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On rational homological stability for block automorphisms of connected sums of products of spheres

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We show rational homological stability for the classifying spaces of the monoid of homotopy self-equivalences and the block diffeomorphism group of iterated connected sums of products of spheres. The spheres can have different dimensions, but need to satisfy a certain connectivity assumption. The main theorems of this paper extend homological stability results for automorphism spaces of connected sums of products of spheres of the same dimension by Berglund and Madsen.

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1 Introduction

Let M be a compact manifold with boundary with a chosen basepoint on the boundary. Denote by $\mathrm{map}_*(M,M)$ the space of pointed maps with the compact-open topology. Denote by $\mathrm{map}_\partial(M,M)$ the subspace of maps fixing the boundary pointwise. We denote by $\mathrm{aut}_\partial(M)$ the submonoid of homotopy self-equivalences fixing the boundary pointwise.

The first object we study in this paper is the rational homology of the classifying space $B\mathrm{aut}_\partial(M)$. More precisely, we show that the topological monoid $\mathrm{aut}_\partial(M)$ satisfies rational homological stability for certain families of manifolds, ie that the rational homology of its classifying space is independent of the manifold in the family in a certain range. For example, for the g–fold connected sum

$$N_g = \Big(\#_g(S^p\times S^q)\Big) \smallsetminus \mathrm{int}(D^{p+q}), \quad \text{where } 3\le p<q<2p-1,$$

we show that $H_i(B\mathrm{aut}_\partial(N_g);\mathbb{Q})$ is independent of g for $g>2i+2$. Moreover we show that $H_i(B\mathrm{aut}_\partial(M\,\#\,N_g);\mathbb{Q})$ is independent of g in the same range, where M is some other connected sum of products of spheres. To make this statement precise, we introduce the following notation. Let I be a finite indexing set for pairs of positive natural numbers p_i and q_i, and let n be a positive natural number such that $3\le p_i\le q_i<2p_i-1$ and $p_i+q_i=n$ for all $i\in I$. Note that this implies that necessarily $n\ge 6$.

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We define a smooth n–dimensional manifold with boundary diffeomorphic to S^{n-1},

$$N_I = \Big(\#_{i\in I}(S^{p_i}\times S^{q_i})\Big)\smallsetminus \mathrm{int}(D^n),$$

and we assume a basepoint chosen on the boundary.

For a given $p\in\mathbb{N}$ we define the "generalized genus" as

$$g_p=\begin{cases}\frac{1}{2}\,\mathrm{rank}(H_p(N_I)) & \text{if } 2p=n,\\ \mathrm{rank}(H_p(N_I)) & \text{otherwise;}\end{cases}$$

ie g_p is the number of $S^p\times S^q$ summands of N_I, where $q=n-p$. In other words we see N_I as the connected sum

$$N_I=\Big(\#_{i\in\{j\in I\mid p_j\neq p\}}(S^{p_i}\times S^{q_i})\Big)\#\Big(\#_{g_p}(S^p\times S^q)\Big)\smallsetminus \mathrm{int}D^n.$$

Write

$$V^{p,q}=S^p\times S^q\smallsetminus \mathrm{int}(D^n_1\sqcup D^n_2)\quad \text{for } 3\le p\le q<2p-1 \text{ and } p+q=n.$$

We define a new manifold

$$N_I'=N_I\cup_{\partial_1}V^{p,q},$$

by identifying one boundary component of $V^{p,q}$ with ∂N_I. Note that N_I' is canonically diffeomorphic to the manifold $N_{I'}$ with $I'=I\cup\{i'\}$, where $p_{i'}=p$ and $q_{i'}=q$. Using this, we define the stabilization map

$$\sigma\colon \mathrm{aut}_\partial(N_I)\to \mathrm{aut}_\partial(N_I\cup_{\partial_1}V^{p,q})\xrightarrow{\cong}\mathrm{aut}_\partial(N_{I'}),$$

by extending a self-map of N_I by the identity on $V^{p,q}$. In this paper we study the induced map on homology of the classifying spaces

$$\sigma_*\colon H_*(B\mathrm{aut}_\partial(N_I);\mathbb{Q})\to H_*(B\mathrm{aut}_\partial(N_{I'});\mathbb{Q}).$$

Theorem A *The map*

$$H_i(B\mathrm{aut}_\partial(N_I);\mathbb{Q})\to H_i(B\mathrm{aut}_\partial(N_{I'});\mathbb{Q})$$

induced by the stabilization map with respect to $V^{p,q}$, where $3\le p\le q<2p-1$, is an isomorphism for $g_p>2i+2$ when $2p\neq n$ and $g_p>2i+4$ if $2p=n$ and an epimorphism for $g_p\ge 2i+2$, respectively $g_p\ge 2i+4$, unless $n=2d$ with $d\neq 3,7$ odd, and $g_d=1$.

In an earlier version of this paper the last condition of the theorem was erroneously not present. The reason the latter is needed is Lemma 3.4, which implies that the rational cohomology of the homotopy classes of $\mathrm{aut}_\partial(N_I)$ is trivial. This is not true for $S^d \times S^d$ when d is odd and $d \neq 3, 7$. We thank Manuel Krannich for pointing this out to us.

The block diffeomorphism group $\widetilde{\mathrm{Diff}}_\partial(X)$ is the realization of the Δ–group, ie a simplicial group without degeneracies, with k–simplices, face-preserving diffeomorphisms

$$\varphi\colon \Delta^k \times X \to \Delta^k \times X$$

such that φ is the identity on a neighborhood of $\Delta^k \times \partial X$. We map a k–simplex φ to the k–simplex in $\widetilde{\mathrm{Diff}}_\partial(N_{I'})$

$$\Delta^k \times (N_I \cup_{\partial_1} V^{p,q}) \to \Delta^k \times (N_I \cup_{\partial_1} V^{p,q})$$

given by φ on $\Delta^k \times N_I$ and the identity on $\Delta^k \times V^{p,q}$, where we use the fact that $N_{I'} \cong N_I \cup_{\partial_1} V^{p,q}$. This induces the stabilization map $B\widetilde{\mathrm{Diff}}_\partial(N_I) \to B\widetilde{\mathrm{Diff}}_\partial(N_{I'})$. Similarly we can define the classifying spaces of the monoid of block homotopy self-equivalences $B\widetilde{\mathrm{aut}}_\partial(N_I)$, the group of block homeomorphisms $B\widetilde{\mathrm{Top}}_\partial(N_I)$ and the group of block PL–homeomorphisms $B\widetilde{\mathrm{PL}}_\partial(N_I)$.

Theorem B *The map*

$$H_i(B\widetilde{\mathrm{Diff}}_\partial(N_I);\mathbb{Q}) \to H_i(B\widetilde{\mathrm{Diff}}_\partial(N_{I'});\mathbb{Q})$$

induced by the stabilization map with respect to $V^{p,q}$, where $3 \le p \le q < 2p-1$, is an isomorphism for $g_p > 2i+2$ when $2p \neq n$ and $g_p > 2i+4$ if $2p = n$ and an epimorphism for $g_p \ge 2i+2$, respectively $g_p \ge 2i+4$, unless $n = 2d$ with $d \neq 3, 7$ odd, and $g_d = 1$.

Remark 1.1 Since $B\widetilde{\mathrm{aut}}_\partial(N_I) \simeq B\mathrm{aut}_\partial(N_I)$, Theorem A also holds for the classifying space of block homotopy automorphisms. Moreover note that the classifying spaces of the universal covers $B\widetilde{\mathrm{Diff}}_\partial(N_I)\langle 1\rangle$, $B\widetilde{\mathrm{Top}}_\partial(N_I)\langle 1\rangle$ and $B\widetilde{\mathrm{PL}}_\partial(N_I)\langle 1\rangle$ have the same rational homology groups. This follows from the fact that the spaces G/O, G/PL and G/Top occurring in the surgery exact sequence to calculate the normal invariants are rationally homotopy equivalent and that the weak equivalence (15) also holds for Top and PL. For Theorem B to also hold for block PL–homeomorphisms one would need a statement about the mapping class groups like Proposition 5.3(2), ie more knowledge about the PL–isotopy classes of PL–homeomorphisms.

The argument is based on work of Berglund and Madsen [4; 5], where homological stability for the (block) automorphism spaces of the manifolds $\#_g(S^d \times S^d) \smallsetminus \mathrm{int}(D^{2d})$ is shown. Berglund and Madsen moreover determine the stable cohomology of the classifying spaces of the monoid of homotopy automorphisms and the block diffeomorphism group.

Outline of the argument The main idea of the proof is to consider the universal covering

$$B\mathrm{aut}_\partial(N_I)\langle 1\rangle \to B\mathrm{aut}_\partial(N_I)$$

or rather the universal covering spectral sequence with E^2–page

$$H_s\big(\pi_1(B\mathrm{aut}_\partial(N_I)); H_t(B\mathrm{aut}_\partial(N_I)\langle 1\rangle; \mathbb{Q})\big) \Rightarrow H_{s+t}(B\mathrm{aut}_\partial(N_I); \mathbb{Q}).$$

If we can show that the stabilization map induces isomorphisms for large generalized genus g_p in a range of s and t, then the spectral sequence comparison theorem implies homological stability for the monoid of homotopy automorphisms. So we have reduced the problem to showing homological stability for the groups $\pi_1(B\mathrm{aut}_\partial(N_I))$ with certain twisted coefficients.

The first step is to determine the group of homotopy classes of homotopy automorphisms (Section 5), or rather the quotient acting nontrivially on the homology of the universal covering. More precisely, we determine the image and kernel of the "reduced homology" map

$$\widetilde{H}\colon \pi_0(\mathrm{aut}_\partial(N_I)) \to \mathrm{Aut}(\widetilde{H}_*(N_I))$$

to the automorphisms of the reduced homology as a graded group. For the manifolds N_I the kernel is finite. The connectivity assumption $3 \le p_i \le q_i < 2p_i - 1$ is necessary here to get the isomorphism (6) in the proof of Proposition 5.1. The image which we call $\Gamma_I \subset \mathrm{Aut}(\widetilde{H}_*(N_I))$ is the subgroup respecting the intersection form and when the dimension of N_I is even, a certain tangential invariant. In particular we show that elements in the kernel of $\widetilde{H}_*$ act trivially on $H_t(B\mathrm{aut}_\partial(N_I)\langle 1\rangle; \mathbb{Q})$. Thus we only have to study the groups Γ_I.

In Section 3 we review hyperbolic modules and give a slight generalization in order to describe the Γ_I as automorphisms of an object with underlying graded $\mathbb{Z}$–module $\widetilde{H}_*(N_I)$.

In Section 4 we use homological stability results by van der Kallen and Charney to show a homological stability result for the groups Γ_I with certain twisted coefficient systems of "finite degree". In Section 6 we review results of Berglund and Madsen in order to show that the homology groups $H_t(B\mathrm{aut}_\partial(N_I)\langle 1\rangle;\mathbb{Q})$ are in fact a twisted coefficient system satisfying homological stability for the Γ_I. A crucial tool for showing twisted homological stability are Schur multifunctors (defined in Section 2), which we use to determine the degree of the coefficient systems. In fact, developing the Schur multifunctors as a tool to handle the different degrees coming from the homology $\widetilde{H}_*(N_I)$ was one of the main technical obstacles in generalizing Berglund and Madsen's result.

To show the homological stability for the block diffeomorphism groups, we consider the homotopy fibration

$$\mathrm{aut}_\partial(N_I)/\widetilde{\mathrm{Diff}}_\partial(N_I) \to B\widetilde{\mathrm{Diff}}_\partial(N_I) \to B\mathrm{aut}_\partial(N_I).$$

We can determine the homology of a component of the homotopy fiber using surgery theoretic methods applied by Berglund and Madsen, which suffices to show homological stability using similar arguments as for the monoid of homotopy automorphisms.

Future research The most interesting question is if it is possible to determine the stable (rational) cohomology of $B\mathrm{aut}_\partial(N_I)$ and $B\widetilde{\mathrm{Diff}}_\partial(N_I)$. The argument by Berglund and Madsen [5] to calculate the stable cohomology for the manifolds $W_g = \#_g(S^d \times S^d) \setminus \mathrm{int}(D^{2d})$ relies on Galatius and Randal-Williams [12], where the stable cohomology of the moduli space of many even-dimensional smooth manifolds is determined. More precisely Berglund and Madsen show that the universal covering spectral sequence for the classifying space of the monoid of homotopy automorphisms collapses at the E_2–page

$$H^*\big(\pi_1(B\mathrm{aut}_\partial(W_g)); H^*(B\mathrm{aut}_\partial(W_g)\langle 1\rangle;\mathbb{Q})\big) \tag{1}$$

in the stable range. Using that $\pi_1(B\mathrm{aut}_\partial(W_g))$ is commensurable with a known arithmetic group Γ_g and results of Borel [6; 7] on the stable real cohomology of arithmetic groups, they conclude that

$$H^*(B\mathrm{aut}_\partial(W_g);\mathbb{Q}) \cong H^*(\Gamma_g;\mathbb{Q}) \otimes H^*(B\mathrm{aut}_\partial(W_g)\langle 1\rangle;\mathbb{Q}))$$

in the stable range. The cohomology of Γ_g in the stable range was calculated explicitly by Borel. Berlung and Madsen link $H^*(B\mathrm{aut}_\partial(W_g)\langle 1\rangle;\mathbb{Q}))$ via certain derivation Lie algebras and Kontsevich's graph complexes to the cohomology of free groups using work of Conant and Vogtmann [9]. They use a similar approach to determine the stable rational cohomology of $B\widetilde{\mathrm{Diff}}_\partial(W_g)$. There is not an analogue of [12] for odd-dimensional manifolds yet, but the homological stability results of Galatius and Randal-Williams [13] have been generalized to odd-dimensional manifolds; see Perlmutter [22; 23]. One could hope that the calculation of the stable cohomology in [5] generalizes to other even-dimensional manifolds, such as

$$\left(\#_f(S^{d-1}\times S^{d+1})\right)\#\left(\#_g(S^d\times S^d)\right)\smallsetminus \operatorname{int}(D^{2d}),$$

especially since Borel's results are also applicable to the mapping class groups considered in this paper. It is however an open question if the analogue of (1) collapses. Using [12] to calculate the stable cohomology of the moduli spaces of even-dimensional manifolds N_I explicitly could help in answering this question.

Note that the manifolds N_I are highly connected in the sense that they are $(m-1)$–connected n–manifolds, where $n\le 3m-2$. In fact every closed highly connected odd-dimensional manifold M is rationally homotopy equivalent to a manifold $N_I\cup_{S^{n-1}}D^n$ (see Proposition 6.1). This does however not imply homological stability starting with an arbitrary highly connected odd-dimensional manifold, since we do not have a twisted homological stability result for their mapping class groups. In general we do not know the image of $\widetilde{H}(-)$ in $\operatorname{Aut}\big(\widetilde{H}_*(M\smallsetminus \operatorname{int}(D))\big)$. This raises the problem of calculating the (homotopy) mapping class groups for arbitrary highly connected manifolds. Another possible approach to show homological stability with an arbitrary starting manifold could be to show twisted versions of the algebraic homological stability results in [13; 22].

Another problem is to determine the PL–isotopy classes of PL–homeomorphisms for the manifolds N_I. As already pointed out in Remark 1.1 this would yield a version of Theorem B for $B\widetilde{\mathrm{Top}}_\partial(N_I)$ and $B\widetilde{\mathrm{PL}}_\partial(N_I)$. As above, determining PL–isotopy classes of PL–homeomorphisms for general highly connected manifolds might lead to a version of Theorem B with more general starting manifolds.

Another direction for future research is to weaken the connectivity assumption in Theorems A and B. In this situation the kernel of $\widetilde{H}\colon \pi_0(\mathrm{aut}_\partial(N_I))\to \operatorname{Aut}(\widetilde{H}_*(N_I))$ might be rationally nontrivial, because we can not identify $[N_I,N_I]_*$ with the endomorphisms of $\widetilde{H}_*(N_I)$ as we do in (6) in the proof of Proposition 5.1. In particular this kernel might act nontrivially on $H_*(B\mathrm{aut}_\partial(N_I)\langle 1\rangle;\mathbb{Q})$. This could lower the stability range or even make it impossible to show homological stability with similar methods, because we have to replace the isomorphism (14), which is the first step in showing Theorem A, by the use of a Lyndon spectral sequence.

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2 Polynomial functors and Schur multifunctors

We start by recalling the definition of polynomial functors in the sense of [11, Section 9] or rather slightly modified as in [10, Section 3]. Let $T\colon \mathcal{A}\to\mathcal{B}$ be a (not necessarily additive) functor between abelian categories. The first cross-effect functor is defined to be

$$T^1(X)=\ker(T(X)\to T(0)),$$

where $X\to 0$ is the natural map to the zero object in $\mathcal{A}$. For $k>1$, the k^{th} cross-effect functor

$$T^k\colon \mathcal{A}^k\to\mathcal{B}$$

is uniquely defined up to isomorphism given T^l for $l<k$ by the properties:

(1) $T^k(A_1,\dots,A_k)=T(0)$ if $A_i=0$ for some i.

(2) There is a natural isomorphism

$$T(A_1\oplus\cdots\oplus A_k)\cong T(0)\oplus\bigoplus_{\{i_1,\dots,i_r\}}T^r(A_{i_1},\dots,A_{i_r}),$$

where the sum runs over all nonempty subsets $\{i_1,\dots,i_r\}\subset\{1,\dots,k\}$.

Definition 2.1 A functor T is *polynomial of degree* $\le k$ if T^l is the constant zero functor for $l>k$.

An immediate consequence is that a functor is of degree ≤ 0 if and only if it is constant.

The higher cross-effects can be defined using deviations. We can define an associative and distributive composition $\top$ on the "integral group ring" of $T(X)\in\mathcal{B}$ by extending $a\top b=ab-a-b$ for $a,b\in T(A)$. Given a map $T(f)\colon T(A)\to T(B)$ we get a new map from $T(A)\oplus T(A)$ defined by

$$(a,b)\mapsto T(f)(a\top b)=T(f)(ab)-T(f)(a)-T(f)(b),$$

which we call the deviation of $T(f)$. The k–fold deviation of k maps

$$T(f_1), \dots, T(f_k)\colon T(A) \to T(B)$$

is the map

$$T(f_1 \top \cdots \top f_k)\colon T(A) \to T(B),$$

given by

$$T(f_1 \top \cdots \top f_k) = T(0) \oplus \bigoplus_{\{i_1,\dots,i_r\}} (-1)^{k-r} T(f_{i_1}, \dots, f_{i_r}),$$

where the sum runs over all nonempty subsets $\{i_1, \dots, i_r\} \subset \{1, \dots, k\}$ and 0 denotes the canonical map $A \to 0 \to B$. Setting $A = A_1 \oplus \cdots \oplus A_k$ and denoting by $\pi_i\colon A \to A_i$ the projections and by $\iota_i\colon A_i \to A$ the inclusions, the k^{th} cross-effect functor is given on objects by

$$T^k(A_1, \dots, A_k) = \mathrm{Image}(T(\iota_1 \circ \pi_1 \top \cdots \top \iota_k \circ \pi_k)).$$

The following properties of polynomial functors follow directly from the definition:

Proposition 2.2 (see eg [10]) (1) *An additive functor is of degree* ≤ 1.

