Algebraic & Geometric Topology 19 (2019) 2989–3032

Quasi-right-veering braids and nonloose links

Tetsuya Ito
Keiko Kawamuro

We introduce a notion of quasi-right-veering for closed braids, which plays an analogous role to right-veering for open books. We show that a transverse link K in a contact 3–manifold (M,ξ) is nonloose if and only if every braid representative of K with respect to every open book decomposition that supports (M,ξ) is quasi-right-veering. We also show that several definitions of right-veering closed braids are equivalent.

57M50; 57M27

1 Introduction

The dichotomy between tight and overtwisted is fundamental to 3–dimensional contact topology. Eliashberg in [8] shows that two homotopic overtwisted contact structures as 2–plane fields are isotopic through contact structures. Therefore, the classification of overtwisted contact structures is reduced to homotopy classification of 2–plane fields. However, this is not the case for tight contact structures, and detecting tightness of a given contact structure often arises as an important problem.

A Legendrian or transverse link in a contact 3–manifold is called *loose* if the complement is overtwisted, and otherwise it is called *nonloose*. In the classification of Legendrian and transverse links in contact 3–manifolds, the nonloose vs loose dichotomy plays a similar role to the tight vs overtwisted dichotomy in the classification of contact structures. Eliashberg and Fraser [9] (resp. Etnyre [12]) show that loose null-homologous Legendrian (resp. transverse) links are coarsely classified by classical invariants called the Thurston–Bennequin number and the rotation number (resp. the self-linking number). Here, coarse classification means up to contactomorphism that is *smoothly* isotopic to the identity.

One useful method to study contact 3–manifolds and transverse links uses *open books* (S,ϕ). Here, S is an oriented compact surface with nonempty boundary ∂S and $\phi=[\varphi]\in\mathcal{MCG}(S)$ is a mapping class represented by a diffeomorphism $\varphi\in$

Published: 20 October 2019 DOI: 10.2140/agt.2019.19.2989

$\mathrm{Diff}^+(S, \partial S)$. This method is depending on Giroux's one-to-one correspondence [17] between contact structures up to isotopy and open books up positive stabilization. See Etnyre's paper [11] for detail. If a contact structure (M, ξ) corresponds with an open book (S, ϕ) under the Giroux correspondence, we say that the contact structure (M, ξ) is *supported* by the open book (S, ϕ).

With the Giroux correspondence, Bennequin [4], Mitsumatsu and Mori [28] and Pavelescu [29; 30] independently show that every transverse link in (M, ξ) can be identified with a closed braid with respect to (S, ϕ). In this paper, we study transverse links via braids.

See the research monograph [27] by LaFountain and Menasco for a gentle introduction to the techniques of *open book foliations* that is central to the new work in this paper. These foliations were first used by the authors in [20] and played an important role in [21; 22; 23; 24; 25; 26]. Open book foliations had their origins in the work of Birman and Menasco in a series of papers about *braid foliations*. See Birman and Finkelstein's article [5] for a useful guide to the work of Birman and Menasco on braid foliations, and [6] for their key paper that is relevant for us. It is the first place where braid foliations were used to solve a major then-open problem in contact topology.

We call ϕ the *monodromy* of the open book (S, ϕ) and we say that the monodromy ϕ is *right-veering* if it turns every properly embedded curve to the right near the boundary (see Definition 3.9). The following result of Honda, Kazez and Matić [18] gives a characterization of tightness in terms of right-veering monodromies.

Theorem 1.1 [18, Theorem 1.1] *A contact* 3*–manifold* (M, ξ) *is tight if and only if for every open book* (S, ϕ) *supporting* (M, ξ)*, the monodromy* ϕ *is right-veering.*

The table below will be helpful for the following discussion:

contact manifold (M, ξ)	transverse link $\mathcal{T}$ in (M, ξ)
tight/overtwisted	nonloose/loose
open book (S, ϕ)	closed braid $\boldsymbol{L}$ with respect to (S, ϕ)
right-veering monodromy	quasi-right-veering braid
monodromy $\phi \in \mathcal{MCG}(S)$	distinguished monodromy $[\varphi_L] \in \mathcal{MCG}(S, P)$
FDCT of ϕ with respect to C $c(\phi, C)$	FDTC of braid $\boldsymbol{L}$ with respect to C $c(\phi, \boldsymbol{L}, C) := c([\varphi_L], C)$

As a natural counterpart of right-veering mapping classes, *right-veering closed braids* (with respect to general open books) have been defined and studied in the literature, for example by Baldwin, Vela-Vick and Vértesi [3], Baldwin and Grigsby [2] and Plamenevskaya [31].

In [18], Honda, Kazez and Matić define the *fractional Dehn twist coefficient* (*FDTC*). The FDTC is an invariant of a monodromy and it detects right-veeringness of the monodromy. Hence the FDTC can be used to determine tight or overtwisted for the supported contact structure; see Colin and Honda [7], Honda, Kazez and Matić [19] and our [23].

In Section 2 we define a closed braid $\boldsymbol{L}$ in open book (S,ϕ) and discuss how to assign an element $[\varphi_L]$ of the mapping class group for $\boldsymbol{L}$. We call $[\varphi_L]$ the distinguished monodromy. Then, as a counterpart of the FDTC $c(\phi,C)$, we define $c(\phi,\boldsymbol{L},C)$, the *FDTC for a closed braid* $\boldsymbol{L}$ *with respect to an open book* (S,ϕ) and a boundary component C of S. The definitions given in this paper are more rigorous than those in our previous paper [25].

In [25], various results on open books and the FDTC are translated to results on closed braids and the FDTC for closed braids. This gives us some hope that open books and closed braids with respect to open books can be treated in a unified manner.

However, this is too optimistic: Note that Theorem 1.1 of Honda, Kazez and Matić implies that any non-right-veering open book supports an overtwisted contact structure, but not every non-right-veering closed braid is loose. A simple example of this fact is a non-right-veering closed braid with respect to an open book supporting a tight contact 3–manifold. In this paper, we find a condition on closed braids to be loose.

In Definition 3.11 we introduce *quasi-right-veering* closed braids. A closed braid L is called quasi-right-veering if for every properly embedded oriented arc $\alpha\subset S$ there does not exist a finite sequence of arcs $\alpha_0=\varphi_L(\alpha),\alpha_1,\alpha_2,\dots,\alpha_n=\alpha$ such that

- they all start from the same point, say $*$,
- α_{i+1} is on the right of α_i near $*$, and
- the interiors of α_i and α_{i+1} are disjoint.

After studying basic properties of quasi-right-veering braids we show that it is the quasi-right-veering condition on closed braids that plays the same role as the right-veering condition on open books in Theorem 1.1. Our first main result is the following:

Theorem 4.1 *A transverse link $\mathcal{T}$ in a contact 3–manifold (M,ξ) is nonloose if and only if every braid representative of $\mathcal{T}$ with respect to every open book decomposition of (M,ξ) is quasi-right-veering.*

In Theorem 4.1 we allow the transverse link $\mathcal{T}$ to be empty. Our definition of quasi-right-veering implies that the empty braid with respect to an open book (S,ϕ) is quasi-right-veering if and only if ϕ is right-veering. A loose empty link can be interpreted as having an overtwisted underlying contact structure. Therefore, Theorem 1.1 follows as a corollary of Theorem 4.1.

Sections 5 and 6 are devoted to more results on nonloose links.

The *depth* of a Legendrian or transverse link $\mathcal{T}\subset(M,\xi)$ is defined by Baker and Onaran in [1]. It is the minimal number of times $\mathcal{T}$ intersects an overtwisted disk in (M,ξ). If $\boldsymbol{L}$ is a braid with respect to (S,ϕ) (ie $\boldsymbol{L}$ is a transverse link), then its associated *axis-augmented transverse link* (Definition 5.3) is the union of $\boldsymbol{L}$ and the binding of (S,ϕ). In Theorem 5.5 we relate depth-one links and non-quasi-right-veering braids.

Theorem 5.5 *Let (S,ϕ) be an open book decomposition of (M,ξ) and let $\boldsymbol{L}$ be a closed braid in the open book (S,ϕ). The depth of the axis-augmented transverse link for $\boldsymbol{L}$ is one if and only if the braid $\boldsymbol{L}$ is not quasi-right-veering.*

Theorem 6.1 below is a result on braids and it can be seen as a generalization of [23, Corollary 1.2] as a result on open books.

Theorem 6.1 *Let (S,ϕ) be a planar (the genus of S is zero) open book of a contact 3–manifold (M,ξ). If a transverse link $\mathcal{T}\subset(M,\xi)$ is represented by a closed braid $\boldsymbol{L}$ such that $c(\phi,\boldsymbol{L},C)>1$ for every boundary component C of S, then $\mathcal{T}$ is nonloose.*

Finally, in Section 7 we address one subtle but important issue on right-veering closed braids. As mentioned above, three different definitions of right-veering closed braids have existed in the literature (see [2; 3; 31]), which we call ∂–$(\partial+P)$, ∂–∂ and ∂–P right-veering (see Definition 7.2). They are not equivalent when focusing on an individual boundary component as stated in Theorem 7.5. However, when all the boundary components are simultaneously considered, they are equivalent:

Corollary 7.6 *For $\psi\in\mathcal{MCG}(S,P)$ the following are equivalent*:

(1) *ψ is ∂–$(\partial+P)$ right-veering with respect to* ***all*** *the boundary components of S.*

(2) *ψ is ∂–∂ right-veering with respect to **all** the boundary components of S.*

(3) *ψ is ∂–P right-veering with respect to **all** the boundary components of S.*

In particular, when S has connected boundary, the three notions of right-veering are equivalent.

2 Closed braids as mapping classes and their FDTC

In this section we review the distinguished monodromy for a closed braid. The distinguished monodromy is an element of the mapping class group of a surface with marked points. We also review the definition of FDTC for closed braids and prove its well-definedness.

Let $S \simeq S_{g,d}$ be an oriented compact surface with genus g and d boundary components. Throughout the paper we assume $d > 0$. Let $P = \{p_1, \dots, p_n\}$ be a (possibly empty) set of n distinct interior points of S. Let $\mathcal{M}CG(S, P)$ (denoted by $\mathcal{M}CG(S)$ if P is empty) be the mapping class group of the punctured surface $S \setminus P$, which is the group of isotopy classes of orientation-preserving homeomorphisms of the surface S fixing ∂S pointwise and fixing P setwise. Let $\mathrm{Diff}^+(S, \partial S)$ denote the group of orientation-preserving diffeomorphisms of S that fix ∂S pointwise. Let $\mathrm{Diff}^+(S, P, \partial S)$ be the group of orientation-preserving diffeomorphisms of S that fix P setwise and ∂S pointwise.

2.1 Three notions of the word open book

Let $\phi \in \mathcal{M}CG(S)$ be a mapping class and $\varphi \in \mathrm{Diff}^+(S, \partial S)$ be a diffeomorphism representing ϕ. In the literature, the term "open book" has been used for closely related several meanings. In general this does not cause much trouble. However, when we discuss the mapping class group of a punctured surface we need a little more care.

In this paper, we call the pair (S, φ) an *abstract open book*, whereas we call the pair (S, ϕ) an *open book*. (In Etnyre's note [11], (S, φ) is called an abstract open book but (S, ϕ) is not assigned any name.) For a 3–manifold M, by *an embedded open book* (or *open book decomposition* in [11]) of M we mean a pair (B, π) of a fibered link $B \subset M$ and fibration $\pi\colon M \setminus B \to S^1$. The link B is called *the binding* and the closure of fiber $\pi^{-1}(t)$ is called a *page* and denoted by S_t.

Given an abstract open book (S, φ) let

$$\Pi\colon S \times [0,1] \to M_{(S,\varphi)} := S \times [0,1]/\sim$$

be the quotient map, where $\sim$ denotes the equivalence relation

$$\begin{aligned} (x,1) &\sim (\varphi(x),0) && \text{for all } x \in S, \\ (x,t) &\sim (x,s) && \text{for all } x \in \partial S \text{ and } t,s \in [0,1]. \end{aligned}$$

The manifold $M_{(S,\varphi)}$ is naturally equipped with an embedded open book $(B,\pi) = (B_{(S,\varphi)}, \pi_{(S,\varphi)})$ as follows: The binding is defined by

$$B = B_{(S,\varphi)} := \Pi(\partial S \times \{t\}),$$

which does not depend on the choice of $t \in [0,1]$. Identify S^1 with the quotient space $[0,1]/0 \sim 1$. The fibration is defined by

$$\pi = \pi_{(S,\varphi)} \colon M_{(S,\varphi)} \setminus B \to S^1, \quad \Pi(x,t) \mapsto t.$$

Then $\pi^{-1}(t)$ is the interior of the page $S_t = \Pi(S \times \{t\})$, which is a compact surface.

2.2 Generalized Birman exact sequence

Definition 2.1 Let $P = \{p_1, \dots, p_n\}$ and $P' = \{p'_1, \dots, p'_n\}$ be finite sets of interior points of S. (We do not require $P = P'$.) Let $\{x_1, \dots, x_n\}$ be an abstract set of n points. A *geometric n–braid* of S joining $P \times \{0\}$ and $P' \times \{1\}$ is embedding of n copies of the interval $[0,1]$ into $S \times [0,1]$,

$$\beta \colon \overbrace{[0,1] \sqcup \cdots \sqcup [0,1]}^{n} \cong \{x_1, \dots, x_n\} \times [0,1] \to S \times [0,1], \quad (x_i, t) \mapsto (\beta^i(t), t),$$

such that $\{\beta^1(0), \dots, \beta^n(0)\} = P$ and $\{\beta^1(1), \dots, \beta^n(1)\} = P'$ as (unordered) sets.

We view the geometric braid β as an isotopy $\{\beta_t \colon P \to S \mid t \in [0,1]\}$ of the set of points P such that $\beta_0 = \mathrm{id}_P$ and $\beta_1(P) = P'$. We extend the isotopy $\{\beta_t\}$ to an ambient smooth isotopy $\{\widehat{\beta}_t \colon S \to S \mid t \in [0,1]\}$ of S that satisfies the following:

- $\widehat{\beta}_0 = \mathrm{id}_S$.
- $\widehat{\beta}_t|_P = \beta_t$ for all $t \in [0,1]$.
- $\widehat{\beta}_t|_{\partial S} = \mathrm{id}_{\partial S}$ for all $t \in [0,1]$.

Given $\{\beta_t\}$, this extension $\{\widehat{\beta}_t\}$ is unique up to isotopy of S fixing ∂S. We call

$$\widehat{\beta}_1 \colon (S,P) \to (S,P') \tag{2-1}$$

a *diffeomorphism associated to the geometric braid* β.

When $P = P'$, the set of isotopy classes of geometric n–braids forms a group. Regardless of the choice of P, the group is isomorphic to $\pi_1(C(S,n))$ the fundamental group of the configuration space of n distinct, unordered points in S. In this paper we denote $\pi_1(C(S,n))$ by $B_n(S)$ and call it the *n–stranded surface braid group* of S. We denote by $[\beta]$ the element of $B_n(S)$ represented by the geometric braid β in $S\times[0,1]$ joining $P\times\{0\}$ and $P\times\{1\}$.

Suppose that β and β' are geometric n–braids joining $P\times\{0\}$ and $P\times\{1\}$ and they are ambient isotopic; namely, $[\beta]=[\beta']\in B_n(S)$. Then their associated diffeomorphisms $\widehat{\beta}_1$ and $\widehat{\beta}'_1 \in \mathrm{Diff}^+(S,P,\partial S)$ are isotopic. Therefore, we obtain a well-defined homomorphism i, which we call the *push map*,

$$i\colon B_n(S)\to \mathcal{MCG}(S,P), \quad [\beta]\mapsto[\widehat{\beta}_1].$$

Suppose that $\varphi\in\mathrm{Diff}^+(S,P,\partial S)$. Forgetting the points P, the diffeomorphism $\varphi\colon (S,P)\to(S,P)$ can be regarded as a diffeomorphism $\varphi\colon S\to S$. This defines a surjective homomorphism $f\colon \mathcal{MCG}(S,P)\to\mathcal{MCG}(S)$, called the *forgetful map*.

