

On properties of Bourgeois contact structures

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The Bourgeois construction associates to every contact open book on a manifold V a contact structure on $V \times \mathbb{T}^2$. We study some of the properties of V that are inherited by $V \times \mathbb{T}^2$ and some that are not.

Giroux has provided recently a better framework to work with contact open books. In the appendix, we quickly review this formalism, and we work out a few classical examples of contact open books to illustrate how to use this new language.

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1 Introduction

In his thesis, Bourgeois [5] used a construction based on work by Lutz [24] that associates to every contact open book on a contact manifold (V, ξ) a contact structure on $V \times \mathbb{T}^2$ that is invariant under the natural \mathbb{T}^2 -action and that restricts on every fiber $V \times \{*\}$ to ξ . Even though all contact structures obtained on $V \times \mathbb{T}^2$ for a given (V, ξ) are homotopic as *almost* contact structures independently of the open book used, Bourgeois proved via contact homology that the resulting contact structures on $V \times \mathbb{T}^2$ often do depend on the open book chosen and not only on ξ itself.

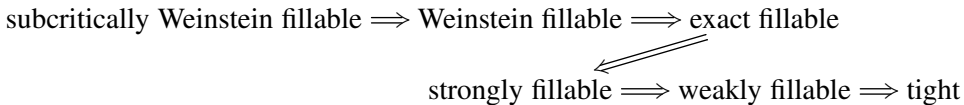
This construction is probably the most interesting explicit method known so far to produce higher-dimensional closed contact manifolds based on lower-dimensional ones.

For this reason we consider it an important question to understand which properties of (V, ξ) are passed on to the associated contact structure on $V \times \mathbb{T}^2$.

For instance, Presas [32] constructed the first examples of higher-dimensional overtwisted contact structures by gluing together two Bourgeois structures associated to overtwisted 3-manifolds. This raised the question of whether the Bourgeois structure associated to an overtwisted structure is overtwisted or not. We will show here that this is not always the case.

The list of properties we will be studying are mostly related to the fillability and tightness of the Bourgeois structures. Note also the recent article by Gironella [17] that studies questions about Bourgeois structures related to ours. We discuss the relation of our work to his in Section 2.

Recall that a general contact structure is either overtwisted or tight; see Borman, Eliashberg and Murphy [4]. Furthermore it is known that overtwisted manifolds are not even weakly fillable; see Massot, Niederkrüger and Wendl [26] and Niederkrüger [29] (to drop the semipositivity condition use Pardon [31]). The different types of fillability can be combined to give the following hierarchy:



For Bourgeois contact structures, we know from [26] (and work related to it):

Theorem A *Let (V, ξ) be a closed contact manifold.*

- (a) *If (V, ξ) is weakly filled by (W, ω) , then independently of the open book decomposition used in the construction, the associated Bourgeois contact structure on $V \times \mathbb{T}^2$ is isotopic to a contact structure that can be weakly filled by $(W \times \mathbb{T}^2, \omega \oplus \text{vol}_{\mathbb{T}^2})$.*
- (b) *If (V, ξ) admits a Weinstein filling that is a k -fold stabilization, and if (K, ϑ) is the canonical open book associated to such a subcritical filling, then the corresponding Bourgeois structure on $V \times \mathbb{T}^2$ will be $(k - 1)$ -subcritically Weinstein fillable.*

We draw the reader’s attention to two different meanings of “stabilization” in this paper. In the context of Weinstein domains, this refers to taking a product with \mathbb{C} (or \mathbb{C}^k); see Section 4 for details. In the context of an (abstract) open book, however, it refers to

a modification of the open book by attaching a handle to the page and also changing the monodromy by a suitable Dehn twist. See, for instance, van Koert [22, Section 4.3].

In Section 2 we explain the Bourgeois construction. The proof of Theorem A is in Section 4.

As already mentioned, the Bourgeois structures do not only depend on the chosen contact manifold (V, ξ) but also on the open book used in the construction; see Bourgeois [5]. On the other hand, two abstract open books with the same page but with mutually inverse monodromies, Ψ and Ψ^{-1} , lead to two contact manifolds that are smoothly (orientation-reversing) diffeomorphic to each other but that, in general, have very different contact properties. For example, from Emmanuel Giroux [19], a contact manifold is Stein/Weinstein fillable if and only if it admits an open book whose monodromy Ψ can be expressed as a product of positive Dehn twists. By contrast, changing the monodromy of an abstract open book to Ψ^{-1} often yields an overtwisted contact structure. Nonetheless we obtain the following unexpected result in Section 3.

Theorem B *Let (V, ξ_+) and (V, ξ_-) be closed contact manifolds supported by abstract Liouville open books that have the same page but inverse monodromy. Then the two corresponding Bourgeois structures on $V \times \mathbb{T}^2$ are contactomorphic.*

This statement shows that the Bourgeois construction is not injective, and combining this result with Theorem A, we also obtain the following corollary.

Corollary 1.1 *There exist examples in every dimension of closed overtwisted contact manifolds (V, ξ) for which at least one of the corresponding Bourgeois structures on $V \times \mathbb{T}^2$ is tight.*

In fact, no example of an overtwisted Bourgeois structure is known to us. Note also that Gironella [17] has recently shown that every contact 3-manifold with nontrivial fundamental group admits an open book whose Bourgeois structure is (hyper)tight. This leads to the following questions.

- Question 1.2** (a) *Can a Bourgeois contact structure ever be overtwisted?*
 (b) *Are there Bourgeois contact structures that are not weakly fillable?*

In both cases, it is an immediate consequence of Theorem B that candidates can only be constructed from open books where both the monodromy and the inverse monodromy lead to overtwisted or not weakly fillable contact structures, respectively. Furthermore,

note that Bowden, Gironella and Moreno [8] have shown recently that *every* Bourgeois contact structure of dimension 5 is tight.

In Section 5, we show that most Bourgeois structures are not subcritically Weinstein fillable. Subcritically fillable contact manifolds are extremely rare — in dimension 3 the only examples are the standard sphere and connected sums of copies of $\mathbb{S}^1 \times \mathbb{S}^2$ with the tight contact structure.

In high dimensions, a general topological characterization of Stein manifolds was obtained by Cieliebak and Eliashberg [13; 11]; based on this, Bowden, Crowley and Stipsicz [7] worked out the formal topological requirements for Stein fillability. Let (V, Ξ_V, ω_Ξ) be an almost contact manifold; that is, V is an oriented manifold with a hyperplane field Ξ_V , and ω_Ξ is a symplectic structure on Ξ_V . An *almost Stein filling* (W, J) of (V, Ξ_V, ω_Ξ) is an almost complex manifold such that

- V is the *oriented* boundary of W ;
- J restricts to Ξ_V , and $J|_{\Xi_V}$ is tamed by ω_Ξ ;
- W admits a handle decomposition with all handles of dimension no more than $\frac{1}{2} \dim W$.

In particular, [7, Proposition 7.1] specializes in our situation to the following.

Theorem 1.3 *Let (V, Ξ_V, ω_Ξ) be an almost contact structure, and let $d\text{vol}$ be a volume form on \mathbb{T}^2 . If $(V \times \mathbb{T}^2, \Xi_V \oplus T\mathbb{T}^2, \omega_\Xi \oplus d\text{vol})$ admits an almost Stein filling, it follows that (V, Ξ_V, ω_Ξ) also admits one. \square*

Conversely, if the almost contact structure (V, Ξ_V, ω_Ξ) admits a subcritical almost Stein filling, then $(V \times \mathbb{T}^2, \Xi_V \oplus T\mathbb{T}^2, \omega_\Xi \oplus d\text{vol})$ admits an almost Stein filling. Compare this also to part (b) of Theorem A. If (V, ξ_V) is only Stein fillable, however, the situation is significantly more involved. For example, \mathbb{T}^3 with the standard contact structure has the Stein filling $T^*\mathbb{T}^2$, but no contact structure on $\mathbb{T}^3 \times \mathbb{T}^2 = \mathbb{T}^5$ can ever be Stein fillable by part (2) of [7, Proposition 6.2].

We give below a few examples of Bourgeois structures that admit subcritical *almost* Stein fillings but no *genuine* subcritical fillings. This is based on obstructions to Weinstein fillability that can easily be deduced from Gromov's 1985 article [21].

Theorem C *A closed contact manifold containing a weakly exact pre-Lagrangian P is not subcritically Weinstein fillable.*

If the dimension of the contact manifold is at least 5 and if P is displaceable then the contact manifold is not even Weinstein fillable.

A limitation of this theorem is that pre-Lagrangians can only be weakly exact in manifolds with a sufficiently large fundamental group (see Lemma 5.2), thus excluding many examples. We believe that the weakly exact condition can be relaxed by using some type of Floer theory, but we refrain from doing so here, to keep this note simple. Note also that with work of Barth, Geiges and Zehmisch [1] and Oancea and Viterbo [30] one can formulate obstructions to subcritical fillability that depend more on the global topology of the contact manifold.

On the other hand, Theorem C leads to the following observation regarding Bourgeois structures:

Corollary 1.4 *Let (V, ξ) be a closed contact manifold and let (K, ϑ) be a compatible open book that contains a closed Legendrian in one of its pages. It then follows that the corresponding Bourgeois structure on $V \times \mathbb{T}^2$ is not subcritically Weinstein fillable.*

This applies in particular to any open book that has been stabilized.

Example 1.5 Consider the contact open book decomposition of the standard contact sphere $(\mathbb{S}^{2n-1}, \xi_0)$ whose page is a ball and whose monodromy is trivial. If $2n - 1 \neq 1$, the corresponding Bourgeois structure on $\mathbb{S}^{2n-1} \times \mathbb{T}^2$ is subcritically Weinstein fillable. If instead we take for example an open book with page the cotangent bundle $T^*\mathbb{S}^{n-1}$ and with monodromy a positive Dehn twist (these examples are classical but they are also explained in depth in the appendix), then the Bourgeois structure will be homotopic to the first one as almost contact structures, but it cannot be contactomorphic¹ to it, since it is not subcritically Weinstein fillable.

This way, we see that the fillability of a Bourgeois structure on $V \times \mathbb{T}^2$ depends not only on the contact manifold (V, ξ) but also on the open book used in the construction. These results should be significantly improved, and in particular it would be nice to find an answer to the following question:

Question 1.6 *Are the Bourgeois structures on $\mathbb{S}^{2n-1} \times \mathbb{T}^2$ obtained from the standard contact sphere $(\mathbb{S}^{2n-1}, \xi_0)$ and the open book decomposition whose page is $T^*\mathbb{S}^{n-1}$ (Examples A.6(b) and A.17(b)) strongly fillable?*

We know from Theorem A that these are weakly fillable. If they were not strongly fillable they would provide the first examples of weakly but not strongly fillable contact

¹These two examples were explicitly excluded in the contact homology computations in Bourgeois's thesis.

manifolds in all dimensions. (Many such examples exist in dimension 3; see for instance Giroux [18] and Eliashberg [14]. In dimension 5, the only ones known so far can be found in Massot, Niederkrüger and Wendl [26]).

Remark 1.7 Example 1.5 generalizes in the following way to toric contact manifolds: Recall that there is an important difference between contact 5–manifolds that have a torus action that is free and those where the \mathbb{T}^3 –action is not free; see Lerman [23].

The open book on (\mathbb{S}^3, ξ_0) with page diffeomorphic to $T^*\mathbb{S}^1$ can be obtained by the map $f(z_1, z_2) = z_1^2 + z_2^2$; see Examples A.6(b) and A.17(b). Both ξ_0 and f are invariant under the free circle action on \mathbb{S}^3 given by matrix multiplication

$$e^{is} * (z_1, z_2) := \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

which implies that not only the contact structure but also the open book is preserved by this action.

Restricting the circle action to a cyclic subgroup $\mathbb{Z}_k \subset \mathbb{S}^1$, we can quotient \mathbb{S}^3 and obtain a lens space L_k carrying the natural contact structure, the induced open book decomposition, and a free circle action. The page of these open books is still diffeomorphic to $T^*\mathbb{S}^1$, and its zero section is a Legendrian submanifold of L_k .

A Bourgeois contact structure on $V \times \mathbb{T}^2$ is clearly invariant under the obvious \mathbb{T}^2 –action (see Definition 2.1), and with $V = \mathbb{S}^3$ or $V = L_k$ as above, it is easy to verify that the initial circle action adds up to give a free \mathbb{T}^3 –action on $V \times \mathbb{T}^2$. With some careful considerations, one obtains that all contact toric 5–manifolds with a free \mathbb{T}^3 –action are either equivariantly contactomorphic to the unit cotangent bundle of \mathbb{T}^3 or to one of the manifolds above. Thus, according to Corollary 1.4, none of the contact toric 5–manifolds with a free \mathbb{T}^3 –action is subcritically fillable.

In the appendix we review the one-to-one correspondence between contact open book decompositions and abstract Liouville open books. For this we use the language of ideal Liouville domains that has been created for this purpose by Giroux [20]. This language requires an initial investment of effort, but provides a suitable framework for discussing the uniqueness of the resulting contact structures up to homotopy and also for addressing problems related to the structure along the binding.

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2 The Bourgeois contact structure

Given a contact manifold (V, ξ) and a symplectic manifold (Σ, ω) , one naively obvious idea of how to construct a contact structure on $V \times \Sigma$ would be to start with the hyperplane field $\xi \oplus T\Sigma$, which is an almost contact structure, and try to deform it to a genuine contact structure.

Bourgeois [6] succeeded in carrying this out in the special case of $\Sigma = \mathbb{T}^2$, using an open book decomposition of (V, ξ) as the input to the construction. Gironella [17] put this construction in a more natural geometric framework in which open books appear organically, generalizing the definition to oriented surfaces Σ . We will describe Bourgeois’s construction, reformulating it using the notion of ideal Liouville domains and comparing with Gironella’s more general framework. The appendix provides background for a reader who might be unfamiliar with the language of ideal Liouville domains.

