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Four-genera of links and Heegaard Floer homology

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For links with vanishing pairwise linking numbers, the link components bound pairwise disjoint oriented surfaces in B^4. We use the h–function which is a link invariant from the Heegaard Floer homology to give lower bounds for the 4–genus of the link. For L–space links, the h–function is explicitly determined by the Alexander polynomials of the link and its sublinks. We show some L–space links where the lower bounds are sharp, and also describe all possible genera of disjoint oriented surfaces bounded by such links.

57M25, 57M27

1 Introduction

Let $\mathcal{L} = L_1 \cup L_2 \cup \cdots \cup L_n$ be an oriented n–component link in S^3 with all linking numbers 0. Recall that a link bounds disjointly embedded oriented surfaces in B^4 if and only if it has vanishing pairwise linking numbers. The 4–*genus* of $\mathcal{L}$ is defined as

$$g_4(\mathcal{L}) = \min\left\{ \sum_{i=1}^{n} g_i \;\middle|\; g_i = g(\Sigma_i),\ \Sigma_1 \sqcup \cdots \sqcup \Sigma_n \hookrightarrow B^4,\ \partial\Sigma_i = L_i \right\}.$$

If $\mathcal{L}$ is a knot, the 4–genus is also known as the slice genus. Powell [17], Murasugi [10] and Livingston [8] showed lower bounds for the 4–genera of links in terms of the Levine–Tristram signatures. Rasmussen [18; 19] defined the h–function (as an analogue of the Frøyshov invariant in Seiberg–Witten theory) for knots, and used it to obtain nontrivial lower bounds for the slice genus of a knot. We generalize Rasmussen's result and obtain lower bounds for the 4–genera of links with vanishing pairwise linking numbers. The h–function for links was introduced by Gorsky and Némethi [3]. It is closely related to d–invariants of large surgeries on links. For details, see Section 2.

We obtain lower bounds for the 4–genera of links in terms of the h–function. When the link has one component, we recover the lower bound for the slice genus given by Rasmussen. Here is our main result:

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Theorem 1.1 *Let $\mathcal{L} = L_1 \cup \cdots \cup L_n \subseteq S^3$ be an oriented link with vanishing pairwise linking numbers. Assume that the link components L_i bound pairwise disjoint, smoothly embedded oriented surfaces $\Sigma_i \subseteq B^4$ of genera g_i. Then, for any $\boldsymbol{v} = (v_1, \dots, v_n) \in \mathbb{Z}^n$,*

$$h(\boldsymbol{v}) \le \sum_{i=1}^{n} f_{g_i}(v_i).$$

where $h(\boldsymbol{v})$ is the h–function of $\mathcal{L}$ and $f_{g_i}\colon \mathbb{Z} \to \mathbb{Z}$ is defined by

$$f_{g_i}(v_i) = \begin{cases} \left\lceil \frac{1}{2}(g_i - |v_i|) \right\rceil & \text{if } |v_i| \le g_i, \\ 0 & \text{if } |v_i| > g_i. \end{cases}$$

Corollary 1.2 *For the link $\mathcal{L}$ in Theorem 1.1, if $\boldsymbol{v} \succeq \boldsymbol{g}$, then $h(\boldsymbol{v}) = 0$, where $\boldsymbol{g} = (g_1, \dots, g_n)$.*

The proof of Theorem 1.1 is inspired by Rasmussen's argument for knots [19]. We construct a nonpositive definite Spin^c–cobordism from large surgeries on the link to the connected sum of circle bundles over closed, oriented surfaces of genera g_i. Ozsváth and Szabó [11] established the behavior of the d–invariants of standard 3–manifolds under negative semidefinite Spin^c–cobordism. We apply this result, and obtain the inequality between the d–invariants of large surgeries on the link and d–invariants of the circle bundles. By using the h–function of the link to compute d–invariants of large surgeries, we prove the inequality.

Theorem 1.3 *If $\mathcal{L} = L_1 \cup \cdots \cup L_n \subset S^3$ is a (smoothly) slice L–space link, then $\mathcal{L}$ is an unlink.*

The idea of the proof goes as follows: The 4–genus for the slice link $\mathcal{L}$ is 0. By Theorem 1.1, the h–function is identically 0. We compute the dual Thurston polytope of $\mathcal{L}$ by using the properties of L–space links and prove that $\mathcal{L}$ is an unlink. For details, see Section 3.2.

As an application of the inequality in Theorem 1.1, we can compare the following two sets. Let

$$\mathfrak{G}(\mathcal{L}) = \{(g(\Sigma_1), g(\Sigma_2), \dots, g(\Sigma_n)) \mid \Sigma_1 \sqcup \cdots \sqcup \Sigma_n \hookrightarrow B^4,\ \partial \Sigma_i = L_i\},$$

where Σ_i are oriented surfaces, and

$$\mathfrak{G}_{\mathrm{HF}}(\mathcal{L}) = \{\boldsymbol{v} = (v_1, \dots, v_n) \mid h(\boldsymbol{v}) = 0 \text{ and } \boldsymbol{v} \succeq \boldsymbol{0}\}.$$

The 4–genus of the link $\mathcal{L}$ equals $\min_{\boldsymbol{g}\in\mathfrak{G}(\mathcal{L})}(g_1+\cdots+g_n)$. By Theorem 1.1, $\mathfrak{G}(\mathcal{L})\subseteq\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$.

If $\mathcal{L}$ is an L–space link (see Definition 2.16), the h–function is explicitly determined by the Alexander polynomials of the link and its sublinks; see Borodzik and Gorsky [1, Section 3.3]. We can describe the set $\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$ in terms of these Alexander polynomials explicitly (see Lemma 2.19). Moreover, let p and q be coprime positive integers, and $L_{(p,q)}$ denote the (p,q)–cable of L_1. Then the link $\mathcal{L}_{p,q}=L_{(p,q)}\cup L_2\cup\cdots\cup L_n$ is also an L–space link if q/p is sufficiently large [1, Proposition 2.8]. The set $\mathfrak{G}_{\mathrm{HF}}(\mathcal{L}_{p,q})$ can be obtained from the set $\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$ by applying the transformation $T\colon\mathbb{Z}^n_{\geq 0}\to\mathbb{Z}^n_{\geq 0}$; see Theorem 4.7. Inductively, let p_i and q_i be coprime positive integers for $1\leq i\leq n$, and let $L_{(p_i,q_i)}$ denote the (p_i,q_i)–cable of L_i. Then the link $\mathcal{L}_{\mathrm{cab}}=L_{(p_1,q_1)}\cup\cdots\cup L_{(p_n,q_n)}$ is also an L–space link if q_i/p_i is sufficiently large for each $1\leq i\leq n$. For example, let $\mathcal{L}$ denote the 2–bridge link $b(4k^2+4k,-2k-1)$, which is an L–space link; see Liu [7]. Then, for sufficiently large surgery coefficients, $\mathcal{L}_{\mathrm{cab}}$ is also an L–space link, and $\mathfrak{G}(\mathcal{L}_{\mathrm{cab}})=\mathfrak{G}_{\mathrm{HF}}(\mathcal{L}_{\mathrm{cab}})$ is as shown in Figure 1. For details, see Section 4.

Proposition 1.4 *If $\mathcal{L}\subset S^3$ is an L–space link such that $\mathfrak{G}(\mathcal{L})=\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$, then, for sufficiently large cables, $\mathcal{L}_{p,q}$ also satisfies that $\mathfrak{G}(\mathcal{L}_{p,q})=\mathfrak{G}_{\mathrm{HF}}(\mathcal{L}_{p,q})$.*

Remark 1.5 For an L–space link $\mathcal{L}$ as in Proposition 1.4, we can prove that $\mathfrak{G}(\mathcal{L}_{\mathrm{cab}})=\mathfrak{G}_{\mathrm{HF}}(\mathcal{L}_{\mathrm{cab}})$ by induction.

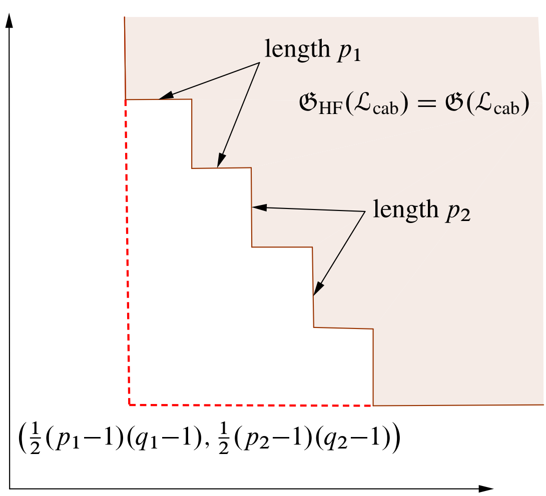

Figure 1: The set $\mathfrak{G}(\mathcal{L}_{\mathrm{cab}})$ for the cable link.

Organization of the paper In Sections 2.1 and 2.2, we review the definitions of the h–function for links in S^3 and the d–invariants for standard 3–manifolds. In Section 2.3, we review the definition of L–space links, and the explicit formula to compute the h–function in terms of the Alexander polynomials of the link and its sublinks. In Section 2.4, we review the Heegaard Floer link homologies of L–space links. In Section 3, we prove Theorems 1.1 and 1.3, and give some lower bounds for the 4–genera of links. In Section 4, we show some examples of L–space links, including the 2–bridge links $b(4k^2+4k,-2k-1)$, where k is some positive integer, and prove that $\mathfrak{G}(\mathcal{L})=\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$ in these examples. Then the 4–genus is determined by the Alexander polynomials. We also show the proof of Proposition 1.4.

Notation and conventions In this paper, all the links and surfaces are assumed to be oriented. We use $\mathcal{L}$ to denote a link in S^3, and $L_1,\dots,L_n$ to denote the link components. Then $\mathcal{L}_1$ and $\mathcal{L}_2$ denote different links in S^3, and L_1 and L_2 denote different components in the same link. We denote vectors in the n–dimension lattice $\mathbb{Z}^n$ by bold letters, and we let $\mathbb{Z}^n_{\geq 0}$ denote the vectors with entries nonnegative. For two vectors $\boldsymbol{u}=(u_1,u_2,\dots,u_n)$ and $\boldsymbol{v}=(v_1,\dots,v_n)$ in $\mathbb{Z}^n$, we write $\boldsymbol{u}\preceq\boldsymbol{v}$ if $u_i\leq v_i$ for each $1\leq i\leq n$, and $\boldsymbol{u}\prec\boldsymbol{v}$ if $\boldsymbol{u}\preceq\boldsymbol{v}$ and $\boldsymbol{u}\neq\boldsymbol{v}$. Let $\boldsymbol{e}_i$ denote the vector in $\mathbb{Z}^n$ where the i^{th} entry is 1 and other entries are 0. For a subset $B\subset\{1,\dots,n\}$, let $\boldsymbol{e}_B=\sum_{i\in B}\boldsymbol{e}_i$. Similarly, we use $L_B\subset\mathcal{L}$ to denote the sublink $\bigcup_{i\in B}L_i$. Assume $\{1,\dots,n\}\setminus B=\{i_1,\dots,i_k\}$. For $\boldsymbol{y}=(y_1,\dots,y_n)\in\mathbb{Z}^n$, let $\boldsymbol{y}\setminus y_B=(y_{i_1},\dots,y_{i_k})$. Let $\Delta_{\mathcal{L}}(t_1,\dots,t_n)$ denote the symmetrized Alexander polynomial of $\mathcal{L}$. Throughout this paper, we work over the field $\mathbb{F}=\mathbb{Z}/2\mathbb{Z}$.

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2 Background

2.1 The h–function

Ozsváth and Szabó associated chain complexes $\mathrm{CF}^-(M)$, $\widehat{\mathrm{CF}}(M)$, $\mathrm{CF}^\infty(M)$ and $\mathrm{CF}^+(M)$ to an admissible Heegaard diagram for a closed oriented connected 3–manifold M [12]. The homologies of these chain complexes are called Heegaard

Floer homologies $\mathrm{HF}^-(M), \widehat{\mathrm{HF}}(M), \mathrm{HF}^\infty(M)$ and $\mathrm{HF}^+(M)$, which are 3–manifold invariants. A nullhomologous link $\mathcal{L} = L_1 \cup \cdots \cup L_n$ in M defines a filtration on the link Floer complex $\mathrm{CF}^-(M)$ [9; 15]. For links in S^3, the filtration is indexed by an n–dimensional lattice $\mathbb{H}$ which is defined as follows:

Definition 2.1 For an oriented link $\mathcal{L} = L_1 \cup \cdots \cup L_n \subset S^3$, define $\mathbb{H}(\mathcal{L})$ to be an affine lattice over $\mathbb{Z}^n$,

$$\mathbb{H}(\mathcal{L}) = \bigoplus_{i=1}^{n} \mathbb{H}_i(\mathcal{L}), \quad \mathbb{H}_i(\mathcal{L}) = \mathbb{Z} + \tfrac{1}{2}\,\mathrm{lk}(L_i, \mathcal{L} \setminus L_i),$$

where $\mathrm{lk}(L_i, \mathcal{L} \setminus L_i)$ denotes the linking number of L_i and $\mathcal{L} \setminus L_i$.

