Algebraic & Geometric Topology 19 (2019) 3315–3334

Upsilon-type concordance invariants

ANTONIO ALFIERI

To a region C of the plane satisfying a suitable convexity condition we associate a knot concordance invariant Υ^C. For appropriate choices of the domain this construction gives back some known knot Floer concordance invariants like Rasmussen's h_i invariants, and the Ozsváth–Stipsicz–Szabó upsilon invariant. Furthermore, to three such regions C, C^+ and C^- we associate invariants $\Upsilon_{C^\pm,C}$ generalizing the Kim–Livingston secondary invariant. We show how to compute these invariants for some interesting classes of knots (including alternating and torus knots), and we use them to obstruct concordances to Floer thin knots and algebraic knots.

57M27

1 Introduction

In [18], Ozsváth and Szabó, by essentially studying the Floer homology [3] of certain Lagrangian tori in the g–fold symmetric product of a genus g Riemann surface, found a package of three-manifold invariants called Heegaard Floer homology. In [17], they used this circle of ideas to define a related package of knot invariants named knot Floer homology. See Ozsváth, Stipsicz and Szabó [13] for an extensive exposition of this topic.

Knot Floer homology has been used to produce knot concordance invariants by many authors; see Rasmussen [23], Ozsváth and Szabó [16], Ozsváth, Stipsicz and Szabó [14] and Kim and Livingston [8]. The purpose of this note is to show that all these constructions can be seen as particular cases of a more general construction. Our investigation is mainly motivated by the following applications.

1.1 In [9], Lidman and Moore characterized L–space pretzel knots. They found that a pretzel knot has an L–space surgery if and only if it is a torus knot $T_{2,2n+1}$ for some $n \geq 1$, or a pretzel knot in the form $P(-2,3,q)$ for some $q \geq 7$ odd. Motivated by the exploration started by Wang [26] and Livingston [11], one may wonder if L–space pretzel knots of the form $P(-2,3,q)$ are concordant to algebraic knots.

Published: 17 December 2019 DOI: 10.2140/agt.2019.19.3315

Theorem 1.1 *None of the L–space pretzel knots $P(-2,3,q)$, with $q \geq 7$ odd, is concordant to a sum of algebraic knots.*

Notice that for these knots the obstruction found in [26, Corollary 3.5] vanishes.

1.2 In [4], Friedl, Livingston and Zentner asked whether a sum of torus knots is concordant to an alternating knot. In [27], Zemke used the involutive Floer homology of Hendricks and Manolescu [6] to prove that certain connected sums of torus knots are not concordant to Floer thin knots. Floer thin knots are upsilon-alternating, meaning that $\Upsilon_K(t) = -\tau(K)\cdot(1-|1-t|)$. A straightforward argument shows that a sum of *positive* torus knots is upsilon-alternating if and only if it is a connected sum of $(2,2n+1)$ torus knots and indeed alternating. However, when both positive and negative torus knots are involved this obstruction can vanish.

Proposition 1.2 *The knot $K = T_{8,5} \,\#\, {-T_{6,5}} \,\#\, {-T_{4,3}}$ is upsilon-alternating but not concordant to a Floer thin knot.*

The connected sum formula (Theorem 6.2) employed in the proof of Proposition 1.2 is used by Aceto and Alfieri [1] to decide which sums of two torus are concordant to alternating knots.

Acknowledgements The author would like to thanks András Stipsicz, András Némethi and Paolo Aceto. The author was partially supported by the NKFIH grant K112735.

2 A quick review of knot Floer homology

An *Alexander filtered, Maslov graded chain complex* is a finitely generated, $\mathbb{Z}$–graded, $(\mathbb{Z}\oplus\mathbb{Z})$–filtered chain complex $C = \left(\bigoplus_{x\in B} \mathbb{Z}_2[U,U^{-1}], \partial\right)$ such that

- ∂ is $\mathbb{Z}_2[U,U^{-1}]$–linear and for a basis element $\boldsymbol{x} \in B$, $\partial \boldsymbol{x} = \sum_{\boldsymbol{y}} n_{\boldsymbol{x},\boldsymbol{y}} U^{m_{x,y}} \cdot \boldsymbol{y}$ for suitable coefficients $n_{\boldsymbol{x},\boldsymbol{y}} \in \mathbb{Z}_2$, and nonnegative exponents $m_{\boldsymbol{x},\boldsymbol{y}} \geq 0$;
- the multiplication by U drops the homological (Maslov) grading M by two, and the filtration levels (denoted by A and j) by one.

An Alexander filtered, Maslov graded chain complex is said to be of *knot type* if in addition $H_*(C,\partial) = \mathbb{Z}_2[U,U^{-1}]$ is graded so that $\deg U = -2$. An Alexander filtered, Maslov graded chain complex can be pictorially described as follows:

(1) picture each $\mathbb{Z}_2$–generator $U^m \cdot \boldsymbol{x}$ of C on the planar lattice $\mathbb{Z}\times\mathbb{Z} \subset \mathbb{R}^2$ in position $(A(\boldsymbol{x})-m, -m) \in \mathbb{Z}\times\mathbb{Z}$;

(2) label each $\mathbb{Z}_2$–generator $U^m \cdot \boldsymbol{x}$ of C with its Maslov grading $M(\boldsymbol{x}) - 2m \in \mathbb{Z}$;

(3) connect two $\mathbb{Z}_2$–generators $U^n \cdot \boldsymbol{x}$ and $U^m \cdot \boldsymbol{y}$ with a directed arrow if in the differential of $U^n \cdot \boldsymbol{x}$ the coefficient of $U^m \cdot \boldsymbol{y}$ is nonzero.

In [17], Ozsváth and Szabó show how to associate to a knot $K \subset S^3$ a knot-type complex $\mathrm{CFK}^\infty(K)$ whose filtered chain homotopy type only depends on the isotopy class of K. For a concise introduction to the background material see [7].

2.1 Hom's invariance principle Denote by $\mathcal{CFK}$ the set of knot-type complexes up to filtered chain homotopy. Say that two knot-type complexes are *stably equivalent*, written $C_1 \sim C_2$, if there exist Alexander filtered, Maslov graded, acyclic chain complexes A_1 and A_2 such that $C_1 \oplus A_1 \simeq C_2 \oplus A_2$. The quotient set $\mathcal{CFK}/\sim$ has a natural group structure: the sum is given by tensor product, the class of zero is the one represented by the Floer chain complex of the unknot $\mathrm{CFK}^\infty(U)$, and the inverse of the class of a complex C is the one represented by its dual complex $\mathrm{Hom}(C, \mathbb{Z}_2[U, U^{-1}])$.

Theorem 2.1 (Hom [7]) *The map $K \mapsto \mathrm{CFK}^\infty(K)$ associating to a knot $K \subset S^3$ its knot Floer complex descends to a group homomorphism $\mathcal{C} \to \mathcal{CFK}/\sim$.*

Summarizing, in order to produce a concordance invariant $\mathcal{C} \to \mathbb{Z}$ one only needs to produce a map $f\colon \mathcal{CFK} \to \mathbb{Z}$ such that $f(C_* \oplus A_*) = f(C_*)$ for every Alexander filtered, Maslov graded, acyclic chain complex A_*.

3 Upsilon-type invariants

Inspired by the exposition in [10], we use knot Floer homology to define some more concordance invariants. We start with a definition.

Definition 3.1 *A region of the plane $C \subset \mathbb{R}^2$ is said to be a south-west region if it is nonempty and $(\overline{x}, \overline{y}) \in C$ implies $\{(x, y) \mid x \leq \overline{x},\ y \leq \overline{y}\} \subseteq C$.*

Examples of south-west regions are shown in Figure 1. Let C be a south-west region of the plane. For $t \in \mathbb{R}$ let $C_t = \{(x, y) \mid (x - t, y - t) \in C\}$ denote the translation of C in the direction of $v_t = (t, t)$. Given a knot-type complex K_* consider the map induced on H_0 by the inclusion $K_*(C_t) \hookrightarrow K_*$, where $K_*(C_t)$ denotes the subcomplex spanned by the generators of K_* lying in C_t. Since $C_t \subseteq C_{t'}$ for $t \leq t'$, and $\bigcup_{t \in \mathbb{R}} C_t = \mathbb{R}^2$, a cycle representing the generator of $H_0(K_*) = \mathbb{Z}_2$ will eventually

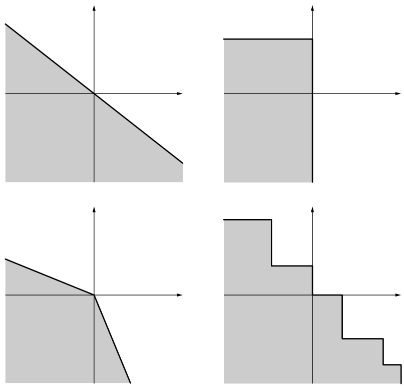

Figure 1: Examples of south-west regions.

be contained in $K_*(C_t)$. Thus, for t big enough the inclusion $H_0(K_*(C_t)) \to H_0(K_*)$ is a surjective map. Let $\Upsilon^C(K_*)$ be the minimum $t \in \mathbb{R}$ such that $K_*(C_t) \hookrightarrow K_*$ induces a surjection on H_0. Here we are using the Maslov grading as homological grading, so that $H_{2i}(K_*) = \mathbb{Z}_2$ and zero otherwise.