(2) *The composition of functors of degrees* $\le k$ *and* $\le l$ *is a functor of degree* $\le kl$.

(3) *Let* $T\colon \mathcal{A} \to \mathcal{B}$ *and* $R\colon \mathcal{C} \to \mathcal{B}$ *be of degrees* $\le k$ *and* $\le l$*, respectively. The levelwise sum*

$$T \oplus R\colon \mathcal{A} \times \mathcal{C} \to \mathcal{B}$$

is polynomial of degree $\le \max\{k, l\}$

Example Let $\mathrm{Mod}(\mathbb{Z})$ be the category of finitely generated $\mathbb{Z}$–modules. An example of a degree $\le k$ functor is the k–fold tensor product

$$\bigotimes\colon \mathrm{Mod}(\mathbb{Z})^k \to \mathrm{Mod}(\mathbb{Z}), \quad (A_1, \dots, A_k) \mapsto \bigotimes_i A_i.$$

Schur functors give examples of polynomial functors. Note that the following definitions also make sense for general commutative rings, but we are going to restrict our presentation to the category of graded rational vector spaces $\mathrm{Vect}_*(\mathbb{Q})$. Schur functors are treated for example in [20]. We could not find any literature on Schur multifunctors and hence state the facts we need here.

Let $\mathscr{M} = \mathscr{M}(n) \in \mathrm{Vect}_*(\mathbb{Q})$ for $n \ge 0$ be a sequence of $\mathbb{Q}[\Sigma_n]$–modules. We refer to them as Σ_n–modules but implicitly use the $\mathbb{Q}[\Sigma_n]$–modules structure; in particular $\otimes_{\Sigma_n}$ refers to the tensor product over $\mathbb{Q}[\Sigma_n]$. The *Schur functor* given by $\mathscr{M}$ is defined to be the endofunctor of $\mathrm{Vect}_*(\mathbb{Q})$ induced by

$$\mathscr{M}(V) = \bigoplus_k \mathscr{M}(k) \otimes_{\Sigma_k} V^{\otimes k} \quad \text{for all } V \in \mathrm{Vect}_*(\mathbb{Q}),$$

where $V^{\otimes k}$ is the left Σ_k–module with action induced by the permutation of the factors by the inverse (with sign according to the Koszul sign convention). Note that $\mathscr{M}(0)$ is just a constant summand. A Schur functor $\mathscr{M}$ with $\mathscr{M}(l)$ trivial for $l > k$ is a polynomial functor of degree $\leq k$.

Let $\eta = (n_1, \dots, n_l)$ with $n_1, \dots, n_l \geq 0$ be a multiindex. Throughout this article we will assume all multiindices to have nonnegative entries. We use the following conventions:

$$\begin{aligned} |\eta| &= \textstyle\sum_{i=1}^{l} n_i, \\ \ell(\eta) &= l, \\ \mu + \eta &= (m_1 + n_1, \dots, m_l + n_l) \quad \text{for } \mu = (m_1, \dots, m_l), \\ (V_i)^{\otimes \eta} &= V_1^{\otimes n_1} \otimes \cdots \otimes V_l^{\otimes n_l}, \quad \text{where } (V_i) \in (\mathrm{Vect}_*(\mathbb{Q}))^l, \\ \Sigma_\eta &= \Sigma_{n_1} \times \cdots \times \Sigma_{n_l}. \end{aligned}$$

Consider a sequence of $\mathbb{Q}[\Sigma_\eta]$–modules $\mathscr{N} = \mathscr{N}(\eta) \in \mathrm{Vect}_*(\mathbb{Q})$ with $\ell(\eta) = l$. As before, we refer to them as Σ_η–modules. We define the *Schur multifunctor* given by $\mathscr{N}$ on objects by

$$\mathscr{N}(V_1, \dots, V_l) = \bigoplus_{\ell(\eta)=l} \mathscr{N}(\eta) \otimes_{\Sigma_\eta} (V_i)^{\otimes \eta} \quad \text{for all } (V_i) \in (\mathrm{Vect}_*(\mathbb{Q}))^l.$$

Similarly a Schur multifunctor $\mathscr{N}$ is a polynomial of degree $\leq k$ if $\mathscr{N}(\eta)$ is trivial for $|\eta| > k$.

Example Consider Schur functors $\mathcal{N}_i\colon \mathrm{Vect}_*(\mathbb{Q}) \to \mathrm{Vect}_*(\mathbb{Q})$ for $i = 1, \dots, l$. The tensor product

$$\bigotimes \mathcal{N}_i\colon \mathrm{Vect}_*(\mathbb{Q})^l \to \mathrm{Vect}_*(\mathbb{Q})$$

is a Schur multifunctor with $\big(\bigotimes \mathcal{N}_i\big)(\eta) = \bigotimes \mathcal{N}_i(n_i)$.

We define the tensor product of $\mathscr{M} = \mathscr{M}(\mu)$ and $\mathscr{N} = \mathscr{N}(\eta)$ as

$$\mathscr{M} \otimes \mathscr{N}(\nu) := \bigoplus_{\mu' + \eta' = \nu} \mathrm{Ind}_{\Sigma_{\mu'} \times \Sigma_{\eta'}}^{\Sigma_\nu} \mathscr{M}(\mu') \otimes \mathscr{N}(\eta').$$

The Schur functor defined by this tensor product is indeed (up to natural isomorphism) the levelwise tensor product of the two functors, as we see by the isomorphisms for

$\mu' + \eta' = \nu$ with $\ell(\mu') = \ell(\eta') = \ell(\nu)$:

$$\begin{aligned}(\mathrm{Ind}_{\Sigma_{\mu'}\times\Sigma_{\eta'}}^{\Sigma_\nu} \mathscr{M}(\mu') \otimes \mathscr{N}(\eta')) \otimes_{\Sigma_\nu} (V_i)^{\otimes \nu} \\ &= \big(\mathscr{M}(\mu') \otimes \mathscr{N}(\eta') \otimes_{\Sigma_{\mu'}\times\Sigma_{\eta'}} \mathbb{Q}[\Sigma_\nu]\big) \otimes_{\Sigma_\nu} (V_i)^{\otimes \nu} \\ &\cong \big(\mathscr{M}(\mu') \otimes \mathscr{N}(\eta')\big) \otimes_{\Sigma_{\mu'}\times\Sigma_{\eta'}} \big((V_i)^{\otimes \mu'} \otimes (V_i)^{\otimes \eta'}\big) \\ &\cong \big(\mathscr{M}(\mu') \otimes_{\Sigma_{\mu'}} (V_i)^{\otimes \mu'}\big) \otimes \big(\mathscr{N}(\eta') \otimes_{\Sigma_{\eta'}} (V_i)^{\otimes \eta'}\big).\end{aligned}$$

The tensor powers of an $\mathscr{N} = \mathscr{N}(\eta)$ are (up to natural isomorphism) explicitly described by

$$\mathscr{N}^{\otimes r}(\nu) = \bigoplus \mathrm{Ind}_{\Sigma_{\eta_1}\times\cdots\times\Sigma_{\eta_r}}^{\Sigma_\nu} \mathscr{N}(\eta_1) \otimes \cdots \otimes \mathscr{N}(\eta_r), \tag{2}$$

where the sum runs over all r–tuples $(\eta_1, \dots, \eta_r)$ with $\ell(\eta_i) = l$ such that $\sum_{i=1}^r \eta_i = \nu$.

Now consider a Schur functor $\mathscr{M} = \mathscr{M}(m)$ and a Schur multifunctor $\mathscr{N} = \mathscr{N}(\eta)$. The composition is a Schur multifunctor isomorphic to the Schur multifunctor given by

$$(\mathscr{M} \circ \mathscr{N})(\nu) = \bigoplus_r \mathscr{M}(r) \otimes_{\Sigma_r} \bigoplus \mathrm{Ind}_{\Sigma_{\eta_1}\times\cdots\times\Sigma_{\eta_r}}^{\Sigma_\nu} \mathscr{N}(\eta_1) \otimes \cdots \otimes \mathscr{N}(\eta_r), \tag{3}$$

where the second sum runs over all r–tuples $(\eta_1, \dots, \eta_r)$ with $\ell(\eta_i) = l$ such that $\sum_{i=1}^r \eta_i = \nu$. The action of Σ_r is by permuting the tuples $(\eta_1, \dots, \eta_r)$ by the inverse. Indeed as we check using (2):

$$\begin{aligned}&(\mathscr{M} \circ \mathscr{N})(V_i) \\ &= \bigoplus_r \mathscr{M}(r) \otimes_{\Sigma_r} \mathscr{N}(V_i)^{\otimes r} \cong \bigoplus_r \mathscr{M}(r) \otimes_{\Sigma_r} \mathscr{N}^{\otimes r}(V_i) \\ &\cong \bigoplus_{r,\nu} \mathscr{M}(r) \otimes_{\Sigma_r} \big(\mathscr{N}^{\otimes r}(\nu) \otimes_{\Sigma_\nu} (V_i)^{\otimes \nu}\big) \\ &\cong \bigoplus_{r,\nu} \mathscr{M}(r) \otimes_{\Sigma_r} \Bigg(\bigoplus_{\sum_{i=1}^r \eta_i = \nu} \mathrm{Ind}_{\Sigma_{\eta_1}\times\cdots\times\Sigma_{\eta_r}}^{\Sigma_\nu} \mathscr{N}(\eta_1) \otimes \cdots \otimes \mathscr{N}(\eta_r) \otimes_{\Sigma_\nu} (V_i)^{\otimes \nu}\Bigg) \\ &\cong \bigoplus_\nu \bigoplus_r \mathscr{M}(r) \otimes_{\Sigma_r} \Bigg(\bigoplus_{\sum_{i=1}^r \eta_i = \nu} \mathrm{Ind}_{\Sigma_{\eta_1}\times\cdots\times\Sigma_{\eta_r}}^{\Sigma_\nu} \mathscr{N}(\eta_1) \otimes \cdots \otimes \mathscr{N}(\eta_r) \otimes_{\Sigma_\nu} (V_i)^{\otimes \nu}\Bigg).\end{aligned}$$

Remark 2.3 We will later use Schur (multi)functors with domain the category of rational vector spaces — just consider them as graded rational vector spaces concentrated in degree 0.

3 Automorphisms of hyperbolic modules over the integers

In this section we review hyperbolic modules in the sense of [1] in the special case with ground ring the integers. Fix a $\lambda \in \{+1, -1\}$. Let $\Lambda \subset \mathbb{Z}$ be an additive subgroup, called the *form parameter*, such that

$$\{z - \lambda z \mid z \in \mathbb{Z}\} \subset \Lambda \subset \{z \in \mathbb{Z} \mid z = -\lambda z\}. \tag{4}$$

Definition 3.1 [1] A Λ–*quadratic module* is a pair (M, μ), where M is a $\mathbb{Z}$–module and μ is a bilinear form, ie a homomorphism

$$\mu\colon M \otimes M \to \mathbb{Z}.$$

To a Λ–quadratic module (M, μ) we associate a Λ–*quadratic form*

$$q_\mu\colon M \to \mathbb{Z}/\Lambda, \quad q_\mu(x) = [\mu(x, x)],$$

and a λ–*symmetric bilinear form*

$$\langle -, - \rangle_\mu\colon M \otimes M \to \mathbb{Z},$$

defined by $\langle x, y\rangle_\mu = \mu(x, y) + \lambda\mu(y, x)$. (Such λ–symmetric bilinear forms are called even.) We call a finitely generated projective Λ–quadratic module (M, μ) *nondegenerate* if the map

$$M \to M^*, \quad x \mapsto \langle x, - \rangle_\mu,$$

is an isomorphism.

Denote by $Q^\lambda(\mathbb{Z}, \Lambda)$ the category of nondegenerate Λ–quadratic modules with morphisms linear maps respecting the associated λ–symmetric bilinear form and the associated Λ–quadratic form.

Given a finitely generated $\mathbb{Z}$–module M, we define a nondegenerate Λ–quadratic module $H(M) = (M \oplus M^*, \mu_M)$, where $\mu_M((x, f), (y, g)) = f(y)$. We call $H(M)$ the *hyperbolic module* on M. A Λ–quadratic module is called *hyperbolic* if it is isomorphic to $H(N)$ for some finitely generated $\mathbb{Z}$–module N. Let $\{e_i\}$ be the standard basis for $\mathbb{Z}^g$ and $\{f_i\}$ the dual basis of $(\mathbb{Z}^g)^*$. Using this basis we consider the automorphisms of the Λ–quadratic module $H(\mathbb{Z}^g)$ as a subgroup of $\mathrm{Gl}_{2g}(\mathbb{Z})$. The subgroups can be described as follows:

Proposition 3.2 [1, Corollary 3.2] *The automorphisms of $H(\mathbb{Z}^g)$ in $Q^\lambda(\mathbb{Z},\Lambda)$ are isomorphic to the subgroup of $\mathrm{Gl}_{2g}(\mathbb{Z})$ consisting of matrices $\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ such that*

$$D^T A + \lambda B^T C = 1,$$
$$D^T B + \lambda B^T D = 0,$$
$$A^t C + \lambda C^T A = 0,$$
$$C^T A \text{ and } D^T B \text{ have diagonal entries in } \Lambda.$$

Note that if $\lambda = 1$ we necessarily have $\Lambda = 0$. When $\lambda = -1$, condition (4) implies that $2\mathbb{Z} \subset \Lambda \subset \mathbb{Z}$, and thus we have the two cases $\Lambda = \mathbb{Z}$ and $\Lambda = 2\mathbb{Z}$. Thus we can list the automorphisms of hyperbolic modules:

(1) If $\lambda = 1$ and $\Lambda = 0$, then $\mathrm{Aut}(H(\mathbb{Z}^g)) = O_{g,g}(\mathbb{Z})$ in $Q^1(\mathbb{Z},0)$.

(2) If $\lambda = -1$ and $\Lambda = \mathbb{Z}$, then $\mathrm{Aut}(H(\mathbb{Z}^g)) = \mathrm{Sp}_{2g}(\mathbb{Z})$ in $Q^{-1}(\mathbb{Z},\mathbb{Z})$.

(3) If $\lambda = -1$ and $\Lambda = 2\mathbb{Z}$, then $\mathrm{Aut}(H(\mathbb{Z}^g))$ in $Q^{-1}(\mathbb{Z},2\mathbb{Z})$ is the subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$ described as

$$\left\{\begin{pmatrix} A & B \\ C & D\end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z}) \;\middle|\; C^T A \text{ and } D^T B \text{ have even entries on the diagonal}\right\}.$$

Let N be a $(d-1)$–connected $2d$–manifold. Wall [29] has shown that the automorphisms of the homology realized by diffeomorphisms are the automorphisms of a Λ–quadratic module with underlying $\mathbb{Z}$–module $H_d(N)$. Later we show a similar statement for connected sums of products of spheres. For this we need a slight variation of Λ–quadratic modules.

Let $n = 2d \in \mathbb{N}$ be even and $\Lambda \subset \mathbb{Z}$ be an additive subgroup such that

$$\{z - (-1)^d z \mid z \in \mathbb{Z}\} \subset \Lambda \subset \{z \in \mathbb{Z} \mid z = -(-1)^d z\}.$$

When $n \in \mathbb{N}$ is odd we set Λ to be the trivial group. It does not play a role in the following definition, but we keep the notion for convenience.

Definition 3.3 A *graded Λ–quadratic module* is a pair (M,μ), where M is a graded $\mathbb{Z}$–module and μ a bilinear n–pairing, ie a degree 0 homomorphism

$$\mu\colon M \otimes M \to \mathbb{Z}[n].$$

We associate to (M,μ) a *symmetric bilinear n–pairing*

$$\langle -,-\rangle_\mu\colon M\otimes M\to\mathbb{Z}[n],$$

defined by $\langle x,y\rangle_\mu=\mu(x,y)+(-1)^{|x||y|}\mu(y,x)$. When $n=2d$, we associate a Λ*–quadratic form*

$$q_\mu\colon M_d\to\mathbb{Z}/\Lambda,\quad q_\mu(x)=[\mu(x,x)].$$

We call a finitely generated projective graded Λ–quadratic module (M,μ) nondegenerate if the map

$$M\to\operatorname{Hom}(M,\mathbb{Z}[n]),\quad x\mapsto\langle x,-\rangle_\mu,$$

is an isomorphism.

For n even, we define $Q^n_*(\mathbb{Z},\Lambda)$ to be the category whose objects are nondegenerate graded Λ–quadratic modules (M,μ), where (M,μ) is finitely generated as a $\mathbb{Z}$–module and the morphisms respect q_μ and $\langle -,-\rangle_\mu$.

For n odd, we define $Q^n_*(\mathbb{Z},\Lambda)$ to be the category whose objects are nondegenerate graded Λ–quadratic modules (M,μ), where (M,μ) is finitely generated as a $\mathbb{Z}$–module and the morphisms respect $\langle -,-\rangle_\mu$.

Let $Q^n_+(\mathbb{Z},\Lambda)$ be the full subcategory with objects concentrated in positive degrees and hence necessarily concentrated in degrees $1,\dots,n-1$.