Let (V, ξ) be a contact manifold with a contact open book decomposition (K, ϑ) (see Definitions A.2 and A.4). From the appendix (see Proposition A.13, we can choose a contact form α_V for ξ and a function $f = f_x + if_y: V \rightarrow \mathbb{C}$ with $\vartheta = f/|f|$ such that $d(\alpha_V/|f|)$ defines an ideal Liouville structure (Definition A.7) on every page of the open book. Clearly, the data α_V and f encode the contact structure and the open book. Accordingly, we call such (α_V, f) a *representation* of the contact open book. (See Lemma A.14 for a justification of this definition.)

Here and in the following, it will often be convenient to write $f = f_x + if_y = \rho e^{i\vartheta}$. We will also consider the 1-form $d\vartheta$ obtained from a map $\vartheta: V \setminus K \rightarrow \mathbb{S}^1$ taking \mathbb{S}^1 to be the unit circle in \mathbb{C} , which we also identify with $\mathbb{R}/2\pi\mathbb{Z}$. Strictly speaking, in

what we write, $d\vartheta$ denotes the differential of the argument of ϑ but we hope that this abuse of notation does not cause any confusion.

Definition 2.1 The *Bourgeois contact structure* associated to a representation (α_V, f) of the contact open book (K, ϑ) on (V, ξ) and the standard orientation $d\varphi_1 \wedge d\varphi_2$ of \mathbb{T}^2 is given by the kernel of the 1-form

$$\alpha = \alpha_V + f_x d\varphi_1 - f_y d\varphi_2$$

on $V \times \mathbb{T}^2$, where (φ_1, φ_2) denotes the standard coordinates on \mathbb{T}^2 .

That a Bourgeois contact structure really is a contact structure will follow directly from the more general result Lemma 2.4. Notice that for a given open book decomposition, (K, ϑ) , the space of all choices of possible representations (α_V, f) is contractible (this is discussed further in the appendix; see the tabular summary on page 3441 and subsequent discussion; also see [20]). In particular then, a choice of contact open book determines an isotopy class of contact structures on $V \times \mathbb{T}^2$. It is also easy to convince oneself that up to contactomorphism the Bourgeois construction does not depend on the chosen identification of \mathbb{T}^2 with $\mathbb{S}^1 \times \mathbb{S}^1$.

Gironella [17] has extended this definition of Bourgeois contact structure as a class of hyperplane fields $\tilde{\xi}_t$ that are deformations of a flat contact fiber bundle $\tilde{\xi}_0$ over Σ . The hyperplane fields $\tilde{\xi}_t$ are contact for $t > 0$. In this paper, we only consider the product case $V \times \Sigma \rightarrow \Sigma$, where Σ is a closed oriented surface, and take the initial flat contact bundle to be $\tilde{\xi}_0 = \xi \oplus T\Sigma$. For deformations of these trivial bundles, Gironella additionally provides a description in more elementary terms, which we repeat here.

Definition 2.2 Let (V, ξ) be a contact manifold and Σ be a closed oriented surface. A *Bourgeois–Gironella contact structure* $\tilde{\xi}$ on $V \times \Sigma$ that deforms the flat contact bundle $\xi \oplus T\Sigma$ is any contact structure that can be written as $\tilde{\xi} = \ker \alpha$ with

$$\alpha = \alpha_V + \beta,$$

where

- (i) α_V is a contact form on V defining ξ ;
- (ii) β is a 1-form on $V \times \Sigma$ that vanishes on vectors that are tangent to the fibers $V \times \{z\}$ for any $z \in \Sigma$;
- (iii) for each fixed $p \in V$, the restriction of α (or, equivalently, of β) to the slice $\{p\} \times \Sigma$ is a closed form;

- (iv) the orientation induced on $V \times \Sigma$ by α is the same one as the product orientation of V with Σ .

Conditions (i) and (ii) are from [17, Proposition 7.1] and condition (iii) is from [17, Claim 7.4]. Note that in the cited reference, this formulation is given in the case $\Sigma = \mathbb{T}^2$. These properties are local in Σ , however, so they remain applicable in this seemingly more general case. We do not know of examples of such structures for $\Sigma \neq \mathbb{T}^2$, however.

In contrast to the definition of the Bourgeois structure, the one of Bourgeois–Gironella structures does not mention open books explicitly; nonetheless, Lemma 2.4 shows that the two notions are intimately linked.

Remark 2.3 Let $d\text{vol}$ denote a volume form on Σ compatible with the orientation. The Bourgeois–Gironella structure $(\ker \alpha, d\alpha)$ is homotopic to $(\xi \oplus T\Sigma, d\alpha_V + d\text{vol})$ as almost contact structures.

This is verified by introducing an ε -factor in the definition of α :

$$\alpha_\varepsilon := \alpha_V + \varepsilon\beta.$$

This allows us to deform $\ker \alpha$ to the flat contact bundle $\ker \alpha_0 = \xi \oplus T\Sigma$. We then expand:

$$\begin{aligned} \alpha_\varepsilon \wedge (d\alpha_\varepsilon + \delta d\text{vol})^{n+1} &= (\alpha_V + \varepsilon\beta) \wedge (d\alpha_V + \varepsilon d\beta + \delta d\text{vol})^{n+1} \\ &= \alpha_\varepsilon \wedge (d\alpha_\varepsilon)^{n+1} + (n+1)\delta\alpha_V \wedge (d\alpha_V)^n \wedge d\text{vol}. \end{aligned}$$

Using that the restriction of $d\beta$ vanishes on every surface slice $\{p\} \times \Sigma$, we check that the first term reduces to

$$\begin{aligned} (1) \quad \alpha_\varepsilon \wedge (d\alpha_\varepsilon)^{n+1} &= \varepsilon^2(n+1)\left(\frac{1}{2}n d\beta^2 \wedge \alpha_V \wedge d\alpha_V^{n-1} + \beta \wedge d\beta \wedge d\alpha_V^n\right) \\ &= \varepsilon^2\alpha \wedge (d\alpha)^{n+1}, \end{aligned}$$

so that if $\ker \alpha$ is contact, any of the $\ker \alpha_t$ for $t > 0$ will be contactomorphic to it.

Furthermore, using (iv) from Definition 2.2, we obtain that $\alpha_\varepsilon \wedge (d\alpha_\varepsilon + \delta d\text{vol})^{n+1}$ is strictly positive for all $\varepsilon \geq 0$ and $\delta \geq 0$ as long as ε and δ do not vanish simultaneously. This shows that $\ker \alpha$ is indeed homotopic to $\xi \oplus T\Sigma$ as almost contact structures.

In the special case of $\Sigma = \mathbb{T}^2$ and α a Bourgeois contact form (so that β is \mathbb{T}^2 -independent), if $\varepsilon < 0$, we have an explicit contactomorphism from α_ε to $\alpha_{|\varepsilon|}$ by applying the orientation-preserving diffeomorphism $(p; \varphi_1, \varphi_2) \mapsto (p; -\varphi_1, -\varphi_2)$ to $V \times \mathbb{T}^2$.

Now, essentially as an application of [17, Proposition 6.9], we obtain the following characterization of Bourgeois–Gironella structures deforming the flat bundle $\xi \oplus T\Sigma$. For the benefit of the reader, we provide a self-contained proof.

Lemma 2.4 *Let $\alpha = \alpha_V + \beta$ be a 1-form on $V \times \Sigma$, where α_V and β satisfy conditions (i)–(iv) of Definition 2.2. Suppose also that V is of dimension at least 3.*

If U is any positively oriented chart of Σ with coordinates (φ_1, φ_2) , then we can write α on $V \times U$ as

$$\alpha|_{V \times U} = \alpha_V + f_x d\varphi_1 - f_y d\varphi_2,$$

where $f = f_x + if_y: V \times U \rightarrow \mathbb{C}$ is a smooth function.

The following two statements are then equivalent:

- (a) α is a contact form.
- (b) For every chart U of Σ and every point $(\varphi_1, \varphi_2) \in U$, the pair $(\alpha_V, f(\cdot, \varphi_1, \varphi_2))$ with f as above is a representation of a contact open book on (V, ξ) .

Proof Let $2n + 1$ be the dimension of V . From (1), we know that the contact condition of $\alpha = \alpha_V + \beta$ is given by

$$\alpha \wedge (d\alpha)^{n+1} = (n + 1) \left(\frac{1}{2} n d\beta^2 \wedge \alpha_V \wedge d\alpha_V^{n-1} + \beta \wedge d\beta \wedge d\alpha_V^n \right) \neq 0.$$

Now replacing β by its representation in a chart, $f_x d\varphi_1 - f_y d\varphi_2$, and writing $f_x + if_y = f = \rho e^{i\vartheta}$, we obtain in these polar coordinates

$$\beta \wedge d\beta = \rho^2 d_V \vartheta \wedge d\varphi_1 \wedge d\varphi_2 \quad \text{and} \quad d\beta^2 = 2\rho d_V \rho \wedge d_V \vartheta \wedge d\varphi_1 \wedge d\varphi_2,$$

where d_V is the exterior derivative only in the V -direction. It therefore follows that

$$\begin{aligned} (2) \quad \alpha \wedge (d\alpha)^{n+1} &= (n + 1) \left[n d_V f_x \wedge d_V f_y \wedge \alpha_V \wedge (d\alpha_V)^{n-1} \right. \\ &\quad \left. + (f_x d_V f_y - f_y d_V f_x) \wedge (d\alpha_V)^n \right] \wedge d\varphi_1 \wedge d\varphi_2 \\ &= (n + 1) \left[n\rho d_V \rho \wedge d_V \vartheta \wedge \alpha_V \wedge (d\alpha_V)^{n-1} \right. \\ &\quad \left. + \rho^2 d_V \vartheta \wedge (d\alpha_V)^n \right] \wedge d\varphi_1 \wedge d\varphi_2. \end{aligned}$$

First, observe that if $(\alpha_V, f(\cdot, \varphi_1, \varphi_2))$ is a representation of a contact open book, then α is a contact form because the term in brackets agrees with the expansion (3) of the volume form Ω_V on V in Lemma A.12. Thus as we wanted to show $\alpha \wedge (d\alpha)^{n+1} = (n + 1)\Omega_V \wedge d\varphi_1 \wedge d\varphi_2$ does not vanish.

To prove the converse, we now suppose instead that α is a contact form. Fix a point $z \in \Sigma$. We must now prove the following statements:

- (i) $0 \in \mathbb{C}$ is a regular value of $p \mapsto f(p, z)$.
- (ii) $K := \{p \in V \mid f(p, z) = 0\}$ is nonempty.
- (iii) $\vartheta := f/|f|: V \setminus K \rightarrow \mathbb{S}^1$ is a fibration.
- (iv) $d(\alpha_V/|f|)$ restricts to each fiber $\vartheta = \vartheta_0$ as an ideal Liouville structure.

By Remark A.3, the first three properties give that $f(\cdot, z)$ defines an open book decomposition on V . The fourth gives that it is a contact open book (see Lemma A.14 for details).

To prove the first statement, let $p \in V$ be such that $f(p, z) = 0$. Since $\alpha \wedge d\alpha^{n+1}$ is a volume form by assumption, it follows from the first equality of (2) that $d_V f_x \wedge d_V f_y$ does not vanish at p , so that 0 is a regular value of $p \mapsto f(p, z)$. For the third statement, we observe that by combining the fact that $\alpha \wedge d\alpha^{n+1}$ is a volume form with the second equality of (2), $d_V \vartheta$ cannot vanish on $V \setminus K$. Hence, $\vartheta(\cdot, z): V \setminus K \rightarrow \mathbb{S}^1$ is a submersion.

In order to show that K is nonempty and also to show that $d(\alpha_V/|f|)$ restricts to the fibers as an ideal Liouville structure, we compute

$$\left(d_V\left(\frac{1}{\rho}\alpha_V\right)\right)^n = \frac{1}{\rho^{n+2}}(-n\rho d_V\rho \wedge \alpha_V \wedge (d\alpha_V)^{n-1} + \rho^2 d\alpha_V^n).$$

Observe that (2) can be rearranged to obtain

$$\begin{aligned} \alpha \wedge (d\alpha)^{n+1} &= (n+1)[-n\rho d_V\rho \wedge \alpha_V \wedge (d\alpha_V)^{n-1} + \rho^2(d\alpha_V)^n] \wedge d_V\vartheta \wedge d\varphi_1 \wedge d\varphi_2 \\ &= (n+1)\rho^{n+2}\left(d_V\left(\frac{1}{\rho}\alpha_V\right)\right)^n \wedge d_V\vartheta \wedge d\varphi_1 \wedge d\varphi_2. \end{aligned}$$

This is a volume form by assumption, so it follows that $d(\alpha_V/|f|)$ is symplectic when restricted to a fiber $\vartheta = \vartheta_0$.

For the sake of contradiction, we suppose that K is empty. In that case, the fiber $\{p \in V \mid \vartheta(p, z) = \vartheta_0\}$ is a closed submanifold of V of dimension $2n$. The restriction of $d(\alpha_V/\rho)$ to this submanifold is symplectic. By Stokes' theorem, this is only possible if the dimension of V is 1. Thus, K is nonempty for $2n + 1 \geq 3$.

Having established that K is nonempty, it follows that $d(\alpha_V/\rho)$ is an ideal Liouville domain structure on the closure of $\vartheta^{-1}(\vartheta_0)$. This then shows that (α_V, f) is a representation of a contact open book on V , as required. □

It follows in particular from this lemma that Bourgeois contact structures as given by Definition 2.1 really are contact structures. In fact, Bourgeois structures are the special Bourgeois–Gironella contact structures on $V \times \mathbb{T}^2$ that are invariant under the canonical torus action.

Gironella shows the nonobvious fact [17, Proposition 6.11] that the \mathbb{T}^2 –average of any Bourgeois–Gironella contact form $\alpha_V + \beta$ is also a contact form. This averaging process gives us a canonical map from Bourgeois–Gironella structures to Bourgeois structures. We do not know of any example of a Bourgeois–Gironella contact form that is not isotopic through Bourgeois–Gironella forms to its \mathbb{T}^2 –average.