Lemma 3.2 *Suppose that C is a south-west region. If K_* and K'_* are two stably equivalent knot-type complexes then $\Upsilon^C(K_*) = \Upsilon^C(K'_*)$.*

Proof The surjectivity of the map induced in homology by the inclusion $K_*(C_t) \hookrightarrow K_*$ is not affected if we sum an acyclic complex A on the right and a subcomplex of the *same* acyclic on the left. □

Corollary 3.3 *Suppose that $C \subset \mathbb{R}^2$ is a south-west region. Given a knot $K \subseteq S^3$, set $\Upsilon^C(K) = \Upsilon^C(\mathrm{CFK}^\infty(K))$. Then $\Upsilon^C(K)$ is a concordance invariant.* □

3.1 The classical upsilon invariant Choose the lower half-space

$$H_t = \left\{\tfrac{1}{2}t \cdot A + \left(1 - \tfrac{1}{2}t\right) \cdot j \leq 0\right\}$$

as the south-west region. As t ranges in $[0,2]$ we get a one-parameter family of invariants of knot-type complexes $\Upsilon_t(K_*) = \Upsilon^{H_t}(K_*)$. According to Corollary 3.3 this provides a one-parameter family of knot concordance invariants. More specifically, set

$$\Upsilon_K(t) = -2 \cdot \Upsilon^{H_t}(\mathrm{CFK}^\infty(K)).$$

In [10, Section 14], Livingston proves that the invariant $\Upsilon_K(t)$ agrees with the upsilon invariant defined by Ozsváth, Stipsicz and Szabó [14].

3.2 Regions for Rasmussen's h_i invariants For $s \geq 0$ choose as south-west region $Q_s = \{A \leq s,\ j \leq 0\}$. This leads to a one-parameter family of knot concordance invariants $V_K(s) = \Upsilon^{Q_s}(K)$. These are the invariants h_i introduced by Rasmussen [23]. They are characterized by the following property:

Proposition 3.4 [23, Section 7.2] *Let $K \subseteq S^3$ be a knot and $q \geq 2g(K)-1$ be an integer. Denote by $W_q(K)$ the q–framed two-handle attachment along K to D^4, so that $S^3_q(K) = \partial W_q(K)$. For any integer $m \in \left[-\frac{1}{2}q, \frac{1}{2}q\right)$ let $\mathfrak{s}_m \in Spin^c(S^3_q(K))$ denote the restriction to $S^3_q(K)$ of a Spin^c structure $\mathfrak{t}_m$ on $W_q(K)$ such that $\langle c_1(\mathfrak{s}), [\widehat{F}]\rangle + q = 2m$, where $\widehat{F} \subset W_q(K)$ denotes a capped-off Seifert surface for K. Then*

$$d(S^3_q(K), \mathfrak{s}_m) = \frac{(q-2m)^2 - q}{4q} - 2V_K(m),$$

where d denotes the Heegaard Floer correction term introduced in [20]. □

3.3 Estimates on the slice genus Suppose that C is a south-west region. Associated to C there is a height function

$$h_C(x) = \min\{t \in \mathbb{R} \mid (x,0) \in C_t\},$$

where C_t denotes as usual the translation of C in the $v_t = (t,t)$ direction. The height function h_C relates the upsilon invariant of the region C to the slice genus.

Theorem 3.5 *Let C be a south-west region. Given a knot $K \subset S^3$ the inequality*

$$\max\{\Upsilon^C(K), \Upsilon^C(-K)\} \leq h_C(g_4(K)) \tag{1}$$

holds, where $g_4(K)$ denotes the slice genus of K.

Proof First of all notice that $h_C(x)$ is a monotone increasing function: Since C is a south-west region, $(x-\delta, 0) \in C_{\Upsilon^C(K)}$ for $\delta > 0$. Thus, $h_C(x-\delta) \leq h_C(x)$.

Let $\nu^+ = \min_i\{V_K(i) = 0\}$. From the definition of the height function $h_C(x)$ and the fact that C is a south-west region, one immediately concludes that $\{A \leq \nu^+,\ j \leq 0\} \subseteq C_{h_C(\nu^+)}$. The fact that $V_K(\nu^+) = 0$ ensures that the south-west region $\{A \leq \nu^+,\ j \leq 0\}$ contains a cycle generating $H_0(\mathrm{CFK}^\infty(K))$ and consequently (because of the inclusion) that so does the translation $C_{h_C(\nu^+)}$. This proves that $\Upsilon^C(K) \leq h_C(\nu^+)$. On the other hand, according to Rasmussen [23, Corollary 7.4] $\nu^+ \leq g_4(K)$, thus $\Upsilon^C(K) \leq h_C(\nu^+) \leq h_C(g_4(K))$.

By the same argument for $-K$ instead of K we get that $\Upsilon^C(-K) \leq h_C(g_4(-K)) = h_C(g_4(K))$, and we are done. □

Example 3.6 If we choose $C = \left\{\frac{1}{2}tA + \left(1 - \frac{1}{2}t\right)j \leq 0\right\}$ as in the classical upsilon invariant (Section 3.1) one has $h_C(x) = \frac{1}{2}t \cdot x$. In this case, equation (1) leads to the inequality $|\Upsilon_K(t)| = 2 \cdot \max\{\Upsilon^C(K), \Upsilon^C_K(-K)\} \leq 2h_C(g_4(K)) = tg_4(K)$, where the first identity is due to the identity $\Upsilon^C_K(-K) = -\Upsilon^C(K)$ (which is not valid for any C). Compare this with [14, Theorem 1.11].

4 Secondary invariants

Roughly speaking, upsilon-type invariants measure how far one needs to travel north-east in the (A, j) plane in order to see a cycle generating $H_0(\mathrm{CFK}^\infty)$ appear. As suggested by Kim and Livingston [8], other concordance invariants could be obtained by measuring how far one should go in order to find some chain realizing a homology in between two such cycles.

More specifically, suppose that two south-west regions C^+ and C^- are given. Given a knot-type complex K_* one can consider the maps induced in homology by the inclusions $K_*(C_t^+) \hookrightarrow K_*$ and $K_*(C_t^-) \hookrightarrow K_*$ (here we are using again the notation of the beginning of Section 3). For $\gamma_\pm = \Upsilon^{C_\pm}(K_*)$ one gets surjections $H_0(K_*(C^+_{\gamma_+})) \to H_0(K_*)$ and $H_0(K_*(C^-_{\gamma_-})) \to H_0(K_*)$. Denote by $\mathcal{Z}^+$ and $\mathcal{Z}^-$ the set of cycles of $K_*(C^+_{\gamma_+})$ and $K_*(C^-_{\gamma_-})$, respectively, projecting on the generator of $H_0(K_*)$. Suppose now that a third south-west region C has been fixed. Since $H_0(K_*) = \mathbb{Z}_2$, for $t \in \mathbb{R}$ large enough there will be a 1–chain $\beta \in K_1$ realizing a homology between a 0–cycle in $\mathcal{Z}^+$ and one in $\mathcal{Z}^-$. We define $\Upsilon_{C\pm,C}(K_*)$ as the minimum $t \in \mathbb{R}$ for which there exist a cycle $z^+ \in \mathcal{Z}^+$ and a cycle $z^- \in \mathcal{Z}^-$ representing the same homology class inside $K_*(C^+_{\gamma_+}) + K_*(C^-_{\gamma_-}) + K_*(C_t)$. We set $\Upsilon_{C\pm,C}(K_*) = -\infty$ in the eventuality that $\mathcal{Z}^+ \cap \mathcal{Z}^- \neq \varnothing$.