For $0<p_i\le q_i$ with $i\in I$ such that $p_i+q_i=n$ and $|I|$ finite, we define a graded Λ–quadratic module H_I by

$$\mathbb{Z}^{g_1}[1]\oplus\cdots\oplus\mathbb{Z}^{g_{\lfloor n/2\rfloor}}\Big[\Big\lfloor\frac{n}{2}\Big\rfloor\Big]\oplus\operatorname{Hom}_{\mathbb{Z}}\Big(\mathbb{Z}^{g_{\lfloor n/2\rfloor}}\Big[\Big\lfloor\frac{n}{2}\Big\rfloor\Big],\mathbb{Z}[n]\Big)\oplus\cdots\oplus\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^{g_1}[1],\mathbb{Z}[n]),$$

where

$$g_k=|\{i\in I:p_i=k\}|.$$

Denote by $\{a_i\}$ the standard basis for $\mathbb{Z}^{g_1}[1]\oplus\cdots\oplus\mathbb{Z}^{g_{\lfloor n/2\rfloor}}[\lfloor n/2\rfloor]$ and by $\{b_i\}$ the dual basis. The pairing $\mu_{H_I}=\mu_I$ is then given by

$$\mu_I(a_i,b_j)=b_j(a_i)=\delta_{i,j}\quad\text{and}\quad\mu_I(a_i,a_j)=\mu_I(b_i,b_j)=0.$$

Denote $\operatorname{Aut}(H_I)$ in $Q^n_+(\mathbb{Z},\Lambda)$ by Γ_I. We get the following cases:

(1) When n is odd we get

$$\Gamma_I\cong\prod_{k=1}^{\lfloor n/2\rfloor}\operatorname{Gl}_{g_k}(\mathbb{Z}).$$

(2) When $n = 2d$ and d is even we necessarily have $\Lambda = 0$ and

$$\Gamma_I \cong O_{g_d,g_d}(\mathbb{Z}) \times \prod_{k=1}^{n/2-1} \mathrm{Gl}_{g_k}(\mathbb{Z}).$$

(3) Similarly for $n = 2d$ with d odd the only cases are $\Lambda = \mathbb{Z}, 2\mathbb{Z}$ and we just get a product of general linear groups and $\mathrm{Sp}_{2g_d}(\mathbb{Z})$, respectively the subgroup described in the list of automorphism groups in item (3) on page 3370.

Berglund and Madsen call a group G *rationally* perfect if $H^1(G; V) = 0$ for any finite-dimensional rational G–representation V. We later need that the automorphism groups of graded hyperbolic modules are rationally perfect.

Lemma 3.4 *The groups Γ_I are rationally perfect, unless they have a summand isomorphic to* $\mathrm{Aut}(H(\mathbb{Z}))$ *in* $Q^{-1}(\mathbb{Z}, 2\mathbb{Z})$.

Remark 3.5 The condition comes from the fact that the rational cohomology of $\mathrm{Aut}(H(\mathbb{Z}))$ in $Q^{-1}(\mathbb{Z}, 2\mathbb{Z})$ is nontrivial. As described in (3) above this is only the case when $n = 2d$ with d odd, $\lambda = -1$, $\Lambda = 2\mathbb{Z}$ and $g_d = 1$.

Proof We begin by observing that being rationally perfect is stable under group extensions; ie if in a group extension

$$0 \to K \to G \to C \to 0$$

K and C are rationally perfect, then so is G. This follows from the Lyndon spectral sequence, since $H^1(C; H^0(K; V))$ and $H^0(C; H^1(K; V))$ are trivial for any finite-dimensional rational G–representation V. In particular products of rationally perfect groups are rationally perfect. Moreover we observe that finite groups are rationally perfect.

It follows from Borel's work on the cohomology of arithmetic groups that the automorphism groups $\mathrm{Aut}(H(\mathbb{Z}^g))$ of the hyperbolic modules $H(\mathbb{Z}^g)$ in $Q^\lambda(\mathbb{Z}, \Lambda)$ are rationally perfect for $g \geq 2$ (see eg [5, Theorem A.1]).

In [2] it is shown that $\mathrm{Sl}_g(\mathbb{Z})$ is rationally perfect for $g \geq 3$. Since $\mathrm{Sl}_g(\mathbb{Z})$ is an index-two normal subgroup of $\mathrm{Gl}_g(\mathbb{Z})$, this now also implies that $\mathrm{Gl}_g(\mathbb{Z})$ is rationally perfect for $g \geq 3$.

Recall that the Γ_I are products of $\mathrm{Aut}(H(\mathbb{Z}^k))$ and instances of $\mathrm{Gl}_l(\mathbb{Z})$. Since $\mathrm{Gl}_1(\mathbb{Z})$ is finite and hence rationally perfect, to finish the proof we have to show that $\mathrm{Aut}(H(\mathbb{Z}))$ and $\mathrm{Gl}_2(\mathbb{Z})$ are rationally perfect.

The group $\mathrm{Sl}_2(\mathbb{Z})$ is an extension

$$0 \to C_2 \to \mathrm{Sl}_2(\mathbb{Z}) \to C_2 * C_3 \to 0,$$

and $C_2 * C_3$ can be seen to be rationally perfect using a Mayer–Vietoris argument. Hence $\mathrm{Sl}_2(\mathbb{Z})$ and also $\mathrm{Gl}_2(\mathbb{Z})$ are rationally perfect.

The group $\mathrm{Aut}(H(\mathbb{Z}))$ in $Q^{-1}(\mathbb{Z},\mathbb{Z})$ is $\mathrm{Sp}_2(\mathbb{Z})$, which is isomorphic to $\mathrm{Sl}_2(\mathbb{Z})$.

Recall that for $\lambda = 1$, we necessarily have $\Lambda = 0$ and $\mathrm{Aut}(H(\mathbb{Z})) \cong O_{1,1}(\mathbb{Z}) \cong C_2 \times C_2$ and hence we are done. □

4 Van der Kallen's and Charney's homological stability results

In this section we recall van der Kallen's homological stability for general linear groups and Charney's homological stability for automorphisms of hyperbolic quadratic modules. We combine them to homological stability for the Γ_I defined above with certain coefficient systems induced by polynomial functors.

Remark 4.1 Charney's results hold for Dedekind domains with involutions and van der Kallen's for associative rings with finite stable range, but we restrict our presentation to $\mathbb{Z}$ (with trivial involution).

We begin by reviewing the notion of coefficient systems as discussed in [10]. A coefficient system for $\{\mathrm{Gl}_g(\mathbb{Z})\}_{g\geq 1}$ is a sequence of $\mathrm{Gl}_g(\mathbb{Z})$–modules $\{\rho_g\}_{g\geq 1}$ together with $\mathrm{Gl}_g(\mathbb{Z})$–maps $F_g\colon \rho_g \to I^*(\rho_{g+1})$, where I^* denotes the restriction via the upper inclusion $I\colon \mathrm{Gl}_g(\mathbb{Z}) \hookrightarrow \mathrm{Gl}_{g+1}(\mathbb{Z})$. We denote the system by ρ and call the maps F_g structure maps. A map of coefficient systems ρ and ρ' is a collection of $\mathrm{Gl}_g(\mathbb{Z})$–maps $\{\tau_g\}_{g\geq 1}$ such that they commute with the structure maps. The levelwise kernels and cokernels are again coefficient systems with the obvious structure maps. Denote by $J\colon \mathrm{Gl}_g(\mathbb{Z}) \to \mathrm{Gl}_{g+1}(\mathbb{Z})$ the lower inclusion map. For a coefficient system ρ we define the shifted system $\Sigma\rho$ by $\Sigma\rho_g := J^*(\rho_{g+1})$ with structure maps $\Sigma F_g := J^* F_{g+1}\colon J^*(\rho_{g+1}) \to I^* J^*(\rho_{g+2})$. Denote by $s_g \in \mathrm{Gl}_g(\mathbb{Z})$ the element permuting the last two standard basis elements. We call a coefficient system central if s_{g+2} acts trivially on the image of $F_{g+1}F_g\colon \rho_g \to \rho_{g+2}$. Denote by $e_{g-1,g} \in \mathrm{Gl}_g(\mathbb{Z})$ the element sending all but the g^{th} standard basis element to itself and the g^{th}, e_g, to $e_{g-1} + e_g$. We call a central coefficient system strongly central if $e_{g+1,g+2}$ acts trivially on the image of $F_{g+1}F_g\colon \rho_g \to \rho_{g+2}$.

Let $c_g \in \mathrm{Gl}_g(\mathbb{Z})$ $(g > 1)$ be the element sending the i^{th} standard basis element to the $(i+1)^{\text{st}}$ and the g^{th} to the first.

Denote by $\mu(c_g)$ the multiplication from the left by c_g. Then the following holds:

Lemma 4.2 [10, Lemma 2.1] *Let ρ be a central coefficient system. Then we have a map of coefficient systems $\tau\colon \rho \to \Sigma\rho$, defined by*

$$\tau_g\colon \rho_g \xrightarrow{F_g} I^*(\rho_{g+1}) \xrightarrow{\mu(c_{g+2})} J^*(\rho_{g+1}) = \Sigma\rho_g.$$

We say that a central coefficient system ρ splits if $\Sigma\rho$ is isomorphic to $\rho \oplus \mathrm{coker}(\tau)$ via τ. We then denote $\mathrm{coker}(\tau)$ by $\Delta\rho$. We now define the notion of degree of a strongly central coefficient system ρ inductively. We say it has degree $k < 0$ if it is constant, and for $k \geq 0$ we say that it has degree $\leq k$ if $\Sigma\rho$ splits and $\Delta\rho$ is a strongly central coefficient system of degree $k-1$.

Theorem 4.3 (van der Kallen [15, page 291]) *Let ρ be a strongly central coefficient system of degree $\leq k$. Then*

$$H_i(\mathrm{Gl}_g(\mathbb{Z}), \rho_g) \to H_i(\mathrm{Gl}_{g+1}(\mathbb{Z}), \rho_{g+1})$$

is an isomorphism for $g > 2i + k + 2$ and an epimorphism for $g \geq 2i + k + 2$.

Denote by λ_g the standard representation of $\mathrm{Gl}_g(\mathbb{Z})$ on $\mathbb{Z}^g$ and by $\overline{\lambda}_g$ the action by the inverse transpose on $\mathbb{Z}^g$. Let $\mathcal{A}$ be an abelian category. Given a functor

$$T\colon \mathrm{Mod}(\mathbb{Z}) \times \mathrm{Mod}(\mathbb{Z}) \to \mathcal{A},$$

we define a coefficient system $\{T(\lambda_g, \overline{\lambda}_g)\}_{g \geq 1}$ with structure maps induced by the standard inclusions and actions induced by λ_g and $\overline{\lambda}_g$.

Lemma 4.4 (compare [15, 5.5] and [10, Lemma 3.1]) *If*

$$T\colon \mathrm{Mod}(\mathbb{Z}) \times \mathrm{Mod}(\mathbb{Z}) \to \mathcal{A}$$

is a polynomial functor of degree $\leq k$, then $\{T(\lambda_g, \overline{\lambda}_g)\}_{g \geq 1}$ is a strongly central coefficient system of degree $\leq k$.

Denote now by G_g the automorphism group of $H(\mathbb{Z}^g)$ in $Q^\lambda(\mathbb{Z}, \Lambda)$. And denote by $e_1, \dots, e_g$ the standard basis for $\mathbb{Z}^g$ and by $f_1, \dots, f_g$ the dual basis of $(\mathbb{Z}^g)^*$.

We see G_g as a subgroup of $\mathrm{Gl}_{2g}(\mathbb{Z})$, by considering the elements of G_g as $2g{\times}2g$–matrices acting on $H(\mathbb{Z}^g)\cong\mathbb{Z}^{2g}$. We define the upper inclusion

$$I\colon G_g\to G_{g+1},\qquad \begin{pmatrix}A & B\\ C & D\end{pmatrix}\mapsto\begin{pmatrix}A & 0 & B & 0\\ 0 & 1 & 0 & 1\\ C & 0 & D & 0\\ 0 & 1 & 0 & 1\end{pmatrix},$$

and similarly the lower inclusion $J\colon G_g\to G_{g+1}$. The definition of a coefficient system is very similar to the one for $\mathrm{Gl}_g(\mathbb{Z})$ and we only briefly summarize it. A coefficient system for $\{G_g\}_{g\ge 1}$ is a sequence of G_g–modules $\{\rho_g\}_{g\ge1}$ together with G_g–maps $F_g\colon \rho_g\to I^*(\rho_{g+1})$. We denote a coefficient system again by ρ and let maps of coefficient systems be as above. The shifted coefficient system $\Sigma\rho$ is the restriction via the lower inclusion as above. A coefficient system is called central if $c_{g+2}\circ c_{2g+4}$ acts trivially on the image of $F_{g+1}F_g\colon \rho_g\to\rho_{g+2}$. For a central coefficient system we define the map of coefficient systems $\tau\colon\rho\to\Sigma\rho$, by

$$\tau_g\colon \rho_g\xrightarrow{F_g} I^*(\rho_{g+1})\xrightarrow{\mu(c_{g+2}\circ c_{2g+4})} J^*(\rho_{g+1})=\Sigma\rho_g.$$

We call a central coefficient system ρ split if τ is injective and $\Sigma\rho\cong\tau(\rho)\oplus\mathrm{coker}(\tau)$. For a central coefficient system we define the degree inductively: we say it has degree $k<0$ if it is constant, and for $k\ge 0$ we say that it has degree $\le k$ if $\Sigma\rho$ splits and $\Delta\rho=\mathrm{coker}(\tau)$ is a strongly central coefficient system of degree $\le k-1$.

Theorem 4.5 (Charney [8, Theorem 4.3]) *Let ρ be a central coefficient system of degree $\le k$. Then*

$$H_i(G_g,\rho_g)\to H_i(G_{g+1},\rho_{g+1})$$

is an isomorphism for $g>2i+k+4$ and an epimorphism for $g\ge 2i+k+4$.

Again we get a central coefficient system of degree $\le k$ by considering the standard G_g–action $\lambda_{g,g}$ on $H(\mathbb{Z}^g)\cong\mathbb{Z}^{2g}$ induced by the inclusion $G_g\subset\mathrm{Gl}_{2g}(\mathbb{Z})$. Let $\mathcal{A}$ be an abelian category. If

$$T\colon \mathrm{Mod}(\mathbb{Z})\to\mathcal{A}$$

is a polynomial functor of degree $\le k$, then $\{T(\lambda_{g,g})\}_{g\ge1}$ is a central coefficient system of degree $\le k$ for $\{G_g\}_{g\ge1}$.

Now we combine the homological stability results above to a result for the groups Γ_I defined in the previous section.

Denote by λ_I the *standard representation of* Γ_I on $(\mathbb{Z}^{g_1},\dots,\mathbb{Z}^{2g_{n/2}},\dots,\mathbb{Z}^{g_{n-1}})$ when n is even and on $(\mathbb{Z}^{g_1},\dots,\mathbb{Z}^{g_{n-1}})$ when n is odd induced by the inclusion $\Gamma_I \subset \prod_{i=1}^{\lfloor n/2\rfloor}\mathrm{Gl}_{r_i}(\mathbb{Z})$ with

$$r_k = \begin{cases} 2g_k & \text{if } k=\frac{1}{2}n, \\ g_k & \text{if } k<\frac{1}{2}n, \end{cases} \qquad \text{where } g_k = |\{i\in I : p_i = k\}|.$$

More explicitly, we assume that an automorphism

$$A = (A_1,\dots,A_{\lfloor n/2\rfloor}) \in \Gamma_I \subset \prod_{i=1}^{\lfloor n/2\rfloor}\mathrm{Gl}_{r_i}(\mathbb{Z}),$$

acts by matrix multiplication of A_i on $\mathbb{Z}^{r_i}$ for $i\le n/2$ and by multiplication by the inverse transpose of A_i on $\mathbb{Z}^{r_{n-i}}$ for $i>n/2$.

Given a functor to an abelian category $T\colon \mathrm{Mod}(\mathbb{Z})^{n-1}\to\mathcal{A}$, we get a Γ_I–module $T(\mathbb{Z}^{r_1},\dots,\mathbb{Z}^{r_{n-1}})$ with the induced action. We denote this Γ_I–module by $T(\lambda_I)$. For a fixed $p\in\mathbb{N}$ such that $0<p\le n/2$, denote by $\Gamma_{I'}$ the automorphism group of $H_{I'}$, where $I' = I\cup\{i'\}$ with $p_{i'}=p$ and $q_{i'}=n-p$. We define the *stabilization map*

$$\sigma_{p,n-p}\colon H_i(\Gamma_I, T(\lambda_I)) \to H_i(\Gamma_{I'}, T(\lambda_{I'}))$$

to be the map induced by the obvious upper inclusion $I_{p,q}\colon \Gamma_I\to\Gamma_{I'}$ and $T(I_{p,q})$.

Proposition 4.6 *Let $\mathcal{A}$ be some abelian category, and let $T\colon \mathrm{Mod}(\mathbb{Z})^{n-1}\to\mathcal{A}$ be a polynomial functor of degree $\le k$. The stabilization map*

$$\sigma_{p,n-p}\colon H_i(\Gamma_I, T(\lambda_I)) \to H_i(\Gamma_{I'}, T(\lambda_{I'}))$$

induces an isomorphism for $g_p > 2i+k+2$ when $2p\ne n$ and $g_p>2i+k+4$ if $2p=n$ and an epimorphism for $g_p\ge 2i+k+2$, respectively $g_p\ge 2i+k+4$, unless $n=2d$ with d odd, $\lambda=-1$, $\Lambda=2\mathbb{Z}$ and $g_d=1$.

Proof Denote by Γ_{g_p} the summand of $\Gamma_I\subset\prod_{i=1}^{\lfloor n/2\rfloor}\mathrm{Gl}_{r_i}(\mathbb{Z})$ that sits in $\mathrm{Gl}_{r_p}(\mathbb{Z})$. Let $\Gamma=\mathrm{Aut}(H_{\bar I})$, where $\bar I = I\smallsetminus\{i\in I \mid p_i=p\}$. Note that $\Gamma\times\Gamma_{g_p}=\Gamma_I$ and $\Gamma_{I'}=\Gamma_{g_p+1}\times\Gamma$, where Γ_{g_p+1} is defined analogously to Γ_{g_p}. Consider the functor

$$\mathfrak{I}_{p,n-p}\colon \begin{cases} \mathrm{Mod}(\mathbb{Z})\to\mathrm{Mod}(\mathbb{Z})^{n-1} & \text{if } 2p=n, \\ \mathrm{Mod}(\mathbb{Z})\times\mathrm{Mod}(\mathbb{Z})\to\mathrm{Mod}(\mathbb{Z})^{n-1} & \text{otherwise,} \end{cases}$$

defined by sending a module M to $(\mathbb{Z}^{g_1},\dots,M,\dots,\mathbb{Z}^{g_{n-1}})$, where the M sits at the $(n/2)^{\text{th}}$ summand, and a pair (M,N) to $(\mathbb{Z}^{g_1},\dots,M,\dots,N,\dots,\mathbb{Z}^{g_{n-1}})$, where the M sits at the p^{th} summand and the N sits at the $(n-p)^{\text{th}}$ summand. This functor is clearly additive and hence of degree ≤ 1. This implies that the composition $T \circ \mathfrak{I}_{p,n-p}$ is of degree $\leq k$ and we get a (strongly) central coefficient system of degree $\leq k$ for

$$\Gamma_{g_p} = \begin{cases} G_{g_p} & \text{if } 2p = n, \\ \mathrm{Gl}_{g_p}(\mathbb{Z}) & \text{otherwise.} \end{cases}$$

This implies that the stabilization maps

$$H_i(\Gamma_{g_p}, T \circ \mathfrak{I}_{n/2,n/2}(\lambda_{g_p,g_p})) \to H_i(\Gamma_{g_p+1}, T \circ \mathfrak{I}_{n/2,n/2}(\lambda_{g_p+1,g_p+1})),$$

respectively

$$H_i(\Gamma_{g_p}, T \circ \mathfrak{I}_{p,n-p}(\lambda_{g_p}, \overline{\lambda}_{g_p})) \to H_i(\Gamma_{g_p+1}, T \circ \mathfrak{I}_{p,n-p}(\lambda_{g_p+1}, \overline{\lambda}_{g_p+1})),$$

are isomorphisms, respectively epimorphisms, in the ranges in the statement of the proposition. Observing that the $T \circ \mathfrak{I}_{n/2,n/2}(\lambda_{g_p,g_p})$, respectively $T \circ \mathfrak{I}_{p,n-p}(\lambda_{g_p}, \overline{\lambda}_{g_p})$, are precisely the restrictions of the Γ_I–representation to the subgroup Γ_{g_p} and using the Lyndon spectral sequence

$$H_k\big(\Gamma_{\bar{I}}, H_l(\Gamma_{g_p}, T(\lambda_I))\big) \Rightarrow H_i(\Gamma_I, T(\lambda_I)),$$

the results follows by comparing spectral sequences. □

5 On mapping class groups

Write

$$N = N_I = \Big(\#_{i \in I}(S^{p_i} \times S^{q_i})\Big) \smallsetminus \mathrm{int}(D^n),$$

where $|I| < \infty$, $3 \leq p_i \leq q_i < 2p_i - 1$ and $p_i + q_i = n$ for $i \in I$.