Given $\boldsymbol{s} = (s_1, \dots, s_n) \in \mathbb{H}(\mathcal{L})$, the *generalized Heegaard Floer complex* $A^-(\mathcal{L}, \boldsymbol{s})$ is defined to be a subcomplex of $\mathrm{CF}^-(S^3)$ corresponding to the filtration indexed by $\boldsymbol{s}$ [9]. For $\boldsymbol{v} \preceq \boldsymbol{s}$, $A^-(\mathcal{L}, \boldsymbol{v}) \subseteq A^-(\mathcal{L}, \boldsymbol{s})$. The link homology HFL^- is defined as the homology of the associated graded complex,

$$\mathrm{HFL}^-(\mathcal{L}, \boldsymbol{s}) = H_*\Big(A^-(\mathcal{L}, \boldsymbol{s}) \Big/ \sum_{\boldsymbol{v} \prec \boldsymbol{s}} A^-(\mathcal{L}, \boldsymbol{v})\Big). \tag{2-1}$$

The complex $A^-(\mathcal{L}, \boldsymbol{s})$ is a finitely generated module over the polynomial ring $\mathbb{F}[U_1, \dots, U_n]$, where the action of U_i drops the homological grading by 2 and drops the i^{th} filtration A_i by 1 [15]. Hence, $U_i A^-(\mathcal{L}, \boldsymbol{s}) \subseteq A^-(\mathcal{L}, \boldsymbol{s} - \boldsymbol{e}_i)$. All the actions U_i are homotopic to each other on each $A^-(\mathcal{L}, \boldsymbol{s})$, and the homology of $A^-(\mathcal{L}, \boldsymbol{s})$ can be regarded as an $\mathbb{F}[U]$–module, where U acts as U_1 [3; 15].

By the large surgery theorem [9], the homology of $A^-(\mathcal{L}, \boldsymbol{s})$ is isomorphic to the Heegaard Floer homology of a large surgery on the link $\mathcal{L}$ equipped with some Spin^c–structure as an $\mathbb{F}[U]$–module [9]. Then the homology of $A^-(\mathcal{L}, \boldsymbol{s})$ consists of one copy of $\mathbb{F}[U]$ and some U–torsion.

Definition 2.2 [1, Definition 3.9] For an oriented link $\mathcal{L} \subseteq S^3$, we define the H–function $H_{\mathcal{L}}(\boldsymbol{s})$ by saying that $-2H_{\mathcal{L}}(\boldsymbol{s})$ is the maximal homological degree of the free part of $H_*(A^-(\mathcal{L}, \boldsymbol{s}))$, where $\boldsymbol{s} \in \mathbb{H}(\mathcal{L})$.

Lemma 2.3 [1, Proposition 3.10] *For an oriented link $\mathcal{L} \subseteq S^3$, the H–function $H_{\mathcal{L}}(\boldsymbol{s})$ takes nonnegative values, and $H_{\mathcal{L}}(\boldsymbol{s} - \boldsymbol{e}_i) = H_{\mathcal{L}}(\boldsymbol{s})$ or $H_{\mathcal{L}}(\boldsymbol{s} - \boldsymbol{e}_i) = H_{\mathcal{L}}(\boldsymbol{s}) + 1$, where $\boldsymbol{s} \in \mathbb{H}(\mathcal{L})$.*

For an n–component link $\mathcal{L}$ with vanishing pairwise linking numbers, $\mathbb{H}(\mathcal{L}) = \mathbb{Z}^n$. The h–function $h_{\mathcal{L}}(\boldsymbol{s})$ is defined as

$$h_{\mathcal{L}}(\boldsymbol{s}) = H_{\mathcal{L}}(\boldsymbol{s}) - H_O(\boldsymbol{s}),$$

where O denotes the unlink with n components and $\boldsymbol{s} \in \mathbb{Z}^n$. Recall that for split links $\mathcal{L}$, the H–function is $H(\mathcal{L}, \boldsymbol{s}) = H_{L_1}(s_1) + \cdots + H_{L_n}(s_n)$, where $H_{L_i}(s_i)$ is the H–function of the link component L_i [1, Proposition 3.11]. Then $H_O(\boldsymbol{s}) = H(s_1) + \cdots + H(s_n)$, where $H(s_i)$ denotes the H–function of the unknot. More precisely, $H_O(\boldsymbol{s}) = \sum_{i=1}^{n} \frac{1}{2}(|s_i| - s_i)$. Then $H_{\mathcal{L}}(\boldsymbol{s}) = h_{\mathcal{L}}(\boldsymbol{s})$ for all $\boldsymbol{s} \succeq \boldsymbol{0}$.

For the rest of this subsection, we use $\mathcal{L} = L_1 \cup \cdots \cup L_n \subset S^3$ to denote links with vanishing pairwise linking numbers. Consider the set

$$\mathfrak{G}_{\mathrm{HF}}(\mathcal{L}) = \{\boldsymbol{s} = (s_1, \ldots, s_n) \in \mathbb{Z}^n \mid h(\boldsymbol{s}) = 0,\ \boldsymbol{s} \succeq \boldsymbol{0}\}.$$

We obtain the following properties of the set $\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$:

Lemma 2.4 *If $\boldsymbol{x} \in \mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$ and $\boldsymbol{y} \succeq \boldsymbol{x}$, then $\boldsymbol{y} \in \mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$. Equivalently, if $\boldsymbol{x} \notin \mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$ and $\boldsymbol{y} \preceq \boldsymbol{x}$, then $\boldsymbol{y} \notin \mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$.*

Proof This is straightforward from Lemma 2.3. □

Lemma 2.5 *If $\boldsymbol{s} = (s_1, \ldots, s_n) \in \mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$, then $\boldsymbol{s} \setminus s_i \in \mathfrak{G}_{\mathrm{HF}}(\mathcal{L} \setminus L_i)$ for all $1 \le i \le n$. Moreover, if $\boldsymbol{s} \setminus s_i \in \mathfrak{G}_{\mathrm{HF}}(\mathcal{L} \setminus L_i)$, then, for s_i sufficiently large, $\boldsymbol{s} = (s_1, \ldots, s_n) \in \mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$.*

Proof For an oriented link $\mathcal{L}$, there exists a natural forgetful map $\pi_i \colon \mathbb{H}(\mathcal{L}) \to \mathbb{H}(\mathcal{L} \setminus L_i)$ [9]. If $\mathcal{L}$ has vanishing pairwise linking numbers, $\pi_i(\boldsymbol{s}) = \boldsymbol{s} \setminus s_i$, where $\boldsymbol{s} \in \mathbb{Z}^n$. Suppose that $\boldsymbol{s} \in \mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$. Then $h_{\mathcal{L}}(\boldsymbol{s}) = H_{\mathcal{L}}(\boldsymbol{s}) = 0$. By Lemma 2.3, $H_{\mathcal{L}}(\boldsymbol{s} + t\boldsymbol{e}_i) = 0$ for all i and $t > 0$. Recall that $H_{\mathcal{L}}(\boldsymbol{s} + t\boldsymbol{e}_i) = H_{\mathcal{L} \setminus L_i}(\boldsymbol{s} \setminus s_i)$ for sufficiently large t [1, Proposition 3.12]. Then $H_{\mathcal{L} \setminus L_i}(\boldsymbol{s} \setminus s_i) = 0$. Thus, $\boldsymbol{s} \setminus s_i \in \mathfrak{G}_{\mathrm{HF}}(\mathcal{L} \setminus L_i)$.

Conversely, if $\boldsymbol{s} \setminus s_i \in \mathfrak{G}_{\mathrm{HF}}(\mathcal{L} \setminus L_i)$ and s_i is sufficiently large, then $H_{\mathcal{L}}(\boldsymbol{s}) = H_{\mathcal{L} \setminus L_i}(\boldsymbol{s} \setminus s_i) = 0$, which implies that $\boldsymbol{s} \in \mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$. □

Definition 2.6 A lattice point $\boldsymbol{s} \in \mathbb{Z}^n$ is *maximal* if $\boldsymbol{s} \notin \mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$ but $\boldsymbol{s} + \boldsymbol{e}_i \in \mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$ for all $1 \le i \le n$.

Lemma 2.7 *The set $\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$ is determined by the set of maximal lattice points and $\mathfrak{G}_{\mathrm{HF}}(\mathcal{L} \setminus L_i)$ for all $1 \le i \le n$.*

Proof We claim that $\boldsymbol{x} \notin \mathfrak{G}_{\mathrm{HF}}(L)$ if and only if either $\boldsymbol{x} \preceq \boldsymbol{z}$ for some maximal lattice point $\boldsymbol{z} \in \mathbb{Z}^n$ or $\boldsymbol{x} \setminus x_i \notin \mathfrak{G}_{\mathrm{HF}}(\mathcal{L} \setminus L_i)$ for some i, where $\boldsymbol{x} = (x_1, \dots, x_n)$. For the "if" part, assume that $\boldsymbol{x} \in \mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$. Then $\boldsymbol{z} \in \mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$ if $\boldsymbol{z} \succeq \boldsymbol{x}$ and $\boldsymbol{x} \setminus x_i \in \mathfrak{G}_{\mathrm{HF}}(\mathcal{L} \setminus L_i)$ for all i by Lemmas 2.4 and 2.5, which contradicts the assumption.

For the "only if" part, assume that $\boldsymbol{x} \notin \mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$ and $\boldsymbol{x} \setminus x_i \in \mathfrak{G}_{\mathrm{HF}}(\mathcal{L} \setminus L_i)$ for all i. It suffices to find a maximal lattice point $\boldsymbol{z}$ such that $\boldsymbol{x} \preceq \boldsymbol{z}$. If $H_{\mathcal{L}}(\boldsymbol{x} + \boldsymbol{e}_i) = 0$ for all i, we let $\boldsymbol{z} = \boldsymbol{x}$. Otherwise, suppose $H_{\mathcal{L}}(\boldsymbol{x} + \boldsymbol{e}_i) \neq 0$ for some $1 \leq i \leq n$. There exists some constant t_i such that $H_{\mathcal{L}}(\boldsymbol{x} + t_i \boldsymbol{e}_i) \neq 0$, and $H_{\mathcal{L}}(\boldsymbol{x} + (t_i + 1)\boldsymbol{e}_i) = 0$ since $\boldsymbol{x} \setminus x_i \in \mathfrak{G}_{\mathrm{HF}}(\mathcal{L} \setminus L_i)$. If, for all $j \neq i$, $H_{\mathcal{L}}(\boldsymbol{x} + t_i \boldsymbol{e}_i + \boldsymbol{e}_j) = 0$, we let $\boldsymbol{z} = \boldsymbol{x} + t_i \boldsymbol{e}_i$. Otherwise, we repeat this process. The process stops after finitely many steps. Thus, there exists a maximal lattice point $\boldsymbol{z}$ such that $\boldsymbol{x} \preceq \boldsymbol{z}$. □

2.2 The d–invariant

For a rational homology sphere M with a Spin^c–structure $\mathfrak{s}$, the Heegaard Floer homology $\mathrm{HF}^\infty(M, \mathfrak{s}) \cong \mathbb{F}[U, U^{-1}]$ and $\mathrm{HF}^+(M, \mathfrak{s})$ is absolutely graded, where the free part is isomorphic to $\mathbb{F}[U^{-1}]$. Define the d–invariant of $(M, \mathfrak{s})$ to be the absolute grading of $1 \in \mathbb{F}[U^{-1}]$ [11].

We define *standard* 3–manifolds following [11, Section 9]:

Definition 2.8 A closed, oriented 3–manifold M is *standard* if, for each torsion Spin^c–structure $\mathfrak{s}$,

$$\mathrm{HF}^\infty(M, \mathfrak{s}) \cong (\Lambda^* H^1(M, \mathbb{F})) \otimes_{\mathbb{F}} \mathbb{F}[U, U^{-1}].$$

Remark 2.9 If M is standard, then $\operatorname{rk} \mathrm{HF}^\infty(M, \mathfrak{s}) = 2^b$ as an $\mathbb{F}[U, U^{-1}]$–module, where $b = b_1(M)$.

Let M_1 and M_2 be a pair of oriented closed 3–manifolds equipped with Spin^c–structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$, respectively. There is a connected sum formula for the Heegaard Floer homology [12, Theorem 6.2],

$$\mathrm{HF}^\infty(M_1 \# M_2, \mathfrak{s}_1 \# \mathfrak{s}_2) \cong H_*(\mathrm{CF}^\infty(M_1, \mathfrak{s}_1) \otimes_{\mathbb{F}[U, U^{-1}]} \mathrm{CF}^\infty(M_2, \mathfrak{s}_2)).$$

By the algebraic Künneth theorem, if M_1 and M_2 are standard, then $M_1 \# M_2$ is also standard.

If a 3–manifold M has a positive first Betti number (ie $b_1(M)>0$), the exterior algebra $\Lambda^*(H_1(M;\mathbb{F}))$ acts on the homology groups $\mathrm{HF}^\infty(M,\mathfrak{s})$, $\mathrm{HF}^+(M,\mathfrak{s})$, $\mathrm{HF}^-(M,\mathfrak{s})$ and $\widehat{\mathrm{HF}}(M,\mathfrak{s})$ [12, Section 4.2.5]. Define the subgroup $\mathfrak{A}_\mathfrak{s}\subset \mathrm{HF}^\infty(M,\mathfrak{s})$ by

$$\mathfrak{A}_\mathfrak{s}=\{x\in \mathrm{HF}^\infty(M,\mathfrak{s}) \mid \gamma\cdot x=0 \text{ for all } \gamma\in H_1(M,\mathbb{F})\}.$$

If M is standard, $\mathfrak{A}_\mathfrak{s}\cong \mathbb{F}[U,U^{-1}]$, and its image under the map $\pi\colon \mathrm{HF}^\infty(M,\mathfrak{s})\to \mathrm{HF}^+(M,\mathfrak{s})$ is isomorphic to $\mathbb{F}[U^{-1}]$.

Definition 2.10 For a standard 3–manifold M equipped with a torsion Spin^c–structure $\mathfrak{s}$, the d–invariant $d(M,\mathfrak{s})$ is defined as the absolute grading of $1\in\pi(\mathfrak{A}_\mathfrak{s})\cong\mathbb{F}[U^{-1}]$.

Ozsváth and Szabó proved an inequality for d–invariants [11, Section 9]. The following theorem is a reformulation of their result, which can be found in [19, Lemma 3.3]:

Proposition 2.11 [11, Section 9] *Suppose that W is a negative semidefinite cobordism from a rational homology sphere Y_1 to a standard 3–manifold Y_2 with $b_1(W)=0$. Let $\mathfrak{s}$ be a Spin^c–structure on W whose restriction $\mathfrak{s}_i$ to Y_i is torsion for $i=1,2$. Then*

$$d(Y_2,\mathfrak{s}_2)-d(Y_1,\mathfrak{s}_1)\geq \tfrac{1}{4}(c_1(\mathfrak{s})^2-2\chi(W)-3\sigma(W)). \tag{2-2}$$

The d–invariants of large surgeries on a link $\mathcal{L}=L_1\cup\cdots\cup L_n\subset S^3$ can be computed in terms of the H–function of the link by the large surgery theorem [9]. Choose a framing vector $\boldsymbol{q}=(q_1,\dots,q_n)\in\mathbb{Z}^n$ where $q_1,\dots,q_n$ are sufficiently large. Let Λ denote the linking matrix where Λ_{ij} is the linking number of L_i and L_j when $i\neq j$, and $\Lambda_{ii}=q_i$.