Lemma 4.1 *Suppose that C^+, C^- and C are given south-west regions. If K_* and K'_* are two stably equivalent knot-type complexes, then $\Upsilon_{C\pm,C}(K_*) = \Upsilon_{C\pm,C}(K'_*)$.*

Proof Suppose that $K'_* = K_* \oplus A$ is obtained from K_* by adding an acyclic complex A. Set $\gamma_\pm = \Upsilon^{C_\pm}(K_*) = \Upsilon^{C_\pm}(K'_*)$ and let $\mathcal{Z}^\pm(K_*)$ and $\mathcal{Z}^\pm(K'_*)$ be the sets of cycles projecting to the generator through $H_0(K_*(C^\pm_{\gamma_\pm})) \to H_0(K_*)$ and $H_0(K'_*(C^\pm_{\gamma_\pm})) \to H_0(K'_*)$, respectively.

We prove that $\Upsilon_{C\pm,C}(K_*) = \Upsilon_{C\pm,C}(K'_*)$ by proving the two inequalities. Suppose by contradiction that there exists $t < \Upsilon_{C\pm,C}(K^*)$ for which a cycle $z^+ \in \mathcal{Z}^+(K'_*)$ gets identified with a cycle in $z^- \in \mathcal{Z}^-(K'_*)$ in $K'_*(C_t) + K'_*(C^+_{\gamma_+}) + K'_*(C^-_{\gamma_-})$.

Pick a 1–chain $\beta'_t \in K'_*(C_t)+K'_*(C^+_{\gamma_+})+K'_*(C^-_{\gamma_-})$ such that $z^+-z^-=\partial\beta'_t$, and write $\beta'_t=\beta_t+a$ with $\beta_t \in K_*(C_t)+K_*(C^+_{\gamma_+})+K_*(C^-_{\gamma_-})$ and $a\in A$. Notice that $z^+=z^+_K+a^+$ and $z^-=z^-_K+a^-$ for some $a^+,a^-\in A$, $z^+_K\in\mathcal{Z}^+(K_*)$ and $z^-_K\in\mathcal{Z}^-(K_*)$. By rewriting the relation $z^+-z^-=\partial\beta'_t$ we get that $(z^+_K-z^-_K-\partial\beta_t)+(a^+-a^--\partial a)=0$, from where we can conclude that $z^+_K-z^-_K=\partial\beta_t$. This contradicts the fact that $\Upsilon_{C\pm,C}(K_*)$ is the minimum t for which such an homology exists, and proves that $\Upsilon_{C\pm,C}(K_*)\le\Upsilon_{C\pm,C}(K'_*)$. The reverse inequality has a similar proof. □

Corollary 4.2 *For a knot $K\subset S^3$ set $\Upsilon_{C\pm,C}(K)=\Upsilon_{C\pm,C}(\mathrm{CFK}^\infty(K))$. Then $\Upsilon_{C\pm,C}(K)$ defines a knot concordance invariant.* □

4.1 Breaking points Summarizing, given south-west regions C^+, C^- and $C\subset\mathbb{R}^2$ we get a map $\Upsilon_{C\pm,C}\colon \mathcal{CFK}/{\sim}\to[-\infty,+\infty)$. In [8], Kim and Livingston produce south-west regions for which the condition $\mathcal{Z}^+\cap\mathcal{Z}^-=\varnothing$ is guaranteed.

Lemma 4.3 (Kim and Livingston) *For $t\in[0,2]$ let $\Upsilon_t\colon\mathcal{CFK}/{\sim}\to\mathbb{R}$ denote the stable equivalence invariant associated to the lower half-space H_t of Section 3.1. Suppose that K_* is a knot-type complex such that $\Upsilon_t(K_*)$ as function of $t\in[0,2]$ is nonsmooth at $t=t^*$. Furthermore, suppose that the derivative of $\Upsilon_t(K_*)$ at $t=t^*$ has a positive jump, meaning that*

$$\Delta\Upsilon'_t(K_*)=\lim_{\epsilon\to 0}(\Upsilon'_{t+\epsilon}(K_*)-\Upsilon'_{t-\epsilon}(K_*))$$

is positive at $t=t^*$. Then, for $\delta>0$ small enough, $C^-=H_{t^*-\delta}$ and $C^+=H_{t^*+\delta}$ give two south-west regions such that $\mathcal{Z}^+\cap\mathcal{Z}^-=\varnothing$. □

We say that the upsilon function $\Upsilon_t(K_*)$ of a knot-type complex K_* has a *breaking point* at $t=t^*$ if, for a small perturbation $\delta>0$, $C^-=H_{t^*-\delta}$ and $C^+=H_{t^*+\delta}$ are two south-west regions such that $\mathcal{Z}^+\cap\mathcal{Z}^-=\varnothing$. The cycles in $\mathcal{Z}^+$ and $\mathcal{Z}^-$ are called the positive and the negative *exceptional cycles* of the breaking point. Lemma 4.3 says that the *singularities* of $\Upsilon_t(K_*)$ (points where $\Upsilon_t(K_*)$ is nonsmooth) at which $\Delta\Upsilon'_t(K_*)>0$ are in fact breaking points.

In the notation of Lemma 4.3 set

$$\Upsilon^{(2)}_{C,t}(K_*)=-2\cdot(\Upsilon_{H_{t\pm\delta},C}(K_*)-\Upsilon_t(K_*)) \tag{2}$$

for $\delta>0$ small enough. This provides a one-parameter family of knot concordance invariants $\Upsilon^{(2)}_{C,t}(K)=\Upsilon^{(2)}_{C,t}(\mathrm{CFK}^\infty(K))$. Notice that the invariant $\Upsilon^{(2)}_{K,t}(s)=\Upsilon^{(2)}_{H_s,t}(K)$ is exactly the secondary upsilon invariant introduced by Kim and Livingston [8].

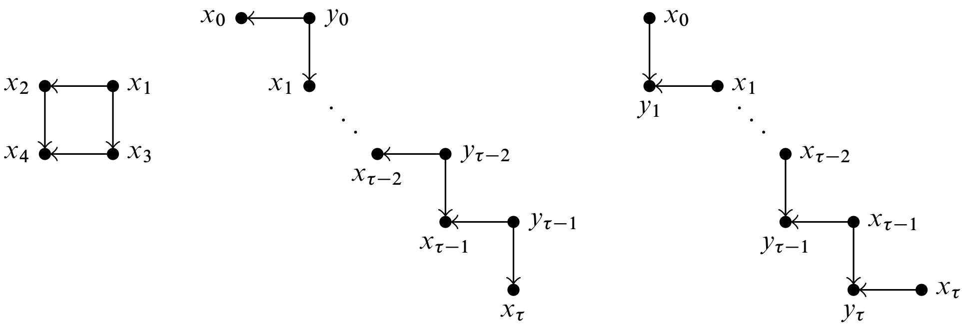

Figure 2: The square complex Q (left), and the staircase complex S_τ ($\tau \le 0$ on the centre, $\tau > 0$ on the right).

5 Floer thin knots

A knot $K \subset S^3$ is called *Floer thin* if its knot Floer homology groups $\widehat{\mathrm{HFK}}_{*,*}(K)$ are concentrated on a diagonal, meaning that $\widehat{\mathrm{HFK}}_{i,j}(K) = 0$ if $i - j \neq \delta$ for a suitable constant δ. Examples of Floer thin knots are alternating and quasialternating knots [15; 12] $\big($in these cases, $\delta = -\frac{1}{2}\sigma$, where σ denotes the knot signature$\big)$. In [22], Petkova shows that for a Floer thin knot the chain homotopy type of $\mathrm{CFK}^\infty(K)$ can be completely reconstructed from its Ozsváth–Szabó tau invariant $\tau = \tau(K)$ and its Alexander polynomial $\Delta_K = a_0 + \sum_{s>0} a_s(T^s + T^{-s})$. More precisely we have that:

- $\mathrm{CFK}^\infty(K)$ has exactly $|a_s|$ generators with $A = s$ and $j = 0$,
- $\mathrm{CFK}^\infty(K) = (S_\tau \otimes \mathbb{Z}_2[U, U^{-1}]) \oplus \big(\bigoplus_i Q_i \otimes \mathbb{Z}_2[U, U^{-1}]\big)$, where S_τ is a staircase complex, and the Q_i are square complexes as the one shown in Figure 2.