3 The Bourgeois structure for open books with inverted monodromy

In this section we will prove Theorem B. To achieve this aim, we first describe an explicit modification of a given contact structure supported by a contact open book. The result of this construction will be a new contact structure that is supported by an open book with identical pages and binding as the first one, but with opposite coorientation. We then show that the monodromies of the two open books are the inverse of each other, and we conclude by studying how this modification affects the Bourgeois construction.

Lemma 3.1 *Let (V, ξ_+) be a contact manifold with a compatible open book decomposition (K, ϑ) , and let α_+ be any contact form that is supported by this open book.*

The space of functions $f = f_x + if_y: V \rightarrow \mathbb{C}$ (writing $|f|^2 d\vartheta = f_x df_y - f_y df_x$) that satisfy the properties below is convex and nonempty:

- (i) (α_+, f) is a representation of a Liouville open book on (K, ϑ) in the sense of Definition A.11.
- (ii) The 1–form

$$\alpha_- := \alpha_+ - C|f|^2 d\vartheta$$

is a contact form for every sufficiently large constant $C \gg 1$.

The contact forms α_+ and α_- induce opposite orientations on V , the contact form α_- is adapted to the open book decomposition $(K, \bar{\vartheta})$, and while its restriction to the binding and pages does not differ from the one of α_+ , the coorientation of pages and binding is reversed.

Proof Let $f = f_x + if_y$ and $g = g_x + ig_y$ be two functions that satisfy the two properties stated above. We know from the appendix that the set of functions F such that (α_+, F) is a representation forms a nonempty convex set, so let us concentrate on property (ii).

Define for a sufficiently large $C \gg 1$ (more precisely there is a suitable constant $C_f \gg 1$ for f and $C_g \gg 1$ for g and we take C to be larger than $\max\{C_f, C_g\}$) the two contact forms

$$\alpha_- := \alpha_+ - C|f|^2 d\vartheta \quad \text{and} \quad \beta_- := \alpha_+ - C|g|^2 d\vartheta.$$

We need to show that the interpolation

$$\alpha_s := (1 - s)\alpha_- + s\beta_-$$

satisfies for all $s \in [0, 1]$ the contact property.

Writing α_s as

$$\alpha_s := \alpha_+ - C((1 - s)|f|^2 + s|g|^2) d\vartheta,$$

it is obvious that all terms of $\alpha_s \wedge (d\alpha_s)^n$ contain at most one $d\vartheta$ -factor, and in particular $|f|^2$ - and $|g|^2$ -terms will never mix. The contact condition simplifies to

$$\alpha_s \wedge (d\alpha_s)^n = (1 - s)\alpha_- \wedge (d\alpha_-)^n + s\beta_- \wedge (\beta_-)^n \neq 0,$$

which is true by assumption, thus proving the desired convexity property. We still need to show that it is not empty.

Let $f = f_x + if_y: V \rightarrow \mathbb{C}$ be a function defining the open book, and write for simplicity $\rho = |f|$, so that $f_x df_y - f_y df_x = \rho^2 d\vartheta$. The condition that $d(\alpha_+ / |f|) = d(\alpha_+ / \rho)$ is a Liouville form on each page can be verified by computing

$$\rho^{n+2} d\vartheta \wedge (d(\alpha_+ / \rho))^n = \rho^2 d\vartheta \wedge (d\alpha_+)^n + n\rho d\rho \wedge d\vartheta \wedge \alpha_+ \wedge (d\alpha_+)^{n-1} \neq 0,$$

and the condition that α_- is a contact form is verified by computing

$$\alpha_- \wedge d\alpha_-^n = \alpha_+ \wedge (d\alpha_+)^n - C[\rho^2 d\vartheta \wedge (d\alpha_+)^n + 2n\rho d\rho \wedge d\vartheta \wedge \alpha_+ \wedge (d\alpha_+)^{n-1}].$$

In both cases, the term $\rho^2 d\vartheta \wedge (d\alpha_+)^n$ is never negative and only vanishes along the binding. The second term $\rho d\rho \wedge d\vartheta \wedge \alpha_+ \wedge (d\alpha_+)^{n-1}$ can be understood as follows: Along the binding the term is positive, since $\rho d\rho \wedge d\vartheta$ is an area form on the disk and because the restriction of α_+ to the binding is by assumption a positive contact form. If ρ is a function that increases linearly in radial direction at the binding K

and that is constant outside a sufficiently small neighborhood of K , then it follows that $\rho d\rho \wedge d\vartheta$ is positive along the binding and everywhere else is nonnegative. This shows that the function ρ can be chosen in such a way that (α_+, f) is a representation and such that α_- will be for any sufficiently large C a contact form.

It remains to show that $\xi_- = \ker \alpha_-$ is supported by $(K, \bar{\vartheta})$. For this, note that $(K, \bar{\vartheta})$ and (K, ϑ) have the same pages and binding. The restrictions of α_- and α_+ agree on both subsets, since the additional term vanishes when restricted to either. The contact forms α_+ and α_- induce opposite orientations on V , which is compatible with the choice of coorientations given by ϑ and $\bar{\vartheta}$ respectively. □

Lemma 3.2 *Assume we are in the setup of the previous lemma. The abstract Liouville open books corresponding to α_- and $(K, \bar{\vartheta})$ and to α_+ and (K, ϑ) have identical ideal Liouville domains as pages, but their monodromies are the inverse of each other.*

Proof Let $f = f_x + if_y$ be the function used in the previous lemma. It is easy to check that $\bar{f} = f_x - if_y$ is a function defining $(K, \bar{\vartheta})$ and since the restrictions of α_+ and α_- agree on all pages, it is clear that $\alpha_-/|\bar{f}| = \alpha_+/|f|$ defines on every page the same ideal Liouville structure as the initial open book. This shows that the pages of the abstract open book corresponding to (K, ϑ) with contact form α_+ and the ones corresponding to $(K, \bar{\vartheta})$ with contact form α_- are identical as ideal Liouville domains.

We set $\lambda_+ := \alpha_+/|f|$, and $\lambda_- := \alpha_-/|\bar{f}|$. Recall that we recover the monodromy of the Liouville open book $(K, \vartheta, d\lambda_+)$ by following the flow of a spinning vector field from an initial page back to itself. By Lemma A.16, we can specify a unique spinning vector field Y_+ by the equations

$$d\vartheta(Y_+) = 2\pi \quad \text{and} \quad \iota_{Y_+} d\lambda_+ = 0.$$

We claim that $Y_- = -Y_+$ is a spinning vector field for the Liouville open book $(K, \bar{\vartheta}, d\lambda_-)$. Clearly Y_- vanishes along the binding and $d\bar{\vartheta}(Y_-) = +2\pi$. It only remains to show that the flow of Y_- preserves the ideal Liouville structure on every page.

For this simply compute

$$\mathcal{L}_{Y_-} d\lambda_- = -\mathcal{L}_{Y_+} \left(d\lambda_+ - C d\left(\frac{\rho^2}{|f|}\right) \wedge d\vartheta \right) = C \mathcal{L}_{Y_+} (d\rho \wedge d\vartheta) = C(\mathcal{L}_{Y_+} d\rho) \wedge d\vartheta.$$

Since $d\vartheta$ vanishes on every page, we see that Y_- is indeed a spinning vector field for $d\lambda_-$.

The time-1 flow of Y_- is obviously the inverse of the time-1 flow of Y_+ , and thus we have shown that the corresponding abstract open books have the same page and that the monodromies are the inverse of each other. \square

We now show that inverting the monodromy of an open book has no influence on the Bourgeois construction.

Theorem B *Let (V, ξ_+) and (V, ξ_-) be closed contact manifolds supported by abstract Liouville open books that have the same page but inverse monodromy. Then the two corresponding Bourgeois structures on $V \times \mathbb{T}^2$ are contactomorphic.*

This is a corollary of Lemma 3.2 combined with the following result.

Lemma 3.3 *Assume we are in the setup of Lemma 3.1, so that (V, ξ_+) is a contact manifold with a compatible open book decomposition (K, ϑ) that is represented by (α_+, f) , and $\xi_- = \ker(\alpha_-)$ with $\alpha_- = \alpha_+ - C|f|^2 d\vartheta$ for sufficiently large C is a contact structure on V that is supported by the open book $(K, \bar{\vartheta})$.*

Then, any Bourgeois contact structure on $V \times \mathbb{T}^2$ associated to the contact open book (ξ_+, K, ϑ) and the standard orientation of \mathbb{T}^2 is isotopic through contact structures to the Bourgeois contact structure on $V \times \mathbb{T}^2$ associated to $(\xi_-, K, \bar{\vartheta})$ and the reversed orientation on \mathbb{T}^2 .

Proof With the notation as in Lemma 3.1, it follows that (α_-, \bar{f}) is a representation of the open book $(\xi_-, K, \bar{\vartheta})$, where $\bar{f} = f_x - if_y$ denotes the complex conjugate.

From Definition 2.1, the Bourgeois contact structure associated to (α_+, f) (and the standard orientation on \mathbb{T}^2) is given by

$$\alpha_+ + f_x d\varphi_1 - f_y d\varphi_2.$$

Consider now the parametric family of 1-forms given by

$$\alpha_\tau = \alpha_+ + f_x d\varphi_1 - f_y d\varphi_2 - \tau C|f|^2 d\vartheta \quad \text{for } 0 \leq \tau \leq 1.$$

A direct computation shows that $\alpha_\tau \wedge (d\alpha_\tau)^{n+1} = \alpha_0 \wedge (d\alpha_0)^{n+1}$, and thus these are all contact forms. Very explicitly, we observe that $\alpha_\tau = \Phi_\tau^* \alpha_0$, with Φ_τ given by

$$\Phi_\tau: V \times \mathbb{T}^2 \rightarrow V \times \mathbb{T}^2, \quad (p; \varphi_1, \varphi_2) \mapsto (p; \varphi_1 - \tau C f_y, \varphi_2 - \tau C f_x).$$

Now, observe that $\alpha_1 = \alpha_- + f_x d\varphi_1 - f_y d\varphi_2$, which is the Bourgeois form on $V \times \mathbb{T}^2$ associated to the representation (α_-, \bar{f}) and the orientation on \mathbb{T}^2 given by $(\partial_{\varphi_1}, -\partial_{\varphi_2})$. \square

Finally we obtain the desired contactomorphism for Theorem B by composing the isotopy from the previous lemma with the diffeomorphism $(p; \varphi_1, \varphi_2) \mapsto (p; \varphi_1, -\varphi_2)$ on $V \times \mathbb{T}^2$.

4 Explicit constructions of fillings

In this section, we will prove Theorem A from the introduction.

Let (V, ξ) be a contact manifold, and let α_V be a contact form for ξ . A symplectic manifold (W, ω) is called a *weak filling* of (V, ξ) (see [26]) if W is compact with (oriented) boundary $\partial W = V$, and if for every $T \in [0, \infty)$,

$$\alpha_V \wedge (T d\alpha_V + \omega)^n > 0,$$

where $\dim V = 2n + 1$. In particular, this gives that the volume form $\alpha_V \wedge d\alpha_V^n$ must induce the same orientation on V as the boundary orientation on $V = \partial W$.

The following argument was inspired by a 3–dimensional proof in [18], and has been sketched in [26, Example 1.1]. A proof mostly identical to ours has recently appeared in [17], but since the argument is relatively short we prefer to restate it here for completeness of our presentation.

Theorem A(a) *Let (V, ξ) be a contact manifold that is weakly filled by (W, ω) , and let (K, ϑ) be any open book that is compatible with ξ . Then the associated Bourgeois contact structure on $V \times \mathbb{T}^2$ is isotopic to a contact structure that can be weakly filled by $(W \times \mathbb{T}^2, \omega \oplus \text{vol}_{\mathbb{T}^2})$.*

Proof First, we verify that the orientation induced by α on $V \times \mathbb{T}^2$ matches the boundary orientation induced on $V \times \mathbb{T}^2 = \partial(W \times \mathbb{T}^2)$. Observe that the boundary orientation on $V \times \mathbb{T}^2$ is the orientation induced by $\alpha_V \wedge d\alpha_V^n \wedge d\text{vol}_{\mathbb{T}^2}$ because V is weakly filled by W . The claim then follows from (iv) in Definition 2.2.

Recall the modified Bourgeois contact form α_ε from Remark 2.3:

$$\alpha_\varepsilon := \alpha_V + \varepsilon\beta.$$

Then, define the following family of polynomials:

$$P_\varepsilon(T) := \alpha_\varepsilon \wedge (T d\alpha_\varepsilon + \omega + \text{vol}_{\mathbb{T}^2})^{n+1}.$$

These have degree at most $n + 1$ in T with coefficients in $\Omega^{2n+1}(V \times \mathbb{T}^2)$ that depend smoothly on ε .

We will show that if $\varepsilon > 0$ is chosen sufficiently small, then $P_\varepsilon(T)$ will be positive for every $T \in [0, \infty)$, so that $(W \times \mathbb{T}^2, \omega \oplus \text{vol}_{\mathbb{T}^2})$ is a weak filling of $\ker \alpha_\varepsilon$, which by Remark 2.3 is isotopic to $\ker \alpha$.

First note from (1) that the leading term of P_ε is $\varepsilon^2 \alpha \wedge d\alpha^{n+1} T^{n+1}$. Its coefficient vanishes for $\varepsilon = 0$, but is strictly positive for $\varepsilon \neq 0$. For $\varepsilon = 0$, we compute

$$P_0(T) = \alpha_V \wedge (T d\alpha_V + \omega + \text{vol}_{\mathbb{T}^2})^{n+1} = (n+1)\alpha_V \wedge (T d\alpha_V + \omega)^n \wedge \text{vol}_{\mathbb{T}^2}.$$

This form is strictly positive for all $T \in [0, \infty)$ by the assumption that (W, ω) is a weak filling of (V, ξ) . Furthermore we see that $P_0(T)$ is of degree n in T with a strictly positive coefficient for the leading term. Any small perturbation of P_0 inside the polynomials of degree n will also be strictly positive on $T \in [0, \infty)$: If we choose a sufficiently large T_0 , the leading term of $P_0(T)$ dominates the remaining terms of the polynomial for $T > T_0$. Thus none of the polynomials of degree n that are close to P_0 will vanish for $T > T_0$. On the other hand, if we only consider a compact interval $[0, T_0]$, it follows by continuity that a small perturbation of P_0 (even in the space of continuous functions) cannot vanish on $[0, T_0]$ either.