Attach n 2–handles to the 4–ball B^4 along $L_1,L_2,\dots,L_n$ with framings $q_1,\dots,q_n$. We obtain a 2–handlebody W with boundary $\partial W=S^3_{\boldsymbol{q}}(\mathcal{L})$ which is the 3–manifold obtained by doing surgery along $L_1,L_2,\dots,L_n$ with surgery coefficients $q_1,\dots,q_n$, respectively. Assume that $\det(\Lambda)\neq 0$; then $S^3_{\boldsymbol{q}}(\mathcal{L})$ is a rational homology sphere with $|H_1(S^3_{\boldsymbol{q}}(\mathcal{L}))|=|\det(\Lambda)|$. Note that if $\mathcal{L}$ has vanishing pairwise linking numbers, then Λ is a diagonal matrix with $\Lambda_{ii}=q_i$, and $\det(\Lambda)=q_1\cdots q_n\neq 0$. The Spin^c–structures on $S^3_{\boldsymbol{q}}(\mathcal{L})$ are enumerated as follows:

Lemma 2.12 [9, Section 9.3] *There are natural identifications*

$$H^2(S^3_{\boldsymbol{q}}(\mathcal{L}))\cong H_1(S^3_{\boldsymbol{q}}(\mathcal{L}))\cong \mathbb{Z}^n/\mathbb{Z}^n\Lambda$$

such that $c_1(\mathfrak{s})=[2\mathfrak{s}]$ for any $\mathfrak{s}\in \mathrm{Spin}^c(S^3_{\boldsymbol{q}}(\mathcal{L}))\cong \mathbb{H}(\mathcal{L})/\mathbb{Z}^n\Lambda$.

Fix $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_n) \in \mathbb{R}^n$ such that the values $\zeta_i > 0$ are very close to 0 and linearly independent over $\mathbb{Q}$. Let $P(\Lambda)$ be the hyperparallelepiped with vertices

$$\zeta + \tfrac{1}{2}(\pm\Lambda_1, \pm\Lambda_2, \dots, \pm\Lambda_n),$$

where all combinations of the signs are used and $\Lambda_1, \dots, \Lambda_n$ are column vectors of the matrix Λ. Let

$$P_{\mathbb{H}}(\Lambda) = P(\Lambda) \cap \mathbb{H}(\mathcal{L}),$$

where $\mathbb{H}(\mathcal{L})$ is the lattice for $\mathcal{L}$.

Proposition 2.13 [9, Section 10.1] *For any $\boldsymbol{v} \in P_{\mathbb{H}}(\Lambda)$ there exists a unique Spin^c–structure $\mathfrak{s}_{\boldsymbol{v}}$ on $S_{\boldsymbol{q}}(\mathcal{L})$ which extends to a Spin^c–structure $\mathfrak{t}_{\boldsymbol{v}}$ on W with $c_1(\mathfrak{t}_{\boldsymbol{v}}) = 2\boldsymbol{v} - (\Lambda_1 + \cdots + \Lambda_n)$.*

Remove a ball B^4 from the 2–handlebody W. We obtain a Spin^c–cobordism $\mathcal{U}$ from $(S^3, \mathfrak{s}_0)$ to $(S^3_{\boldsymbol{q}}(\mathcal{L}), \mathfrak{s}_{\boldsymbol{v}})$. By reversing the orientation of $\mathcal{U}$, we obtain a Spin^c–cobordism $\mathcal{U}'$ equipped with the Spin^c–structure $\mathfrak{t}_{\boldsymbol{v}}$ from $(S^3_{\boldsymbol{q}}(\mathcal{L}), \mathfrak{s}_{\boldsymbol{v}})$ to $(S^3, \mathfrak{s}_0)$.

Theorem 2.14 [1; 9] *For $\boldsymbol{v} \in P_{\mathbb{H}}(\Lambda)$, the d–invariant of a large surgery with surgery coefficients $\boldsymbol{q}$ on $\mathcal{L}$ is given by*

$$d(S^3_{\boldsymbol{q}}(\mathcal{L}), \mathfrak{s}_{\boldsymbol{v}}) = -\deg F_{(\mathcal{U}', \mathfrak{t}_{\boldsymbol{v}})} - 2H(\boldsymbol{v}),$$

where $\deg F_{\mathcal{U}', \mathfrak{t}_{\boldsymbol{v}}}$ is the grading shift of the cobordism $\mathcal{U}'$ with Spin^c–structure $\mathfrak{t}_{\boldsymbol{v}}$. The degree does not depend on the link, but depends on the linking matrix Λ.

2.3 The h–function of L–space links

In [13], Ozsváth and Szabó introduced the concept of L–spaces.

Definition 2.15 A 3–manifold M is an L–space if it is a rational homology sphere and its Heegaard Floer homology has minimal possible rank: for any Spin^c–structure $\mathfrak{s}$, $\widehat{\mathrm{HF}}(M, \mathfrak{s}) = \mathbb{F}$, and $\mathrm{HF}^-(Y, \mathfrak{s})$ is a free $\mathbb{F}[U]$–module of rank 1.

In terms of the large surgery, Gorsky and Némethi defined L–space links in [3].

Definition 2.16 An oriented n–component link $\mathcal{L} \subset S^3$ is an L–space link if there exists $\mathbf{0} \prec \boldsymbol{p} \in \mathbb{Z}^n$ such that the surgery manifold $S^3_{\boldsymbol{q}}(\mathcal{L})$ is an L–space for any $\boldsymbol{q} \succeq \boldsymbol{p}$.

For L–space links $\mathcal{L}$, $H_*(A^-(\mathcal{L},\boldsymbol{s}))=\mathbb{F}[U]$ [7]. By equation (2-1) and the inclusion–exclusion formula, one can write [1]

$$\chi(\mathrm{HFL}^-(\mathcal{L},\boldsymbol{s}))=\sum_{B\subset\{1,\dots,n\}}(-1)^{|B|-1}H_{\mathcal{L}}(\boldsymbol{s}-\boldsymbol{e}_B). \tag{2-3}$$

The Euler characteristic $\chi(\mathrm{HFL}^-(\mathcal{L},\boldsymbol{s}))$ was computed in [15], and we follow the normalization convention in [1],

$$\widetilde{\Delta}_{\mathcal{L}}(t_1,\dots,t_n)=\sum_{\boldsymbol{s}\in\mathbb{H}(\mathcal{L})}\chi(\mathrm{HFL}^-(\mathcal{L},\boldsymbol{s}))t_1^{s_1}\cdots t_n^{s_n}, \tag{2-4}$$

where $\boldsymbol{s}=(s_1,\dots,s_n)$, and

$$\widetilde{\Delta}_{\mathcal{L}}(t_1,\dots,t_n):=\begin{cases}(t_1\cdots t_n)^{1/2}\Delta_{\mathcal{L}}(t_1,\dots,t_n) & \text{if } n>1,\\ \Delta_{\mathcal{L}}(t)/(1-t^{-1}) & \text{if } n=1.\end{cases}$$

Note that we regard $1/(1-t^{-1})$ as an infinite power series.

Theorem 2.17 [3] *The H–function of an L–space link is determined by the Alexander polynomials of its sublinks via*

$$H_{\mathcal{L}}(\boldsymbol{s})=\sum_{\mathcal{L}'\subset\mathcal{L}}(-1)^{\#\mathcal{L}'-1}\sum_{\boldsymbol{u}'\succeq\pi_{\mathcal{L}'}(\boldsymbol{s}+\boldsymbol{1})}\chi(\mathrm{HFL}^-(\mathcal{L}',\boldsymbol{u}')), \tag{2-5}$$

where $\boldsymbol{1}=(1,\dots,1)$ and $\pi_{\mathcal{L}'}\colon\mathbb{H}(\mathcal{L})\to\mathbb{H}(\mathcal{L}')$ is the projection to the entries corresponding to link components $L_i\subset\mathcal{L}'$.

Remark 2.18 For L–space links with two components, the explicit formula for the H–function can also be found in [7].

Consider L–space links $\mathcal{L}$ with vanishing pairwise linking numbers. The set $\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$ can also be described in terms of the Alexander polynomials of the link and its sublinks.

Lemma 2.19 *For an n–component L–space link $\mathcal{L}\subseteq S^3$ with vanishing pairwise linking numbers, $\boldsymbol{s}\in\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$ if and only if for all $\boldsymbol{y}=(y_1,\dots,y_n)\succ\boldsymbol{s}$, the coefficients of $t_1^{y_1}\cdots t_n^{y_n}$ in $\widetilde{\Delta}_{\mathcal{L}}(t_1,\dots,t_n)$ are 0, and the coefficients corresponding to $\boldsymbol{y}\setminus y_B$ in $\widetilde{\Delta}_{\mathcal{L}\setminus L_B}(t_{i_1},\dots,t_{i_k})$ are also 0 for all $B\subset\{1,\dots,n\}$.*

Proof For the "if" part, note that $\chi(\mathrm{HFL}^-(\mathcal{L},\boldsymbol{y}))=0$ and $\chi(\mathrm{HFL}^-(\mathcal{L}\setminus L_B),\boldsymbol{y}\setminus y_B)=0$ for all $\boldsymbol{y}\succ\boldsymbol{s}$ and $B\subset\{1,\dots,n\}$ by (2-4). Then $H_{\mathcal{L}}(\boldsymbol{s})=0$ by Theorem 2.17, and $\boldsymbol{s}\in\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$. For the "only if" part, suppose that $\boldsymbol{s}\notin\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$. By Lemma 2.7, either there exists a maximal vector $z \notin \mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$ such that $s \preceq z$ or there exists some $1 \le j \le n$ such that $s \setminus s_j \notin \mathfrak{G}_{\mathrm{HF}}(\mathcal{L} \setminus L_j)$. We claim that for all maximal lattice points z, $\chi(\mathrm{HFL}^-(\mathcal{L}, z+\mathbf{1})) \neq 0$. Since z is maximal, $h_{\mathcal{L}}(z) = 1$, and for any subset $B \subset \{1, \dots, n\}$, $h_{\mathcal{L}}(z + e_B) = 0$. By (2-3), $\chi(\mathrm{HFL}^-(\mathcal{L}, z+\mathbf{1})) = (-1)^n \neq 0$. If $s \preceq z$, the coefficient of $z + \mathbf{1} \succ s$ in $\widetilde{\Delta}_{\mathcal{L}}(t_1, \dots, t_n)$ equals 0, which contradicts our assumption. If $s \setminus s_i \notin \mathfrak{G}_{\mathrm{HF}}(\mathcal{L} \setminus L_i)$, we use the induction to get a contradiction. □

2.4 The Heegaard Floer link homology

Ozsváth and Szabó associated the multigraded link invariants $\mathrm{HFL}^-(\mathcal{L})$ and $\widehat{\mathrm{HFL}}(\mathcal{L})$ to links $\mathcal{L} \subset S^3$, where $\mathrm{HFL}^-(L)$ is as defined in (2-1), and $\widehat{\mathrm{HFL}}(L)$ is defined as follows [2; 15]:

$$\widehat{\mathrm{HFL}}(\mathcal{L}, s) = H_*\left(A^-(\mathcal{L}, s) \Big/ \left[\sum_{i=1}^{n} A^-(s - e_i) \oplus \sum_{i=1}^{n} U_i A^-(s + e_i)\right]\right).$$

If $\mathcal{L}$ is an L–space link, there exist spectral sequences converging to $\mathrm{HFL}^-(\mathcal{L})$ and $\widehat{\mathrm{HFL}}(\mathcal{L})$, respectively [2; 3].

Proposition 2.20 [3, Theorem 1.5.1] *For an oriented L–space link $\mathcal{L} \subset S^3$ with n components and $s \in \mathbb{H}(\mathcal{L})$, there exists a spectral sequence with $E_\infty = \mathrm{HFL}^-(\mathcal{L}, s)$ and*

$$E_1 = \bigoplus_{B \subset \{1,\dots,n\}} H_*(A^-(\mathcal{L}, s - e_B)),$$

where the differential in E_1 is induced by inclusions.

Remark 2.21 Precisely, the differential ∂_1 in the E_1–page is

$$\partial_1(z(s - e_B)) = \sum_{i \in B} U^{H(s - e_B) - H(s - e_B + e_i)} z(s - e_B + e_i),$$

where $z(s - e_B)$ denotes the unique generator in $H_*(A^-(\mathcal{L}, s - e_B))$ with the homological grading $-2H(s - e_B)$.

Proposition 2.22 [2, Proposition 3.8] *For an L–space link $\mathcal{L} \subset S^3$ with n components and $s \in \mathbb{H}(\mathcal{L})$, there exists a spectral sequence converging to $\widehat{E}_\infty = \widehat{\mathrm{HFL}}(\mathcal{L}, s)$ with E_1–page*

$$\widehat{E}_1 = \bigoplus_{B \subset \{1,\dots,n\}} \mathrm{HFL}^-(\mathcal{L}, s + e_B).$$

There is a nice symmetric property of $\widehat{\mathrm{HFL}}(\mathcal{L})$. Ozsváth and Szabó proved

$$\widehat{\mathrm{HFL}}_*(\mathcal{L},\boldsymbol{s}) \cong \widehat{\mathrm{HFL}}_*(\mathcal{L},-\boldsymbol{s}) \tag{2-6}$$

up to some grading shift in [14].