The push map i and the forgetful map f give the generalized Birman exact sequence [15, Theorem 9.1],

$$1\to B_n(S)\xrightarrow{i}\mathcal{MCG}(S,P)\xrightarrow{f}\mathcal{MCG}(S)\to 1. \tag{2-2}$$

2.3 The distinguished monodromy

Definition 2.2 A *closed n–braid L with respect to an abstract open book* (S,φ) is an oriented link in the 3–manifold $M_{(S,\varphi)}$ with the embedded open book $(B,\pi)=(B_{(S,\varphi)},\pi_{(S,\varphi)})$ such that $L\subset M_{(S,\varphi)}\setminus B$ and L intersects every page S_t positively and transversely at n points.

Definition 2.3 We say that two closed braids L and L' with respect to the same abstract open book (S,φ) are *braid isotopic* if they are isotopic through closed braids.

In the following, we discuss how to assign an element of the mapping class group, which we call the distinguished monodromy, to a closed braid (Definition 2.5), and study how braid isotopy affects the distinguished monodromy (Proposition 2.7).

The quotient map $\Pi\colon S\times[0,1]\to M_{(S,\varphi)}$ restricted on $S\times\{t\}$ gives a diffeomorphism $\Pi|_{S\times\{t\}}\colon S\times\{t\}\to S_t$. Composing $(\Pi|_{S\times\{t\}})^{-1}$ and the projection pr: $S\times[0,1]\to S$,

$(x,t) \mapsto x$, every page S_t of the embedded open book (B,π) of $M_{(S,\varphi)}$ is diffeomorphic to S:

$$S_t \xrightarrow{(\Pi|_{S\times\{t\}})^{-1}} S\times\{t\} \xrightarrow{\mathrm{pr}} S, \quad \Pi(x,t)\mapsto (x,t)\mapsto x.$$

Denote the diffeomorphism by $p_t := \mathrm{pr}\circ(\Pi|_{S\times\{t\}})^{-1}\colon S_t \to S$. We view $M_{(S,\varphi)}$ as the union of the pages S_t for $t\in[0,1)$ and extend p_t to

$$p\colon M_{(S,\varphi)} \to S \tag{2-3}$$

by setting $p|_{S_t} = p_t$ for $t\in[0,1)$. Thus, $p(\Pi(x,t)) = x$ for $t\in[0,1)$. We call p a *projection*. The projection p is clearly not continuous near the page S_0 since, in general,

$$\lim_{t\to 0^-} p(\Pi(x,t)) = \varphi(x) \neq x = \lim_{t\to 0^+} p(\Pi(x,t)).$$

Take a collar neighborhood $\nu(\partial S)$ of the boundary ∂S. In the following we impose the following conditions to braids, which can always be achieved by braid isotopy:

Definition 2.4 We say that a closed braid L with respect to an abstract open book (S,φ) is *admissible* if the following conditions are satisfied:

$$\varphi|_{\nu(\partial S)} = \mathrm{id}_{\nu(\partial S)}, \tag{2-4}$$

$$P := p(L\cap S_0) \subset \nu(\partial S). \tag{2-5}$$

Recall that under the quotient map $\Pi\colon S\times[0,1]\to M_{(S,\varphi)}$, a point $(x,1)$ is identified with the point $(\varphi(x),0)$. Cutting the manifold $M_{(S,\varphi)}$ along the page S_0, the admissible closed braid L gives rise to a geometric n–braid, denoted by $\beta_L \subset S\times[0,1]$, joining $P\times\{0\}$ and $P\times\{1\}$ such that $\Pi(\beta_L) = L$.

By (2-4) and (2-5) we have $\varphi|_P = \mathrm{id}$, hence we may view φ as an element of $\mathrm{Diff}^+(S,P,\partial S)$. In order to distinguish φ in $\mathrm{Diff}^+(S,\partial S)$ and φ in $\mathrm{Diff}^+(S,P,\partial S)$, we denote the latter by $j(\varphi)$. The map j can be regarded as the right inverse of the forgetful map f in the generalized Birman exact sequence (2-2) since $f([j(\varphi)] = [\varphi]$.

Choose a diffeomorphism $\widehat{\beta}_{L1} \in \mathrm{Diff}^+(S,P,\partial S)$ associated to the geometric braid β_L. Let

$$\varphi_L := \widehat{\beta}_{L1}\circ j(\varphi) \in \mathrm{Diff}^+(S,P,\partial S). \tag{2-6}$$

Then we have

$$(M_{(S,\varphi)}, L) \simeq ((S,P)\times[0,1])/\!\sim_{\varphi_L}, \tag{2-7}$$

where the equivalence relation $\sim_{\varphi_L}$ satisfies $(x,1)\sim(\varphi_L(x),0)$ for all $x\in S$ and $(x,1)\sim(x,t)$ for all $x\in\partial S$ and $t\in[0,1]$. Since $\widehat{\beta}_{L1}$ is unique up to isotopy, so is φ_L.

Definition 2.5 Let L be an admissible closed braid with respect to an abstract open book (S,φ). The *distinguished monodromy* of L is the mapping class

$$[\varphi_L]\in\mathcal{MCG}(S,P).$$

In [25] the above $[\varphi_L]$ is denoted by ϕ_L. In this paper, we do not use the notation ϕ_L because we want to distinguish closed braids with respect to (S,φ) (Definition 2.2) and closed braids with respect to (S,ϕ) (Definition 2.8), and L is only defined with respect to (S,φ).

Definition 2.6 When $|P|=|P'|$, the groups $\mathcal{MCG}(S,P)$ and $\mathcal{MCG}(S,P')$ are isomorphic. We say that an isomorphism $\Theta\colon\mathcal{MCG}(S,P)\to\mathcal{MCG}(S,P')$ is *point-changing* if Θ is defined by $\Theta([\psi])=[\theta^{-1}\circ\psi\circ\theta]$ for some orientation-preserving diffeomorphism $\theta\colon(S,P')\to(S,P)$ such that $\theta|_{\partial S}=\mathrm{id}_{\partial S}$ and θ is isotopic to id_S if we forget the marked points of P and P'. If $P=P'$, every point-changing isomorphism is an inner automorphism of $\mathcal{MCG}(S,P)$.

Proposition 2.7 *Let L and L' be admissible closed n–braids with respect to an abstract open book (S,φ). Write $P:=p(L\cap S_0)$ and $P':=p(L'\cap S_0)\subset\nu(\partial S)$. If L and L' are braid isotopic then there exists a point-changing isomorphism*

$$\gamma^*\colon\mathcal{MCG}(S,P)\to\mathcal{MCG}(S,P')$$

such that $[\varphi_{L'}]=\gamma^*([\varphi_L])$.

Proof Cutting the manifold $M_{(S,\varphi)}$ along the page S_0, the closed braids L and L' give rise to geometric n–braids $\beta:=\beta_L$ and $\beta':=\beta_{L'}\subset S\times[0,1]$, respectively. Since L and L' are braid isotopic we have

$$[\beta']=[\gamma^{-1}\bullet\beta\bullet\gamma^{\varphi}]\quad\text{(read from the right to left)}\tag{2-8}$$

for some geometric n–braid $\gamma\subset S\times[0,1]$ connecting $P'\times\{0\}$ and $P\times\{1\}$ and specified by an isotopy $\{\gamma_t\colon P'\to S\mid t\in[0,1]\}$ such that $\gamma_0=\mathrm{id}_{P'}$ and $\gamma_1(P')=P$. Here

- the bullet "$\bullet$" is concatenation of geometric braids;

- γ^{-1} is the geometric n–braid joining $P \times \{0\}$ and $P' \times \{1\}$ defined by
$$(\gamma^{-1})_t := \gamma_{1-t};$$
- γ^{φ} is the geometric n–braid joining $P' \times \{0\}$ and $P \times \{1\}$ defined by
$$(\gamma^{\varphi})_t := \varphi \circ \gamma_t.$$

As done in (2-1), we extend $\{\gamma_t\}$ to a smooth isotopy $\{\widehat{\gamma}_t \colon S \to S \mid t \in [0,1]\}$. The diffeomorphism $\widehat{\gamma}_1 \colon (S, P') \to (S, P)$ associated to γ gives rise to a point-changing isomorphism

$$\gamma^* \colon \mathcal{MCG}(S,P) \to \mathcal{MCG}(S,P'), \quad [\psi] \mapsto [\widehat{\gamma}_1^{-1} \circ \psi \circ \widehat{\gamma}_1]. \tag{2-9}$$

Similarly, we extend $\{\gamma^{\varphi}{}_t \colon P' \to S \mid t \in [0,1]\}$ to an isotopy $\{\widehat{\gamma^{\varphi}}_t \colon S \to S \mid t \in [0,1]\}$. Let $i' \colon B_n(S) \to \mathcal{MCG}(S,P')$ be the push map in the generalized Birman exact sequence (2-2) where P is replaced with P'. Then by the definition of push map we have

$$i'([\gamma^{-1} \bullet \beta \bullet \gamma^{\varphi}]) = [\widehat{\gamma}_1^{-1} \circ \widehat{\beta}_1 \circ \widehat{\gamma^{\varphi}}_1]. \tag{2-10}$$

Let $j'(\varphi)$ denote the diffeomorphism φ, viewed as an element of $\mathrm{Diff}^+(S, P', \partial S)$. By the definition of the braid γ^{φ} we have

$$\widehat{\gamma^{\varphi}}_1 = \varphi \circ \widehat{\gamma}_1 \circ \varphi^{-1} = j(\varphi) \circ \widehat{\gamma}_1 \circ j'(\varphi)^{-1}. \tag{2-11}$$

The following calculation concludes the proposition:

$$\begin{aligned}
[\varphi_{L'}] &= [\widehat{\beta'}_1 \circ j'(\varphi)] = i'([\beta'])[j'(\varphi)] && \text{(by Definition 2.5)}\\
&= i'([\gamma^{-1} \bullet \beta \bullet \gamma^{\varphi}])[j'(\varphi)] && \text{(by (2-8))}\\
&= [\widehat{\gamma}_1^{-1} \circ \widehat{\beta}_1 \circ \widehat{\gamma^{\varphi}}_1 \circ j'(\varphi)] && \text{(by (2-10))}\\
&= [\widehat{\gamma}_1^{-1} \circ \widehat{\beta}_1 \circ j(\varphi) \circ \widehat{\gamma}_1 \circ j'(\varphi)^{-1} \circ j'(\varphi)] && \text{(by (2-11))}\\
&= [\widehat{\gamma}_1^{-1} \circ (\widehat{\beta}_1 \circ j(\varphi)) \circ \widehat{\gamma}_1]\\
&= \gamma^*([\varphi_L]) && \text{(by Definition 2.5 and (2-9)).} \quad \square
\end{aligned}$$

In our previous papers, such as [25], by a closed braid with respect to the open book (S, ϕ) we mean a closed braid with respect to some *abstract* open book (S, φ) with $[\varphi] = \phi$. As long as we have geometric argument (eg open book foliations), this causes no trouble since we implicitly fix an abstract open book (S, φ) throughout the discussion. However, when we discuss connection to mapping class groups (eg the distinguished monodromy), we need to understand what happens if we take another diffeomorphism φ' with $[\varphi'] = \phi$.

Assume that φ and φ' are isotopic. Let $\{\rho_t \in \mathrm{Diff}^+(S, \partial S) \mid t \in [0,1]\}$ be a smooth isotopy between $\rho_0 = \mathrm{id}_S$ and $\rho_1 = \varphi' \circ \varphi^{-1}$, and define

$$\overline{\rho}\colon S \times [0,1] \to S \times [0,1] \quad \text{by } \overline{\rho}(x,t) = (\rho_{1-t}(x), t).$$

Since $\overline{\rho}(x,1) = (x,1)$ and $\overline{\rho}(\varphi(x),0) = (\varphi'(x),0)$, the map $\overline{\rho}$ induces a diffeomorphism

$$\rho\colon M_{(S,\varphi)} \to M_{(S,\varphi')} \tag{2-12}$$

preserving the embedded open books associated to (S,φ) and (S,φ'). That is, the binding is mapped to the binding $\rho(B_{(S,\varphi)}) = B_{(S,\varphi')}$ and for each $t \in [0,1]$ the page S_t of $M_{(S,\varphi)}$ is mapped to the page S_t of $M_{(S,\varphi')}$. In particular, if L is a closed braid with respect to (S,φ) then $\rho(L)$ is a closed braid with respect to (S,φ').

Definition 2.8 Let $L \subset M_{(S,\varphi)}$ (resp. $L' \subset M_{(S,\varphi')}$) be a closed braid with respect to (S,φ) (resp. (S,φ')).

1. We say that triples $((S,\varphi),L)$ and $((S,\varphi'),L')$ are *equivalent* if $[\varphi] = [\varphi']$ and the closed braid $\rho(L) \subset M_{(S,\varphi')}$ is braid isotopic to L'.
2. The equivalence class of $((S,\varphi),L)$ with $[\varphi] = \phi$ is called a *closed braid with respect to the open book* (S,ϕ) and denoted by $[(S,\varphi),L]$ or simply $\boldsymbol{L}$.

We remark that L and L' are braid isotopic if and only if $\varphi = \varphi'$ and $((S,\varphi),L)$ and $((S,\varphi),L')$ are equivalent

The next theorem states that the distinguished monodromy $[\varphi_L]$ is an invariant of the equivalence class of $((S,\varphi),L)$ up to point-changing isomorphism:

Theorem 2.9 *Let L (resp. L') be an admissible closed braid with respect to an abstract open book (S,φ) (resp. (S,φ')). Let $P := p(L \cap S_0)$ and $P' := p(L' \cap S_0)$. If $((S,\varphi),L)$ and $((S,\varphi'),L')$ are equivalent then there is a point-changing isomorphism $\gamma^*\colon \mathcal{MCG}(S,P) \to \mathcal{MCG}(S,P')$ such that $[\varphi'_{L'}] = \gamma^*([\varphi_L])$.*

Proof Since $\varphi|_{\nu(\partial S)} = \varphi'|_{\nu(\partial S)} = \mathrm{id}_{\nu(\partial S)}$, we may assume that the above isotopy $\{\rho_t\colon S \to S \mid t \in [0,1]\}$ between id_S and $\varphi' \circ \varphi^{-1}$ satisfies $\rho_t|_{\nu(\partial S)} = \mathrm{id}_{\nu(\partial S)}$ for all $t \in [0,1]$. Define $\overline{\rho}$ and ρ as above. Then $p(L \cap S_0) = p(\rho(L) \cap S_0) =: P \subset \nu(\partial S)$. This together with $[\varphi] = [\varphi']$ implies that

$$[j(\varphi)] = [j(\varphi')] \in \mathcal{MCG}(S, P). \tag{2-13}$$

Let β_L (resp. $\beta_{\rho(L)}$) denote the geometric braid obtained from L (resp. $\rho(L)$) by cutting the manifold $M_{(S,\varphi)}$ (resp. $M_{(S,\varphi')}$) along the page S_0. Note that $\beta_{\rho(L)} = \overline{\rho}(\beta_L)$. For $(x,t) \in S \times [0,1]$ and $s \in [0,1]$ let

$$\overline{\rho}_s(x,t) := (\rho_{s(1-t)}(x), t).$$

Then $\{\overline{\rho}_s \in \mathrm{Diff}^+(S \times [0,1]) \mid s \in [0,1]\}$ gives an isotopy between $\overline{\rho}_0 = \mathrm{id}_{S\times[0,1]}$ and $\overline{\rho}_1 = \overline{\rho}$. Therefore, the geometric braids β_L and $\overline{\rho}(\beta_L)$ are isotopic through the family of geometric braids $\{\overline{\rho}_s(\beta_L) \mid s \in [0,1]\}$ having the same endpoints, and in, the surface braid group,

$$[\beta_{\rho(L)}] = [\overline{\rho}(\beta_L)] = [\beta_L] \in B_n(S). \tag{2-14}$$

By (2-13) and (2-14) we have an identity between the distinguished monodromies of L and $\rho(L)$,

$$[\varphi'_{\rho(L)}] = i([\beta_{\rho(L)}])[j(\varphi')] = i([\beta_L])[j(\varphi)] = [\varphi_L] \in \mathcal{MCG}(S, P).$$

Since $\rho(L)$ and L' are braid isotopic, by Proposition 2.7 we get $[\varphi'_{L'}] = \gamma^*([\varphi'_{\rho(L)}]) = \gamma^*([\varphi_L])$. $\square$

Remark 2.10 One can develop the distinguished monodromy in more a general setting by weakening the definition of admissible closed braids. In the definition of admissible closed braids, if we replace the conditions (2-4) and (2-5) with a weaker one like $\varphi(P) = P$, we can still define the diffeomorphism $j(\varphi) \in \mathrm{Diff}^+(S, P, \partial S)$ which plays an essential role in the definition of the distinguished monodromy. However, given $\varphi \in \mathrm{Diff}^+(S, \partial S)$, finding an n–point set $P \subset S$ with $\varphi(P) = P$ is difficult since it amounts to a concrete search of periodic orbits of φ. This is why the assumptions (2-4) and (2-5) make it easier to define and use the distinguished monodromy of a closed braid.

