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On the coarse geometry of certain right-angled Coxeter groups

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Let Γ be a connected, triangle-free, planar graph with at least five vertices that has no separating vertices or edges. If the graph Γ is $\mathcal{CFS}$, we prove that the right-angled Coxeter group G_Γ is virtually a Seifert manifold group or virtually a graph manifold group and we give a complete quasi-isometry classification of these groups. Furthermore, we prove that G_Γ is hyperbolic relative to a collection of $\mathcal{CFS}$ right-angled Coxeter subgroups of G_Γ. Consequently, the divergence of G_Γ is linear, quadratic or exponential. We also generalize right-angled Coxeter groups which are virtually graph manifold groups to certain high-dimensional right-angled Coxeter groups (our families exist in every dimension) and study the coarse geometry of this collection. We prove that strongly quasiconvex, torsion-free, infinite-index subgroups in certain graph of groups are free and we apply this result to our right-angled Coxeter groups.

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1 Introduction

For each finite simplicial graph Γ the associated *right-angled Coxeter group* G_Γ has generating set S equal to the vertices of Γ, relations $s^2 = 1$ for each s in S and relations $st = ts$ whenever s and t are adjacent vertices. The graph Γ is the *defining graph* of the right-angled Coxeter group G_Γ and its flag complex $K = K(\Gamma)$ is the *defining nerve* of the group. Therefore, we also denote the right-angled Coxeter group G_Γ by G_K, where K is the flag complex of Γ.

In geometric group theory, groups acting on $\mathrm{CAT}(0)$ cube complexes are fundamental objects and right-angled Coxeter groups provide a rich source of these groups. The geometry of right-angled Coxeter groups was studied by Caprace [8], Davis and Okun [17], Dani and Thomas [13; 14], Dani, Stark and Thomas [12], Behrstock, Hagen and Sisto [5], Levcovitz [27], Haulmark, Nguyen and Tran [24], Tran [30] and others. In this paper, we first study the geometry of right-angled Coxeter groups G_Γ whose

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defining graph Γ is connected, triangle-free, planar, has at least 5 vertices and has no separating vertices or edges (we call them *the standing assumptions*). Then we generalize a part of work on the such group to certain high-dimensional right-angled Coxeter groups.

1.1 Right-angled Coxeter groups with $\mathcal{CFS}$ defining graphs

It is well known from the work of Davis and Januszkiewicz [16] that every right-angled Artin group is commensurable (hence quasi-isometric) to some right-angled Coxeter group and therefore we are especially interested in right-angled Coxeter groups whose coarse geometry is "similar" to the one of a right-angled Artin group. Behrstock and Charney [2] prove that the divergence of a one-ended right-angled Artin group is linear or quadratic. Therefore, the divergence of a one-ended right-angled Coxeter which is quasi-isometric to some right-angled Artin group must be linear or quadratic. It has been shown by Dani and Thomas [13] and Levcovitz [27] that the divergence of a right-angled Coxeter group G_Γ is linear or quadratic if and only if Γ is $\mathcal{CFS}$ (see Definition 2.12 for the concept of $\mathcal{CFS}$ graphs). Thus, studying right-angled Coxeter groups with $\mathcal{CFS}$ defining graphs is one of the main goals in this paper.

1.1.1 Quasi-isometric classification of two-dimensional, right-angled Coxeter groups Quasi-isometric classification of groups is one of most essential programs in geometric group theory. A complete solution for quasi-isometric classification of the class of right-angled Coxeter groups is unknown (even in the case of $\mathcal{CFS}$ graphs). Behrstock observed that the question on quasi-isometric classification of $\mathcal{CFS}$ right-angled Coxeter groups is appealing but likely difficult (see [1, Question 4.2]). In this paper, we partially answer that question when $\mathcal{CFS}$ defining graphs Γ satisfy the standing assumptions.

The key idea here is that after doing a tree-like decomposition on the graph Γ (see Section 3), we obtain a tree which we call the *visual decomposition tree*. We will give the precise definition of visual decomposition tree later in Section 3. Currently, the reader only needs to know that each piece of this decomposition is a suspension of distinct points. We observe that the right-angled Coxeter group associated to a piece of this decomposition resembles Seifert fibered space. We then glue these pieces in the pattern of the visual decomposition tree to get a graph manifold where G_Γ acts properly and cocompactly. Using the work of Behrstock and Neumann on quasi-isometric classification of graph manifolds, we obtain a quasi-isometric classification theorem for right-angled Coxeter groups with $\mathcal{CFS}$ defining graphs.

Theorem 1.1 *Let Γ be a graph satisfying the standing assumptions. Then*:

(1) *The right-angled Coxeter group G_Γ is virtually a Seifert manifold group if and only if Γ is a suspension of some distinct vertices.*

(2) *The right-angled Coxeter group G_Γ is virtually a graph manifold group if and only if Γ is $\mathcal{CFS}$ and it is not a suspension of distinct vertices.*

(3) *Let Γ and Γ' be two $\mathcal{CFS}$ graphs satisfying the standing assumptions. Let T_r and T_r' be two visual decomposition trees of Γ and Γ', respectively. Then two groups G_Γ and $G_{\Gamma'}$ are quasi-isometric if and only if T_r and T_r' are bisimilar.*

As we discussed above every right-angled Artin group is quasi-isometric to some $\mathcal{CFS}$ right-angled Coxeter group. A natural question that arises is which $\mathcal{CFS}$ right-angled Coxeter groups are quasi-isometric to some right-angled Artin group. In [1], Behrstock gives an example of $\mathcal{CFS}$ right-angled Coxeter group which is not quasi-isometric to any right-angled Artin group by using Morse boundary. More precisely, the Morse boundary of the right-angled Coxeter group in his examples contains a circle. Meanwhile, Morse boundaries of all right-angled Artin groups are empty or totally disconnected, this is implicit in Charney and Sultan [9] and also follows immediately from Theorem F in Cordes and Hume [11]. Therefore, the right-angled Coxeter group in his example is not quasi-isometric to any right-angled Artin group since Morse boundary is a quasi-isometry invariant (see [9] and also Cordes [10]). However, it would be natural to conjecture that a one-ended right-angled Coxeter group G_Γ is quasi-isometric to some right-angled Artin group if and only if Γ is $\mathcal{CFS}$ and the Morse boundary of G_Γ is empty or totally disconnected. However, we show that this is not true.

In fact, let Γ be a $\mathcal{CFS}$, nonjoin graph which satisfies the standing assumptions. By work implicit in [9] and the fact that right-angled Coxeter group G_Γ can be decomposed as a tree of groups with empty Morse boundary, we observe that G_Γ has totally disconnected Morse boundary. However, G_Γ is not necessarily quasi-isometric to a right-angled Artin group. More precisely, we give a characterization on the defining graph Γ for G_Γ to be quasi-isometric to a right-angled Artin group. Moreover, we also specify types of right-angled Artin groups which are quasi-isometric to such right-angled Coxeter groups.

Theorem 1.2 *Let Γ be a $\mathcal{CFS}$, nonjoin graph satisfying the standing assumptions and T_r a visual decomposition tree of Γ. Then the following are equivalent*:

(1) *The right-angled Coxeter group G_Γ is quasi-isometric to a right-angled Artin group.*

(2) *The right-angled Coxeter group G_Γ is quasi-isometric to the right-angled Artin group of a tree of diameter at least 3.*

(3) *The right-angled Coxeter group G_Γ is quasi-isometric to the right-angled Artin group of a tree of diameter exactly 3.*

(4) *All vertices of the tree T_r are black.*

We remark that a visual decomposition tree of such a graph Γ as above is a colored tree whose vertices are colored by black and white and it is constructed in Construction 3.13. By the above theorem, if the defining graph Γ that has a visual decomposition tree T_r containing at least one white vertex (see Example 4.2), then the right-angled Coxeter group G_Γ is not quasi-isometric to any right-angled Artin group.

1.1.2 Quasi-isometric classification of high-dimensional, right-angled Coxeter groups As we discuss above, the key tool of the proof of quasi-isometric classification of $\mathcal{CFS}$ right-angled Coxeter groups G_Γ with defining graphs satisfying the standing assumptions (see Theorem 1.1(3)) is to decompose Γ into a tree of suspensions of distinct points. We develop this idea to study right-angled Coxeter groups whose nerve belongs to a collection $\mathbb{K}_n$ with $n \geq 1$ of certain n–dimensional flag complexes which can be decomposed as a tree of simpler flag complexes (see Definition 5.6). We remark that the 1–skeleton of each flag complex in $\mathbb{K}_n$ is always $\mathcal{CFS}$ and $\mathbb{K}_1$ is actually the collection of all $\mathcal{CFS}$, nonjoin graphs satisfying the standing assumptions.

Each flag complex K in $\mathbb{K}_n$ (by definition) can be constructed from a $\boldsymbol{p}/\boldsymbol{f}$–bipartite T in a collection $\mathbb{T}_n$ (see Definitions 5.1 and 5.6). The tree T is colored in a way to be described in Section 5.3 and we apply the concept of bisimilarity on such tree T to give a complete quasi-isometric classification of each collection of right-angled Coxeter groups $\{G_K\}_{K \in \mathbb{K}_n}$.

Theorem 1.3 *Let K and K' be two flag complexes in $\mathbb{K}_n$ and we assume that K and K' can be constructed from two trees T and T' in $\mathbb{T}_n$. Then two right-angled Coxeter groups G_K and $G_{K'}$ are quasi-isometric if and only if two colored trees T and T' are bisimilar after possibly reordering the $\boldsymbol{p}$–colors by an element of the symmetric group on $2n+2$ elements.*

In [6], Behrstock, Januszkiewicz and Neumann study quasi-isometry classification of some high-dimensional RAAGs. The nerves of these groups can also be constructed from a tree of certain flag complexes of high dimension. Behrstock, Januszkiewicz and Neumann use the tree structure of the nerves to construct geometric models of the corresponding RAAGs to study the quasi-isometry classification of these groups. The reader can observe that the strategy of the proof of Theorem 1.3 (see Section 5.3) is similar to the one for quasi-isometry classification of RAAGs in [6]. In fact, we also study quasi-isometry classes of our RACGs by constructing their geometric models. However, the such geometric models are not totally identical to the ones in [6] and they are actually required certain nontrivial techniques. Moreover, our collection of RACGs is "richer" and it "includes" the collection of RAAGs in [6] in term of quasi-isometry classes of both collections (see Theorem 5.11).

1.1.3 Strongly quasiconvex subgroups of $\mathcal{CFS}$ right-angled Coxeter groups One method to understand the structure of a finitely generated group G is to investigate subgroups of G whose geometry reflects that of G. Quasiconvex subgroups of hyperbolic groups is a successful application of this approach. However, quasiconvexity is not as useful for arbitrary finitely generated groups since quasiconvexity depends on a choice of generating set and, in particular, is not preserved under quasi-isometry. In [19], Durham and Taylor introduce a strong notion of quasiconvexity in finitely generated groups, called *stability*, which is preserved under quasi-isometry.

Stability agrees with quasiconvexity when ambient groups are hyperbolic. However, a stable subgroup of a finitely generated group is always hyperbolic whether the ambient group is hyperbolic or not (see [19]). In some sense, the geometry of a stable subgroup does not reflect completely that of the ambient group. In July 2017, the second author in [31] introduced another concept of quasiconvexity, called *strong quasiconvexity*, which is strong enough to be preserved under quasi-isometry and reflexive enough to capture the geometry of ambient groups. This notion was also introduced independently by Genevois [20] in September 2017 under the name Morse subgroup.

There is a strong connection between strong quasiconvexity and stability. More precisely, a subgroup is stable if and only if it is strongly quasiconvex and hyperbolic (see [31]). Moreover, these notions agree in the hyperbolic setting. Outside the hyperbolic setting, there are many strongly quasiconvex subgroups that are not stable.

A natural question arises of which nonhyperbolic group G whose all strongly quasiconvex subgroups of infinite index of G are hyperbolic (ie stable). In [31], the second author proves that all strongly quasiconvex subgroups of infinite index of one-ended right-angled Artin groups are stable. In a recent paper (see [26]), Kim proves that all strongly quasiconvex subgroups of infinite index of mapping class group of an oriented, connected, finite-type surface with negative Euler characteristic are stable. We prove this fact is true for G_K where K is a flag complex in $\mathbb{K}_n$

Theorem 1.4 *Let K be a flag complex in $\mathbb{K}_n$ and H a strongly quasiconvex subgroup of infinite index of the right-angled Coxeter group G_K. Then H is virtually free. In particular, H is stable.*

We remark here that not all $\mathcal{CFS}$ right-angled Coxeter groups have the property that all infinite-index strongly quasiconvex subgroups are virtually free (or even hyperbolic). We refer the reader to Example 6.13 for this fact.

The main ingredient for the proof of Theorem 1.4 is the tree of groups structure of the right-angled Coxeter group G_Γ with vertex groups and edge groups satisfying certain conditions. Actually, we prove a stronger result that is applied to such a tree of groups in general. More precisely:

Proposition 1.5 *Assume a group G is decomposed as a finite graph T of groups that satisfies the following*:

1. *For each vertex v of T the vertex group G_v is finitely generated and undistorted. Moreover, any strongly quasiconvex, infinite subgroup of G_v is of finite index.*
2. *Each edge group is infinite.*

Then, if H is a strongly quasiconvex, torsion-free subgroup of G of infinite index, then H is a free subgroup.

1.2 Right-angled Coxeter groups with arbitrary defining graphs satisfying the standing assumptions

In general case (when the graph Γ is not necessarily $\mathcal{CFS}$), we prove that if Γ satisfies the standing assumptions, the associated right-angled Coxeter group G_Γ is hyperbolic relative to a certain collection of $\mathcal{CFS}$ right-angled Coxeter subgroups. More precisely:

Theorem 1.6 *Let Γ be a graph satisfying the standing assumptions. There is a collection $\mathbb{J}$ of $\mathcal{CFS}$ subgraph of Γ such that the right-angled Coxeter group G_Γ is relatively hyperbolic with respect to the collection $\mathbb{P} = \{G_J \mid J \in \mathbb{J}\}$.*

For the proof of Theorem 1.6 we carefully investigate the tree structure of the defining graph and use results in [8, Theorem A$'$] and Druțu and Sapir [18, Corollary 1.14] to figure out the relatively hyperbolic structure of group G_Γ. The investigation of the tree structure for proof of Theorem 1.6 is quite technical and we refer the reader to Section 4.2 for the details.