3 The proof of the main theorem

3.1 The Spin^c–cobordism

In this section, we use $\mathcal{L} = L_1 \cup \cdots \cup L_n \subset S^3$ to denote an oriented link with vanishing pairwise linking numbers. Suppose that link components L_i bound pairwise disjoint smoothly embedded surfaces Σ_i of genera g_i in B^4 for all $1 \le i \le n$. Attach n 2–handles to the 4–ball B^4 along $L_1, L_2, \dots, L_n$ with framings $-p_1, -p_2, \dots, -p_n$. We obtain a 2–handlebody W with boundary $\partial W = S^3_{-p_1,\dots,-p_n}(\mathcal{L})$ which is the 3–manifold obtained by doing surgeries along $L_1, L_2, \dots, L_n$ with surgery coefficients $-p_1, -p_2, \dots, -p_n$, respectively. The linking matrix Λ is a diagonal matrix with $\lambda_{ii} = -p_i$. Observe that $\det(\Lambda) \neq 0$, so $S^3_{-p_1,\dots,-p_n}(\mathcal{L})$ is a rational homology sphere. For our purpose, we assume that $p_i \gg 0$ for all $1 \le i \le n$ in this section.

Let Σ'_i be the closed surface in W which is the union of Σ_i and the core of the 2–handle attached along L_i. Then Σ'_i are also pairwise disjoint. Observe that W is homotopy equivalent to the wedge of n copies of S^2. Thus, $H_2(W) = \mathbb{Z}^n$ and $[\Sigma'_i]$ are generators of $H_2(W)$. The self-intersection number of each Σ'_i in W is $-p_i$.

Take small tubular neighborhoods $\mathrm{nd}(\Sigma'_i)$ of Σ'_i such that they are also pairwise disjoint. Then $\mathrm{nd}(\Sigma'_i)$ is a disk bundle over Σ'_i and its boundary $\partial(\mathrm{nd}(\Sigma'_i))$ is a circle bundle B_{-p_i} with Euler number $-p_i$. The boundary connected sum $\mathfrak{D}$ of the disk bundles over Σ'_i in W is obtained by identifying smoothly embedded balls $B^3_i \subset B_{-p_i}$ and $B^3_{i+1} \subset B_{-p_{i+1}}$ for $1 \le i \le n-1$, and $\mathfrak{D}$ is also a smooth oriented manifold [4, Section 6.3]. Observe that $\mathfrak{D}$ has the homotopy type of $D_{-p_1} \vee \cdots \vee D_{-p_n}$, where D_{-p_i} denotes the disk bundle over Σ'_i. Since D_{-p_i} is homotopy equivalent to Σ'_i,

$$\widetilde{H}_j(\mathfrak{D}) \cong \bigoplus_{i=1}^{n} \widetilde{H}_j(\Sigma'_i).$$

Let X denote the complement of $\mathfrak{D}$ in W. It is a cobordism from $B_{-p_1} \# \cdots \# B_{-p_n}$ to $S^3_{-p_1,\dots,-p_n}(\mathcal{L})$. Let $\overline{X}$ be the cobordism from $S^3_{-p_1,\dots,-p_n}(\mathcal{L})$ to $B_{-p_1} \# \cdots \# B_{-p_n}$ obtained by reversing the orientation of X.

Proposition 3.1 *For a circle bundle B_{-m} over a closed oriented surface of genus g and Euler number $-m<0$, its cohomology is*

$$H^1(B_{-m})\cong\mathbb{Z}^{2g},\quad H^2(B_{-m})\cong\mathbb{Z}^{2g}\oplus\mathbb{Z}_m,\quad H^3(B_{-m})\cong\mathbb{Z}.$$

Proof For the circle bundle B_{-m}, we have the following long exact sequence by using the Gysin sequence:

$$0\to H^1(\Sigma_g)\to H^1(B_{-m})\to H^0(\Sigma_g)\xrightarrow{\cup e} H^2(\Sigma_g)\to H^2(B_{-m})\to H^1(\Sigma_g)\to 0,$$

where e is the Euler class. Then we compute that

$$0\to\mathbb{Z}^{2g}\to H^1(B_{-m})\to\mathbb{Z}\xrightarrow{\cdot m}\mathbb{Z}\to H^2(B_{-m})\to\mathbb{Z}^{2g}\to 0.$$

Thus, $H^1(B_{-m})\cong\mathbb{Z}^{2g}$ and we have the short exact sequence

$$0\to\mathbb{Z}_m\to H^2(B_{-m})\to\mathbb{Z}^{2g}\to 0.$$

Since $\mathbb{Z}^{2g}$ is free, the exact sequence splits and $H^2(B_{-m})\cong\mathbb{Z}^{2g}\oplus\mathbb{Z}_m$. The circle bundle B_{-m} is oriented and closed, so $H^3(B_{-m})\cong\mathbb{Z}$. □

Lemma 3.2 *Suppose that M_1 and M_2 are closed, connected and oriented smooth n–dimensional manifolds. Then*

$$H^i(M_1\# M_2)\cong H^i(M_1)\oplus H^i(M_2)\quad\text{for } i\neq 0 \text{ and } n,$$

and $H^0(M_1\# M_2)\cong H^n(M_1\# M_2)\cong\mathbb{Z}$.

Corollary 3.3 *The cohomology of $\#_{i=1}^n B_{-p_i}$ is*

$$H^1\Bigl(\mathop{\#}_{i=1}^{n} B_{-p_i}\Bigr)\cong\mathbb{Z}^{2g_1+\cdots+2g_n},\quad H^2\Bigl(\mathop{\#}_{i=1}^{n} B_{-p_i}\Bigr)\cong\mathbb{Z}^{2g_1+\cdots+2g_n}\oplus\mathbb{Z}_{p_1}\cdots\oplus\mathbb{Z}_{p_n}$$

and $H^0(\#_{i=1}^n B_{-p_i})\cong H^3(\#_{i=1}^n B_{-p_i})\cong\mathbb{Z}$.

Proposition 3.4 *For the cobordism $\overline{X}$, we have*

$$H^2(\overline{X})\cong H^2\Bigl(\mathop{\#}_{i=1}^{n} B_{-p_i}\Bigr)\cong\mathbb{Z}^{2g_1+\cdots+2g_n}\oplus\mathbb{Z}_{p_1}\oplus\cdots\oplus\mathbb{Z}_{p_n},\quad H^1(\overline{X})\cong 0.$$

Proof We use the Mayer–Vietoris sequence to compute the cohomology of $\overline{X}$. Observe that W is the union of $\overline{X}$ and $\mathfrak{D}$, and the intersection of $\overline{X}$ and $\mathfrak{D}$ is $\#_{i=1}^n B_{-p_i}$.

Then we have the long exact sequence

$$
\begin{aligned}
0 \to H^1(W) \to H^1(\overline{X}) \oplus H^1(\Sigma_1') \cdots \oplus H^1(\Sigma_n') \xrightarrow{i^*} \bigoplus_{i=1}^{n} H^1(B_{-p_i}) \\
\to H^2(W) \to \bigoplus_{i=1}^{n} H^2(\Sigma_i') \oplus H^2(\overline{X}) \to H^2\Big(\mathop{\#}_{i=1}^{n} B_{-p_i}\Big) \\
\to H^3(W) \to H^3(\overline{X}) \to H^3\Big(\mathop{\#}_{i=1}^{n} B_{-p_2}\Big) \to 0.
\end{aligned}
$$

Recall that W is homotopy equivalent to $S^2 \vee \cdots \vee S^2$. Then $H^1(W) \cong H^3(W) \cong H^4(W) \cong 0$. Thus, we have

$$
H^3(\overline{X}) \cong H^3\Big(\mathop{\#}_{i=1}^{n} B_{-p_i}\Big) \cong \mathbb{Z}, \quad H^4(\overline{X}) = 0
$$

and

$$
\begin{aligned}
0 \to H^1(\overline{X}) \oplus \mathbb{Z}^{2g_1} \cdots \oplus \mathbb{Z}^{2g_n} \xrightarrow{i^*} \bigoplus_{i=1}^{n} \mathbb{Z}^{2g_i} \to \mathbb{Z}^n \to H^2(\overline{X}) \oplus \mathbb{Z}^n \to \bigoplus_{i=1}^{n} (\mathbb{Z}^{2g_i} \oplus \mathbb{Z}_{p_i}) \\
\to 0.
\end{aligned}
$$

We claim that the map $j^*\colon H^1(\Sigma_i') \to H^1(B_{-p_i})$ is an isomorphism. Observe that $H^1(\Sigma_i') \cong H_1(\Sigma_i')$ and $H^1(B_{-p_i}) \cong H_2(B_{-p_i})$ by the Poincaré duality. Each generator in $H_1(\Sigma_i')$ is represented by a simple closed curve in Σ_i'. The curve along with its circle fiber is a generator in $H_2(B_{-p_i})$, which is precisely the image of the curve under j^*. Therefore, j^* is an isomorphism. Note that the map i^* restricted to the summand $H^1(\Sigma_i')$ is exactly j^*, mapping $H^1(\Sigma_i')$ isomorphically onto the summand $H^1(B_{-p_i})$. Hence, i^* is an isomorphism when restricted to $\mathbb{Z}^{2g_1} \oplus \cdots \oplus \mathbb{Z}^{2g_n}$. Then $H^1(\overline{X}) = 0$. We have the short exact sequence

$$
0 \to \mathbb{Z}^n \xrightarrow{g} H^2(\overline{X}) \oplus \mathbb{Z}^n \xrightarrow{f} \bigoplus_{i=1}^{p} (\mathbb{Z}^{2g_i} \oplus \mathbb{Z}_{p_i}) \to 0. \tag{3-1}
$$

Note that each $\mathbb{Z}$–summand in $H_2(W, \partial W) \cong H^2(W)$ is represented by the surface Σ_i' and it corresponds to the generator of $H^2(\Sigma_i') \cong H_0(\Sigma_i')$. Then g maps $\mathbb{Z}^n$ identically to the summand $\mathbb{Z}^n$ of $H^2(\overline{X}) \oplus \mathbb{Z}^n$. This implies that the map f is an isomorphism when restricted to $H^2(\overline{X})$. Thus $H^2(\overline{X}) \cong H^2\big(\#_{i=1}^n B_{-p_i}\big) \cong \bigoplus_{i=1}^n (\mathbb{Z}^{2g_i} \oplus \mathbb{Z}_{p_i})$. □

Remark 3.5 From the computation in the proof, $\chi(X) = 2g_1 + \cdots 2g_n$.

Proposition 3.6 *The intersection form* $\mathcal{Q}\colon H^2(\overline{X})/\mathrm{Tor}\times H^2(\overline{X})/\mathrm{Tor}\to\mathbb{Q}$ *vanishes.*

Proof For two elements $s,t\in H^2(\overline{X})/\mathrm{Tor}\cong H_2(\overline{X})$, we have $\mathcal{Q}(s,t)=\langle\overline{s},\mathrm{PD}(t)\rangle$, where $\overline{s}$ is the image of s under the map $p_*\colon H_2(\overline{X})\to H_2(\overline{X},\partial\overline{X})$ induced by the projection and $\mathrm{PD}(t)\in H^2(\overline{X},\partial\overline{X})$. We claim the map $i_*\colon H_2(\partial\overline{X})\to H_2(\overline{X})$ induced by the inclusion is surjective. Consider the Mayer–Vietoris sequence of homology similar to the argument in the proof of Proposition 3.4. We have

$$0\to H_2\Big(\mathop{\#}_{i=1}^{n} B_{-p_i}\Big)\xrightarrow{f'}\bigoplus_{i=1}^{n} H_2(\Sigma_i')\oplus H_2(\overline{X})\xrightarrow{g'} H_2(W)\to H_1\Big(\mathop{\#}_{i=1}^{n} B_{-p_i}\Big)\to\cdots.$$

Observe that $H_2(W)$ is generated by the surfaces Σ_i'. Then g' is injective when restricted to $\bigoplus_{i=1}^n H_2(\Sigma_i')$. From the proof of Proposition 3.4, $H_2(\overline{X})\cong\bigoplus_{i=1}^n\mathbb{Z}^{2g_i}\cong H_2\big(\#_{i=1}^n B_{-p_i}\big)$. Since f' is injective, f' maps to $H_2(\overline{X})$ and it is surjective. Note that $H_2(\overline{X})\cong H_2\big(\#_{i=1}^n B_{-p_i}\big)\oplus H_2(S^3_{-p_1,\dots,-p_n}(\mathcal{L}))$. The map i_* equals the map f' when restricted to the summand $H_2\big(\#_{i=1}^n B_{-p_i}\big)$. Hence, i_* is surjective. Then $p_*=0$ by the long exact sequence of homology induced by the inclusion $\partial\overline{X}$ to $\overline{X}$. Hence, $\overline{s}=0$ and $\mathcal{Q}(s,t)=0$. Therefore the intersection form $\mathcal{Q}$ vanishes in $\overline{X}$. □

Corollary 3.7 *The signature* $\sigma(\overline{X})$ *equals* 0.

By Proposition 3.4, $H^2(\overline{X})\cong H^2\big(\#_{i=1}^n B_{-p_i}\big)\cong\mathbb{Z}^{2g_1\cdots+2g_n}\oplus\mathbb{Z}_{p_1}\cdots\oplus\mathbb{Z}_{p_n}$. We can identify the generator of a $\mathbb{Z}$–summand in $H^2(B_{-p_i})$ to be the Poincaré dual of a simple closed curve which is a generator of $H_1(\Sigma_i')$, and we identify the generator of $\mathbb{Z}_{p_i}$ to be the Poincaré dual of the fiber. Then this will give an isomorphism from $H^2(\overline{X})$ to $\mathbb{Z}^{2g_1\cdots+2g_n}\oplus\mathbb{Z}_{p_1}\cdots\oplus\mathbb{Z}_{p_n}$. The restriction map from $H^2(\overline{X})$ to $H^2(S^3_{-p_1,\dots,-p_n}(\mathcal{L}))\cong\mathbb{Z}_{p_1}\oplus\cdots\oplus\mathbb{Z}_{p_n}$ is the projection onto the summand $\mathbb{Z}_{p_1}\oplus\cdots\oplus\mathbb{Z}_{p_n}$. Note that the fiber of the circle bundle B_{-p_i} is the meridian of the link component L_i, which corresponds to the generator of $\mathbb{Z}_{p_i}$ in $H^2(S^3_{-p_1,\dots,-p_n}(\mathcal{L}))$.