2.4 The fractional Dehn twist coefficient for closed braids

We briefly review the definition of the *fractional Dehn twist coefficient* (*FDTC*, in short) with respect to a boundary component $C \subset \partial S$. We follow the original definition of Honda, Kazez and Matić [18], which uses the Nielsen–Thurston classification of

$\phi \in \mathcal{M}CG(S, P)$. For a different but equivalent definition that avoids the Nielsen–Thurston classification, see [25].

The FDTC is a map

$$c(-, C)\colon \mathcal{M}CG(S, P) \to \mathbb{Q}$$

from the mapping class group to rational numbers.

If ϕ is periodic, then $\phi^N = T_C^M T_{C_1}^{M_1} \cdots T_{C_n}^{M_n}$ for some integers N, M, $M_1, \ldots, M_n$, where $C_1, \ldots, C_n$ are connected components of ∂S other than C and T_X denotes the mapping class represented by the right-handed Dehn twist along a simple closed curve X. Then we define the FDTC by $c(\phi, C) = \frac{M}{N}$.

If ϕ is pseudo-Anosov, ϕ is represented by a pseudo-Anosov homeomorphism φ with a stable measured geodesic lamination Λ^s of S. The connected component X of $S \setminus \Lambda^s$ that contains C is homeomorphic to an annulus $S^1 \times [0,1]$ with finitely many points $\{(x_1, 1), \ldots, (x_N, 1)\}$ removed, where C is identified with $S^1 \times \{0\}$. Let $\lambda_i = \{x_i\} \times [0, 1) \subset X$. Then $\varphi(\lambda_i)$ is an arc that starts from $(x_i, 0)$ and approaches $(x_{i+k}, 1)$ for some $k > 0$, after winding around C clockwise $\ell \in \mathbb{Z}$ times. Then we define $c(\phi, C) = \ell + \frac{k}{n}$ (see Figure 1).

Finally, when ϕ is reducible there is a representative homeomorphism φ and a subsurface $S' \subset S$ that contains C such that $\varphi|_{S'}\colon S' \to S'$ is periodic or pseudo-Anosov. Then we define $c(\phi, C) = c(\phi|_{S'}, C)$.

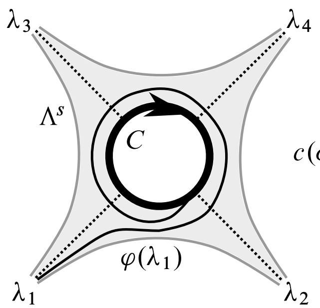

Figure 1: The FDTC for pseudo-Anosov case: the arc $\varphi(\lambda_1)$ approaches λ_2 after winding once around C clockwise.

Proposition 2.11 *Let L and L' be admissible closed braids with respect to (S, φ) and (S, φ'), respectively. If $((S, \varphi), L)$ and $((S, \varphi'), L')$ are equivalent then for every boundary component C we have*

$$c([\varphi_L], C) = c([\varphi'_{L'}], C).$$

Proof Let $P := p(L \cap S_0)$ and $P' := p(L' \cap S_0)$. By Theorem 2.9 there is a point-changing isomorphism $\gamma^*\colon \mathcal{MCG}(S,P) \to \mathcal{MCG}(S,P')$ such that $[\varphi'_{L'}] = \gamma^*([\varphi_L])$. By the properties of the isotopy $\{\widehat{\gamma}_t\}$ in the proof of Proposition 2.7, the following diagram commutes:

$$\begin{tikzcd} \mathcal{MCG}(S,P) \arrow[rr, "\gamma^*"] \arrow[dr, "c(-{,}C)"'] & & \mathcal{MCG}(S,P') \arrow[dl, "c(-{,}C)"] \\ & \mathbb{Q} & \end{tikzcd}$$

Therefore,

$$c([\varphi_L], C) = c(\gamma^*([\varphi_L]), C) = c([\varphi'_{L'}], C). \qquad \square$$

When $P' = P$, the isomorphism γ^* is an inner automorphism of $\mathcal{MCG}(S,P)$ and the commutativity implies invariance of the FDTC under conjugation: $c(\psi[\varphi_L]\psi^{-1}, C) = c([\varphi_L], C)$ for any $\psi \in \mathcal{MCG}(S,P)$.

Now we are ready to define the FDTC for a braid.

Definition 2.12 Let L be a closed braid (not necessarily admissible) with respect to an abstract open book (S,φ). Suppose that L' is an admissible closed braid with respect to (S,φ) such that L and L' are braid isotopic. The *fractional Dehn twist coefficient* (*FDTC*) *of the equivalence class* $\boldsymbol{L} = [(S,\varphi), L]$ *with respect to* C is the FDTC of the distinguished monodromy $[\varphi_{L'}]$ with respect to C and denote it by $c(\phi, \boldsymbol{L}, C)$ (in [25] it is denoted by $c(\phi, L, C)$). Namely,

$$c(\phi, \boldsymbol{L}, C) := c([\varphi_{L'}], C).$$

Thanks to Proposition 2.11, the FDTC $c(\phi, \boldsymbol{L}, C)$ is well defined.

If a braid L is empty, we set $P = \varnothing$ and define the distinguished monodromy $[\varphi_L] := [\varphi] = \phi$. Hence the FDTC of the empty closed braid is equal to the FDTC of the monodromy of the open book.

3 Quasi-right-veering maps

In this section we introduce a partial ordering $\ll_{\mathsf{right}}$ and quasi-right-veering closed braids, then we compare right-veering and quasi-right-veering. We use the same notation as in the previous section.

3.1 Strongly right-veering partial ordering $\ll_{\mathsf{right}}$

Definition 3.1 For each boundary component C of S, we choose a basepoint $*_C \in C$. Let $\mathcal{A}_C(S, P)$ be the set of isotopy classes of properly embedded arcs $\gamma\colon [0,1] \to S \setminus P$ satisfying $\gamma(0) = *_C$ and $\gamma(1) \in \partial S \setminus \{*_C\}$. Here, by isotopy we mean isotopy fixing the endpoints $\gamma(0)$ and $\gamma(1)$.

For simplicity of notation, an actual arc $\gamma\colon [0,1] \to S$ representing its isotopy class $[\gamma] \in \mathcal{A}_C(S, P)$ may be denoted by the same symbol, γ. We may call an element of $\mathcal{A}_C(S, P)$ simply an *arc* γ instead of the isotopy class of γ.

We say that two arcs α and β intersect *efficiently* if they attain the geometric intersection number between their isotopy classes.

Definition 3.2 Let $\alpha, \beta \in \mathcal{A}_C(S, P)$. Suppose that (arcs representing) α and β intersect efficiently. We write $\alpha \prec_{\mathsf{right}} \beta$ and say that β lies on the *right side* of α if the arc β lies on the right side of α in a small neighborhood of the basepoint $*_C$.

Remark In [18; 25], where the set P is empty, the symbol $>$ is used in place of $\prec_{\mathsf{right}}$.

The order $\prec_{\mathsf{right}}$ is a total ordering. For any family of arcs $\{\alpha_i\} \subset \mathcal{A}_C(S, P)$ we can always put them in a position simultaneously so that α_i and α_j intersect efficiently for any pairs (i, j). This can be done, for example, by choosing a hyperbolic metric on $S \setminus P$ and realizing the arcs as geodesics.

We introduce another ordering $\ll_{\mathsf{right}}$ which plays a central role in this paper.

Definition 3.3 For arcs $\alpha, \beta \in \mathcal{A}_C(S, P)$, we define $\alpha \ll_{\mathsf{right}} \beta$ if there exists a sequence of arcs $\alpha_0, \dots, \alpha_k \in \mathcal{A}_C(S; P)$ such that

$$\alpha = \alpha_0 \prec_{\mathsf{right}} \alpha_1 \prec_{\mathsf{right}} \cdots \prec_{\mathsf{right}} \alpha_k = \beta, \tag{3-1}$$

$$\mathrm{Int}(\alpha_i) \cap \mathrm{Int}(\alpha_{i+1}) = \varnothing \quad \text{for all } i = 0, \dots, k-1. \tag{3-2}$$

In the rest of the subsection, we study properties of $\ll_{\mathsf{right}}$.

By the definition it is easy to see that $\ll_{\mathsf{right}}$ is a partial ordering, ie $\alpha \ll_{\mathsf{right}} \beta$ and $\beta \ll_{\mathsf{right}} \gamma$ imply $\alpha \ll_{\mathsf{right}} \gamma$. If the puncture set P is empty, then [18, Lemma 5.2] shows that the ordering $\ll_{\mathsf{right}}$ coincides with $\prec_{\mathsf{right}}$. However, when P is nonempty, $\ll_{\mathsf{right}}$ is *not* a total ordering and there is a difference between $\prec_{\mathsf{right}}$ and $\ll_{\mathsf{right}}$, as shown in Proposition 3.5. To see the difference we first introduce the following notion:

Definition 3.4 Let $\alpha, \beta \in \mathcal{A}_C(S, P)$ with $\alpha \prec_{\mathrm{right}} \beta$. Assume that there exist subarcs $\delta_\alpha \subset \alpha$ and $\delta_\beta \subset \beta$ such that

- $*_C \in \delta_\alpha \cap \delta_\beta$;
- $\delta_\alpha \cup \delta_\beta$ bounds a (possibly immersed) bigon $D \subset S$ which lies on the right side of α (ie the orientation of δ_α, as a subarc of α, disagrees with the orientation of ∂D); and
- $D \cap P \neq \varnothing$ (D contains some marked points).

We call the bigon D a *boundary right P–bigon from α to β*.

A boundary right P–bigon gives an obstruction for $\alpha \ll_{\mathrm{right}} \beta$:

Proposition 3.5 *Let $\alpha, \beta \in \mathcal{A}_C(S, P)$ be arcs with $\alpha \prec_{\mathrm{right}} \beta$. If there is a boundary right P–bigon from α to β then $\alpha \not\ll_{\mathrm{right}} \beta$.*

Proof If there is a boundary right P–bigon D from α to β then every arc $\gamma \in \mathcal{A}_C(S; P)$ that satisfies $\alpha \prec_{\mathrm{right}} \gamma \prec_{\mathrm{right}} \beta$ must intersect D and yields either a boundary right P–bigon from α to γ or from γ to β (see Figure 2, left). Thus, for any sequence of arcs $\alpha = \gamma_0 \prec_{\mathrm{right}} \gamma_1 \prec_{\mathrm{right}} \cdots \prec_{\mathrm{right}} \gamma_n = \beta$ there exists an $i \in \{0, \ldots, n-1\}$ such that γ_i and γ_{i+1} forms a boundary right P–bigon, which means $\mathrm{Int}(\gamma_i)$ and $\mathrm{Int}(\gamma_{i+1})$ cannot be disjoint. $\square$

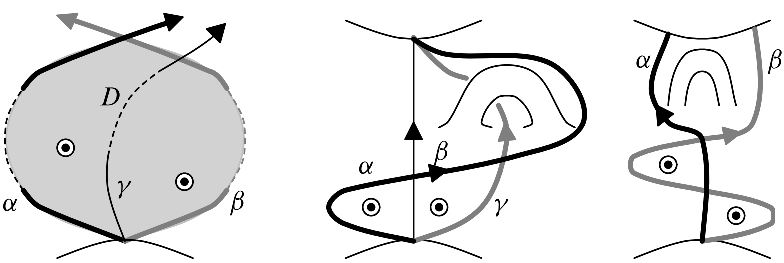

Figure 2: Left: The arc γ with $\alpha \prec_{\mathrm{right}} \gamma \prec_{\mathrm{right}} \beta$ cuts the boundary right P–bigon D, yielding a boundary right P–bigon from γ to β. Center: $\alpha \prec_{\mathrm{right}} \beta$ and $\beta \ll_{\mathrm{right}} \gamma$, but $\alpha \not\ll_{\mathrm{right}} \gamma$. Right: $f(\alpha) \prec_{\mathrm{right}} f(\beta)$ and $\alpha \prec_{\mathrm{right}} \beta$, but $\alpha \not\ll_{\mathrm{right}} \beta$.

As a corollary, we observe that conditions $\alpha \prec_{\mathrm{right}} \beta$ and $\beta \ll_{\mathrm{right}} \gamma$ may *not* imply $\alpha \ll_{\mathrm{right}} \gamma$ in general. (Also $\alpha \ll_{\mathrm{right}} \beta$ and $\beta \prec_{\mathrm{right}} \gamma$ may not imply $\alpha \ll_{\mathrm{right}} \gamma$.) For example, the arcs depicted in Figure 2, center, satisfy $\alpha \prec_{\mathrm{right}} \beta$ and $\beta \ll_{\mathrm{right}} \gamma$ but, by Proposition 3.5, $\alpha \not\ll_{\mathrm{right}} \gamma$.

We conjecture the converse of Proposition 3.5:

Conjecture 3.6 *We have $\alpha \ll_{\mathrm{right}} \beta$ if and only if $\alpha \prec_{\mathrm{right}} \beta$ and there exist no boundary right P–bigons from α to β.*

We study more properties of $\ll_{\mathrm{right}}$:

Lemma 3.7 *Let $f\colon \mathcal{A}_C(S,P) \to \mathcal{A}_C(S)$ be the forgetful map. If $\alpha \ll_{\mathrm{right}} \beta$ in $\mathcal{A}_C(S;P)$ then we have $f(\alpha) \prec_{\mathrm{right}} f(\beta)$ in $\mathcal{A}_C(S)$.*

Proof Since $\alpha \ll_{\mathrm{right}} \beta$, there is a sequence of arcs $\alpha = \gamma_0 \prec_{\mathrm{right}} \gamma_1 \prec_{\mathrm{right}} \cdots \prec_{\mathrm{right}} \gamma_n = \beta$ in $\mathcal{A}_C(S,P)$ with $\mathrm{Int}(\gamma_i) \cap \mathrm{Int}(\gamma_{i+1}) = \varnothing$ for all i. This implies that γ_i and γ_{i+1} do not cobound any marked bigons. Therefore, $\mathrm{Int}(f(\gamma_i)) \cap \mathrm{Int}(f(\gamma_{i+1})) = \varnothing$ and we can conclude $f(\alpha) = f(\gamma_0) \prec_{\mathrm{right}} f(\gamma_1) \prec_{\mathrm{right}} \cdots \prec_{\mathrm{right}} f(\gamma_n) = f(\beta)$ in $\mathcal{A}_C(S)$; that is, $f(\alpha) \prec_{\mathrm{right}} f(\beta)$ in $\mathcal{A}_C(S)$. □

Remark The converse of Lemma 3.7 does not hold in general, even if we assume $\alpha \prec_{\mathrm{right}} \beta$. See Figure 2, right.