Denote by $\mathrm{Aut}(\widetilde{H}_*(N_I))$ the automorphisms of the graded group $\widetilde{H}_*(N_I)$. In this section we study the map

$$H_*\colon \pi_0 \,\mathrm{aut}_\partial(N_I) \to \mathrm{Aut}(\widetilde{H}_*(N_I)).$$

In particular we are going to determine its image and show that the kernel is finite.

Denote by $\mathrm{incl}\colon \partial N \hookrightarrow N$ the inclusion of the boundary. We observe that $V_I = \bigvee_{i \in I}(S^{p_i} \vee S^{q_i}) \subset N$ is a deformation retract and denote by

$$\alpha_j\colon S^{p_j} \hookrightarrow \bigvee_{i \in I}(S^{p_i} \vee S^{q_i}) \quad \text{and} \quad \beta_j\colon S^{q_j} \hookrightarrow \bigvee_{i \in I}(S^{p_i} \vee S^{q_i})$$

the inclusions. We consider incl as an element of $\pi_{n-1}\big(\bigvee_{i\in I}(S^{p_i}\vee S^{q_i})\big)$ and we observe that it is given by the sum of Whitehead products $\sum_{i\in I}[\alpha_i,\beta_i]$. Denote by

$$\langle -,-\rangle\colon H_*(N)\otimes H_{n-*}(N)\to\mathbb{Z},\quad x\otimes y\mapsto(\mathrm{PD}^{-1}(x)\cup\mathrm{PD}^{-1}(y))([N,\partial N]),$$

the intersection form, where $\mathrm{PD}^{-1}\colon H_*(N)\to H^{n-*}(N,\partial N)$ denotes the Poincaré duality isomorphisms and we evaluate on the fundamental class $[N,\partial N]$. The $\{\alpha_i\}$ and $\{\beta_i\}$ define a basis for $\widetilde{H}_*(N)$ via the Hurewicz homomorphism, which we denote by $\{a_i\}$, respectively $\{b_i\}$. Note that b_i is dual to a_i.

In the case $n=2d$ is even we need to recall a further piece of structure from Wall's classification of highly connected even-dimensional manifolds [29]. The elements $x\in H_d(N)$ can be represented by embedded S^d. Denote by $\nu_x\in\pi_{d-1}(\mathrm{SO}(d))$ the clutching function of the normal bundle of this embedding. It is independent of the choice of embedding, since homotopic embeddings are isotopic in this case. This defines a function

$$q\colon H_d(N)\to\pi_{d-1}(\mathrm{SO}(d)),\quad x\mapsto[\nu_x].$$

Denote by ι_d the class of the identity in $\pi_d(S^d)$ and by

$$\partial\colon \pi_d(S^d)\to\pi_{d-1}(\mathrm{SO}(d))$$

the boundary map in the fibration $\mathrm{SO}(d)\to\mathrm{SO}(d+1)\to S^d$. The function q satisfies

$$\langle x,x\rangle=HJq(x)\quad\text{and}\quad q(x+y)=q(x)+q(y)+\langle x,y\rangle\partial\iota_d,$$

where $\pi_{d-1}(\mathrm{SO}(d))\xrightarrow{J}\pi_{2d-1}(S^d)\xrightarrow{H}\mathbb{Z}$ denote the J–homomorphism and the Hopf invariant. There is also a purely homotopy theoretic description of Jq in [16, Section 8].

Note that for $a_i,b_j\in H_d(N)$, we have $q(a_i)=q(b_j)=0$. Hence Image(q) is contained in the subgroup $\langle\partial\iota_d\rangle$ generated by $\partial\iota_d$. The J–homomorphism restricts to an isomorphism

$$J|_{\langle\partial\iota_d\rangle}\colon\langle\partial\iota_d\rangle\to J(\langle\partial\iota_d\rangle)\cong\begin{cases}\langle[\iota_d,\iota_d]\rangle\xrightarrow[H]{\cong}2\mathbb{Z} & \text{if } d \text{ is even,}\\ 0 & \text{if } d=1,3,7,\\ \langle[\iota_d,\iota_d]\rangle\cong\mathbb{Z}/2\mathbb{Z} & \text{if } d \text{ is odd and not } 1,3 \text{ or } 7,\end{cases}$$

where the second isomorphism is induced by the Hopf invariant. Let

$$\mathrm{Aut}(\widetilde{H}_*(N),\langle -,-\rangle,Jq)\quad\text{and}\quad\mathrm{Aut}(\widetilde{H}_*(N),\langle -,-\rangle,q)$$

be the automorphisms of the reduced homology respecting the intersection form and the function Jq (respectively q). Note that

$$\langle x, y\rangle = \mu(x, y) + (-1)^{|x||y|}\mu(y, x),$$

where $\mu(-,-)$ is determined by

$$\mu(a_i, b_j) = \delta_{i,j} \quad \text{and} \quad \mu(b_i, a_j) = \mu(a_i, a_j) = \mu(b_i, b_j) = 0.$$

Now let

$$\Lambda = \begin{cases} 0 & \text{if } n = 2d \text{ and } d \text{ is even,} \\ \mathbb{Z} & \text{if } n = 2d \text{ and } d \text{ is 3 or 7,} \\ 2\mathbb{Z} & \text{if } n = 2d \text{ and } d \text{ is odd and not 3 or 7.} \end{cases}$$

Moreover $Jq = q_\mu$, where q_μ is the Λ–quadratic form associated to μ, where we identify $\langle [\iota_d, \iota_d]\rangle$ with $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ respectively. It suffices to check this for the elements $a_i + b_i$, and for these $Jq(a_i + b_i) = [\iota_d, \iota_d]$ and $q_\mu(a_i + b_i) = 1$. By the discussion above we see that

$$\mathrm{Aut}(\widetilde{H}_*(N), \langle -,-\rangle, q) \cong \mathrm{Aut}(\widetilde{H}_*(N), \langle -,-\rangle, Jq) \cong \Gamma_I = \mathrm{Aut}(H_I) \quad \text{in } Q^n_*(\mathbb{Z}, \Lambda).$$

For a representative f of $[f] \in \pi_0(\mathrm{aut}_\partial(N))$ it is clear that $\widetilde{H}_*(f) \in \Gamma_I$. We are now going to show that all elements of Γ_I can be realized by a homotopy self-equivalence, fixing the boundary pointwise.

Proposition 5.1 *The group homomorphism*

$$\pi_0(\mathrm{aut}_\partial(N)) \xrightarrow{\widetilde{H}_*} \mathrm{Aut}(\widetilde{H}_*(N), \langle -,-\rangle, Jq)$$

is surjective and has finite kernel.

Proof Compare [4, Proof of Theorem 2.10]. The cofibration incl: $\partial N \hookrightarrow N$ induces a fibration

$$\mathrm{map}_\partial(N, N) \to \mathrm{map}_*(N, N) \to \mathrm{map}_*(\partial N, N),$$

where $\mathrm{map}_\partial(N, N)$ is the fiber over incl. Let $\mathrm{map}_\partial(N, N)$ and $\mathrm{map}_*(N, N)$ be based at the identity. Restricting the total space to invertible elements, we also get the fibration

$$\mathrm{aut}_\partial(N) \to \mathrm{aut}_*(N) \to \mathrm{map}_*(\partial N, N).$$

We are going to analyze the long exact homotopy sequences

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \pi_1(\mathrm{map}_*(\partial N, N), \mathrm{incl}) & \longrightarrow & \pi_0(\mathrm{aut}_\partial(N)) & \longrightarrow & \pi_0(\mathrm{aut}_*(N)) & \longrightarrow & [\partial N, N]_* \\ & \| & & \downarrow & & \downarrow & & \| \\ \cdots \longrightarrow & \pi_1(\mathrm{map}_*(\partial N, N), \mathrm{incl}) & \longrightarrow & [N, N]_\partial & \longrightarrow & [N, N]_* & \longrightarrow & [\partial N, N]_* \end{array} \tag{5}$$

We consider the monoid homomorphism

$$\widetilde{H}_*\colon [N, N]_* \to \mathrm{End}(\widetilde{H}_*(N))$$

and show that it is onto and with finite kernel. Using the relative Hurewicz isomorphism, it is easy to see that $V_I \hookrightarrow \prod_{i\in I}(S^{p_i} \times S^{q_i})$ is $(2\min_{i\in I}\{p_i\}-1)$–connected and hence more than $\max_{i\in I}\{q_i\}$–connected (note that we use the connectivity assumption $p_i \le q_i < 2p_i - 1$ here). Thus we get a bijection

$$\begin{aligned} [N, N]_* \cong [V_I, V_I]_* &\cong \Big[V_I, \prod_{i\in I}(S^{p_i} \times S^{q_i})\Big]_* \\ &\cong \prod [S^{p_i}, S^{p_j}]_* \times \prod [S^{q_i}, S^{q_j}]_* \times \prod [S^{q_i}, S^{p_j}]_* \times \prod [S^{p_i}, S^{q_j}]_*, \end{aligned} \tag{6}$$

where the products in the last line are over $(i, j) \in I \times I$. We write $I = \bigcup_l I_l$, where $I_l = \{i \mid p_i = l\}$. The only nonfinite factors of the product above are

$$\prod_l \prod_{(i,j)\in I_l\times I_l} ([S^{p_i}, S^{p_j}]_* \times [S^{q_i}, S^{q_j}]_*). \tag{7}$$

We make the identification $\mathrm{End}(\widetilde{H}_*(N)) \cong \prod \mathrm{Mat}_{r_l}(\mathbb{Z})$, where $r_l = \mathrm{rank}(H_l(N))$, using the basis $\{a_i\} \cup \{b_i\}$. Note that for $l = n/2$, a b_i becomes a $(r_l/2+i)^{\text{th}}$ basis element. Denote by $\alpha_1^l, \dots, \alpha_{r_l}^l$ and $\beta_1^l, \dots, \beta_{r_l}^l$ the inclusions $S^l \hookrightarrow V_I$ and $S^{n-l} \hookrightarrow V_I$ respectively. There is a multiplicative section of $\widetilde{H}_*$,

$$\prod \mathrm{Mat}_{r_l}(\mathbb{Z}) \to [N, N]_*, \quad (M^l) = (m_{i,j}^l) \mapsto f_{(M^l)} = \bigvee_{l=1}^{\lfloor n/2 \rfloor} f_{M^l}, \tag{8}$$

where $f_{M^l}\colon \bigvee_{i\in I_l} S^{p_i} \vee S^{q_i} \to V_I$ is given by

$$f_{M^l} = \begin{cases} \bigvee_{i=1}^{r_l} \big(\sum_{j=1}^{r_l} m_{i,j}^l \alpha_j^l \vee \sum_{j=1}^{r_l} m_{i,j}^{n-l} \beta_j^l\big) & \text{if } l \ne \frac{1}{2}n, \\ \bigvee_{i=1}^{r_l/2} \big(\sum_{j=1}^{r_l/2} m_{i,j}^l \alpha_j^l + \sum_{j=r_l/2+1}^{r_l} m_{i,j}^l \beta_{(j-r_l/2)}^l\big) & \\ \quad \vee \bigvee_{i=r_l/2+1}^{r_l} \big(\sum_{j=1}^{r_l/2} m_{i,j}^l \alpha_j^l + \sum_{j=r_l/2+1}^{r_l} m_{i,j}^l \beta_{(j-r_l/2)}^l\big) & \text{if } l = \frac{1}{2}n. \end{cases}$$

We observe that the image of this section is precisely the submonoid of $[N,N]_*$ corresponding to the nonfinite factors (7). Hence we get that $\widetilde{H}_*\colon [N,N]_* \to \mathrm{End}(\widetilde{H}_*(N))$ is surjective and has finite kernel. Restricting to the submonoids of invertible elements this implies upon using the section (8) that $\pi_0(\mathrm{aut}_*(N)) \to \mathrm{Aut}(\widetilde{H}_*(N))$ is surjective with finite kernel. The image of $\pi_0(\mathrm{aut}_\partial(N)) \to \pi_0(\mathrm{aut}_*(N))$ consists of the elements $[f] \in \pi_0(\mathrm{aut}_*(N))$ such that $f \circ \mathrm{incl} \simeq \mathrm{incl}$ (we assume all homotopy equivalences in this proof to be pointed). Since (8) restricts to a section $\mathrm{Aut}(\widetilde{H}_*(N)) \to \pi_0(\mathrm{aut}_*(N))$ we get that the image of

$$\widetilde{H}_*\colon \pi_0(\mathrm{aut}_\partial(N)) \to \mathrm{Aut}(\widetilde{H}_*(N))$$

is given by the (M^l) such that $f_{(M^l)} \circ \mathrm{incl} \simeq \mathrm{incl}$. Using the Hilton–Milnor theorem we make the identification

$$\pi_{n-1}(N) \cong \pi \oplus \bigoplus_l \pi_{n-1}\Bigl(\bigvee_{i\in I_l} (S^{p_i} \vee S^{q_i})\Bigr), \tag{9}$$

where π is some subgroup of $\pi_{n-1}(N)$. We observe that

$$f_{(M^l)} \circ \mathrm{incl} \simeq \sum_{i\in I} [f_{(M^l)} \circ \alpha_i, f_{(M^l)} \circ \beta_i] \simeq \sum_{l=1}^{\lfloor n/2\rfloor} \sum_{i\in I_l} [f_{M^l} \circ \alpha_i^l, f_{M^l} \circ \beta_i^l],$$

ie that the action of $f_{(M^l)}$ respects the summands of the identification (9). Thus it suffices to check that $f_{M^l} \circ \sum_{i\in I_l} [\alpha_i^l, \beta_i^l] \simeq \sum_{i\in I_l} [\alpha_i^l, \beta_i^l]$ for all l. We use that left homotopy composition is distributive for suspensions [32, page 126], ie that $(x+y)\circ \Sigma z \simeq x \circ \Sigma z + y \circ \Sigma z$. For $l \neq n/2$ we calculate

$$\begin{aligned} f_{M^l} \circ \sum_{i=1}^{r_l} [\alpha_i^l, \beta_i^l] &\simeq \sum_{i=1}^{r_l} [f_{M^l} \circ \alpha_i^l, f_{M^l} \circ \beta_i^l] \simeq \sum_{i,j,k} [m_{i,j}^l \alpha_j^l, m_{i,k}^{n-l} \beta_k^l] \\ &\simeq \sum_{i,j,k} m_{i,j}^l m_{i,k}^{n-l} [\alpha_j^l, \beta_k^l] \simeq \sum_{j,k} ((M^l)^T M^{n-l})_{j,k} [\alpha_j^l, \beta_k^l]. \end{aligned}$$

This expression is homotopic to $\sum_{i=1}^{r_l} [\alpha_i^l, \beta_i^l]$ if

$$(M^l)^T M^{n-l} = \mathrm{id}_{\mathrm{Mat}_{r_l}(\mathbb{Z})}. \tag{10}$$

For $l = n/2$ we write

$$M^l = \begin{pmatrix} A^l & B^l \\ C^l & D^l \end{pmatrix}.$$

We calculate

$$
\begin{aligned}
f_{M^l} \circ \sum_{i=1}^{r_l/2} [\alpha_i^l, \beta_i^l] &\simeq \sum_{i=1}^{r_l/2} [f_{M^l} \circ \alpha_i^l, f_{M^l} \circ \beta_i^l] \\
&\simeq \sum_{i=1}^{r_l/2} \Bigl[\sum_{j=1}^{r_l/2} (a_{i,j}^l \alpha_j^l + b_{i,j}^l \beta_j^l), \sum_{k=1}^{r_l/2} (c_{i,k}^l \alpha_k^l + d_{i,k}^l \beta_k^l) \Bigr] \\
&\simeq \sum_{i=1}^{r_l/2} \sum_{j=1}^{r_l/2} \sum_{k=1}^{r_l/2} a_{i,j}^l d_{i,k}^l [\alpha_j^l, \beta_k^l] + \sum_{i=1}^{r_l/2} \sum_{j=1}^{r_l/2} \sum_{k=1}^{r_l/2} b_{i,j}^l c_{i,k}^l [\beta_j^l, \alpha_k^l] \\
&\qquad + \sum_{i=1}^{r_l/2} \sum_{j=1}^{r_l/2} \sum_{k=1}^{r_l/2} a_{i,j}^l c_{i,k}^l [\alpha_j^l, \alpha_k^l] + \sum_{i=1}^{r_l/2} \sum_{j=1}^{r_l/2} \sum_{k=1}^{r_l/2} b_{i,j}^l d_{i,k}^l [\beta_j^l, \beta_k^l] \\
&\simeq \sum_{j=1}^{r_l/2} \sum_{k=1}^{r_l/2} \bigl((A^l)^T D^l + (-1)^{n/2}((C^l)^T B^l)\bigr)_{j,k} [\alpha_j^l, \beta_k^l] \\
&\quad + \sum_{j=1}^{r_l/2} \sum_{k=1}^{r_l/2} ((A^l)^T C^l)_{j,k} [\alpha_j^l, \alpha_k^l] + \sum_{j=1}^{r_l/2} \sum_{k=1}^{r_l/2} ((B^l)^T D^l)_{j,k} [\beta_j^l, \beta_k^l].
\end{aligned}
$$

This expression is homotopic to $\sum_{i=1}^{r_l/2} [\alpha_i^l, \beta_i^l]$ if

$$
\begin{gathered}
(A^l)^T D^l + (-1)^{n/2} (C^l)^T B^l = 1, \\
(A^l)^T C^l + (-1)^{n/2} (C^l)^T A^l = 0, \\
(B^l)^T D^l + (-1)^{n/2} (D^l)^T B^l = 0, \\
(A^l)^T C^l \text{ and } (B^l)^T D^l \text{ have diagonal entries in } \Lambda,
\end{gathered}
$$

where

$$
\Lambda = \begin{cases} 2\mathbb{Z} & \text{if } n = 2d \text{ and } d \text{ is odd and not 3 or 7}, \\ \mathbb{Z} & \text{if } n = 2d \text{ and } d \text{ is 3 or 7}, \\ 0 & \text{if } n = 2d \text{ and } d \text{ is even}. \end{cases}
$$

The diagonal entries of $(A^l)^T C^l$ and $(B^l)^T D^l$ have to be in Λ to kill the elements $[\alpha_i^l, \alpha_i^l]$ and $[\beta_i^l, \beta_i^l]$. These are exactly the conditions to be an automorphism of $H(\mathbb{Z}^{g_{n/2}})$ in $Q^{(-1)^{n/2}}(\mathbb{Z}, \Lambda)$. Combining this with the condition in (10) we see that the image of $\widetilde{H}_*$ in $\mathrm{Aut}(\widetilde{H}_*(N))$ is given by

$$
\Gamma_I \subset \prod_{k=1}^{\lfloor n/2 \rfloor} \mathrm{Gl}_{r_k}(\mathbb{Z}).
$$

Thus we proved that

$$\widetilde{H}_*\colon \pi_0(\mathrm{aut}_\partial(N)) \to \mathrm{Aut}(\widetilde{H}_*(N), \langle -,- \rangle, Jq)$$

is surjective. To show that the kernel is finite it suffices to check that $\pi_0(\mathrm{aut}_\partial(N)) \to \pi_0(\mathrm{aut}_*(N))$ has finite kernel. This follows from the fact that

$$\pi_1(\mathrm{incl}^*)\colon \pi_1(\mathrm{aut}_*(N), \mathrm{id}_N) \otimes \mathbb{Q} \to \pi_1(\mathrm{map}_*(\partial N, N), \mathrm{incl}) \otimes \mathbb{Q}$$

is surjective for the manifolds N_I with $3 \le p_i \le q_i < 2p_i - 1$, as we will see in Remark 6.7. □

In fact it suffices to know Γ_I for our purposes, since the action of elements of the kernel of $\widetilde{H}_*$ by conjugation is trivial up to homotopy.