An $\boldsymbol{s}=(s_1,\dots,s_n)\in\mathbb{Z}^n/\mathbb{Z}^n\Lambda$ corresponds to a Spin^c–structure $\mathfrak{s}$ on $S^3_{-p_1,\dots,-p_n}(\mathcal{L})$ which can be extended to W by Proposition 2.13. We denote its restrictions to $\overline{X}$ and $\#_{i=1}^n B_{-p_i}$ both by $\mathfrak{s}'$. Moreover, we let s_i' denote the restriction of the Spin^c–structure on $\overline{X}$ to B_{-p_i}. By an argument similar to the one in [19, Lemma 3.1], we have $c_1(s_i')=2s_i$. So s_i' is a torsion Spin^c–structure on B_{-p_i}, which indicates that $\mathfrak{s}'$ is a torsion Spin^c–structure on $\#_{i=1}^n B_{-p_i}$.

Lemma 3.8 *The three-manifolds* $S^3_{-p_1,\dots,-p_n}(\mathcal{L})$ *and* $\#_{i=1}^n B_{-p_i}$ *are both standard.*

Proof Recall that we assume that $p_i \gg 0$ for all i in this section. Then $S^3_{-p_1,\dots,-p_n}(\mathcal{L})$ is a rational homology sphere. So $H^1(S^3_{-p_1,\dots,-p_n}(\mathcal{L}))$ is trivial and

$$\mathrm{HF}^\infty(S^3_{-p_1,\dots,-p_n}(\mathcal{L}),\mathfrak{s}) \cong \mathbb{F}[U,U^{-1}]$$

for any Spin^c–structure $\mathfrak{s}$ on $S^3_{-p_1,\dots,-p_n}(\mathcal{L})$. Hence, $S^3_{-p_1,\dots,-p_n}(\mathcal{L})$ is a standard three-manifold.

For the circle bundle B_{-p_i} with a torsion Spin^c–structure s'_i, Rasmussen proved that

$$\mathrm{HF}^\infty(B_{-p_i},s'_i) \cong \mathrm{HF}^\infty\big(\#^{2g}\, S^1\times S^2,\mathfrak{s}_0\big),$$

where $\mathfrak{s}_0$ is the unique torsion Spin^c–structure on the manifold $\#^{2g}(S^1\times S^2)$ [19]. Thus, $\mathrm{HF}^\infty\big(\#_{i=1}^n B_{-p_i},\mathfrak{s}'\big)$ is also standard by the connected sum formula for Heegaard Floer homology [12, Theorem 6.2]. □

Remark 3.9 By the additivity property of the d–invariants [5, Proposition 4.3],

$$d\Big(\mathop{\#}_{i=1}^{n} B_{-p_i},\mathfrak{s}'\Big) = d(B_{-p_1},s'_1)+\cdots+d(B_{-p_n},s'_n).$$

Next, we can use Proposition 2.11 to prove the following d–invariant inequality:

Proposition 3.10
$$d(S^3_{-p_1,\dots,-p_n}(\mathcal{L}),\mathfrak{s}) \le \sum_{i=1}^{n} d(B_{-p_i},s'_i)+g_1+\cdots+g_n.$$

Proof By Proposition 3.6 and Lemma 3.8, the 4–manifold $\overline{X}$ is negative semidefinite and bounds standard 3–manifolds. Let $\mathfrak{s}$ be a Spin^c–structure on $S^3_{-p_1,\dots,-p_n}(\mathcal{L})$ and $\mathfrak{s}'$ be the corresponding Spin^c–structure on $\overline{X}$ and $\#_{i=1}^n B_{-p_i}$, which is a torsion Spin^c–structure on $\#_{i=1}^n B_{-p_i}$. By Proposition 3.6, $c_1^2(\mathfrak{s}') = \mathcal{Q}(c_1(\mathfrak{s}'),c_1(\mathfrak{s}')) = 0$, $b_1(\overline{X})=0$ and $b_2^+(\overline{X})=0$. By (2-2),

$$0 \le 4\Big(-d(S^3_{-p_1,\dots,-p_n},\mathfrak{s})+d\Big(\mathop{\#}_{i=1}^{n} B_{-p_i},\mathfrak{s}'\Big)\Big)+2(2g_1+\cdots+2g_n).$$

This implies

$$0 \le -4d(S^3_{-p_1,\dots,-p_n},\mathfrak{s})+4d\Big(\mathop{\#}_{i=1}^{n} B_{-p_i},\mathfrak{s}'\Big)+4g_1+\cdots+4g_n.$$

Thus,

$$d(S^3_{-p_1,\dots,-p_n}(\mathcal{L}),\mathfrak{s}) \le \sum_{i=1}^{n} d(B_{-p_i},s'_i)+g_1+\cdots+g_n.$$ □

Let $\mathcal{L}^*$ denote the mirror of $\mathcal{L}$. Observe that $S^3_{-p_1,\dots,-p_n}(\mathcal{L})$ is obtained from $S^3_{p_1,\dots,p_n}(\mathcal{L}^*)$ by reversing the orientation. For any $\boldsymbol{s}\in\mathbb{Z}^n$, choose sufficiently large $p_i\gg 0$ so that $\boldsymbol{s}\in P_{\mathbb{H}}(\Lambda)$. Let $\mathfrak{s}$ denote the Spin^c–structure on $S^3_{-p_1,\dots,-p_n}(\mathcal{L})$ corresponding to $\boldsymbol{s}$. By Theorem 2.14,

$$d(S^3_{-p_1,\dots,-p_n}(\mathcal{L}),\mathfrak{s}) = -d(S^3_{p_1,\dots,p_n}(\mathcal{L}^*),\mathfrak{s}) = \deg F_{\mathbb{U}',\mathfrak{s}} + 2H_{L^*}(\boldsymbol{s}),$$

where $H_{\mathcal{L}^*}$ is the H–function of $\mathcal{L}^*$. Let O denote the unlink with n components. Similarly, we have

$$d(S^3_{-p_1,\dots,-p_n}(O),\mathfrak{s}) = -d(S^3_{p_1,\dots,p_n}(O),\mathfrak{s}) = \deg F_{\mathbb{U}',\mathfrak{s}} + 2H_O(\boldsymbol{s}).$$

Thus,

$$d(S^3_{-p_1,\dots,-p_n}(\mathcal{L}),\mathfrak{s}) - d(S^3_{-p_1,\dots,-p_n}(O),\mathfrak{s}) = 2H_{L^*}(\boldsymbol{s}) - 2H_O(\boldsymbol{s}) = 2h_{\mathcal{L}^*}(\boldsymbol{s}).$$

Recall that for a circle bundle B_{-m} with Euler number $-m$ over a closed, oriented genus g surface, $H^2(B_{-m})\cong\mathbb{Z}^{2g}\oplus\mathbb{Z}_m$. We label the torsion Spin^c–structures on B_{-m} following the convention in [19]. Note that B_{-m} can be obtained by doing $-m$ surgery on the "Borromean knot" $B\subset \#^{2g}(S^1\times S^2)$. Let X_2 be the surgery cobordism from $\#^{2g}(S^1\times S^2)$ to B_{-m}. The restriction map $H^2(X_2)\to H^2\big(\#^{2g}(S^1\times S^2)\big)$ has kernel isomorphic to $\mathbb{Z}$, which corresponds to the 2–handle attached along B. If x denotes a generator of $\mathbb{Z}$, we let $\mathfrak{t}_k$ denote the Spin^c–structure on X_2 such that $c_1(\mathfrak{t}_k)=(-m+2k)x$. For simplicity, we still let $\mathfrak{t}_k$ be its restriction on B_{-m}. For the lens space $L(m,1)$ in Proposition 3.11, the labeling of Spin^c–structures on $L(m,1)$ is similar. We consider the surgery cobordism from S^3 to $L(m,1)$. For details, see [19, Section 2.1].

Proposition 3.11 [19, Proposition 3.4] *Let B_{-m} denote a circle bundle equipped with a torsion Spin^c–structure $\mathfrak{t}_k$ over a closed oriented surface Σ_g. For $m\gg 0$,*

$$d(B_{-m},\mathfrak{t}_k) = \begin{cases} E(m,k) - g + 2\left\lceil \frac{1}{2}(g-|k|)\right\rceil & \text{if } |k|\le g,\\ E(m,k)-g & \text{if } |k|>g,\end{cases}$$

where $\{E(m,k)\mid k\in\mathbb{Z}_m\}$ is the set of d–invariants of the lens space $L(m,1)$.

3.2 Proofs of the main theorems

We prove Theorems 1.1 and 1.3 in this subsection.

Proof of Theorem 1.1 By Propositions 3.10 and 3.11,

$$d(S^3_{-p_1,\dots,-p_n}(\mathcal{L}),\mathfrak{s}) \le \sum_{i=1}^{n}(E(p_i,s_i)-g_i+2f_{g_i}(s_i))+g_1+\cdots+g_n.$$

Recall that $d(S^3_{-p_1,\dots,-p_n}(\mathcal{L}),\mathfrak{s}) = 2h_{\mathcal{L}^*}(\boldsymbol{s})+d(S^3_{-p_1,\dots,-p_n}(O),\mathfrak{s})$. For lens spaces, our orientation convention is the one used in [19], namely that $-p$ surgery on the unknot produces the oriented space $L(p,1)$. Then $S^3_{-p_1,\dots,-p_n}(O)=L(p_1,1)\#\cdots\#L(p_n,1)$, and

$$d(S^3_{-p_1,\dots,-p_n}(O),\mathfrak{s}) = \sum_{i=1}^{n} d(L(p_i,1),s_i) = \sum_{i=1}^{n} E(p_i,s_i).$$

Hence,

$$h_{\mathcal{L}^*}(\boldsymbol{s}) \le \sum_{i=1}^{n} f_{g_i}(s_i).$$

The surfaces Σ_i bounded by L_i are pairwise disjoint. Then the corresponding link components of the mirror link $\mathcal{L}^*$ bound the mirrors of Σ_i which have the same genera as Σ_i. Thus we have $h_L(\boldsymbol{s}) \le \sum_{i=1}^{n} f_{g_i}(s_i)$. □

Corollary 3.12 *For an oriented n–component link $\mathcal{L}\subset S^3$, $\mathfrak{G}(\mathcal{L})\subset \mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$.*

Proof Suppose that the link components of $\mathcal{L}$ bound pairwise disjoint surfaces in B^4 of genera g_i. By Theorem 1.1, $h_{\mathcal{L}}(\boldsymbol{s})=0$ if $\boldsymbol{s}\succeq \boldsymbol{g}$, where $\boldsymbol{g}=(g_1,\dots,g_n)$. □

Definition 3.13 An oriented n–component link $\mathcal{L}\subset S^3$ is (smoothly) *slice* if there exist n disjoint, smoothly embedded disks in B^4 with boundary $\mathcal{L}$.

Proof of Theorem 1.3 If $\mathcal{L}$ is slice, then $h_{\mathcal{L}}=0$ by Theorem 1.1. Thus $H_{\mathcal{L}}(\boldsymbol{v}) = H_O(\boldsymbol{v}) = \sum_{i=1}^{n} H(v_i)$, where $H(v_i)$ is the H–function for the unknot and $\boldsymbol{v} = (v_1,\dots,v_n)\in\mathbb{Z}^n$. We claim that $\mathrm{HFL}^-(\mathcal{L},\boldsymbol{v})=0$ if there exists a component $v_j>0$. By Proposition 2.20, there exists a spectral sequence converging to $\mathrm{HFL}^-(\mathcal{L})$ with the E_1–page

$$E_1(\boldsymbol{v}) = \bigoplus_{B\subset\{1,\dots,n\}} H_*(A^-(\mathcal{L},\boldsymbol{v}-\boldsymbol{e}_B))$$

and differential ∂_1 which is induced by inclusions.

Let $\mathcal{K}=\{1,\dots,n\}\setminus\{j\}$, and

$$E'(\boldsymbol{v}) = \bigoplus_{B\subset\mathcal{K}} H_*(A^-(\mathcal{L},\boldsymbol{v}-\boldsymbol{e}_B)),\qquad E''(\boldsymbol{v}) = \bigoplus_{B\subset\mathcal{K}} H_*(A^-(\mathcal{L},\boldsymbol{v}-\boldsymbol{e}_B-\boldsymbol{e}_j)).$$

Then $E_1(\boldsymbol{v}) = E'(\boldsymbol{v}) \oplus E''(\boldsymbol{v})$. Recall that for L–space links $\mathcal{L}$ and each $B \subset \{1, \dots, n\}$, $H_*(A^-(\mathcal{L}, \boldsymbol{v} - \boldsymbol{e}_B)) \cong \mathbb{F}[U]$ [7]. Let ∂' and ∂'' denote the differentials in $E'(\boldsymbol{v})$ and $E''(\boldsymbol{v})$ which are induced by ∂_1. Let z denote the generator of $H_*(A^-(\mathcal{L}, \boldsymbol{v} - e_B - e_j)) \in E''(\boldsymbol{v})$ with homological grading $-2H(\boldsymbol{v} - e_B - e_j)$. Observe that $H(\boldsymbol{v} - e_B - e_j) = H(\boldsymbol{v} - e_B)$ since $H(v_j - 1) = H(v_j)$ for $v_j > 0$. Then $\partial_1(z) = \partial''(z) + z'$, where z' is the generator of $H_*(A^-(\mathcal{L}, \boldsymbol{v} - \boldsymbol{e}_B))$ with homological grading $-2H(\boldsymbol{v} - e_B)$. Let $\mathcal{D}$ be an acyclic chain complex with two generators a and b, and the differential $\partial_D(a) = b$. Then the chain complex $(E_1(\boldsymbol{v}), \partial_1)$ is isomorphic to $(E''(\boldsymbol{v}) \otimes \mathcal{D}, \partial'' \otimes \partial_D)$. Thus $E_2 = 0$, and the spectral sequence collapses at E_2. Therefore, $\mathrm{HFL}^-(\mathcal{L}, \boldsymbol{v}) = 0$ if there exists $v_j > 0$.

