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Ropelength, crossing number and finite-type invariants of links

RAFAL KOMENDARCZYK
ANDREAS MICHAELIDES

Ropelength and *embedding thickness* are related measures of geometric complexity of classical knots and links in Euclidean space. In their recent work, Freedman and Krushkal posed a question regarding lower bounds for embedding thickness of n–component links in terms of the Milnor linking numbers. The main goal of the current paper is to provide such estimates, and thus generalize the known linking number bound. In the process, we collect several facts about finite-type invariants and ropelength/crossing number of knots. We give examples of families of knots where such estimates behave better than the well-known knot–genus estimate.

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1 Introduction

Given an n–component link (we assume class C^1 embeddings) in 3–space

$$L\colon S^1 \sqcup \cdots \sqcup S^1 \to \mathbb{R}^3, \qquad L = (L_1, L_2, \dots, L_n), \quad L_i = L|_{\text{the } i^{\text{th}} \text{ circle}}, \tag{1-1}$$

its *ropelength* $\mathsf{rop}(L)$ is the ratio $\mathsf{rop}(L) = \ell(L)/r(L)$ of length $\ell(L)$, which is a sum of lengths of individual components of L, to *reach* or *thickness* $r(L)$, ie the largest radius of the tube embedded as a normal neighborhood of L. The *ropelength within the isotopy class* $[L]$ of L is defined as

$$\mathsf{Rop}(L) = \inf_{L' \in [L]} \mathsf{rop}(L'), \quad \mathsf{rop}(L') = \frac{\ell(L')}{r(L')}, \tag{1-2}$$

(in Cantarella, Kusner and Sullivan [10] it is shown that the infimum is achieved within $[L]$ and the minimizer is of class $C^{1,1}$). A related measure of complexity, called *embedding thickness*, was introduced recently in Freedman and Krushkal [20], in the general context of embeddings' complexity. For links, the embedding thickness $\tau(L)$ of L is given by the value of its reach $r(L)$ assuming that L is a subset of the unit ball

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B_1 in $\mathbb{R}^3$ (note that any embedding can be scaled and translated to fit in B_1). Again, the embedding thickness of the isotopy class $[L]$ is given by

$$\mathcal{T}(L) = \sup_{L' \in [L]} \tau(L'). \tag{1-3}$$

For a link $L \subset B_1$, the volume of the embedded tube of radius $\tau(L)$ is $\pi \ell(L)\tau(L)^2$ — see Gray [23] — and the tube is contained in the ball of radius $r = 2$, yielding

$$\mathsf{rop}(L) = \frac{\pi \ell(L)\tau(L)^2}{\pi \tau(L)^3} \leq \frac{\frac{4}{3}\pi 2^3}{\pi \tau(L)^3} \implies \tau(L) \leq \left(\frac{11}{\mathsf{rop}(L)}\right)^{\frac{1}{3}}. \tag{1-4}$$

In turn a lower bound for $\mathsf{rop}(L)$ gives an upper bound for $\tau(L)$ and vice versa. For other measures of complexity of embeddings such as distortion or Gromov–Guth thickness, see eg Gromov [24] or Gromov and Guth [25].

Bounds for the ropelength of knots, and in particular the lower bounds, have been studied by many researchers; we only list a small fraction of these works here: Buck and Simon [5; 6], Cantarella, Kusner and Sullivan [10], Diao, Ernst, Janse van Rensburg and Por [16; 14; 13; 17], Litherland, Simon, Durumeric and Rawdon [32; 40] and Ricca, Maggioni and Moffatt [41; 33; 42]. Many of the results are applicable directly to links, but the case of links is treated in more detail by Cantarella, Kusner and Sullivan [10] and in the earlier work of Diao, Ernst, and Janse Van Rensburg [15] concerning the estimates in terms of the pairwise linking number. In [10], the authors introduce a cone surface technique and show the following estimate, for a link L (defined as in (1-1)) and a given component L_i [10, Theorem 11]:

$$\mathsf{rop}(L_i) \geq 2\pi + 2\pi\sqrt{\mathsf{Lk}(L_i, L)}, \tag{1-5}$$

where $\mathsf{Lk}(L_i, L)$ is the maximal total linking number between L_i and the other components of L. A stronger estimate was obtained in [10] by combining the Freedman–He [19] asymptotic crossing number bound for energy of divergence-free fields and the cone surface technique as follows:

$$\mathsf{rop}(L_i) \geq 2\pi + 2\pi\sqrt{\mathsf{Ac}(L_i, L)}, \quad \mathsf{rop}(L_i) \geq 2\pi + 2\pi\sqrt{2g(L_i, L) - 1}, \tag{1-6}$$

where $\mathsf{Ac}(L_i, L)$ is the *asymptotic crossing number* (see [19]) and the second inequality is a consequence of the estimate $\mathsf{Ac}(L_i, L) \geq 2g(L_i, L) - 1$, where $g(L_i, L)$ is a minimal genus among surfaces embedded in $\mathbb{R}^3 \setminus \{L_1 \cup \cdots \cup \widehat{L}_i \cup \cdots \cup L_n\}$ [19, page 220] (in fact, the estimate (1-6) subsumes (1-5) since $\mathsf{Ac}(L_i, L) \geq \mathsf{Lk}(L_i, L)$). A relation between $\mathsf{Ac}(L_i, L)$ and the higher linking numbers of Milnor [35; 36] is unknown and appears difficult. The following question, concerning the embedding thickness, is stated in [20, page 1424]:

Question A *Let L be an n–component link which is Brunnian* (ie almost trivial in the sense of Milnor [35]). *Let M be the maximum value among Milnor's $\overline{\mu}$–invariants with distinct indices*, *ie among $|\overline{\mu}_{\mathtt{I};j}(L)|$. Is there a bound*

$$\varnothing(L) \leq c_n M^{-1/n} \tag{1-7}$$

for some constant $c_n > 0$, independent of the link L? Is there a bound on the crossing number $\mathsf{Cr}(L)$ in terms of M?

Recall that the Milnor $\overline{\mu}$–invariants $\{\overline{\mu}_{\mathtt{I};j}(L)\}$ of L are indexed by an ordered tuple $(\mathtt{I};j) = (i_1, i_2, \dots, i_k; j)$ from the index set $\{1, \dots, n\}$, where the last index j has a special role (see below). If all the indexes in $(\mathtt{I};j)$ are distinct, $\{\overline{\mu}_{\mathtt{I};j}\}$ are link homotopy invariants of L and are often referred to simply as *Milnor linking numbers* or *higher linking numbers* [35; 36]. The definition $\{\overline{\mu}_{\mathtt{I};j}\}$ begins with coefficients $\mu_{\mathtt{I};j}$ of the Magnus expansion of the j^{th} longitude of L in $\pi_1(\mathbb{R}^3 - L)$. Then

$$\overline{\mu}_{\mathtt{I};j}(L) \equiv \mu_{\mathtt{I};j}(L) \mod \Delta_\mu(\mathtt{I};j), \qquad \Delta_\mu(\mathtt{I};j) = \gcd(\Gamma_\mu(\mathtt{I};j)), \tag{1-8}$$

where $\Gamma_\mu(\mathtt{I};j)$ is a certain subset of lower-order Milnor invariants; see [36]. Regarding $\overline{\mu}_{\mathtt{I};j}(L)$ as an element of $\mathbb{Z}_d = \{0, 1, \dots, d-1\}$, $d = \Delta_\mu(\mathtt{I};j)$ (or $\mathbb{Z}$, whenever $d = 0$), let us set

$$[\overline{\mu}_{\mathtt{I};j}(L)] := \begin{cases} \min(\overline{\mu}_{\mathtt{I};j}, d - \overline{\mu}_{\mathtt{I};j}) & \text{for } d > 0, \\ |\overline{\mu}_{\mathtt{I};j}| & \text{for } d = 0. \end{cases} \tag{1-9}$$

Our main result addresses Question A for general n–component links (deposing of the Brunnian assumption) as follows:

Theorem A *Let L be an n–component link $n \geq 2$ and $\overline{\mu}(L)$ one of its top Milnor linking numbers*; *then*

$$\mathsf{rop}(L)^{4/3} \geq \sqrt[3]{n}([\overline{\mu}(L)])^{1/(n-1)}, \quad \mathsf{Cr}(L) \geq \tfrac{1}{3}(n-1)([\overline{\mu}(L)])^{1/(n-1)}. \tag{1-10}$$

In the context of Question A, the estimate of Theorem A transforms, using (1-4), as

$$\tau(L)\left(\frac{11}{[4]n}\right)^{\frac{1}{3}} M^{-1/4(n-1)}.$$

Naturally, Question A can be asked for knots and links and lower bounds in terms of finite-type invariants in general. Such questions have been raised for instance by

Cantarella [8; 9], where the Bott–Taubes integrals [4] — see also Volić [43] — have been suggested as a tool for obtaining estimates.