Combining this with the positivity of the coefficient for T^{n+1} -term in P_ε we obtain the desired result. □

Before proving part (b) of Theorem A, we will briefly recall the basic definitions on Weinstein manifolds.

A *Weinstein manifold* (W, ω, X, f) is a symplectic manifold (W, ω) without boundary, together with

- (i) a complete vector field X such that $\mathcal{L}_X \omega = \omega$, a so-called complete *Liouville vector field*, and
- (ii) a proper Morse function $f: W \rightarrow [0, \infty)$ that is a Lyapunov function for X , meaning that there is a positive constant δ such that $df(X) \geq \delta \cdot (\|X\|^2 + \|df\|^2)$ with respect to some Riemannian metric.

Other definitions may not require f to be a Morse function, but we follow [11] and just note that a given Weinstein manifold is symplectomorphic to one whose Lyapunov function is Morse.

The topology of a Weinstein manifold (W, ω, X, f) is relatively restricted, because the index of every critical point of f is less than or equal to half the dimension of W . If f has only critical points of index strictly less than $\frac{1}{2} \dim W$, then we say

that (W, ω, X, f) is a *subcritical Weinstein manifold*; and if f has only critical points of index not more than $\frac{1}{2} \dim W - k$, then we say that (W, ω, X, f) is *k-subcritical*.

The complex plane with the standard symplectic form $\omega_0 = dx \wedge dy$, Liouville vector field $X_0 = \frac{1}{2}(x\partial_x + y\partial_y)$, and Morse function $f_0(x + iy) = x^2 + y^2$ is a Weinstein manifold. The *stabilization of a Weinstein manifold* (W, ω, X, f) is the product Weinstein manifold $(W \times \mathbb{C}, \omega \oplus \omega_0, X \oplus X_0, f + f_0)$.

The stabilization of any Weinstein manifold is subcritical, and according to the following result by Cieliebak [10; 11, Section 14.4], subcritical Weinstein manifolds are essentially stabilizations.

Theorem 4.1 (Cieliebak) *Every subcritical Weinstein manifold of dimension $2n$ is symplectomorphic to the stabilization of a Weinstein manifold of dimension $2n - 2$.*

Note that we only use this theorem to obtain that any compact Lagrangian in a subcritical Weinstein domain is Hamiltonian displaceable. This simpler result is given by [2, Lemma 3.2].

A regular level set $M_c = f^{-1}(c)$ of a Weinstein manifold (W, ω, X, f) carries a natural contact structure given by the kernel of the 1-form $\alpha_c := \omega(X, \cdot)|_{TM_c}$. We say that a contact manifold (V, ξ) is (*subcritically*) *Weinstein fillable* if it is contactomorphic to a regular level set $(M_c, \ker \alpha_c)$ of a (subcritical) Weinstein manifold (W, ω, X, f) such that all critical values of f are strictly smaller than c .

Let (V, ξ) be a contact manifold that is subcritically filled by a stabilized Weinstein manifold $(W \times \mathbb{C}, \omega \oplus \omega_0, X \oplus X_0, f + f_0)$. A computation shows that (V, ξ) is supported by the open book with binding $K_0 = V \cap (W \times \{0\})$ and fibration $\vartheta_0: V \setminus K_0 \rightarrow \mathbb{S}^1$ given by $(p, z) \mapsto z/|z|$. The corresponding abstract open book has page W and trivial monodromy. The details of this are carried out in Example A.19.

The following proposition finishes the proof of Theorem A. It shows that certain Bourgeois contact structures are Weinstein fillable.

Theorem A(b) *Let (V, ξ) be a closed contact manifold that is subcritically filled by the Weinstein manifold $(W \times \mathbb{C}, \omega \oplus \omega_0, X \oplus X_0, f + f_0)$. Let (K_0, ϑ_0) be the associated open book with trivial monodromy.*

The Bourgeois contact structure on $V \times \mathbb{T}^2$ obtained by using the contact open book (K_0, ϑ_0) can be filled by the Weinstein manifold

$$(W \times T^*\mathbb{T}^2, \omega \oplus d\lambda_{\text{can}}, X \oplus X_{\mathbb{T}^2}, \tilde{f}),$$

where the cotangent bundle of \mathbb{T}^2 is written with coordinates $(q_1, q_2; p_1, p_2) \in \mathbb{T}^2 \times \mathbb{R}^2$, the vector field $X_{\mathbb{T}^2}$ equals $p_1 \partial_{p_1} + p_2 \partial_{p_2}$, and \tilde{f} is a small perturbation of $f + f_{\mathbb{T}^2}$, where $f_{\mathbb{T}^2} = p_1^2 + p_2^2$, as described in [11, Example 11.12(2)].

The perturbation of \tilde{f} is necessary to convert the Morse–Bott function to a Morse function.

Proof Identify (V, ξ) with the regular level set M_c in $(W \times \mathbb{C}, J \oplus i)$. The Bourgeois structure on $M_c \times \mathbb{T}^2$ is given by the contact form

$$\alpha = \lambda_W + x \, dy - y \, dx + x \, d\varphi_1 - y \, d\varphi_2,$$

where λ_W is the Liouville form $\iota_X \omega$ on W , $z = x + iy$ are the coordinates on \mathbb{C} , and (φ_1, φ_2) are the coordinates on the torus.

The diffeomorphism from $W \times \mathbb{C} \times \mathbb{T}^2$ to $W \times T^*\mathbb{T}^2$ that sends $(x, y; \varphi_1, \varphi_2) \in \mathbb{C} \times \mathbb{T}^2$ to $(q_1, q_2; p_1, p_2) = (-\varphi_1 - y, \varphi_2 + x; x, y) \in T^*\mathbb{T}^2$ and keeps the W -factor unchanged is the desired contactomorphism. Note in particular that it pulls back $f + f_{\mathbb{T}^2}$ to $f + f_0$. □

5 Obstructions to subcritical fillings

The aim of this section is to show that most Bourgeois structures are not subcritically fillable. We will first introduce the necessary preliminaries to prove Theorem C.

Let (V, ξ) be a contact manifold.

Definition 5.1 A submanifold P of a contact manifold (V, ξ) is called *pre-Lagrangian*

- if $\dim P = \frac{1}{2}(\dim V + 1)$, and
- if there exists a contact form α for ξ such that $d\alpha|_{TP} = 0$.

It is easy to see that ξ induces a *regular* Legendrian foliation on such a P .

The symplectization $(SV, d\lambda_{\text{can}})$ of (V, ξ) is the submanifold

$$SV := \{(p, \eta_p) \in T^*V \mid \ker \eta_p = \xi_p \text{ and } \eta_p \text{ agrees with the coorientation of } \xi_p\}$$

of the cotangent bundle of V , where λ_{can} denotes the restriction of the canonical Liouville 1-form of T^*V . We denote by $\pi_V: SV \rightarrow V$ the projection $\pi_V(p, \eta_p) = p$. The choice of a contact form α for ξ allows us to identify $(SV, d\lambda_{\text{can}})$ with $(\mathbb{R} \times V, d(e^t \alpha))$ via the map $(t, p) \in \mathbb{R} \times V \mapsto e^t \alpha_p \in SV$, making use of the tautological property $\beta^* \lambda_{\text{can}} = \beta$ for $\beta \in \Omega^1(V)$.

An equivalent definition of $P \subset V$ being pre-Lagrangian is to say that the symplectization contains a Lagrangian $L \subset SV$ such that the projection $\pi_V: SV \rightarrow V$ restricts to a diffeomorphism $\pi_V|_L: L \rightarrow P$ (see [15, Proposition 2.2.2]). Every such Lagrangian is called a *Lagrangian lift of P* . These lifts are related to the choice of a contact form α with $d\alpha|_{TP} = 0$ by $L = \alpha(P)$, where α is regarded as a section $V \rightarrow SV$.

Gromov [21, 2.3.B₃] calls a Lagrangian L in a symplectic manifold (W, ω) *weakly exact* if $\int_{\mathbb{D}^2} u^* \omega$ vanishes for every smooth map

$$u: (\mathbb{D}^2, \partial\mathbb{D}^2) \rightarrow (W, L).$$

In the spirit of [25] we use the same notion for pre-Lagrangians: a pre-Lagrangian P in a contact manifold (V, ξ) is called *weakly exact* if for every contact form α with $d\alpha|_{TP} = 0$ and for every smooth map $u: (\mathbb{D}^2, \partial\mathbb{D}^2) \rightarrow (V, P)$, the integral $\int_{\mathbb{D}^2} u^* d\alpha$ vanishes. In fact, if this integral is zero for one such form, then it is zero for every α' for which $d\alpha'|_{TP} = 0$.

In contrast to the Lagrangian case where weak exactness is a rather subtle symplectic property, the weak exactness for pre-Lagrangians reduces to the following topological observation:

Lemma 5.2 *A closed pre-Lagrangian $P \subset (V, \xi)$ is weakly exact if and only if every smooth loop in P that is positively transverse to the foliation $\mathcal{F} := \xi \cap TP$ is nontrivial in $\pi_1(V)$.*

Remark 5.3 In dimension 3, the only type of closed pre-Lagrangian is an embedded torus whose characteristic foliation is linear. In this case, Lemma 5.2 states that weak exactness is equivalent to the incompressibility of the torus, because the transverse loops generate the full fundamental group of the torus.

Tight contact manifolds with positive Giroux torsion contain “many” incompressible pre-Lagrangians and are at the same time not even strongly fillable. Theorem C requires the existence of only one incompressible pre-Lagrangian, but this weaker condition only contradicts a more specific type of filling.

Proof of Lemma 5.2 Let α be a contact form on V such that $d\alpha|_{TP} = 0$.

Assume that P is weakly exact and that $\gamma \subset P$ is a smooth loop that is positively transverse to \mathcal{F} . If $[\gamma]$ were trivial in $\pi_1(V)$, we could choose a (smooth) map $u: (\mathbb{D}^2, \partial\mathbb{D}^2) \rightarrow (V, P)$ with $u|_{\partial\mathbb{D}^2} = \gamma$, so that by Stokes’ theorem,

$$\int_u d\alpha = \int_\gamma \alpha.$$

Since P is weakly exact, the left integral had to be 0, while the right integral has to be strictly positive, because $\alpha(\gamma') > 0$ everywhere. Thus it follows that γ cannot be contractible in V .

For the opposite direction, assume now that every smooth loop in P that is positively transverse to the foliation is nontrivial in $\pi_1(V)$. To show that P is weakly exact, we have to prove that for any smooth map $u: (\mathbb{D}^2, \partial\mathbb{D}^2) \rightarrow (V, P)$ the integral $\int_{\mathbb{D}^2} u^* \alpha = \int_{\partial\mathbb{D}^2} u^* \alpha$ is 0.

We show below that every loop γ in P with $\int_{\gamma} \alpha > 0$ can be homotoped to one that is positively transverse to \mathcal{F} . Our starting assumption then implies that none of the loops $\gamma \subset P$ with $\int_{\gamma} \alpha \neq 0$ can be contractible in V , and since the boundary of a disk u clearly is contractible, we obtain $\int_{\partial\mathbb{D}^2} u^* \alpha = 0$ as we wanted to show.

It remains to prove that every smooth loop $\gamma: \mathbb{S}^1 \rightarrow P$ satisfying $\int_{\gamma} \alpha > 0$ is isotopic to a smooth loop $\tilde{\gamma}: \mathbb{S}^1 \rightarrow P$ that is everywhere positively transverse to \mathcal{F} .

By assumption, $C = \int_{\gamma} \alpha$ is positive, and we define $g(t) = \alpha(\gamma'(t))$, so that

$$\int_0^{2\pi} g(t) dt = \int_{\gamma} \alpha = C.$$

Set

$$f(t) = \frac{Ct}{2\pi} - \int_0^t g(s) ds,$$

and observe that $f(0) = 0 = f(2\pi)$ and that $f'(t) = C/(2\pi) - g(t)$.

Choose any vector field Y on P such that $\alpha(Y) = 1$, and let $\Phi_t(x) = \Phi_t^Y(x)$ denote its time- t flow. Note that Φ_t preserves $\alpha|_{TP}$.

For every $\tau \in [0, 1]$, the map

$$t \mapsto \Phi_{\tau f(t)}(\gamma(t))$$

provides a smooth loop $\mathbb{S}^1 \rightarrow P$, and for $\tau = 1$ we obtain

$$\alpha\left(\frac{d}{dt}[\Phi_{f(t)}(\gamma(t))]\right) = \alpha(f'(t)Y + \Phi_{f(t)*}\gamma'(t)) = f'(t) + g(t) = \frac{C}{2\pi} > 0.$$

This thus constructs the desired isotopy. □

The link between weakly exact pre-Lagrangians and weakly exact Lagrangians is established by the following lemma whose proof is an easy exercise using the tautological property and Stokes' theorem (see also [25, Lemma 2.2]).

Lemma 5.4 *A pre-Lagrangian $P \subset (V, \xi)$ is weakly exact if and only if its Lagrangian lifts are weakly exact in the symplectization $(SV, d\lambda_{\text{can}})$.*

Recall that a Lagrangian L is called *displaceable* if there is a compactly supported Hamiltonian isotopy ϕ_t such that $\phi_1(L) \cap L = \emptyset$. Accordingly, a pre-Lagrangian P is called *displaceable* if there is a contact isotopy ϕ_t such that $\phi_1(P) \cap P = \emptyset$.

The proof of Theorem C uses the following result by Gromov as an essential ingredient:

Theorem 5.5 *Let $(W, d\lambda)$ be an exact symplectic manifold, convex at infinity.*

- (a) *There are no closed, weakly exact Lagrangians in $(W \times \mathbb{C}, d\lambda \oplus dz \wedge d\bar{z})$; see [21, Section 2.3.B₃].*
- (b) *There are no closed, displaceable weakly exact Lagrangians in $(W, d\lambda)$; see [21, Section 2.3.B'₃].*

With this we are ready to prove Theorem C, which simply translates the statements above to certain pre-Lagrangians to give obstructions to (subcritical) Weinstein fillability.