The next proposition gives a sufficient condition for $\alpha \ll_{\mathrm{right}} \beta$.

Proposition 3.8 *Let $\alpha, \beta \in \mathcal{A}_C(S,P)$ be arcs with $\alpha \prec_{\mathrm{right}} \beta$. If α and β do not cobound bigons with marked points, then $\alpha \ll_{\mathrm{right}} \beta$.*

Proof If α and β do not cobound bigons with marked points then following the proof of [18, Lemma 5.2] one can construct an arc $\gamma \in \mathcal{A}_C(S;P)$ such that $\alpha \prec_{\mathrm{right}} \gamma \prec_{\mathrm{right}} \beta$ with $\#(\alpha,\gamma) < \#(\alpha,\beta)$ and $\#(\gamma,\beta) < \#(\alpha,\beta)$. Here $\#(-,-)$ denotes the geometric intersection number of the interiors of the two arcs. Moreover, the construction of γ shows that the pair α and γ and the pair γ and β do not cobound bigons with marked points. Thus, iterating this interpolation process, we get a sequence of arcs satisfying the conditions (3-1) and (3-2). □

3.2 Definition of quasi-right-veering

The mapping class group $\mathcal{MCG}(S,P)$ acts on the set $\mathcal{A}_C(S;P)$ naturally. Let $\phi \in \mathcal{MCG}(S,P)$ be represented by $\varphi \in \mathrm{Diff}^+(S,P,\partial S)$ and $\alpha \in \mathcal{A}_C(S;P)$ be represented by an arc $a \in S \setminus P$. Then $\phi(\alpha)$ denotes the isotopy class of the arc $\varphi(a)$.

Naturally extending the notion of right-veering mapping classes in [18], we define the following (see Baldwin, Vela-Vick and Vértesi [3, Section 4]):

Definition 3.9 We say that $\psi \in \mathcal{MCG}(S, P)$ is *right-veering* with respect to the boundary component C if $\alpha \prec_{\mathsf{right}} \psi(\alpha)$ or $\alpha = \psi(\alpha)$ for every $\alpha \in \mathcal{A}_C(S, P)$.

Remark 3.10 Baldwin and Grigsby [2] and Plamenevskaya [31] use a slightly different definition of "right-veering". In Section 7 we discuss the relationship between these two superficially different notions of right-veering.

Since $\prec_{\mathsf{right}}$ is a total ordering on the set $\mathcal{A}_C(S, P)$, $\psi \in \mathcal{MCG}(S, P)$ is right-veering if and only if $\psi(\alpha) \not\prec_{\mathsf{right}} \alpha$ for every $\alpha \in \mathcal{A}_C(S, P)$.

With this alternative definition of right-veering in mind, we introduce quasi-right-veering mapping classes.

Definition 3.11 We say that $\psi \in \mathcal{MCG}(S, P)$ is *quasi-right-veering* with respect to the boundary component C of S if every arc $\alpha \in \mathcal{A}_C(S, P)$ satisfies $\psi(\alpha) \not\ll_{\mathsf{right}} \alpha$. (Warning: since $\ll_{\mathsf{right}}$ is not a total ordering, $\psi(\alpha) \not\ll_{\mathsf{right}} \alpha$ is not equivalent to $\alpha \ll_{\mathsf{right}} \psi(\alpha)$ or $\alpha = \psi(\alpha)$.)

We note that the definitions of right-veering and quasi-right-veering are independent of a choice of the distinguished point $*_C$.

Next we show that right-veering and quasi-right-veering are invariant properties for equivalence classes $\boldsymbol{L} = [(S, \varphi), L]$.

Proposition 3.12 *Let L (resp. L') be an admissible closed braid with respect to an abstract open book (S, φ) (resp. (S, φ')). Suppose that $((S, \varphi), L)$ and $((S, \varphi'), L')$ are equivalent. Then, for every boundary component C of S, the distinguished monodromy $[\varphi_L]$ is right-veering (resp. quasi-right-veering) with respect to C if and only if the distinguished monodromy $[\varphi'_{L'}]$ is right-veering (resp. quasi-right-veering) with respect to C.*

Proof A diffeomorphism $\theta \in \mathrm{Diff}^+(S, \partial S)$ induces a map

$$\theta_*\colon \mathcal{A}_C(S, P) \to \mathcal{A}_C(S, \theta(P)).$$

By definition of $\prec_{\mathsf{right}}$ and $\ll_{\mathsf{right}}$, both $\prec_{\mathsf{right}}$ and $\ll_{\mathsf{right}}$ are preserved by θ_*. That is, $\alpha \prec_{\mathsf{right}} \beta$ (resp. $\alpha \ll_{\mathsf{right}} \beta$) if and only if $\theta_*(\alpha) \prec_{\mathsf{right}} \theta_*(\beta)$ (resp. $\theta_*(\alpha) \ll_{\mathsf{right}} \theta_*(\beta)$).

This implies that, if $\Theta\colon \mathcal{MCG}(S, P) \to \mathcal{MCG}(S, P')$ is a point-changing isomorphism (Definition 2.6) then $\phi \in \mathcal{MCG}(S, P)$ is right-veering (resp. quasi-right-veering) if and only if $\Theta(\phi) \in \mathcal{MCG}(S, P')$ is right-veering (resp. quasi-right-veering). By Theorem 2.9, this means that the distinguished monodromy $[\varphi_L]$ is right-veering (resp. quasi-right-veering) if and only if $[\varphi'_{L'}]$ is right-veering (resp. quasi-right-veering). $\square$

Now we define a (quasi-)right-veering closed braid, which is a central object in the paper.

Definition 3.13 Let C be a boundary component of S. Let L be a closed braid with respect an abstract open book (S, φ). We say that the closed braid $\boldsymbol{L} := [(S, \varphi), L]$ with respect to the open book (S, ϕ) is

- *right-veering with respect to* C (resp. *quasi-right-veering with respect to* C) if there exists an admissible closed braid L' with respect to (S, φ) that represents $\boldsymbol{L}$ such that $[\varphi_{L'}] \in \mathcal{MCG}(S, P)$ is right-veering (resp. quasi-right-veering) with respect to C.
- *right-veering* (resp. *quasi-right-veering*) if $\boldsymbol{L}$ is right-veering (resp. quasi-right-veering) with respect to every boundary component of S.

Well-definedness follows from Proposition 3.12.

3.3 Comparison of quasi-right-veering and right-veering

In this section, we discuss relation (Proposition 3.14) and difference (Proposition 3.16 and Corollary 3.17) between quasi-right-veering and right-veering.

First, if L is empty then by identifying $[\varphi_L]$ with ϕ, the empty closed braid is quasi-right-veering if and only if the monodromy ϕ is right-veering.

In general, we have the following:

Proposition 3.14 *Let $\psi \in \mathcal{MCG}(S, P)$ be a mapping class.*

(1) *If ψ is right-veering then ψ is quasi-right-veering.*

(2) *If $f(\psi) \in \mathcal{MCG}(S)$ is right-veering then ψ is quasi-right-veering, where $f\colon \mathcal{MCG}(S, P) \to \mathcal{MCG}(S)$ is the forgetful map in the generalized Birman exact sequence (2-2).*

We rephrase Proposition 3.14 as follows, in terms of open books and closed braids.

Corollary 3.15 *Every closed braid with respect to an open book (S, ϕ) is quasi-right-veering if $\phi \in \mathcal{MCG}(S)$ is right-veering.*

Proof of Proposition 3.14 The first statement immediately follows from the definition of quasi-right-veering.

To prove the second statement, assume that $\psi \in \mathcal{MCG}(S, P)$ is not quasi-right-veering with respect to some boundary component C of S. Then there exists an arc $\alpha \in \mathcal{A}_C(S, P)$ such that $\psi(\alpha) \ll_{\text{right}} \alpha$. By Lemma 3.7 we get $f(\psi)(f(\alpha)) = f(\psi(\alpha)) \prec_{\text{right}} f(\alpha)$ in $\mathcal{A}_C(S)$; that is, $f(\psi) \in \mathcal{MCG}(S)$ is not right-veering. $\square$

It is proved in [18, Section 3] that the right-veeringness of $\phi \in \mathcal{MCG}(S)$ is almost equivalent to positivity of its FDTC. We say "almost" because the statement is slightly complicated when ϕ is not pseudo-Anosov and its FDTC is 0. If $\phi \in \mathcal{MCG}(S)$ is pseudo-Anosov, ϕ is right-veering with respect to a boundary component C if and only if $c(\phi, C) > 0$. We remark that parallel statements on positivity and right-veeringness hold for elements $\psi \in \mathcal{MCG}(S, P)$. Namely if ψ is right-veering then $c(\psi, C) \geq 0$. Moreover, if ψ is pseudo-Anosov then ψ is right-veering with respect to C if and only if $c(\psi, C) > 0$.

The next proposition shows significant difference between quasi-right-veering and right-veering. In particular, quasi-right-veering is much less related to positivity of the FDTC.

Proposition 3.16 *Let (S, ϕ) be an open book.*

(1) *For every boundary component C of S and integers $N < 0$ and $n > 1$, there exists a closed n–braid $\boldsymbol{L} = [(S, \varphi), L]$ with respect to (S, ϕ) such that*
   - *$\boldsymbol{L}$ is quasi-right-veering with respect to C, and*
   - *$c(\phi, \boldsymbol{L}, C) \leq N < 0$; ie $\boldsymbol{L}$ is non-right-veering with respect to C.*

(2) *For every negative integer N there exists a closed braid $\boldsymbol{L}$ with respect to (S, ϕ) such that*
   - *$\boldsymbol{L}$ is quasi-right-veering, and*
   - *$c(\phi, \boldsymbol{L}, C) \leq N < 0$ for every boundary component C; ie $\boldsymbol{L}$ is non-right-veering.*

Proof Fix a boundary component C of S. Take $\varphi \in \mathrm{Diff}^+(S, \partial S)$ representing ϕ such that $\varphi|_{\nu(\partial S)} = \mathrm{id}_{\nu(\partial S)}$. Let $\nu(C)$ denote the connected component of $\nu(\partial S)$ that contains C. We identify $\nu(C)$ with the annulus $A = \{z \in \mathbb{C} \mid 1 \leq |z| < 2\}$ so that the boundary component C is identified with $\{z \in \mathbb{C} \mid |z| = 1\}$.

We put

$$P = \left\{ p_i \in \mathbb{C} \mid i = 1, \dots, n \text{ and } p_i = 1 + \frac{i}{n+1} \right\} \subset A \cong \nu(C) \subset S.$$

For $k \in \mathbb{N}$ let $\beta_{C,k} \subset S \times [0,1]$ be the geometric n–braid whose i^{th} strand $\gamma_{k,i} \colon [0,1] \to A \times [0,1] \subset S \times [0,1]$ is given by (see Figure 3, left)

$$\gamma_{k,i}(t) = \begin{cases} \left(\left(1 + \frac{1}{n+1}\right) \exp(2\pi\sqrt{-1}kt), t\right) & (i = 1), \\ \left(\left(1 + \frac{2}{n+1}\right) \exp(-2\pi\sqrt{-1}kt), t\right) & (i = 2), \\ \left(1 + \frac{i}{n+1}, t\right) & (i = 3, \dots, n). \end{cases}$$

Thus, the 1^{st} strand of $\beta_{C,k}$ winds k times around C counterclockwise and the 2^{nd} strand winds k times clockwise. Let $L_{C,k} := \Pi(\beta_{C,k}) \subset M_{(S,\varphi)}$ be the closed n–braid with respect to the abstract open book (S, φ) obtained by taking the braid closure of $\beta_{C,k}$, where $\Pi \colon S \times [0,1] \to M_{(S,\varphi)} := S \times [0,1]/\sim$ is the quotient map.

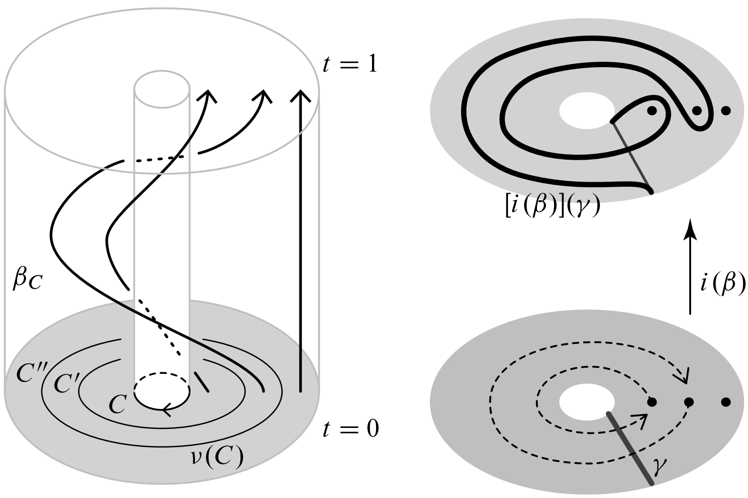

Figure 3: Left: The braid $L_{C,1}$ is not right-veering but quasi-right-veering. Right: The map $(T_C)^{-1}(T_{C'})^2(T_{C''})^{-1}$ forces to form a boundary right P–bigon.

With the push map $i \colon B_n(S) \to \mathcal{MCG}(S, P)$ in the generalized Birman exact sequence (2-2) we have $i([\beta_{C,1}]) = (T_C)^{-1}(T_{C'})^2(T_{C''})^{-1}$, where T_C, $T_{C'}$ and $T_{C''}$ are the right-handed Dehn twists about the curves C, $C' = \left\{ z \in A \mid |z| = \frac{3}{2n+2} \right\}$ and $C'' = \left\{ z \in A \mid |z| = \frac{5}{2n+2} \right\}$. The distinguished monodromy of the closed braid $L := L_{C,k}$ is

$$[\varphi_L] = [\varphi_{L_{C,k}}] = i([\beta_{C,k}])[j(\varphi)] = (T_C)^{-k}(T_{C'})^{2k}(T_{C''})^{-k}[j(\varphi)].$$

Since $j(\varphi)|_{\nu(C)} = \mathrm{id}|_{\nu(C)}$ we have $c(\phi, \boldsymbol{L}, C) = c([\varphi_L], C) = -k < 0$. This shows that $\boldsymbol{L}$ is not right-veering.

For any $\gamma \in \mathcal{A}_C(S; P)$ the factor $(T_C)^{-k}(T_{C'})^{2k}(T_{C''})^{-k}$ of $[\varphi_L]$ forces to form a boundary right P–bigon from $[\varphi_L](\gamma)$ to γ. See Figure 3, right. Thus, by Proposition 3.5, $[\varphi_L](\gamma) \not\ll_{\mathrm{right}} \gamma$ for every $\gamma \in \mathcal{A}_C(S; P)$, which means $\boldsymbol{L}$ is quasi-right-veering with respect to C. This proves (1).

Next we prove (2). Let $\{C_1, \dots, C_d\}$ be the set of boundary components of S. For each component C_i we take a closed braid $L_{C_i,k}$ as given in the proof of (1), and let $L = \bigsqcup_{i=1}^{d} L_{C_i,k}$ be the disjoint union of the $L_{C_i,k}$. By (1) we see that $\boldsymbol{L}$ is quasi-right-veering. By Proposition 2.11 we obtain $c(\phi, \boldsymbol{L}, C_i) \leq -k$ for all $i = 1, \dots, d$. $\square$

The set of right-veering mapping classes in $\mathcal{MCG}(S, P)$ forms a monoid. However, this is not the case for quasi-right-veering mapping classes:

Corollary 3.17 *The set of quasi-right-veering mapping classes in $\mathcal{MCG}(S, P)$ does* not *form a monoid.*

Proof We use the same notation as in Proposition 3.16. Let $\chi = (T_{C'})^{-1} i([\beta_{C,1}])^{-1} = T_C T_{C'}^{-3} T_{C''}$ and $\psi = i([\beta_{C,1}])$. Both χ and ψ are quasi-right-veering but $\chi\psi = (T_{C'})^{-1}$ is not quasi-right-veering. $\square$

3.4 Transverse links are right-veering and quasi-right-veering

We have defined right-veering and quasi-right-veering and studied their properties. In this section, we study transverse links in contact manifolds from the viewpoint of right-veering and quasi-right-veering and obtain Propositions 3.20 and 3.22.