Question B *Can we find estimates for ropelength of knots/links, in terms of their finite-type invariants?*

In the remaining part of this introduction let us sketch the basic idea behind our approach to Question B, which relies on the relation between the finite-type invariants and the crossing number.

Note that since $\mathsf{rop}(K)$ is scale invariant, it suffices to consider *unit thickness knots*, ie K together with the unit radius tube neighborhood (ie $r(K)=1$). In this setting, $\mathsf{rop}(K)$ just equals the *length* $\ell(K)$ of K. From now on we assume unit thickness, unless stated otherwise. In [5], Buck and Simon gave the following estimates for $\ell(K)$, in terms of the crossing number $\mathsf{Cr}(K)$ of K:

$$\ell(K) \geq \Big(\frac{4\pi}{11}\,\mathsf{Cr}(K)\Big)^{\frac{3}{4}}, \quad \ell(K) \geq 4\sqrt{\pi\,\mathsf{Cr}(K)}. \tag{1-11}$$

Clearly, the first estimate is better for knots with large crossing number, while the second one can be sharper for low crossing number knots (which manifests itself for instance in the case of the trefoil). Recall that $\mathsf{Cr}(K)$ is a minimal crossing number over all possible knot diagrams of K within the isotopy class of K. The estimates in (1-11) are a direct consequence of the ropelength bound for the *average crossing number*[1] $\mathsf{acr}(K)$ of K (proven in [5, Corollary 2.1]), ie

$$\ell(K)^{4/3} \geq \frac{4\pi}{11}\,\mathsf{acr}(K), \quad \ell(K)^2 \geq 16\pi\,\mathsf{acr}(K). \tag{1-12}$$

In Section 3, we obtain an analog of (1-11) for n–component links ($n \geq 2$) in terms of the *pairwise crossing number*[2] $\mathsf{PCr}(L)$,

$$\ell(L) \geq \frac{1}{\sqrt{n-1}}\big(\tfrac{3}{2}\,\mathsf{PCr}(L)\big)^{3/4}, \quad \ell(L) \geq \frac{n\sqrt{16\pi}}{\sqrt{n^2-1}}(\mathsf{PCr}(L))^{1/2}. \tag{1-13}$$

For low crossing number knots, the Buck and Simon bound (1-11) was further improved by Diao[3] [13]:

$$\ell(K) \geq \tfrac{1}{2}\big(d_0 + \sqrt{d_0^2 + 64\pi\,\mathsf{Cr}(K)}\big), \quad d_0 = 10 - 6(\pi + \sqrt{2}) \approx 17.334. \tag{1-14}$$

[1] That is, an average of the crossing numbers of diagrams of K over all projections of K; see (3-2).
[2] See (3-14) and Corollary I; generally $\mathsf{PCr}(L) \leq \mathsf{Cr}(L)$, as the individual components can be knotted.
[3] More precisely, $16\pi\,\mathsf{Cr}(K) \leq \ell(K)(\ell(K) - 17.334)$ [13].

On the other hand, there are well-known estimates for $\mathsf{Cr}(K)$ in terms of finite-type invariants of knots. For instance,

$$\tfrac{1}{4}\,\mathsf{Cr}(K)(\mathsf{Cr}(K)-1)+\tfrac{1}{24}\geq|c_2(K)|,\quad \tfrac{1}{8}(\mathsf{Cr}(K))^2\geq|c_2(K)|. \tag{1-15}$$

Lin and Wang [31] considered the second coefficient of the Conway polynomial $c_2(K)$ (ie the first nontrivial type 2 invariant of knots) and proved the first bound in (1-15). The second estimate of (1-15) can be found in Polyak and Viro's work [39]. Further, Willerton, in his thesis [44], obtained estimates for the "second", after $c_2(K)$, finite-type invariant $V_3(K)$ of type 3, as

$$\tfrac{1}{4}\,\mathsf{Cr}(K)(\mathsf{Cr}(K)-1)(\mathsf{Cr}(K)-2)\geq|V_3(K)|. \tag{1-16}$$

In the general setting, Bar-Natan [3] shows that if $V(K)$ is a type n invariant then $|V(K)|=O(\mathsf{Cr}(K)^n)$. All these results rely on the arrow diagrammatic formulas for Vassiliev invariants developed in the work of Goussarov, Polyak and Viro [22].

Clearly, combining (1-15) and (1-16) with (1-11) or (1-14) immediately yields lower bounds for ropelength in terms of a given Vassiliev invariant. One may take these considerations one step further and extend the above estimates to the case of the $2n^{\text{th}}$ coefficient of the Conway polynomial $c_{2n}(K)$, with the help of arrow diagram formulas for $c_{2n}(K)$, obtained recently in Chmutov, Duzhin and Mostovoy [11] and Chmutov, Khoury and Rossi [12]. In Section 2, we follow Polyak and Viro's argument of [39] to obtain:

Theorem B *Given a knot K, we have the crossing number estimate*

$$\mathsf{Cr}(K)\geq((2n)!\,|c_{2n}(K)|)^{1/2n}\geq\tfrac{2}{3}n\,|c_{2n}(K)|^{1/2n}. \tag{1-17}$$

Combining (1-17) with Diao's lower bound (1-14) one obtains:

Corollary C *For a unit thickness knot K,*

$$\ell(K)\geq\tfrac{1}{2}\Big(d_0+\big(d_0^2+\tfrac{128}{3}n\pi\,|c_{2n}(K)|^{1/2n}\big)^{1/2}\Big). \tag{1-18}$$

Recall that a somewhat different approach to ropelength estimates is presented in [10], where the authors introduce a cone surface technique, which, combined with the asymptotic crossing number, $\mathsf{Ac}(K)$, bound of Freedman and He [19] gives

$$\ell(K)\geq 2\pi+2\pi\sqrt{\mathsf{Ac}(K)},\quad \ell(K)\geq 2\pi+2\pi\sqrt{2g(K)-1}, \tag{1-19}$$

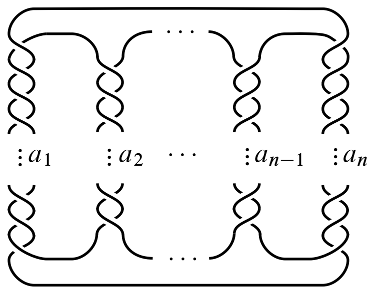

Figure 1: $P(a_1,\dots,a_n)$ pretzel knots.

where the second bound follows from the knot genus estimate $\mathsf{Ac}(K) \geq 2g(K)-1$ of [19].

When comparing estimates (1-19) and (1-18), in favor of (1-18), we may consider a family of *pretzel knots* $P(a_1,\dots,a_n)$, where a_i is the number of signed crossings in the i^{th} tangle of the diagram; see Figure 1. Additionally, for a diagram $P(a_1,\dots,a_n)$, to represent a knot one needs to assume either both n and all a_i are odd or one of the a_i is even; see Kawauchi [26].

Genera of selected subfamilies of pretzel knots are known, for instance Manchon [21, Theorem 13] implies

$$g(P(a,b,c)) = 1, \quad c_2(P(a,b,c)) = \tfrac{1}{4}(ab+ac+bc+1),$$

where a, b and c are odd integers with the same sign (for the value of $c_2(P(a,b,c))$; see the table in [21, page 390]). Concluding, the lower bound in (1-18) can be made arbitrarily large by letting $a,b,c \to +\infty$, while the lower bound in (1-19) stays constant for any values of a, b and c, under consideration. Yet another[4] example of a family of pretzel knots with constant genus one and arbitrarily large c_2–coefficient is

$$D(m,k) = P(m,\underbrace{\varepsilon,\dots,\varepsilon}_{|k|\ \text{times}}),$$

with $m>0,k$, where $\varepsilon = k/|k|$ is the sign of k (eg $D(3,-2) = P(3,-1,-1)$). For any such $D(m,k)$, we have $c_2(D(m,k)) = \frac{1}{4}mk$.

