dots x_p\}$ and $V^{\mathrm{odd}} = \mathbb{K}\{y_1, \dots y_q\}$. Recall the relative Sullivan algebra $\mathcal{M}_{\mathrm{P}} = (\wedge V^{\otimes 2} \otimes \wedge sV, d)$ over $(\wedge V, d)^{\otimes 2}$ from Section 6. Note that the relative Sullivan algebra $(\wedge V^{\otimes 2} \otimes \wedge sV, d)$ is a relative Sullivan model of the multiplication map $(\wedge V, d)^{\otimes 2} \to (\wedge V, d)$, Hence, using this as a semifree resolution, we have

$$\mathrm{Ext}_{\wedge V^{\otimes 2}}(\wedge V, \wedge V^{\otimes 2}) = H^*(\mathrm{Hom}_{\wedge V^{\otimes 2}}(\wedge V^{\otimes 2} \otimes \wedge sV, \wedge V^{\otimes 2})).$$

By [11, Corollary 5.5], we have a cocycle $f \in \mathrm{Hom}_{\wedge V^{\otimes 2}}(\wedge V^{\otimes 2} \otimes \wedge sV, \wedge V^{\otimes 2})$ satisfying

$$f(sx_1 \cdots sx_p) = \prod_{j=1}^{j=q} (1 \otimes y_j - y_j \otimes 1) + u$$

for some $u \in (y_1 \otimes y_1, \dots, y_q \otimes y_q)$. Consider the evaluation map

$$\mathrm{ev}\colon \mathrm{Ext}_{\wedge V^{\otimes 2}}(\wedge V, \wedge V^{\otimes 2}) \otimes \mathrm{Tor}_{\wedge V^{\otimes 2}}(\wedge V, \wedge V/I_V) \to \mathrm{Tor}_{\wedge V^{\otimes 2}}(\wedge V^{\otimes 2}, \wedge V/I_V) \xrightarrow{\cong} H^*(\wedge V/I_V),$$

where $(\wedge V, d)^{\otimes 2}$, $(\wedge V, d)$ and $(\wedge V/I_V, d)$ are $(\wedge V, d)^{\otimes 2}$–module via id, $\varepsilon \cdot \mathrm{id}$ and $\mathrm{pr} \circ (\varepsilon \cdot \mathrm{id})$, respectively. Here, we use $(\wedge V^{\otimes 2} \otimes \wedge sV, d)$ as a semifree resolution of $(\wedge V, d)$. Then, we have

$$\mathrm{ev}([f] \otimes [sx_1 \cdots sx_p]) = [y_1 \cdots y_q] \neq 0,$$

and hence $[f] \neq 0$ in $\mathrm{Ext}_{\wedge V^{\otimes 2}}(\wedge V, \wedge V^{\otimes 2})$. So it suffices to calculate $\mathrm{Ext}_t(\mathrm{id}, t)([f])$ to determine the sign in (7.12), where $t\colon (\wedge V, d)^{\otimes 2} \to (\wedge V, d)$ is the dga homomorphism defined by $t(v \otimes 1) = 1 \otimes v$ and $t(1 \otimes v) = v \otimes 1$.

Proof of (7.12) By definition, $\mathrm{Ext}_t(\mathrm{id}, t)$ is induced by the map

$$\mathrm{Hom}_t(\tilde{t}, t)\colon \mathrm{Hom}_{\wedge V^{\otimes 2}}(\wedge V^{\otimes 2} \otimes \wedge sV, \wedge V^{\otimes 2}) \to \mathrm{Hom}_{\wedge V^{\otimes 2}}(\wedge V^{\otimes 2} \otimes \wedge sV, \wedge V^{\otimes 2}),$$

where $\tilde{t}$ is the dga automorphism defined by $\tilde{t}|_{\wedge V^{\otimes 2}} = t$ and $\tilde{t}(sv) = -sv$. Since $\tilde{t}(sx_1 \cdots sx_p) = (-1)^p sx_1 \cdots sx_p$ and

$$t\Big(\prod_{j=1}^{j=q}(1 \otimes y_j - y_j \otimes 1)\Big) = (-1)^q \prod_{j=1}^{j=q}(1 \otimes y_j - y_j \otimes 1),$$

we have

$$\mathrm{ev}([\mathrm{Hom}_t(\tilde{t}, t)(f)] \otimes [sx_1 \cdots sx_p]) = \mathrm{ev}([t \circ f \circ \tilde{t}] \otimes [sx_1 \cdots sx_p]) = (-1)^{p+q}[y_1 \cdots y_q].$$

Since the parity of $p+q$ is the same as that of the dimension of $(\wedge V, d)$ as a Gorenstein algebra, the sign in (7.12) is proved to be $(-1)^m$. □

9 Proof of (7.11)

In this section, we give the proof of (7.11), using the spectral sequence constructed in the proof of Theorem 3.1. Although the key idea of the proof of Theorem 3.1 is the same as Theorem 2.3 due to Félix and Thomas, we give the proof here for the convenience of the reader.