By exploring the relatively hyperbolic structure of groups in Theorem 1.6 we can take advantage of Theorem 1.1 to study quasi-isometry classification of right-angled Coxeter groups even in the case of non-$\mathcal{CFS}$ defining graphs. In fact, by Theorem 1.6 these groups are relatively hyperbolic with respect to collections of $\mathcal{CFS}$ right-angled Coxeter groups. Therefore, if we know the difference in term of quasi-isometry between two such peripheral structures of two relatively hyperbolic groups G_Ω and $G_{\Omega'}$ then, by Theorem 1.1, we can distinguish G_Ω and $G_{\Omega'}$ also in terms of quasi-isometry. We refer the reader to Example 4.5 for this application.

Theorem 1.6 also contributes to study the divergence of right-angled Coxeter groups. Behrstock, Hagen and Sisto [5] show that the divergence of a one-ended right-angled Coxeter group is either exponential or bounded above by a polynomial. Dani and Thomas [13] also show that for every positive integer d, there is a right-angled Coxeter group with divergence x^d. However, by combining Theorem 1.6 with results in [13, Theorem 1.1] and Sisto [29, Theorem 1.3], the divergence functions of one-ended right-angled Coxeter groups G_Γ of planar, triangle-free graphs Γ are quite simple. More precisely:

Corollary 1.7 *Let Γ be a graph satisfying the standing assumptions. Then the divergence of the right-angled Coxeter group G_Γ is linear, quadratic or exponential.*

Overview

In Section 2 we review some concepts in geometric group theory and 3–manifold theory. In Section 3 we study the "tree structure" of graphs satisfying the standing assumptions. In Section 4 we study right-angled Coxeter groups with planar defining graph. We give the proof of Theorems 1.1 and 1.2 in Section 4.1. The proof of Theorem 1.6 is given in Section 4.2. In Section 5 we generalize Theorem 1.1 to a certain high-dimensional right-angled Coxeter groups. We give the proof of Theorem 1.3 in Section 5.3. In Section 6 we study strongly quasiconvex subgroups of $\mathcal{CFS}$ right-angled Coxeter groups. We give proofs of Theorem 1.4 and Proposition 1.5 in Section 6.2.

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2 Preliminaries

In this section, we review some concepts in geometric group theory and 3–manifold theory: right-angled Coxeter groups, Davis complexes, right-angled Artin groups, relatively hyperbolic groups, graph manifolds and mixed manifolds. We discuss the work of Caprace [8], Behrstock, Hagen and Sisto [5] and Dani and Thomas [13] on peripheral structures of relatively hyperbolic right-angled Coxeter groups and divergence of right-angled Coxeter groups. We also discuss the work of Gersten [21] and Kapovich and Leeb [25] on divergence of 3–manifold groups. We also mention the concept of colored graphs and the bisimilarity equivalence relation on these graphs. Lastly, we review the work of Behrstock and Neumann [7] and Gordon [23] on connections between right-angled Artin groups and 3–manifold groups.

2.1 Right-angled Coxeter groups and their relatively hyperbolic structures

We first review the concepts of right-angled Coxeter groups and Davis complexes.

Definition 2.1 Given a finite simplicial graph Γ, the associated *right-angled Coxeter group* G_Γ is generated by the set S of vertices of Γ and has relations $s^2=1$ for all s in S and $st=ts$ whenever s and t are adjacent vertices. The graph Γ is the *defining graph* of the right-angled Coxeter group G_Γ and its flag complex $K=K(\Gamma)$ is the *defining nerve* of the group. Sometimes, we also denote the right-angled Coxeter group G_Γ by G_K, where K is the flag complex of Γ.

Let S_1 be a subset of S. The subgroup of G_Γ generated by S_1 is a right-angled Coxeter group G_{Γ_1}, where Γ_1 is the induced subgraph of Γ with vertex set S_1 (ie Γ_1 is the union of all edges of Γ with both endpoints in S_1). The subgroup G_{Γ_1} is called a *special subgroup* of G_Γ.

Definition 2.2 Given a finite simplicial graph Γ, the associated *Davis complex* Σ_Γ is a cube complex constructed as follows. For every k–clique $T \subset \Gamma$, the special subgroup G_T is isomorphic to the direct product of k copies of Z_2. Hence, the Cayley graph of G_T is isomorphic to the 1–skeleton of a k–cube. The Davis complex Σ_Γ has 1–skeleton the Cayley graph of G_Γ, where edges are given unit length. Additionally, for each k–clique $T \subset \Gamma$ and coset gG_T, we glue a unit k–cube to $gG_T \subset \Sigma_\Gamma$. The Davis complex Σ_Γ is a CAT(0) space and the group G_Γ acts properly and cocompactly on the Davis complex Σ_Γ (see [15]).

We now review the concept of relatively hyperbolic groups.

Definition 2.3 For a finitely generated group G with Cayley graph $\Gamma(G,S)$ equipped with the path metric and a finite collection $\mathbb{P}$ of subgroups of G, one can construct the *coned off Cayley graph* $\widehat{\Gamma}(G,S,\mathbb{P})$ as follows: for each left coset gP where $P \in \mathbb{P}$, add a vertex v_{gP}, called a *peripheral vertex*, to the Cayley graph $\Gamma(G,S)$ and for each element x of gP, add an edge $e(x,gP)$ of length $\frac{1}{2}$ from x to the vertex v_{gP}. This results in a metric space that may not be proper (ie closed balls need not be compact).

Definition 2.4 (relatively hyperbolic group) A finitely generated group G is *hyperbolic relative to a finite collection* $\mathbb{P}$ *of subgroups of* G if the coned off Cayley graph is δ–hyperbolic and *fine* (ie for each positive number n, each edge of the coned off Cayley graph is contained in only finitely many circuits of length n). Each group $P \in \mathbb{P}$ is a *peripheral subgroup* and its left cosets are *peripheral left cosets*, and we denote the collection of all peripheral left cosets by Π.

Theorem 2.5 [18, Corollary 1.14] *If a group G is hyperbolic relative to the collection $\{H_1,\dots,H_m\}$ and each H_i is hyperbolic relative to a collection of subgroups $\{H_i^1, H_i^2,\dots,H_i^{n_i}\}$, then G is hyperbolic relative to the collection*

$$\big\{H_i^j \mid i \in \{1,2,\dots,m\},\ j \in \{1,2,\dots,n_i\}\big\}.$$

In the rest of this subsection, we discuss the work of Caprace [8] and Behrstock, Hagen and Sisto [5] on peripheral structures of relatively hyperbolic right-angled Coxeter groups.

Theorem 2.6 [8, Theorem A′] *Let Γ be a simplicial graph and $\mathbb{J}$ be a collection of induced subgraphs of Γ. Then the right-angled Coxeter groups G_Γ is hyperbolic relative to the collection $\mathbb{P} = \{G_J \mid J \in \mathbb{J}\}$ if and only if the following three conditions hold*:

(1) *If σ is an induced 4–cycle of Γ, then σ is an induced 4–cycle of some $J \in \mathbb{J}$.*

(2) *For all J_1 and J_2 in $\mathbb{J}$ with $J_1 \neq J_2$, the intersection $J_1 \cap J_2$ is empty or $J_1 \cap J_2$ is a complete subgraph of Γ.*

(3) *If a vertex s commutes with two nonadjacent vertices of some J in $\mathbb{J}$, then s lies in J.*

Theorem 2.7 [8, Theorem B] *Let Γ be a simplicial graph. If G_Γ is relatively hyperbolic with respect to finitely generated subgroups $H_1, \dots, H_m$, then each H_i is conjugate to a special subgroup of G_Γ.*

Theorem 2.8 [5, Theorem I] *Let $\mathcal{T}$ be the class consisting of the finite simplicial graphs Λ such that G_Λ is strongly algebraically thick. Then for any finite simplicial graph Γ either $\Gamma \in \mathcal{T}$, or there exists a collection $\mathbb{J}$ of induced subgraphs of Γ such that $\mathbb{J} \subset \mathcal{T}$ and G_Γ is hyperbolic relative to the collection $\mathbb{P} = \{G_J \mid J \in \mathbb{J}\}$ and this peripheral structure is minimal.*

Remark 2.9 In Theorem 2.8 we use the notion of *strong algebraic thickness*, which is introduced in [3] and is a sufficient condition for a group to be nonhyperbolic relative to any collection of proper subgroups. We refer the reader to [3] for more details. The following theorem from [5] characterizes all strongly algebraically thick right-angled Coxeter groups and it will prove useful for studying peripheral subgroups of relatively hyperbolic right-angled Coxeter groups.

Theorem 2.10 [5, Theorem II] *Let $\mathcal{T}$ be the class of finite simplicial graphs whose corresponding right-angled Coxeter groups are strongly algebraically thick. Then $\mathcal{T}$ is the smallest class of graphs satisfying the following conditions*:

(1) *The 4–cycle lies in $\mathcal{T}$.*

(2) *Let $\Gamma \in \mathcal{T}$ and let $\Lambda \subset \Gamma$ be an induced subgraph which is not a complete graph. Then the graph obtained from Γ by coning off Λ is in $\mathcal{T}$.*

(3) *Let $\Gamma_1, \Gamma_2 \in \mathcal{T}$ and suppose there exists a graph Γ which is not a complete graph and which arises as a subgraph of each of the Γ_i. Then the union Λ of Γ_1, Γ_2 along Γ is in $\mathcal{T}$, and so is any graph obtained from Λ by adding any collection of edges joining vertices in $\Gamma_1 - \Gamma$ to vertices of $\Gamma_2 - \Gamma$.*

2.2 Divergence of right-angled Coxeter groups and 3–manifold groups

Roughly speaking, divergence is a quasi-isometry invariant that measures the circumference of a ball of radius n as a function of n. We refer the reader to [22] for a precise definition. In this section, we state some theorems about divergence of certain right-angled Coxeter groups and 3–manifold groups which will be used later in this paper.

2.2.1 Divergence of right-angled Coxeter groups

Theorem 2.11 [5] *The divergence of a right-angled Coxeter group is either exponential* (*if the group is relatively hyperbolic*) *or bounded above by a polynomial* (*if the group is strongly algebraically thick*).

Definition 2.12 Given a graph Γ, define the associated *four-cycle* graph Γ^4 as follows. The vertices of Γ^4 are the induced loops of length four (ie four-cycles) in Γ. Two vertices of Γ^4 are connected by an edge if the corresponding four-cycles in Γ share a pair of nonadjacent vertices. Given a subgraph K of Γ^4, we define the *support* of K to be the collection of vertices of Γ (ie generators of G_Γ) that appear in the four-cycles in Γ corresponding to the vertices of K. A graph Γ is $\mathcal{CFS}$ if $\Gamma = \Omega * K$, where K is a (possibly empty) clique and Ω is a nonempty subgraph such that Ω^4 has a connected component whose support is the entire vertex set of Ω.

Theorem 2.13 [13, Theorem 1.1] *Let Γ be a finite, simplicial, connected, triangle-free graph which has no separating vertices or edges. Let G_Γ be the associated right-angled Coxeter group.*

(1) *The group G_Γ has linear divergence if and only if Γ is a join.*
(2) *The group G_Γ has quadratic divergence if and only if Γ is $\mathcal{CFS}$ and is not a join.*

2.2.2 Divergence of 3–manifold groups Let M be a compact, orientable 3–manifold with empty or toroidal boundary. The 3–manifold M is *geometric* if its interior admits a geometric structure in the sense of Thurston which is one of the 3–sphere, Euclidean 3–space, hyperbolic 3–space, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, Nil or Sol. We note that a geometric 3–manifold M is Seifert fibered if its geometry is neither Sol nor hyperbolic. A nongeometric 3–manifold can be cut into hyperbolic and Seifert fibered “blocks” along a JSJ decomposition. It is called a *graph manifold* if all the pieces are Seifert fibered; otherwise it is a *mixed manifold*.

Theorem 2.14 (Gersten [21] and Kapovich and Leeb [25]) *Let M be a nongeometric manifold. Then M is a graph manifold if and only if the divergence of $\pi_1(M)$ is quadratic, and M is a mixed manifold if and only if the divergence of $\pi_1(M)$ is exponential.*

Remark 2.15 Let M be a compact, orientable 3–manifold with linear divergence. We note that M is geometric, otherwise its divergence is at least quadratic. Also, M is not a hyperbolic manifold because the divergence of a hyperbolic manifold is exponential. If the universal cover $\widetilde{M}$ of M is the direct product with $\mathbb{R}$ of a fattening of a tree with all vertex degrees at least 3, then M is not homeomorphic to $D^2 \times S^1$, $T^2 \times I$ or $K^2 \mathbin{\hat{\times}} I$ (twisted I–bundle over the Klein bottle). Also M is not a Sol manifold, otherwise M is a closed manifold (because we excluded $D^2 \times S^1$, $T^2 \times I$ and $K^2 \mathbin{\hat{\times}} I$), which contradicts the fact $\widetilde{M}$ is the direct product with $\mathbb{R}$ of a fattening of a tree with all vertex degrees at least 3. Therefore, M must be a Seifert manifold excluding $D^2 \times S^1$, $T^2 \times I$ and $K^2 \mathbin{\hat{\times}} I$.

2.3 Colored graphs and bisimilarity

In this section, we review the concepts of colored graphs and bisimilarity in [7; 6]. We will use them to classify certain right-angled Coxeter groups in this paper.

Definition 2.16 A *colored graph* is a graph Γ, a set C and a "vertex coloring" $c\colon V(\Gamma) \to C$.

A *weak covering* of colored graphs is a graph homomorphism $f\colon \Gamma \to \Gamma'$ which respects colors and has the property that for each $v \in V(\Gamma)$ and for each edge $e' \in E(\Gamma')$ at $f(v)$, there exists an $e \in E(\Gamma)$ at v with $f(e) = e'$.

Definition 2.17 Colored graphs Γ_1 and Γ_2 are *bisimilar*, written $\Gamma_1 \sim \Gamma_2$, if Γ_1 and Γ_2 weakly cover some common colored graph.

Proposition 2.18 [7] *The bisimilarity relation $\sim$ is an equivalence relation. Moreover, each equivalence class has a unique minimal element up to isomorphism.*

2.4 Right-angled Artin groups and connection to 3–manifold groups

We now review the concept of right-angled Artin groups and the works of Behrstock and Neumann [7] and Gordon [23] on connections between right-angled Artin groups and 3–manifold groups.