Remark D A natural question can be raised about the reverse situation: can we find a family of knots with constant c_{2n}–coefficient (or any finite-type invariant; see

[4] Out of a few such examples given in [21].

Remark L), but arbitrarily large genus? For instance, there exist knots with $c_2 = 0$ and nonzero genus (such as 8_2); in these cases (1-19) still provides a nontrivial lower bound.

The paper is structured as follows: Section 2 is devoted to a review of arrow polynomials for finite-type invariants and Kravchenko–Polyak tree invariants in particular; it also contains the proof of Theorem B. Section 3 contains information on the average overcrossing number for links and link ropelength estimates analogous to the ones obtained by Buck and Simon [5] (see (1-12)). The proof of Theorem A is presented in Section 4, together with final comments and remarks.

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The results of this paper grew out of considerations in Michaelides's doctoral thesis [34], where a weaker versions of the estimates (in the Borromean case) were obtained.

2 Arrow polynomials and finite-type invariants

Recall from [11] the *Gauss diagram* of a knot K is a way of representing signed overcrossings in a knot diagram, by arrows based on a circle (*Wilson loops* [2]) with signs encoding the sign of the crossings (see Figure 2, showing the 5_2 knot and its Gauss diagram). More precisely, the Gauss diagram G_K of a knot $K\colon S^1 \to \mathbb{R}^3$ is constructed by marking pairs of points in the domain S^1, endpoints of a corresponding arrow in G_K, which are mapped to crossings in a generic planar projection of K. The

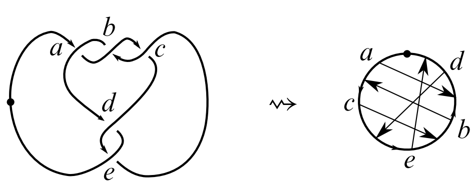


Figure 2: 5_2 knot and its Gauss diagram (all crossings are positive).

arrow always points from the under- to the over–crossing and the orientation of the circle S^1 in G_K agrees with the orientation of the knot.

Given a Gauss diagram G, the *arrow polynomials* of [22; 38] are defined simply as a signed count of selected subdiagrams in G. For instance, the second coefficient of the Conway polynomial $c_2(K)$ is given by the signed count of in G, denoted as

$$c_2(K) = \langle \,\cdot\,, G\rangle = \sum_{\phi\colon \,\cdot\, \to G} \operatorname{sign}(\phi), \quad \operatorname{sign}(\phi) = \prod_{\alpha\in \,\cdot\,} \operatorname{sign}(\phi(\alpha)), \tag{2-1}$$

where the sum is over all basepoint-preserving graph embeddings $\{\phi\}$ of into G, and the sign is a product of signs of corresponding arrows in $\phi(\,\cdot\,) \subset G$. For example, in the Gauss diagram of 5_2 knot in Figure 2, there are two possible embeddings of into the diagram. One matches the pair of arrows $\{a, d\}$ and another pair $\{c, d\}$; since all crossings are positive, we obtain $c_2(5_2) = 2$.

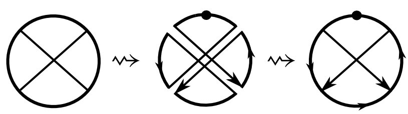
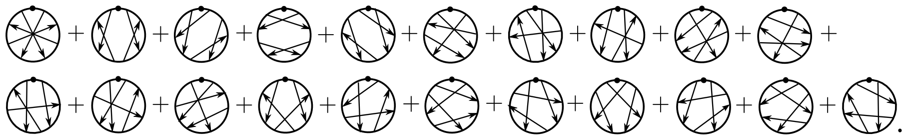


Figure 3: Turning a one-component chord diagram with a basepoint into an arrow diagram.

For other even coefficients of the Conway polynomial, the work in [12] provides the following recipe for their arrow polynomials. Given $n > 0$, consider any chord diagram D, on a single circle component with $2n$ chords, such as , and . A chord diagram D is said to be a k*–component diagram* if, after parallel doubling of each chord according to $\rightsquigarrow$, the resulting curve will have k components. For instance, $\rightsquigarrow$ is a 1–component diagram and $\rightsquigarrow$ is a 3–component diagram. For the coefficients c_{2n}, only one component diagram will be of interest and we turn a one-component chord diagram with a basepoint into an arrow diagram according to the following rule [12]:

> *Starting from the basepoint we move along the diagram with doubled chords. During this journey we pass both copies of each chord in opposite directions. Choose an arrow on each chord which corresponds to the direction of the first passage of the copies of the chord* (*see Figure 3 for the illustration*).

We call the arrow diagram obtained according to this method the *ascending arrow diagram* and denote by C_{2n} the sum of all based one-component ascending arrow diagrams with $2n$ arrows. For example, $C_2 = \otimes$ and C_4 is (see [12, page 777])

$C_4 =$ 

In [12], the authors show for $n \geq 1$ that the $c_{2n}(K)$ coefficient of the Conway polynomial of K equals

$$c_{2n}(K) = \langle C_{2n}, G_K \rangle. \tag{2-2}$$

Theorem B *Given a knot K, we have the crossing number estimate*

$$\mathsf{Cr}(K) \geq ((2n)!|c_{2n}(K)|)^{1/2n} \geq \tfrac{2}{3} n |c_{2n}(K)|^{1/2n}. \tag{2-3}$$

Proof Given K and its Gauss diagram G_K, let $X = \{1, 2, \ldots, \mathsf{cr}(K)\}$ index arrows of G_K (ie crossings of a diagram of K used to obtain G_K). For diagram term A_i in the sum $C_{2n} = \sum_i A_i$, an embedding $\phi\colon A_i \mapsto G_K$ covers a certain $2n$–element subset of crossings in X, which we denote by $X_\phi(i)$. Let $\mathcal{E}(i; G_K)$ be the set of all possible embeddings $\phi\colon A_i \mapsto G_K$, and

$$\mathcal{E}(G_K) = \bigsqcup_i \mathcal{E}(i; G_K).$$

Note that $X_\phi(i) \neq X_\xi(j)$ for $i \neq j$ and $X_\phi(i) \neq X_\xi(i)$ for $\phi \neq \xi$, thus for each i we have an injective map

$$F_i\colon \mathcal{E}(i; G_K) \mapsto \mathcal{P}_{2n}(X), \quad F_i(\phi) = X_\phi(i),$$

where $\mathcal{P}_{2n}(X) = \{2n\text{–element subsets of } X\}$. F_i extends in an obvious way to the whole disjoint union $\mathcal{E}(G_K)$, as $F\colon \mathcal{E}(G_K) \to \mathcal{P}_{2n}(X)$, $F = \bigsqcup_i F_i$, and remains injective. In turn, for every i we have

$$|\langle A_i, G_K \rangle| \leq \#\mathcal{E}(i; G_K)$$

and therefore

$$|\langle C_{2n}, G_K \rangle| \leq \#\mathcal{E}(G_K) < \#\mathcal{P}_{2n}(X) = \binom{\mathsf{cr}(K)}{2n}.$$

If $\mathsf{cr}(K) < 2n$ then the left-hand side vanishes. Since $\binom{\mathsf{cr}(K)}{2n} \leq \mathsf{cr}(K)^{2n}/(2n)!$, we obtain

$$|c_{2n}(K)| \leq \frac{\mathsf{cr}(K)^{2n}}{(2n)!} \implies ((2n)!|c_{2n}(K)|)^{1/2n} \leq \mathsf{cr}(K),$$

which gives the first part of (2-3). Using the upper lower bound for $m!$ (Stirling's approximation [1])

$$m! \geq \sqrt{2\pi} m^{m+1/2} e^{-m},$$

applying $e^{-1} \geq \frac{1}{3}$, $(\sqrt{2\pi})^{1/m} \geq 1$ and $(m^{m+1/2})^{1/m} \geq m(\sqrt{m})^{1/m} \geq m$ yields

$$(m!)^{1/m} \geq (\sqrt{2\pi}(m)^{m+1/2} e^{-m})^{1/m} \geq \tfrac{1}{3} m \tag{2-4}$$

for $m = 2n$, so one obtains the second part of (2-3). □

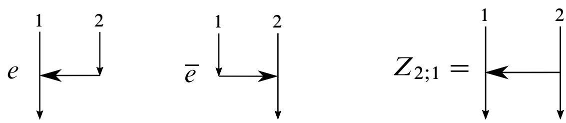


Figure 4: Elementary trees e and $\overline{e}$ and the $Z_{2;1}$ arrow polynomial.