Proof of Theorem 3.1 Take a $(A \otimes B, d)$–semifree resolution $\eta\colon (P, d) \xrightarrow{\cong} (A, d)$. Define $(C, d) = (\mathrm{Hom}_{A \otimes B}(P, A \otimes B), d)$. Then $\mathrm{Ext}_{A \otimes B}(A, A \otimes B) = H^*(C, d)$. We fix a nonnegative integer N, and define a complex

$$(C_N, d) = \big(\mathrm{Hom}_{A \otimes B}(P, (A/A^{>n}) \otimes B), d\big).$$

We will compute the cohomology of (C_N, d). Define a filtration $\{F^p C_N\}_{p \geq 0}$ on (C_N, d) by $F^p C_N = \mathrm{Hom}_{A \otimes B}(P, (A/A^{>n})^{\geq p} \otimes B)$. Then we obtain a spectral sequence $\{E_r^{p,q}\}_{r \geq 0}$ converging to $H^*(C_N, d)$.

Claim 9.1
$$E_2^{p,q} = \begin{cases} H^p(A/A^{>N}) & \text{if } q = m, \\ 0 & \text{if } q \neq m. \end{cases}$$

Proof of Claim 9.1 We may assume $p \leq N$. Then we have an isomorphism of complexes

$$(A^{\geq p}/A^{\geq p+1}, 0) \otimes (\mathrm{Hom}_B(B \otimes_{A \otimes B} P, B), d) \xrightarrow{\cong} (E_0^p, d_0),$$

hence

$$(A^{\geq p}/A^{\geq p+1}) \otimes H^*(\mathrm{Hom}_B(B \otimes_{A \otimes B} P, B), d) \xrightarrow{\cong} E_1^p.$$

Define

$$\overline{\eta}\colon (B,\overline{d})\otimes_{A\otimes B}(P,d)\xrightarrow{1\otimes\eta}(B,\overline{d})\otimes_{A\otimes B}(A,d)\cong\mathbb{K}.$$

Note that the last isomorphism follows from the assumption (a). Then, since η is a quasi-isomorphism, so is $\overline{\eta}$. Hence we have

$$H^q(\mathrm{Hom}_B(B\otimes_{A\otimes B}P,B),d)\cong\mathrm{Ext}_B^q(\mathbb{K},B)\cong\begin{cases}\mathbb{K} & \text{if } q=m,\\ 0 & \text{if } q\neq m,\end{cases}$$

by the assumption (b).

Hence we have

$$E_1^{p,q}\cong(A^{\geq p}/A^{\geq p+1})\otimes H^q(\mathrm{Hom}_B(B\otimes_{A\otimes B}P,B),d)\cong A^p\otimes\mathrm{Ext}_B^q(\mathbb{K},B).$$

Moreover, using the assumption (c) and the above isomorphisms, we can compute the differential d_1 and have an isomorphism of complexes

(9.2) $$(E_1^{*,q},d_1)\cong(A^*,d)\otimes\mathrm{Ext}_B^q(\mathbb{K},B).$$

This proves Claim 9.1. □

Now we return to the proof of Theorem 3.1. We will recover $H^*(C)$ from $H^*(C_N)$ by taking a limit. Since $\varprojlim^1_N C_N=0$, we have an exact sequence

$$0\to\varprojlim_N{}^1 H^*(C_N)\to H^*(\varprojlim_N C_N)\to H^*(\varprojlim_N H^*(C_N))\to 0.$$

By Claim 9.1, the sequence $\{H^*(C_N)\}_N$ satisfies the (degreewise) Mittag-Leffler condition, and hence $\varprojlim^1_N H^*(C_N)=0$. Thus, we have

$$H^l(C)\cong H^l(\varprojlim_N C_N)\cong\varprojlim_N H^l(C_N)\cong H^{l-m}(A).$$

This proves Theorem 3.1. □

Next, using the above spectral sequence, we determine the sign in (7.11).

Proof of (7.11) If $k=1$, (7.11) is the same as (7.12), which was proved in Section 8. Hence we assume $k\geq 2$. As in Section 7, let M be a k–connected $\mathbb{K}$–Gorenstein space of finite type with $\dim\pi_*(M)\otimes\mathbb{K}<\infty$, and $(\wedge V,d)$ its minimal Sullivan model. Using the Sullivan models constructed in Section 5, we have that the automorphism $\mathrm{Ext}_{\tau^*}(\mathrm{id},\tau^*)$ on

$$\mathrm{Ext}_{C^*(L^{k-1}M)}(C^*(M),C^*(L^{k-1}M))$$

is induced by the automorphism $\mathrm{Hom}_t(\widetilde{t},t)$ on

$$\mathrm{Hom}_{\wedge V\otimes\wedge s^{k-1}V}(\wedge V\otimes\wedge s^{k-1}V\otimes\wedge s^k V,\wedge V\otimes\wedge s^{k-1}V),$$

where t and $\widetilde{t}$ are the dga automorphisms on

$$(\wedge V\otimes\wedge s^{k-1}V,d) \quad\text{and}\quad (\wedge V\otimes\wedge s^{k-1}V\otimes\wedge s^k V,d),$$

respectively, defined by

$$\begin{aligned} &t(v)=v, \quad t(s^{k-1}v)=-s^{k-1}v,\\ &\widetilde{t}(v)=v, \quad \widetilde{t}(s^{k-1}v)=-s^{k-1}v, \quad \widetilde{t}(s^k v)=-s^k v. \end{aligned}$$