We also have $\widehat{\mathrm{HFL}}(\mathcal{L}, \boldsymbol{v}) = 0$ if there exists $v_j > 0$ by the spectral sequence in Proposition 2.22. By the symmetric property [14], $\widehat{\mathrm{HFL}}(\mathcal{L}, -\boldsymbol{v}) = \widehat{\mathrm{HFL}}(\mathcal{L}, \boldsymbol{v}) = 0$. Hence, $\widehat{\mathrm{HFL}}(\mathcal{L}, \boldsymbol{v}) = 0$ if $\boldsymbol{v} \neq \boldsymbol{0}$. If $\mathcal{L}$ has no trivial component (an unknotted component which is also unlinked from the rest of the link), the dual Thurston polytope of $\mathcal{L}$ is a point at the origin [16, Theorem 1.1]. Then the link $\mathcal{L}$ bounds disjoint disks in S^3, and $\mathcal{L}$ is an unlink. Otherwise, the split unknotted components bound disjoint disks and we apply the same argument to the rest of the link components. Then $\mathcal{L}$ bounds disjoint disks in S^3 and it is still an unlink. □

3.3 Lower bounds for the 4–genera

In this subsection, we use $\mathcal{L} \subset S^3$ to denote an n–component link with vanishing pairwise linking numbers. The inequality in Theorem 1.1 produces some lower bounds for the 4–genus of $\mathcal{L}$.

Corollary 3.14 *For the link $\mathcal{L}$,*

$$g_4(\mathcal{L}) \geq \min\{s_1 + \cdots + s_2 \mid h(\boldsymbol{x}) = 0 \text{ if } \boldsymbol{x} \succeq \boldsymbol{s} = (s_1, \dots, s_n)\}. \tag{3-2}$$

Proof This is straightforward from Corollary 3.12 □

Corollary 3.15 *For the link $\mathcal{L}$, $g_4(\mathcal{L}) \geq 2\max_{\boldsymbol{s} \in \mathbb{Z}^n} h_{\mathcal{L}}(\boldsymbol{s}) - n$. In particular,*

$$g_4(\mathcal{L}) \geq 2h_{\mathcal{L}}(\boldsymbol{0}) - n. \tag{3-3}$$

Proof By Theorem 1.1, for all $\boldsymbol{s} \in \mathbb{Z}^n$, $h_{\mathcal{L}}(\boldsymbol{s}) \leq \lceil \tfrac{1}{2} g_1 \rceil + \cdots + \lceil \tfrac{1}{2} g_n \rceil$. Observe that $\lceil \tfrac{1}{2} g_i \rceil \leq \tfrac{1}{2}(g_i + 1)$. Then

$$g_1 + \cdots + g_n + n \geq 2 \max_{\boldsymbol{s} \in \mathbb{Z}^n} h_{\mathcal{L}}(\boldsymbol{s}).$$

Hence $g_4(\mathcal{L}) \geq 2\max_{\boldsymbol{s} \in \mathbb{Z}^n} h_{\mathcal{L}}(\boldsymbol{s}) - n$. □

Corollary 3.16 *Let $g_4(L_i)$ denote the 4–genus of the link component L_i. Then*

$$g_4(\mathcal{L}) \geq 2h_{\mathcal{L}}(\boldsymbol{s}) - n + |s_1| + \cdots + |s_2|, \tag{3-4}$$

where $\boldsymbol{s} = (s_1, \ldots, s_n)$ and $|s_i| \leq g_4(L_i)$.

Proof Suppose that $\mathcal{L}$ bounds pairwise disjoint surfaces Σ_i in B^4 of genera g_i. Then $g_i \geq g_4(L_i)$ for all i. If $|s_i| \leq g_4(L_i)$, then, by Theorem 1.1,

$$h_{\mathcal{L}}(\boldsymbol{s}) \leq \sum_{i=1}^{n} \left\lceil \tfrac{1}{2}(g_i - |s_i|) \right\rceil.$$

Since $\lceil \frac{1}{2}(g_i - |s_i|) \rceil \leq \frac{1}{2}(g_i - |s_i| + 1)$, we have

$$g_1 + \cdots + g_n \geq 2h_{\mathcal{L}}(\boldsymbol{s}) - n + |s_1| + \cdots + |s_2|.$$

Hence, $g_4(\mathcal{L}) \geq 2h_{\mathcal{L}}(\boldsymbol{s}) - n + |s_1| + \cdots + |s_2|$. □

For the rest of the subsection, we prove that the analogues of Lemmas 2.4, 2.5 and 2.7 hold for the set $\mathfrak{G}(\mathcal{L})$. For an oriented link $\mathcal{L}$ with vanishing pairwise linking numbers, we use the *cancellation process* to find pairwise disjoint surfaces in B^4 bounded by $\mathcal{L}$. Let $\Sigma_i \subset S^3$ denote a Seifert surface bounded by L_i. Then Σ_i and Σ_j intersect transversely at an even number of points in B^4 since the linking number equals 0. We remove the tubular neighborhoods of a positive crossing and a negative crossing in Σ_i and obtain a new surface with two punctures. Add a tube along an arc in Σ_j which connects the two intersection points to the punctured surface where the attaching circles are boundaries of these two punctures, as in Figure 2. Then we obtain a new surface Σ_i' with fewer intersection points with Σ_j and higher genus compared with Σ_i. The tube can also be attached to the surface Σ_j along an arc connecting the intersection points in Σ_i. We repeat the process until we get pairwise disjoint surfaces in B^4 bounded by $\mathcal{L}$. We call the process of adding tubes to eliminate intersection points the *cancellation process*.

Lemma 3.17 *If $\boldsymbol{g} \in \mathfrak{G}(\mathcal{L})$ and $\boldsymbol{y} \succeq \boldsymbol{g}$, then $\boldsymbol{y} \in \mathfrak{G}(\mathcal{L})$. Equivalently, if $\boldsymbol{g} \notin \mathfrak{G}(\mathcal{L})$ and $\boldsymbol{y} \preceq \boldsymbol{g}$, then $\boldsymbol{y} \notin \mathfrak{G}(\mathcal{L})$.*

Proof If $\boldsymbol{g} = (g_1, \ldots, g_n) \in \mathfrak{G}(\mathcal{L})$, there exist pairwise disjoint surfaces Σ_i embedded in B^4 of genera g_i and $\partial \Sigma_i = L_i$. We can attach tubes to the surfaces Σ_i to increase the genera. Thus $\boldsymbol{y} \in \mathfrak{G}(\mathcal{L})$ if $\boldsymbol{y} \succeq \boldsymbol{g}$. □

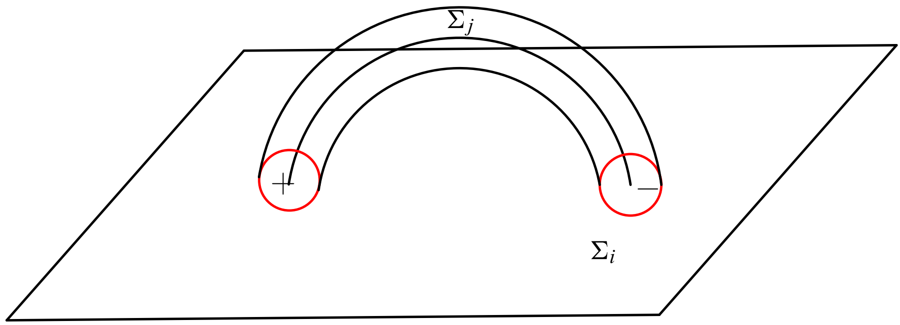

Figure 2: Cancellation process.

Lemma 3.18 *If $\boldsymbol{g}=(g_1,\dots,g_n)\in\mathfrak{G}(\mathcal{L})$, then $\boldsymbol{g}\setminus g_i\in\mathfrak{G}(\mathcal{L}\setminus L_i)$ for all $1\le i\le n$. Moreover, if $\boldsymbol{g}\setminus g_i\in\mathfrak{G}(\mathcal{L}\setminus L_i)$, then, for g_i sufficiently large, $\boldsymbol{g}=(g_1,\dots,g_n)\in\mathfrak{G}(\mathcal{L})$.*

Proof If $\boldsymbol{g}\in\mathfrak{G}(\mathcal{L})$, it is easy to obtain that $\boldsymbol{g}\setminus g_i\in\mathfrak{G}(\mathcal{L}\setminus L_i)$. Conversely, if $\boldsymbol{g}\setminus g_i\in\mathfrak{G}(\mathcal{L}\setminus L_i)$ for sufficiently large $g_i\gg 0$, we claim that $\boldsymbol{g}\in\mathfrak{G}(\mathcal{L})$. Suppose that $\mathcal{L}\setminus L_i$ bounds pairwise disjoint surfaces Σ_j in B^4. Let Σ_i in S^3 denote a Seifert surface bounded by L_i. Then Σ_i intersects with Σ_j transversely at an even number of points in B^4 since the linking number equals 0. By the cancellation process, we add tubes to Σ_i until the new surface is disjoint from all the surfaces Σ_j. Thus, for sufficiently large g_i, $\boldsymbol{g}\in\mathfrak{G}(\mathcal{L})$. □

Lemma 3.19 *The set $\mathfrak{G}(\mathcal{L})$ is determined by the set of maximal lattice points and $\mathfrak{G}(\mathcal{L}\setminus L_i)$ for all $1\le i\le n$.*

Proof The proof is similar to the one in Lemma 2.7 by using Lemmas 3.17 and 3.18. □

4 Examples

4.1 Examples

For L–space links, the H–function can be computed explicitly by the Alexander polynomials of the link and sublinks. The lower bound for 4–genus of the link in Section 3 can also be computed explicitly. In this section, we will show examples of L–space links where $\mathfrak{G}(\mathcal{L})=\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$.

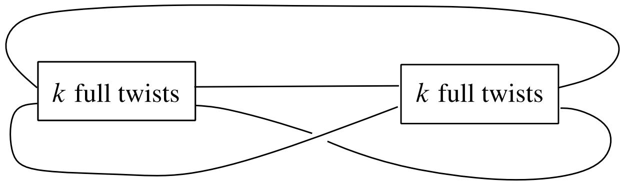

Figure 3: Two-bridge link $b(4k^2+4k,-2k-1)$.

Example 4.1 For k a positive integer, the two-bridge link $\mathcal{L}_k = b(4k^2+4k,-2k-1)$ is a 2–component L–space link with linking number 0 [7], and both link components are unknots; see Figure 3. The Alexander polynomial of $\mathcal{L}_k$ can be obtained with the help of a computer program [7, Section 6] (see also [6, Section 3]):

$$\Delta_{\mathcal{L}_k}(t_1,t_2) = (-1)^k \sum_{|i+1/2|+|j+1/2|\le k} (-1)^{i+j} t_1^{i+1/2} t_2^{j+1/2}.$$

The H–function of $\mathcal{L}_k$ is computed in [7, Proposition 6.12]. Note that $n^{+L_2}_{s_1,s_2}$ in [7] equals $H(s_1,s_2) - \frac{1}{2}(|s_1| - s_1)$ in our notation. Then the h–function of $\mathcal{L}_k$ is shown in Figure 4, where $h(k,0)=0$ and $h(k-1,0)=1$. For the shaded area bounded by

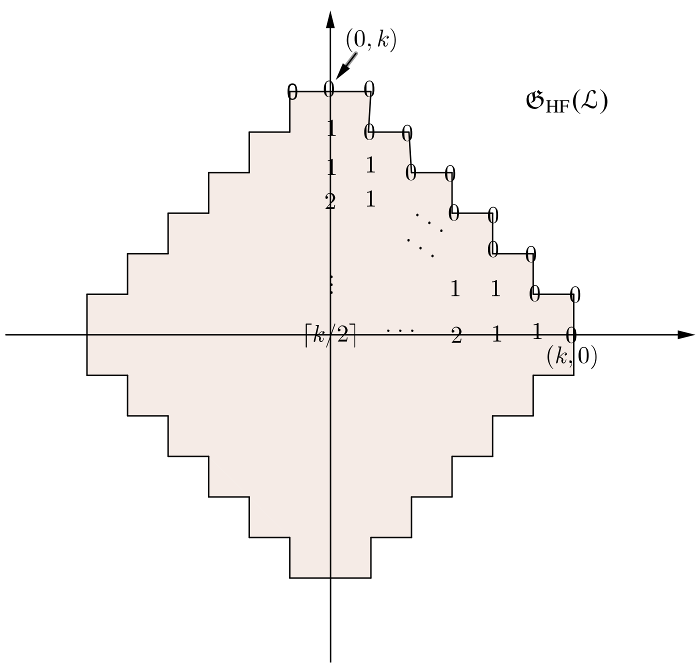

Figure 4: The h–function of $\mathcal{L}_k$.

the "stairs", the h–function is nonzero, and $h(\boldsymbol{s}) = 0$ for all lattice points $\boldsymbol{s}$ on the "stairs" and outside of the shaded area in Figure 4. Thus, $\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$ consists of all the lattice points on the "stairs" and outside the shaded area in the first quadrant. By the inequality (3-2),

$$g_4(\mathcal{L}_k) \geq \min\{s_1 + s_2 \mid h(\boldsymbol{x}) = 0 \text{ if } \boldsymbol{x} \succeq \boldsymbol{s} = (s_1, s_2)\} = k.$$

Observe that the components of $\mathcal{L}_k$ bound disks D_1 and D_2 in S^3. Push the disks into B^4. Then they intersect transversely at $2k$ points in B^4. By the cancellation process of crossings, we obtain disjoint surfaces Σ_1' and Σ_2' in B^4 bounded by the link components. Assume that the genus of Σ_1' is k and Σ_2' is still a disk of genus 0. Then $g_4(\mathcal{L}_k) \leq k$. Thus, $g_4(\mathcal{L}_k) = k$. We can add tubes to either D_1 or D_2 in the cancellation process. Thus, for all $\boldsymbol{g} = (g_1, g_2)$ with $g_1 + g_2 = k$, we find disjoint surfaces in B^4 of genera g_1 and g_2, respectively. Therefore, $\mathfrak{G}(\mathcal{L}_k) = \mathfrak{G}_{\mathrm{HF}}(\mathcal{L}_k)$.