Next, we turn to arrow polynomials for Milnor linking numbers. In [29], Kravchenko and Polyak introduced tree invariants of string links and established their relation to Milnor linking numbers via the skein relation of [37]. In the recent paper, the authors[5] [27] showed that the arrow polynomials of Kravchenko and Polyak, applied to Gauss diagrams of closed based links, yield certain $\overline{\mu}$–invariants (as defined in (1-8)). For the purpose of the proof of Theorem A, it suffices to give a recursive definition, provided in [27], for the arrow polynomial of $\overline{\mu}_{23\cdots n;1}(L)$ denoted by $Z_{n;1}$. Changing the convention, adopted for knots, we follow [29; 27] and use vertical segments (strings) oriented downwards in place of circles (Wilson loops) as components. The

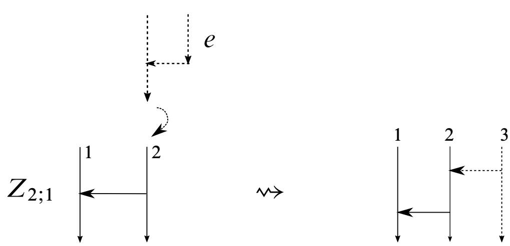


Figure 5: Obtaining a term in $Z_{3;1}$ via stacking e on the second component of $Z_{2;1}$, ie $Z_{2;1} \prec_2 e$.

polynomial $Z_{n;1}$ is obtained inductively from $Z_{n-1;1} = \sum_k \pm A_k$ by expanding each term A_k of $Z_{n;1}$ through stacking elementary tree diagrams e and $\overline{e}$, shown in

[5] Consult [28] for a related result.

Figure 4; the sign of a resulting term is determined accordingly. The stacking operation is denoted by $\prec_i$, where $i = 1, \dots, n$ tells which component is used for stacking. Figure 5 shows $Z_{2;1} \prec_2 e$. The inductive procedure is defined as follows:

(i) $Z_{2;1}$ is shown in Figure 4 (right).

(ii) For each term A_k in $Z_{n-1;1}$ produce terms in $Z_{n;1}$ by stacking[6] e and $\overline{e}$ on each component, ie $A_k \prec_i e$ for $i = 1, \dots, n$ and $A_k \prec_i \overline{e}$ for $i = 2, \dots, n$; see Figure 5. Eliminate isomorphic (duplicate) diagrams.

(iii) The sign of each term in $Z_{n;1}$ equals to $(-1)^q$, where q is the number of arrows pointing to the right.

As an example consider $Z_{3;1}$; we begin with the initial tree $Z_{2;1}$, and expand by stacking e and $\overline{e}$ on the strings of $Z_{2;1}$; this is shown in Figure 6, and we avoid stacking $\overline{e}$ on the first component (called the *trunk* [27]). Thus, $Z_{3;1}$ is obtained as $A + B - C$, where $A = Z_{2;1} \prec_2 e$, $B = Z_{2;1} \prec_1 e$ and $C = Z_{2;1} \prec_2 \overline{e}$.

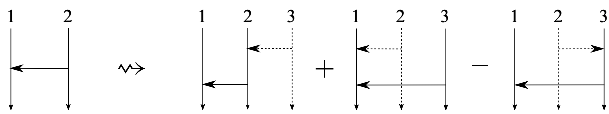


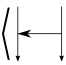
Figure 6: $Z_{3;1} = A + B - C$ obtained from $Z_{2;1}$ via (i)–(iii).

Given $Z_{n;1}$, the main result of [27] (see also [28] for a related result) yields the formula

$$\overline{\mu}_{n;1}(L) \equiv \langle Z_{n;1}, G_L \rangle \mod \Delta_\mu(n;1), \tag{2-5}$$

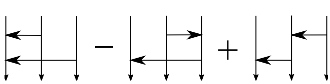
where $\overline{\mu}_{n;1}(L) := \overline{\mu}_{2\cdots n;1}(L)$, G_L is a Gauss diagram of an n–component link L and the indeterminacy $\Delta_\mu(n;1)$ is as defined in (1-8). Recall that $\langle Z_{n;1}, G_L \rangle = \sum_k \pm \langle A_k, G_L \rangle$, where $Z_{n;1} = \sum_k \pm A_k$ and $\langle A_k, G_L \rangle = \sum_{\phi\colon A_k \to G_L} \operatorname{sign}(\phi)$ is a signed count of subdiagrams isomorphic to A_k in G_L.

For $n = 2$, we obtain the usual linking number

(2-6) $\overline{\mu}_{2;1}(L) = \langle Z_{2;1}, G_L \rangle = \Big\langle$  $, G_L \Big\rangle.$

For $n = 3$ and $n = 4$ the arrow polynomials can be obtained following the stacking procedure

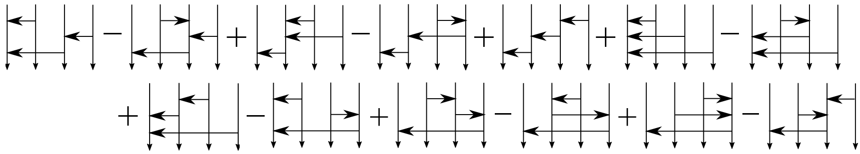
$$\overline{\mu}_{3;1}(L) = \langle Z_{3;1}, G_L \rangle \mod \gcd\{\overline{\mu}_{2;1}(L), \overline{\mu}_{3;1}(L), \overline{\mu}_{3;2}(L)\},$$

$Z_{3;1} =$  ,

[6] Note that $\overline{e}$ is not allowed to be stacked on the first component.

and

$$\bar{\mu}_{4;1}(L) = \langle Z_{4;1}, G_L \rangle \mod \Delta_\mu(4;1),$$

$Z_{4;1} =$ 

Given a formula for $\bar{\mu}_{n;1}(L) = \bar{\mu}_{23\cdots n;1}(L)$, all remaining $\bar{\mu}$–invariants with distinct indices can be obtained from the permutation identity (for $\sigma \in \Sigma(1,\dots,n)$)

$$\text{(2-7)} \quad \bar{\mu}_{\sigma(2)\sigma(3)\cdots\sigma(n);\sigma(1)}(L) = \bar{\mu}_{23\cdots n;1}(\sigma(L)), \quad \sigma(L) = (L_{\sigma(1)}, L_{\sigma(2)}, \dots, L_{\sigma(n)}).$$

By (2-5), (2-7) and (1-8) we have

$$\text{(2-8)} \quad \bar{\mu}_{\sigma(2)\sigma(3)\cdots\sigma(n);\sigma(1)}(L) = \langle \sigma(Z_{n;1}), G_L \rangle \mod \Delta_\mu(\sigma(2)\sigma(3)\cdots\sigma(n);\sigma(1)),$$

where $\sigma(Z_{n;1})$ is the arrow polynomial obtained from $Z_{n;1}$ by permuting the strings according to σ.

Remark E One of the properties of $\bar{\mu}$–invariants is their cyclic symmetry [36, (21)], ie given a cyclic permutation ρ, we have

$$\bar{\mu}_{\rho(2)\rho(3)\cdots\rho(n);\rho(1)}(L) = \bar{\mu}_{23\cdots n;1}(L).$$

3 Overcrossing number of links

We will denote by D_L a regular diagram of a link L, and by $D_L(v)$ the diagram obtained by the projection of L onto the plane normal to a vector[7] $v \in S^2$. For a pair of components L_i and L_j in L, define the *overcrossing number* in the diagram and the *pairwise crossing number* of components L_i and L_j in D_L, ie

$$\begin{aligned} \mathsf{ov}_{i,j}(D_L) &= \{\text{number of times } L_i \text{ overpasses } L_j \text{ in } D_L\}, \\ \text{(3-1)} \qquad \mathsf{cr}_{i,j}(D_L) &= \{\text{number of times } L_i \text{ overpasses and underpasses } L_j \text{ in } D_L\} \\ &= \mathsf{ov}_{i,j}(D_L) + \mathsf{ov}_{j,i}(D_L) = \mathsf{cr}_{j,i}(D_L). \end{aligned}$$

In the following, we also use the *average overcrossing number* and *average pairwise crossing number* of components L_i and L_j in L, defined as an average over all $D_L(v)$

[7] Unless otherwise stated we assume that v is generic and thus $D_L(v)$ is a regular diagram.

for $v \in S^2$, ie

(3-2)
$$\mathsf{aov}_{i,j}(L) = \frac{1}{4\pi}\int_{S^2} \mathsf{ov}_{i,j}(v)\,dv,$$
$$\mathsf{acr}_{i,j}(L) = \frac{1}{4\pi}\int_{S^2} \mathsf{cr}_{i,j}(v)\,dv = 2\,\mathsf{aov}_{i,j}(L).$$

The following result is based on the work in [8; 9; 5]; the idea of using the rearrangement inequality comes from [8; 9].