Recall that an abstract open book (S, φ) gives an embedded open book $(B_{(S,\varphi)}, \pi_{(S,\varphi)})$ of the manifold $M_{(S,\varphi)}$ (see Section 2.1). We say that a contact structure ξ on $M_{(S,\varphi)}$ is *supported by* (S, φ) if ξ is isotoped through contact structures so that there is a contact 1–form α for ξ such that $d\alpha$ is a positive area form on each page S_t of the embedded open book and $\alpha > 0$ on the binding $B_{(S,\varphi)}$. By Thurston and Winkelnkemper [32], for every (S, φ) there exists a contact structure $\xi_{(S,\varphi)}$ on $M_{(S,\varphi)}$ supported by (S, φ). Such a contact structure is unique up to isotopy due to Giroux [17].

Definition 3.18 Let $\xi_{(S,\varphi)}$ be a contact structure on $M_{(S,\varphi)}$ supported by (S, φ). In this paper, we say that

- a contact 3–manifold (M,ξ) is *supported by* (S,φ) if the manifolds (M,ξ) and $(M_{(S,\varphi)},\xi_{(S,\varphi)})$ are contactomorphic;
- (S,ϕ) is an *open book* of (M,ξ) if (M,ξ) is supported by an abstract open book (S,φ) with $[\varphi]=\phi$.

Next we list basic facts about transverse links and closed braids. The fact (3) was discovered by Bennequin [4] (for the $(S,\phi)=(D^2,[\mathrm{id}])$ case), Mitsumatsu and Mori [28] and Pavelescu [30, Theorem 3.2]:

(1) Every closed braid with respect to an abstract open book (S,φ) is a transverse link when viewed in $(M_{(S,\varphi)},\xi_{(S,\varphi)})$ for some contact structure $\xi_{(S,\varphi)}$ supported by (S,φ).

(2) The transverse link type in (1) only depends on the equivalence class of the closed braid, in the following sense:

Let L and L' be closed braids with respect to (S,φ) and (S,φ'), respectively, and assume that $((S,\varphi),L)$ and $((S,\varphi'),L')$ are equivalent. Suppose that $\xi_{(S,\varphi)}$ (resp. $\xi'_{(S,\varphi)}$) is a contact structure on $M_{(S,\varphi)}$ (resp. $M_{(S,\varphi')}$) supported by (S,φ) (resp. (S,φ')) so that L (resp. L') is regarded as a transverse link in $(M_{(S,\varphi)},\xi_{(S,\varphi)})$ (resp. $(M_{(S,\varphi')},\xi_{(S,\varphi')})$).

Since φ and φ' are isotopic, an isotopy between φ and φ' induces a diffeomorphism $\rho\colon M_{(S,\varphi)}\to M_{(S,\varphi')}$ (see (2-12)) that preserves the embedded open book structure. In particular, $\rho_*(\xi_{(S,\varphi)})$ is supported by the open book (S,φ'), hence it is isotopic to $\xi_{(S,\varphi')}$. By Gray stability we have a diffeomorphism $\theta\colon M_{(S,\varphi')}\to M_{(S,\varphi')}$ isotopic to the identity satisfying $\theta_*(\rho_*\xi_{(S,\varphi)})=\xi_{(S,\varphi')}$. Consequently, we have a contactomorphism

$$\varrho=\theta\circ\rho\colon (M_{(S,\varphi)},\xi_{(S,\varphi)})\to(M_{(S,\varphi')},\xi_{(S,\varphi')}) \tag{3-3}$$

and $\varrho(L)$ and L' are transversely isotopic. Note that $\varrho(L)$ is a transverse link in $(M_{(S,\varphi')},\xi_{(S,\varphi')})$ but may not be a closed braid with respect to (S,φ') since θ may not preserve the pages.

(3) Any transverse link in a contact 3–manifold $(M_{(S,\varphi)},\xi_{(S,\varphi)})$ can be transversely isotoped to a closed braid with respect to (S,φ).

Definition 3.19 Suppose that (S,ϕ) is an open book of (M,ξ). We say that *a transverse link $\mathcal{T}$ in (M,ξ) is represented by a closed braid $\boldsymbol{L}$ with respect to (S,ϕ)*, if there is an abstract open book (S,φ) with $[\varphi]=\phi$ and a closed braid L with respect to (S,φ) such that $\boldsymbol{L} = [(S,\varphi), L]$ and there is a contactomorphism $\tau\colon (M,\xi) \to (M_{(S,\varphi)}, \xi_{(S,\varphi)})$ such that $L = \tau(\mathcal{T})$.

We prove two propositions. The first one is on quasi-right-veeringness of transverse links.

Proposition 3.20 *Every transverse link in a contact manifold (M,ξ) admits a quasi-right-veering closed braid representative with respect to some open book of (M,ξ).*

Proof Honda, Kazez and Matić [18, Proposition 6.1] show that every contact 3–manifold admits an open book decomposition (S,ϕ) with right-veering monodromy. This fact and our Corollary 3.15 yield the proposition. □

The second proposition is about right-veeringness of transverse links.

To state the proposition, we recall a positive stabilization of a closed braid. Here we present an algebraic formulation so that the connection to distinguished monodromy is clear. For a geometric formulation based on open book foliation machinery, we refer to [26].

Definition 3.21 Let $\varphi \in \mathrm{Diff}^+(S,\partial S)$ with $[\varphi] = \phi \in \mathcal{MCG}(S)$ be such that $\varphi|_{\nu(\partial S)} = \mathrm{id}_{\nu(\partial S)}$. Let L be an admissible closed n–braid with respect to an abstract open book (S,φ) such that $P = p(L \cap S_0) \subset \nu(\partial S)$, and let $\beta_L \subset S \times [0,1]$ be the geometric n–braid obtained from L. Let C be a boundary component of S and $\nu(C)$ be the connected component of $\nu(\partial S)$ that contains C. Take $\nu'(C) \subset \nu(C)$, a subcollar neighborhood of C such that

$$\beta_L \cap (\nu'(C) \times [0,1]) = \varnothing. \tag{3-4}$$

Let $q \in \nu'(C)$ and γ be a properly embedded arc in $S \setminus (P \cup \{q\})$ that connects a point $p \in P$ and q. The disjoint union of the strand $\{q\} \times [0,1]$ and β_L yields a geometric $(n+1)$–braid

$$\overline{\beta}_L := \beta_L \sqcup (\{q\} \times [0,1]) \subset S \times [0,1].$$

Let $H_\gamma \in \mathcal{MCG}(S, P \cup \{q\})$ be the positive half twist about γ, and $h_\gamma \subset S \times [0,1]$ be a geometric $(n+1)$–braid that represents H_γ in the sense that $i([h_\gamma]) = H_\gamma$, where $i\colon B_{n+1}(S) \to \mathcal{MCG}(S, P \cup \{q\})$ denotes the push map in the generalized Birman exact sequence (2-2). A *positive stabilization* of the closed braid L about the arc γ

(and the binding corresponding to C) is a closed $(n+1)$–braid L' obtained by taking the braid closure of the geometric $(n+1)$–braid

$$h_\gamma \bullet \overline{\beta}_L \quad \text{(read from right to left)},$$

where the bullet $\bullet$ denotes the concatenation of geometric braids.

Since $q \in \nu'(C)$, the property (3-4) implies that L' is obtained by connecting L and a meridian circle of C with a positively twisted band. Therefore, L and L' are transversely isotopic. See [30, Theorem 4.2], where Pavelescu proves that closed braids are transversely isotopic if and only if they differ by braid isotopies and positive stabilizations and their inverses.

Recall the diffeomorphism $\varphi_L \in \mathrm{Diff}^+(S, P, \partial S)$ in (2-6) that represents the distinguished monodromy $[\varphi_L] \in \mathcal{MCG}(S, P)$. By (3-4), we may assume that $\varphi_L|_{\nu'(C)} = \mathrm{id}_{\nu'(C)}$. Since $q \in \nu'(C)$, we may view φ_L as an element of $\mathrm{Diff}^+(S, P \cup \{q\}, \partial S)$ and denote it by $\overline{\varphi}_L$. We obtain $[\overline{\varphi}_L] \in \mathcal{MCG}(S, P \cup \{q\})$ and the distinguished monodromy for L' satisfies

$$[\varphi_{L'}] = H_\gamma[\overline{\varphi}_L] \in \mathcal{MCG}(S, P \cup \{q\}). \tag{3-5}$$

Here is the second proposition:

Proposition 3.22 *Every closed braid $\boldsymbol{L} = [L]$ with respect to an open book (S, ϕ) can be made right-veering after a sequence of positive stabilizations of L.*

When $(S, \phi) = (D^2, [\mathrm{id}])$ the same statement is proved by Plamenevskaya [31, Proposition 3.1].

Proof Suppose that L is an admissible closed braid with respect to an abstract open book (S, φ) that represents $\boldsymbol{L}$.

Let C be a boundary component of S. Let $\nu'(C) \subset \nu(C)$ be a subcollar neighborhood of C that does not contain (or intersect) $P = p(L \cap S_0) \subset \nu(\partial S)$. Choose points q and q' in $\nu'(C)$. See Figure 4. Let $\gamma_1 \subset S \setminus (P \cup \{q, q'\})$ be a properly embedded arc that connects one of the points in P and the point q. Let $\gamma_2 \subset \nu'(C) \setminus \{q, q'\}$ be an arc that connects q and q'. Assume the following:

1. The interiors of γ_1 and γ_2 intersect exactly at one point in $\nu'(C)$. We name it r.
2. Let $\gamma_1' \subset \gamma_1$ and $\gamma_2' \subset \gamma_2$ be the subarcs connecting r and q. Then the simple closed curve $\gamma_1' \cup \gamma_2'$ is homotopic to C in $S \setminus (P \cup \{q'\})$.

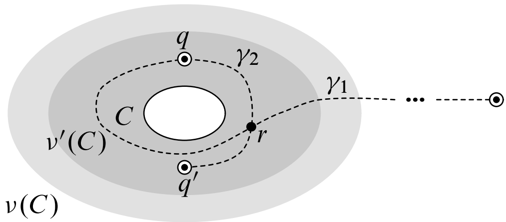

Figure 4: Twice stabilizing about C makes a closed braid right-veering with respect to C.

Let L' be a closed $(n+2)$–braid obtained from L by positive stabilizations first about γ_1 and then γ_2 as constructed in Definition 3.21.

The diffeomorphism $\varphi_L \in \mathrm{Diff}^+(S, P, \partial S)$ satisfies $\varphi_L|_{\nu'(C)} = \mathrm{id}_{\nu'(C)}$. Let $\overline{\varphi}_L$ denote the diffeomorphism φ_L viewed as an element of $\mathrm{Diff}^+(S, P \cup \{q, q'\}, \partial S)$. By (3-5) the distinguished monodromy of L' satisfies $[\varphi_{L'}] = H_{\gamma_2} H_{\gamma_1}[\overline{\varphi}_L] \in \mathcal{MCG}(S, P \cup \{q, q'\})$.

Since $\varphi_L|_{\nu'(C)} = \mathrm{id}_{\nu'(C)}$ and every essential arc in $\mathcal{A}_C(S, P \cup \{q, q'\})$ intersects either γ_1 or γ_2, the monodromy $[\varphi_{L'}]$ is right-veering with respect to C.

Applying this operation for every boundary component, we get a right-veering closed braid that is transversely isotopic to the original braid L. □

4 Characterization of nonloose links

We now prove our main theorem:

Theorem 4.1 *A transverse link $\mathcal{T}$ in a contact 3–manifold (M, ξ) is nonloose if and only if every braid representative of $\mathcal{T}$ with respect to every open book of (M, ξ) is quasi-right-veering.*

Our proof of Theorem 4.1 is a generalization of the proof of [22, Theorem 2.4]. We may assume that the readers are familiar with basic definitions and properties of open book foliations that can be found in [20; 21; 25]. As stated in Section 1, the monograph [27] and the article [5] are helpful introductions of braid foliations and open book foliations.

Proof of Theorem 4.1 ($\Longrightarrow$) First we show that non-quasi-right-veering braids are loose.

Assume that a transverse link $\mathcal{T}$ can be represented by an admissible non-quasi-right-veering closed braid L with respect to an abstract open book (S,φ) that supports (M,ξ). Let $P := p(L \cap S_0)$. That is, there exist a boundary component $C \subset \partial S$ and an arc $\alpha \in \mathcal{A}_C(S,P)$ such that there is a sequence of arcs $[\varphi_L](\alpha) = \alpha_0 \prec_{\text{right}} \alpha_1 \prec_{\text{right}} \cdots \prec_{\text{right}} \alpha_k = \alpha$ with $\mathrm{Int}(\alpha_i) \cap \mathrm{Int}(\alpha_{i+1}) = \varnothing$ for all $i = 0, \dots, k-1$.

We explicitly construct a transverse overtwisted disk D_{trans} in $M_{(S,\varphi)} \setminus L$ by giving its movie presentation. A similar construction can be found in [22]. Here, a *transverse overtwisted disk* (see [20, Definition 4.1] for the precise definition) is a disk admitting a certain type of open book foliation and is bounded by a transverse pushoff of a usual overtwisted disk.

For $i = 0, \dots, k$ denote the endpoint $\alpha_i(1) \in \partial S$ of the arc α_i by w_i. Slightly moving w_i along ∂S, if necessary, we may assume that all the points $w_0, \dots, w_{k-1}$ are distinct and still satisfying $\mathrm{Int}(\alpha_i) \cap \mathrm{Int}(\alpha_{i+1}) = \varnothing$. Since $[\varphi_L](\alpha) = \alpha_0$ we get $w_0 = w_k$. Fix a sufficiently small $\varepsilon > 0$.

The open book foliation of D_{trans} contains one negative elliptic point at $*_C$ and k positive elliptic points at $w_0, \dots, w_{k-1}$. See Figure 5.

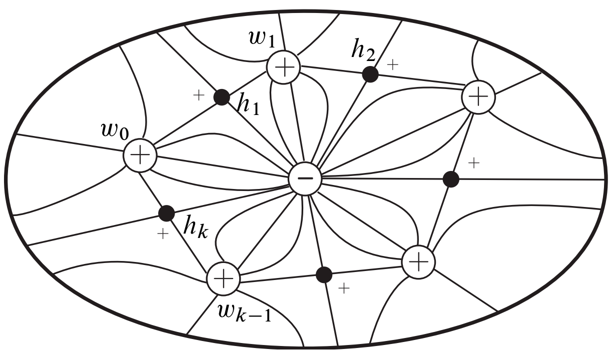

Figure 5: Transverse overtwisted disk D_{trans}.

The movie presentation of D_{trans} on the page S_0 consists of $k-1$ a–arcs emanating from $w_1, \dots, w_{k-1}$ and a b–arc that is a copy of α_0 joining w_0 and $*_C$. For $t \in \left[0, \frac{1}{k+1}\right)$ the movie presentation on the page S_t is the same as S_0.