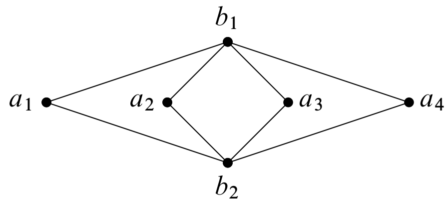

Figure 1: The 4–cycle with vertices a_2, a_3, b_1 and b_2 is separating but not strongly separating with respect to the current choice of planar embedding.

Definition 2.19 Given a finite simplicial graph Γ, the associated *right-angled Artin group* A_Γ has generating set S the vertices of Γ, and relations $st = ts$ whenever s and t are adjacent vertices.

The following two theorems show some connections between right-angled Artin groups and 3–manifold groups.

Theorem 2.20 (Gordon [23]) *The following are equivalent for a one-ended right-angled Artin group* A_Γ:

(1) A_Γ *is virtually a* 3*–manifold group*;

(2) A_Γ *is a* 3*–manifold group*; *and*

(3) Γ *is either a tree or a triangle.*

Theorem 2.21 (Behrstock and Neumann [7]) *A right-angled Artin group* A_Γ *is quasi-isometric to a* 3*–manifold group if and only if it is a* 3*–manifold group* (*and is hence as in Theorem 2.20*).

3 Graph decomposition

In this section, we study the "tree structure" of graphs Γ satisfying the standing assumptions. This structure will help us study corresponding right-angled Coxeter groups G_Γ in next section.

Definition 3.1 A 4–cycle σ of a graph Γ *separates* Γ if $\Gamma - \sigma$ has at least two components.

We now talk about a stronger notion of "separating 4–cycle" of planar graph. This notion depends on the choice of embedding map of the ambient graph into the plane and the notion is based on the Jordan curve theorem.

Definition 3.2 Let Γ be a graph satisfying the standing assumptions and let $f\colon \Gamma \to \mathbb{R}^2$ be an embedding. A 4–cycle σ of Γ *strongly separates* Γ *with respect to* f if $f(\Gamma)$ has nonempty intersection with both components of $\mathbb{R}^2 - f(\sigma)$.

Remark 3.3 If the map f in Definition 3.2 is clear from the context, we just say the 4–cycle σ strongly separates Γ. It is clear that if a 4–cycle σ strongly separates a graph Γ with respect to some embedding map f, then σ separates Γ in the usual sense. However, if we fix an embedding f of the graph Γ into the plane, then a separating 4–cycle of Γ is not necessarily strongly separating with respect to f. In fact, let Γ be a planar graph with the choice of embedding f in the plane as in Figure 1, the 4–cycle with vertices a_2, a_3, b_1 and b_2 is separating but not strongly separating with respect to f.

Definition 3.4 Assume a 4–cycle σ strongly separates a graph Γ with respect to an embedding f. Let U_1 and U_2 be two components of $\mathbb{R}^2 - f(\sigma)$. Let Γ_i be σ together with components of $\Gamma - \sigma$ that are mapped into U_i via f. Then, $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \sigma$. We call the pair (Γ_1, Γ_2) a *strong visual decomposition* of Γ along σ with respect to f. If the embedding f is clear from the context, we just say the pair (Γ_1, Γ_2) is a strong visual decomposition of Γ along σ

Basically, the following lemma shows that each such subgraph Γ_i in a strong visual decomposition of the graph Γ above inherits important properties of the ambient graph Γ.

Lemma 3.5 *Let Γ be a graph satisfying the standing assumptions. Let (Γ_1, Γ_2) be a strong visual decomposition of Γ along a 4–cycle σ with respect to some embedding f. Then each subgraph Γ_i also satisfies the standing assumptions. Moreover, if Γ is $\mathcal{CFS}$, then each subgraph Γ_i is also $\mathcal{CFS}$.*

Proof It is clear that each graph Γ_i is connected, triangle-free, planar and has at least 5 vertices. We now prove that if either Γ_1 or Γ_2 (say Γ_1) has a separating vertex or a separating edge C, then C is also a separating vertex or separating edge of Γ. Let v be a vertex in $\sigma - C$. Since C is a separating vertex or separating edge of Γ_1, there is a vertex u in $\Gamma_1 - C$ such that there is no path in $\Gamma_1 - C$ connecting u and v. We observe that $\sigma - C$ is a connected set in Γ_1. Then u is not a vertex of σ. We will prove that there is no path in $\Gamma - C$ connecting u and v. Assume for contradiction that there is a path α in $\Gamma - C$ connecting u and v. We can choose a connected subpath β

of α connecting u and some vertex v' of σ such that $\beta \cap \sigma = \{v'\}$. It is clear that β is a path in Γ_1. Again $\sigma - C$ is a connected set in Γ_1 and two vertices v and v' both lie in $\sigma - C$. There is a path in Γ_1 connecting u and v, which is a contradiction. This implies that there is no path in $\Gamma - C$ connecting u and v. Therefore, C is a separating vertex or separating edge of Γ, which is a contradiction. Thus, each subgraph Γ_i has no separating vertex and no separating edge.

We now assume that Γ is $\mathcal{CFS}$ and we will prove that each Γ_i is also $\mathcal{CFS}$. We only need to prove Γ_1 is $\mathcal{CFS}$ and the proof for Γ_2 is analogous. Let K be a component of Γ^4 whose support is the entire vertex set of Γ. Let K_1 be an induced subgraph of K that contains all vertices which are 4–cycles of Γ_1. It suffices to prove that K_1 is connected and its support is the entire vertex set of Γ_1.

We first prove that the 4–cycle σ is a vertex of K. Let u_1 be a vertex in $\Gamma_1 - \sigma$ and let u_2 be a vertex in $\Gamma_2 - \sigma$. Then there is a sequence of 4–cycles $Q_1, Q_2, \dots, Q_n$ which are vertices of K such that Q_1 contains u_1 and Q_n contains u_2 and $Q_i \cap Q_{i+1}$ is the union of two adjacent edges for each i. We now prove that some Q_k contains two nonadjacent vertices of σ. Assume for contradiction that no Q_i contains two nonadjacent vertices of σ. Therefore, each Q_i is contained in Γ_1 or Γ_2. It is clear that Q_1 is contained in Γ_1 and Q_n is contained in Γ_2. Then there are Q_ℓ and $Q_{\ell+1}$ such that Q_ℓ is contained in Γ_1 and $Q_{\ell+1}$ is contained in Γ_2. Therefore, $Q_\ell \cap Q_{\ell+1}$ is contained in the 4–cycle σ. This implies that both Q_ℓ and $Q_{\ell+1}$ contain two nonadjacent vertices of σ, which is a contradiction. Therefore, some Q_k contains two nonadjacent vertices of σ. Thus, there is a path in Γ^4 connecting Q_k and σ. This implies that σ is a vertex of K. Therefore, σ is also a vertex of K_1.

We now prove K_1 is connected; it suffices to prove each vertex in K_1 is connected to σ by a path in K_1. Let γ be an arbitrary 4–cycle which is a vertex of K_1. If γ contains two nonadjacent vertices of σ, then it is clear that there is a path in K_1 of length at most 2 connecting γ and σ. Otherwise, let $\gamma = P_0, P_1, P_2, \dots, P_m = \sigma$ be the sequence of vertices of K such that $P_i \cap P_{i+1}$ is the union of two adjacent edges. Let k be the smallest number such that P_k contains two nonadjacent vertices of σ. Therefore, P_i is contained in Γ_1 for each $i \leq k-1$. Thus, P_i is a vertex in K_1 for each $i \leq k-1$. Let b and c be two nonadjacent vertices of $P_{k-1} \cap P_k$. Then it is clear that b and c are not nonadjacent vertices of $P_k \cap \sigma$. This implies that P_k is also contained in Γ_1. Therefore, P_k is also a vertex of K_1. Since P_k contains two nonadjacent vertices of σ, there is a path of length at most 2 in K_1 connecting P_k and σ. Thus, there is a path in K_1 connecting γ and σ. Therefore, K_1 is connected.

We now prove that the support of K_1 is the entire vertex set of Γ_1. Let u be a vertex in Γ_1. If u is a vertex of σ or u is adjacent to nonadjacent vertices of σ, then u is in the support of K_1 clearly. Otherwise, let P be a vertex of K that contains u. Then P does not contain two nonadjacent vertices of σ. Therefore, P is contained in Γ_1. Thus, P is a vertex of K_1. Thus, u belongs to the support of K_1. This implies that the support of K_1 is the entire vertex set of Γ_1. Therefore, Γ_1 is $\mathcal{CFS}$. □

Definition 3.6 Let Γ be a graph satisfying the standing assumptions and $f\colon \Gamma \to \mathbb{R}^2$ be an embedding. We denote $n(\Gamma, f)$ the number of 4–cycles in Γ that strongly separates Γ with respect to f.

The graph Γ is called *prime* if Γ is not a 4–cycle and $n(\Gamma, f) = 0$ for some embedding $f\colon \Gamma \to \mathbb{R}^2$.

The following lemma helps us understand the structure of prime graphs.

Lemma 3.7 *Let Γ be a graph satisfying the standing assumptions. Assume that Γ is a prime graph. Then Γ is the suspension of 3 distinct points or Γ does not contain the suspension of 3 distinct points. In particular, if Γ is $\mathcal{CFS}$, then it must be the suspension of 3 distinct points.*

Proof We assume that Γ contains subgraph K which is a suspension of three vertices called a_1, a_2 and a_3. Let b_1 and b_2 be suspension vertices of K. We will show that $\Gamma = K$. Let $f\colon \Gamma \to \mathbb{R}^2$ be an embedding. Let C_1 be the image of the 4–cycle with vertices b_1, b_2, a_2 and a_3. Let C_2 be the image of the 4–cycle with vertices b_1, b_2, a_1 and a_3. Let C_3 be the image of the 4–cycle with vertices b_1, b_2, a_1 and a_2. We can assume that $f(a_2)$ lies in the bounded component of $\mathbb{R}^2 - C_2$.

Assume for contradiction that $\Gamma \neq K$. Then there is a vertex d of Γ that does not belong to the set $\{b_1, b_2, a_1, a_2, a_3\}$. If $f(d)$ lies in the unbounded component of $\mathbb{R}^2 - C_2$, then $f(\Gamma)$ intersects with both components of $\mathbb{R}^2 - C_2$. Therefore, the 4–cycle with vertices b_1, b_2, a_1 and a_3 strongly separates Γ, which is a contradiction. If $f(d)$ lies in the bounded component of $\mathbb{R}^2 - C_2$, then $f(d)$ lies in the bounded component of $\mathbb{R}^2 - C_1$ or $\mathbb{R}^2 - C_3$ (say $\mathbb{R}^2 - C_1$). Also $f(a_1)$ lies in the unbounded component of $\mathbb{R}^2 - C_1$. Therefore, $f(\Gamma)$ intersects with both components of $\mathbb{R}^2 - C_1$. This implies that the 4–cycle with vertices b_1, b_2, a_2 and a_3 strongly separates Γ, which is a contradiction. Therefore, $\Gamma = K$. □

In the following two lemmas, we discuss some behaviors of 4–cycles in a strong decomposition of a graph.

Lemma 3.8 *Let Γ be a graph satisfying the standing assumptions and $f\colon \Gamma \to \mathbb{R}^2$ be an embedding. Assume that (Γ_1, Γ_2) be a strong visual decomposition of Γ with respect to f along some 4–cycle σ. Then for each i the 4–cycle σ does not strongly separate any subgraph K of Γ_i that contains σ with respect to $f|_K$. Moreover, if a 4–cycle α in some Γ_i that strongly separates Γ_i with respect to $f|_{\Gamma_i}$, then α also strongly separates Γ with respect to f.*

Proof Let V_b and V_u be the two components of $\mathbb{R}^2 - f(\sigma)$. By relabeling if necessary, we assume that $f(\Gamma_1) \subset V_b \cup f(\sigma)$ and $f(\Gamma_2) \subset V_u \cup f(\sigma)$. Let K be any subgraph of Γ_i such that K contains σ. We will show that σ does not strongly separate K with respect to $f|_K$. Without losing generality, we can assume that $i = 1$ (the case $i = 2$ is similar). It follows that $f(K) \subset f(\Gamma_1)$. We now show that $f(K) \cap V_u = \varnothing$. Indeed, we know that $f(\Gamma_2) - f(\sigma) \subset V_u$ and $f(\sigma) = f(\Gamma_1) \cap f(\Gamma_2)$. It follows that $f(K) \cap (f(\Gamma_2) - f(\sigma)) \subset f(\Gamma_1) \cap (f(\Gamma_2) - f(\sigma)) = \varnothing$, thus $f(K) \cap V_u = \varnothing$ because $f(K) \cap V_u = f(K) \cap (f(\Gamma_2) - f(\sigma))$.

We are now going to prove that if α is a 4–cycle in some Γ_i which strongly separates Γ_i with respect to $f|_{\Gamma_i}$, then α also strongly separates Γ with respect to f. Let U_u and U_b be two components of $\mathbb{R}^2 - f(\alpha)$. Since α strongly separates Γ_i with respect to $f|_{\Gamma_i}$, we have $f(\Gamma_i) \cap U_b$ and $f(\Gamma_i) \cap U_u$ are nonempty sets. Of course, it implies that $f(\Gamma) \cap U_b$ and $f(\Gamma) \cap U_u$ are nonempty sets as well, thus α strongly separates Γ with respect to f. □

Lemma 3.9 *Let Γ be a graph satisfying the standing assumptions and $f\colon \Gamma \to \mathbb{R}^2$ be an embedding. Assume that (Γ_1, Γ_2) is a strong visual decomposition of Γ with respect to f along some 4–cycle σ. If α is a 4–cycle that does not strongly separate Γ with respect to f, then α is contained in Γ_1 or Γ_2.*

Proof If $\alpha \cap \sigma$ does not contain two nonadjacent vertices, then α is contained in Γ_1 or Γ_2, clearly. We now assume that $\alpha \cap \sigma$ contains two nonadjacent vertices. Let (a_1, a_2) and (b, c) be two pairs of nonadjacent vertices of σ. Let (a_3, a_4) and (b, c) be two pairs of nonadjacent vertices of α. Assume for contradiction that α is not contained in Γ_1 or Γ_2. Then $f(a_3)$ and $f(a_4)$ lie in different components of $\mathbb{R}^2 - f(\sigma)$. Therefore, $f(a_1)$ and $f(a_2)$ lie in different components of $\mathbb{R}^2 - f(\alpha)$. This implies that α strongly separates Γ with respect to f, which is a contradiction. Therefore, α is contained in Γ_1 or Γ_2. □

The following lemma is a key step to decomposing a graph satisfying the standing assumptions into a tree of subgraphs.