Lemma 5.2 *Let f represent an element of the kernel of*

$$\widetilde{H}_*\colon \pi_0(\mathrm{aut}_\partial(N_I)) \to \Gamma_I.$$

Then $f^{-1} \circ g \circ f \simeq g$ for all $g \in \mathrm{aut}_\partial(N_I)$.

Proof Note that if $[f]$ is in the kernel of $\widetilde{H}_*$ then by exactness of (5) it is given by an element $\partial\alpha$, where $\alpha \in \pi_1(\mathrm{map}_*(\partial N_I, N_I), \mathrm{incl}) \cong \pi_n(N_I)$. We represent α as a map

$$\alpha(x,t)\colon \partial N_I \times I \to N_I \quad \text{such that } \alpha(x,0) = \alpha(x,1) = \mathrm{id}_{\partial N_I}.$$

Choosing a collar neighborhood of ∂N_I allows us to make the identification

$$N_I \xrightarrow{\cong} N_I \cup_{\mathrm{id}_{\partial N_I}} \partial N_I \times I.$$

Now we represent f by the composite

$$N_I \xrightarrow{\cong} N_I \cup_{\mathrm{id}_{\partial N_I}} \partial N_I \times I \xrightarrow{\mathrm{id}_{N_I} \cup \alpha} N_I.$$

We represent f^{-1} similarly using $-\alpha$. If we now represent $f^{-1} \circ g \circ f$ by the composition

$$N_I \xrightarrow{\cong} N_I \cup_{\mathrm{id}_{\partial N_I}} \partial N_I \times I \xrightarrow{\mathrm{id}_{N_I} \cup \alpha} N_I \xrightarrow{\cong} N_I \cup_{\mathrm{id}_{\partial N_I}} \partial N_I \times I$$
$$\xrightarrow{g \cup \mathrm{id}_{\partial N_I \times I}} N_I \cup_{\mathrm{id}_{\partial N_I}} \partial N_I \times I \xrightarrow{\mathrm{id}_{N_I} \cup -\alpha} N_I,$$

ie consider α and $-\alpha$ as having domain two iteratively attached closed cylinders and see that it is homotopic to

$$N_I \xrightarrow{\cong} N_I \cup_{\mathrm{id}_{\partial N_I}} \partial N_I \times I \cup_{\mathrm{id}_{\partial N_I}} \partial N_I \times I \xrightarrow{g \cup \alpha \cup -\alpha} N_I.$$

This is homotopic (rel ∂N_I) to g, since

$$\partial N_I \times I \cup_{\mathrm{id}_{\partial N_I}} \partial N_I \times I \xrightarrow{\alpha \cup -\alpha} N_I$$

is homotopic to the inclusion of $\partial N_I \times I \cup_{\mathrm{id}_{\partial N_I}} \partial N_I \times I$ as a collar. □

Denote by $\mathrm{Diff}_\partial(N)$ the group of self-diffeomorphisms of N fixing a collar neighborhood of the boundary pointwise with the Whitney C^∞–topology. Let $J\colon \mathrm{Diff}_\partial(N) \to \mathrm{aut}_\partial(N)$ be the inclusion. To show homological stability for the block diffeomorphism group the following fact about the mapping class group suffices.

Proposition 5.3 (1) *The map* $\widetilde{H}_*\colon \pi_0(\mathrm{Diff}_\partial(N)) \to \mathrm{Aut}(\widetilde{H}_*(N), \langle -, - \rangle, q)$ *is surjective.*

(2) *The image of* $\pi_0(J)\colon \pi_0(\mathrm{Diff}_\partial(N)) \to \pi_0(\mathrm{aut}_\partial(N))$ *has finite index.*

Proof The first part follows from [18] and [30, Lemma 17]. Kreck shows that all elements of $\mathrm{Aut}(\widetilde{H}_{n/2}(N), \langle -, - \rangle, q)$ can be realized as self-diffeomorphisms of

$$\big(\#_{g_{n/2}}(S^{n/2} \times S^{n/2})\big) \smallsetminus \mathrm{int}(D^n)$$

fixing the boundary pointwise. Wall shows that for manifolds $\natural_g(D^{q+1} \times S^p)$, where $3 \le p \le q$ and $\natural_g$ denotes the g–fold boundary connected sum, all automorphisms of the homology are realized by diffeomorphisms. Hence it follows for manifolds $\#_{g_i}(S^{p_i} \times S^{q_i})$. Since we can assume that a diffeomorphism fixes a disk up to isotopy, we get it in particular for $\big(\#_{g_i}(S^{p_i} \times S^{q_i})\big) \smallsetminus \mathrm{int}(D^n)$. Using the diffeomorphisms above and extending them by the identity on the complement of the manifolds above the claim follows. The second part follows from the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_0(\mathrm{S\,Diff}_\partial(N)) & \longrightarrow & \pi_0(\mathrm{Diff}_\partial(N)) & \xrightarrow{\widetilde{H}_*} & \mathrm{Aut}(\widetilde{H}_*(N), \langle -, - \rangle, q) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \pi_0(J)} & & \downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & \pi_0(\mathrm{S\,aut}_\partial(N)) & \longrightarrow & \pi_0(\mathrm{aut}_\partial(N)) & \xrightarrow{\widetilde{H}_*} & \mathrm{Aut}(\widetilde{H}_*(N), \langle -, - \rangle, Jq) & \longrightarrow & 0
\end{array}$$

where $\pi_0(\mathrm{S\,Diff}_\partial(N))$ and $\pi_0(\mathrm{S\,aut}_\partial(N))$ denote the kernels of the maps $\widetilde{H}_*$ and the fact that $\pi_0(\mathrm{S\,aut}_\partial(N))$ is finite by Proposition 5.1. □

Remark 5.4 There is much literature on the groups of components of mapping spaces of closed manifolds in different categories. Highly connected even-dimensional manifolds are for example studied in [18] and [14]. Products of spheres are studied in [19; 26; 28]. Homotopy self-equivalences of manifolds and in particular of connected sums of products of spheres are treated in [3].

For later use we need the following lemma.

Lemma 5.5 *The groups* $\pi_0\mathrm{aut}_\partial(N_I)$ *and* $\mathrm{Image}(\pi_0(J))$ *are rationally perfect, unless* $n = 2d$ *with* $d \neq 3, 7$ *odd, and* $g_d = 1$.

Proof This follows from Lemma 3.4 and the fact that the groups are finite extensions of Γ_I. □

6 On the rational homotopy type of homotopy automorphisms

In the last section we determined the group

$$\pi_I = \pi_1(B\mathrm{aut}_\partial(N_I), \mathrm{id}_{N_I}) \cong \pi_0(\mathrm{aut}_\partial(N_I))$$

up to finite extensions. It acts on the simply connected covering $X_I = B\mathrm{aut}_\partial(N_I)\langle 1\rangle$ by deck transformations. This section has two goals:

(1) Describe the π_I–modules $H_*(X_I;\mathbb{Q})$ algebraically (Proposition 6.9).

(2) Make sure the algebraic model is appropriate for showing homological stability using the results in Section 4 (Proposition 6.3).

All results in this section are either contained in [5] or straightforward generalizations.

We assume some familiarity with Quillen's approach to rational homotopy theory [24], ie the functor

$$\lambda\colon \mathrm{Top}_1 \to \mathrm{dgL}_0$$

from the category of simply connected based topological spaces to the category of reduced differential graded (dg) Lie algebras. It induces an equivalence of homotopy categories, where the weak equivalences in Top_1 are isomorphisms in rational homotopy groups and in dgL_0 quasi-isomorphisms. The homology of $\lambda(X)$ allows us to recover the rational homotopy groups of X. More precisely, there is an isomorphism of graded Lie algebras

$$H_*(\lambda(X)) \cong \pi_*(\Omega X) \otimes \mathbb{Q},$$

where the Lie bracket on the right-hand side is given by the Samelson product. The rational homology of X is given by the Chevalley–Eilenberg homology of $\lambda(X)$, which we will explain later.

For a given simply connected space X the value $\lambda(X)$ is in general very complicated and one considers dg Lie models instead. A *dg Lie model* for a simply connected topological space X is a free differential graded Lie algebra $(\mathbb{L}(V),\partial)$, together with a quasi-isomorphism

$$(\mathbb{L}(V),\partial)\xrightarrow{\simeq}\lambda(X).$$

When $\pi_*(\Omega X)\otimes\mathbb{Q}$ is quasi-isomorphic to $\lambda(X)$, the space X is called *coformal.*

6.1 On a dg Lie model for the simply connected covering of the homotopy automorphisms

Since $N_I\simeq\bigvee_{i\in I}(S^{p_i}\vee S^{q_i})$, the free Lie algebra $\mathbb{L}(s^{-1}\widetilde{H}_*(N_I,\mathbb{Q}))$ with trivial differential is a dg Lie model for N_I, where s^{-1} denotes the desuspension. We are going to write

$$\mathbb{L}_I=\mathbb{L}(s^{-1}\widetilde{H}_*(N_I,\mathbb{Q})).$$

Recall that we denoted the homology classes represented by the inclusions

$$\alpha_i\colon S^{p_i}\hookrightarrow N_I\quad\text{and}\quad\beta_i\colon S^{q_i}\hookrightarrow N_I,$$

by a_i, respectively b_i. Write

$$\omega_I=\sum_{i\in I}-(-1)^{|a_i|}[s^{-1}a_i,s^{-1}b_i].$$

We model the inclusion of the boundary incl: $\partial N_I\to N_I$ by

$$\mathbb{L}(\gamma)\to\mathbb{L}_I,\quad\gamma\mapsto\omega_I,$$

where $\mathbb{L}(\gamma)$ is generated by a single generator of degree $n-2$.

Note that the manifolds N_I are highly connected in the sense that they are $(m-1)$–connected n–manifolds, where $n\leq 3m-2$. In fact we can obtain every closed highly connected odd-dimensional manifold up to rational homotopy equivalence by attaching a disk to N_I.

Proposition 6.1 *Let M be a closed $(m-1)$–connected $(2n+1)$–manifold, where $n\leq\frac{1}{2}(3m-3)$. Then there exists an N_I such that*

$$M\simeq_{\mathbb{Q}}N_I\cup_{S^{2n}}D^n.$$

Proof There is a differential graded Lie model of M generated by the desuspended reduced rational homology $s^{-1}\widetilde{H}_*(M,\mathbb{Q})$ and a differential ∂. For degree reasons the differential can only be nonzero on the desuspension of the fundamental class. Moreover it can be chosen to be of the form

$$\partial(s^{-1}[M]) = \frac{1}{2}\sum \pm[s^{-1}e_i, s^{-1}(e_i^{\#})],$$

for some basis $\{e_i\}$ of $\widetilde{H}_*(M \setminus \{*\}, \mathbb{Q})$ because of Poincaré duality, where $^{\#}$ denotes the dual with respect to the intersection pairing (see [27, Theorem 2]). This however is the differential graded Lie model of a manifold $N_I \cup_{S^{2n}} D^{2n+1}$ for some I, where S^{2n} is included as the boundary in N_I and D^{2n+1}. □

Let $f\colon L \to K$ be a map of differential graded Lie algebras. We say that a degree n linear map $\theta \in \mathrm{Hom}_n(L, K)$ is an *f–derivation* of degree n if

$$\theta[x, y] = [\theta(x), f(y)] + (-1)^{n|x|}[f(x), \theta(y)] \quad \text{for all } x, y \in L.$$

The f–derivations form a differential graded vector space $\mathrm{Der}_f(L, K)$, with differential given by

$$D(\theta) = d_K \circ \theta - (-1)^{|\theta|}\theta \circ d_L.$$

The *derivations* of a differential graded Lie algebra L are the special case

$$\mathrm{Der}(L) = \mathrm{Der}_{\mathrm{id}_L}(L, L).$$

We define a bracket for $\theta, \eta \in \mathrm{Der}(L)$, by

$$[\theta, \eta] = \theta \circ \eta - (-1)^{|\theta||\eta|}\eta \circ \theta,$$

which makes $\mathrm{Der}(L)$ into a differential graded Lie algebra.

Denote by $\mathrm{Der}_\omega(\mathbb{L}_I)$ the derivation Lie algebra annihilating ω_I, ie the kernel of the evaluation map $\mathrm{ev}_{\omega_I}\colon \mathrm{Der}(\mathbb{L}_I) \to \mathbb{L}_I$ at ω_I. The *positive truncation* L^+ of a dg Lie algebra L is given by

$$L_i^+ = \begin{cases} L_i & \text{for } i \geq 2, \\ \ker(d_L\colon L_1 \to L_0) & \text{for } i = 1, \\ 0 & \text{for } i \leq 0 \end{cases}$$

with its obvious differential and Lie bracket.

Proposition 6.2 (special case of [5, Corollary 3.11]) *The simply connected covering $B\mathrm{aut}_\partial(N_I)\langle 1\rangle$ is coformal and there is an isomorphism of graded Lie algebras*

$$\pi_*(\Omega B\mathrm{aut}_\partial(N_I)\langle 1\rangle) \otimes \mathbb{Q} \cong \mathrm{Der}_\omega^+(\mathbb{L}_I).$$

Proof The manifolds N_I are formal (in the sense of Sullivan's rational homotopy theory) and have trivial reduced rational cohomology rings since they are homotopy equivalent to wedges of spheres. Thus we can apply [5, Corollary 3.11]. □

6.2 Derivations and the cyclic Lie operad

In this section we collect the results from [5, Sections 6.1 and 6.2]. Let V be a graded finite-dimensional rational vector space. By an inner product of degree m we mean a degree $-m$ morphism

$$\langle -,-\rangle\colon V\otimes V\to\mathbb{Q}$$

that is nondegenerate in the sense that the adjoint

$$V\to\operatorname{Hom}(V,\mathbb{Q}),\qquad v\mapsto\langle x,-\rangle,$$

is an isomorphism of graded vector spaces. The inner product is graded antisymmetric if

$$\langle x,y\rangle=-(-1)^{|x||y|}\langle y,x\rangle\quad\text{for all }\ x,y\in V.$$

Denote by Sp^m the category with objects graded finite-dimensional rational vector spaces V together with a graded antisymmetric inner product $\langle -,-\rangle_V$ of degree m. The morphisms in Sp^m are linear maps that respect the inner product. For a morphism $f\colon V\to W$, there is a unique linear map $f^!$ such that

$$\langle x,f^!(y)\rangle_V=\langle f(x),y\rangle_W\quad\text{for all }\ x\in V\text{ and }y\in W.$$

Since $f^!f=\mathrm{id}_V$ we get that morphisms in Sp^m are injective.

Given a object V of Sp^m, consider the inner product $\langle -,-\rangle_V$ as an element of $\operatorname{Hom}(V^{\otimes 2},\mathbb{Q})$. We make the identification $V^{\otimes 2}\cong\operatorname{Hom}(V^{\otimes 2},\mathbb{Q})$ using the inner product on $V^{\otimes 2}$ defined by

$$\langle x\otimes y,x'\otimes y'\rangle=(-1)^{|x'||y|}\langle x,x'\rangle\langle y,y'\rangle.$$

Thus $\langle -,-\rangle_V$ gives rise to an element $\omega_V\in V^{\otimes 2}$. The antisymmetry of $\langle -,-\rangle$ implies that we can consider ω_V as an element $\omega_V\in\mathbb{L}(V)$. If we choose a graded basis $\iota_1,\dots,\iota_r$ with dual basis $\iota_1^\#,\dots,\iota_r^\#$ for V, then

$$\omega_V=\pm\frac{1}{2}\sum_i[\iota_i^\#,\iota_i].$$

Example We consider $s^{-1}H_I\otimes\mathbb{Q}$ as an element of $\mathrm{Sp}^{(n-2)}$. Then $\omega_{s^{-1}H_I\otimes\mathbb{Q}}$ is equal to ω_I up to sign.

A Sp^m–module in a category $\mathcal{V}$ is a functor $\mathrm{Sp}^m \to \mathcal{V}$. Our goal in this section is to show that we can describe $\mathrm{Der}_\omega(\mathbb{L}(-))$ as a Sp^m–module in a category of graded Lie algebras gL. Moreover we are going to see that this functor is in fact naturally equivalent to a Schur functor.