Let (V, ξ) be a regular level set $f^{-1}(c)$ of a Weinstein manifold (W, ω, X, f) such that all critical values of f are smaller than c . Using that the flow Φ_t^X of the Liouville field is by assumption complete, we construct a symplectic embedding $j: SV \hookrightarrow W$ of the symplectization of V in W . The image $j(SV)$ is dense in W , and its complement consists only of the Lagrangian skeleton of W ; that is, $W \setminus j(SV)$ is the union of the stable manifolds of the critical points of X . These are all of dimension $n \leq \frac{1}{2} \dim W$ and of dimension $n < \frac{1}{2} \dim W$ if W is subcritical. Also the image of a closed Lagrangian $L \subset SV$ is clearly a closed Lagrangian $j(L)$ in W .

By Lemma 5.4, a weakly exact pre-Lagrangian in V gives rise to a weakly exact Lagrangian in the symplectization SV . In the context of Weinstein fillings, we have the following stronger result (see also the proof of [3, Proposition 5.1]):

Lemma 5.6 *Let (W, ω, X, h) be the Weinstein filling of a contact manifold (V, ξ) , and let $P \subset (V, \xi)$ be a pre-Lagrangian with Lagrangian lift $L \subset (SV, d\lambda)$. Assume either that $\dim W \geq 6$ or that W is subcritical and $\dim W = 4$.*

The pre-Lagrangian P is weakly exact if and only if $j(L) \subset W$ is weakly exact in W .

Proof Since every map $v: (\mathbb{D}^2, \partial\mathbb{D}^2) \rightarrow (SV, L)$ can be viewed as a map into W , the weak exactness of $j(L) \subset W$ implies directly the one of $L \subset SV$. By Lemma 5.4 it then follows that P is also weakly exact.

Assume now that $L \subset SV$ is a weakly exact Lagrangian and let $u: (\mathbb{D}^2, \partial\mathbb{D}^2) \rightarrow (W, j(L))$ be a smooth map. Since the skeleton of W is only n -dimensional, and

since $\dim W = 2n \geq 6$, we see that $n + 2 < 2n$, so that the image of u will generically not intersect the skeleton of W . After a homotopy, we may assume that the image of u lies in the complement of the Lagrangian skeleton and thus in $j(SV)$, and we may apply the weak exactness assumption of L in SV .

If $\dim W = 4$, but if W is subcritical, then we arrive to the same conclusion because the skeleton of W is only 1-dimensional. □

Theorem C *A closed contact manifold containing a weakly exact pre-Lagrangian P is not subcritically Weinstein fillable.*

If the dimension of the contact manifold is at least 5 and if P is displaceable then it follows that the contact manifold is not even Weinstein fillable.

Proof Combining Theorem 4.1 by Cieliebak with Theorem 5.5(a) by Gromov, we see that subcritical Weinstein manifolds do not contain any weakly exact Lagrangians. As a consequence of Lemma 5.6, it then follows that contact manifolds that are subcritically fillable may not contain weakly exact pre-Lagrangians. This proves the first statement of the theorem.

According to [25, Lemma 2.4] a contact isotopy that displaces a pre-Lagrangian lifts to a Hamiltonian isotopy with compact support in the symplectization that displaces a Lagrangian lift of the pre-Lagrangian. Using Theorem 5.5(b) combined with Lemma 5.6 we then see that a Weinstein fillable contact manifold of dimension at least 5 may not contain any displaceable weakly exact pre-Lagrangians. □

Example 5.7 Let (W, ω) be a closed manifold with an integral symplectic form, so that we can find a principal circle bundle $\pi: V \rightarrow W$ with Euler class $[\omega]$. The prequantization (V, α) is a contact manifold where we choose a contact form α such that $d\alpha = \pi^*\omega$ and $\alpha(Z) = 1$ for Z the infinitesimal generator of the circle action (this implies that α is invariant under the circle action and Z is its associated Reeb vector field).

Any Lagrangian L in W is covered by a pre-Lagrangian $P = \pi^{-1}(L)$ in V , because $\alpha|_{TP}$ is not singular and $d\alpha|_{TP} = \pi^*(\omega|_{TL}) = 0$; see [15]. Furthermore if L is weakly exact so is P , because if $u: (\mathbb{D}^2, \partial\mathbb{D}^2) \rightarrow (V, P)$ is any smooth map, then we obtain with a simple calculation

$$\int_{\mathbb{D}^2} u^* d\alpha = \int_{\mathbb{D}^2} u^* \pi^* \omega = \int_{\mathbb{D}^2} (\pi \circ u)^* \omega = 0,$$

using that $\pi \circ u$ is a smooth map in W with boundary in L .

A prequantization over a symplectic manifold containing a weakly exact Lagrangian is thus not subcritically Weinstein fillable.

Lemma 5.8 *Let (V, ξ) be a contact manifold with compatible open book (K, ϑ) , and equip $V \times \mathbb{T}^2$ with the Bourgeois structure corresponding to this open book. We have the following two constructions of pre-Lagrangians:*

- (a) *If there is a closed Legendrian $L \subset V$ contained in one page of (K, ϑ) then $L \times \mathbb{T}^2 \subset V \times \mathbb{T}^2$ is a weakly exact pre-Lagrangian.*
- (b) *Any closed pre-Lagrangian $P \subset K$ in the binding $(K, \xi \cap TK)$ yields a pre-Lagrangian $P \times \mathbb{T}^2$ in the Bourgeois manifold. This pre-Lagrangian is weakly exact if and only if every loop in P that is positively transverse to the characteristic foliation is noncontractible in V .*

Part (b) also applies directly to more general Bourgeois–Gironella structures on $V \times \Sigma$.

Proof (a) We will first show that $L \times \mathbb{T}^2$ is a pre-Lagrangian. Let α_V be a contact form for ξ that is supported by the open book (K, ϑ) . The Bourgeois structure on $V \times \mathbb{T}^2$ is given as the kernel of the form $\alpha = \alpha_V + f_x d\varphi_1 - f_y d\varphi_2$, where $(\alpha_V, f_x + if_y)$ is a representation of (K, ϑ) and ξ . If L is Legendrian, then $\alpha_V|_{TL} = 0$.

Since L is contained in the interior of one of the pages, either f_x or f_y does not vanish anywhere on L . Suppose it is f_x , then we extend $f_x|_L$ to a nowhere-vanishing function \hat{f}_x on $V \times \mathbb{T}^2$ that we use to rescale α . For this new contact form, we have $\alpha|_{T(L \times \mathbb{T}^2)} = d\varphi_1 - c d\varphi_2$, where c is a constant $f_y/f_x = \tan \vartheta|_L$. This implies that $L \times \mathbb{T}^2$ is pre-Lagrangian.

To see that $L \times \mathbb{T}^2$ is weakly exact choose any loop γ that is positively transverse to the foliation of $L \times \mathbb{T}^2$ given by $\ker \alpha$. According to Lemma 5.2, $L \times \mathbb{T}^2$ is weakly exact if γ is not contractible in $V \times \mathbb{T}^2$, ie nontrivial in $\pi_1(V \times \mathbb{T}^2) = \pi_1(V) \times \pi_1(\mathbb{T}^2)$. Since the characteristic foliation on $L \times \mathbb{T}^2$ is the lift of the linear foliation on \mathbb{T}^2 , it follows that γ projects to a nontrivial loop in $\pi_1(\mathbb{T}^2)$.

(b) Let P be a pre-Lagrangian in the binding K . Notice that both functions f_x and f_y vanish along $K \times \mathbb{T}^2$ so in particular along $P \times \mathbb{T}^2$. Notice also that $P \times \mathbb{T}^2$ is of the correct dimension. It follows therefore that $P \times \mathbb{T}^2$ is pre-Lagrangian. The statement about weak exactness follows immediately from Lemma 5.2, and that $\pi_1(V \times \mathbb{T}^2)$ decomposes as a product. \square

Corollary 1.4 from the introduction now follows immediately from Theorem C and Lemma 5.8(a). As an application of Theorem C we are able to show that even though some Bourgeois contact structures are subcritically fillable, most are not. In particular, we see that changing the open book for a given contact structure may destroy the subcritical fillability of the resulting Bourgeois structure.

Appendix Contact open books and ideal Liouville domains

The aim of this appendix is to give a short overview on the *ideal Liouville domains* introduced by Giroux [20] and illustrate their use by working out a few classical examples of contact open books.

Even though the relation between contact open books and abstract open books is by now well known and has been discussed in several sources (eg [19; 16; 22]), the ideal Liouville domain machinery of Giroux [20] encapsulates the fussy technical details (such as smoothing corners and modifying monodromy maps near the boundary of the page), thus leading to a significantly cleaner exposition.

An abstract Liouville open book consists of an ideal Liouville domain together with a monodromy map (see Definition A.7).

Theorem A.1 [20, Propositions 17 and 21] *Given a contact manifold (V, ξ) with contact open book (K, ϑ) that carries the contact structure, we may construct an abstract open book (F, ω, ϕ) , where (F, ω) is an ideal Liouville domain and $\phi \in \text{Diff}_\partial(F, \omega)$. This construction is well defined up to homotopy of the pair (ω, ϕ) through ideal Liouville structures on F and symplectomorphisms relative to the boundary.*

Two contact structures on V homotopic through contact structures carried by (K, ϑ) induce the same (F, ω, ϕ) likewise up to homotopy of the pair (ω, ϕ) .

Furthermore, given an abstract open book (F, ω, ϕ) , there exists a (V, K, ϑ) , well-defined up to diffeomorphism and a contact structure ξ carried by (K, ϑ) and unique up to homotopy of carried contact structures, whose associated abstract open book is (F, ω, ϕ) .

Note that two contact structures on V homotopic through contact structures carried by (K, ϑ) are clearly contactomorphic, but there may not exist a contactomorphism preserving the open book (K, ϑ) .

In the next three sections, we explain the formalism introduced by Giroux and illustrate it by applying it to the two most elementary open books on the standard contact sphere.

We also verify that an open book with trivial monodromy is explicitly fillable by the Weinstein manifold obtained by stabilizing the page (in the sense of Weinstein domains; see Section 4).

A.1 Contact open books

Definition A.2 Let V be a closed manifold. An *open book* on V is a pair (K, ϑ) , where

- $K \subset V$ is a nonempty codimension-2 submanifold with trivial normal bundle;
- $\vartheta: V \setminus K \rightarrow \mathbb{S}^1$ is a fibration that agrees in a tubular neighborhood $K \times \mathbb{D}^2$ of $K = K \times \{0\}$ with a normal angular coordinate.

We call K the *binding* of the open book, and we call the closure $F_\varphi := \vartheta^{-1}(e^{i\varphi}) \cup K$ of every fiber a *page* of the open book.

Note that the pages are smooth compact submanifolds with boundary K .

Remark A.3 It is easy to see that an open book can equivalently be specified by a smooth function $h: V \rightarrow \mathbb{C}$ for which 0 is a regular value such that $K_h := h^{-1}(0)$ is not empty, and such that

$$\vartheta_h: V \setminus K_h \rightarrow \mathbb{S}^1, \quad p \mapsto \frac{h(p)}{|h(p)|},$$

is a submersion. The set of smooth functions defining a given open book is a nonempty convex subset of $C^\infty(V, \mathbb{C})$.

If V is an oriented manifold, the coorientations specified by ϑ orient both the pages and the binding. From a practical viewpoint it is helpful to formulate these orientations using volume forms.

- A vector $R \in T_p V$ at a point $p \in V \setminus K$ is positively transverse to a page if and only if $d\vartheta(R) > 0$. Given a positive volume form Ω_V on V , it follows that $\iota_R \Omega_V$ determines the positive orientation for the page. A volume form Ω_F on a page F_φ is thus positive if and only if $d\vartheta \wedge \Omega_F$ is positive on $TV|_{\text{Int } F_\varphi}$.
- Identify the neighborhood of K with $K \times \mathbb{D}^2$ such that the angular coordinate φ agrees with ϑ and such that the disk has the canonical orientation with coordinates $(x, y) \in \mathbb{D}^2$. Then it follows that for a positive volume form Ω_V on V the restriction of $\iota_{\partial_y} \iota_{\partial_x} \Omega_V$ is a positive volume form on the binding. Conversely, a volume form Ω_K on K is positive if and only if $dx \wedge dy \wedge \Omega_K = r dr \wedge d\varphi \wedge \Omega_K$ is a positive volume element on $TV|_K$.

Note that with these orientations the binding is oriented as the boundary of the pages.

Definition A.4 [19] Let (V, ξ) be a closed contact manifold. We say that ξ is supported by an open book decomposition (K, ϑ) of V if ξ admits a contact form α such that

- (i) the binding K is a contact submanifold with positive contact form $\alpha_K := \alpha|_{TK}$;
- (ii) the restriction of $d\alpha$ to the interior $\text{Int } F_\varphi$ of every page is a positive symplectic form.

In both cases, “positive” refers to the orientation induced on K and the pages by the open book decomposition. We call a contact form α as above, adapted to the open book, and we call (ξ, K, ϑ) a contact open book decomposition.

The following remark is in a way an extension of Remark A.3 to the contact category.

Remark A.5 Let V be a closed manifold, let α be a contact form with Reeb field R_α , and let $h = h_x + ih_y: V \rightarrow \mathbb{C}$ be a smooth function.

To show that $K_h := h^{-1}(0)$ and $\vartheta_h = h/|h|$ define an open book (K_h, ϑ_h) and that α is adapted to it, it suffices to verify that

- (i) K_h is nonempty and $\alpha \wedge (d\alpha)^{n-1} \wedge dh_x \wedge dh_y$ is positive along K_h ;
- (ii) $\frac{1}{2}i(h d\bar{h}(R_\alpha) - \bar{h} dh(R_\alpha)) = h_x dh_y(R_\alpha) - h_y dh_x(R_\alpha) > 0$ on all of $V \setminus K_h$.