Notice that $A = \bigoplus_i Q_i \otimes \mathbb{Z}_2[U, U^{-1}]$ is acyclic. Consequently, up to acyclics, for a Floer thin knot K we have that $\mathrm{CFK}^\infty(K) = S_{\tau(K)} \otimes \mathbb{Z}_2[U, U^{-1}]$.

5.1 Three-parameter upsilon invariants of thin knots We show how to compute some upsilon-type invariants in the case of Floer thin knots. Choose as south-west region

$$C = \left\{\tfrac{1}{2}t \cdot A + \left(1 - \tfrac{1}{2}t\right) \cdot j \le 0\right\} \cup \left\{\tfrac{1}{2}s \cdot A + \left(1 - \tfrac{1}{2}s\right) \cdot j \le q\right\}.$$

As the parameters $s, t \in [0, 1]$ and $q \in \mathbb{R}$ vary, the concordance invariant Υ^C gives rise to a three-parameter family of concordance invariants $\Upsilon_K(t, s, q)$ collapsing to the classical upsilon invariant when $t = s$ and $q = 0$. Let us compute $\Upsilon^C(K) = \Upsilon^C(\mathrm{CFK}^\infty(K))$ for a Floer thin knot K.

Suppose first that $\tau = \tau(K)$ is positive. Since $\mathrm{CFK}^\infty(K) = S_\tau \otimes \mathbb{Z}_2[U, U^{-1}] \oplus A$ with S_τ staircase shaped as in Figure 2 and A acyclic, Hom's principle shows that $\Upsilon^C(\mathrm{CFK}^\infty(K)) = \Upsilon^C(S_\tau \otimes \mathbb{Z}_2[U, U^{-1}])$.

Let C_γ denote the translation of C in the $v_\gamma = (\gamma, \gamma)$ direction. The south-west region C_γ contains a generator of $H_0(S_\tau \otimes \mathbb{Z}_2[U, U^{-1}]) = \mathbb{Z}_2$ as soon as it contains one of the x_i generators of Figure 2. Thus, $\Upsilon^C(S_\tau \otimes \mathbb{Z}_2[U, U^{-1}])$ is the minimum γ such that

$$\tfrac{1}{2}t \cdot A(x_i) + \left(1 - \tfrac{1}{2}t\right) \cdot j(x_i) \leq \gamma \quad \text{or} \quad \tfrac{1}{2}s \cdot A(x_i) + \left(1 - \tfrac{1}{2}s\right) \cdot j(x_i) - q \leq \gamma$$

for at least one of the x_i generators, hence $\Upsilon^C(S_\tau \otimes \mathbb{Z}_2[U, U^{-1}])$ is computed by the expression

$$\min_i \min\left\{\tfrac{1}{2}tA(x_i) + \left(1 - \tfrac{1}{2}t\right)j(x_i),\ \tfrac{1}{2}sA(x_i) + \left(1 - \tfrac{1}{2}s\right)j(x_i) - q\right\}.$$

Plugging in $A(x_i) = \tau - i$ and $j(x_i) = i$ we get

$$\Upsilon^C(S_\tau \otimes \mathbb{Z}_2[U, U^{-1}]) = \min_i \min\{(1-t)i + \tau,\ (1-s)i + \tau - q\},$$

from where the identity

$$\Upsilon^C(S_\tau \otimes \mathbb{Z}_2[U, U^{-1}]) = \min\left\{\tfrac{1}{2}t\tau,\ (1-s)\left\lceil \tfrac{1}{2}\tau + \frac{q}{t-s} \right\rceil + \tfrac{1}{2}(s\tau - 2q)\right\}$$

follows for $t \neq s$. For $t = s$ one can easily see that

$$\Upsilon^C(S_\tau \otimes \mathbb{Z}_2[U, U^{-1}]) = \tfrac{1}{2}t\tau - \max\{0, q\}.$$

If $\tau < 0$ then the situation is somehow easier: there is only one 0–cycle generating $H_0(S_\tau \otimes \mathbb{Z}_2[U, U^{-1}]) = \mathbb{Z}_2$, namely $z = \sum_i x_i$. Thus, in this case, the invariant $\Upsilon^C(S_\tau \otimes \mathbb{Z}_2[U, U^{-1}])$ is computed by the expression

$$\max_i \min\left\{\tfrac{1}{2}tA(x_i) + \left(1 - \tfrac{1}{2}t\right)j(x_i),\ \tfrac{1}{2}sA(x_i) + \left(1 - \tfrac{1}{2}s\right)j(x_i) - q\right\}.$$

By substituting the values of $A(x_i)$ and $j(x_i)$ we get

$$\Upsilon^C(S_\tau \otimes \mathbb{Z}_2[U, U^{-1}]) = \min\left\{(1-s)\left\lfloor \tfrac{1}{2}\tau + \frac{q}{t-s} \right\rfloor + \tfrac{1}{2}(s\tau - 2q),\ \tfrac{1}{2}(2-t)\tau\right\}$$

if $t \neq s$. If $t = s$, we have the identity

$$\Upsilon^C(S_\tau \otimes \mathbb{Z}_2[U, U^{-1}]) = \tfrac{1}{2}(2-t)\tau - \min\{0, q\}.$$

As an immediate corollary of this discussion we get the following proposition:

Proposition 5.1 *Suppose that $K \subset S^3$ is a Floer thin knot. Then*

$$\Upsilon_K(t) = -\tau(K)\cdot(1-|t-1|). \qquad \square$$

Notice that the Ozsváth–Stipsicz–Szabó upsilon function of a Floer thin knot $K \subset S^3$ has only one singularity at $t=1$, where it actually has a breaking point if $\Delta\Upsilon'_{t=1}(K) = 2\tau(K) > 0$. We now compute the Kim–Livingston secondary invariant of these singularities.

Proposition 5.2 *Suppose that $K \subset S^3$ is a Floer thin knot. Then*

$$\Upsilon^{(2)}_{K,1}(s) = (1-\tau(K))\cdot|1-s| - 1$$

if $\tau(K) > 0$, and $\Upsilon^{(2)}_{K,1}(s) = -\infty$ otherwise.

Proof In the notation of Section 4.1, we would like to compute $\Upsilon^{(2)}_{C,1}(K_*)$ for $C = \left\{\frac{1}{2}sA + \left(1-\frac{1}{2}s\right)j \le 0\right\}$ and $K_* = S_\tau \otimes \mathbb{Z}_2[U,U^{-1}]$.

If $\tau > 0$, the upsilon function $\Upsilon_t(S_\tau \otimes \mathbb{Z}_2[U,U^{-1}])$ has a breaking point at $t=1$. The exceptional sets $\mathcal{Z}^+$ and $\mathcal{Z}^-$ of this breaking point are easy to identify: there is only one positive and one negative exceptional cycle, namely $z^+ = x_0$ and $z^- = x_\tau$ (see Figure 2 again). A quick inspection of the same figure reveals that a 1–chain realizing a homology between z^+ and z^- is given by $b = \sum_i y_i$. Notice that there is exactly one such chain since $H_1(S_\tau \otimes \mathbb{Z}_2[U,U^{-1}]) = 0$, and ∂ vanishes on chains with even Maslov grading. Thus,

$$\Upsilon^{(2)}_{C,1}(S_\tau \otimes \mathbb{Z}_2[U,U^{-1}]) = -2\left(\max_i\left(\tfrac{1}{2}sA(y_i) + \left(1-\tfrac{1}{2}s\right)j(y_i)\right) - \tfrac{1}{2}\tau\right).$$

Plugging in $A(y_i) = \tau - i$ and $j(y_i) = i+1$, the claimed identity can be deduced by algebraic manipulation.

If $\tau \le 0$, there is only one 0–cycle generating $H_0(S_\tau \otimes \mathbb{Z}_2[U,U^{-1}])$, namely $z = \sum_i x_i$. Thus $\mathcal{Z}^+ \cap \mathcal{Z}^- = \{z\}$ and we conclude that $\Upsilon^{(2)}_{K,1}(s) = -\infty$. $\square$

6 L–space knots

Another interesting class of knots is provided by L–space knots. Recall that a rational homology sphere Y is an *L–space* if $\widehat{\mathrm{HF}}(Y,\mathfrak{s}) = \mathbb{Z}_2$ in every Spin^c structure. This happens for example in the case of a lens space $Y = L(p,q)$, whence the name. A knot $K \subset S^3$ is said to be an L*–space knot* if it has a positive surgery $S^3_p(K)$ that is an L–space. Basic examples of L–space knots are positive torus knots.