The movie presentation on the page $S_{1/(k+1)}$ contains one hyperbolic point, h_1, whose describing arc joining α_0 and the a–arc from w_1 is a parallel copy of α_1 in $S_{1/(k+1)-\varepsilon}$. Since $\mathrm{Int}(\alpha_0) \cap \mathrm{Int}(\alpha_1) = \varnothing$ the interior of the describing arc is disjoint from all the

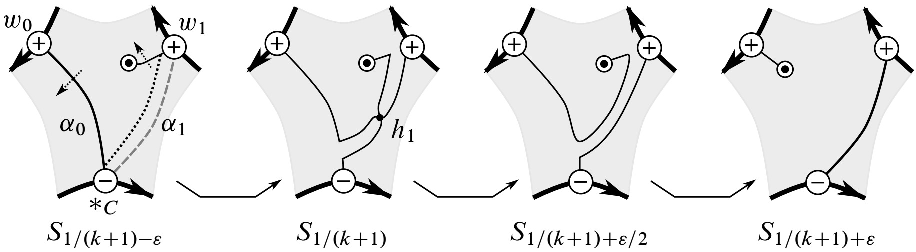

Figure 6: Movie for $t \in \left[\frac{1}{k+1}-\varepsilon, \frac{1}{k+1}+\varepsilon\right]$: the b–arc α_0 disappears and the new b–arc α_1 appears at $t=\frac{1}{k+1}$. The black dashed arc is the describing arc and the gray dashed arc is α_1. Black solid arrows indicate the orientation of ∂S. Dashed arrows are positive normals to D_{trans}.

a–arcs and the b–arc in the page $S_{1/(k+1)-\varepsilon}$. Since $\alpha_0 \prec_{\mathsf{right}} \alpha_1$, the normal vectors of D_{trans} point *out* of the describing arc, thus by [23, Observation 2.5] the sign of the hyperbolic point h_1 is positive. The movie presentation on the page $S_{1/(k+1)+\varepsilon}$ consists of one b–arc which is a copy of α_1 connecting w_1 and $*_C$ and $k-1$ a–arcs emanating from $w_0, w_2, \dots, w_{k-1}$.

We inductively apply the above procedure. Let $j=1,\dots,k$. The above paragraph describes the $j=1$ case.

On the page $S_{j/(k+1)}$ ($j>1$) we put a positive hyperbolic point h_j whose describing arc is a parallel copy of α_j. As a consequence the page $S_{j/(k+1)+\varepsilon}$ has one b–arc which is a copy of α_j connecting w_j and $*_C$ and $k-1$ a–arcs emanating from w_i for $i=1,\dots,j-1,j+1,\dots,k-1$.

On the page S_1 the movie presentation consists of one b–arc which is a copy of $\alpha_k=\alpha$ and $k-1$ a–arcs emanating from $w_0,\dots,w_{k-1}$. Since $[\varphi_L](\alpha)=\alpha_0$, the slices $D_{\mathsf{trans}} \cap S_1$ and $D_{\mathsf{trans}} \cap S_0$ of D_{trans} can be identified under the diffeomorphism $\varphi_L \in \mathrm{Diff}^+(S, P, \partial S)$. In other words, the movie presentation gives rise to an embedded surface in $M_{(S,\varphi)} \setminus L$. The construction tells us that the surface is topologically a disk, and moreover it is a transverse overtwisted disk (see [22]).

($\Longleftarrow$) Assume that a transverse link $\mathcal{T} \subset (M,\xi)$ is loose. By taking a neighborhood of an overtwisted disk $D \subset M \setminus \mathcal{T}$, we may regard (M,ξ) as the connected sum $(M',\xi')\#(S^3,\xi'_{\mathrm{ot}})$ such that $\mathcal{T} \subset (M',\xi')$. Here ξ'_{ot} denotes some overtwisted contact structure on S^3. Applying the argument of Honda, Kazez and Matić in [18, page 444] to (S^3,ξ'_{ot}) we may regard (M,ξ) as $(N,\xi_N)\#(S^3,\xi_{\mathrm{ot}})$ such that $\mathcal{T} \subset (N,\xi_N)$, where (S^3,ξ_{ot}) denotes the overtwisted contact structure supported by (A, T_A^{-1}), an annulus open book with a left-handed Dehn twist about a core curve of A.

Take an abstract open book (S_N, φ_N) supporting (N, ξ_N) and a closed braid L_N representing $\mathcal{T}$. Then the original contact 3–manifold (M,ξ) is supported by the open book $(S,\varphi) := (S_N, \varphi_N) * (A, T_A^{-1})$, where $*$ represents the Murasugi sum of the open books (see [11, Definition 2.16]) and L_N is closed braid with respect to (S,φ).

Let γ be the isotopy class of a cocore of the attached 1–handle $S \setminus S_N$. We have

$$[\varphi_{L_N}]\gamma = [T_A^{-1}]\gamma \ll_{\mathrm{right}} \gamma,$$

hence L_N is not quasi-right-veering. □

Corollary 4.2 *A transverse link $\mathcal{T}$ in a contact 3–manifold (M,ξ) is nonloose if and only if for every closed braid representative L of $\mathcal{T}$ with respect to every abstract open book (S,φ) that supports (M,ξ) and for every boundary component C and $\gamma \in \mathcal{A}_C(S;P)$, where $P := p(L\cap S)$, at least one of the following holds:*

(1) $\gamma = [\varphi_L](\gamma)$.

(2) $\gamma \prec_{\mathrm{right}} [\varphi_L](\gamma)$.

(3) γ *and* $[\varphi_L](\gamma)$ *cobound bigons that contain points of P.*

Proof $(\Longrightarrow)$ If there exists γ such that $[\varphi_L](\gamma) \prec_{\mathrm{right}} \gamma$ and no marked bigons are cobounded by $[\varphi_L](\gamma)$ and γ, then Proposition 3.8 shows that $[\varphi_L](\gamma) \ll_{\mathrm{right}} \gamma$. Thus $[\varphi_L] \in \mathcal{MCG}(S,P)$ is not quasi-right-veering. Then Theorem 4.1 shows that $\mathcal{T}$ is loose.

$(\Longleftarrow)$ This implication holds by exactly the same proof of (the $\Longleftarrow$ part of) Theorem 4.1. □

5 Depth of transverse links

Theorem 4.1 can be used to study the *depth* introduced by Baker and Onaran [1], which measures nonlooseness of transverse links.

Let F be an oriented surface in an oriented 3–manifold M and $K \subset M$ be an oriented link that transversely intersects F. We denote the number of intersection points of K and F by $\#(K\cap F)$, which does not necessarily realize the geometric intersection number. We also denote the number of positive and negative intersection points of K and F by $\#^+(K\cap F)$ and $\#^-(K\cap F)$, respectively. We have $\#(K\cap F) = \#^+(K\cap F) + \#^-(K\cap F)$.

Definition 5.1 [1] Let K be a transverse link or a Legendrian link in (M,ξ). The depth $d(K)$ of K is defined by

$$d(K)=\min\{\#(K\cap D)\mid D \text{ is an overtwisted disk in } (M,\xi)\}.$$

Assuming that (M,ξ) is overtwisted, we see that K is loose if and only if $d(K)=0$.

In the following K represents a transverse link. First we give a new interpretation of the depth $d(K)$ in terms of open book foliations. Let (S,ϕ) be an open book supporting a contact 3–manifold (M,ξ). Recall that existence of a *transverse* overtwisted disk in the open book (S,ϕ) (see [20, Definition 4.1]) is equivalent to existence of an overtwisted disk in (M,ξ).

Theorem 5.2 *For a transverse link K in (M,ξ) let*

$$d^-_{\mathsf{trans}}(K):=\min\left\{\#^-(K'\cap D)\;\middle|\; \begin{array}{l} K' \textit{ is a link transversely isotopic to } K,\\ D \textit{ is a transverse overtwisted disk in } (S,\phi).\end{array}\right\}$$

Then $d(K)=d^-_{\mathsf{trans}}(K)$.

In a special case, where K is the binding of an open book, the equality is proved in [24].

The theorem highlights the difference between a transverse overtwisted disk (whose boundary is a transverse unknot) and an ordinary overtwisted disk (whose boundary is a Legendrian unknot).

Applications of the theorem can be found in Theorems 5.5 and 6.1.

Proof We first show that $d(K)\le d_{\mathsf{trans}}(K)$.

Let D_{trans} and K_0 be a transverse overtwisted disk and transverse link which attain $d_{\mathsf{trans}}(K)$. Therefore, $d_{\mathsf{trans}}(K)=\#^-(K_0\cap D_{\mathsf{trans}})$. By the structural stability theorem [20, Theorem 2.21], we may assume that

(a) the characteristic foliation $\mathcal{F}_\xi(D_{\mathsf{trans}})$ and the open book foliation $\mathcal{F}_{\mathrm{ob}}(D_{\mathsf{trans}})$ are topologically conjugate.

Let $G_{++}(\mathcal{F}_\xi(D_{\mathsf{trans}}))$ (resp. $G_{--}(\mathcal{F}_\xi(D_{\mathsf{trans}}))$) be the Giroux graph [17, page 646] consisting of the positive (resp. negative) elliptic points and the stable (resp. unstable) separatrices of positive (resp. negative) hyperbolic points. By the assumption (a), these graphs are identified with the corresponding graphs $G_{++}:=G_{++}(\mathcal{F}_{\mathrm{ob}}(D_{\mathsf{trans}}))$ and

$G_{--} := G_{--}(\mathcal{F}_{\mathrm{ob}}(D_{\mathsf{trans}}))$ in the open book foliation $\mathcal{F}_{\mathrm{ob}}(D_{\mathsf{trans}})$; see Definition 2.17 of [20] for the definitions.

Take small neighborhoods $N_+, N_- \subset D_{\mathsf{trans}}$ of the graphs $G_{++}(\mathcal{F}_\xi(D_{\mathsf{trans}}))$ and $G_{--}(\mathcal{F}_\xi(D_{\mathsf{trans}}))$, respectively. By transverse isotopy we move K_0 without introducing new intersection points with D_{trans} so that

(b) the intersection $K_0 \cap D_{\mathsf{trans}}$ is disjoint from the region $N_+ \cup N_-$.

We apply the Giroux elimination lemma [16, Lemma 3.3] to remove all the positive elliptic and positive hyperbolic points of $\mathcal{F}_\xi(D_{\mathsf{trans}})$ (see Figure 7). Call the resulting disk D'. By (a) and the definition of a transverse overtwisted disk, the characteristic foliation $\mathcal{F}_\xi(D')$ has a unique negative elliptic point enclosed by a circle leaf. We can find an ordinary overtwisted disc $D \subset D'$. Since the Giroux elimination is supported on $N_+ \cup N_-$, the condition (b) implies that this process does not produce new intersections, ie $K_0 \cap D_{\mathsf{trans}} = K_0 \cap D'$.

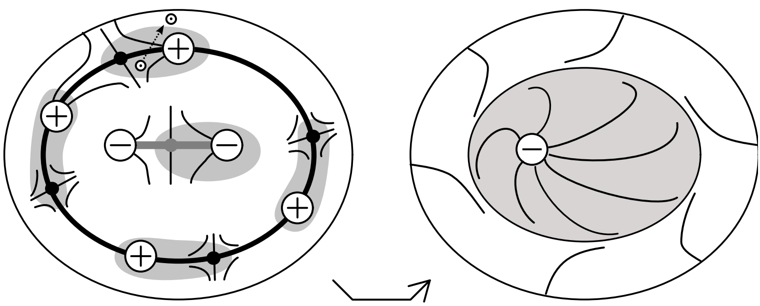

Figure 7: Left: From a transverse overtwisted disk to a usual overtwisted disk. The graphs G_{++} and G_{--} are depicted by black and gray bold lines, respectively. A dot $\odot$ represents an intersection of K and D_{trans} which is moved away from the gray regions before applying the Giroux elimination lemma to the gray regions. Right: Disk D' and an overtwisted disk D' (highlighted in gray).

Due to Epstein, Fuchs and Meyer [10] and Etnyre and Honda [13], the set of transverse links up to transverse isotopy is naturally identified, through positive transverse pushoff, with the set of Legendrian links up to Legendrian isotopy and negative stabilization.

Baker and Onaran in the proof of [1, Theorem 4.1.4] show that every positive intersection of a Legendrian link and an overtwisted disk can be removed by a negative stabilization of the Legendrian link.

Therefore, each positive intersection of K_0 and the overtwisted disk D can be removed by a suitable transverse isotopy. That is, there exists a link K_1 that is transversely isotopic to K_0 such that $\#(K_1 \cap D) = \#^-(K_1 \cap D) = \#^-(K_0 \cap D)$. We conclude

$$d(K) \le \#(K_1 \cap D) = \#^-(K_0 \cap D) \le \#^-(K_0 \cap D') = \#^-(K_0 \cap D_{\mathsf{trans}}) = d_{\mathsf{trans}}(K).$$

Next we show that $d(K) \ge d_{\mathsf{trans}}(K)$. Let D be an overtwisted disk in (M, ξ) that intersects K at $d(K)$ points.

Take a slightly larger disc, D', which contains D in its interior and is bounded by a positive transverse pushoff of the Legendrian unknot ∂D such that $D' \cap K = D \cap K$.

Using transverse isotopy we make K disjoint from the binding of the embedded open book. Following Pavelescu's proof of Alexander's theorem [30, Theorem 3.2], one can find an isotopy of M preserving each page of the embedded open book setwise and taking the nonbraided part of $\partial D' \cup K$ (subsets which are not positively transverse to pages) into a neighborhood of the binding.

Inside the neighborhood of the binding we make $\partial D' \cup K$ braided with respect to the open book using Bennequin's techniques from [4]. We call the resulting link and disk K' and D'', respectively. It is possible that new positive intersection points of D'' and K' may be created if a component of K is transversely isotopic to a binding component. However, no new negative intersection points will be introduced. Hence, $\#^-(K' \cap D'') = \#^-(K \cap D') \le d(K)$.

Fixing $\partial D''$ and K', and following the proof of [25, Theorem 3.3], we perturb D'' so that the resulting disk, D''', admits an essential open book foliation. This process can be done without introducing new intersection points with K', hence $\#^-(K' \cap D''') = \#^-(K' \cap D'')$.

Since the Bennequin–Eliashberg inequality does not hold, since

$$\begin{aligned} \mathrm{sl}(\partial D''', [D''']) = \mathrm{sl}(\partial D'', [D'']) = \mathrm{sl}(\partial D', [D']) &= \mathrm{tb}(\partial D, [D]) - \mathrm{rot}(\partial D, [D]) \\ &= 1 \not\le -\chi(D'''), \end{aligned}$$

we can apply the proof of [20, Theorem 4.3] to D''' and obtain a transverse overtwisted disc, D_{trans}. By the nature of this construction we have

$$\begin{aligned} \#^-(K' \cap D_{\mathsf{trans}}) &= \#^-(K' \cap D'''), \\ \#^+(K' \cap D_{\mathsf{trans}}) &\ge \#^+(K' \cap D'''), \end{aligned} \tag{5-1}$$

where strict inequality in (5-1) may hold only when a component of K' is transversely isotopic to a binding component. Summing up, we have

$$d_{\mathsf{trans}}(K) \le \#^-(K' \cap D_{\mathsf{trans}}) = \#^-(K' \cap D''') = \#^-(K' \cap D'') = \#^-(K \cap D') \le d(K). \quad \square$$

Many properties of quasi-right-veering braids are studied in Sections 3.3 and 3.4. Theorem 5.5 below gives another property of quasi-right-veering. One may also apply Theorem 5.5 to the study of knots and links of large depth.

Definition 5.3 Let (S, ϕ) be an open book of a contact 3–manifold (M, ξ) and $\boldsymbol{L} = [(S, \varphi), L]$ be a closed braid with respect to (S, ϕ). The *axis-augmented transverse link for* $\boldsymbol{L}$ is a transverse link represented by $B \cup L$, where $B = B_{(S,\varphi)}$ denotes the binding of the embedded open book supporting (M, ξ).