It is clear that $\mathbb{L}(-)\colon \mathrm{Sp}^m \to gL$ defines a functor. Moreover using the adjoint $f^!$ of a morphism $f\colon V \to W$ in Sp^m, it follows that we can consider $\mathrm{Der}(\mathbb{L}(-))\colon \mathrm{Sp}^m \to gL$ as a functor, where $\mathrm{Der}(\mathbb{L}(f))(\theta)$ for $\theta \in \mathrm{Der}(\mathbb{L}(V))$ is given by the unique derivation defined by

$$\mathrm{Der}(\mathbb{L}(f))(\theta)(x) = \mathbb{L}(f)\theta(f^!(x)) \quad \text{for } x \in W.$$

(see [5, Proposition 6.1], where it is also shown that $\mathrm{Der}(\mathbb{L}(f))(\theta)$ is injective). Proposition 6.2 in [5] now states that for a morphism $f\colon V \to W$ in Sp^m, the diagram

$$\begin{array}{ccc}
\mathrm{Der}(\mathbb{L}(V)) & \xrightarrow{\mathrm{ev}_{\omega_V}} & \mathbb{L}(V) \\
{\scriptstyle \mathrm{Der}(\mathbb{L}(f))}\downarrow & & \downarrow{\scriptstyle \mathbb{L}(f)} \\
\mathrm{Der}(\mathbb{L}(W)) & \xrightarrow{\mathrm{ev}_{\omega_W}} & \mathbb{L}(W)
\end{array}$$

commutes. This implies, in particular, that we get a functor

$$\mathrm{Der}_\omega(\mathbb{L}(-))\colon \mathrm{Sp}^m \to gL,$$

given by the kernel $\mathrm{Der}_\omega(\mathbb{L}(V))$ of the evaluation map $\mathrm{ev}_{\omega_V}\colon \mathrm{Der}(\mathbb{L}(V)) \to \mathbb{L}(V)$ for $V \in \mathrm{Sp}^m$.

We identify the Sp^m–module $\mathrm{Der}(\mathbb{L}(-))$ with the Sp^m–module $\mathbb{L}(V) \otimes V$, upon using the map

$$\theta_{-,-}\colon \mathbb{L}(V) \otimes V \to \mathrm{Der}(\mathbb{L}(-)), \qquad \theta_{\chi,x}(y) = \chi\langle x, y\rangle \quad \text{for } x \in V \text{ and } \theta \in \mathbb{L}(V)$$

(see [5, Proposition 6.3]). Under this identification the evaluation map becomes the map induced by the Lie bracket; ie the diagram

$$\begin{array}{ccc}
\mathbb{L}(V) \otimes V & \xrightarrow{[-,-]} & \mathbb{L}(V) \\
{\scriptstyle \theta_{-,-}}\downarrow & & \Vert \\
\mathrm{Der}(\mathbb{L}(V)) & \xrightarrow{\mathrm{ev}_{\omega_V}} & \mathbb{L}(V)
\end{array}$$

commutes. Denote by $\mathfrak{g}(V)$ the kernel of $[-,-]$, and observing that $[-,-]$ surjects onto the graded Lie subalgebra $\mathbb{L}^{\geq 2}(V)$ of elements of bracket length ≥ 2, we get the commutative diagram of Sp^m–modules (11) at the top of the next page.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{g}(V) & \longrightarrow & \mathbb{L}(V)\otimes V & \xrightarrow{[-,-]} & \mathbb{L}^{\geq 2}(V) & \longrightarrow & 0 \\
 & & \cong \downarrow & & \downarrow \theta_{-,-} & & \| & & \\
0 & \longrightarrow & \operatorname{Der}_\omega(\mathbb{L}(V)) & \longrightarrow & \operatorname{Der}(\mathbb{L}(V)) & \xrightarrow{\mathrm{ev}_{\omega_V}} & \mathbb{L}^{\geq 2}(V) & \longrightarrow & 0
\end{array} \tag{11}
$$

Note that in the top row we do not use the inner product, and thus it defines in fact a functor $\mathfrak{g}$ from the category of graded vector spaces.

Denote by $\mathscr{L}ie = \{\mathscr{L}ie(n)\}_{n\geq 0}$ the graded Lie operad and by $\mathscr{L}ie(\!(n)\!)$ the cyclic Lie operad. Denote by $t = (1\,2\,3\cdots n) \in \Sigma_n$ the cyclic permutation and denote by $t * \xi$ the action of t on $\xi \in \mathscr{L}ie(\!(n)\!)$. There are short exact sequences of Σ_n–modules

$$0 \to \mathscr{L}ie(\!(n)\!) \xrightarrow{\mu} \mathbb{Q}[\Sigma_n] \otimes_{\Sigma_{n-1}} \mathscr{L}ie(n-1) \xrightarrow{\epsilon} \mathscr{L}ie(n) \to 0,$$

where $\mu(\xi) = \sum_i t^i \otimes t^{-i} * \xi$ and $\epsilon(\sigma \otimes \zeta) = \sigma[\zeta, x_n]$ (see [5, Proposition 6.4]).

Using the exact sequence we identify the rows of (11) with

$$s^{-m} \bigoplus_{n\geq 2} \mathscr{L}ie(\!(n)\!) \otimes_{\Sigma_n} V^{\otimes n} \to s^{-m} \bigoplus_{n\geq 2} \mathscr{L}ie(n) \otimes_{\Sigma_n} V^{\otimes n} \to \bigoplus_{n\geq 2} \mathscr{L}ie(n) \otimes_{\Sigma_n} V^{\otimes n}.$$

Motivated by this we define

$$\mathscr{L}ie(\!(V)\!) = s^{-m} \bigoplus_{n\geq 2} \mathscr{L}ie(\!(n)\!) \otimes_{\Sigma_n} V^{\otimes n}.$$

The Lie algebra structure on $\mathscr{L}ie(\!(V)\!)$ can in fact be explicitly described using the cyclic operad structure of $\mathscr{L}ie(\!(n)\!)$ and the one on $V^{\otimes n}$ given by contractions, but we are not going to need it. We summarize the above as:

Proposition 6.3 [5, Proposition 6.6] *There is an isomorphism of* Sp^m*–modules in* gLie,

$$\mathscr{L}ie(\!(-)\!) \cong \operatorname{Der}_\omega(\mathbb{L}(-)).$$

Remark 6.4 As a composition of the Schur functors s^{-m} and the one given by the Σ–module $\{\mathscr{L}ie(\!(n)\!)\}_{n\geq 1}$, we see that $\mathscr{L}ie(\!(-)\!)$ is also a Schur functor.

6.3 The action of the homotopy mapping class group

To identify the action induced by deck transformations on $\operatorname{Der}^+_\omega(\mathbb{L}_I)$ we begin by noting that the Lie algebras (both with Samelson product) $\pi_*(\Omega B\mathrm{aut}_\partial(N_I)) \otimes \mathbb{Q}$ and $\pi_*(\mathrm{aut}_\partial(N_I)) \otimes \mathbb{Q}$ are naturally isomorphic since $\mathrm{aut}_\partial(N_I)$ is a group-like monoid.

Moreover the deck transformation action can be described in terms of the Samelson product; ie when we make the identification $\pi_*(B\mathrm{aut}_\partial(N_I)) \cong \pi_{*-1}(\mathrm{aut}_\partial(N_I))$, the deck transformation action of $\pi_1(B\mathrm{aut}_\partial(N_I)\langle 1\rangle)$ corresponds to the action of $\pi_0(\mathrm{aut}_\partial(N_I))$ on $\pi_{*-1}(\mathrm{aut}_\partial(N_I))\otimes\mathbb{Q}$ given by conjugation. The main tool to identify the action is the following theorem.

Theorem 6.5 ([21]; stated as in [5, Theorem 3.6]) *Let $f\colon X \to Y$ be a map of simply connected CW–complexes with X finite and $\phi_f\colon \mathbb{L}_X \to \mathbb{L}_Y$ a Quillen model. There are natural bijections*

$$\pi_k(\mathrm{map}_*(X,Y), f)\otimes\mathbb{Q} \cong H_k(\mathrm{Der}_{\phi_f}(\mathbb{L}_X,\mathbb{L}_Y)) \quad \text{for } k \geq 1,$$

which are vector space isomorphisms for $k > 1$. In the case $X = Y$ and $f = \mathrm{id}_X$, there are isomorphisms of vector spaces

$$\pi_k(\mathrm{aut}_*(X), \mathrm{id}_X)\otimes\mathbb{Q} \cong H_k(\mathrm{Der}(\mathbb{L}_X)) \quad \text{for } k \geq 1,$$

and the Samelson product corresponds to the Lie bracket.

Proposition 6.6 (compare [5, Proposition 5.5]) *There is a $\pi_0(\mathrm{aut}_\partial(N_I))$–equivariant isomorphism of graded Lie algebras*

$$\pi_*^+(\mathrm{aut}_\partial(N_I))\otimes\mathbb{Q} \cong \mathrm{Der}_\omega^+(\mathbb{L}_I),$$

where the action on the right-hand side is through the canonical action of Γ_I on H_I.

Proof We are going to study the long exact sequence of rational homotopy groups of the fibration

$$\mathrm{aut}_\partial(N_I) \to \mathrm{aut}_*(N_I) \to \mathrm{map}_*(\partial N_I, N_I).$$

Denote by $\varphi\colon \mathbb{L}(\omega) \to \mathbb{L}_I$ the inclusion of the graded Lie subalgebra of $\mathbb{L}_I$ generated by ω_I. Using Theorem 6.5 we see that the map

$$\pi_k(\mathrm{aut}_*(N_I))\otimes\mathbb{Q} \to \pi_k(\mathrm{map}_*(\partial N_I, N_I))\otimes\mathbb{Q}$$

is given by

$$\mathrm{Der}(\mathbb{L}_I)_k \xrightarrow{\varphi_k^*} \mathrm{Der}_\varphi(\mathbb{L}(\omega_I),\mathbb{L}_I)_k,$$

where φ_k^* is the restriction to $\mathbb{L}(\omega_I)$. Note that $\mathrm{Der}_\varphi(\mathbb{L}(\omega_I),\mathbb{L}_I) \cong s^{(n-2)}\mathbb{L}_I$. Under this identification the map φ^* becomes the evaluation map, which is surjective as discussed for the diagram (11). Hence the long exact sequence of rational homotopy groups splits as

$$0 \to \operatorname{Der}_\omega(\mathbb{L}_I)_* \to \operatorname{Der}(\mathbb{L}_I)_* \xrightarrow{\varphi_k^*} \operatorname{Der}_\varphi(\mathbb{L}(\omega_I), \mathbb{L}_I)_* \to 0,$$

where we use that $\operatorname{Der}_\omega(\mathbb{L}_I)$ is the kernel of the evaluation map. The resulting isomorphism

$$\pi_*^+(\operatorname{aut}_\partial(N_I), \operatorname{id}_{N_I}) \otimes \mathbb{Q} \cong \operatorname{Der}_\omega^+(\mathbb{L}_I)$$

is in fact an isomorphism of graded Lie algebras. Indeed, since the inclusion $\operatorname{aut}_\partial(N_I) \to \operatorname{aut}_*(N_I)$ is a map of topological monoids, the induced maps on rational homotopy groups respect the Samelson product and $\operatorname{Der}^+(\mathbb{L}_I) \cong \pi_*^+(\operatorname{aut}(N_I)) \otimes \mathbb{Q}$ is an isomorphism of Lie algebras. Hence we can calculate the Samelson product of $\pi_*^+(\operatorname{aut}_\partial(N_I))$ in $\operatorname{Der}^+(\mathbb{L}_I)$.

Now let $f, g \in \operatorname{aut}_*(N_I)$. The action of $[f] \in \pi_0(\operatorname{aut}_*(N_I))$ on $\pi_k(\operatorname{aut}_*(N_I))$ is induced by pointwise conjugation $g \mapsto fgf^{-1}$, where f^{-1} is some choice of homotopy inverse. Let ϕ_f be a Quillen model for f and $\theta \in \operatorname{Der}(\mathbb{L}_I)_k$. The action of $[f]$ on $\operatorname{Der}(\mathbb{L}_I)_k$ is given by

$$\theta \mapsto \phi_f \circ \theta \circ \phi_f^{-1},$$

by the naturality of the identification $\pi_k(\operatorname{aut}_*(N_I)) \otimes \mathbb{Q} \cong \operatorname{Der}(\mathbb{L}_I)_k$. For a homotopy self-equivalence f consider the induced map $f_* \in \operatorname{Aut}(\widetilde{H}_*(N_I))$. The map $\mathbb{L}(s^{-1}(f_* \otimes \mathbb{Q}))$ is in fact a Lie model for f, which shows that we can identify the conjugation action with the induced action of $\operatorname{Aut}(\widetilde{H}_*(N_I))$ on $\operatorname{Der}(\mathbb{L}_I)_k$.

Using that $\operatorname{Der}_\omega^+(\mathbb{L}_I)_k \to \operatorname{Der}^+(\mathbb{L}_I)_k$ is injective, we calculate the conjugation action of $\pi_0(\operatorname{aut}_\partial(N_I))$ on $\pi_k(\operatorname{aut}_\partial(N_I), \operatorname{id}_{N_I})$ in terms of $\operatorname{Der}_\omega(\mathbb{L}_I)_k$. Let f be an element of $\operatorname{aut}_\partial(N_I)$; it is in particular also an element of $\operatorname{aut}_*(N_I)$ and we know that its homotopy class $[f]$ in $\pi_0(\operatorname{aut}_*(N_I))$ gives us an element in Γ_I. Considering $\theta \in \operatorname{Der}_\omega(\mathbb{L}_I)_k$ as an element in $\operatorname{Der}(\mathbb{L}_I)_k$ we see that $[f]$ acts by the action induced by $f_* \in \Gamma_I$ on $\widetilde{H}_*(N_I)$. □

Remark 6.7 Observe that we discussed in the proof that the map

$$\pi_1(\operatorname{aut}_*(N_I)) \otimes \mathbb{Q} \to \pi_1(\operatorname{map}_*(\partial N_I, N_I)) \otimes \mathbb{Q}$$

is surjective, and hence the kernel of $\pi_0(\operatorname{aut}_\partial(N_I)) \to \Gamma_I$ is finite.

We need to identify the maps induced by the stabilization map on rational homotopy groups. Given an element of $\theta \in \operatorname{Der}_\omega^+(\mathbb{L}_I)$ we define an element $\theta' \in \operatorname{Der}_\omega^+(\mathbb{L}_{I'})$ by letting $\theta' = \theta$ on generators ι_i and κ_i, where $i \in I$ and $\theta(\iota_{i'}) = \theta(\kappa_{i'}) = 0$. Using that $\mathbb{L}_{I'}$ is free we get a derivation θ', which is indeed an element of $\mathrm{Der}^+_\omega(\mathbb{L}_{I'})$, since $\omega_{I'} = \omega_I + (-1)^{|\iota_{i'}|}[\iota_{i'}, \kappa_{i'}]$. We refer to this map again as the stabilization map.

Proposition 6.8 *The isomorphism*

$$\pi_*(\mathrm{aut}_\partial(N_I)\langle 1\rangle) \otimes \mathbb{Q} \cong \mathrm{Der}^+_\omega(\mathbb{L}_I)$$

is compatible with the stabilization maps.

Proof This works exactly as in [5, Proposition 7.7]. □

Ultimately we describe the rational homology $H_*(B\mathrm{aut}_\partial(N_I)\langle 1\rangle, \mathbb{Q})$ as π_I–modules. The link between a dg Lie model and the rational homology of a space is given by the Chevalley–Eilenberg homology. The Chevalley–Eilenberg complex of a dg Lie algebra L with differential d_L is the chain complex $C^{\mathrm{CE}}_*(L) = \Lambda_* sL$ with differential $\delta^{\mathrm{CE}} = \delta^{\mathrm{CE}}_0 + \delta^{\mathrm{CE}}_1$, where s denotes the suspension and Λ_* the free graded commutative algebra. The differentials are given by

$$\delta^{\mathrm{CE}}_0(sx_1 \wedge \cdots \wedge sx_n) = -\sum_{1 \le i \le n} (-1)^{n_i} sx_1 \wedge \cdots \wedge sd_L x_i \wedge \cdots \wedge sx_n,$$

$$\delta^{\mathrm{CE}}_1(sx_1 \wedge \cdots \wedge sx_n) = \sum_{1 \le i < j \le n} (-1)^{|sx_1| + \eta_{i,j}} s[x_i, x_j] \wedge sx_1 \wedge \cdots \wedge \widehat{sx_i} \wedge \cdots \wedge \widehat{sx_i} \wedge \cdots \wedge sx_n,$$

where $n_i = \sum_{j<i} |sx_j|$ and $\eta_{i,j}$ is such that

$$sx_1 \wedge \cdots \wedge sx_n = (-1)^{\eta_{i,j}} sx_i \wedge sx_j \wedge sx_1 \wedge \cdots \wedge \widehat{sx_i} \wedge \cdots \wedge \widehat{sx_i} \wedge \cdots \wedge sx_n.$$

Quillen showed that for a dg Lie model L_X of a space X the Chevalley–Eilenberg homology gives the rational homology groups of X, ie that $H^{\mathrm{CE}}_*(L_X) \cong H_*(X; \mathbb{Q})$.

Grade the Chevalley–Eilenberg chains by word length; ie let $(\Lambda^p(L))_q$ be the elements of word length p and degree q. Denote by $H^{\mathrm{CE}}_{p,q}(L)$ the homology of the chain complex

$$\cdots \to (\Lambda^{p+1}(L))_q \xrightarrow{\delta_1} (\Lambda^p(L))_q \xrightarrow{\delta_1} (\Lambda^{p-1}(L))_q \to \cdots.$$

The Quillen spectral sequence is the spectral sequence coming from the filtration by word length. In case that the dg Lie algebra L_X is a model for a space X, we can identify the E^2–page with

$$E^2(L)_{p,q} = H^{\mathrm{CE}}_{p,q}(L_X) \cong H^{\mathrm{CE}}_{p,q}(\pi_*(\Omega X) \otimes \mathbb{Q}) \Rightarrow H^{\mathrm{CE}}_*(L_X) \cong H_*(X; \mathbb{Q}).$$

The Quillen spectral sequence collapses on the E^2–page for coformal spaces, and hence we get isomorphisms

$$H_r(B\mathrm{aut}_\partial(N_I)\langle 1\rangle, \mathbb{Q}) \cong \bigoplus_{p+q=r} H^{\mathrm{CE}}_{p,q}\big(\pi_*(\Omega B\mathrm{aut}_\partial(N_I)\langle 1\rangle)\otimes\mathbb{Q}\big). \tag{12}$$

The Quillen spectral sequence is in fact natural with respect to unbased maps of simply connected spaces [5, Proposition 2.1]. Since π_I is rationally perfect (see Lemma 5.5), we do not have any extension problems, and hence the isomorphism above is in fact an isomorphism of π_I–modules (see [5, Proposition 2.3]).