Lemma 3.10 *Let Γ be a graph satisfying the standing assumptions and $f\colon \Gamma \to \mathbb{R}^2$ be an embedding. Assume there is a finite tree T that encodes the structure of Γ as follows*:

1. *Each vertex v of T is associated to an induced connected subgraph Γ_v of Γ that satisfies the standing assumptions. Moreover, $\Gamma_v \neq \Gamma_{v'}$ if $v \neq v'$ and $\bigcup_{v\in V(T)} \Gamma_v = \Gamma$.*
2. *Each edge e of T is associated to a 4–cycle Γ_e of Γ. Moreover, $\Gamma_e \neq \Gamma_{e'}$ if $e \neq e'$.*
3. *Two vertices v_1 and v_2 of T are endpoints of the same edge e if and only if $\Gamma_{v_1} \cap \Gamma_{v_2} = \Gamma_e$. Moreover, if V_1 and V_2 are vertex sets of two components of T minus the midpoint of e, then $\bigl(\bigcup_{v\in V_1} \Gamma_v, \bigcup_{v\in V_2} \Gamma_v\bigr)$ is a strong visual decomposition of Γ along Γ_e with respect to f.*
4. *The number $m = \max_{v\in V(T)}(n(\Gamma_v, f|_{\Gamma_v}))$ is positive.*

Then there is another tree $\overline{T}$ that encodes the structure of Γ as in conditions (1), (2) *and* (3) *as above and $n(\Gamma_v, f|_{\Gamma_v}) \leq m-1$ for each vertex v of $\overline{T}$. Moreover, if the subgraph Γ_v is $\mathcal{CFS}$ for each vertex v of T, then the subgraph Γ_w is also $\mathcal{CFS}$ for each vertex w of $\overline{T}$.*

Proof Let v_0 be an arbitrary vertex of T such that $m = n(\Gamma_{v_0}, f|_{\Gamma_{v_0}})$. Since $n(\Gamma_{v_0}, f|_{\Gamma_{v_0}}) > 0$, the graph Γ_{v_0} has a 4–cycle σ that strongly separates Γ_{v_0} with respect to $f|_{\Gamma_{v_0}}$. Let (Γ_1, Γ_2) be a strong visual decomposition of Γ_{v_0} along σ with respect to $f|_{\Gamma_{v_0}}$. Let e be an arbitrary edge of T that contains v_0 as an endpoint. Then the 4–cycle Γ_e does not strongly separates Γ_{v_0} with respect to $f|_{\Gamma_{v_0}}$ by Lemma 3.8. Therefore, the 4–cycle Γ_e is contained in Γ_1 or Γ_2 by Lemma 3.9. Thus, we can modify the tree T to obtain another tree T' as follows.

We first replace the vertex v_0 of T by an edge e_0 with two endpoints v_1 and v_2. We associate the new edge e_0 to the 4–cycle $\Gamma_{e_0} = \sigma$. We associate to the new vertex v_1 the graph $\Gamma_{v_1} = \Gamma_1$ and each edge e of T satisfying $\Gamma_e \subset \Gamma_1$ is attached to v_1 in the new tree $\overline{T}$. Similarly, we associate to the new vertex v_2 the graph $\Gamma_{v_2} = \Gamma_2$ and each edge e of T satisfying $\Gamma_e \subset \Gamma_2$ is attached to v_2 in the new tree $\overline{T}$. It is not hard to see the new tree $\overline{T}$ encodes the structure of the graph Γ carrying conditions (1), (2) and (3) in the lemma. Moreover, the numbers $n(\Gamma_{v_1}, f|_{\Gamma_{v_1}})$ and $n(\Gamma_{v_2}, f|_{\Gamma_{v_2}})$ are less than or equal to $m-1$ by Lemma 3.8 and the number $n(\Gamma_v, f|_{\Gamma_v})$ does not change for other vertices. Also the new vertex graphs Γ_{v_1} and Γ_{v_2} also satisfy the standing assumptions by Lemma 3.5. Also by this lemma, the two new vertex graphs Γ_{v_1} and Γ_{v_2} are $\mathcal{CFS}$ if Γ_{v_0} is $\mathcal{CFS}$. Repeating this process to any vertex v satisfying $n(\Gamma_v, f|_{\Gamma_v}) = m$, we can obtain the desired tree $\overline{T}$. Moreover, if the subgraph Γ_v is $\mathcal{CFS}$ for each vertex v of T, then the subgraph Γ_w is also $\mathcal{CFS}$ for each vertex w of $\overline{T}$. □

The following proposition is a direct result of Lemma 3.10.

Proposition 3.11 *Let Γ be a graph satisfying the standing assumptions and $f\colon \Gamma \to \mathbb{R}^2$ be an embedding. Then there is a finite tree T that encodes the structure of Γ as follows:*

(1) *Each vertex v of T is associated to an induced prime subgraph Γ_v of Γ. Moreover, $\Gamma_v \neq \Gamma_{v'}$ if $v \neq v'$ and $\bigcup_{v \in V(T)} \Gamma_v = \Gamma$.*

(2) *Each edge e of T is associated to a 4–cycle Γ_e of Γ. Moreover, $\Gamma_e \neq \Gamma_{e'}$ if $e \neq e'$.*

(3) *Two vertices v_1 and v_2 of T are endpoints of the same edge e if and only if $\Gamma_{v_1} \cap \Gamma_{v_2} = \Gamma_e$. Moreover, if V_1 and V_2 are vertex sets of two components of T minus the midpoint of e, then $\left(\bigcup_{v \in V_1} \Gamma_v, \bigcup_{v \in V_2} \Gamma_v\right)$ is a strong visual decomposition of Γ along Γ_e with respect to f.*

Moreover, if the graph Γ is $\mathcal{CFS}$, then the subgraph Γ_v is also $\mathcal{CFS}$ for each vertex v of T (therefore, Γ_v is a suspension of exactly three points by Lemma 3.7).

Using the "tree structure" on a defining graph Γ as in Proposition 3.11 can help us understand the structure of the corresponding right-angled Coxeter group G_Γ.

Corollary 3.12 *Let Γ be a graph satisfying the standing assumptions. Then the right-angled Coxeter group G_Γ is a tree of groups that satisfies the following conditions:*

(1) *Each vertex group T_v is G_C, where C is the suspension of three distinct points or T_v is a relatively hyperbolic group with respect to a collection of $D_\infty \times D_\infty$ subgroups of T_v.*

(2) *Each edge group is $D_\infty \times D_\infty$.*

Moreover, all vertex groups are isomorphic to a right-angled Coxeter group of the suspension of three distinct points if and only if Γ is $\mathcal{CFS}$.

Proof We decompose the defining graph Γ as a tree T of subgraphs as in Proposition 3.11. This decomposition induces the corresponding decomposition of G_Γ as a tree of groups. Since each edge graph in Proposition 3.11 is an induced 4–cycle, each edge group in the corresponding decomposition of G_Γ is $D_\infty \times D_\infty$, which proves (2). We now prove (1).

Let v be an arbitrary vertex of T such that the corresponding vertex graph Γ_v is not a suspension of three points. Therefore Γ_v does not contain any suspension of three points by Lemma 3.7. Let $\mathbb{J}_v$ be the collection of all 4–cycles in Γ_v. Then $\mathbb{J}_v$ satisfies condition (1) in Theorem 2.6 clearly. Since Γ_v does not contains suspension of three points, the intersection of two 4–cycles in Γ_v is either empty or a point. Moreover, if a vertex u of Γ_v is adjacent to a 4–cycle σ of Γ_v, then u must be a vertex of σ. Therefore, $\mathbb{J}_v$ satisfies conditions (2) and (3) in Theorem 2.6. This implies that the corresponding subgroup $T_v = G_{\Gamma_v}$ is a relatively hyperbolic group with respect to a collection of $D_\infty \times D_\infty$ subgroups of T_v. □

In the rest of this section, we will assume that the ambient graph Γ is $\mathcal{CFS}$. Therefore, it is shown in Proposition 3.11 that each vertex subgraph Γ_v is a suspension of exactly three points. For our purpose of obtaining a quasi-isometric classification of right-angled Coxeter groups with $\mathcal{CFS}$ graph, the tree structure T in Proposition 3.11 is not the right one to look at. We now modify the tree T to obtain a two-colored new tree that encodes structure of Γ by doing the following construction. We refer the reader to Example 4.2 for some explicit constructions.

Construction 3.13 We proceed in four steps:

Step 1 We color an edge of T by two colors, red and blue, as follows. Let e be an edge of Γ with two vertices v_1 and v_2. If Γ_{v_1} and Γ_{v_2} have the same suspension points, then we color the edge e red. Otherwise, we color e blue.

Step 2 Let $\mathcal{R}$ be the union of all red edges of T. We remark that $\mathcal{R}$ is not necessarily connected. We form a new tree T_r from the tree T by collapsing each component C of $\mathcal{R}$ to a vertex labeled by v_C and we associate each such new vertex v_C to the graph $\Gamma_{v_C} = \bigcup_{v \in V(C)} \Gamma_v$. For each vertex v of T_r which is also a vertex of T we still assign v the graph Γ_v as in the previous tree T structure. It is clear that for each vertex v in the new tree T_r, the vertex graph Γ_v is also the suspension of a vertex set called A_v. However, the number of elements in A_v may be greater than three and we call this number the weight of v, denoted by $w(v)$. It is also clear that the new tree T_r encodes the structure of Γ carrying conditions (1), (2) and (3) of Lemma 3.10. Moreover, if v_1 and v_2 are two adjacent vertices in T_r, then the suspension vertices of Γ_{v_1} are elements in A_{v_2} and similarly the suspension vertices of Γ_{v_2} are elements in A_{v_1}.

Step 3 We now choose an appropriate cyclic ordering on the set A_v for the vertex v of T_r. Two vertices a and a' in A_v are adjacent if the pair $\{a, a'\}$ together with two suspension points of Γ_v form a 4–cycle that does not strongly separates Γ_v with respect to $f|_{\Gamma_v}$ (see Figure 2). We note that if v_1 and v_2 are endpoints of an edge e of T_r, then by Lemma 3.8 the 4–cycle Γ_e does not strongly separate each graph Γ_{v_i} with respect to $f|_{\Gamma_{v_i}}$. Therefore, the suspension vertices of Γ_{v_1} are two adjacent elements in A_{v_2} and similarly the suspension vertices of Γ_{v_2} are two adjacent elements in A_{v_1}.

Step 4 We now color vertices of T_r. For each vertex v of T_r, the graph Γ_v is a suspension of a vertex set A_v of T_r. We recall that the weight of v, denoted by $w(v)$, is the number of elements of A_v. It is clear that $w(v)$ is also the number of pairs of adjacent elements in A_v with respect to the above cyclic ordering on A_v. Since, for each edge e of the tree T_r that contains v as an endpoint, the 4–cycle Γ_e does not strongly separate Γ_v, the 4–cycle Γ_e contains a unique pair of nonadjacent elements of A_v. Moreover, if e' is another edge of T_r that contains v as an endpoint, $\Gamma_{e'}$ must contain a different pair of nonadjacent elements of A_v. Therefore, the weight $w(v)$ is always greater than or equal to the degree of v in T_r. We now color v black if its weight is strictly greater than its degree. Otherwise, we color v white.

We now summarize some key properties of the tree T_r in the above construction:

(1) Each vertex v of T_r is associated to an induced subgraph Γ_v of Γ that is a suspension of a vertex set A_v with at least 3 elements and there is some cyclic ordering on A_v. We call the number of elements in A_v the *weight* of the vertex v, denoted by $w(v)$. The weight $w(v)$ of each vertex v is greater than or equal to its degree. We color v black if its weight is strictly greater than its degree. Otherwise, we color v white.

(2) $\Gamma_v \neq \Gamma_{v'}$ if $v \neq v'$ and $\bigcup_{v \in V(T_r)} \Gamma_v = \Gamma$.

(3) Each edge e of T_r is associated to a 4–cycle Γ_e of Γ. Moreover, $\Gamma_e \neq \Gamma_{e'}$ if $e \neq e'$.

(4) Two vertices v_1 and v_2 of T_r are endpoints of the same edge e if and only if $\Gamma_{v_1} \cap \Gamma_{v_2} = \Gamma_e$. Moreover, if v_1 and v_2 are two adjacent vertices of T_r,

the suspension vertices of Γ_{v_1} are two adjacent elements in A_{v_2}, and similarly the suspension vertices of Γ_{v_2} are two adjacent elements in A_{v_1}. Lastly, if V_1 and V_2 are vertex sets of two components of T_r minus the midpoint of e, then $\left(\bigcup_{v\in V_1}\Gamma_v\right)\cap\left(\bigcup_{v\in V_2}\Gamma_v\right)=\Gamma_e$.

Definition 3.14 (visual decomposition trees) Let Γ be a $\mathcal{CFS}$ graph satisfying the standing assumptions. A tree T_r that encodes the structure of Γ carrying properties (1), (2), (3) and (4) as above is called a *visual decomposition tree* of Γ.

Remark 3.15 The existence of a visual decomposition tree for a $\mathcal{CFS}$ graph Γ satisfying the standing assumptions is guaranteed by Construction 3.13. We do not know whether or not this visual decomposition tree for Γ is unique. However, we only need the existence part of such a tree for our purposes. Moreover, it is not hard to draw a visual decomposition tree for a given $\mathcal{CFS}$ graph Γ satisfying the standing assumptions.