The homotopy type of the master complex of an L–space knot can be reconstructed from its Alexander polynomial. More precisely, suppose that $K \subset S^3$ is a genus g L–space knot. According to [19], its Alexander polynomial can be written in the form $\Delta_K(t) = 1 - t^{\alpha_1} + \cdots - t^{\alpha_{2k-1}} + t^{\alpha_{2k}}$, with $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{2k} = 2g$. Starting from the sequence $a_i = \alpha_i - \alpha_{i-1}$ recording the jumps between consecutive exponents of the monomials appearing in the Alexander polynomial, construct a chain complex $S_*(K) = S_*(a_1, \dots, a_{2k})$ as follows. Set

$$S_*(a_1, \dots, a_{2k}) = \mathbb{Z}_2\{x_0, \dots, x_k, y_0, \dots, y_{k-1}\} \otimes \mathbb{Z}_2[U, U^{-1}],$$

and consider the differential

$$\begin{cases} \partial x_i = 0, & i = 0, \dots, k, \\ \partial y_i = x_i + x_{i+1}, & i = 0, \dots, k-1. \end{cases}$$

Define

$$\begin{cases} A(x_i) = n_i, \\ j(x_i) = m_i, \\ M(x_i) = 0, \end{cases} \qquad \text{and} \qquad \begin{cases} A(y_i) = n_i, \\ j(y_i) = m_{i+1}, \\ M(y_i) = 1, \end{cases} \tag{3}$$

where

$$\begin{cases} n_i = g - \sum_{j=0}^{i} a_{2j}, \\ n_0 = 0, \end{cases} \qquad \begin{cases} m_i = \sum_{j=1}^{i} a_{2j-1}, \\ m_0 = 0, \end{cases}$$

and coherently extend these gradings to $\mathbb{Z}_2\{x_0, \dots, x_k, y_0, \dots, y_{k-1}\} \otimes \mathbb{Z}_2[U, U^{-1}]$ so that multiplication by U drops the Maslov grading M by -2, and the Alexander filtration A as well as the algebraic filtration j by -1. In [21], Peters proves that there is a chain homotopy equivalence $\mathrm{CFK}^\infty(K) \simeq S_*(K)$.

6.1 Kim–Livingston secondary invariant of L–space knots Let us compute the upsilon invariant $\Upsilon_t(S_*)$ of a staircase complex $S_* = S_*(a_1, \dots, a_{2k})$. Since the lower half-space $\frac{1}{2}t \cdot A + \left(1 - \frac{1}{2}t\right) \cdot j \leq \gamma$ contains a cycle generating $H_0(S_*)$ as soon as it contains one of the x_i generators, we have that

$$\Upsilon_t(S_*) = \min_i \left\{\tfrac{1}{2}tn_i + \left(1 - \tfrac{1}{2}t\right)m_i\right\}. \tag{4}$$

Thus, for an L–space knot $K \subset S^3$ one has

$$\Upsilon_K(t) = -2 \cdot \min_i \left\{(n_i - m_i) \cdot \tfrac{1}{2}t + m_i\right\} = -\min_i \{\alpha_i t + m_i\},$$

as already pointed out by Ozsváth, Stipsicz and Szabó [14].

From (4) it is clear where the upsilon function $\Upsilon_t(S_*)$ of a staircase complex S_* has its breaking points. A parameter t is a singularity for $\Upsilon_t(S_*)$ if and only if the min on the left-hand side of (4) is realised by more than one index i. These singularities are breaking points since, at these points, $\Delta\Upsilon'_t > 0$. Notice that there are no other breaking points since at a regular parameter t the half space $\frac{1}{2}t \cdot A + \left(1-\frac{1}{2}t\right) \cdot j \le \Upsilon_t(S_*)$ contains (on its boundary line) exactly one x_i generator.

Proposition 6.1 *Let $S_* = S_*(a_1, \dots, a_{2k})$ be a staircase complex. Suppose that t is a breaking point of $\Upsilon_t(S_*)$; then*

$$\Upsilon^{(2)}_{S_*,t}(s) = -2\Big(\max_{i_- \le j < i_-} \left\{\tfrac{1}{2}sn_j + \left(1-\tfrac{1}{2}s\right)m_{j+1}\right\} - \Upsilon_t(S_*)\Big),$$

where i_- and i_+ denote respectively the minimum and the maximum index realizing the minimum in (4).

Proof If t is a breaking point of $\Upsilon_t(S_*)$ then the half-space

$$\tfrac{1}{2}t \cdot A + \left(1-\tfrac{1}{2}t\right) \cdot j \le \Upsilon_t(S_*)$$

contains (actually on its boundary line) exactly those x_i generators which have index $i \in \{0, \dots, k\}$ realizing the minimum in the expression of (4).

The exceptional sets $\mathcal{Z}^+$ and $\mathcal{Z}^-$ of such a singularity both contain exactly one 0–cycle: $z^+ = x_{i_+}$ and $z_- = x_{i_-}$, respectively. Notice that since $H_1(S_*) = 0$, there is only one 1–chain realizing a homology between these cycles, namely $\beta = \sum_{j=i_-}^{i_+-1} y_j$. Thus, in the notation of equation (2) of Section 4.1, we have that

$$\Upsilon_{H_{t\pm\delta},H_s}(S_*) = \max_{i_- \le j < i_+} \left\{\tfrac{1}{2}sn_j + \left(1-\tfrac{1}{2}s\right)m_{j+1}\right\},$$

from which the formula follows. □

6.2 A connected sum formula One of the fundamental properties of the Ozsváth–Stipsicz–Szabó upsilon invariant is its additivity property

$$\Upsilon_t(A_* \otimes B_*) = \Upsilon_t(A_*) + \Upsilon_t(B_*),$$

turning Υ_t into a group homomorphism from $\mathcal{CFK}/{\sim}$ to the group of piecewise linear functions $[0,2] \to \mathbb{R}$. General upsilon-type invariants and their secondary counterparts do not enjoy this property. In this section we prove a connected sum formula for the Kim–Livingston secondary invariant of staircase complexes.

Theorem 6.2 *Let $A_* = S_*(a_1, \dots, a_{2n})$ and $B_* = S_*(b_1, \dots, b_{2m})$ be staircase complexes. Suppose that $\Upsilon_t(A_*)$ has a breaking point at a point $t = s$ where $\Upsilon_t(B_*)$ is smooth. Then*

$$\Upsilon^{(2)}_{A_* \otimes B_*, s}(s) = \Upsilon^{(2)}_{A_*, s}(s).$$

Proof Denote by $x_0, \dots, x_n$ and $z_0, \dots, z_m$ the Maslov grading zero generators of the staircases of A_* and B_*, respectively. Similarly, denote by $y_0, \dots, y_{n-1}$ and $w_0, \dots, w_{m-1}$ their Maslov grading one generators.

The fact that $\Upsilon_t(B_*)$ is smooth at $t = s$ guarantees that the half-space

$$\tfrac{1}{2}s \cdot A + \left(1 - \tfrac{1}{2}s\right) \cdot j \le \Upsilon_s(B_*)$$

only contains (actually on its boundary line) one 0–cycle $z = z_r$ generating $H_0(B_*)$. Since A_* is a staircase complex, the sets of its exceptional cycles $\mathcal{Z}^+$ and $\mathcal{Z}^-$ at $t = s$ both include exactly one 0–cycle. Denote those cycles by $x^+ = x_k$ and $x^- = x_h$, respectively. In this notation, the sets of exceptional cycles of $A_* \otimes B_*$ at $t = s$ are given by $\mathcal{Z}^+ = \{x_k \otimes z_r\}$ and $\mathcal{Z}^- = \{x_h \otimes z_r\}$.

Given a chain $\xi = \sum_i \xi_i$ with $\xi_1, \dots, \xi_n$ homogeneous with respect to both the Alexander and the algebraic grading, set

$$E_s(\xi) = \max_i \left\{\tfrac{1}{2}s \cdot A(\xi_i) + \left(1 - \tfrac{1}{2}s\right) \cdot j(\xi_i)\right\}.$$

In this notation, $E_s(x) \le \gamma$ if and only if the chain ξ is contained in the subcomplex of the lower half-space $\frac{1}{2}s \cdot A + \left(1 - \frac{1}{2}s\right) \le \gamma$.