Proposition 6.9 *There are isomorphisms of π_I–modules*

$$H^{\mathrm{CE}}_r(\mathrm{Der}_\omega(\mathbb{L}_I)) \cong H_r(B\mathrm{aut}_\partial(N_I)\langle 1\rangle; \mathbb{Q})$$

compatible with the stabilization maps

Proof We use the isomorphism of π_I–modules (12). By Proposition 6.6, we make the identification

$$H^{\mathrm{CE}}_{p,q}\big(\pi_*(\Omega B\mathrm{aut}_\partial(N_I)\langle 1\rangle)\otimes\mathbb{Q}\big) \cong H^{\mathrm{CE}}_{p,q}(\mathrm{Der}_\omega(\mathbb{L}_I))$$

as π_I–modules. This in turn gives the isomorphism in the claim.

The compatibility with the stabilization maps follows from Proposition 6.8. □

7 Homological stability

7.1 An algebraic homological stability result

Recall the graded hyperbolic modules $H_I \cong \widetilde{H}_*(N_I)$ from Section 3, which have the groups Γ_I as their automorphism groups. Recall that we denoted by λ_I the Γ_I–module H_I with standard action which we considered as an object

$$((H_I)_1, \dots, (H_I)_{n-1}) \in \mathrm{Mod}(\mathbb{Z})^{n-1}.$$

Moreover recall that we defined

$$g_p = \begin{cases} \mathrm{rank}((H_I)_p) & \text{if } 2p < n, \\ \mathrm{rank}\,\frac{1}{2}((H_I)_p) & \text{if } 2p = n, \end{cases}$$

$$\mathfrak{g}_I = \mathrm{Der}^+_\omega(\mathbb{L}_I) = \mathrm{Der}^+_\omega(\mathbb{L}(s^{-1}H_I\otimes\mathbb{Q})).$$

For a fixed $p \le \lfloor n/2\rfloor$ we set

$$H_{I'} = H_I \oplus \mathbb{Z}[p] \oplus \mathrm{Hom}(\mathbb{Z}[p], \mathbb{Z}[n])$$

with the pairing induced by evaluation, where the $\mathbb{Z}[p]$ indicates the graded abelian group with a $\mathbb{Z}$ concentrated in degree p. This corresponds to the reduced homology $\widetilde{H}_*(N_I\#(S^p\times S^{n-p}))$ with the intersection pairing. Denote by $\Gamma_{I'}$ the automorphisms of the graded hyperbolic module $H_{I'}$. Recall that we denoted by $I_{p,n-p}\colon H_I\to H_{I'}$ the upper inclusion.

Proposition 7.1 *Let $n>3$. For all $l\geq 0$ there are polynomial functors*

$$\mathscr{C}_l\colon \mathrm{Mod}(\mathbb{Z})^{n-1}\to \mathrm{Vect}(\mathbb{Q})$$

of degree $\leq l/2$ and isomorphisms of Γ_I–modules

$$\mathscr{C}_l(\lambda_I)\cong C_l^{\mathrm{CE}}(\mathfrak{g}_I)$$

compatible with the maps induced by inclusions.

Proof In Proposition 6.3 we described the derivations $\mathrm{Der}_\omega(\mathbb{L}_I)$ as a Schur functor

$$\mathscr{L}ie(\!(-)\!)\colon \mathrm{Sp}^{n-2}\to \mathrm{gLie},\quad V\mapsto s^{-n+2}\bigoplus_{k\geq 2}\mathscr{L}ie(\!(k)\!)\otimes_{\Sigma_k}V^{\otimes k},$$

that extended to the category of graded vector spaces $\mathrm{Vect}_*(\mathbb{Q})$. Consider the inclusion

$$\mathcal{I}\colon \prod_{i=0}^{n-2}\mathrm{Vect}(\mathbb{Q})\to \mathrm{Vect}_*(\mathbb{Q}),\quad (V_i)_{i=0}^{n-2}\mapsto \bigoplus_{i=0}^{n-2}V_i[i].$$

The composition $\mathscr{L}ie(\!(-)\!)\circ\mathcal{I}$ is a Schur multifunctor, which we are denoting by $\widetilde{\mathscr{U}}$, with

$$\widetilde{\mathscr{U}}(\mu)=s^{[1m_1+2m_2+\cdots+(n-2)m_{n-2}-n+2]}\mathscr{L}ie(\!(|\mu|)\!),$$

where $\mu=(m_0,m_1,\dots,m_{n-2})$, given the Σ_μ–module structure induced by the inclusion $\Sigma_\mu\subset\Sigma_{|\mu|}$. Thus the positive-degree derivations are given by the Schur multifunctor

$$\mathscr{U}\colon \prod_{i=0}^{n-2}\mathrm{Vect}(\mathbb{Q})\to \mathrm{gLie},$$

where $\mathscr{U}(\mu)=\widetilde{\mathscr{U}}(\mu)$, when

$$1m_1+2m_2+\cdots+(n-2)m_{n-2}-n+2\geq 1\iff \sum_{i=1}^{n-2}m_i\frac{i}{n-1}\geq 1,$$

and 0 otherwise. The Chevalley–Eilenberg chains are given by the Schur functor Λ with $\Lambda(r)$ the trivial Σ_r–module concentrated in degree r. The composition $\widetilde{\mathscr{C}}=\Lambda\circ\mathscr{U}$

is now given by (using (3))

$$\widetilde{\mathscr{C}_r}(\mu) \cong \bigoplus_r \Lambda(r) \otimes_{\Sigma_r} \bigoplus \mathrm{Ind}_{\Sigma_{\mu_1}\times\cdots\times\Sigma_{\mu_r}}^{\Sigma_\mu} \mathscr{U}(\mu_1) \otimes \cdots \otimes \mathscr{U}(\mu_r),$$

where the second sum runs over all r–tuples $(\mu_1, \dots, \mu_r)$ such that $\sum_{i=1}^r \mu_i = \mu$, and the action of Σ_r is by permuting the r–tuples by the inverse. For a fixed r we get that the summand corresponding to $(\mu_1, \dots, \mu_r)$, where $\mu_s = (m_{1,s}, \dots, m_{r,s})$, is only nonzero if $\sum_{i=1}^{n-2} \big(m_{i,s}(i/(n-1))\big) \geq 1$ for all $s = 1, \dots, r$. That implies that

$$\sum_{i=1}^{n-2} m_i \frac{i}{n-1} = \sum_{s=1}^{r} \left(\sum_{i=1}^{n-2} m_{i,s} \frac{i}{n-1} \right) \geq r. \tag{13}$$

If the summand is nonzero it is of degree

$$\begin{aligned} r + \sum_{s=1}^{r} \left(\left(\sum_{i=1}^{n-2} m_{i,s} i \right) - n + 2 \right) &= r + \left(\sum_{i=1}^{n-2} m_i i \right) - nr + 2r \\ &= \left(\sum_{i=1}^{n-2} m_i i \right) + r(3-n) \\ &\geq \left(\sum_{i=1}^{n-2} m_i i \right) + \left(\sum_{i=1}^{n-2} m_i \frac{i}{n-1} \right)(3-n) \\ &= \frac{2}{n-1} \left(\sum_{i=1}^{n-2} m_i i \right), \end{aligned}$$

where we used that $3-n$ is negative and (13). This implies now that the Chevalley–Eilenberg l–chains are a Schur multifunctor $\widetilde{\mathscr{C}_l}$, where $\widetilde{\mathscr{C}_l}(\mu)$ vanishes for

$$l < \frac{2}{n-1} \left(\sum_{i=1}^{n-2} m_i i \right) \leq 2 \sum_{i=1}^{n-2} m_i = 2|\mu|.$$

Hence it is of degree $\leq l/2$. The functor $\mathscr{C}_l$ in the statement is given by the precomposition with the additive functor

$$- \otimes \mathbb{Q} \colon \mathrm{Mod}(\mathbb{Z})^{n-1} \to \prod_{i=0}^{n-2} \mathrm{Vect}(\mathbb{Q}).$$

The compatibility with the action follows from the fact the functor lifts to

$$Q_+^n(\mathbb{Z}, \Lambda) \to \mathrm{Sp}^{(n-2)}, \quad M \mapsto s^{-1}(M \otimes \mathbb{Q}). \qquad \square$$

As an immediate consequence of Propositions 7.1 and 4.6 we get:

Corollary 7.2 *The stabilization map*

$$H_k(\Gamma_I, C_l^{\mathrm{CE}}(\mathfrak{g}_I) \to H_k(\Gamma_{I'}, C_l^{\mathrm{CE}}(\mathfrak{g}_{I'}))$$

is an isomorphism for $g_p > 2k+l+2$ when $2p \neq n$ and $g_p > 2k+l+4$ if $2p = n$ and an epimorphism for $g_p \geq 2k+l+2$, respectively $g_p \geq 2k+l+4$, unless $n = 2d$ with $d \neq 3, 7$ odd, and $g_d = 1$.

Before we proof the main proposition of this section, ie deduce homological stability for the Chevalley–Eilenberg homology from the stability for the chains, we need the following observation about the Chevalley–Eilenberg chains.

Lemma 7.3 *There exists a chain homotopy equivalences $C_*^{\mathrm{CE}}(\mathfrak{g}_I) \xrightarrow{\simeq} H_*^{\mathrm{CE}}(\mathfrak{g}_I)$ sending cycles $z \mapsto [z]$ such that*

$$\begin{array}{ccc} C_*^{\mathrm{CE}}(\mathfrak{g}_I) & \xrightarrow{\sigma_*} & C_*^{\mathrm{CE}}(\mathfrak{g}_{I'}) \\ \simeq \downarrow & & \simeq \downarrow \\ H_*^{\mathrm{CE}}(\mathfrak{g}_I) & \xrightarrow{\sigma_*} & H_*^{\mathrm{CE}}(\mathfrak{g}_{I'}) \end{array}$$

commutes up to chain homotopy of $\mathbb{Q}[\Gamma_I]$–chain complexes.

Proof This is true for all degreewise finite-dimensional $\mathbb{Q}[G]$–chain complexes for G rationally perfect groups by [5, Lemma B.1] and [5, Proposition B.5]. The groups Γ_I are rationally perfect (see Lemma 3.4) and the $C_*^{\mathrm{CE}}(\mathfrak{g}_I)$ are degreewise finite-dimensional, since the $\mathfrak{g}_I$ are. □

Proposition 7.4 *The stabilization map*

$$H_k(\Gamma_I, H_l^{\mathrm{CE}}(\mathfrak{g}_I)) \to H_k(\Gamma_{I'}, H_l^{\mathrm{CE}}(\mathfrak{g}_{I'}))$$

is an isomorphism for $g_p > 2k+2l+2$ when $2p \neq n$ and $g_p > 2k+2l+4$ if $2p = n$ and an epimorphism for $g_p \geq 2k+2l+2$, respectively $g_p \geq 2k+2l+4$, unless $n = 2d$ with $d \neq 3, 7$ odd, and $g_d = 1$.

Proof Consider the first hyperhomology spectral sequence with E^1–page

$$E^1_{k,l}(I) = H_l\big((\Gamma_I; C_k^{\mathrm{CE}}(\mathfrak{g}_I))\big) \Rightarrow \mathbb{H}_{k+l}(\Gamma_I; C_*^{\mathrm{CE}}(\mathfrak{g}_I)).$$

By Corollary 7.2, $E^1_{k,l}(I) \to E^1_{k,l}(I')$ is an isomorphism for

$$g_p > \begin{cases} 2k+2l+4 \geq k+2l+4 & \text{if } p = \frac{1}{2}n, \\ 2k+2l+2 \geq k+2l+2 & \text{otherwise,} \end{cases}$$

and an epimorphism for $\geq$. By the spectral sequence comparison theorem we get that the map

$$\mathbb{H}_i(\Gamma_I, C_*^{\mathrm{CE}}(\mathfrak{g}_I)) \to \mathbb{H}_i(\Gamma_{I'}, C_*^{\mathrm{CE}}(\mathfrak{g}_{I'}))$$

induced by the stabilization map is an isomorphism for $g_p > 2i+2$ when $2p \neq n$ and $g_p > 2i+4$ if $2p = n$ and an epimorphism for $\geq$. Upon using Lemma 7.3 and the chain homotopy invariance of hyperhomology we get that the map

$$\mathbb{H}_i(\Gamma_I, C_*^{\mathrm{CE}}(\mathfrak{g}_I)) \to \mathbb{H}_i(\Gamma_{I'}, C_*^{\mathrm{CE}}(\mathfrak{g}_{I'}))$$

induced by the stabilization map is an isomorphism and epimorphism in the same range as above. Ultimately we use the natural splitting for hyperhomology groups with coefficients in a chain complex with trivial differential,

$$\begin{array}{ccc} \mathbb{H}_i(\Gamma_I; H_*^{\mathrm{CE}}(\mathfrak{g}_I)) & \xrightarrow{\sigma_i} & \mathbb{H}_i(\Gamma_{I'}; H_*^{\mathrm{CE}}(\mathfrak{g}_{I'})) \\ \cong \downarrow & & \cong \downarrow \\ \bigoplus_{k+l=i} H_k(\Gamma_I; H_l^{\mathrm{CE}}(\mathfrak{g}_I)) & \xrightarrow{\sigma_{k,l}} & \bigoplus_{k+l=i} H_k(\Gamma_{I'}; H_l^{\mathrm{CE}}(\mathfrak{g}_{I'})) \end{array}$$

Hence we see that the maps $\sigma_{k,l}$ are isomorphisms and epimorphisms in the range in the statement of Proposition 7.4. □

7.2 Homological stability for monoid of homotopy automorphisms

The first main result of this article now easily follows from the previous results.

Theorem A *The map*

$$H_i(B\mathrm{aut}_\partial(N_I); \mathbb{Q}) \to H_i(B\mathrm{aut}_\partial(N_{I'}); \mathbb{Q})$$

induced by the stabilization map is an isomorphism for $g_p > 2i+2$ when $2p \neq n$ and $g_p > 2i+4$ if $2p = n$ and an epimorphism for $g_p \geq 2i+2$, respectively $g_p \geq 2i+4$, unless $n = 2d$ with $d \neq 3, 7$ odd, and $g_d = 1$.

Proof We begin by observing that

$$H_k(\Gamma_I, H_l^{\mathrm{CE}}(\mathfrak{g}_I)) \cong H_k(\pi_I, H_l^{\mathrm{CE}}(\mathfrak{g}_I)), \tag{14}$$

because the action of π_I is through $\widetilde{H}\colon \pi_I \to \Gamma_I$ and the kernel of $\widetilde{H}_*$ is finite (Propositions 5.1 and 6.6). The result now follows from Proposition 7.4 combined with Proposition 6.9 upon using the spectral sequence comparison theorem. □

7.3 Homological stability for the block diffeomorphism group

Denote by $\widetilde{\mathrm{aut}}_\partial(X)$ the Δ–monoid of block homotopy equivalences, with k–simplices face-preserving homotopy equivalences

$$\varphi\colon \Delta^k \times X \to \Delta^k \times X,$$

such that $\varphi|_{\Delta^k \times \partial X}$ is the identity. The block diffeomorphism group $\widetilde{\mathrm{Diff}}_\partial(X)$ is the Δ–subgroup with k–simplices, face-preserving diffeomorphisms

$$\varphi\colon \Delta^k \times X \to \Delta^k \times X,$$

such that φ is the identity on a neighborhood of $\Delta^k \times \partial X$. We do not distinguish between Δ–objects and their realizations. Denote the inclusion $\widetilde{\mathrm{Diff}}_\partial(X) \hookrightarrow \widetilde{\mathrm{aut}}_\partial(X)$ by $\widetilde{J}$. The inclusion

$$\mathrm{aut}_\partial(X) \hookrightarrow \widetilde{\mathrm{aut}}_\partial(X)$$

is a homotopy equivalence and hence we are going to consider them as identified. The block diffeomorphism group $\widetilde{\mathrm{Diff}}_\partial(X)$ and the diffeomorphism group $\mathrm{Diff}_\partial(X)$ with the Whitney C^∞–topology on the other hand are not homotopy equivalent — the difference is related to algebraic K–theory (see [31]). The homogeneous space $\widetilde{\mathrm{aut}}_\partial(X)/\widetilde{\mathrm{Diff}}_\partial(X)$ is by definition the homotopy fiber of the map $\widetilde{J}\colon B\widetilde{\mathrm{Diff}}_\partial(X) \to B\widetilde{\mathrm{aut}}_\partial(X)$. It is related to surgery theory as we explain now.

Let X be a smooth manifold of dimension ≥ 5 with boundary ∂X. Quinn [25] shows that there is a quasifibration of Kan Δ–sets

$$\mathcal{S}_\partial^{G/O}(X) \to \mathrm{map}_*(X/\partial X, G/O) \to \mathbb{L}(X)$$

and that its homotopy exact sequence is the surgery exact sequence. The space $\mathcal{S}_\partial^{G/O}(X)$ is the realization of a Δ–set with k–simplices pairs (W, f), where W is a smooth $(k+3)$–ad (see eg [25, Section 2]) and $f\colon W \to \Delta^k \times X$ is a face-preserving homotopy equivalence such that f restricts to a diffeomorphism $f|_{\partial_{k+1}W}\colon \partial_{k+1}W \to \Delta^k \times \partial X$. There is a map

$$\widetilde{\mathrm{aut}}_\partial(X)/\widetilde{\mathrm{Diff}}_\partial(X) \to \mathcal{S}_\partial^{G/O}(X),$$

which by the h–cobordism theorem induces a weak homotopy equivalence

$$\widetilde{\mathrm{aut}}_\partial(X)/\widetilde{\mathrm{Diff}}_\partial(X)_{(1)} \simeq_{\text{w.e.}} \mathcal{S}_\partial^{G/O}(X)_{(1)} \tag{15}$$

of the identity components (see [4, Section 3.2]). Now assume X is simply connected.

Since

$$G/O \simeq_{\mathbb{Q}} \mathrm{BO} \simeq_{\mathbb{Q}} \prod_{i\geq 1} K(\mathbb{Q},4i),$$

we understand the rational homotopy groups:

$$\pi_i(\mathrm{map}_*(X/\partial X, G/O)) \otimes \mathbb{Q} \cong \bigoplus_k H^k(X,\partial X;\mathbb{Q}) \otimes \pi_{k+i}(G/O).$$

Note that if X is simply connected,

$$\pi_i(\mathbb{L}(X)) \otimes \mathbb{Q} \cong L_{\dim(X)+i}(X) \cong \begin{cases} \mathbb{Q} & \text{if } \dim(X)+i \equiv 0 \bmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

We now specialize to N_I.