4 Right-angled Coxeter groups with planar defining graph

In this section, we divide the collection of graphs Γ satisfying the standing assumptions into two types: $\mathcal{CFS}$ and non-$\mathcal{CFS}$. For a $\mathcal{CFS}$ graph Γ, we prove that the corresponding right-angled Coxeter group G_Γ is virtually a Seifert manifold group if Γ is a join and virtually a graph manifold group otherwise (see (1) and (2) in Theorem 1.1). We then use the work of Behrstock and Neumann [7] to classify all such groups G_Γ up to quasi-isometry (see (3) in Theorem 1.1). When a graph Γ is nonjoin, $\mathcal{CFS}$ and satisfies the standing assumptions, we give a characterization on Γ for G_Γ to be quasi-isometric to right-angled Artin groups and we also specify types of right-angled Artin groups which are quasi-isometric to such right-angled Coxeter groups (see Theorem 1.2). For a non-$\mathcal{CFS}$ graph Γ, we prove that the corresponding right-angled Coxeter groups G_Γ is relatively hyperbolic with respect to a collection of $\mathcal{CFS}$ right-angled Coxeter subgroups of G_Γ (see Theorem 1.6). These results have some applications on divergence of right-angled Coxeter groups.

4.1 Right-angled Coxeter groups with $\mathcal{CFS}$ graphs

In this subsection, we will give the proof of Theorems 1.1 and 1.2.

Let Γ be a $\mathcal{CFS}$ graph satisfying the standing assumptions. Let T_r be a two-colored visual decomposition tree of Γ (see Section 3). Since Γ is planar, it follows that G_Γ

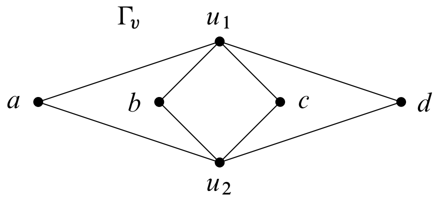

Figure 2: The graph Γ_v is a suspension of the set $A_v = \{a, b, c, d\}$ with two suspension points u_1 and u_2. Since the 4–cycles generated by $\{a, b, u_1, u_2\}$, $\{b, c, u_1, u_2\}$, $\{c, d, u_1, u_2\}$ and $\{d, a, u_1, u_2\}$ are not strongly separating, all pairs of adjacent elements in A_v with respect to the cyclic ordering are $\{a, b\}$, $\{b, c\}$, $\{c, d\}$ and $\{d, a\}$.

is virtually a 3–manifold group. The fact G_Γ is virtually Seifert manifold or graph manifold may not be surprising to experts. However for the purpose of obtaining a quasi-isometric classification (see Theorem 1.1(3)) we will construct explicitly a 3–manifold Y where the right-angled Coxeter group G_Γ acts properly and cocompactly. We then elaborate the work of Kapovich and Leeb [25] and Gersten [21] to get the proof of Theorem 1.1. We note that the construction of the manifold Y is associated to the graph T_r; we then import the work of Behrstock and Neumann [7] to get the proof of Theorem 1.1(3).

Construction 4.1 We now construct a 3–manifold Y on which the right-angled Coxeter group G_Γ acts properly and cocompactly. For each vertex v of T_r, the graph Γ_v is a suspension of a finite set A_v of vertices of Γ. Let b and c be suspension vertices and assume A_v has n elements labeled cyclically by a_i for $i \in \mathbb{Z}_n$. The Davis complex of the right-angled Coxeter group G_{A_v} is an n–regular tree T_n with edges labeled by a_i. We now construct a "fattened tree" $F(T_n)$ of T_n as follows:

We replace each vertex of T_n by a regular n–gon with sides labeled cyclically by $\overline{a}_i$ and we also assume the length side of the n–gon is $\frac{1}{2}$. We replace each edge E labeled by a_i by a strip $E \times \left[-\frac{1}{4}, \frac{1}{4}\right]$. We label each side of length 1 of the strip $E \times \left[-\frac{1}{4}, \frac{1}{4}\right]$ by a_i and we identify the edge E to $E \times \{0\}$ of the strip. Moreover, if u is an endpoint of the edge E of T_n, then the edge $\{u\} \times \left[-\frac{1}{4}, \frac{1}{4}\right]$ is identified to the side labeled by a_i of the n–gon that replaces u. It is clear that the right-angled Coxeter group G_{A_v} acts properly and cocompactly on the fattened tree $F(T_n)$ analogously to how it acts on the Davis complex T_n. By the construction, for each $i \in \mathbb{Z}_n$ there is a bi-infinite boundary geodesic, denoted by $\ell_{\{i-1,i\}}$, in $F(T_n)$ that is a concatenation of edges labeled by a_{i-1} and a_i.

The right-angled Coxeter group $G_{\{b,c\}}$ acts on the line ℓ that is a concatenation of edges labeled by b and c by edge reflections. Let $P_v = F(T_n) \times \ell$ and we equip on P_v the product metric. Then, the right-angled Coxeter group G_{Γ_v} acts properly and cocompactly on P_v in the obvious way. The space P_v is also a 3–manifold with boundaries. Moreover, for each $i \in \mathbb{Z}_n$ the right-angled Coxeter groups generated by $\{a_{i-1}, a_i, b, c\}$ acts on the Euclidean plane $\ell_{\{i-1,i\}} \times \ell$ as an analogous way it acts on its Davis complex. We label this plane by $\{a_{i-1}, a_i, b, c\}$.

If v_1 and v_2 are two adjacent vertices in T_r, then the pair of suspension vertices (a_1, a_2) of Γ_{v_1} are a pair of adjacent elements in A_{v_2} and the pair of suspension vertices (b_1, b_2) of Γ_{v_2} are a pair of adjacent elements in A_{v_1}. Therefore, the spaces P_{v_1} and P_{v_2} have two Euclidean planes that are both labeled by $\{a_1, a_2, b_1, b_2\}$ as we constructed above. Thus, using the Bass–Serre tree $\widetilde{T}_r$ of the decomposition of G_Γ as a tree T_r of subgroups we can form a three manifold Y by gluing copies of P_v appropriately and we obtain a proper, cocompact action of G_Γ on Y.

We first give a proof of Theorem 1.1.

Proof of Theorem 1.1 Let Y be the manifold in Construction 4.1. For each vertex v of T_r, let P_v be the associated space in Construction 4.1. We now are going to prove the necessity of (1) and (2). Since Γ is $\mathcal{CFS}$, the divergence of G_Γ is either linear or quadratic by Theorem 2.13. If the divergence of G_Γ is linear, then Γ is a join $\Gamma_1 * \Gamma_2$ of two induced subgraphs Γ_1 and Γ_2 by Theorem 2.13. Since Γ is triangle-free, has at least 5 vertices and has no separating vertices, each graph Γ_i contains no edges but at least two vertices. Also Γ is planar. Therefore, either Γ_1 or Γ_2 must contain exactly two vertices and Γ must be a suspension of at least 3 vertices. Therefore, the tree T_r constructed as in Construction 3.13 consists of one vertex v and G_Γ acts properly and cocompactly on P_v. Let H be a finite-index, torsion-free subgroup of G_Γ. Then H has linear divergence and acts freely and cocompactly on P_v. Therefore, H is the fundamental group of the compact manifold $M = P_v/H$. By possibly passing to a finite cover of M, we can assume that M is orientable. Moreover, the boundary components of M are tori, thus M is a Seifert manifold by Remark 2.15.

We now assume that the divergence of G_Γ is quadratic. Let H be a finite-index, torsion-free subgroup of G_Γ. Then H acts freely and cocompactly on the 3–manifold Y. Thus, H is the fundamental group of the compact manifold $M = Y/H$. By possibly passing to a finite cover of M, we can assume that M is orientable. We note that ∂M consists of tori. Since the divergence of H is quadratic, it follows that the divergence of $\pi_1(M)$ is quadratic. It follows M is a nongeometric manifold, otherwise the divergence of $\pi_1(M)$ is either linear or exponential. Thus M is a graph manifold by Theorem 2.14.

We are going to prove the sufficiency of (1) and (2). Let Γ be just a graph satisfying the standing assumptions. If G_Γ is virtually a Seifert manifold group. Then the divergence of G_Γ is linear since the divergence of a Seifert manifold group is linear. Therefore, Γ is a join by Theorem 2.13. Also, Γ is planar and triangle-free. Therefore, Γ is a suspension of some distinct vertices.

If G_Γ is virtually a graph manifold group, then the divergence of G_Γ is quadratic since the divergence of a graph manifold group is quadratic (see Theorem 2.14). Therefore, Γ is $\mathcal{CFS}$ and it is not a join by Theorem 2.13. Again, Γ is planar and triangle-free. Thus, Γ is $\mathcal{CFS}$ and it is not a suspension of distinct vertices.

We are now going to prove (3). Since the Bass–Serre tree $\widetilde{T}_r$ weakly covers T_r, two trees $\widetilde{T}_r$ and T_r are bisimilar. Also, we can color vertices of $\widetilde{T}_r$ using its weakly covering on T_r. We observe that a vertex v of $\widetilde{T}_r$ is colored black if and only if the corresponding copy of some P_v includes the boundary of Y. Using the proof of Theorem 3.2 in [7], we obtain the proof of theorem. □

Example 4.2 Let Γ and Γ' be the graphs in Figure 3. It is not hard to see a visual decomposition tree T_r of Γ is shown in the same figure with the following information. The graph Γ_{u_1} is the suspension of three vertices a_1, a_3 and a_5 with two suspension vertices a_6 and a_7. The graph Γ_{u_2} is the suspension of three vertices a_2, a_6 and a_7 with two suspension vertices a_1 and a_3. The graph Γ_{u_3} is the suspension of three vertices a_4, a_6 and a_7 with two suspension vertices a_3 and a_5. The graph Γ_{u_4} is the suspension of three vertices a_6, a_7 and a_8 with two suspension vertices a_1 and a_5. We observe that each u_i has weight 3. Therefore, three vertices u_2, u_3 and u_4 are colored black and u_1 is colored white.

Similarly, a visual decomposition tree T'_r of Γ' is also shown in the Figure 3 with the following information. The graph Γ_{v_1} is the suspension of four vertices b_1, b_3, b_5 and b_9 with two suspension vertices b_6 and b_7. The graph Γ_{v_2} is the suspension of three vertices b_2, b_6 and b_7 with two suspension vertices b_1 and b_3. The graph Γ_{v_3} is the suspension of three vertices b_4, b_6 and b_7 with two suspension vertices b_3 and b_5. The graph Γ_{v_4} is the suspension of three vertices b_6, b_7 and b_8 with two suspension vertices b_1 and b_9. We observe that each v_i has weight 3 excepts v_1 has

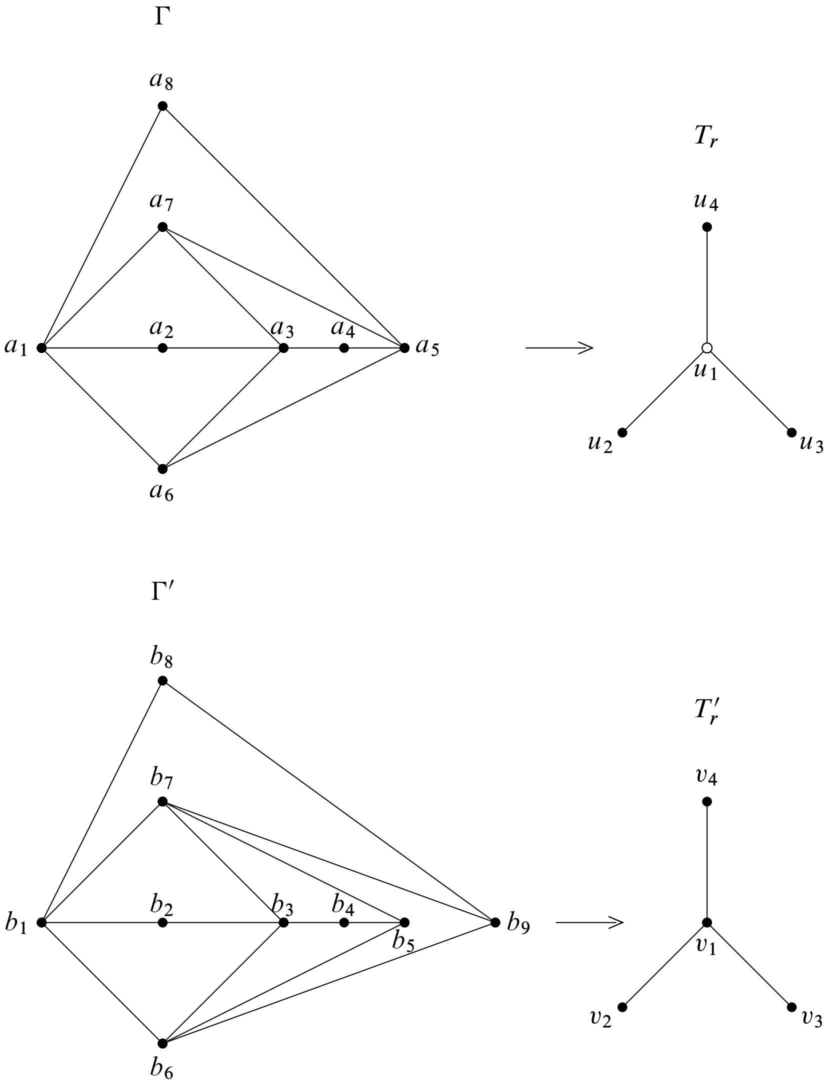

Figure 3: Two groups G_Γ and $G_{\Gamma'}$ are not quasi-isometric because two corresponding decomposition trees T_r and T'_r are not bisimilar.

weight 4. Therefore, all four vertices v_i are colored by black. Therefore, two visual decomposition trees T_r and T'_r are not bisimilar although they are isomorphic if we ignore the vertex colors. Therefore, the two groups G_Γ and $G_{\Gamma'}$ are not quasi-isometric.

We now discuss the connection between right-angled Coxeter groups G_Γ of nonjoin, $\mathcal{CFS}$ graphs Γ satisfying the standing assumptions and right-angled Artin groups.

Proof of Theorem 1.2 We first prove that (1) and (2) are equivalent, and it suffices to prove that (1) implies (2). Assume the right-angled Coxeter group G_Γ is quasi-isometric to a right-angled Artin group A_Ω. Then A_Ω is one-ended and quasi-isometric to a 3–manifold group by Theorem 1.1. Therefore, A_Ω is a one-ended, 3–manifold group by Theorem 2.21. Thus, Ω is a tree or a triangle by Theorem 2.20. Since G_Γ is virtually a graph manifold group by Theorem 1.1, the graph Ω must be a tree of diameter at least 3. Therefore, (1) and (2) are equivalent.