Lemma F *Given a unit thickness link L and any 2–component sublink (L_i, L_j),*

(3-3)
$$\min(\ell_i \ell_j^{1/3}, \ell_j \ell_i^{1/3}) \geq 3\,\mathsf{aov}_{i,j}(L), \quad \ell_i \ell_j \geq 16\pi\,\mathsf{aov}_{i,j}(L)$$

for $\ell_i = \ell(L_i)$ and $\ell_j = \ell(L_j)$ the lengths of L_i and L_j, respectively.

Proof Consider the Gauss map of $L_i = L_i(s)$ and $L_j = L_j(t)$,

$$F_{i,j}\colon S^1 \times S^1 \mapsto \mathrm{Conf}_2(\mathbb{R}^3) \mapsto S^2, \quad F_{i,j}(s,t) = \frac{L_i(s) - L_j(t)}{\|L_i(s) - L_j(t)\|}.$$

If $v \in S^2$ is a regular value of $F_{i,j}$ (which happens for the set of full measure on S^2) then

$$\mathsf{ov}_{i,j}(v) = \#\{\text{points in } F_{i,j}^{-1}(v)\},$$

ie $\mathsf{ov}_{i,j}(v)$ stands for number of times the i–component of L passes over the j–component in the projection of L onto the plane in $\mathbb{R}^3$ normal to v. As a direct consequence of Federer's coarea formula [18] (see eg [34] for a proof),

(3-4)
$$\int_{L_i \times L_j} |F_{i,j}^{*}\omega| = \frac{1}{4\pi}\int_{S^1\times S^1} \frac{|\langle L_i(s) - L_j(t), L_i'(s), L_j'(t)\rangle|}{\|L_i(s) - L_j(t)\|^3}\,ds\,dt$$
$$= \frac{1}{4\pi}\int_{S^2} \mathsf{ov}_{i,j}(v)\,dv,$$

where $\omega = \frac{1}{4\pi}(x\,dy \wedge dz - y\,dx \wedge dz + z\,dx \wedge dy)$ is the normalized area form on the unit sphere in $\mathbb{R}^3$ and

(3-5)
$$\langle v, w, z\rangle := \det(v, w, z) \quad \text{for } v, w, z \in \mathbb{R}^3.$$

Assuming the arc-length parametrization by $s \in [0, \ell_i]$ and $t \in [0, \ell_j]$ of the components, we have $\|L_i'(s)\| = \|L_j'(t)\| = 1$ and therefore

(3-6)
$$\left|\frac{\langle L_i(s) - L_j(t), L_i'(s), L_j'(t)\rangle}{\|L_i(s) - L_j(t)\|^3}\right| \leq \frac{1}{\|L_i(s) - L_j(t)\|^2}.$$

Combining equations (3-4) and (3-6) yields

$$\int_0^{\ell_j}\int_0^{\ell_i}\frac{1}{\|L_i(s)-L_j(t)\|^2}\,ds\,dt=\int_0^{\ell_j} I_i(L_j(t))\,dt, \tag{3-7}$$

where

$$I_i(p)=\int_0^{\ell_i}\frac{1}{\|L_i(s)-p\|^2}ds=\int_0^{\ell_i}\frac{1}{r(s)^2}ds,\quad r(s)=\|L_i(s)-p\|,$$

is often called the *illumination* of L_i from the point $p\in\mathbb{R}^3$; see [5]. Following the approach of [5; 8; 9], we estimate $I_i(t)=I_i(p)$ for $p=L_j(t)$. Denote by $B_a(p)$ the ball at $p=L_j(t)$ of radius a, and $s(z)$ the length of a portion of L_i within the spherical shell $\mathrm{Sh}(z)=B_z(p)\setminus B_1(p)$ for $z\geq 1$. Note that, because the distance between L_i and L_j is at least 2, the unit thickness tube about L_i is contained entirely in $\mathrm{Sh}(z)$ for big enough z. Clearly, $s(z)$ is nondecreasing. Since the volume of a unit thickness tube of length a is πa, comparing the volumes we obtain

$$\pi s(z)\leq \mathrm{Vol}(\mathrm{Sh}(z))=\tfrac{4}{3}\pi(z^3-1^3)\qquad\text{and}\qquad s(z)\leq\tfrac{4}{3}z^3\quad\text{for } z\geq 1. \tag{3-8}$$

Next, using the monotone rearrangement $(1/r^2)^*$ of $1/r^2$ (Remark G),

$$\left(\frac{1}{r^2}\right)^*(s)\leq\left(\tfrac{4}{3}\right)^{2/3}s^{-2/3}, \tag{3-9}$$

and, by the monotone rearrangement inequality [30],

$$\begin{aligned}I_i(p)=\int_0^{\ell_i}\frac{1}{r^2(s)}\,ds&\leq\int_0^{\ell_i}\left(\frac{1}{r^2}\right)^*(s)\,ds\\&\leq\int_0^{\ell_i}\left(\tfrac{4}{3}\right)^{2/3}s^{-2/3}\,ds=3\left(\tfrac{4}{3}\right)^{2/3}\ell_i^{1/3}.\end{aligned} \tag{3-10}$$

Integrating (3-10) with respect to the t–parameter, we obtain

$$\mathsf{aov}(L)\leq\frac{1}{4\pi}\int_0^{\ell_j}\int_0^{\ell_i}\frac{1}{\|L_i(s)-L_j(t)\|^2}\,ds\,dt\leq 3\left(\tfrac{4}{3}\right)^{2/3}\tfrac{1}{4\pi}\ell_j\ell_i^{\frac{1}{3}}<\tfrac{1}{3}\ell_j\ell_i^{1/3}.$$

Since the argument works for any choice of i and j, the estimates in (3-3) are proven. The second estimate in (3-3) follows immediately from $1/\|L_i(s)-L_j(t)\|^2\leq\frac{1}{4}$. □

Remark G Recall that for a nonnegative real-valued function f (on $\mathbb{R}^n$), vanishing at infinity, the *rearrangement f^* of f* is given by

$$f^*(x)=\int_0^{\infty}\chi^*_{\{f>u\}}(x)\,du,$$

where $\chi^*_{\{f>u\}}(x) = \chi_{B_\rho}(x)$ is the characteristic function of the ball B_ρ centered at the origin, determined by the volume condition $\mathrm{Vol}(B_\rho) = \mathrm{Vol}(\{x \mid f(x) > u\})$; see [30, page 80] for further properties of the rearrangements. In particular, the rearrangement inequality states [30, page 82] $\int_{\mathbb{R}^n} f(x)\,dx \le \int_{\mathbb{R}^n} f^*(x)\,dx$. For one-variable functions, we may use the interval $[0,\rho]$ in place of the ball B_ρ; then f^* is a decreasing function on $[0,+\infty)$. Specifically, for $f(s) = 1/r^2(s) = 1/\|L_i(s)-p\|^2$, we have

$$\left(\frac{1}{r^2}\right) * (s) = u \quad \text{for length}\left(\{x \mid u < \frac{1}{r^2(x)} \le 1\}\right) = s,$$

where $\mathrm{length}(\{x \mid u < 1/r^2(x) \le 1\})$ stands for the length of the portion of L_i satisfying the given condition. Further, by the definition of $s(z)$, from the previous paragraph and (3-8), we obtain

$$\begin{aligned} s &= \mathrm{length}\left(\left\{x \;\middle|\; \frac{1}{r^2(x)} > u\right\}\right) = \mathrm{length}\left(\left\{x \;\middle|\; 1 \le r(x) < \frac{1}{\sqrt{u}}\right\}\right) \\ &= s\left(\frac{1}{\sqrt{u}}\right) \le \frac{4}{3}\left(\frac{1}{\sqrt{u}}\right)^3, \end{aligned}$$

and (3-9) as a result.