Lemma 7.5 [4, Lemma 3.5] *The surgery obstruction map induces an isomorphism*

$$H^n(N_I,\partial N_I;\mathbb{Q}) \otimes \pi_{n+k}(G/O) \to L_{n+k}(\mathbb{Z}) \otimes \mathbb{Q}$$

for $n+k \equiv 0 \bmod 4$.

Proof Consider the smooth and topological surgery exact sequences

$$\begin{array}{ccccccc} \cdots \longrightarrow & N_\partial^{G/O}(N_I \times D^k) \otimes \mathbb{Q} & \longrightarrow & L_{n+k}(\mathbb{Z}) \otimes \mathbb{Q} & \longrightarrow \cdots \\ & \downarrow & & \| & \\ \cdots \longrightarrow & N_\partial^{G/\mathrm{Top}}(N_I \times D^k) \otimes \mathbb{Q} & \longrightarrow & L_{n+k}(\mathbb{Z}) \otimes \mathbb{Q} & \longrightarrow \cdots \end{array}$$

The left-hand vertical map is an isomorphism since $\pi_i(\mathrm{Top}/O)$ is finite (see eg [17]). Milnor's plumbing construction ensures that for $k+n$ even there is an element in $N_\partial^{G/\mathrm{Top}}(N_I \times D^k)$ with nontrivial surgery obstruction. Since

$$N_\partial^{G/\mathrm{Top}}(N_I \times D^k) \cong \pi_k(\mathrm{map}_*(N_I/\partial N_I, G/\mathrm{Top}))$$

and

$$\pi_k(\mathrm{map}_*(N_I/\partial N_I, G/O)) \otimes \mathbb{Q} \cong H^n(N_I,\partial N_I;\mathbb{Q}) \otimes \pi_{n+k}(G/O) \quad \text{for } n+k \equiv 0 \bmod 4,$$

the claim follows because both sides are just one-dimensional rational vector spaces. □

This now implies that we have a natural isomorphism

$$\pi_k(\mathcal{S}_\partial^{G/O}(N_I)) \otimes \mathbb{Q} \cong \bigoplus_i H^i(N_I;\mathbb{Q}) \otimes \pi_{i+k}(G/O) \quad \text{for } k > 0 \tag{16}$$

(compare [5, Corollary 4.6]).

Recall that $J_0\pi_0(\mathrm{Diff}_\partial(N_I))$ has finite index in $\pi_0(\mathrm{aut}_\partial(N_I))$ (Proposition 5.3). By Cerf's pseudoisotopy theorem thus also $\widetilde{J}_1\pi_1(B\widetilde{\mathrm{Diff}}_\partial(N_I))$ in $\pi_1(B\mathrm{aut}_\partial(N_I))$. Denote by $\overline{B}\mathrm{aut}_\partial(N_I)$ the finite cover corresponding to $\mathrm{Image}(\widetilde{J}_1)$. Note that it has the same (higher) rational homotopy groups. By construction $\widetilde{J}$ lifts to a map $B\widetilde{\mathrm{Diff}}_\partial(N_I) \to \overline{B}\mathrm{aut}_\partial(N_I)$. Instead of $\widetilde{\mathrm{aut}}_\partial(N_I)/\widetilde{\mathrm{Diff}}_\partial(N_I)$ we consider

$$\mathcal{F}_I = \mathrm{hofib}(B\widetilde{\mathrm{Diff}}_\partial(N_I) \to \overline{B}\mathrm{aut}_\partial(N_I)).$$

We reduced the problem of showing rational homological stability for the block diffeomorphisms to the study of the Serre spectral sequence of the homotopy fibration above. The only missing ingredient is now to understand the rational homology groups of $\mathcal{F}_I$ as $\overline{\pi}_I = \pi_1(\overline{B}\mathrm{aut}_\partial(N_I))$–modules. Observe that by Proposition 5.3 there is a surjection

$$\pi_1(\overline{B}\mathrm{aut}_\partial(N_I)) \to \Gamma_I$$

induced by $\widetilde{H}_*$.

Denote by $\eta\colon S_\partial^{G/O}(X) \to \pi_0\mathrm{map}_*(X/\partial X, G/O)$ the normal invariant, and denote by $\sigma\colon \pi_0\mathrm{map}_*(X/\partial X, G/O) \to L_{\dim(X)}(X)$ the surgery obstruction. Using the surgery exact sequence, we see that for n odd $\pi_1(\mathcal{F}_I)$ is abelian, since it is a subgroup of the abelian group $[\Sigma(N_I/\partial N_I), G/O]_*$. For n even it is a finite extension of the abelian group $[\Sigma(N_I/\partial N_I), G/O]_*$ by a finite cyclic group (in case $L_{n+2}(\mathbb{Z}) \cong \mathbb{Z}$, the proof of Lemma 7.5 makes sure that the map to $L_{n+2}(\mathbb{Z})$ is nonzero and hence the kernel of σ is a finite cyclic group). Write

$$\pi_k^{\mathrm{ab}}(\mathcal{F}_I) = \begin{cases} \pi_1(\mathcal{F}_I)/\,\mathrm{Image}(L_{n+2}(\mathbb{Z}) \to \pi_1(\mathcal{F}_I)) & \text{if } k = 1, \\ \pi_k(\mathcal{F}_I) & \text{if } k > 1. \end{cases}$$

Proposition 7.6 *There are isomorphisms of $\overline{\pi}_I$–modules compatible with the stabilization maps*

(1) $\pi_k^{\mathrm{ab}}(\mathcal{F}_I)\otimes\mathbb{Q} \cong (\widetilde{H}_*(N_I,\mathbb{Q})\otimes\pi_*(G/O))_k$, *where* $|a\otimes\alpha| = |\alpha|-|a|$ *and* $k \geq 1$,

(2) $H_*(\mathcal{F}_I,\mathbb{Q}) \cong \Lambda(\pi_*^{\mathrm{ab}}(\mathcal{F}_I)\otimes\mathbb{Q})$,

where the actions on the left-hand side are induced by the standard actions of Γ_I *on* $\widetilde{H}_*(N_I)$.

Proof Compare [4, page 26 and Theorem 3.6] and [5, Proposition 7.15]. Observe that the rationalization $(\mathcal{F}_I)_\mathbb{Q}$ has rational homotopy groups $\pi_k^{\mathrm{ab}}(\mathcal{F}_I)\otimes\mathbb{Q}$. Consider the splitting of the homotopy exact sequence of the surgery fibration as

$$0 \to L_{n+k+1}(\mathbb{Z})/\,\mathrm{Image}(\sigma) \to \pi_k(\mathcal{S}_\partial^{G/O}(N_I)) \to \mathrm{Image}(\eta) \to 0.$$

By Lemma 7.5 we get

$$L_{n+k+1}(\mathbb{Z})/\operatorname{Image}(\sigma)\otimes\mathbb{Q}\cong 0 \quad \text{and} \quad \operatorname{Image}(\eta)\otimes\mathbb{Q}\cong(\widetilde{H}^*(N_I,\mathbb{Q})\otimes\pi_*(G/O))_k.$$

Using the isomorphism

$$\widetilde{H}_*(N_I;\mathbb{Q})\cong \operatorname{Hom}_{\mathbb{Q}}(\widetilde{H}_*(N_I;\mathbb{Q});\mathbb{Q})\cong \widetilde{H}^*(N_I;\mathbb{Q})$$

we get the isomorphism (1). We see that the action on the right-hand side is induced by the standard action of Γ_I as follows: Use the identification

$$\pi_k(\mathcal{S}_\partial^{G/O}(X))\cong S_\partial^{G/O}(N_I\times D^k).$$

An element of $S_\partial^{G/O}(N_I\times D^k)$ is represented by a manifold $(X,\partial X)$ together with a homotopy equivalence $f\colon X\to N_I\times D^k$ such that $f|_{\partial X}\colon \partial X\to\partial(N_I\times D^k)$ is a diffeomorphism. The action of a

$$[\phi]\in\pi_1(\bar{B}\widetilde{\operatorname{aut}}_\partial(N_I))\cong\operatorname{Image}(\widetilde{J}_1)\cong\operatorname{Image}(J_0)\subset\pi_0(\operatorname{aut}_\partial(N_I))$$

on f is given by the composition

$$X\xrightarrow{f}N_I\times D^k\xrightarrow{\phi\times\mathrm{id}_{D^k}}N_I\times D^k,$$

where ϕ is a diffeomorphism representing $[\phi]$ considered as an element of $\operatorname{Image}(J_0)$. [4, Lemma 3.3] now implies that

$$\eta((\phi\times\mathrm{id}_{D^k})\circ f)=((\phi\times\mathrm{id}_{D^k})^*)^{-1}(\eta(f))+\eta(\mathrm{id}_{D^k})=((\phi\times\mathrm{id}_{D^k})^*)^{-1}(\eta(f)),$$

using that the normal invariant of a diffeomorphism is trivial. This implies that $[\phi]$ acts on $\widetilde{H}^*(N;\mathbb{Q})\otimes\pi_*(G/O)_k$ via $(\phi^{-1})^*\otimes\mathrm{id}_{\pi_*(G/O)}$. But this exactly corresponds to the standard action under the isomorphism

$$\widetilde{H}^*(N_I;\mathbb{Q})\cong\operatorname{Hom}_{\mathbb{Q}}(\widetilde{H}_*(N_I;\mathbb{Q});\mathbb{Q})\cong\widetilde{H}_*(N_I;\mathbb{Q}).$$

If ϕ lies in the kernel of the map

$$\pi_1(\bar{B}\operatorname{aut}_\partial(N_I))\to\Gamma_I,$$

then it is in the kernel of $\widetilde{H}_*$ and a similar argument as before shows that it acts trivially on $\pi_k^{\mathrm{ab}}(\mathcal{F}_I)$ (compare Lemma 5.2). The compatibility with the stabilization maps follows from the fact that the isomorphisms (16) are natural.

The statement for (2) follows from that for (1) by using the fact that G/O and hence also the mapping-space $\operatorname{map}_*(N_I/\partial N_I,G/O)$ are infinite loop spaces. Thus all rational k–invariants vanish for $\mathrm{map}_*(N_I/\partial N_I, G/O)$. Equivalently, for each element $\alpha \in \pi_k(\mathrm{map}_*(N_I/\partial N_I, G/O))\otimes\mathbb{Q}$ there is an element $c \in H^k(\mathrm{map}_*(N_I/\partial N_I, G/O);\mathbb{Q})$ such that $c(h(\alpha)) \neq 0$, where h denotes the rational Hurewicz homomorphism. Since

$$\pi_k((\mathcal{F}_I)_{\mathbb{Q}}) \otimes \mathbb{Q} \to \pi_k(\mathrm{map}_*(N_I/\partial N_I, G/O)) \otimes \mathbb{Q}$$

is injective, it follows that all rational k–invariants also vanish for $(\mathcal{F}_I)_{\mathbb{Q}}$. This shows that $(\mathcal{F}_I)_{\mathbb{Q}}$ is a product of Eilenberg–Mac Lane spaces and hence its homology is given by the free graded commutative algebra on its homotopy groups. Moreover the $\pi_1(\bar{B}\mathrm{aut}_\partial(N_I))$–action is induced by the standard action. □

We use the previous proposition to give a Schur multifunctor description of $H_r(\mathcal{F}_I;\mathbb{Q})$. For a multiindex μ with $\ell(\mu) = n-1$, consider the Σ_μ–modules $\Pi(\mu)$ given by

$$\Pi(0,\dots,1,\dots,0) = s^{-i}\pi_*(G/O)\otimes\mathbb{Q},$$

where the 1 sits in the i^{th} position and all other entries are 0. The corresponding Schur multifunctor

$$\Pi\colon \mathrm{Mod}(\mathbb{Z})^{n-1} \to \mathrm{Vect}_*(\mathbb{Q}),$$

has the property that there is an isomorphism of the induced Γ_I–modules

$$\Pi(H_I) \cong (\widetilde{H}_*(N_I,\mathbb{Q}) \otimes \pi_*(G/O))^+.$$

It follows now that we get an isomorphism of Γ_I–modules

$$\Lambda\circ\Pi(H_I) \cong \Lambda((\widetilde{H}_*(N_I,\mathbb{Q}) \otimes \pi_*(G/O)^+)) \cong H_*(\mathcal{F}_I,\mathbb{Q}),$$

where the left-hand Λ denotes the free graded commutative algebra endofunctor of $\mathrm{Vect}_*(\mathbb{Q})$. Recall that Λ is given as the Schur functor with $\Lambda(n) = \mathbb{Q}[n]$ and trivial Σ_n–action. Now setting $\mathscr{H}_r = \Lambda_r\circ\Pi$ and observing that Λ_r is of degree $\leq r$ and Π additive, we get the following:

Proposition 7.7 *There is an isomorphism of Γ_I–modules*

$$H_r(\mathcal{F}_I;\mathbb{Q}) \cong \bigoplus_{\ell(\mu)=n-1} \mathscr{H}_r(\mu)\otimes_{\Sigma_\mu} H_I^{\otimes\mu},$$

compatible with the stabilization maps, where the $\mathscr{H}_r(\mu)$ are zero for multiindexes μ such that $|\mu| > r$.

Now we prove the second main theorem of this paper.

Theorem B *The stabilization map*

$$H_i(B\widetilde{\mathrm{Diff}}_\partial(N_I);\mathbb{Q}) \to H_i(B\widetilde{\mathrm{Diff}}_\partial(N_{I'});\mathbb{Q})$$

is an isomorphism for $g_p > 2i+2$ when $2p \neq n$ and $g_p > 2i+4$ if $2p = n$ and an epimorphism for $g_p \geq 2i+2$, respectively $g_p \geq 2i+4$, unless $n = 2d$ with $d \neq 3, 7$ odd, and $g_d = 1$.

Proof Write $Y_I = B\widetilde{\mathrm{Diff}}_\partial(N_I)$ and $\overline{X}_I = \overline{B}\widetilde{\mathrm{aut}}_\partial(N_I)$. Consider the Serre spectral sequences of the homotopy fibration

$$\mathcal{F}_I \to Y_I \to \overline{X}_I$$

and the analogue for I'. The stabilization map induces maps on the E_2–pages

$$\sigma_*\colon H_k(\overline{X}_I; H_l(\mathcal{F}_I;\mathbb{Q})) \to H_k(\overline{X}_{I'}; H_l(\mathcal{F}_{I'};\mathbb{Q})).$$

The theorem follows upon showing that these are isomorphisms for $g_p > 2k+2l+2$ ($+4$ if $p = n/2$) and epimorphisms for $g_p \geq 2k+2l+2$ ($+4$ if $p = n/2$). For this we consider the universal covering spectral sequence

$$H_r\big(\pi_1(\overline{X}_I); H_s(\overline{X}_I\langle 1\rangle; H_l(\mathcal{F}_I;\mathbb{Q}))\big) \Rightarrow H_{r+s}(\overline{X}_I; H_l(\mathcal{F}_I,\mathbb{Q})).$$

The condition above would follow upon showing that the maps induced by the stabilization map on the E^2–page are isomorphisms for $g_p > 2r+2s+2l+2$ (4 if $p = n/2$) and epimorphisms for $g_p \geq 2r+2s+2l+2$ ($+4$ if $p = n/2$). To show this we observe that there are isomorphism of Γ_I–modules compatible with the stabilization maps,

$$H_s(\overline{X}_I\langle 1\rangle; H_l(\mathcal{F}_I;\mathbb{Q})) \cong H_s(\overline{X}_I\langle 1\rangle) \otimes H_l(\mathcal{F}_I;\mathbb{Q}) \cong H_s^{\mathrm{CE}}(\mathfrak{g}_I) \otimes H_l(\mathcal{F}_I;\mathbb{Q}),$$

where Γ_I acts on the 2nd and 3rd terms diagonally. Note that $\overline{X}_I$ and X_I have the same universal cover, which is moreover homotopy equivalent to $B\mathrm{aut}_\partial(N_I)\langle 1\rangle$. The stability for

$$H_r(\pi_1(\overline{X}_I); H_s^{\mathrm{CE}}(\mathfrak{g}_I) \otimes H_l(\mathcal{F}_I;\mathbb{Q})) \tag{17}$$

follows from stability for

$$H_r(\pi_1(\overline{X}_I); C_s^{\mathrm{CE}}(\mathfrak{g}_I) \otimes H_l(\mathcal{F}_I;\mathbb{Q})),$$

exactly as in Proposition 7.4 upon using the two hyperhomology spectral sequences and that $\overline{\pi}_I$ is rationally perfect (Lemma 5.5). Hence we are left with showing that the stabilization maps

$$H_r(\pi_1(\overline{X}_I); C_s^{\mathrm{CE}}(\mathfrak{g}_I) \otimes H_l(\mathcal{F}_I;\mathbb{Q})) \to H_r(\pi_1(\overline{X}_{I'}); C_s^{\mathrm{CE}}(\mathfrak{g}_{I'}) \otimes H_l(\mathcal{F}_{I'};\mathbb{Q}))$$

are isomorphisms for $g_p > 2r + 2s + 2l + 2$ ($+4$ if $p = n/2$) and epimorphisms for $g_p \geq 2r + 2s + 2l + 2$ ($+4$ if $p = n/2$). The Lyndon spectral sequence reduces this to the corresponding statement for

$$H_r(\Gamma_I; C_s^{\mathrm{CE}}(\mathfrak{g}_I) \otimes H_l(\mathcal{F}_I; \mathbb{Q})) \to H_r(\Gamma_{I'}; C_s^{\mathrm{CE}}(\mathfrak{g}_{I'}) \otimes H_l(\mathcal{F}_{I'}; \mathbb{Q})).$$

Propositions 7.7 and 6.8 give us isomorphisms of Γ_I–modules compatible with the stabilization map $C_s^{\mathrm{CE}}(\mathfrak{g}_I) \otimes H_l(\mathcal{F}_I; \mathbb{Q})) \cong \mathscr{C}_s(H_I) \otimes \mathscr{H}_l(H_I)$. The functor $\mathscr{C}_s$ is polynomial of degree $\leq s/2$ and the functor $\mathscr{H}_l$ is polynomial of degree $\leq l$. The tensor product (in the sense of Schur multifunctors) $\mathscr{C}_s \otimes \mathscr{H}_l$ is of degree $\leq s/2 + l$. By Proposition 4.6 the stabilization maps

$$H_r(\Gamma_I; \mathscr{C}_s \otimes \mathscr{H}_l(H_I)) \to H_r(\Gamma_{I'}; \mathscr{C}_s \otimes \mathscr{H}_l(H_{I'}))$$

are isomorphisms for $g_p > 2r + s/2 + l + 2$ ($+4$ if $p = n/2$) and epimorphisms for $g_p \geq 2r + s/2 + l + 2$ ($+4$ if $p = n/2$), which finishes the proof. □

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