(Here, as above, $\bar{h} = h_x - ih_y$ denotes the complex conjugate.)

Proof By condition (i), 0 is a regular value of h . Recall that

$$d\vartheta_h = \frac{i}{2|h|^2}(h d\bar{h} - \bar{h} dh) = \frac{1}{|h|^2}(h_x dh_y - h_y dh_x);$$

thus condition (ii) simply implies that $d\vartheta_h(R_\alpha) > 0$, and it follows that ϑ_h is a submersion. Thus h defines an open book by Remark A.3. Let us now show that α is adapted to the open book (K_h, ϑ_h) .

Since $\alpha \wedge (d\alpha)^n$ is a volume form, $\iota_{R_\alpha} \alpha \wedge (d\alpha)^n = (d\alpha)^n$ cannot be degenerate on any hyperplane transverse to R_α . In particular, because $d\vartheta_h(R_\alpha) > 0$, the Reeb field is positively transverse to the interior of the pages, and $d\alpha$ restricts to a positive symplectic form on them.

Because TK_h lies in the kernel of the 2-form $d(|h|^2 d\vartheta_h) = 2 dh_x \wedge dh_y$, condition (i) implies then that α restricts to a contact form on K_h . Furthermore, $d(|h|^2 d\vartheta_h)$ defines the positive coorientation for the binding, and thus $\alpha|_{TK_h}$ is by (i) positive. \square

We now describe two elementary examples of contact open book decompositions of the standard sphere that we will study in detail in the next two sections of this appendix using the language of [20].

Example A.6 We assume that the unit sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ is equipped with the standard contact structure ξ_0 , which is the hyperplane field of complex tangencies. Equivalently, this is given as the kernel of the 1-form

$$\alpha_0 = \frac{1}{2} \sum_{j=1}^n (x_j dy_j - y_j dx_j),$$

where we write the coordinates of \mathbb{C}^n as $\mathbf{z} = (z_1, \dots, z_n) = (x_1 + iy_1, \dots, x_n + iy_n)$.

Every holomorphic function $g: \mathbb{C}^n \rightarrow \mathbb{C}$ with an isolated singularity at the origin induces a contact open book decomposition of the standard sphere (after possibly shrinking the radius of the sphere; see [28] for the topological and [9] for the contact case). For the concrete applications we have in mind here, we will not appeal to this general result, and instead study the following two very explicit situations.

(a) Let $g_1(z_1, \dots, z_n) = z_1$. Then the binding is the submanifold $K = \{z_1 = 0\}$ and the fibration is $\vartheta: (z_1, \dots, z_n) \mapsto z_1/|z_1|$. The binding K is just the standard contact sphere and the Reeb vector field $R_0 = (iz_1, \dots, iz_n)$ for α_0 generates the Hopf fibration which is transverse to the (the interior) of every page $F_\varphi = \{\arg z_1 = \varphi\}$, so that $d\alpha_0$ will restrict to a symplectic form defining the correct orientation on every page. Furthermore, since the binding K is connected, it follows from Stokes' theorem that α induces the boundary orientation of the pages on K . It follows that (K, ϑ) is a contact open book decomposition.

(b) Let us now study the case of $g_2(z_1, \dots, z_n) = z_1^2 + \dots + z_n^2$. The complex hypersurface $V_{g_2} = g_2^{-1}(0)$ is everywhere smooth except at the origin and since it is invariant under linear scaling $\lambda \cdot (z_1, \dots, z_n) = (\lambda \cdot z_1, \dots, \lambda \cdot z_n)$ with $\lambda \in \mathbb{R}_+$, it follows that V_{g_2} is transverse to \mathbb{S}^{2n-1} , so that the binding $K = g_2^{-1}(0) \cap \mathbb{S}^{2n-1}$ is a smooth codimension 2 submanifold of the standard sphere. To check the contact condition note that the restriction of a plurisubharmonic function to a complex submanifold preserves this property. In our case, the restriction of $\mathbf{z} \mapsto \|\mathbf{z}\|^2$ to V_{g_2} is such a function, and K is one of its regular level sets, so that K is a contact submanifold.

For the pages, notice that the Reeb field R_0 increases the argument of g_2 everywhere g_2 does not vanish. This implies that R_0 is positively transverse to the pages, and in particular $d\alpha_0$ defines a symplectic structure on them.

It is well known from the “classical” treatment that the page in the first example is a ball with the standard symplectic structure and that its monodromy is the identity. In the second example, the page is the cotangent bundle of the sphere and the monodromy is a generalized Dehn twist. In Examples A.10 and A.17 we will work out the abstract open books in these two cases using the formalism of ideal Liouville domains.

A.2 Ideal Liouville domains

As we already mentioned above, the pages of an abstract open book will be described by an ideal Liouville domain.

Definition A.7 Let F be a compact manifold with boundary $K := \partial F$, and let ω be an exact symplectic form on the interior $\text{Int } F = F \setminus K$.

The pair (F, ω) is an *ideal Liouville domain* if there exists a primitive $\lambda \in \Omega^1(\text{Int } F)$ for ω such that for any smooth function $u: F \rightarrow [0, \infty)$ with regular level set $K = u^{-1}(0)$, the 1-form $u\lambda$ extends to a smooth 1-form λ_u on all of F whose restriction to K is a (positive) contact form. Every such primitive λ is called a *Liouville form* of (F, ω) .

The intuitive picture of an ideal Liouville domain is that of a classical Liouville domain that has been completed by attaching a cylindrical end and has then been compactified by fixing a certain asymptotic information at “infinity” that is captured in the boundary of the ideal Liouville domain. Below we give a formal description of this completion process.

For the many properties shared by these objects, we refer to [20]. In particular we point out that the contact structure induced on the boundary is, as observed by Courte [12, Proposition 2.4], already determined by (F, ω) itself and does not depend on the auxiliary Liouville form chosen [20, Proposition 2]. We denote the space of all diffeomorphisms of F that keep the boundary pointwise fixed and that preserve ω on the interior by $\text{Diff}_\partial(F, \omega)$.

Definition A.8 An *abstract Liouville open book* consists of an ideal Liouville domain (F, ω) and a diffeomorphism $\phi \in \text{Diff}_\partial(F, \omega)$.

We will now describe the *completion of classical Liouville domains* allowing us to do the transition from classical to ideal Liouville domains. Recall that a “classical” *Liouville domain* (F, λ_c) is a compact manifold with boundary K such that

- $\omega_c := d\lambda_c$ is a symplectic form;

- the Liouville vector field X_λ defined by the equation $\iota_{X_\lambda} \omega_c = \lambda_c$ points along K transversely out of the domain F .

In particular it follows that λ_c restricts on K (oriented as the boundary of (F, ω_c)) to a positive contact form.

Following [20, Example 9], we will convert (F, λ_c) into an ideal Liouville domain (F, ω) , keeping the smooth manifold F unchanged, but modifying $d\lambda_c$ to a new symplectic form ω on $\text{Int } F$ (that will be related to but different from ω_c !).

Lemma A.9 *The space of all functions $u: F \rightarrow [0, \infty)$ satisfying*

- $u^{-1}(0) = K$ is a regular level set;
- $X_\lambda(\ln u) < 1$ on $\text{Int } F$ (or equivalently $du(X_\lambda) < u$ on all of F)

is convex and nonempty.

Proof Convexity is a basic calculation; for the existence use a collar neighborhood $(-\varepsilon, 0] \times K$ with coordinates (t, x) defined by the flow of X_λ , and let $u(t, x)$ be a function that agrees with $-t$ close to $t = 0$, and flattens out to be constant on a slightly larger neighborhood of K . □

With a function u as in the previous lemma, we claim that $\omega := d(\lambda_c/u)$ is an ideal Liouville structure on F . Firstly, ω is symplectic on $\text{Int } F$: At points where $\lambda_c = 0$ we have $\omega = (1/u)\omega_c$; to check the nondegeneracy of ω at the remaining points note first that λ_c vanishes if and only if X_λ does, then compute

$$\omega^n = \left(\frac{1}{u} d\lambda_c - \frac{1}{u^2} du \wedge \lambda_c \right)^n = \frac{1}{u^n} (\omega_c^n - n d(\ln u) \wedge \lambda_c \wedge \omega_c^{n-1}).$$

Plugging X_λ into ω^n and using that $\iota_{X_\lambda} \omega_c = \lambda_c$ and $\iota_{X_\lambda} \lambda_c = 0$, we see that

$$\iota_{X_\lambda} \omega^n = \frac{n}{u^n} (1 - X_\lambda(\ln u)) \lambda_c \wedge \omega_c^{n-1} = \frac{1}{u^n} (1 - X_\lambda(\ln u)) \iota_{X_\lambda} \omega_c^n,$$

which is nondegenerate.

This implies now that (F, ω) is an ideal Liouville domain, because $\lambda := (1/u)\lambda_c$ is a primitive of ω for which $u\lambda$ clearly restricts to a contact form on K . The contact structure on the boundary of (F, ω) is equal to the initial contact structure.

As explained in [20, Example 9], one can equivalently obtain (F, ω) by attaching an infinite cylindrical end to the boundary and then compactifying the result. Also note

that by the convexity of the admissible choices for u the completion is unique up to isotopy.

The completion of a classical Liouville domain to an ideal Liouville domain is particularly straightforward when applied to a Weinstein domain, or equivalently for our purposes, a Weinstein manifold (W, ω, X, f) of finite type. In this case, choose a regular value C such that $F := \{f \leq C\}$ is a nonempty domain with smooth boundary. In particular, (F, λ) with the Liouville form $\lambda := \iota_X \omega$ is a classical Liouville domain, the function $u := C - f$ is nonnegative on F and has the boundary $K = \partial F$ as regular level set, and since f is a Lyapunov function for X , we check that $X(\ln u) = -(1/u)X(f) \leq 0$ is always smaller than 1. The interior of the ideal Liouville domain F is symplectomorphic to cutting off the part $\{f > C\}$ from W and replacing it by the cylindrical end of the level set $\{f = C\}$. By choosing C sufficiently large, this then recovers the Weinstein manifold of finite type W .

We will now illustrate the notion of an ideal Liouville domain with two basic examples obtained via this completion procedure. As we will see in the next section, these two examples correspond to the pages of the open books from Example A.6.

Example A.10 (a) Let $\overline{\mathbb{D}}^{2n}$ be the closed unit disk in $(\mathbb{C}^n, \omega_0 = d\lambda_0)$ with coordinates $z = x + iy = (x_1 + iy_1, \dots, x_n + iy_n)$ and let $\lambda_0 = \frac{1}{2} \sum_{j=1}^n (x_j dy_j - y_j dx_j)$ be the standard Liouville form. We could of course use the fact that \mathbb{C}^n is a Weinstein manifold with Liouville vector field $X_{\lambda_0} = \frac{1}{2} \sum_{j=1}^n (x_j (\partial/\partial x_j) + y_j (\partial/\partial y_j))$ and Lyapunov function $f(z) = \|z\|^2$ to apply the remark we just made.

Instead we will perform the completion procedure using the function

$$u: \overline{\mathbb{D}}^{2n} \rightarrow [0, \infty), \quad z \mapsto 1 - \|z\|^4.$$

The reason we make this unexpected choice for u is to recover the page of the abstract open book in Example A.17. Recall that up to symplectomorphism, the ideal Liouville domain does not depend on the particular choice of the function satisfying the properties of Lemma A.9.

Note first that the boundary of the closed disk is a regular level set of u since u factors as $u(z) = (1 - \|z\|)(1 + \|z\|)(1 + \|z\|^2)$. Furthermore, $X_{\lambda_0}(\|z\|^4) \geq 0$, so that $X_{\lambda_0}(\ln(1 - \|z\|^4)) \leq 0$.

Setting $\lambda = \lambda_0/(1 - \|z\|^4)$, we obtain that $(\overline{\mathbb{D}}^{2n}, d\lambda)$ is the desired completion of $(\overline{\mathbb{D}}^{2n}, \lambda_0)$ to an ideal Liouville domain.

The interior of the ideal Liouville domain $(\mathbb{D}^{2n}, d\lambda)$ is symplectomorphic to

$$(\mathbb{C}^n, \omega_0 = d\lambda_0);$$

simply use the diffeomorphism

$$\mathbb{D}^{2n} \rightarrow \mathbb{C}^n, \quad z \mapsto \frac{1}{\sqrt{1 - \|z\|^4}} z,$$

to pull back λ_0 .

(b) Let us now see how to associate an ideal Liouville domain to a unit cotangent bundle. For this, let (L, g) be a closed Riemannian manifold, and let λ_{can} be the canonical 1-form on T^*L . It is well known that $(T^*L, d\lambda_{\text{can}}, X_\lambda, f)$ with $X_\lambda = \mathbf{p} \cdot \partial_{\mathbf{p}}$ and $f(\mathbf{q}, \mathbf{p}) = \|\mathbf{p}\|^2$ is a Weinstein manifold.

As described above we can apply the completion using the function $u = 1 - f$, so that the (closed) unit disk bundle $\overline{\mathbb{D}}(T^*L) = \{(\mathbf{q}, \mathbf{p}) \in T^*L \mid \|\mathbf{p}\| \leq 1\}$ is an ideal Liouville domain with the symplectic structure given by $d\lambda$ where we have set $\lambda = \lambda_{\text{can}}/(1 - \|\mathbf{p}\|^2)$.

In this case, we can identify the interior of $(\overline{\mathbb{D}}(T^*L), d\lambda)$ with $(T^*L, d\lambda_{\text{can}})$ using the map $\mathbb{D}(T^*L) \rightarrow T^*L$ given by $(\mathbf{q}, \mathbf{p}) \mapsto (\mathbf{q}, \mathbf{p}/(1 - \|\mathbf{p}\|^2))$.

A.3 From contact open book decompositions to abstract Liouville open books and back

The link between abstract Liouville open books and contact open books is established by the following intermediate object.