Lemma 5.4 *The axis-augmented transverse link for a closed braid* $\boldsymbol{L}$ *is well defined up to contactomorphism.*

Proof Suppose that φ and $\varphi' \in \mathrm{Diff}^+(S, \partial S)$ are isotopic. Fix a contact structure $\xi_{(S,\varphi)}$ on $M_{(S,\varphi)}$ (resp. $\xi_{(S,\varphi')}$ on $M_{(S,\varphi')}$) that is supported by (S, φ) (resp. (S, φ')). Starting with an isotopy between φ and φ' we have a contactomorphism

$$\varrho\colon (M_{(S,\varphi)}, \xi_{(S,\varphi)}) \to (M_{(S,\varphi')}, \xi_{(S,\varphi')})$$

as constructed in (3-3). Let B (resp. B') be the binding of the embedded open book on $M_{(S,\varphi)}$ (resp. $M_{(S,\varphi')}$). When closed braids $((S, \varphi), L)$ and $((S, \varphi'), L')$ are equivalent, the link $\varrho(B \cup L)$ is transversely isotopic to $B' \cup L'$. Thus, up to choice of identification $(M_{(S,\varphi)}, \xi_{(S,\varphi)}) \cong (M, \xi)$, the transverse link type of $B \cup L$ is uniquely determined. $\square$

Theorem 5.5 *Let* $\boldsymbol{L}$ *be a closed braid in the open book* (S, ϕ)*. The depth of the axis-augmented transverse link for* $\boldsymbol{L}$ *is* 1 *if and only if the braid* $\boldsymbol{L}$ *is not quasi-right-veering.*

When the closed braid L is empty we can reprove the following:

Corollary 5.6 [24, Corollary 1] *Let* B *be the binding of the embedded open book on* $M_{(S,\varphi)}$*. The depth* $d(B) = 1$ *if and only if* $\phi = [\varphi]$ *is not right-veering.*

Proof of Theorem 5.5 In the following, we take an abstract open book (S, φ) and a closed braid L such that $\boldsymbol{L} = [((S, \varphi), L)]$, and let $K = L \cup B$, where B denotes the binding of the embedded open book on $M_{(S,\varphi)}$.

($\Longleftarrow$) Suppose that the braid L is not quasi-right-veering. As in the proof of Theorem 4.1, we can construct a transverse overtwisted disk with only one negative elliptic point in the complement of L. By Theorem 5.2 we have $d(K) \leq 1$. On the other hand, since K contains the binding B and the binding of any open book is nonloose, which is proved by Etnyre and Vela-Vick [14], we have $d(K) \geq d(B) \geq 1$.

($\Longrightarrow$) Assume that $d(K) = 1$. Let D be an overtwisted disk in (M, ξ) satisfying $\#(K \cap D) = d(K) = 1$. Since the complement of the binding of a supporting open book decomposition is tight [14], $\#(D \cap K) = \#(D \cap B) = 1$ and $\#(D \cap L) = 0$.

Following the proof of Theorem 5.2 (the second half showing $d(K) \geq d_{\mathsf{trans}}(K)$) we can construct starting from D a transverse overtwisted disk D_{trans} in the complement of L such that

$$\#^{-}(K \cap D_{\mathsf{trans}}) = \#^{-}(B \cap D_{\mathsf{trans}}) = 1.$$

Let $v \in B \cap D_{\mathsf{trans}}$ denote the unique negative intersection point. That is, v is the unique negative elliptic point in the open book foliation $\mathcal{F}_{\mathrm{ob}}(D_{\mathsf{trans}})$ of D_{trans}. Assume that v lies on a boundary component C of S. For a regular page S_t of the embedded open book let $b_t \subset S_t$ denote the unique b–arc in $\mathcal{F}_{\mathrm{ob}}(D_{\mathsf{trans}})$ that ends at v. We use v as the basepoint $*_C$ of C. Recall the projection map (2-3),

$$p\colon M_{(S,\varphi)} \to S.$$

We view the image $p(b_t)$ as an element of $\mathcal{A}_C(S, P)$, where $P = p(L \cap S_0)$.

Let $S_{t_1}, \dots, S_{t_k}$ $(0 < t_1 < \cdots < t_k < 1)$ be the singular pages of the open book foliation $\mathcal{F}_{\mathrm{ob}}(D_{\mathsf{trans}})$ and $\varepsilon > 0$ be a sufficiently small number such that S_{t_i} is the only singular page in the interval $(t_i - \varepsilon, t_i + \varepsilon)$. Since D_{trans} is a transverse overtwisted disk with one negative elliptic point, by the definition of a transverse overtwisted disk [20, Definition 4.1], all the hyperbolic points of $\mathcal{F}_{\mathrm{ob}}(D_{\mathsf{trans}})$ are positive. This shows that $p(b_{t_i-\varepsilon}) \prec_{\mathsf{right}} p(b_{t_i+\varepsilon})$ with $\mathrm{Int}(p(b_{t_i-\varepsilon})) \cap \mathrm{Int}(p(b_{t_i+\varepsilon})) = \varnothing$ for all $i = 1, \dots, k$ (see Figure 8, top-right, or consult Observation 2.5 of [23]). Let us put

$$\gamma_i := p(b_{t_i+\varepsilon}) = p(b_{t_{i+1}-\varepsilon}) \in \mathcal{A}_C(S; P).$$

Then the sequence of arcs satisfies

$$[\varphi_L](p(b_1)) = p(b_0) = \gamma_0 \prec_{\mathsf{right}} \gamma_1 \prec_{\mathsf{right}} \cdots \prec_{\mathsf{right}} \gamma_k = p(b_1)$$

and $\mathrm{Int}(\gamma_i) \cap \mathrm{Int}(\gamma_{i+1}) = \varnothing$; hence, $[\varphi_L] \in \mathcal{MCG}(S, P)$ is not quasi-right-veering. $\square$

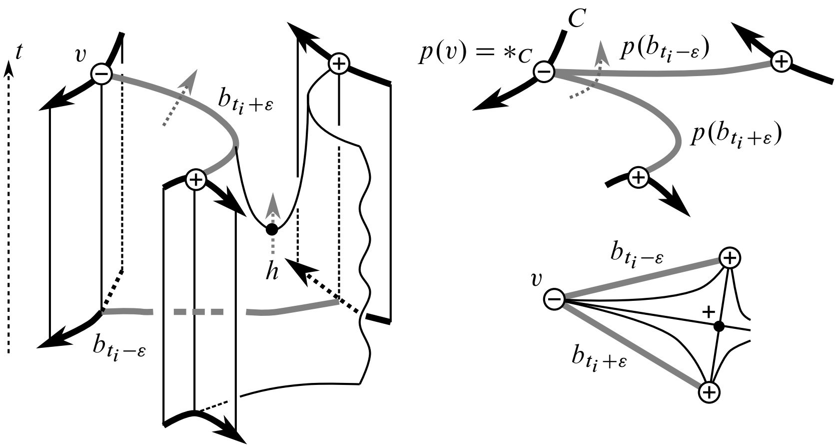

Figure 8: Left: A positive hyperbolic point h (saddle tangency). The gray dashed arrows indicate positive normal vectors to the surface. Black arrows indicate the orientations of the binding. Top-right: Comparison of the b–arcs $p(b_{t_i-\varepsilon})$ and $p(b_{t_i+\varepsilon})$ projected to S. Bottom-right: Corresponding portion of the open book foliation $\mathcal{F}_{\text{ob}}(D_{\text{trans}})$.

6 Very positive FDTC and nonloose links

Proposition 3.16 and Theorem 4.1 show that negative FDTC $c(\phi, \boldsymbol{L}, C) < 0$ does not always imply looseness of the closed braid $\boldsymbol{L}$. This makes a sharp contrast to the empty braid case, where the negative FDTC $c(\phi, C) < 0$ implies that the contact structure $\xi_{(S,\phi)}$ is overtwisted.

On the other hand, if the FDTC is very positive then there is some similarity between the nonempty braid case and the empty braid case. In [23, Corollary 1.2] it is proved that a planar open book (S, ϕ) with $c(\phi, C) > 1$ for every boundary component C supports a tight contact structure. We may regard this as a special case ($\boldsymbol{L} = \varnothing$) of the following theorem:

Theorem 6.1 *Let (S, ϕ) be a planar open book of a contact 3–manifold (M, ξ). If a transverse link $\mathcal{T} \subset (M, \xi)$ is represented by a closed braid $\boldsymbol{L} = [(S, \varphi), L]$ such that $c(\phi, \boldsymbol{L}, C) > 1$ for every boundary component C of S, then $\mathcal{T}$ is nonloose.*

Proof Let (S, φ) be an abstract open book that supports (M, ξ) and L be an admissible closed braid with respect to (S, φ) that represents $\boldsymbol{L}$. By (2-7) we have

$$((S \setminus P) \times [0, 1])/\sim_{\varphi_L} \simeq M_{(S,\varphi)} \setminus L.$$

Assume that L is loose. By Theorem 5.2 there exists a transverse overtwisted disk D in $M_{(S,\varphi)} \setminus L$. Applying the proof of [23, Theorem 1.1] to the diffeomorphism $\varphi_L \in \mathrm{Diff}^+(S, P, \partial S)$, we can construct a transverse overtwisted disk D' in $M_{(S,\varphi)} \setminus L$ such that every b–arc of $\mathcal{F}_{\mathrm{ob}}(D')$ ending at a valence ≤ 1 vertex of the graph $G_{--}(D')$ is projected to an essential arc in the punctured surface $S \setminus P$ under the projection map $p\colon M_{(S,\varphi)} \to S$ in (2-3). Using [25, Lemma 5.11] the existence of such a disk D' implies that $c(\phi, L, C) = c([\varphi_L], C) \leq 1$ for some boundary component C of S. $\square$

7 Comparison of three definitions of right-veering

In this section we compare several proposed definitions of a right-veering mapping class in $\mathcal{MCG}(S, P)$.

Definition 7.1 We say that an arc $\gamma\colon [0, 1] \to S$ is a ∂–P (resp. ∂–∂) arc if the following are all satisfied:

(1) $\gamma(0) \in \partial S$ and γ is transverse to ∂S at $\gamma(0)$.

(2) $\gamma(t) \in \mathrm{Int}(S) \setminus P$ for $t \in (0, 1)$.

(3) $\gamma(1) \in P$ (resp. $\gamma(1) \in \partial S$ and γ is transverse to ∂S at $\gamma(1)$).

(4) $\mathrm{Int}(\gamma)$ is embedded in $S \setminus P$ and not boundary-parallel.

For a boundary component C of S, we say that a ∂–P or ∂–∂ arc is *based on* C if $\gamma(0) \in C$.

As natural generalizations of the right-veeringness for $\phi \in \mathcal{MCG}(S)$ to $\psi \in \mathcal{MCG}(S, P)$ there are three candidates.

Definition 7.2 For a boundary component C of S we say that $\psi \in \mathcal{MCG}(S, P)$ is

(1) ∂–$(\partial + P)$ *right-veering with respect to* C if $\gamma \prec_{\mathrm{right}} \psi(\gamma)$ or $\gamma = \psi(\gamma)$ for all ∂–∂ and ∂–P arcs γ based on C;

(2) ∂–∂ *right-veering with respect to* C if $\gamma \prec_{\mathrm{right}} \psi(\gamma)$ or $\gamma = \psi(\gamma)$ for all ∂–∂ arcs γ based on C;

(3) ∂–P *right-veering with respect to* C if $\gamma \prec_{\mathrm{right}} \psi(\gamma)$ or $\gamma = \psi(\gamma)$ for all ∂–P arcs γ based on C;

(4) ∂–$(\partial + P)$, ∂–∂ or ∂–P *right-veering*, respectively, if ψ is ∂–$(\partial + P)$, ∂–∂ or ∂–P right-veering, respectively, with respect to every boundary component of S.

The ∂–∂ right-veering is used by Baldwin, Vela-Vick and Vértesi in [3]. It is easy to see that our Definition 3.9 of right-veering is equivalent to the ∂–∂ right-veering. Recall that in Definition 3.9 we only consider ∂–∂ arcs starting from the distinguished basepoint $*_C \in C$. This restriction is just to define the orderings $\prec_{\mathsf{right}}$ and $\ll_{\mathsf{right}}$ on $\mathcal{A}_C(S;P)$.

On the other hand, Baldwin and Grigsby [2] and Plamenevskaya [31] use the notion of ∂–P right-veering to study the classical braid group $\mathcal{MCG}(D^2,P)$. Baldwin and Grigsby ask in [2, Remark 3.3] whether these two superficially different notions of right-veering are equivalent or not. The following example shows that the notions (2) and (3) are in general not exactly the same:

Example 7.3 Assume that S has more than one boundary component and nonempty marked points $P \subset \mathrm{Int}(S)$. Let C and C' be distinct boundary components. Clearly $T_{C'}^{-1} \in \mathcal{MCG}(S,P)$ is not ∂–∂ right-veering with respect to C. On the other hand, $T_{C'}^{-1}$ preserves all ∂–P arcs based on C. This means that $T_{C'}^{-1}$ is ∂–P right-veering with respect to C.

More generally, let $\psi \in \mathcal{MCG}(S,P)$ be a ∂–P right-veering map with respect to C, and suppose that $\psi(\gamma)=\gamma$ for some ∂–∂ arc γ connecting distinct C and C'. Then $T_{C'}^{-1}\psi$ is still ∂–P right-veering with respect to C, but is not ∂–∂ right-veering with respect to C since $T_{C'}^{-1}\psi(\gamma) = T_{C'}^{-1}(\gamma) \prec_{\mathsf{right}} \gamma$.

It turns out that the difference between ∂–∂ right-veering and ∂–P right-veering only shows up when $\psi \in \mathcal{MCG}(S,P)$ involves negative Dehn twists along boundary components like in Example 7.3.

Definition 7.4 We say that $\psi \in \mathcal{MCG}(S,P)$ is *special* with respect to C if the following two conditions are satisfied:

- ψ is not ∂–∂ right-veering with respect to C.
- If a ∂–∂ arc γ that is based on C and ending at C' has $\psi(\gamma) \prec_{\mathsf{right}} \gamma$, then $C' \neq C$ and $\psi(\gamma) = T_{C'}^{-n}(\gamma)$ for some $n>0$.

That is, a special map ψ is not ∂–∂ right-veering with respect to C only because of negative Dehn twists about some other boundary component C'.

Theorem 7.5 *Let $\psi \in \mathcal{MCG}(S,P)$.*

(1) *If ψ is ∂–∂ right-veering with respect to C, then ψ is ∂–P right-veering with respect to C.*

(2) *If ψ is ∂–P right-veering with respect to C then either*

- *ψ is ∂–∂ right-veering with respect to C, or*
- *ψ is special with respect to C.*

Proof We prove both (1) and (2) by showing the contrapositives.

First we prove (1). Assume to the contrary that there is a ∂–P arc γ based on C with $\psi(\gamma) \prec_{\mathsf{right}} \gamma$. Let $\kappa \in \mathcal{A}_C(S; P)$ be a properly embedded arc which is the boundary of a regular neighborhood of γ in S. Then we see that κ is a ∂–∂ arc with $\psi(\kappa) \prec_{\mathsf{right}} \kappa$.

To see (2), assume to the contrary that ψ is not ∂–∂ right-veering with respect to C and is not special with respect to C. Then there exists a ∂–∂ arc γ based on C such that $\psi(\gamma) \prec_{\mathsf{right}} \gamma$. We put $\psi(\gamma)$ and γ so that they intersect efficiently. Our goal is to show that there exists a ∂–P arc κ based on C with $\kappa(0) = \gamma(0)$ and

$$\psi(\gamma) \prec_{\mathsf{right}} \kappa \prec_{\mathsf{right}} \gamma.$$

This shows $\psi(\kappa) \prec_{\mathsf{right}} \psi(\gamma) \prec_{\mathsf{right}} \kappa$; hence, ψ cannot be ∂–P right-veering with respect to C.

If $|\gamma \cap \psi(\gamma)| = m > 0$, we name $\mathrm{Int}(\gamma) \cap \mathrm{Int}(\psi(\gamma)) = \{p_1, \dots, p_m\} = \{q_1, \dots, q_m\}$, where $p_i = \gamma(t_i)$ with $0 < t_1 < t_2 < \cdots < t_m < 1$ and $q_i = (\psi(\gamma))(s_i)$ with $0 < s_1 < s_2 < \cdots < s_m < 1$. If $|\gamma \cap \psi(\gamma)| = m = 0$, we put $t_1 = s_1 = 1$ and $p_1 = q_1 = \gamma(1)$.