It is easy to find a 1–chain realizing a homology between $x_k \otimes z_r$ and $x_h \otimes z_r$:

$$\partial\left(\sum_{\ell=a}^{b-1} y_\ell \otimes z_r\right) = \sum_{\ell=a}^{b-1} \partial y_\ell \otimes z_r = \sum_{\ell=a}^{b-1} (x_\ell + x_{\ell+1}) \otimes z_r = x_k \otimes z_r - x_h \otimes z_r.$$

If we prove that between the 1–cycles realizing a homology between $x_k \otimes z_r$ and $x_h \otimes z_r$ this is the one with minimal E_s, then we conclude that

$$\begin{aligned}\Upsilon^{(2)}_{A_* \otimes B_*, s}(s) &= -2\left(E_s\left(\sum_{\ell=a}^{b-1} y_\ell \otimes z_r\right) - \Upsilon_t(A_* \otimes B_*)\right) \\ &= -2\left(E_s\left(\sum_{\ell=a}^{b-1} y_\ell\right) + E_s(z_r) - \Upsilon_t(A_*) - \Upsilon_t(B_*)\right)\end{aligned}$$

$$= -2\left(E_s\left(\sum_{\ell=a}^{b-1} y_\ell\right) - \Upsilon_t(A_*)\right) = \Upsilon^{(2)}_{A_*,s}(s)$$

and we are done. Let us prove that $\beta = \sum_{\ell=a}^{b-1} y_\ell \otimes z_r$ is a cycle minimizing $E_s(\beta)$ in the class of 1–cycles realizing homologies between $x_k \otimes z_r$ and $x_h \otimes z_r$.

From the fact that, for a staircase complex, $\partial(x) = 0$ for those x's with homogeneous even Maslov degree, one can conclude that any 1–chain realizing an homology between $x_k \otimes z_r$ and $x_h \otimes z_r$ differs from β by the boundary of an element in $A_1 \otimes B_1$. In other words, such a 1–chain should be of the form

$$\partial\left(\sum_{i,j} \epsilon_{ij}\, y_i \otimes w_j\right) + \sum_{\ell=a}^{b-1} y_\ell \otimes z_r$$

for some coefficients $\epsilon_{ij} \in \mathbb{Z}_2$. Obviously, we have that

$$E_s\left(\partial\left(\sum_{i,j} \epsilon_{ij}\, y_i \otimes w_j\right) + \sum_{\ell=a}^{b-1} y_\ell \otimes z_r\right) \geq E_s\left(\sum_{\ell=a}^{b-1} y_\ell \otimes z_r\right) \tag{5}$$

provided that none of the generators $y_a \otimes z_r,\ y_{a+1} \otimes z_r,\ \dots,\ y_{b-1} \otimes z_r$ appears as a component of

$$\partial\left(\sum_{i,j} \epsilon_{ij}\, y_i \otimes w_j\right) = \sum_{i,j} \epsilon_{ij}\, \partial y_i \otimes w_j + \sum_{i,j} \epsilon_{ij}\, y_i \otimes \partial w_j.$$

On the other hand, it so happens that for some $y_i \otimes z_r$, after cancellation, the summand

$$\sum_j \epsilon_{ij}\, y_i \otimes \partial w_j = \sum_j \epsilon_{ij}\, y_i \otimes z_j + \epsilon_{ij}\, y_i \otimes z_{j+1}$$

has a component of the form $y_i \otimes z_\mu$ for some $\mu \neq k$. Thus, since $E_s(y_i \otimes z_\mu) = E_s(y_i) + E_s(z_\mu) > E_s(y_i) + \Upsilon_s(B_*) = E_s(y_i) + E_s(z_r) = E_s(y_i \otimes z_r)$, also in this case the inequality in (5) holds, and we are done. □

Proof of Proposition 1.2 According to Feller and Krcatovitch [2], the Ozsváth–Stipsicz–Szabó upsilon function $\Upsilon_{p,q}(t)$ of the (p,q) torus knot can be computed recursively by means of the formula $\Upsilon_{p,q}(t) = \Upsilon_{p-q,q}(t) + \Upsilon_{q+1,q}(t)$. Thus, $\Upsilon_K(t) = \Upsilon_{8,5}(t) - \Upsilon_{6,5}(t) - \Upsilon_{4,3}(t) = \Upsilon_{6,5}(t) + \Upsilon_{4,3}(t) + \Upsilon_{3,2}(t) - \Upsilon_{6,5}(t) - \Upsilon_{4,3}(t) = \Upsilon_{3,2}(t)$, proving that K is an upsilon-alternating knot.

Now suppose by contradiction that there exists a Floer thin knot J such that $T_{6,5} \# T_{4,3} \sim T_{8,5} \# J$. The upsilon function of the torus knot $T_{6,5}$ has its singularities at $t = \frac{2}{5}, \frac{4}{5}, \frac{6}{5}, \frac{8}{5}$

while the one of J has its only singularity at $t=1$. The upsilon functions of the torus knot $T_{4,3}$ and $T_{8,5}$ on the other hand both have a singularity at $t=\frac{2}{3}$. Thus, as a consequence of Theorem 6.2 we have that

$$\Upsilon^{(2)}_{T_{4,3},2/3}\left(\tfrac{2}{3}\right)=\Upsilon^{(2)}_{T_{6,5}\#T_{4,3},2/3}\left(\tfrac{2}{3}\right)=\Upsilon^{(2)}_{T_{8,5}\#J,2/3}\left(\tfrac{2}{3}\right)=\Upsilon^{(2)}_{T_{8,5},2/3}\left(\tfrac{2}{3}\right).$$

We claim that $\Upsilon^{(2)}_{T_{4,3},2/3}\left(\frac{2}{3}\right)\neq\Upsilon^{(2)}_{T_{8,5},2/3}\left(\frac{2}{3}\right)$. In fact, by Proposition 6.1 we have that $\Upsilon^{(2)}_{T_{4,3},2/3}\left(\frac{2}{3}\right)=-\frac{4}{3}$ and $\Upsilon^{(2)}_{T_{8,5},2/3}\left(\frac{2}{3}\right)=-\frac{20}{3}$. □

7 Algebraic knots

Suppose that $Z\subset\mathbb{C}^2$ is a planar complex curve given by the equation $f(x,y)=0$. Recall that a point $p\in Z$ is said to be *regular* if the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ do not both vanish at p. A point that is not regular is said to be *singular*. In what follows, by an isolated plane curve singularity (Z,p) we mean a planar complex curve Z with an isolated singularity at $p\in Z$. Without loss of generality we can always suppose p to be the origin of $\mathbb{C}^2$.

Let $(Z,0)$ be an isolated plane curve singularity. A small sphere $S^3_\epsilon(0)$ centred at the origin intersects Z transversally in a link $K=S^3_\epsilon(0)\cap Z$. This is the *link* of the plane curve singularity and, in a neighbourhood of the origin, Z looks like a cone over it. If the link $K\subset S^3$ is actually a knot, we say that $(Z,0)$ is *cuspidal*. Knots arising from this construction are called *algebraic knots*.

Naturally attached to a plane curve singularity $(Z,0)$ there is an arithmetic object capturing information about the complex geometry of its germ. Given an analytic parametrization $\varphi(z)$ of Z around 0, consider the pullback homomorphism $\varphi^*\colon\mathbb{C}[\![x,y]\!]\to\mathbb{C}[\![z]\!]$ defined by $g\mapsto g\circ\varphi$. Set

$$S=\{s\in\mathbb{Z}_{\geq 0}\mid g(\varphi(z))=z^s h(z)\text{ for some }g\in\mathbb{C}[\![x,y]\!]\text{ and }h\in\mathbb{C}[\![z]\!]\text{ with }h(0)\neq 0\}.$$

It is easy to see that S is a *semigroup*, meaning that $0\in S$ and, if $a,b\in S$, so is $a+b$. The semigroup of a cuspidal singularity $(Z,0)$ is related to its knot K via the Alexander polynomial

$$\Delta_K(t)=\sum_{s\in S}t^s-t^{s+1}. \tag{6}$$

Notice that this is a finite sum since the semigroup of a plane curve singularity eventually covers all the positive integers.