Definition A.11 A *Liouville open book* (K, ϑ, ω_t) on a closed manifold V is an open book (K, ϑ) , each of whose pages F_t is equipped with a (positive) ideal Liouville structure $\omega_t \in \Omega^2(\text{Int } F_t)$. To guarantee a certain compatibility between the ω_t , we require that there be

- a global smooth 1-form β on V called a *binding form* and
- a function $f: V \rightarrow \mathbb{C}$ defining the open book (as in Remark A.3)

such that $\beta/|f|$ restricts on the interior of each page $\text{Int } F_t = F_t \setminus K$ to a Liouville form of ω_t .

We say that the pair (β, f) is a *representation* of the Liouville open book.

Note in particular that a binding form induces a positive contact form on the binding, since β restricts on the boundary of each page to a contact form.

We often make use of the following technical lemma.

Lemma A.12 *Let (K, ϑ, ω_t) be a Liouville open book on a manifold V . Choose a representation (β, f) such that $\lambda := \beta/|f|$ restricts on the interior of each page $\text{Int } F_t = F_t \setminus K$ to a Liouville form of $\omega_t|_{\text{Int } F_t}$.*

Then it follows that

$$\Omega_V := |f|^{n+2} d\vartheta \wedge (d\lambda)^n$$

extends to a well-defined volume form on all of V .

Furthermore, writing $f = \rho e^{i\vartheta}$, we have

$$(3) \quad \Omega_V = n\rho d\rho \wedge d\vartheta \wedge \beta \wedge (d\beta)^{n-1} + \rho^2 d\vartheta \wedge (d\beta)^n.$$

Proof It is clear that Ω_V is a volume form on $V \setminus K$, so it only remains to analyze its behavior along the binding. Writing f in polar coordinates, and replacing λ by β/ρ , we obtain (3), whose right-hand side is defined on all of V . Its second term vanishes along the binding while the first one is positive, since the binding itself is a positive contact submanifold of (V, ξ) . This proves that Ω_V is a volume form. □

Here is a tabular summary of the three notions introduced so far and their relationships:

abstract Liouville open book		Liouville open book		contact open book
ideal Liouville domain (F, ω)	\Leftrightarrow	open book (K, ϑ) on V	\Leftrightarrow	open book (K, ϑ) on V
$\phi \in \text{Diff}_\partial(F, \omega)$	\Leftrightarrow	ideal Liouville structure ω_t on each page F_t	\Leftrightarrow	contact structure ξ on V carried by (K, ϑ)

The connection between contact open books and Liouville open books is the following: a contact structure is said to be *symplectically supported* by a Liouville open book if it admits a contact form that is a binding form.

Proposition A.13 [20, Proposition 18] *If (K, ϑ) is a contact open book on V supporting the contact structure ξ , and $f: V \rightarrow \mathbb{C}$ is any defining function, then there exists a contact form α such that $d(\alpha/|f|)$ restricts to each page as an ideal Liouville structure. Furthermore, for fixed f , the set of such forms α is a nonempty convex cone.* □

In other words, for each defining function f , there is a contact form α such that (α, f) is the representation of a Liouville open book on V . (Notice also that the space of defining functions for a given open book is also convex and nonempty, so the space of pairs is contractible.)

We also have a converse:

Lemma A.14 [20, page 19] *If the contact structure ξ on V is symplectically supported by a Liouville open book, then ξ is supported (in the sense of Definition A.4) by the underlying smooth open book.*

These two facts justify our earlier definition that a pair (α, f) is a representation of a contact open book decomposition when α is a contact form and (α, f) is a representation of a Liouville open book with defining function f .

Additionally, from [20, Proposition 21], the symplectically supported contact structures form a nonempty and weakly contractible subset in the space of all hyperplane fields. To describe now the connection between Liouville open books and abstract Liouville open books, we need the following notion.

Definition A.15 Let (K, ϑ, ω_t) be a Liouville open book on V . A smooth vector field X on V is called a *spinning vector field* if it satisfies the following properties:

- X vanishes along the binding K , and $d\vartheta(X) = 2\pi$ on $V \setminus K$.
- The flow of X preserves the ideal Liouville structure on every page.

The *monodromy* of the Liouville open book is a diffeomorphism $\phi: F_0 \rightarrow F_0$ obtained by restricting the time-1 flow of a spinning vector field X to the page F_0 . Clearly, ϕ is an element of $\text{Diff}_\partial(F_0, \omega_0)$.

Lemma A.16 *Let (K, ϑ, ω_t) be a Liouville open book with a representation (β, f) such that $\lambda = \beta/|f|$ restricts to a Liouville form on the interior of each page.*

There exists a unique vector field Y satisfying the two equations

$$d\vartheta(Y) = 2\pi \quad \text{and} \quad \iota_Y d\lambda = 0.$$

This field is a spinning vector field of the Liouville open book.

Proof It is clear that Y is defined on $V \setminus K$ and that its flow preserves the Liouville structure on the pages, but the properties of Y along the binding are less obvious. Plug Y into the volume form Ω_V from Lemma A.12:

$$\iota_Y \Omega_V = 2\pi |f|^{n+2} (d\lambda)^n = 2\pi |f|^2 (d\beta)^n - \pi n d(|f|^2) \wedge \beta \wedge (d\beta)^{n-1}.$$

Since the right-hand side is defined on all of V , and since Ω_V is a volume form, we have found a defining equation for Y that shows that Y is everywhere smooth and vanishes along K . □

One can easily obtain an abstract open book from a proper Liouville open book by keeping only one of its pages and choosing the monodromy with respect to any spinning vector field. All pages are isomorphic, and spinning vector fields form a convex subset, so that all choices will lead to homotopic abstract Liouville open books.

Starting from an abstract Liouville open book (F, ω) with diffeomorphism $\phi \in \text{Diff}_\partial(F, \omega)$, Giroux constructs first a mapping torus and then blows down its boundary to obtain the binding, producing this way a Liouville open book, and thus the desired bijection.

Example A.17 Let us now come back to the contact open book decompositions introduced in Example A.6 and illustrate how to recover their abstract Liouville open books by applying the formalism explained above.

(a) Recall that the function $g_1(z_1, \dots, z_n) = z_1$ determines a contact open book (K, ϑ) on the standard sphere. We will see that the page of the corresponding abstract open book is the ideal Liouville domain given by Example A.10(a) and that the monodromy is the identity. This is then a special case of the more general Example A.19. We present this example first because of its concreteness.

Every page $F_t = \{\arg z_1 = t\} \cup K$ is diffeomorphic to the closed unit disk $\overline{\mathbb{D}}^{2n-2} = \{(q_1 + ip_1, \dots, q_{n-1} + ip_{n-1}) \in \mathbb{C}^{n-1} \mid \|\mathbf{q} + i\mathbf{p}\| \leq 1\}$, which can be embedded into \mathbb{S}^{2n-1} using the inverse of the stereographic projection²

$$\iota_t: (\mathbf{q} + i\mathbf{p}) \mapsto \frac{1}{1 + \|\mathbf{q} + i\mathbf{p}\|^2} ((1 - \|\mathbf{q} + i\mathbf{p}\|^2)e^{it}; 2(\mathbf{q} + i\mathbf{p})).$$

The 1-form $\beta := \alpha_0 / |g_1|$ is the binding form for the Liouville open book $(K, \vartheta, d\beta|_{F_t})$. That this really defines a Liouville structure on the pages can be verified by pulling back β with ι_t to \mathbb{D}^{2n-2} . We obtain

$$|\iota_t^* \alpha_0| = \frac{1 - \|\mathbf{q} + i\mathbf{p}\|^2}{1 + \|\mathbf{q} + i\mathbf{p}\|^2} \quad \text{and} \quad \iota_t^* \alpha_0 = \frac{2}{(1 + \|\mathbf{q} + i\mathbf{p}\|^2)^2} \lambda_0,$$

so that

$$\iota_t^* \beta = \frac{\lambda_0}{1 - \|\mathbf{q} + i\mathbf{p}\|^4}.$$

This is precisely the Liouville form on the unit disk given in Example A.10(a). It follows that the page of the abstract open book is $(\overline{\mathbb{D}}^{2n-2}, \omega)$, just as we wanted to show.

²A map of the type $(x, y) \mapsto (x, y, \sqrt{1 - (x^2 + y^2)})$, the most “obvious” candidate for such an embedding, fails to be smooth along the boundary and is thus not suitable for our purposes!

Consider now the vector field

$$Y = 2\pi(x_1 \partial_{y_1} - y_1 \partial_{x_1})$$

on \mathbb{S}^{2n-1} . Clearly, Y vanishes along $K = \{x_1 = y_1 = 0\}$, and it satisfies $d\vartheta(Y) = 2\pi$. Furthermore $\mathcal{L}_Y \beta = 0$, so that its flow preserves the Liouville structures induced by $d\beta$ on each page. We obtain that Y is a spinning vector field and since its time-1 flow is the identity on all of \mathbb{S}^{2n-1} , it follows in particular that the monodromy of the Liouville open book is trivial.

(b) The second open book decomposition (K, ϑ) of the sphere is determined by the function $g_2(z) = z_1^2 + \dots + z_n^2$.

Remember that in Example A.10(b) we described ideal Liouville domains on the closed unit disk cotangent bundles $\overline{\mathbb{D}}(T^*L)$ with Liouville form $\lambda_{\text{can}}/(1 - \|\mathbf{p}\|^2)$. For the special case $L = \mathbb{S}^{n-1}$ we can significantly simplify these manifolds by using the identification $\overline{\mathbb{D}}(T^*\mathbb{S}^{n-1}) := \{(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \|\mathbf{q}\| = 1, \mathbf{q} \perp \mathbf{p}, \|\mathbf{p}\| \leq 1\}$ with Liouville form $-(1/(1 - \|\mathbf{p}\|^2)) \sum_{j=1}^n p_j dq_j$.

We will now show that this ideal Liouville domain is the page of the abstract open book corresponding to (K, ϑ) . We embed the unit disk bundle into \mathbb{S}^{2n-1} via

$$\iota_t: (\mathbf{q}, \mathbf{p}) \mapsto \frac{(\mathbf{q} + i\mathbf{p})e^{it/2}}{\sqrt{1 + \|\mathbf{p}\|^2}}.$$

The image of each such map is one of the pages.

Pulling back α_0 , we obtain $\iota_t^* \alpha_0 = (1/(2(1 + \|\mathbf{p}\|^2))) \sum_{j=1}^{n-1} (q_j dp_j - p_j dq_j)$. This can be simplified using that the differential of $\langle \mathbf{q}, \mathbf{p} \rangle = 0$ is $\sum_{j=1}^{n-1} (q_j dp_j + p_j dq_j) = 0$, so that $\iota_t^* \alpha_0 = -(1/(1 + \|\mathbf{p}\|^2)) \sum_{j=1}^{n-1} p_j dq_j$.

We claim that $\beta := \alpha_0/|g_2|$ is the binding form for a Liouville open book. Note that $|g_2 \circ \iota_t| = (1 - \|\mathbf{p}\|^2)/(1 + \|\mathbf{p}\|^2)$, so that $\iota_t^* \beta = -(1/(1 - \|\mathbf{p}\|^2)) \sum_{j=1}^{n-1} p_j dq_j$, which is the Liouville form given on the domain above. This shows that the page of the abstract open book is indeed $(\overline{\mathbb{D}}(T^*\mathbb{S}^{n-1}), \omega)$.

It remains to show that the monodromy is a generalized Dehn twist as we had already claimed in Example A.6. Recall first that a *Dehn twist* on $T^*\mathbb{S}^{n-1}$ can be written as follows: Identify the cotangent bundle of \mathbb{S}^{n-1} with the submanifold of $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{C}^n$ consisting of pairs of points (\mathbf{q}, \mathbf{p}) such that $\|\mathbf{q}\| = 1$ and $\mathbf{p} \perp \mathbf{q}$. The canonical 1-form λ_{can} on $T^*\mathbb{S}^{n-1}$ is simply the restriction of the 1-form $-\sum_{j=1}^n p_j dq_j$

to $T^*\mathbb{S}^{n-1}$. Then we can write down the following type of symplectomorphisms with compact support:

$$\Phi: T^*\mathbb{S}^{n-1} \rightarrow T^*\mathbb{S}^{n-1}, \quad \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{q} \cos \rho + (\mathbf{p}/\|\mathbf{p}\|) \sin \rho \\ -\|\mathbf{p}\|\mathbf{q} \sin \rho + \mathbf{p} \cos \rho \end{pmatrix},$$

where $\rho(\mathbf{q}, \mathbf{p}) := \rho(\|\mathbf{p}\|)$ is any smooth function that is 0 for very large values of $\|\mathbf{p}\|$ and is equal to $-\pi$ on a neighborhood of 0. Dividing by $\|\mathbf{p}\|$ in the definition of Φ is not problematic, because $\sin \rho$ vanishes close to the zero section of $T^*\mathbb{S}^{n-1}$.

A direct verification shows that Φ preserves the length $\|\mathbf{p}\|$ and that it has compact support, and pulling back the canonical 1-form using that $d\|\mathbf{q}\|^2 = d1 = 0$ and that $\sum_j p_j dq_j = -\sum_j q_j dp_j$, we obtain

$$\Phi^*\lambda_{\text{can}} = \lambda_{\text{can}} - \|\mathbf{p}\| d\rho,$$

which implies that Φ is indeed a symplectomorphism, because ρ only depends on $\|\mathbf{p}\|$.

Next, let us pull back the generalized Dehn twist to the ideal Liouville domain $(\overline{\mathbb{D}}(T^*\mathbb{S}^{n-1}), \omega)$. As in Example A.10(b), we stretch out the interior of the unit disk bundle to cover the full cotangent bundle using the map $(\mathbf{q}, \mathbf{p}) \mapsto (\mathbf{q}, \mathbf{p}/(1 - \|\mathbf{p}\|^2))$. The inverse of this map is

$$(\mathbf{q}, \mathbf{p}) \mapsto \left(\mathbf{q}, \frac{\sqrt{4\|\mathbf{p}\|^2 + 1} - 1}{2\|\mathbf{p}\|^2} \mathbf{p} \right).$$

We pull back the Dehn twist to the ideal Liouville domain and find that it still is of the same form as on $T^*\mathbb{S}^{n-1}$, only that the function $\rho(\|\mathbf{p}\|)$ needs to be replaced by $\rho(\|\mathbf{p}\|/(1 - \|\mathbf{p}\|^2))$.