Suppose that $q_1 = p_k$. Let

$$\delta := \gamma|_{[0,t_k]} * (\psi(\gamma)|_{[0,s_1]})^{-1};$$

then δ is an oriented simple closed curve in $S \setminus P$. Here $*$ denotes concatenation of paths read from left to right, and $(-)^{-1}$ means the arc with reversed orientation. If δ is separating, we denote by R the connected component of $S \setminus (\delta \cup P)$ that lies on the left side of δ with respect to the orientation of δ. If δ is nonseparating, let $R := S \setminus (\delta \cup P)$.

Let us call the arc γ *bad* if the following two properties are satisfied:

- R is an annulus (possibly a pinched annulus if $m = 0$) with no punctures. (In particular, δ is separating.)
- The sign of the intersection of γ and $\psi(\gamma)$ (in this order) at q_1 is positive.

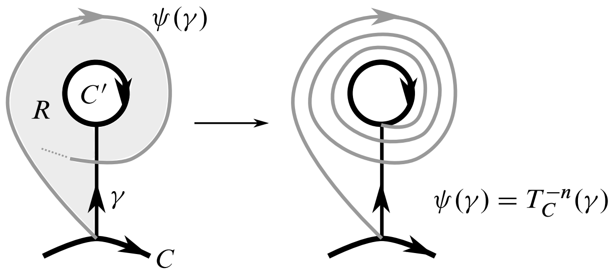

Figure 9: A bad arc γ and its image $\psi(\gamma)$.

Assume that γ is bad. Let $C' = \partial R \setminus \delta$. Note that C' is a boundary component of S. Since γ and $\psi(\gamma)$ intersect efficiently and δ is separating, we can see that $\psi(\gamma)$ cannot exit out of the annulus R and $C \neq C'$. See Figure 9. Therefore, we observe that if γ is bad then $C' \neq C$ and $\psi(\gamma) = T_{C'}^{-n}(\gamma)$ for some $n > 0$.

Since we assume that ψ is not ∂–∂ right-veering with respect to C and is not special with respect to C, the above observation implies that γ is not bad.

Knowing that γ is not bad, we consider two cases to construct κ:

Case 1 R is an annulus with punctures or a nonannulus surface with or without punctures.

The sign of the intersection of γ and $\psi(\gamma)$ at q_1 can be either positive or negative. Take an arc γ' in $S \setminus (P \cup \gamma \cup \delta)$ which connects q_1 and some puncture point and efficiently intersects $\psi(\gamma)|_{[s_1,1]}$.

Case 1A There exists such an arc γ' which lies on the left side of γ near q_1.

Define $\kappa := \gamma|_{[0,t_k]} * \gamma'$. See Figure 10, left.

Case 1B The arc γ' cannot exist on the left side of γ near q_1, so γ' lies on the right side of γ near q_1.

See Figure 10, right. If R contains punctures then let $\kappa \subset R$ be an arc connecting $\gamma(0)$ and one of the punctures in R and satisfying $\psi(\gamma) \prec_{\mathsf{right}} \kappa \prec_{\mathsf{right}} \gamma$ and $\mathrm{Int}(\kappa) \cap \delta = \varnothing$. (We do not use γ' here.)

Now we may assume that R is a nonannular surface with no punctures. We can take an arc γ'' in $R \setminus (R \cap \gamma')$ such that:

- $\gamma''(0) = \gamma''(1) = \gamma(0)$.
- $\psi(\gamma) \prec_{\mathsf{right}} \gamma'' \prec_{\mathsf{right}} \gamma$.

- $\mathrm{Int}(\gamma'') \cap \delta = \varnothing$.
- γ'' is not parallel to δ.
- γ'' and γ efficiently intersect.

Let $q'' := \gamma''(u) = \gamma(t) \in \gamma'' \cap \gamma$ be the intersection point such that $\gamma''|_{(0,u)}$ is disjoint from γ. If $\mathrm{Int}(\gamma'') \cap \mathrm{Int}(\gamma) = \varnothing$ then we take $q'' := \gamma''(1) = \gamma(0)$. Namely, $u = 1$ and $t = 0$. Define

$$\kappa := \begin{cases} \gamma''|_{[0,u]} * \gamma|_{[t,t_k]} * \gamma' & \text{if } t < t_k, \\ \gamma''|_{[0,u]} * \gamma|_{[t_k,t]} * \gamma' & \text{if } t_k < t. \end{cases}$$

Case 2 R is an annulus with no punctures, and the sign of the intersection of γ and $\psi(\gamma)$ at q_1 is negative.

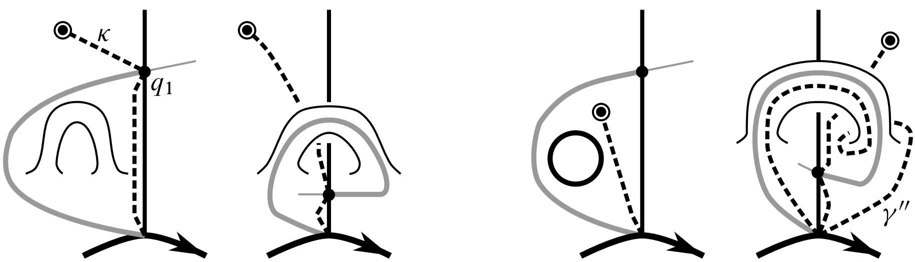

Figure 10: Case 1A (left) and Case 1B (right). A ∂–P arc κ (dashed arc) is chosen so that it does not intersect γ (black bold line) and $\psi(\gamma)|_{[0,s_1]}$ (gray bold arc), possibly with one exceptional point q_1.

Let k' $(\neq k)$ be the number satisfying $q_2 = p_{k'}$.

Case 2A $(k' < k)$ Since δ is separating the sign of the intersection of γ and $\psi(\gamma)$ at q_2 is positive. Take an arc γ' in $S \setminus (P \cup \gamma \cup \psi(\gamma)|_{[0,s_2]})$ which connects q_2 and a puncture point and efficiently intersects $\psi(\gamma)|_{[s_2,1]}$. Then put $\kappa := \psi(\gamma)|_{[0,s_1]} * (\gamma|_{[t_{k'},t_k]})^{-1} * \gamma'$. See Figure 11, left.

Case 2B $(k < k')$ Let γ' be an arc in $S \setminus (P \cup \gamma \cup \psi(\gamma)|_{[0,s_2]})$ that connects $\gamma(0)$ and a puncture point. Put

$$\kappa := \begin{cases} \gamma|_{[0,t_{k'}]} * (\psi(\gamma)|_{[s_1,s_2]})^{-1} * (\gamma|_{[0,s_1]})^{-1} * \gamma' & \text{if } \gamma \prec_{\mathrm{right}} \gamma', \\ \gamma|_{[0,t_{k'}]} * (\psi(\gamma)|_{[s_1,s_2]})^{-1} * C * (\gamma|_{[0,s_1]})^{-1} * \gamma' & \text{if } \gamma' \prec_{\mathrm{right}} \gamma. \end{cases}$$

In the second case, in order to make κ embedded, it turns along C. See Figure 11, right. □

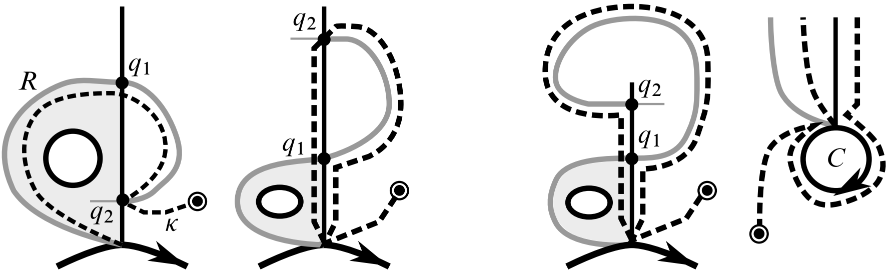

Figure 11: Case 2A (left) and Case 2B (right), with the case $\gamma' \prec_{\mathrm{right}} \gamma$ rightmost: construction of a ∂–P arc κ (dashed); κ does not intersect γ (black bold line) and $\psi(\gamma)|_{[0,s_2]}$ (gray bold arc), possibly with exceptions near q_1, q_2 and $\gamma(1)$ (if $\gamma(1) \in C$).

As a consequence of Theorem 7.5, we conclude the three notions of right-veering are equivalent when all the boundary components are considered simultaneously that prevent ψ being special.

Corollary 7.6 *For $\psi \in \mathcal{MCG}(S, P)$ the following are equivalent*:

(1) *ψ is ∂–$(\partial + P)$ right-veering with respect to **all** the boundary components of S.*

(2) *ψ is ∂–∂ right-veering with respect to **all** the boundary components of S.*

(3) *ψ is ∂–P right-veering with respect to **all** the boundary components of S.*

In particular, when S has connected boundary the three notions of right-veering are equivalent.

Therefore, in the case of the braid group $B_n = \mathcal{MCG}(D^2, \{n \text{ points}\})$ the proposed definitions of right-veering in [3] and [2; 31] are the same because D^2 has connected boundary. Also, we remark that the subtle difference between ∂–P right-veering with respect to C and ∂–∂ right-veering with respect to C (existence of a special mapping class ψ) only appears when $c(\psi, C) = 0$.

Remark 7.7 One may come up with further different candidates of right-veering. Instead of using embedded arcs, one may use immersed arcs. However, one can check that immersed ∂–$(\partial+P)$ (resp. ∂–∂, ∂–P) right-veering with respect to C is equivalent to the (embedded) ∂–$(\partial + P)$ (resp. ∂–∂, ∂–P) right-veering with respect to C.

Acknowledgements

The authors would like to thank John Etnyre and the anonymous referee for many useful comments and suggestions. Ito was partially supported by JSPS KAKENHI grant number 15K17540 and 16H02145. Kawamuro was partially supported by NSF grant DMS-1206770 and Simons Foundation Collaboration Grants for Mathematicians.

References

[1] **K L Baker**, **S Onaran**, *Nonlooseness of nonloose knots*, Algebr. Geom. Topol. 15 (2015) 1031–1066 MR

[2] **J A Baldwin**, **J E Grigsby**, *Categorified invariants and the braid group*, Proc. Amer. Math. Soc. 143 (2015) 2801–2814 MR

[3] **J A Baldwin**, **D S Vela-Vick**, **V Vértesi**, *On the equivalence of Legendrian and transverse invariants in knot Floer homology*, Geom. Topol. 17 (2013) 925–974 MR

[4] **D Bennequin**, *Entrelacements et équations de Pfaff*, from "Third Schnepfenried geometry conference, I", Astérisque 107, Soc. Mat. de France, Paris (1983) 87–161 MR

[5] **J S Birman**, **E Finkelstein**, *Studying surfaces via closed braids*, J. Knot Theory Ramifications 7 (1998) 267–334 MR

[6] **J S Birman**, **W W Menasco**, *Stabilization in the braid groups, II: Transversal simplicity of knots*, Geom. Topol. 10 (2006) 1425–1452 MR

[7] **V Colin**, **K Honda**, *Reeb vector fields and open book decompositions*, J. Eur. Math. Soc. 15 (2013) 443–507 MR

[8] **Y Eliashberg**, *Classification of overtwisted contact structures on 3–manifolds*, Invent. Math. 98 (1989) 623–637 MR

[9] **Y Eliashberg**, **M Fraser**, *Topologically trivial Legendrian knots*, J. Symplectic Geom. 7 (2009) 77–127 MR

[10] **J Epstein**, **D Fuchs**, **M Meyer**, *Chekanov–Eliashberg invariants and transverse approximations of Legendrian knots*, Pacific J. Math. 201 (2001) 89–106 MR

[11] **J B Etnyre**, *Lectures on open book decompositions and contact structures*, from "Floer homology, gauge theory, and low-dimensional topology" (D A Ellwood, P S Ozsváth, A I Stipsicz, Z Szabó, editors), Clay Math. Proc. 5, Amer. Math. Soc., Providence, RI (2006) 103–141 MR

[12] **J B Etnyre**, *On knots in overtwisted contact structures*, Quantum Topol. 4 (2013) 229–264 MR

[13] **J B Etnyre**, **K Honda**, *Knots and contact geometry, I: Torus knots and the figure eight knot*, J. Symplectic Geom. 1 (2001) 63–120 MR

[14] **J B Etnyre**, **D S Vela-Vick**, *Torsion and open book decompositions*, Int. Math. Res. Not. 2010 (2010) 4385–4398 MR

[15] **B Farb**, **D Margalit**, *A primer on mapping class groups*, Princeton Mathematical Series 49, Princeton Univ. Press (2012) MR

[16] **E Giroux**, *Convexité en topologie de contact*, Comment. Math. Helv. 66 (1991) 637–677 MR

[17] **E Giroux**, *Géométrie de contact: de la dimension trois vers les dimensions supérieures*, from "Proceedings of the International Congress of Mathematicians" (T Li, editor), volume 2, Higher Ed. Press, Beijing (2002) 405–414 MR

[18] **K Honda**, **W H Kazez**, **G Matić**, *Right-veering diffeomorphisms of compact surfaces with boundary*, Invent. Math. 169 (2007) 427–449 MR

[19] **K Honda**, **W H Kazez**, **G Matić**, *Right-veering diffeomorphisms of compact surfaces with boundary, II*, Geom. Topol. 12 (2008) 2057–2094 MR

[20] **T Ito**, **K Kawamuro**, *Open book foliation*, Geom. Topol. 18 (2014) 1581–1634 MR

[21] **T Ito**, **K Kawamuro**, *Operations on open book foliations*, Algebr. Geom. Topol. 14 (2014) 2983–3020 MR

[22] **T Ito**, **K Kawamuro**, *Visualizing overtwisted discs in open books*, Publ. Res. Inst. Math. Sci. 50 (2014) 169–180 MR

[23] **T Ito**, **K Kawamuro**, *Overtwisted discs in planar open books*, Internat. J. Math. 26 (2015) art. id. 1550027 MR

[24] **T Ito**, **K Kawamuro**, *Coverings of open books*, from "Advances in the mathematical sciences" (G Letzter, K Lauter, E Chambers, N Flournoy, J E Grigsby, C Martin, K Ryan, K Trivisa, editors), Assoc. Women Math. Ser. 6, Springer (2016) 139–154 MR

[25] **T Ito**, **K Kawamuro**, *Essential open book foliations and fractional Dehn twist coefficient*, Geom. Dedicata 187 (2017) 17–67 MR

[26] **T Ito**, **K Kawamuro**, *The defect of Bennequin–Eliashberg inequality and Bennequin surfaces*, Indiana Univ. Math. J. 68 (2019) 799–833

[27] **D J LaFountain**, **W W Menasco**, *Braid foliations in low-dimensional topology*, Graduate Studies in Mathematics 185, Amer. Math. Soc., Providence, RI (2017) MR

[28] **Y Mitsumatsu**, **A Mori**, *On Bennequin's isotopy lemma* MR Appendix to Y Mitsumatsu, *Convergence of contact structures to foliations*, from "Foliations 2005" (P Walczak, R Langevin, S Hurder, T Tsuboi, editors), World Sci., Hackensack, NJ (2006) 353–371

[29] **E Pavelescu**, *Braids and open book decompositions*, PhD thesis, University of Pennsylvania (2008) MR Available at `http://search.proquest.com/docview/304492097`

[30] **E Pavelescu**, *Braiding knots in contact 3–manifolds*, Pacific J. Math. 253 (2011) 475–487 MR

[31] **O Plamenevskaya**, *Braid monodromy, orderings and transverse invariants*, Algebr. Geom. Topol. 18 (2018) 3691–3718 MR

[32] **W P Thurston**, **H E Winkelnkemper**, *On the existence of contact forms*, Proc. Amer. Math. Soc. 52 (1975) 345–347 MR

Department of Mathematics, Kyoto University
Kyoto, Japan

Department of Mathematics, The University of Iowa
Iowa City, IA, United States

tetitoh@math.kyoto-u.ac.jp, keiko-kawamuro@uiowa.edu

Received: 15 May 2018 Revised: 10 December 2018