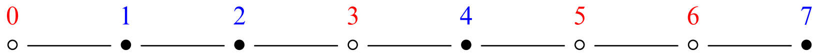

Figure 3: The semigroup of the plane curve singularity $x^5 + y^3 = 0$ is generated by 5 and 3. Its link is the torus knot $T_{5,3}$. The associated staircase can be computed from the colouring above by counting the gaps between blue $(1, 2, 4, 7)$ and red $(0, 3, 5, 6)$ numbers. In this case $r_1 = 1$, $r_2 = 1$, $r_3 = 2$, $b_1 = 2$, $b_2 = 1$, $b_3 = 1$ and $\mathrm{CFK}^\infty(T_{5,3}) = S_*(1, 2, 1, 1, 2, 1)$.

Any cuspidal plane curve singularity $(Z, 0)$ has a parametrization of the form $x = z^a$, $y = z^{q_1} + \cdots + z^{q_n}$ for some positive integers $q_1 < q_2 < \cdots < q_n$. Such a representation is unique if we further assume $\gcd(a, q_1, \dots, q_i)$ to not divide q_{i+1} and $\gcd(a, q_1, \dots, q_n) = 1$. The sequence $(a; q_1, \dots, q_n)$ is the *Puiseaux characteristic sequence* of the cuspidal singularity $(Z, 0)$, and the number a is its *Puiseaux exponent*. It is a fundamental fact of the theory of plane curve singularities [24, Chapter 5] that starting from the Puiseaux characteristic sequence of a cuspidal singularity one can reconstruct both its semigroup and the topology of its link.

Theorem 7.1 *Let $(Z, 0)$ be a cuspidal plane curve singularity with Puiseaux characteristic sequence $(a; q_1, \dots, q_n)$. Set $D_i = \gcd(a, q_1, \dots, q_i)$, $s_1 = q_1$ and*

$$s_i = \frac{aq_1 + D_1(q_2 - q_1) + \cdots + D_{i-1}(q_i - q_{i-1})}{D_{i-1}}$$

for $i = 0, \dots, n-1$. Then the link K of $(Z, 0)$ is the $(n-1)$–fold iterated cable of the $(a/D_1, q_1/D_1)$ torus knot with cabling coefficients $(D_{i-1}/D_i, s_{i-1}/D_i)$ for $i = 2, \dots, n$. Furthermore, the semigroup of $(Z, 0)$ is generated by $\{a, s_1, \dots, s_n\}$. □

From the viewpoint of Heegaard Floer theory, algebraic knots are interesting since they provide a good source of examples of L–space knots [5]. Because of (6), the staircase of an algebraic knot can be recovered from the semigroup of its singularity. More precisely, suppose that $(Z, 0)$ is a plane curve singularity giving rise to a genus g algebraic knot K. The semigroup S of $(Z, 0)$ determines a colouring of $\{0, \dots, 2g-1\}$: colour by red the numbers in $S \cap \{0, \dots, 2g-1\}$ and by blue the ones in its complement $(\mathbb{Z} \setminus S) \cap \{0, \dots, 2g-1\}$. By counting the gaps between blue and red numbers as suggested by Figure 3, we get two sequences of numbers $r_1, \dots r_g$ and $b_1, \dots, b_g$. As a consequence of the general recipe discussed at the beginning of Section 6 one can see that $\mathrm{CFK}^\infty(K) \simeq S_*(r_1, b_1, \dots, r_g, b_g)$.

7.1 L–space pretzel knots We now proceed to the proof of Theorem 1.1. Suppose that C is a south-west region. For every $x \in \mathbb{R}$ we can consider the truncated south-west region $C_x = C \cap \{A \leq x\}$. This leads to a one-parameter family of upsilon-type invariants $\Upsilon^{C_x}(K_*)$. Since $C_x \subseteq C$ we have that $\Upsilon^{C_x}(K_*) \leq \Upsilon^C(K_*)$. Furthermore, for x large enough, $\Upsilon^{C_x}(K_*) = \Upsilon^C(K_*)$. Set

$$\eta_C(K_*) = \min\{x \mid \Upsilon^{C_x}(K_*) = \Upsilon^C(K_*)\}.$$

Obviously, this is an invariant of stable equivalence. For a knot $K \subset S^3$ denote by $\eta_C(K) = \eta_C(\mathrm{CFK}^\infty(K))$ the associated knot concordance invariance.

Lemma 7.2 *Suppose that $(Z,0)$ is a cuspidal plane curve singularity with Puiseaux sequence $(a; q_1, \dots, q_n)$. Denote by K the algebraic knot associated to $(Z,0)$ and by S its semigroup. Let $n(S)$ be the maximum among the integers $n \geq 0$ such that*

$$S \cap \mathbb{Z}_{\leq na} = \{0, a, 2a, \dots, na\}.$$

Choose $C = \{1/a \cdot A + (1-1/a) \cdot j \leq 0\}$. Then

$$\eta_C(K) = \left(1 - \frac{1}{a}\right)\tau(K) - (a-1)n(S).$$

Proof Let g be the genus of K. Colour red and blue the numbers in $\{0, 1, \dots, 2g-1\}$ as specified by $(R = S \cap \{0, 1, \dots, 2g-1\}, B = (\mathbb{Z} \setminus S) \cap \{0, 1, \dots, 2g-1\})$ and by recording the gaps between red and blue numbers form the sequences $r_1, \dots, r_g$ and $b_1, \dots, b_g$ suggested by Figure 3. By the definition of $n(S)$ we have that $r_1 = \cdots = r_n = 1$, $b_1 = \cdots = b_n = a-1$ and $1 \leq b_i < a-1$ for $i = n+1, \dots, g$. Thus, $\mathrm{CFK}^\infty(K) \simeq S_*(1, a-1, \dots, 1, a-1, r_{n+1}, b_{n+1}, \dots, r_g, b_g)$, where the pair $(1, a-1)$ repeats $n = n(S)$ times. Denote by $x_0, \dots, x_g$ the Maslov grading zero generators of the staircase for $\mathrm{CFK}^\infty(K)$. Similarly, denote by $y_1, \dots, y_g$ its Maslov grading one generators.

Consider the half-space C_γ of the (A, j) plane defined by $1/a \cdot A + (1-1/a) \cdot j \leq \gamma$. We claim that for $\gamma = \Upsilon^C(K) = -2 \cdot \Upsilon_{2/a}(K) = \tau(K)/a$ the only Maslov grading zero generators contained in C_γ are $x_1, \dots, x_n$. For this purpose define

$$E(x_i) = 1/a \cdot A(x_i) + (1-1/a) \cdot j(x_i).$$

Obviously $E(x_i) \leq \gamma$ if and only if x_i is in C_γ. A quick computation reveals that $E(x_i) = \gamma$ for $i = 1, \dots, n$ and consequently that $x_1, \dots, x_n$ are actually contained in the boundary line of C_γ. We claim that $E(x_i) > \gamma$ for any $i > n$. For $k \geq 1$ we have that

$$E(x_{n+k}) = E(x_n) - \frac{1}{a}\sum_{i=1}^{k} b_{n+i} + \Big(1 - \frac{1}{a}\Big)\sum_{i=1}^{k} r_{n+i}$$
$$= \gamma + \frac{1}{a}\Big(-\sum_{i=1}^{k} b_{n+i} + (a-1)\sum_{i=1}^{k} r_{n+i}\Big).$$

On the other hand,

$$(a-1)\sum_{i=1}^{k} r_{n+i} \geq k\cdot(a-1) > \sum_{i=1}^{k} b_{n+i},$$

proving that $E(x_{n+k}) \geq \gamma + 1/a \cdot (\text{something positive}) > \gamma$, and we are done.