From the Gauss linking integral (3-4),

$$|\mathrm{Lk}(L_i, L_j)| \le \mathrm{aov}_{i,j}(L),$$

thus we immediately recover the result of [15] (but with a specific constant),

$$3|\mathrm{Lk}(L_i,L_j)| \le \min(\ell_i \ell_j^{1/3}, \ell_j \ell_i^{1/3}), \quad 16\pi|\mathrm{Lk}(L_i,L_j)| \le \ell_i \ell_j. \tag{3-11}$$

Summing up over all possible pairs i and j and using the symmetry of the linking number, we have

$$6\sum_{i<j}|\mathrm{Lk}(L_i,L_j)| = 3\sum_{i\ne j}|\mathrm{Lk}(L_i,L_j)| \le \sum_{i\ne j}\ell_i\ell_j^{1/3} = \left(\sum_i \ell_i\right)\left(\sum_j \ell_j^{1/3}\right) - \sum_i \ell_i^{4/3}.$$

From Jensen's inequality [30], we know that

$$\frac{1}{n}\left(\sum_i \ell_i^{1/3}\right) \le \left(\frac{1}{n}\sum_i \ell_i\right)^{\frac{1}{3}} \quad \text{and} \quad \frac{1}{n}\left(\sum_i \ell_i^{4/3}\right) \ge \left(\frac{1}{n}\sum_i \ell_i\right)^{\frac{4}{3}},$$

therefore

$$\Big(\sum_i \ell_i\Big)\Big(\sum_j \ell_j^{1/3}\Big)-\sum_i \ell_i^{4/3} \leq n^{2/3}\,\mathsf{rop}(L)\,\mathsf{rop}(L)^{1/3}-n^{-1/3}\,\mathsf{rop}(L)^{4/3} = \frac{n-1}{n^{1/3}}\,\mathsf{rop}(L)^{4/3}. \tag{3-12}$$

Analogously, using the second estimate in (3-11) and Jensen's inequality yields

$$32\pi \sum_{i<j} |\mathsf{Lk}(L_i,L_j)| = 16\pi \sum_{i\neq j} |\mathsf{Lk}(L_i,L_j)| \leq \sum_{i\neq j} \ell_i\ell_j \leq \Big(1-\frac{1}{n^2}\Big)\Big(\sum_i \ell_i\Big)^2.$$

Corollary H *Let L be an n–component link ($n \geq 2$); then*

$$\mathsf{rop}(L)^{4/3} \geq \frac{6n^{1/3}}{n-1}\sum_{i<j}|\mathsf{Lk}(L_i,L_j)|, \quad \mathsf{rop}(L)^2 \geq \frac{32\pi n^2}{n^2-1}\sum_{i<j}|\mathsf{Lk}(L_i,L_j)|. \tag{3-13}$$

In terms of growth of the pairwise linking numbers $|\mathsf{Lk}(L_i,L_j)|$, for a fixed n, the above estimate performs better than the one in (1-5). One may also replace $\sum_{i<j}|\mathsf{Lk}(L_i,L_j)|$ with the isotopy invariant

$$\mathsf{PCr}(L) = \min_{D_L}\Big(\sum_{i\neq j} \mathsf{cr}_{i,j}(D_L)\Big) \tag{3-14}$$

(satisfying $\mathsf{PCr}(L) \leq \mathsf{Cr}(L)$), which we call the *pairwise crossing number* of L. This conclusion can be considered as an analog of the Buck and Simon estimate (1-11) for knots.

Corollary I *Let L be an n–component link ($n \geq 2$) and $\mathsf{PCr}(L)$ its pairwise crossing number; then*

$$\mathsf{rop}(L)^{4/3} \geq \frac{3n^{1/3}}{n-1}\,\mathsf{PCr}(L), \quad \mathsf{rop}(L)^2 \geq \frac{16\pi n^2}{n^2-1}\,\mathsf{PCr}(L). \tag{3-15}$$

4 Proof of Theorem A

The following auxiliary lemma will be useful:

Lemma J *Given nonnegative numbers $a_1,\dots,a_N$, we have, for $k \geq 2$,*

$$\sum_{1\leq i_1<i_2<\cdots<i_k\leq N} a_{i_1}a_{i_2}\cdots a_{i_k} \leq \frac{1}{N^k}\binom{N}{k}\Big(\sum_{i=1}^N a_i\Big)^k. \tag{4-1}$$

Proof It suffices to observe that for $a_i \geq 0$ the ratio

$$\frac{\sum_{1\leq i_1<i_2<\cdots<i_k\leq N} a_{i_1}a_{i_2}\cdots a_{i_k}}{\left(\sum_{i=1}^N a_i\right)^k}$$

achieves its maximum for $a_1 = a_2 = \cdots = a_N$. □

Recall from (1-9) that $\bar{\mu}_{n;1} := \bar{\mu}_{23\cdots n;1}$, and

(4-2) $$[\bar{\mu}_{n;1}(L)] := \begin{cases} \min(\bar{\mu}_{n;1}(L), d - \bar{\mu}_{n;1}(L)) & \text{for } d > 0, \\ |\bar{\mu}_{n;1}(L)| & \text{for } d = 0, \end{cases} \qquad d = \Delta_\mu(n;1).$$

For convenience, recall the statement of Theorem A:

Theorem A *Let L be an n–component link of unit thickness and $\bar{\mu}(L)$ one of its top Milnor linking numbers; then*

(4-3) $$\ell(L) \geq \sqrt[4]{n}\left(\sqrt[n-1]{[\bar{\mu}(L)]}\right)^{3/4}, \quad \mathsf{Cr}(L) \geq \tfrac{1}{3}(n-1)\sqrt[n-1]{[\bar{\mu}(L)]}.$$

Proof Let G_L be a Gauss diagram of L obtained from a regular link diagram D_L. Consider any term A of the arrow polynomial $Z_{n;1}$ and index the arrows of A by (i_k, j_k) for $k = 1, \ldots, n-1$ in such a way that i_k is the arrowhead and j_k is the arrowtail; we have the obvious estimate

(4-4) $$|\langle A, G_L\rangle| \leq \prod_{k=1}^{n-1} \mathsf{ov}_{i_k,j_k}(D_L) \leq \prod_{k=1}^{n-1} \mathsf{cr}_{i_k,j_k}(D_L).$$

Let $N = \binom{n}{2}$; since every term (a tree diagram) of $Z_{n;1}$ is uniquely determined by its arrows indexed by string components, $\binom{N}{n-1}$ gives an upper bound for the number of terms in $Z_{n;1}$. Using Lemma J, with $k = n-1$, N as above and $a_k = \mathsf{cr}_{i_k,j_k}(D_L)$, $k = 1, \ldots, N$, one obtains, from (4-4),

(4-5) $$|\langle Z_{n;1}, G_L\rangle| \leq \frac{1}{N^{n-1}}\binom{N}{n-1}\left(\sum_{i<j} \mathsf{cr}_{i,j}(D_L)\right)^{n-1}.$$

Remark K The estimate (4-5) is valid for any arrow polynomial in place of $Z_{n;1}$ which has arrows based on different components and no parallel arrows on a given component.