The equivalence between (2) and (3) is proved by Behrstock and Neumann in [7]. We now prove that (3) and (4) are equivalent. We first prove (3) implies (4). Assume that the right-angled Coxeter group G_Γ is quasi-isometric to the right-angled Artin group A_Ω of a tree Ω of diameter exactly 3. We now assume for contradiction that the tree T_r contains a white vertex. As we discussed above, G_Γ is virtually a fundamental group of a graph manifold M such that M has at least one Seifert component that does not contain any boundary component of M. Therefore, the group A_Ω is quasi-isometric to $\pi_1(M)$. On the other hand, Behrstock and Neumann in [7] shows that A_Ω is the fundamental group of a graph manifold M' with boundary components in each Seifert piece and the fundamental group of such a manifold M' is not quasi-isometric to $\pi_1(M)$, this is a contradiction. Therefore, all vertices of the tree T_r are black.

We now prove that (4) implies (3). In fact, if all vertices of the tree T_r are black, the group G_Γ is virtually the fundamental group of a graph manifold M_1 with boundary components in each Seifert piece. Also, the right-angled Artin group A_Ω of a tree Ω of diameter exactly 3 is the fundamental group of a graph manifold M_2 with boundary components in each Seifert piece. Moreover, two groups $\pi_1(M_1)$ and $\pi_1(M_2)$ are quasi-isometric by Behrstock and Neumann [7]. Therefore, the right-angled Coxeter group G_Γ is quasi-isometric to the right-angled Artin group A_Ω. □

4.2 Right-angled Coxeter groups with non-$\mathcal{CFS}$ graphs

In this subsection, we are going to prove Theorem 1.6.

Let Γ be a non-$\mathcal{CFS}$ graph satisfying the standing assumptions. Let $f\colon \Gamma \to \mathbb{R}^2$ be an embedding. Let T be a tree that encodes the structure of Γ as in Proposition 3.11. Since Γ is not a $\mathcal{CFS}$ graph, there is a vertex v_0 of T such that Γ_{v_0} does not contain a suspension of three points.

For each adjacent edge e of v_0 let V_e^1 and V_e^2 be vertex sets of two components of T minus the midpoint of e and we assume that V_e^2 contains the vertex v_0. Let $K_e = \bigcup_{v \in V_e^1} \Gamma_v$ and $L_e = \bigcup_{v \in V_e^2} \Gamma_v$. Then $K_e \cap L_e = \Gamma_e$ by Proposition 3.11.

Let e_1 and e_2 be two arbitrary adjacent edges of v_0. Then it is clear that $V^1_{e_1} \subset V^2_{e_2}$ and $V^1_{e_2} \subset V^2_{e_1}$. Therefore, $K_{e_1} \subset L_{e_2}$ and $K_{e_2} \subset L_{e_1}$. Therefore, $K_{e_1} \cap K_{e_2} \subset L_{e_2} \cap K_{e_2} \subset \Gamma_{e_2}$. Similarly, we also have $K_{e_1} \cap K_{e_2} \subset \Gamma_{e_1}$. This implies that $K_{e_1} \cap K_{e_2} \subset \Gamma_{e_1} \cap \Gamma_{e_2}$. Also Γ_{e_1} and Γ_{e_2} are both 4–cycles in Γ_{v_0}, which does not contain a suspension of three points. Thus, $\Gamma_{e_1} \cap \Gamma_{e_2}$ is empty or a vertex or an edge. Therefore, $K_{e_1} \cap K_{e_2}$ is empty or a vertex or an edge.

Let $\mathbb{J}^1_{v_0}$ be the collection of all graphs K_e for edges e adjacent to v_0. Then $\mathbb{J}^1_{v_0}$ satisfies condition (2) of Theorem 2.6 by the above argument. Let $\mathbb{J}^2_{v_0}$ be the collection of all 4–cycles in Γ_{v_0} which are distinct from Γ_e for any adjacent edge e of v_0. Since Γ_{v_0} does not contain a suspension of three points, $\mathbb{J}^2_{v_0}$ also satisfies condition (2) of Theorem 2.6. Let $\mathbb{J}_{v_0} = \mathbb{J}^1_{v_0} \cup \mathbb{J}^2_{v_0}$.

We use the following proposition in the proof of Theorem 1.6.

Proposition 4.3 *The right-angled Coxeter group G_Γ is relatively hyperbolic with respect to the collection $\mathbb{P}_{v_0} = \{G_J \mid J \in \mathbb{J}_{v_0}\}$.*

Proof We will prove that $\mathbb{J}_{v_0}$ also satisfies condition (2) of Theorem 2.6. It suffices to show the intersection between a graph K_e in $\mathbb{J}^1_{v_0}$ and a 4–cycle σ in $\mathbb{J}^2_{v_0}$ is empty or a vertex or an edge. Indeed, $K_e \cap \sigma = K_e \cap (\Gamma_{v_0} \cap \sigma) = (K_e \cap \Gamma_{v_0}) \cap \sigma = \Gamma_e \cap \sigma$, which is empty or a vertex or an edge since Γ_{v_0} does not contain a suspension of three points. Therefore, $\mathbb{J}_{v_0}$ satisfies condition (2) of Theorem 2.6.

We now prove that $\mathbb{J}_{v_0}$ satisfies condition (3) of Theorem 2.6. We first prove that $\mathbb{J}^2_{v_0}$ satisfies condition (3) of Theorem 2.6. Let σ be a 4–cycle in $\mathbb{J}^2_{v_0}$ and d be a vertex that is adjacent to nonadjacent vertices b and c of σ. We now prove that d is a vertex of Γ_{v_0}. Assume for contradiction that d does not belong to Γ_{v_0}. Therefore, d is a vertex of $K_e - \Gamma_e$ for some adjacent edge e of v_0. Since $\Gamma_e \cap \sigma$ does not contain nonadjacent vertices, either b or c (say b) does not belong to Γ_e. Therefore, two vertices b and d lie in the same component of $\Gamma - \Gamma_e$. This implies that $K_e - \Gamma_e$ and $\Gamma_{v_0} - \Gamma_e$ are contained in the same component of $\Gamma - \Gamma_e$, which is a contradiction. Therefore, d is a vertex of Γ_{v_0}. Since Γ_{v_0} does not contain a suspension of three points, d is a vertex of σ. Therefore, $\mathbb{J}^2_{v_0}$ satisfies condition (3) of Theorem 2.6. We now prove that $\mathbb{J}^1_{v_0}$ satisfies condition (3) of Theorem 2.6. Let K_e be a subgraph in $\mathbb{J}^1_{v_0}$ and d a vertex that are adjacent to nonadjacent vertices b and c of K_e. Assume for contradiction that d is not a vertex K_e. Using a similar argument as above, two points b and c are vertices of Γ_e. Therefore, if d is a vertex of Γ_{v_0}, then Γ_{v_0} contains a

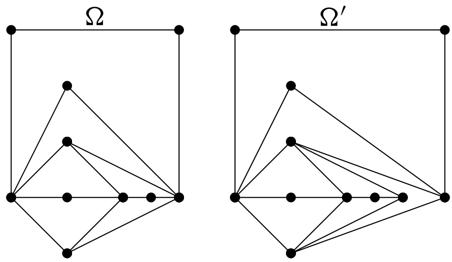

Figure 4: Two relatively hyperbolic right-angled Coxeter groups G_Ω and $G_{\Omega'}$ are not quasi-isometric because their peripheral subgroups are not quasi-isometric.

suspension of three points, which is a contradiction. Thus, d is a vertex of $K_{e_1} - \Gamma_{e_1}$ for some adjacent edge e_1 of v_0 other than e. Also $K_e \subset L_{e_1}$ as we observe above, so the two points b and c are vertices of L_{e_1}. Therefore, using a similar argument as above, the two points b and c are vertices of Γ_{e_1}. Therefore, $\Gamma_e \cap \Gamma_{e_1}$ contains two nonadjacent vertices b and c. This implies that Γ_{v_0} contains a suspension of three points, which is a contradiction. Therefore, $\mathbb{J}^1_{v_0}$ satisfies condition (3) of Theorem 2.6. Thus, $\mathbb{J}_{v_0}$ satisfies condition (3) of Theorem 2.6.

Finally, we prove that $\mathbb{J}_{v_0}$ satisfies condition (1) of Theorem 2.6. Let σ be an arbitrary 4–cycle of Γ. It is clear that if $\sigma \cap \Gamma_e$ does not contain nonadjacent vertices for any adjacent edge e of v_0, then σ is either a 4–cycle in $\mathbb{J}^2_{v_0}$ or a 4–cycle in a subgraph of $\mathbb{J}^1_{v_0}$. Now we assume that there is an adjacent edge e of v_0 such that $\sigma \cap \Gamma_e$ contains two nonadjacent vertices b_1 and b_2. Let a_1 and a_2 be the remaining vertices of σ. Since both a_1 and a_2 are adjacent to both vertices of K_e, they are all vertices of K_e, as we prove above. Thus, σ is a 4–cycle of K_e. Therefore, $\mathbb{J}_{v_0}$ satisfies condition (1) of Theorem 2.6. □

Proof of Theorem 1.6 Let T_1 be the subgraph of T induced by all vertices v with Γ_v a suspension of three points (T_1 is not necessarily connected). Let $\mathcal{T}$ be the set of all components of T_1. For each component C in $\mathcal{T}$, let $\Gamma_C = \bigcup_{v \in V(C)} \Gamma_v$. Then, it is clear that Γ_C is a $\mathcal{CFS}$ graph. Let $\mathbb{J}_1$ be the collection of all Γ_C for all components C in $\mathcal{T}$. Let $\mathbb{J}_2$ be the collection of all 4–cycles which are not part of any suspension of three vertices of Γ. Let $\mathbb{J} = \mathbb{J}_1 \cup \mathbb{J}_2$.

Let n be the number of vertices v of the tree T such that Γ_v is not a suspension of three points. We can prove the above proposition easily by induction on n using Theorem 2.5 and Proposition 4.3. We leave the details to the reader. □

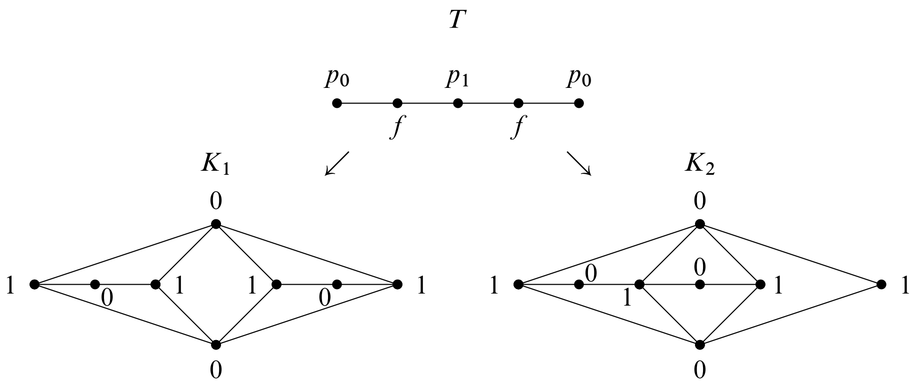

Figure 5: Two nonisomorphic 1–dimensional flag complexes (triangle-free graphs) K_1 and K_2 in the collection $\mathbb{K}_1$ can be constructed from the same tree T in $\mathbb{T}_1$.

Remark 4.4 In the above theorem, if the defining graph Γ is $\mathcal{CFS}$, then the right-angled Coxeter group G_Γ is trivially relatively hyperbolic with respect to itself.

Example 4.5 Let Ω and Ω' be two graphs as in Figure 4. Then Ω (resp. Ω') contains the subgraph Γ (resp. Γ') in Figure 3. Moreover, the group G_Ω (resp. $G_{\Omega'}$) is relatively hyperbolic with respect to the group G_Γ (resp. $G_{\Gamma'}$) by Theorem 2.6. However, two groups G_Γ and $G_{\Gamma'}$ are not quasi-isometric by Example 4.2. Therefore, the two groups G_Ω and $G_{\Omega'}$ are not quasi-isometric by Theorem 4.1 in [4].

5 On generalization to certain high-dimensional right-angled Coxeter groups

The main ingredient in the proof of quasi-isometric classification of $\mathcal{CFS}$ right-angled Coxeter groups with defining graphs satisfying the standing assumptions is the decomposition of defining graphs as tree structures. Exploiting this idea we study certain high-dimensional right-angled Coxeter groups.

5.1 Tree structure of the nerves of certain high-dimensional RAAGs and RACGs

In this section, we introduce a collection of bipartite trees with certain structures and we will use this collection to construct two different collections of flag complexes. The first collection of flag complexes is used to describe high-dimensional RAAGs introduced in [6] and the second one is used to construct certain high-dimensional RACGs.

Definition 5.1 For each integer $n \geq 1$ we define $\mathbb{T}_n$ be the collection of $\boldsymbol{p}/\boldsymbol{f}$–bipartite tree T satisfying the following:

(1) The valence of each $\boldsymbol{f}$–vertex is at least 2 and at most $n+1$.

(2) Each $\boldsymbol{p}$–vertex is labeled by a number in $\mathbb{I}_n = \{0, 1, 2, \ldots, n\}$ such that if v and v' are two different $\boldsymbol{p}$–vertices that are both adjacent to an $\boldsymbol{f}$–vertex, then v and v' are labeled by different numbers in $\mathbb{I}_n$.

(3) Each $\boldsymbol{p}$–vertex v is assigned to an integer $w(v)$, which we call the *weight* of v, that is greater than or equal to the valence of v.

We now use each collection tree $\mathbb{T}_n$ $(n \geq 1)$ to construct a collection of flag complexes.

Definition 5.2 For each integer $n \geq 1$ and T a $\boldsymbol{p}/\boldsymbol{f}$–bipartite tree in the collection $\mathbb{T}_n$ we construct a flag complex L $(= L(T))$ as follows:

(1) Each $\boldsymbol{p}$–vertex v of T is associated to a flag complex $L_v = \Delta_v^{n-1} * B_v$, where Δ_v^{n-1} is an $(n-1)$–simplex, B_v is the set of $w(v)$ distinct points, and $*$ denotes a join of two complexes. Moreover, if v is labeled by a number i in $\mathbb{I}_n$, then each point in B_v is also labeled by i and all n vertices in Δ_v^{n-1} are labeled distinctly by elements in $\mathbb{I}_n - \{i\}$.

(2) Each $\boldsymbol{f}$–vertex u of T is associated to an n–simplex L_u and we label all $n+1$ vertices of L_u distinctly by elements in $\mathbb{I}_n$.