Since the only generators with Maslov grading zero in C_γ are $x_1, \dots, x_n$, we conclude that $C_\gamma \cap \{A \leq x + \gamma\}$ contains a cycle generating $H_0(\mathrm{CFK}^\infty(K))$ provided $x + \gamma \geq \min\{A(x_1), \dots, A(x_n)\} = g - n(a-1)$. Thus, $\eta_C(K) + \gamma = g - n(a-1)$. Plugging in $\gamma = 1/a \cdot \tau(K)$, $g = \tau(K)$ and $n = n(S)$, the claim follows. □

Proof of Theorem 1.1 Using the skein relation at a negative crossing we find that the symmetrized Alexander polynomial $\Delta_q(t)$ of a $P(-2,3,q)$ pretzel knot ($q \geq 7$ odd) is given by $\Delta_q(t) = (t - 1 + t^{-1})\Delta_{2,q}(t) + (t^{1/2} - t^{-1/2})\Delta_{2,q+3}(t)$, where $\Delta_{2,p}(t)$ denotes the Alexander polynomial of the $(2,p)$ torus link. Since $P(-2,3,q)$ is an L–space knot, this leads to the conclusion that

$$\mathrm{CFK}^\infty(P(-2,3,q)) \simeq S_*(1,2,1,1,\dots,1,1,2,1),$$

from where one computes $\tau(P(-2,3,q)) = \frac{1}{2}(q+3)$, and $\eta_C(P(-2,3,q)) = \frac{1}{3}(q-3)$ for $C = \big\{\frac{1}{3}\cdot A + \frac{2}{3}\cdot j \leq 0\big\}$. Notice that $\Upsilon_{P(-2,3,q)}(t) = -2\cdot\Upsilon_t(S_*(1,2,1,\dots,1,2,1))$ has its only singularities at $t = \frac{2}{3}, 1, \frac{4}{3}$.

Suppose by contradiction that, for some $q \geq 7$ odd, the pretzel knot $P(-2,3,q)$ is concordant to a sum of algebraic knots $K_1 \# \cdots \# K_m$. For $i = 1, \dots, m$ let $(Z_i, 0)$ be a plane curve singularity with knot K_i. Denote by S_i the semigroup of $(Z_i, 0)$ and by a_i its Puiseaux exponent. According to Wang [25], the Ozsváth–Stipsicz–Szabó upsilon invariant $\Upsilon_K(t)$ of an algebraic knot has its first singularity at $t = 2/a$, where a denotes its Puiseaux exponent. Since $\Upsilon_{P(-2,3,q)}(t) = \Upsilon_{\#_i K_i}(t) = \sum_i \Upsilon_{K_i}(t)$, and $\Delta\Upsilon'_{K_i}(t) \geq 0$, this leads to the conclusion that either $a_i = 3$ or $a_i = 2$.

Notice that, as a consequence of Theorem 7.1, if $K = ((T_{p,q})_{p_1,q_1} \dots)_{p_n,q_n}$ is the knot of a cuspidal plane curve singularity $(Z,0)$ with Puiseaux exponent a then

$a = p \cdot (p_1 \dots p_n)$. Since for every i the Puiseaux exponent of K_i is either 3 or 2, we conclude that K_i is either a $(3, p)$ torus knot, or a $(2, k)$ torus knot.

An argument along the lines of [10, Theorem 6.2] reveals that $\eta_C(K_1 \# \cdots \# K_m) = \eta_C(K_1) + \cdots + \eta_C(K_m)$. A direct computation shows that, for the $(2, 2k+1)$ torus knot, $\eta_C = \frac{2}{3} \cdot k$. Thus, as a consequence of (5) we have that

$$\eta_C(P(-2,3,q)) = \eta_C(K_1 \# \cdots \# K_m) = \frac{2}{3} \sum_i \tau(K_i) - 2 \sum_j n(S_j),$$

where the second sum is extended only to the $(3, p)$ torus knot summands. Plugging in $\sum_i \tau(K_i) = \tau(K_1 \# \cdots \# K_m) = \tau(P(-2,3,q)) = \frac{1}{2}(q+3)$ and $\eta_C(P(-2,3,q)) = \frac{1}{3}(q-3)$, we get that $-2 = -2\sum_j n(S_j)$ and consequently that $K_1 \# \cdots \# K_m$ is either of the form $T_{3,4} \# J$ or $T_{3,5} \# J$, where J is a sum of $(2, n)$ torus knots and hence alternating. This leads to a contradiction since a knot of this form has $\tau = -\frac{1}{2}\sigma$ while for a pretzel knot of the form $P(-2,3,q)$ we have $\frac{1}{2}(q+3) = \tau \neq -\frac{1}{2}\sigma = \frac{1}{2}(q+1)$. □

References

[1] **P Aceto**, **A Alfieri**, *On sums of torus knots concordant to alternating knots*, Bull. Lond. Math. Soc. 51 (2019) 327–343 MR

[2] **P Feller**, **D Krcatovich**, *On cobordisms between knots, braid index, and the upsilon-invariant*, Math. Ann. 369 (2017) 301–329 MR

[3] **A Floer**, *Morse theory for Lagrangian intersections*, J. Differential Geom. 28 (1988) 513–547 MR

[4] **S Friedl**, **C Livingston**, **R Zentner**, *Knot concordances and alternating knots*, Michigan Math. J. 66 (2017) 421–432 MR

[5] **E Gorsky**, **A Némethi**, *Links of plane curve singularities are L–space links*, Algebr. Geom. Topol. 16 (2016) 1905–1912 MR

[6] **K Hendricks**, **C Manolescu**, *Involutive Heegaard Floer homology*, Duke Math. J. 166 (2017) 1211–1299 MR

[7] **J Hom**, *A survey on Heegaard Floer homology and concordance*, J. Knot Theory Ramifications 26 (2017) 1740015, 24 MR

[8] **S-G Kim**, **C Livingston**, *Secondary upsilon invariants of knots*, Q. J. Math. 69 (2018) 799–813 MR

[9] **T Lidman**, **A H Moore**, *Pretzel knots with L–space surgeries*, Michigan Math. J. 65 (2016) 105–130 MR

[10] **C Livingston**, *Notes on the knot concordance invariant upsilon*, Algebr. Geom. Topol. 17 (2017) 111–130 MR

[11] **C Livingston**, *Concordances from connected sums of torus knots to L–space knots*, New York J. Math. 24 (2018) 233–239 MR

[12] **C Manolescu**, **P Ozsváth**, *On the Khovanov and knot Floer homologies of quasi-alternating links*, from "Proceedings of Gökova Geometry–Topology Conference 2007" (S Akbulut, T Önder, R J Stern, editors), Gökova Geometry/Topology Conference, Gökova (2008) 60–81 MR

[13] **P S Ozsváth**, **A I Stipsicz**, **Z Szabó**, *Grid homology for knots and links*, Mathematical Surveys and Monographs 208, Amer. Math. Soc., Providence, RI (2015) MR

[14] **P S Ozsváth**, **A I Stipsicz**, **Z Szabó**, *Concordance homomorphisms from knot Floer homology*, Adv. Math. 315 (2017) 366–426 MR

[15] **P Ozsváth**, **Z Szabó**, *Heegaard Floer homology and alternating knots*, Geom. Topol. 7 (2003) 225–254 MR

[16] **P Ozsváth**, **Z Szabó**, *Knot Floer homology and the four-ball genus*, Geom. Topol. 7 (2003) 615–639 MR

[17] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and knot invariants*, Adv. Math. 186 (2004) 58–116 MR

[18] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. 159 (2004) 1027–1158 MR

[19] **P Ozsváth**, **Z Szabó**, *On knot Floer homology and lens space surgeries*, Topology 44 (2005) 1281–1300 MR

[20] **P Ozsváth**, **Z Szabó**, *Holomorphic triangles and invariants for smooth four-manifolds*, Adv. Math. 202 (2006) 326–400 MR

[21] **T D Peters**, *A concordance invariant from the Floer homology of* ± 1 *surgeries*, preprint (2010) arXiv

[22] **I Petkova**, *Cables of thin knots and bordered Heegaard Floer homology*, Quantum Topol. 4 (2013) 377–409 MR

[23] **J A Rasmussen**, *Floer homology and knot complements*, PhD thesis, Harvard University (2003) MR arXiv

[24] **C T C Wall**, *Singular points of plane curves*, London Mathematical Society Student Texts 63, Cambridge Univ. Press (2004) MR

[25] **S Wang**, *On the first singularity for the upsilon invariant of algebraic knots*, Bull. Lond. Math. Soc. 48 (2016) 349–354 MR

[26] **S Wang**, *Semigroups of* L*–space knots and nonalgebraic iterated torus knots*, Math. Res. Lett. 25 (2018) 335–346 MR

[27] **I Zemke**, *Connected sums and involutive knot Floer homology*, Proc. Lond. Math. Soc. 119 (2019) 214–265 MR

Alfréd Rényi Institute of Mathematics
Budapest, Hungary

alfieriantonio90@gmail.com

Received: 7 December 2017 Revised: 14 November 2018