By (2-5), we can find $k \in \mathbb{Z}$ such that $\langle Z_{n;1}, G_L\rangle = \bar{\mu}_{n;1} + kd$. Since

$$[\bar{\mu}_{n;1}(D_L)] \leq |\bar{\mu}_{n;1}(D_L) + kd| = |\langle Z_{n;1}, G_L\rangle| \quad \text{for all } k \in \mathbb{Z},$$

replacing D_L with a diagram obtained by projection of L in a generic direction $v \in S^2$, we rewrite the estimate (4-5) as

$$\alpha_n \sqrt[n-1]{[\bar{\mu}_{n;1}(D_L(v))]} \le \sum_{i<j} \mathsf{cr}_{i,j}(v), \quad \alpha_n = \left(\frac{1}{N^{n-1}}\binom{N}{n-1}\right)^{\frac{-1}{n-1}}. \tag{4-6}$$

Integrating over the sphere of directions and using invariance[8] of $[\bar{\mu}_{n;1}]$ yields

$$4\pi\alpha_n \sqrt[n-1]{[\bar{\mu}_{n;1}(L)]} \le \sum_{i<j} \int_{S^2} \mathsf{cr}_{i,j}(v)\, dv.$$

By Lemma F, we obtain

$$\begin{aligned}\alpha_n \sqrt[n-1]{[\bar{\mu}_{n;1}(L)]} &\le \sum_{i<j} \mathsf{acr}_{i,j}(L) = 2\sum_{i<j} \mathsf{aov}_{i,j}(L) \le 2\sum_{i<j} \tfrac{1}{3}\min(\ell_i \ell_j^{1/3}, \ell_j \ell_i^{1/3}) \\ &\le \frac{1}{3}\sum_{i\neq j} \ell_i \ell_j^{1/3},\end{aligned}$$

since $\sum_{i<j} 2\min(\ell_i\ell_j^{1/3}, \ell_j\ell_i^{1/3}) \le \sum_{i\neq j} \ell_i\ell_j^{1/3}$. As in the derivation of (3-12) (see Corollary H), by Jensen's inequality,

$$\mathsf{rop}(L)^{4/3} \ge \frac{3n^{1/3}\alpha_n}{n-1} \sqrt[n-1]{[\bar{\mu}_{n;1}(L)]}. \tag{4-7}$$

Now, let us estimate the constant α_n. Note that

$$\frac{N^{n-1}}{\binom{N}{n-1}} = \frac{N^{n-1}}{N(N-1)\cdots(N-(n-1)+1)}(n-1)! \ge (n-1)!.$$

Again, by Stirling's approximation (letting $m = n-1$ in (2-4)) we obtain, for $n \ge 2$,

$$\alpha_n \ge ((n-1)!)^{1/(n-1)} \ge \tfrac{1}{3}(n-1); \tag{4-8}$$

thus, (4-7) can be simplified to

$$\mathsf{rop}(L)^{4/3} \ge \sqrt[3]{n} \sqrt[n-1]{[\bar{\mu}_{n;1}(L)]}, \tag{4-9}$$

as claimed in the first inequality of (4-3). For a minimal diagram $D_L^{\min}$ of L,

$$\mathsf{Cr}(L) \ge \sum_{i<j} \mathsf{cr}_{i,j}(D_L^{\min});$$

[8] Both $\bar{\mu}_{n;1}$ and d are isotopy invariants.

thus the second inequality of (4-3) is an immediate consequence of (4-6) (with $D_L(v)$ replaced by $D_L^{\min}$) and (4-8). Using the permutation identity (2-7) and the fact that $\mathsf{rop}(\sigma(L)) = \mathsf{rop}(L)$ for any $\sigma \in \Sigma(1,\dots,n)$, we may replace $\bar{\mu}_{n;1}(L)$ with any other[9] top $\bar{\mu}$–invariant of L. □

In the case of almost trivial (Borromean) links, $d = 0$, and we may slightly improve the estimate in (4-5) of the above proof, by using the cyclic symmetry of $\bar{\mu}$–invariants noted in Remark E. We have, in particular,

$$n\bar{\mu}_{23\cdots n;1}(L) = \sum_{\rho,\rho \text{ is cyclic}} \bar{\mu}_{\rho(2)\rho(3)\cdots\rho(n);\rho(1)}(L) = \sum_{\rho,\rho \text{ is cyclic}} \langle \rho(Z_{n;1}), G_L\rangle. \tag{4-10}$$

Since cyclic permutations applied to the terms of $Z_{n;1}$ produce distinct arrow diagrams,[10] by Remark K, we obtain the bound

$$n|\bar{\mu}_{n;1}(L)| \leq \sum_{\rho,\rho \text{ is cyclic}} |\langle \rho(Z_{n;1}), G_L\rangle| \leq \frac{1}{N^{n-1}}\binom{N}{n-1}\Big(\sum_{i<j} \mathsf{cr}_{i,j}(D_L)\Big)^{n-1}. \tag{4-11}$$

Disregarding Stirling's approximation, we have

$$\mathsf{rop}(L)^{4/3} \geq \frac{3\sqrt[3]{n}\,\tilde{\alpha}_n}{(n-1)} \sqrt[n-1]{|\bar{\mu}_{n;1}(L)|}, \quad \tilde{\alpha}_n = \bigg(\frac{1}{nN^{n-1}}\binom{N}{n-1}\bigg)^{\frac{-1}{n-1}}, \tag{4-12}$$

or, using the second bound in (3-3),

$$\mathsf{rop}(L)^2 \geq 4^3\pi\tilde{\alpha}_n\Big(\frac{n^2}{n^2-1}\Big) \sqrt[n-1]{|\bar{\mu}_{n;1}(L)|}.$$

In particular, for $n = 3$, we have $N = 3$ and $\tilde{\alpha}_3 = 3$ and the estimates read

$$\mathsf{rop}(L) \geq (5\sqrt[3]{3}\sqrt{|\bar{\mu}_{23;1}(L)|}\,)^{3/4}, \quad \mathsf{rop}(L) \geq 6\sqrt{6\pi}\,\sqrt[4]{|\bar{\mu}_{23;1}(L)|}. \tag{4-13}$$

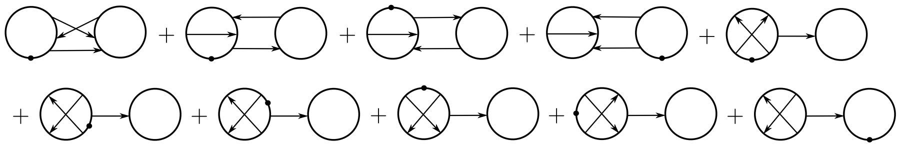
Since $6\sqrt{6\pi} \approx 26.049$, the second estimate is better for Borromean rings ($\mu_{23;1} = 1$) and improves the linking number bound of (1-5), $6\pi \approx 18.85$, but falls short of the genus bound (1-6), $12\pi \approx 37.7$. Numerical simulations suggest that the ropelength of Borromean rings is ≈ 58.05 [10; 7].

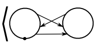
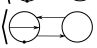
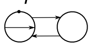
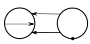
Remark L This methodology can be easily extended to other families of finite-type invariants of knots and links. For illustration, let us consider the third coefficient of the

[9] There are $(n-2)!$ different top Milnor linking numbers [35].

[10] Since the trunk of a tree diagram is unique; see [29; 27].

Conway polynomial, ie $c_3(L)$ of a two-component link L. The arrow polynomial C_3 of $c_3(L)$ is [12, page 779]

$C_3 =$ 

Let G_L be the Gauss diagram obtained from a regular link diagram D_L, and D_{L_k} the subdiagram of the k^{th} component of L for $k = 1, 2$. The absolute value of the first term $\langle$$, G_L\rangle$ of $\langle C_3, G_L\rangle$ does not exceed $\binom{\mathrm{cr}_{1,2}(D_L)}{3}$, the absolute value of the sum $\langle$ $+$  $+$ $, G_L\rangle$ does not exceed $\mathrm{cr}(D_{L_1})\binom{\mathrm{cr}_{1,2}(D_L)}{2}$, and, for the remaining terms, a bound is $\binom{\mathrm{cr}(D_{L_1})}{2}\,\mathrm{cr}_{1,2}(D_L)$. Therefore, a rough upper bound for $|\langle C_3, G_L\rangle|$ can be written as

$$|\langle C_3, G_L\rangle| \leq (\mathrm{cr}_{1,2}(D_L) + \mathrm{cr}(D_{L_1}))^3.$$

Similarly, as in (4-6), replacing D_L with $D_L(v)$ and integrating over the sphere of directions we obtain

$$|c_3(L)|^{1/3} \leq \mathrm{acr}_{1,2}(L) + \mathrm{acr}(L_1).$$

For a unit thickness link L, (1-12) and (3-3) give

$$\mathrm{acr}_{1,2}(L) + \mathrm{acr}(L_1) \leq \tfrac{1}{3}\ell_1^{1/3}\ell_2 + \tfrac{1}{3}\ell_2^{1/3}\ell_1 + \tfrac{4}{11}\ell_1^{1/3}\ell_1,$$
$$\mathrm{acr}_{1,2}(L) + \mathrm{acr}(L_1) \leq \tfrac{1}{16\pi}\ell_1^2 + \tfrac{1}{8\pi}\ell_1\ell_2.$$