Remark 4.2 For $k = 1$ we get the Whitehead link $\mathcal{L}_1$, and the 4–genus $g_4(\mathcal{L}_1)$ equals 1.

Example 4.3 The Borromean link $\mathcal{L} = L_1 \cup L_2 \cup L_3$ is a 3–component L–space link with vanishing pairwise linking numbers [7]. Its Alexander polynomial equals

$$\Delta_{\mathcal{L}}(t_1, t_2, t_3) = (t_1^{1/2} - t_1^{-1/2})(t_2^{1/2} - t_2^{-1/2})(t_3^{1/2} - t_3^{-1/2}).$$

By (2-5), $h_{\mathcal{L}}(\boldsymbol{v}) = 0$ if $\boldsymbol{v} \succ \boldsymbol{0}$, and $h_{\mathcal{L}}(\boldsymbol{0}) = 1$. Thus, $\mathfrak{G}_{\mathrm{HF}}(\mathcal{L}) = \{\boldsymbol{v} \in \mathbb{Z}^n \mid \boldsymbol{v} \succ \boldsymbol{0}\}$ and $g_4(\mathcal{L}) \geq 1$.

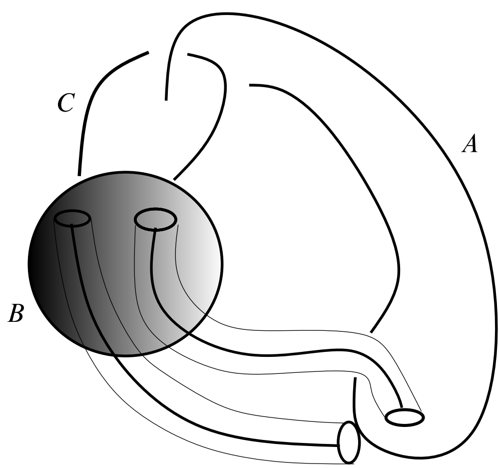

Figure 5: Borromean link.

We claim that $g_4(\mathcal{L})=1$. In Figure 5, link components A and C bound pairwise disjoint disks D_1 and D_3, respectively, in S^3. We push the disk D_1 in B^4. Note that the link component B bounds a disk D_2 in S^3 which is disjoint from D_1, but intersects with the disk D_3. After pushing D_1 and D_3 in B^4, these two disks intersect transversely at two points. By adding a tube to cancel these intersection points, we obtain three disjoint surfaces bounded by the Borromean link with genera 0, 1 and 0. Thus, $g_4(\mathcal{L})=1$, and $\mathfrak{G}(\mathcal{L})=\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$.

Example 4.4 The mirror of $L7a3$ is a 2–component L–space link $\mathcal{L}=L_1\cup L_2$ with linking number 0, where L_1 is the right-handed trefoil and L_2 is the unknot [7]. Its Alexander polynomial equals

$$\Delta_{\mathcal{L}}(t_1,t_2)=-(t_1^{1/2}-t_1^{-1/2})(t_2^{1/2}-t_2^{-1/2})(t_2+t_2^{-1}).$$

The h–function in the first quadrant is shown as in Figure 6 by (2-5) or the formula in [7]. Then the shaded area is $\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$ and $g_4(\mathcal{L})\geq 2$.

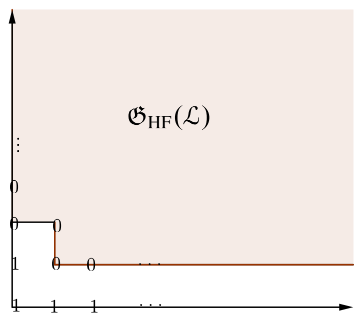

Figure 6: The h–function for the mirror of $L7a3$.

Observe that the right-handed trefoil and the unknot bound Seifert surfaces of genera 1 and 0, respectively, in S^3. They intersect transversely at two points after pushing them in B^4. By the cancellation process, we can obtain disjoint surfaces of genera $(2,0)$ or $(1,1)$ bounded by the link. Thus, $g_4(\mathcal{L})=2$, and $\mathfrak{G}(\mathcal{L})=\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$.

Example 4.5 Let $\mathcal{L}$ denote the disjoint union of two links $\mathcal{L}_1$ and $\mathcal{L}_2$. Then $\mathfrak{G}(\mathcal{L})=\mathfrak{G}(\mathcal{L}_1)\times\mathfrak{G}(\mathcal{L}_2)$ and $\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})=\mathfrak{G}_{\mathrm{HF}}(\mathcal{L}_1)\times\mathfrak{G}_{\mathrm{HF}}(\mathcal{L}_2)$.

Proof Suppose that $\mathcal{L}_1$ has n_1 components and $\mathcal{L}_2$ has n_2 components. If $\boldsymbol{g}_1\in\mathfrak{G}(\mathcal{L}_1)$ and $\boldsymbol{g}_2\in\mathfrak{G}(\mathcal{L}_2)$, then $(\boldsymbol{g}_1,\boldsymbol{g}_2)\in\mathfrak{G}(\mathcal{L})$, where $\mathcal{L}=\mathcal{L}_1\sqcup\mathcal{L}_2$. Conversely, if

$$\boldsymbol{g}=(g_1,\dots,g_{n_1},\dots,g_{n_1+n_2})\in\mathfrak{G}(\mathcal{L}),$$

it is straightforward to obtain that $(g_1,\dots,g_{n_1})\in\mathfrak{G}(\mathcal{L}_1)$ and $(g_{n_1+1},\dots,g_{n_1+n_2})\in\mathfrak{G}(\mathcal{L}_2)$. Thus, $\mathfrak{G}(\mathcal{L})=\mathfrak{G}(\mathcal{L}_1)\times\mathfrak{G}(\mathcal{L}_2)$.

For the set $\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$, we first prove that $H_{\mathcal{L}}(\boldsymbol{s})=H_{\mathcal{L}_1}(\boldsymbol{s}_1)+H_{\mathcal{L}_2}(\boldsymbol{s}_2)$, where $\boldsymbol{s}_1=(s_1,\dots,s_{n_1})$, $\boldsymbol{s}_2=(s_{n_1+1},\dots,s_{n_1+n_2})$ and $\boldsymbol{s}=(s_1,\dots,s_{n_1},\dots,s_{n_1+n_2})$. The argument is similar to the one in [1, Proposition 3.11]. For the link $\mathcal{L}$ by [15, Section 11], one has

$$A^-(\mathcal{L},\boldsymbol{s})\cong A^-(\mathcal{L}_1,\boldsymbol{s}_1)\otimes_{\mathbb{F}[U]}A^-(\mathcal{L}_2,\boldsymbol{s}_2),$$

and the isomorphism preserves the homological gradings. Since H–functions take nonnegative values, $H_{\mathcal{L}}(\boldsymbol{s})=0$ if and only if $H_{\mathcal{L}_1}(\boldsymbol{s}_1)=H_{\mathcal{L}_2}(\boldsymbol{s}_2)=0$. Thus, $\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})=\mathfrak{G}_{\mathrm{HF}}(\mathcal{L}_1)\times\mathfrak{G}_{\mathrm{HF}}(\mathcal{L}_2)$. □

4.2 Cables of L–space links

Let $\mathcal{L}=L_1\cup\cdots\cup L_n\subset S^3$ be an L–space link with vanishing pairwise linking numbers. Let p and q be coprime positive integers. The link $\mathcal{L}_{p,q}=L_{(p,q)}\cup L_2\cup\cdots\cup L_n$ is an L–space link if q/p is sufficiently large [1, Proposition 2.8]. Here $L_{p,q}$ denotes the (p,q)–cable on L_1. By induction, we can consider the links obtained by cabling any link components. In particular, we let $\mathcal{L}_{\mathrm{cab}}=L_{(p_1,q_1)}\cup\cdots\cup L_{(p_n,q_n)}$, where $L_{(p_i,q_i)}$ is the (p_i,q_i)–cable on L_i. If for all i, q_i/p_i is sufficiently large, then $\mathcal{L}_{\mathrm{cab}}$ is also an L–space link [1, Proposition 2.8].

Given coprime positive integers p and q, define the map $T\colon \mathbb{Z}^n_{\geq 0}\to\mathbb{Z}^n_{\geq 0}$ as

$$T(\boldsymbol{s})=\boldsymbol{p}\cdot\boldsymbol{s}+\bigl(\tfrac{1}{2}(p-1)(q-1),0,\dots,0\bigr),$$

where $\boldsymbol{p}=(p,1,\dots,1)$ and $\boldsymbol{p}\cdot\boldsymbol{s}=(ps_1,s_2,\dots,s_n)$.

Lemma 4.6 *Given coprime positive integers p and q, $\boldsymbol{s}\succeq\boldsymbol{s}'$ if and only if $T(\boldsymbol{s})\succeq T(\boldsymbol{s}')$.*

Proof The proof is straightforward. □

Theorem 4.7 *Let $\mathcal{L}=L_1\cup\cdots\cup L_n$ be an L–space link with vanishing pairwise linking numbers, and let $\mathcal{L}_{p,q}=L_{(p,q)}\cup L_2\cup\cdots\cup L_n$, where p and q are coprime positive integers with q/p sufficiently large and $L_{(p,q)}$ is the (p,q)–cable on L_1. Then*

$$\mathfrak{G}_{\mathrm{HF}}(\mathcal{L}_{p,q})=\{\boldsymbol{u}\in\mathbb{Z}^n_{\geq 0}\mid \boldsymbol{u}\succeq T(\boldsymbol{s})\text{ for some }\boldsymbol{s}\in\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})\}.$$

Proof The normalizations for knots and links with at least two components are different. We first prove the theorem in the case that $\mathcal{L}$ is a knot. Let $\Delta_{\mathcal{L}}(t)$ denote the symmetrized Alexander polynomial of $\mathcal{L}$. Then the symmetrized Alexander polynomial of the cable knot $\mathcal{L}_{p,q}$ is computed by Turaev in [20, Theorem 1.3.1]:

$$\Delta_{\mathcal{L}_{p,q}}(t) = \frac{\Delta_{\mathcal{L}}(t^p)(t^{1/2}-t^{-1/2})}{t^{p/2}-t^{-p/2}} \cdot \frac{t^{pq/2}-t^{-pq/2}}{t^{q/2}-t^{-q/2}}. \tag{4-1}$$

Here we are multiplying $\Delta_{\mathcal{L}}(t^p)$ by a Laurent polynomial of degree $T(\mathbf{0})$.

Recall that in Section 2.3 we normalize the polynomial $\widetilde{\Delta}_{\mathcal{L}}(t) = \Delta_{\mathcal{L}}(t)/(1-t^{-1})$ and regard $1/(1-t^{-1})$ as the power series $1+t^{-1}+t^{-2}+\cdots$. So the monomial with the highest degree in $\Delta_{\mathcal{L}}(t)$ is also the highest-degree term in $\widetilde{\Delta}_{\mathcal{L}}(t)$. Suppose that t^b is the highest-degree term in $\Delta_{\mathcal{L}}(t)$, where $b \geq 0$. We claim that $\mathfrak{G}_{\mathcal{L}} = [b,\infty)$. By equations (2-3) and (2-4), $\chi(\mathrm{HFL}^-(\mathcal{L},s)) = H(s-1)-H(s)$ equals 0 for all $s > b$ and equals 1 for $s=b$. Recall that $H(s)=0$ if s is sufficiently large by Theorem 2.17. Hence, $H(b-1)=1$ and $H(s)=0$ for all $s \geq b$, which proves the claim. Observe that the highest-degree term in $\Delta_{\mathcal{L}_{p,q}}(t)$ is $t^{bp+(p-1)(q-1)/2} = T(b)$. By a similar argument, we prove that $\mathfrak{G}_{\mathrm{HF}}(\mathcal{L}_{p,q}) = [T(b),\infty)$.

Now we consider the case that $\mathcal{L}$ has at least two components. Let $\Delta_{\mathcal{L}}(t_1,\ldots,t_n)$ denote the symmetrized Alexander polynomial of $\mathcal{L}$. Then the Alexander polynomial of the cable link $\mathcal{L}_{p,q}$ is computed by Turaev in [20, Theorem 1.3.1]:

$$\Delta_{\mathcal{L}_{p,q}}(t_1,\ldots,t_n) = \Delta_{\mathcal{L}}(t_1^p,t_2,\ldots,t_n)\frac{t_1^{pq/2}-t_1^{-pq/2}}{t_1^{q/2}-t_1^{-q/2}}. \tag{4-2}$$

Then

$$\widetilde{\Delta}_{\mathcal{L}_{p,q}}(t_1,\ldots,t_n) = t_1^{1/2-p/2}\widetilde{\Delta}_{\mathcal{L}}(t_1^p,\ldots,t_n)\frac{t_1^{pq/2}-t_1^{-pq/2}}{t_1^{q/2}-t_1^{-q/2}}. \tag{4-3}$$

Here $(t_1^{pq/2}-t_1^{-pq/2})/(t_1^{q/2}-t_1^{-q/2})$ is a Laurent polynomial of degree $\frac{1}{2}pq-\frac{1}{2}q$.