(3) If an $\boldsymbol{f}$–vertex u is adjacent to a $\boldsymbol{p}$–vertex v, then we identify the n–simplex L_u with an n–simplex in L_v such that their vertex labels are matched (therefore, we have exactly $w(v)$ different ways for the identification). Moreover, if u and u' are two different $\boldsymbol{f}$–vertices that are both adjacent to a $\boldsymbol{p}$–vertex v, then L_u and $L_{u'}$ are identified to two different n–simplices of L_v.

The proof for the following proposition is easy and we leave it to the reader.

Proposition 5.3 *Each tree T in $\mathbb{T}_n$ defines a unique flag complex $L(T)$ up to simplicial complex isomorphism.*

We now review the collection of RAAG nerves studied in [6].

Definition 5.4 [6] For each integer $n \geq 1$ we define $\mathbb{L}_n$ to be the smallest class of n–dimensional simplicial complexes satisfying:

(1) The n–simplex is in $\mathbb{L}_n$.

(2) If L_1 and L_2 are complexes in $\mathbb{L}_n$ then the union of L_1 and L_2 along any $(n-1)$–simplex is in $\mathbb{L}_n$.

The following proposition shows that each collection $\mathbb{L}_n$ of RAAG nerves can be characterized by using the corresponding collection $\mathbb{T}_n$ of bipartite trees.

Proposition 5.5 [6] *For each integer $n \geq 1$, a complex L belongs to the collection $\mathbb{L}_n$ if and only if L can be constructed from a tree T in the collection $\mathbb{T}_n$ as in Definition 5.2.*

In [6], Behrstock, Januszkiewicz and Neumann study quasi-isometry classification of collection of RAAGs $\{A_L\}_{L \in \mathbb{L}_n}$ for each $n \geq 1$.

We now discuss a different collection of simplicial complexes and we will use it to introduce certain high-dimensional RACGs.

Definition 5.6 For each integer $n \geq 1$ and T a $\boldsymbol{p}/\boldsymbol{f}$–bipartite tree in the collection $\mathbb{T}_n$ we construct a flag complex K $(= K(T))$ as follows:

(1) Each $\boldsymbol{p}$–vertex v of T is associated to a flag complex $K_v = S_v^{n-1} * A_v$, where S_v^{n-1} is an $(n-1)$–sphere $S_0 * S_0 * \cdots * S_0$ (n factors S_0) and A_v is the set of $w(v)$ distinct points with some cyclic ordering. Moreover, if v is labeled by a number i in $\mathbb{I}_n$, then each point in A_v is labeled by i and each pair of nonadjacent vertices in S_v^{n-1} is labeled by the same numbers in $\mathbb{I}_n - \{i\}$ such that two different pairs of nonadjacent vertices in S_v^{n-1} are labeled by different numbers.

(2) Each $\boldsymbol{f}$–vertex u of T is associated to an n–sphere $K_u = S_0 * S_0 * \cdots * S_0$ ($n+1$ factors S_0) and we label two nonadjacent vertices in K_u by the same numbers in $\mathbb{I}_n$ such that two different pairs of nonadjacent vertices in K_u are labeled by different numbers.

(3) If an $\boldsymbol{f}$–vertex u is adjacent to a $\boldsymbol{p}$–vertex v, then we identify the complex K_u with a subcomplex in K_v such that their vertex labels are matched. Moreover, if the $\boldsymbol{p}$–vertex v is labeled by a number i in $\mathbb{I}_n$, then two nonadjacent vertices of the complex K_u labeled by i are identified to two adjacent elements in the set A_v of K_v with respect to the cyclic ordering on A_v. Lastly, if u and u' are two different $\boldsymbol{f}$–vertices that are both adjacent to a $\boldsymbol{p}$–vertex v, then K_u and $K_{u'}$ are identified to two different subcomplexes of K_v.

Let $\mathbb{K}_n$ be the collection of all flag complexes each of which can be constructed from some tree in $\mathbb{T}_n$ as above.

Remark 5.7 Two nonisomorphic flag complexes in $\mathbb{K}_n$ can be constructed from the same tree T in $\mathbb{T}_n$ (see Figure 5). In this paper, we study the coarse geometry including quasi-isometry classification of collection of RACGs $\{G_K\}_{K \in \mathbb{K}_n}$ for each $n \geq 1$

5.2 Quasi-isometry classification of some high-dimensional right-angled Artin groups

In this section, we briefly review the work of Behrstock, Januszkiewicz and Neumann on quasi-isometry classification of RAAGs with nerves in $\mathbb{L}_n$. We first review the construction of Behrstock, Januszkiewicz and Neumann of geometric models for their RAAGs.

Construction 5.8 Fix a flag complex L in $\mathbb{L}_n$ and we assume that L can be constructed from a tree T in $\mathbb{T}_n$ as in Definition 5.2. For each $\boldsymbol{p}$–vertex v of T the vertex complex $L_v = \Delta_v^{n-1} * B_v$ defines a right-angled Artin group A_{L_v} which is the product of a free group of rank $k = w(v)$ with $\mathbb{Z}^n$.

Let $u_1, u_2, \dots, u_k$ be all vertices of B_v. Giving the free group of rank k induced by B_v the redundant presentation

$$\langle u_0, u_1, \dots, u_k \mid u_0 u_1 \cdots u_k = 1 \rangle$$

helps us consider this free group as the fundamental group of a $(k+1)$–punctured sphere S_{k+1}. Therefore, the right-angled Artin group A_{L_v} is the fundamental group of the $(n+1)$–manifold $M_v = S_{k+1} \times T^n$. Moreover, L_v is the union of k n–simplices of the form $\Delta_v^{n-1} * b$ ($b \in B_v$) and the right-angled Artin subgroups induced by these simplices are the fundamental groups of k of the $k+1$ boundary components of M_v.

When two vertex spaces L_v and $L_{v'}$ of L intersects in an n–simplex, this corresponds to gluing the corresponding manifolds, M_v and $M_{v'}$, along a boundary component by a *flip* (ie a map that switches the base coordinate of one piece with one of the S^1 factors of the torus fiber of the other piece). Therefore, we can associate to any flag complex L in $\mathbb{L}_n$ a space X_L with $\pi_1(X_L) = A_L$. Thus, the right-angled Artin group A_L acts properly and cocompactly on the universal cover $\widetilde{X}_L$ of X_L. We call $\widetilde{X}_L$ the *geometric model* of the right-angled Artin group A_L.

By the above construction, the space $\widetilde{X}_L$ can be decomposed as copies of $\widetilde{M}_v = \widetilde{S}_{k+1} \times \mathbb{R}^n$, which we call *geometric pieces* with $\boldsymbol{p}$–vertex v of T, and they are glued accordingly. Moreover, the geometric pieces have boundaries which are not shared with other geometric pieces in the decomposition.

In [6], Behrstock, Januszkiewicz and Neumann use the above geometric models to classify such right-angled Artin groups A_L up to quasi-isometry. Before giving a complete quasi-isometry classification for their RAAGs, for each tree $T \in \mathbb{T}_n$ Behrstock, Januszkiewicz and Neumann colored it using a color set

$$C_1 = \{c, b_0, b_1, b_2, \dots, b_{n-1}, b_n\}$$

in the identical way of labeling vertices of T. More precisely, we color each $\boldsymbol{f}$–vertex by c and color each $\boldsymbol{p}$–vertex labeled by i in $\mathbb{I}_n$ by b_i. Although it seems to be redundant to color the tree T in the way that is identical to their vertex labels, we still want to differentiate coloring and labeling so we can compare this coloring with another coloring on T we will construct later. The following theorem talks about a complete quasi-isometry classification of the collection of RAAGs $\{A_L\}_{L \in \mathbb{L}_n}$ for each $n \geq 1$.

Theorem 5.9 [6, Theorem 1.1] *Let L and L' be two flag complexes in $\mathbb{L}_n$. Assume that L and L' are constructed from the corresponding trees T and T' as in Definition 5.2 and we color these trees using the color set C_1. Then two right-angled Artin groups A_L and $A_{L'}$ are quasi-isometric if and only if the two trees T and T' are bisimilar after possibly reordering the $\boldsymbol{p}$–colors by an element of the symmetric group on $n+1$ elements.*

5.3 Geometric models for high-dimensional right-angled Coxeter groups with nerves in $\mathbb{K}_n$ and quasi-isometry classification

In this section, we will construct a geometric model for the right-angled Coxeter group G_K where K is a flag complex in $\mathbb{K}_n$. We then apply this geometric model and line by line argument as in Sections 3 and 4 of [6] to get the proof of Theorem 1.3. Before we construct a geometric model for G_K we need a new coloring for each tree T in $\mathbb{T}_n$ as follows.

New coloring of each tree T in $\mathbb{T}_n$ Let C_1 be the color set given by Section 5.2. Let

$$C_2 = \{c, b_0, b_1, b_2, \dots, b_{n-1}, b_n, w_0, w_1, w_2, \dots, w_{n-1}, w_n\},$$

which contains the color set C_1.

The new coloring is similar to the previous coloring except we will consider vertex weight in the coloring process. We first color each f–vertex of T by c in this coloring as we did with the previous coloring.

We color a p–vertex as follows. Assume that a p–vertex v is labeled by a number i in $\mathbb{I}_n$. We color v by b_i if the weight of v is strictly greater than its valence and we color v by w_i if the weight of v is the same as its valence. Therefore, two different ways of coloring C_1 and C_2 are identical if and only if the weight of each p–vertex is strictly greater than its valence.

Construction of geometric models We now construct geometric models for our RACGs. Let K be a flag complex in $\mathbb{K}_n$ and we assume that K can be constructed from a tree T in $\mathbb{T}_n$ as in Definition 5.6. Let Σ_K be the Davis complex of the right-angled Coxeter group G_K. We now construct a "fattened" Davis complex Y_K on which G_K acts properly and cocompactly.

For each p–vertex v of T we have the associated flag complex $K_v = S_v^{n-1} * A_v$, where S_v^{n-1} is an $(n-1)$–sphere $S_0 * S_0 * \cdots * S_0$ and A_v is the set of $w(v)$ distinct points with some cyclic ordering. Assume that elements in A_v are labeled cyclically by a_i where $i \in \mathbb{Z}_n$ $(n = w(v))$. The Davis complex of the right-angled Coxeter group G_{A_v} is an n–regular tree T_n with edges labeled by a_i. We first construct a "fattened tree" $F(T_n)$ of T_n as follows:

We replace each vertex of T_n by a regular n–gon with sides labeled cyclically by $\overline{a}_i$ and we also assume the length side of the n–gon is $\frac{1}{2}$. We replace each edge E labeled by a_i by a strip $E \times \left[-\frac{1}{4}, \frac{1}{4}\right]$. We label each side of length 1 of the strip $E \times \left[-\frac{1}{4}, \frac{1}{4}\right]$ by a_i and we identify the edge E to $E \times \{0\}$ of the strip. Moreover, if w is an endpoint of the edge E of T_n, then the edge $\{w\} \times \left[-\frac{1}{4}, \frac{1}{4}\right]$ is identified to the side labeled by $\overline{a}_i$ of the n–gon that replaces w. It is clear that the right-angled Coxeter group G_{A_v} acts properly and cocompactly on the fattened tree $F(T_n)$ as an analogous way its acts on the Davis complex T_n. Moreover, the fattened tree $F(T_n)$ is a 2–dimensional manifold and each boundary component is a line which is labeled concatenatively by $\{a_{i-1}, a_i\}$ for some $i \in \mathbb{Z}_n$.

The Davis complex $\Sigma_{S_v^{n-1}}$ of the right-angled Coxeter group $G_{S_v^{n-1}}$ is isometric to $\mathbb{R}^n$. Let $P_v = \Sigma_{S_v^{n-1}} \times F(T_n)$. Then the right-angled Coxeter group G_{K_v} acts properly and cocompactly on P_v, obviously. Moreover, P_v is an $(n+1)$–manifold and the boundary components of P_v are copies of the Davis complexes of right-angled Coxeter groups $G_{S_v^{n-1} * \{a_{i-1}, a_i\}}$ for $i \in \mathbb{Z}_n$.

For each $\boldsymbol{f}$–vertex u that is adjacent to a $\boldsymbol{p}$–vertex v, the flag complex K_u is identified to a subcomplex of the form $S_v^{n-1} * \{a_{i-1}, a_i\}$ in $K_v = S_v^{n-1} * A_v$. Therefore, each boundary component of P_v that is a copy of the Davis complex of right-angled Coxeter group $G_{S_v^{n-1} * \{a_{i-1}, a_i\}}$ can also be considered as a copy of the Davis complex Σ_{K_u}. Thus, using the Bass–Serre tree $\widetilde{T}$ of the decomposition of the right-angled Coxeter group G_K as the tree T of subgroups, we can form a space Y_K by gluing copies of each space P_v appropriately and we obtain a proper, cocompact action of G_K on the new space Y_K. We call each copy of P_v for some $\boldsymbol{p}$–vertex v of T a *geometric piece of type* v and we call the space Y_K a *geometric model* for the right-angled Coxeter group G_K.

Remarks 5.10 (1) For each $\boldsymbol{p}$–vertex v a geometric piece of type v has boundary components which are not shared with other geometric pieces if and only if the weight of the vertex v is strictly greater than its valence (ie the vertex v is colored by some color b_i when we color the tree T using the color set C_2 as above).

(2) The geometric model Y_K of a right-angled Coxeter group G_K $(K \in \mathbb{K}_n)$ have a similar structure with the geometric model $\widetilde{X}_L$ of a right-angled Artin group A_L $(L \in \mathbb{L}_n)$ except Y_K may contains geometric pieces such that all its boundary components are shared with other geometric pieces.

Proof of Theorem 1.3 We use the geometric model Y_K in the construction above for each right-angled Coxeter group G_K $(K \in \mathbb{K}_n)$ and line by line argument as in Sections 3 and 4 of [6] to get the proof. □

We can also use an almost identical proof as in Sections 3 and 4 in [6] to prove the following theorem:

Theorem 5.11 *Let L be a flag complex in $\mathbb{L}_n$ and let K be a flag complex in $\mathbb{K}_n$. Assume that L and K can be constructed from two trees T_L and T_K in $\mathbb{T}_n$, respectively. We color the tree T_L using the color set C_1 and the tree T_K using the color set C_2. Then the RAAG A_L and RACG G_K are quasi-isometric if and only if the $\boldsymbol{p}$–vertices of T_K are only colored by colors in the set C_1 and two colored trees T_L and T_K are bisimilar after possibly reordering the $\boldsymbol{p}$–colors by an element of the symmetric group on $n+1$ elements.*

6 Strongly quasiconvex subgroups of $\mathcal{CFS}$ right-angled Coxeter groups

6.1 Background on strongly quasiconvex subgroups and stable subgroups

In this subsection, we review two notions of quasiconvex subgroups and stable subgroups. We also recall some results related to these two notions.