Thus, for some constants $\alpha, \beta > 0$, we have

$$\ell(L)^2 \geq A|c_3(L)|^{1/3}, \quad \ell(L)^{4/3} \geq B|c_3(L)|^{1/3}.$$

In general, given a finite type-n invariant $V_n(L)$ and a unit thickness m–link L, we may expect constants $\alpha_{m,n}$ and $\beta_{m,n}$ such that

$$\ell(L)^2 \geq \alpha_{m,n}|V_n(L)|^{1/n}, \quad \ell(L)^{4/3} \geq \beta_{m,n}|V_n(L)|^{1/n}.$$

References

[1] **M Abramowitz**, **I A Stegun**, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series 55, US Government Printing Office, Washington, DC (1964) MR

[2] **D Bar-Natan**, *Perturbative aspects of the Chern–Simons topological quantum field theory*, PhD thesis, Princeton University (1991) MR Available at `https://search.proquest.com/docview/303979053`

[3] **D Bar-Natan**, *Polynomial invariants are polynomial*, Math. Res. Lett. 2 (1995) 239–246 MR

[4] **R Bott**, **C Taubes**, *On the self-linking of knots*, J. Math. Phys. 35 (1994) 5247–5287 MR

[5] **G Buck**, **J Simon**, *Thickness and crossing number of knots*, Topology Appl. 91 (1999) 245–257 MR

[6] **G R Buck**, **J K Simon**, *Total curvature and packing of knots*, Topology Appl. 154 (2007) 192–204 MR

[7] **R V Buniy**, **J Cantarella**, **T W Kephart**, **E J Rawdon**, *Tight knot spectrum in QCD*, Phys. Rev. D 89 (2014) art. id. 054513

[8] **J Cantarella**, *Introduction to geometric knot theory, I: Knot invariants defined by minimizing curve invariants*, talk slides, ICTP Knot Theory Summer School (2009) Available at http://indico.ictp.it/event/a08157/session/102/contribution/56

[9] **J Cantarella**, *Introduction to geometric knot theory, II: Ropelength and tight knots*, talk slides, ICTP Knot Theory Summer School (2009) Available at http://indico.ictp.it/event/a08157/session/104/contribution/57

[10] **J Cantarella**, **R B Kusner**, **J M Sullivan**, *On the minimum ropelength of knots and links*, Invent. Math. 150 (2002) 257–286 MR

[11] **S Chmutov**, **S Duzhin**, **J Mostovoy**, *Introduction to Vassiliev knot invariants*, Cambridge Univ. Press (2012) MR

[12] **S Chmutov**, **M C Khoury**, **A Rossi**, *Polyak-viro formulas for coefficients of the Conway polynomial*, J. Knot Theory Ramifications 18 (2009) 773–783 MR

[13] **Y Diao**, *The lower bounds of the lengths of thick knots*, J. Knot Theory Ramifications 12 (2003) 1–16 MR

[14] **Y Diao**, **C Ernst**, *Total curvature, ropelength and crossing number of thick knots*, Math. Proc. Cambridge Philos. Soc. 143 (2007) 41–55 MR

[15] **Y Diao**, **C Ernst**, **E J Janse Van Rensburg**, *Upper bounds on linking numbers of thick links*, J. Knot Theory Ramifications 11 (2002) 199–210 MR

[16] **Y Diao**, **C Ernst**, **E J Janse van Rensburg**, *Thicknesses of knots*, Math. Proc. Cambridge Philos. Soc. 126 (1999) 293–310 MR

[17] **C Ernst**, **A Por**, *Average crossing number, total curvature and ropelength of thick knots*, J. Knot Theory Ramifications 21 (2012) art. id. 1250028 MR

[18] **H Federer**, *Geometric measure theory*, Grundl. Math. Wissen. 153, Springer (1969) MR

[19] **M H Freedman**, **Z-X He**, *Divergence-free fields: energy and asymptotic crossing number*, Ann. of Math. 134 (1991) 189–229 MR

[20] **M Freedman**, **V Krushkal**, *Geometric complexity of embeddings in* $\mathbb{R}^d$, Geom. Funct. Anal. 24 (2014) 1406–1430 MR

[21] **P M González Manchón**, *Homogeneous links and the Seifert matrix*, Pacific J. Math. 255 (2012) 373–392 MR

[22] **M Goussarov**, **M Polyak**, **O Viro**, *Finite-type invariants of classical and virtual knots*, Topology 39 (2000) 1045–1068 MR

[23] **A Gray**, *Tubes*, 2nd edition, Progress in Mathematics 221, Birkhäuser, Basel (2004) MR

[24] **M Gromov**, *Filling Riemannian manifolds*, J. Differential Geom. 18 (1983) 1–147 MR

[25] **M Gromov**, **L Guth**, *Generalizations of the Kolmogorov–Barzdin embedding estimates*, Duke Math. J. 161 (2012) 2549–2603 MR

[26] **A Kawauchi**, *A survey of knot theory*, Birkhäuser, Basel (1996) MR

[27] **R Komendarczyk**, **A Michaelides**, *Tree invariants and Milnor linking numbers with indeterminacy*, preprint (2016) arXiv

[28] **Y Kotorii**, *The Milnor* $\overline{\mu}$ *invariants and nanophrases*, J. Knot Theory Ramifications 22 (2013) art. id. 1250142 MR

[29] **O Kravchenko**, **M Polyak**, *Diassociative algebras and Milnor's invariants for tangles*, Lett. Math. Phys. 95 (2011) 297–316 MR

[30] **E H Lieb**, **M Loss**, *Analysis*, 2nd edition, Graduate Studies in Mathematics 14, Amer. Math. Soc., Providence, RI (2001) MR

[31] **X-S Lin**, **Z Wang**, *Integral geometry of plane curves and knot invariants*, J. Differential Geom. 44 (1996) 74–95 MR

[32] **R A Litherland**, **J Simon**, **O Durumeric**, **E Rawdon**, *Thickness of knots*, Topology Appl. 91 (1999) 233–244 MR

[33] **F Maggioni**, **R L Ricca**, *On the groundstate energy of tight knots*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 465 (2009) 2761–2783 MR

[34] **A V Michaelides**, *Lower bounds for ropelength of links via higher linking numbers and other finite type invariants*, PhD thesis, Tulane University School of Science and Engineering (2015) MR Available at `https://search.proquest.com/docview/1825278543`

[35] **J Milnor**, *Link groups*, Ann. of Math. 59 (1954) 177–195 MR

[36] **J Milnor**, *Isotopy of links: algebraic geometry and topology*, from "A symposium in honor of S Lefschetz" (R H Fox, D C Spencer, A W Tucker, editors), Princeton Univ. Press (1957) 280–306 MR

[37] **M Polyak**, *On the algebra of arrow diagrams*, Lett. Math. Phys. 51 (2000) 275–291 MR

[38] **M Polyak**, **O Viro**, *Gauss diagram formulas for Vassiliev invariants*, Int. Math. Res. Not. 1994 (1994) 445–453 MR

[39] **M Polyak**, **O Viro**, *On the Casson knot invariant*, J. Knot Theory Ramifications 10 (2001) 711–738 MR

[40] **E J Rawdon**, *Approximating the thickness of a knot*, from "Ideal knots" (A Stasiak, V Katritch, L H Kauffman, editors), Ser. Knots Everything 19, World Sci., River Edge, NJ (1998) 143–150 MR

[41] **R L Ricca**, **F Maggioni**, *On the groundstate energy spectrum of magnetic knots and links*, J. Phys. A 47 (2014) art. id. 205501 MR

[42] **R L Ricca**, **H K Moffatt**, *The helicity of a knotted vortex filament*, from "Topological aspects of the dynamics of fluids and plasmas" (H K Moffatt, G M Zaslavsky, P Comte, M Tabor, editors), NATO Adv. Sci. Inst. Ser. E Appl. Sci. 218, Kluwer, Dordrecht (1992) 225–236 MR

[43] **I Volić**, *A survey of Bott–Taubes integration*, J. Knot Theory Ramifications 16 (2007) 1–42 MR

[44] **S Willerton**, *On the first two Vassiliev invariants*, Experiment. Math. 11 (2002) 289–296 MR

Department of Mathematics, Tulane University
New Orleans, LA, United States

Department of Mathematics, University of South Alabama
Mobile, AL, United States

rako@tulane.edu, amichaelides@southalabama.edu

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