Now, consider the spectral sequence $\{E_r^{p,q}\}$ in the proof of Theorem 3.1 by taking $(A\otimes B,d)=(\wedge V\otimes\wedge s^{k-1}V,d)$ and $(P,d)=(\wedge V\otimes\wedge s^{k-1}V\otimes\wedge s^k V,d)$. Since $k\geq 2$, $\mathrm{Hom}_t(\widetilde{t},t)$ induces automorphisms on the complexes C_N and F^pC_N, and hence on the spectral sequence $\{E_r^{p,q}\}$. By the isomorphism (9.2), we have

$$E_2^{p,q}\cong H^p(A)\otimes\mathrm{Ext}^q_{\wedge s^{k-1}V}(\mathbb{K},\wedge s^{k-1}V),$$

and that the automorphism induced on E_2 is the same as $\mathrm{id}\otimes\mathrm{Ext}_{\overline{t}}(\mathrm{id},\overline{t})$, where $\overline{t}$ is defined by $\overline{t}(s^{k-1}v)=-s^{k-1}v$ for $v\in V$. Since the differential is zero on $\wedge s^{k-1}V$, we have an isomorphism

$$\mathrm{Ext}^*_{\wedge s^{k-1}V}(\mathbb{K},\wedge s^{k-1}V)\cong\bigotimes_i\mathrm{Ext}^*_{\wedge s^{k-1}v_i}(\mathbb{K},\wedge s^{k-1}v_i),$$

where $\{v_1,\dots,v_l\}$ is a basis of V. Using this isomorphism, we can identify

$$\mathrm{Ext}_{\overline{t}}(\mathrm{id},\overline{t})=\bigotimes_i\mathrm{Ext}_{\overline{t}_i}(\mathrm{id},\overline{t}_i),$$

where $\overline{t}_i$ is defined by $\overline{t}_i(s^{k-1}v_i)=-s^{k-1}v_i$.

Since $(-1)^{\dim V}=(-1)^{\overline{m}}$, it suffices to show $\mathrm{Ext}_{\overline{t}_i}(\mathrm{id},\overline{t}_i)=-1$. Taking a resolution, we have

$$\begin{aligned} \mathrm{Ext}^*_{\wedge s^{k-1}v_i}(\mathbb{K},\wedge s^{k-1}v_i)&=H^*(\mathrm{Hom}_{\wedge s^{k-1}v_i}(\wedge s^{k-1}v_i\otimes\wedge s^k v_i,\wedge s^{k-1}v_i)),\\ \mathrm{Ext}_{\overline{t}_i}(\mathrm{id},\overline{t}_i)&=H^*(\mathrm{Hom}_{\overline{t}_i}(\widehat{t}_i,\overline{t}_i)), \end{aligned}$$

where the differential d on $\wedge s^{k-1}v_i\otimes\wedge s^k v_i$ is defined by $d(s^{k-1}v_i)=0$ and $d(s^k v_i)=s^{k-1}v_i$, and the dga homomorphism $\widehat{t}_i$ is defined by $\widehat{t}_i(s^{k-1}v_i)=-s^{k-1}v_i$ and $\widehat{t}_i(s^k v_i)=-s^k v_i$. Using this resolution, we have the generator $[f]$ of the cohomology $H^*(\mathrm{Hom}_{\wedge s^{k-1}v_i}(\wedge s^{k-1}v_i\otimes\wedge s^k v_i,\wedge s^{k-1}v_i))\cong\mathbb{K}$ as follows:

- If $|s^{k-1}v_i|$ is odd, define f by $f(1) = s^{k-1}v_i$ and $f((s^k v_i)^l) = 0$ for $l \geq 1$.
- If $|s^{k-1}v_i|$ is even, define f by $f(1) = 0$ and $f((s^k v_i)) = 1$.

In both cases, we have $\mathrm{Hom}_{\overline{t}_i}(\widehat{t}_i, \overline{t}_i)(f) = \overline{t}_i \circ f \circ \widehat{t}_i = -f$. This proves $\mathrm{Ext}_{\overline{t}_i}(\mathrm{id}, \overline{t}_i) = -1$ and completes the determination of the sign in (7.11). □

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Graduate School of Mathematical Sciences, The University of Tokyo
Tokyo, Japan

swaka@ms.u-tokyo.ac.jp

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