Definition 6.1 A subset A of a geodesic metric space X is *Morse* if for every $K \geq 1$ and $C \geq 0$ there is some $M = M(K,C)$ such that every (K,C)–quasi-geodesic with endpoints on A is contained in the M–neighborhood of A. We call the function M a *Morse gauge*.

Definition 6.2 Let $\Phi\colon A \to X$ be a quasi-isometric embedding between geodesic metric spaces. We say A is *strongly quasiconvex* in X if the image $\Phi(A)$ is Morse in X. We say A is *stable* in X if for any $K \geq 1$ and $L \geq 0$, there is an $R = R(K,L) \geq 0$ such that if α and β are two (K,L)–quasi-geodesics with the same endpoints in $\Phi(A)$, then the Hausdorff distance between α and β is less than R.

When we say A is strongly quasiconvex (stable) in X we mean that A is strongly quasiconvex (stable) in X with respect to a particular quasi-isometric embedding $\Phi\colon A \to X$. Such a quasi-isometric embedding will always be clear from context, for example an undistorted subgroup H of a finitely generated group G. We now recall the concepts of strongly quasiconvex subgroups and stable subgroups.

Definition 6.3 Let G be a finite generated group and S an arbitrary finite generating set of G. Let H be a finite generated subgroup of G and T an arbitrary finite generating set of H. The subgroup H is *undistorted* in G if the natural inclusion $i\colon H \to G$ induces a quasi-isometric embedding from the Cayley graph $\Gamma(H,T)$ into the Cayley graph $\Gamma(G,S)$. We say H is *stable* in G if $\Gamma(H,T)$ is stable in $\Gamma(G,S)$.

Stable subgroups were proved to be independent of the choice of finite generating sets (see Section 3 in [19]).

Definition 6.4 Let G be a finite generated group and H a subgroup of G. We say H is *strongly quasiconvex* in G if H is a Morse subset in the Cayley graph $\Gamma(G,S)$ for some (any) finite generating set S.

Strongly quasiconvex subgroups were proved to be independent of the choice of finite generating sets of the ambient groups. Moreover, strongly quasiconvex subgroups are all finitely generated and undistorted. We refer the reader to the work of the second author in Section 4 in [31] for more details. The following proposition tells us a relation between strongly quasiconvex subgroups and stable subgroups.

Proposition 6.5 [31, Proposition 4.3] *Let G be a finitely generated group. A subgroup H of G is stable if and only if H is strongly quasiconvex and hyperbolic.*

The following proposition gives us a way to get another quasiconvex subgroup from a strongly quasiconvex subgroup.

Proposition 6.6 [31, Proposition 4.11] *Let G be a finitely generated group and A an undistorted subgroup of G. If H is a strongly quasiconvex subgroup of G, then $H_1 = H \cap A$ is a strongly quasiconvex subgroup of A. In particular, H_1 is finitely generated and undistorted in A.*

We now discuss the height and the width of subgroups.

Definition 6.7 Let G be a group and H a subgroup.

(1) Conjugates $g_1 H g_1^{-1}, \dots, g_k H g_k^{-1}$ are *essentially distinct* if the associated cosets $g_1 H, \dots, g_k H$ are distinct.

(2) H has height at most n in G if the intersection of any $n+1$ essentially distinct conjugates is finite. The least n for which this is satisfied is called the height of H in G.

(3) The *width* of H is the maximal cardinality of the set

$$\{g_i H : |g_i H {g_i}^{-1} \cap g_j H {g_j}^{-1}| = \infty\},$$

where $\{g_i H\}$ ranges over all collections of distinct cosets.

Finite subgroups and subgroups of finite index have finite height and width, and infinite normal subgroups of infinite index have infinite height and width. Hence, the next proposition states that strongly quasiconvex subgroups are far from being normal.

Theorem 6.8 [31, Theorem 1.2] *Let G be a finitely generated group and let H be a strongly quasiconvex subgroup. Then H has finite height and finite width.*

6.2 Strongly quasiconvex subgroups and stable subgroups in certain tree of groups and application to right-angled Coxeter groups

In this subsection, we prove that torsion-free, strongly quasiconvex subgroups of infinite index of certain finite graphs of groups are free. This result can be applied to our right-angled Coxeter groups.

Lemma 6.9 *Assume a group G is decomposed as a finite graph T of groups such that each edge group is infinite. Let G_v be a vertex subgroup. Then, for each g_1 and g_2 in G, there is a finite sequence of conjugates of vertex subgroups $g_1G_vg_1^{-1} = Q_0, Q_1, \dots, Q_m = g_2G_vg_2^{-1}$ such that $Q_{i-1} \cap Q_i$ is infinite for each $i \in \{1, 2, \dots, m\}$.*

Proof Let $\widetilde{T}$ be the Bass–Serre tree of the decomposition of G. Then conjugates of vertex groups (resp. edge groups) correspond to vertices (edges) of $\widetilde{T}$. Therefore, the lemma follows from the facts that $\widetilde{T}$ is connected and each edge group is infinite. □

The following proposition shows some properties of strongly quasiconvex subgroups in certain graphs of groups.

Proposition 6.10 *Assume a group G is decomposed as a finite graph T of groups that satisfies the following*:

(1) *For each vertex v of T, the vertex group G_v is finitely generated and undistorted. Moreover, any strongly quasiconvex, infinite subgroup of G_v is of finite index.*

(2) *Each edge group is infinite.*

Then, if H is a strongly quasiconvex subgroup of G of infinite index, then $gHg^{-1} \cap G_v$ is finite for each vertex group G_v and each group element g.

Proof We assume for contradiction that $g_0Hg_0^{-1} \cap G_v$ is infinite for some vertex group G_v and some $g_0 \in G$. We claim that $gHg^{-1} \cap G_v$ has finite index in G_v for all $g \in G$. In fact, since $g_0Hg_0^{-1}$ is a strongly quasiconvex subgroup and G_v is an undistorted subgroup, $g_0Hg_0^{-1} \cap G_v$ is a strongly quasiconvex subgroup of G_v by Proposition 6.6. Therefore, $g_0Hg_0^{-1} \cap G_v$ has finite index in G_v by the hypothesis.

We now prove that $gHg^{-1} \cap G_v$ has finite index in G_v for all $g \in A_\Gamma$. By Lemma 6.9, there is a finite sequence of conjugates of vertex subgroups $g_0^{-1}G_vg_0 = Q_0, Q_1, \dots, Q_m = g^{-1}G_vg$ such that $Q_{i-1} \cap Q_i$ is infinite for each $i \in \{1, 2, \dots, m\}$. Since $g_0Hg_0^{-1} \cap G_v$ has finite index in G_v, $H \cap g_0^{-1}G_vg_0$ has finite index in $Q_0 = g_0^{-1}G_vg_0$.

Also, the subgroup $Q_0 \cap Q_1$ is infinite. Then $H \cap Q_1$ is infinite. Using a similar argument as above, we obtain that $H \cap Q_1$ has finite index in Q_1. Repeating this process, we have that $H \cap g^{-1} G_v g$ has finite index in $g^{-1} G_v g$. In other words, $gHg^{-1} \cap G_v$ has finite index in G_v.

By Theorem 6.8, there is a number n such that the intersection of any $n+1$ essentially distinct conjugates of H is finite. Since H has infinite index in G, there are $n+1$ distinct elements $g_1, g_2, \dots g_{n+1}$ such that $g_i H \neq g_j H$ for each $i \neq j$. Also, $g_i H g_i^{-1} \cap G_v$ has finite index in G_v for each i. Then $\left(\bigcap g_i H g_i^{-1}\right) \cap G_v$ also has finite index in G_v. In particular, $\bigcap g_i H g_i^{-1}$ is infinite, which is a contradiction. Therefore, $gHg^{-1} \cap G_v$ is finite for each vertex group G_v. □

Proposition 6.11 *Assume a group G is decomposed as a finite graph T of groups. Let H be a subgroup of G such that $gHg^{-1} \cap G_v$ is trivial for each vertex group G_v and each group element g. Then H is free.*

Proof Let $\widetilde{T}$ be the Bass–Serre tree of the decomposition of G. Then G acts on $\widetilde{T}$ such that the stabilizer of a vertex of T is a conjugate of a vertex group. To show H is free, it is enough to show that H acts freely on $\widetilde{T}$. To see H acts freely on $\widetilde{T}$, it suffices to show that for each vertex $v \in \widetilde{T}$ then $\mathrm{Stab}_H(v) = \{e\}$. Note that $\mathrm{Stab}_H(v) = \mathrm{Stab}_G(v) \cap H$. By the assumption, we have that $\mathrm{Stab}_G(v) \cap H = \{e\}$, thus $\mathrm{Stab}_H(v) = \{e\}$. The proposition is proved. □

Proof of Proposition 1.5 The proof is a combination of Propositions 6.10 and 6.11. □

Proposition 6.12 *If G is a finitely generated group that has infinite center and H is an infinite strongly quasiconvex subgroup of G, then H is of finite index.*

Proof Let Z be the center of the group G. We first prove that the subgroup $Z \cap H$ has finite index in Z. Assume for a contradiction that the subgroups $Z \cap H$ has infinite index in Z. Then there is an infinite sequence (z_n) of elements in Z such that $z_i(Z \cap H) \neq z_j(Z \cap H)$ for $i \neq j$. Therefore, $z_i H \neq z_j H$ for $i \neq j$. However, we also have $z_i H z_i^{-1} = z_j H z_j^{-1}$ for all $i \neq j$, which contradicts Theorem 1.2 in [31], namely that a strongly quasiconvex subgroup has finite height. Therefore, the subgroup $Z \cap H$ has finite index in Z. In particular, the subgroup $Z \cap H$ is infinite.

We now assume for a contradiction that the subgroup H has infinite index in G. Then there is an infinite sequence (g_n) of elements in G such that $g_i H \neq g_j H$ for $i \neq j$.

However, $Z \cap H$ is an infinite subgroup of $g_i H g_i^{-1}$ for all i, which contradicts Theorem 1.2 in [31], namely that a strongly quasiconvex subgroup has finite height. Therefore, the subgroup H has finite index in G. $\square$

By combining the above proposition with Proposition 1.5, we obtain the proof of Theorem 1.4.

Proof of Theorem 1.4 Obviously, the right-angled Coxeter group G_K is a tree of groups whose vertex groups have infinite center and whose edge groups are infinite. Let G_1 be a finite-index, torsion-free subgroup of the right-angled Coxeter group G_K and $H_1 = H \cap G_1$. Then H_1 is a strongly quasiconvex, torsion-free subgroup of G_K of infinite index. Therefore, H_1 is a free group by Propositions 1.5 and 6.12. Also, H_1 is a finite-index subgroup of H. Therefore, the subgroup H is virtually free. $\square$

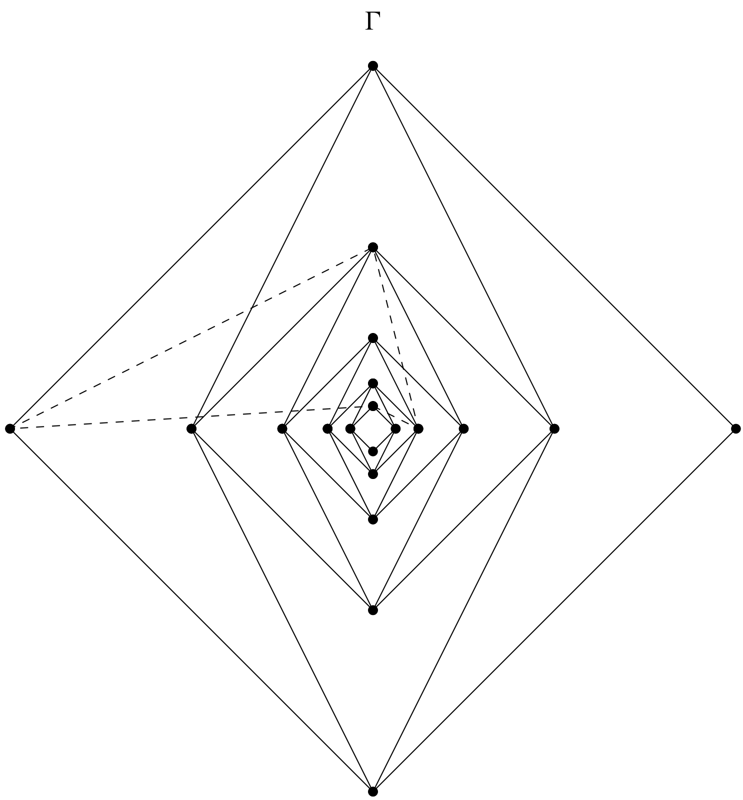

Figure 6: The special subgroup H generated by the dashed 4–cycle is a nonstable, strongly quasiconvex subgroup of infinite index of the right-angled Coxeter group G_Γ.

Example 6.13 We now construct a connected, triangle-free, $\mathcal{CFS}$ graph Γ with no separating vertices or edges such that the corresponding right-angled Coxeter group G_Γ has a nonstable, strongly quasiconvex subgroup of infinite index.

Let Γ be the graph in Figure 6 and K be the dashed 4–cycle of Γ. It is not hard to check Γ is connected, triangle-free, $\mathcal{CFS}$ and has no separating vertices or edges. Moreover, the 4–cycle K does not contain any pair of nonadjacent vertices of 4–cycle other than itself. Therefore, the subgroup $H = G_K$ is strongly quasiconvex by Theorem 1.11 in [31]. Note that H has infinite index in G_Γ. Also H is not hyperbolic and therefore H is not stable.

Remark 6.14 The existence of the subgroup $H \le G_\Gamma$ in Example 6.13 implies that the group G_Γ is not commensurable to any right-angled Artin group because all strongly quasiconvex subgroups of infinite index of a one-ended right-angled Artin group are free. We note that G_Γ is not even quasi-isometric to any right-angled Artin group by the very recent work of Russell, Spriano and Tran (see Example 7.7 in [28]).

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