Since the monodromy map of an abstract Liouville open book needs to be the identity only on the boundary of the domain, we weaken our definition by requiring that ρ vanish for $\|\mathbf{p}\| = 1$, but not necessarily also on a neighborhood of 1. Also instead of imposing that $\rho = -\pi$ on a small neighborhood of the zero section, it is sufficient for us that this condition be met *on* the zero section itself (being careful to preserve the smoothness of Φ). We thank the referee for providing us with the following concise definition:

Definition A.18 Choose any smooth function $g: [0, 1] \rightarrow \mathbb{R}$ such that $g(1) = \pi$. A *Dehn twist on the ideal Liouville domain* $(\overline{\mathbb{D}}(T^*\mathbb{S}^{n-1}), \omega)$ is a map of the form

$$\Phi: \overline{\mathbb{D}}(T^*\mathbb{S}^{n-1}) \rightarrow \overline{\mathbb{D}}(T^*\mathbb{S}^{n-1}), \quad \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{q} \cos \rho + (\mathbf{p}/\|\mathbf{p}\|) \sin \rho \\ -\|\mathbf{p}\|\mathbf{q} \sin \rho + \mathbf{p} \cos \rho \end{pmatrix},$$

where $\rho(\mathbf{q}, \mathbf{p}) := \rho(\|\mathbf{p}\|)$ can be written as $\rho(r) = rg(r^2) - \pi$.

To show that this more general definition is suitable, we must verify that Φ is smooth along the zero section of $T^*\mathbb{S}^{n-1}$. For this, observe that if $f: \mathbb{R} \rightarrow \mathbb{R}$ is a smooth even function, the composition $\mathbf{p} \mapsto f(\|\mathbf{p}\|)$ will also be smooth.

Expand the trigonometric functions to see that

$$\sin \rho(r) = -\sin(rg(r^2)) \quad \text{and} \quad \cos \rho(r) = -\cos(rg(r^2)).$$

The second function is clearly even. For the first one, notice that $-\sin(rg(r^2))$ is odd and vanishes at $r = 0$. We may therefore write it as $rh(r)$ for a smooth h that necessarily needs to be even. This then implies that both $(1/r) \sin \rho(r) = h(r)$ and $r \sin \rho(r) = r^2h(r)$ are well defined and even, and thus Φ is everywhere smooth.

From this, every Dehn twist lies in $\text{Diff}_\partial(\overline{\mathbb{D}}(T^*\mathbb{S}^{n-1}), \omega)$, and the space of Dehn twists is contractible.

Recall that the monodromy of the open book is obtained as the restriction to a page of the time-1 flow of a spinning vector field. A long but straightforward computation shows that the vector field Y specified by Lemma A.16 is

$$Y = \pi \operatorname{Re}(g_2) \sum_{j=1}^n \left(y_j \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial y_j} \right) + \pi \operatorname{Im}(g_2) \sum_{j=1}^n \left(y_j \frac{\partial}{\partial y_j} - x_j \frac{\partial}{\partial x_j} \right),$$

or equivalently using the Wirtinger formalism, we can write Y as

$$Y = \pi i g_2 \cdot \sum_{j=1}^n \bar{z}_j \frac{\partial}{\partial z_j} - \pi i \bar{g}_2 \cdot \sum_{j=1}^n z_j \frac{\partial}{\partial \bar{z}_j},$$

where we have used that $\partial/\partial z_j := \frac{1}{2}(\partial/\partial x_j - i(\partial/\partial y_j))$. The arguments explained in [27, Exercise 6.20], allow us to find the flow Φ_t^Y of this vector field: Combining $\vartheta = g_2/|g_2|$ with the normalization of Y we obtain the equation

$$\frac{g_2(\Phi_t^Y(z))}{|g_2(\Phi_t^Y(z))|} = e^{2\pi i t}.$$

We also see easily that the flow of Y preserves $|g_2|$, because $\mathcal{L}_Y |g_2|^2 = 0$.

Let $\mathbf{z}(t)$ be the trajectory of Y starting at a point $\mathbf{z}(0)$. To simplify the notation write g_0 instead of $|g_2(\mathbf{z}(0))|$. Then $\mathbf{z}(t)$ is the solution to the ordinary differential equation

$$\dot{\mathbf{z}}(t) = \pi i g_2(\mathbf{z}(t)) \cdot \bar{\mathbf{z}}(t) = \pi i e^{2\pi i t} g_0 \cdot \bar{\mathbf{z}}(t).$$

Defining $\mathbf{u}(t) := e^{-\pi i t} \mathbf{z}(t)$, we compute

$$\dot{\mathbf{u}}(t) = -\pi i \mathbf{u}(t) + e^{-\pi i t} \dot{\mathbf{z}}(t) = -\pi i \mathbf{u}(t) + \pi i g_0 \cdot \bar{\mathbf{u}}(t).$$

Splitting \mathbf{u} into real and imaginary parts as $\mathbf{u} = \mathbf{u}_x + i \mathbf{u}_y$, the previous equation can be written as $\dot{\mathbf{u}}_x(t) = \pi(1 + g_0)\mathbf{u}_y(t)$ and $\dot{\mathbf{u}}_y(t) = -\pi(1 - g_0)\mathbf{u}_x(t)$, which combine back to

$$\ddot{\mathbf{u}}(t) = -\pi^2(1 - g_0^2)\mathbf{u}(t).$$

The general solution of this equation is $\mathbf{u}(t) = A_+ e^{\pi i c t} + A_- e^{-\pi i c t}$ with $c := \sqrt{1 - g_0^2}$, so that

$$\mathbf{z}(t) = A_+ e^{\pi i(c+1)t} + A_- e^{-\pi i(c-1)t},$$

where $A_+, A_- \in \mathbb{C}^n$ are complex vectors. The coefficients are

$$A_{\pm} = \frac{1}{2} \left(1 \mp \sqrt{\frac{1 - g_0}{1 + g_0}} \right) \mathbf{x}(0) + \frac{i}{2} \left(1 \mp \sqrt{\frac{1 + g_0}{1 - g_0}} \right) \mathbf{y}(0),$$

so that we find

$$\mathbf{z}(1) = -\mathbf{z}(0) \cos \pi \sqrt{1 - g_0^2} + i \left(\sqrt{\frac{1 - g_0}{1 + g_0}} \mathbf{x}(0) + i \sqrt{\frac{1 + g_0}{1 - g_0}} \mathbf{y}(0) \right) \sin \pi \sqrt{1 - g_0^2}.$$

To recover the monodromy of the abstract open book, restrict the time-1 flow of Y to the 0–page of the contact open book, and pull back the diffeomorphism obtained this way to the abstract page via the embedding ι_0 . Recall first that $g_2(\iota_0(\mathbf{q} + i \mathbf{p})) = (1 - \|\mathbf{p}\|^2)/(1 + \|\mathbf{p}\|^2)$. Then we get

$$\begin{aligned} \iota_0^{-1} \circ \Phi_1^Y \circ \iota_0(\mathbf{q} + i \mathbf{p}) = & - \left(\mathbf{q} \cos \frac{2\pi \|\mathbf{p}\|}{1 + \|\mathbf{p}\|^2} + \frac{\mathbf{p}}{\|\mathbf{p}\|} \sin \frac{2\pi \|\mathbf{p}\|}{1 + \|\mathbf{p}\|^2} \right) \\ & - i \left(-\|\mathbf{p}\| \mathbf{q} \sin \frac{2\pi \|\mathbf{p}\|}{1 + \|\mathbf{p}\|^2} + \mathbf{p} \cos \frac{2\pi \|\mathbf{p}\|}{1 + \|\mathbf{p}\|^2} \right). \end{aligned}$$

This is just a Dehn twist as in Definition A.18 with the function

$$\rho(\mathbf{q}, \mathbf{p}) := \frac{2\pi \|\mathbf{p}\|}{1 + \|\mathbf{p}\|^2} - \pi = \|\mathbf{p}\| g(\|\mathbf{p}\|^2) - \pi,$$

where $g(r) = 2\pi/(1 + r)$. Clearly $g(1) = 2\pi/2 = \pi$, as desired.

Example A.19 We will now consider the case of a contact open book with trivial monodromy, and show that this manifold is symplectically filled by the stabilization of one of its pages. Using Theorem A.1, we will argue in the opposite direction, namely

we take the stabilization of an arbitrary Liouville domain, and show that it contains a contact-type hypersurface that is supported by an open book with trivial monodromy and whose pages are isomorphic to the initial Liouville domain. We conclude by using that any contact manifold with an open book decomposition with trivial monodromy can be obtained via this construction.

Let $(F, d\beta)$ be a (classical) Liouville domain with boundary $\partial F = K$ and with associated Liouville vector field X_L , so that $d\beta(X_L, \cdot) = \beta$. Choose a function $u: F \rightarrow [0, \infty)$ as in Lemma A.9; that is, 0 is a regular value, $u^{-1}(0) = K$, and $du(X_L) < u$. In the case that $(F, d\beta)$ is a Weinstein domain with a Lyapunov function f for X_L with $f^{-1}(C) = \partial F$, we can simply set $u := C - f$.

Let now $V \subset F \times \mathbb{C}$ be the hypersurface defined by

$$V = \{(p, z) \in F \times \mathbb{C} \mid u(p) - |z|^2 = 0\}.$$

From our hypothesis on u it follows that 0 is a regular value of $u(p) - |z|^2$, so that V is a closed embedded submanifold (touching the boundary of $F \times \mathbb{C}$ from the inside).

The manifold $F \times \mathbb{C}$ has an exact symplectic structure given by $d(\beta + \frac{1}{2}(x dy - y dx))$, where $z = x + iy$ denotes the coordinate on \mathbb{C} . The corresponding Liouville field is $X_L + \frac{1}{2}(x\partial_x + y\partial_y)$. This vector field is transverse to V , because

$$du(X_L) - |z|^2 = du(X_L) - u < 0,$$

by the properties in Lemma A.9, and it follows that the restriction of $\beta + \frac{1}{2}(x dy - y dx)$ to V defines a contact form α on V .

The open book we consider is obtained by taking the binding to be $K = K \times \{(0, 0)\} \subset F \times \mathbb{C}$ and the defining map $f: V \rightarrow \mathbb{C}$ by $f(p, z) = z$. It is not very difficult to check that f really defines a smooth open book decomposition on V , so that we will only show that the pair (α, f) is a representation of a *contact* open book.

The closure of any page is diffeomorphic to F , since it is then the set of points $\{(p, r e^{i\vartheta}) \in V \mid r \geq 0, p \in F\}$. This admits an “obvious” identification with F , given by

$$p \mapsto (p, \sqrt{u(p)} e^{i\vartheta}),$$

but unfortunately this map fails to be smooth up to the boundary (compare to the footnote of Example A.17). Instead, we will need to precompose it with a homeomorphism $\varphi: F \rightarrow F$ that is a diffeomorphism on the interior, and that maps the collar neighborhood $(-\varepsilon, 0] \times K \rightarrow (-\varepsilon, 0] \times K$ by $(s, x) \mapsto (g(s), x)$, where $g(s) = s^2$

for s near 0, $g(s) = s$ for s near $-\varepsilon$ and $g'(s) > 0$ for $s < 0$ (this then extends as the identity of F away from the collar neighborhood). We denote the composition $\sqrt{u \circ \varphi}$ by \tilde{u} , and observe that

$$\Phi: F \hookrightarrow V \times \mathbb{C}, \quad p \mapsto (p, \tilde{u}(p) e^{i\vartheta}),$$

is a smooth embedding of F into V . Equivalently, we could have treated this as a change of smooth structure at the boundary of F . (This is related to the discussion of smoothness immediately preceding Proposition 21 in [20].)

The resulting ideal Liouville form on the page is given by the restriction of $\alpha/|z|$, which pulls back to the 1-form

$$\lambda := \Phi^* \left(\frac{1}{|z|} \alpha \right) = \frac{1}{\tilde{u}} \beta$$

on $\text{Int } F$. We now claim this gives $(F, d\lambda)$ an ideal Liouville structure. This requires that $v\lambda$ extend to a contact form on K for any smooth function $v: F \rightarrow [0, \infty)$ for which $K = v^{-1}(0)$ is a regular level set. The function \tilde{u} introduced above is such a function, and clearly $\tilde{u}\lambda$ agrees with the 1-form β that is a contact form on K .

To verify that $d\lambda$ is indeed symplectic on $F \setminus \partial F$, write $r = |z| = \sqrt{u}$ and compute in the interior of F :

$$r^{n+2} \left[d \left(\frac{1}{r} \beta \right) \right]^n = r^2 (d\beta)^n - \frac{1}{2} n d(r^2) \wedge \beta \wedge (d\beta)^{n-1} = u (d\beta)^n - \frac{1}{2} n du \wedge \beta \wedge (d\beta)^{n-1}.$$

Now, contracting X_L with $0 = du \wedge (d\beta)^n$, we obtain the identity

$$0 = du(X_L) (d\beta)^n - n du \wedge \beta \wedge (d\beta)^{n-1}.$$

It now follows that

$$r^{n+2} \left[d \left(\frac{1}{r} \beta \right) \right]^n = \frac{1}{2} (2u - du(X_L)) (d\beta)^n$$

is a positive volume form on F , because u satisfies the assumptions in Lemma A.9, and we have that $u - du(X_L) > 0$.

This shows that $(F, d\lambda)$ is symplectomorphic to the completion of $(F, d\beta)$. Finally, to compute the monodromy, we notice that by our construction, β itself is a binding form on V , and thus $2\pi\partial_\vartheta$ is a spinning vector field. Its monodromy is indeed the identity map.

Any Liouville page F can be used as the starting point for this construction. If, additionally, F is a Weinstein domain, we obtain V as the boundary of an explicit subcritical filling.

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