Observe that $T(\boldsymbol{s}) = \boldsymbol{p}\cdot\boldsymbol{s} + \bigl(\frac{1}{2}-\frac{1}{2}p,\ldots,0\bigr) + \bigl(\frac{1}{2}(pq-q),\ldots,0\bigr)$ for any $\boldsymbol{s} \in \mathbb{Z}^n$. We claim that the coefficients of $t_1^{y_1}\cdots t_n^{y_n}$ in $\widetilde{\Delta}_{\mathcal{L}}(t_1,\ldots,t_n)$ are 0 for all $\boldsymbol{y} \succ \boldsymbol{s}$ if and only if for all $\boldsymbol{y}' \succ T(\boldsymbol{s})$, the coefficients of $t_1^{y_1'}\cdots t_n^{y_n'}$ are 0 in $\widetilde{\Delta}_{\mathcal{L}_{p,q}}(t_1,\ldots,t_n)$. The "only if " part is straightforward by observing that every monomial $t_1^{y_1}\cdots t_n^{y_n}$ in $\widetilde{\Delta}_{\mathcal{L}}(t_1,\ldots,t_n)$ can contribute only to monomials of degree less than or equal to $T(\boldsymbol{y})$ in $\widetilde{\Delta}_{\mathcal{L}_{p,q}}(t_1,\ldots,t_n)$. For the "if" part, we assume that for all $\boldsymbol{y}' \succ T(\boldsymbol{s})$, the coefficients of $t_1^{y_1'}\cdots t_n^{y_n'}$ are 0 in $\widetilde{\Delta}_{\mathcal{L}_{p,q}}(t_1,\ldots,t_n)$. Suppose there exists $\boldsymbol{y} \succ \boldsymbol{s}$ such that the coefficient of $t_1^{y_1}\cdots t_n^{y_n}$ is nonzero. Then there exists a maximal lattice point $\boldsymbol{y}''\succeq\boldsymbol{y}$ with associated nonzero coefficient. By Lemma 4.6, $T(\boldsymbol{y}'')\succ T(\boldsymbol{s})$, and the coefficient corresponding to $T(\boldsymbol{y}'')$ in $\widetilde{\Delta}_{\mathcal{L}_{p,q}}(t_1,\dots,t_n)$ is nonzero by (4-3), which contradicts our assumption. This proves the claim.

For all subsets $B\subset\{1,\dots,n\}$, the similar statement holds for the Alexander polynomials $\widetilde{\Delta}_{\mathcal{L}\setminus L_B}(t_{i_1},\dots,t_{i_k})$ and $\widetilde{\Delta}_{\mathcal{L}_{p,q}\setminus(\mathcal{L}_{p,q})_B}(t_{i_1},\dots,t_{i_k})$. By Lemma 2.19, $\boldsymbol{y}'\in\mathfrak{G}_{\mathrm{HF}}(\mathcal{L}_{p,q})$ if $\boldsymbol{y}'\succeq T(\boldsymbol{s})$ for some $\boldsymbol{s}\in\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$. Thus,

$$\mathfrak{G}_{\mathrm{HF}}(\mathcal{L}_{p,q})\supset\{\boldsymbol{u}\in\mathbb{Z}^n\mid \boldsymbol{u}\succeq T(\boldsymbol{s})\text{ for some }\boldsymbol{s}\in\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})\}.$$

Conversely, suppose $\boldsymbol{y}'\in\mathfrak{G}_{\mathrm{HF}}(\mathcal{L}_{p,q})$. If $\boldsymbol{y}'=T(\boldsymbol{s})$ for some $\boldsymbol{s}\in\mathbb{Z}^n$, by Lemma 2.19 and the claim, $\boldsymbol{s}\in\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$. If $\boldsymbol{y}'$ is not in the image of T, then there exists $\boldsymbol{s}\in\mathbb{Z}^n$ such that $\boldsymbol{y}'\succ T(\boldsymbol{s})$ and $\boldsymbol{y}'\prec T(\boldsymbol{y})$ for all $\boldsymbol{y}\succ\boldsymbol{s}$. We claim that $\boldsymbol{s}\in\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$. If there exists $\boldsymbol{y}\succ\boldsymbol{s}$ such that the coefficient corresponding to $\boldsymbol{y}$ in $\widetilde{\Delta}_{\mathcal{L}}(t_1,\dots,t_n)$ is not 0, then there exists a maximal lattice point $\boldsymbol{y}''\succeq\boldsymbol{y}$ with associated nonzero coefficient. So the coefficient corresponding to $T(\boldsymbol{y}'')$ in $\widetilde{\Delta}_{\mathcal{L}_{p,q}}(t_1,\dots,t_n)$ is also not 0, which contradicts our assumption. Similarly, we prove that for all subsets $B\subset\{1,\dots,n\}$ and all $\boldsymbol{y}\succ\boldsymbol{s}$, the coefficients corresponding to $\boldsymbol{y}\setminus y_B$ in $\widetilde{\Delta}_{\mathcal{L}\setminus L_B}(t_{i_1},\dots,t_{i_k})$ are all 0. By Lemma 2.19, $\boldsymbol{s}\in\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$. Thus, $\mathfrak{G}_{\mathrm{HF}}(\mathcal{L}_{p,q})=\{\boldsymbol{u}\in\mathbb{Z}^n\mid\boldsymbol{u}\succeq T(\boldsymbol{s})\text{ for some }\boldsymbol{s}\in\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})\}$. □

Lemma 4.8 *For such cable links $\mathcal{L}_{p,q}$, $\mathfrak{G}(\mathcal{L}_{p,q})\supset\{\boldsymbol{u}\in\mathbb{Z}^n\mid\boldsymbol{u}\succeq T(\boldsymbol{g})$ for some $\boldsymbol{g}\in\mathfrak{G}(\mathcal{L})\}$.*

Proof Suppose that the link components in $\mathcal{L}$ bound pairwise disjoint surfaces Σ_i in B^4 of genera g_i. The cable knot $L_{(p,q)}$ bounds a surface of genus $pg_1+\frac{1}{2}(p-1)(q-1)$: We start with p copies of Σ_1 and use $(p-1)q$ half-twisted bands to connect them. Since Σ_i are pairwise disjoint, the new surfaces are also pairwise disjoint. □

Proof of Proposition 1.4 Let $\mathfrak{G}'=\{\boldsymbol{u}\in\mathbb{Z}^n\mid\boldsymbol{u}\succeq T(\boldsymbol{g})\text{ for some }\boldsymbol{g}\in\mathfrak{G}(\mathcal{L})\}$. By assumption, $\mathfrak{G}(\mathcal{L})=\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$. Then $\mathfrak{G}'=\mathfrak{G}_{\mathrm{HF}}(\mathcal{L}_{p,q})$ by Theorem 4.7. Since $\mathfrak{G}_{\mathrm{HF}}(\mathcal{L}_{p,q})\supset\mathfrak{G}(\mathcal{L}_{p,q})\supset\mathfrak{G}'$ by Lemma 4.8, we have $\mathfrak{G}_{\mathrm{HF}}(\mathcal{L}_{p,q})=\mathfrak{G}(\mathcal{L}_{p,q})$. □

Remark 4.9 By induction, Proposition 1.4 also holds if we replace some link components in $\mathcal{L}$ by their cables. We need to choose an appropriate transformation map T correspondingly.

By Proposition 1.4, we can apply cables on all L–space links in the examples of Section 4.1.

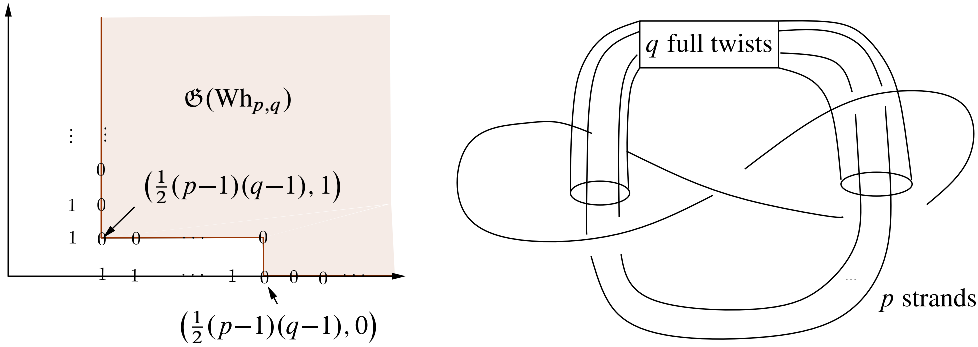

Figure 7: Left: the h–function of $\mathrm{Wh}_{p,q}$. Right: disjoint surfaces.

Example 4.10 (cables on the Whitehead link) Let $\mathrm{Wh}_{p,q}$ denote the link consisting of the (p,q)–cable on one component of the Whitehead link and the unchanged second component. The linking number is 0, and $\mathrm{Wh}_{p,q}$ is an L–space link if p and q are coprime with $q/p \geq 3$ [1].

By Theorem 2.17, one can compute the h–function of the Whitehead link $\mathcal{L}$ in the first quadrant: $h_{\mathcal{L}}(\boldsymbol{s}) = 0$ for all $\boldsymbol{s} \succ \boldsymbol{0}$ and $h_{\mathcal{L}}(\boldsymbol{0}) = 1$. By (3-2), $g_4(\mathcal{L}) \geq 1$. It is not hard to find disjoint surfaces bounded by the Whitehead link in B^4 with genera 0 and 1, respectively. Hence $g_4(\mathcal{L}) = 1$ and $\mathfrak{G}_{\mathrm{HF}}(\mathcal{L}) = \mathfrak{G}(\mathcal{L})$. By Theorem 4.7, $\mathfrak{G}_{\mathrm{HF}}(\mathrm{Wh}_{p,q})$ can be obtained from $\mathfrak{G}_{\mathrm{HF}}(\mathcal{L})$ by applying the transformation T, which is shown in Figure 7 (left). By Proposition 1.4, $\mathfrak{G}_{\mathrm{HF}}(\mathrm{Wh}_{p,q}) = \mathfrak{G}(\mathrm{Wh}_{p,q})$. Thus $g_4(\mathrm{Wh}_{p,q}) = g_1 + g_2 = \frac{1}{2}(p-1)(q-1)+1$. The link $\mathrm{Wh}_{p,q}$ bounds disjoint surfaces of genera $\frac{1}{2}(p-1)(q-1)$ and 1 as in Figure 7 (right). Note that the unknot component bounds a disk and $T_{p,q}$ bounds a surface with genus $\frac{1}{2}(p-1)(q-1)$ in S^3. These two surfaces intersect transversely at $2p$ intersection points in B^4. We add a tube to the disk which contains all the p strands to construct two disjoint surfaces bounded by $\mathrm{Wh}_{p,q}$ with genera 1 and $\frac{1}{2}(p-1)(q-1)$, respectively. We can also add p tubes to the Seifert surface bounded by $T_{p,q}$ to cancel each pair of intersection points of a strand and the disk. Then we obtain two disjoint surfaces with genera 0 and $\frac{1}{2}(p-1)(q-1)+p$, which corresponds to the point $\left(\frac{1}{2}(p-1)(q-1)+p, 0\right)$ in Figure 7 (left). Hence, we have realized all points in $\mathfrak{G}(\mathrm{Wh}_{p,q})$.

Example 4.11 For the 2–bridge link $\mathcal{L}_k = b(4k^2+4k, -2k-1) = L_1 \cup L_2$, consider the cable link $\mathcal{L}_{\mathrm{cab}} = L_{(p_1,q_1)} \cup L_{(p_2,q_2)}$, where p_i and q_i are coprime positive integers with q_i/p_i sufficiently large. In Example 4.1, we compute the h–function of $\mathcal{L}_k$ in the first quadrant and proved $\mathfrak{G}(\mathcal{L}_k) = \mathfrak{G}_{\mathrm{HF}}(\mathcal{L}_k)$. By Theorem 4.7 and the induction,

$\mathfrak{G}_{\mathrm{HF}}(\mathcal{L}_{\mathrm{cab}})$ can be obtained from $\mathfrak{G}_{\mathrm{HF}}(\mathcal{L}_k)$ by applying the appropriate transformation T, which is shown in Figure 1. By Proposition 1.4, $\mathfrak{G}_{\mathrm{HF}}(\mathcal{L}_{\mathrm{cab}}) = \mathfrak{G}(\mathcal{L}_{\mathrm{cab}})$. In Figure 1, all the horizontal segments in the "stair" have length p_1, vertical segments have length p_2 and there are k steps.

References

[1] **M Borodzik**, **E Gorsky**, *Immersed concordances of links and Heegaard Floer homology*, Indiana Univ. Math. J. 67 (2018) 1039–1083 MR

[2] **E Gorsky**, **J Hom**, *Cable links and L–space surgeries*, Quantum Topol. 8 (2017) 629–666 MR

[3] **E Gorsky**, **A Némethi**, *Lattice and Heegaard Floer homologies of algebraic links*, Int. Math. Res. Not. 2015 (2015) 12737–12780 MR

[4] **A A Kosinski**, *Differential manifolds*, Pure and Applied Mathematics 138, Academic, Boston, MA (1993) MR

[5] **A S Levine**, **D Ruberman**, *Generalized Heegaard Floer correction terms*, from "Proceedings of the Gökova Geometry–Topology Conference 2013" (S Akbulut, D Auroux, T Önder, editors), GGT, Gökova (2014) 76–96 MR

[6] **Y Liu**, *Heegaard Floer homology of surgeries on two-bridge links*, preprint (2014) arXiv

[7] **Y Liu**, *L–space surgeries on links*, Quantum Topol. 8 (2017) 505–570 MR

[8] **C Livingston**, *Knot 4–genus and the rank of classes in $W(\mathbb{Q}(t))$*, Pacific J. Math. 252 (2011) 113–126 MR

[9] **C Manolescu**, **P S Ozsváth**, *Heegaard Floer homology and integer surgeries on links*, preprint (2010) arXiv

[10] **K Murasugi**, *On a certain numerical invariant of link types*, Trans. Amer. Math. Soc. 117 (1965) 387–422 MR

[11] **P Ozsváth**, **Z Szabó**, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math. 173 (2003) 179–261 MR

[12] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. 159 (2004) 1159–1245 MR

[13] **P Ozsváth**, **Z Szabó**, *On knot Floer homology and lens space surgeries*, Topology 44 (2005) 1281–1300 MR

[14] **P Ozsváth**, **Z Szabó**, *Heegaard diagrams and Floer homology*, from "International Congress of Mathematicians" (M Sanz-Solé, J Soria, J L Varona, J Verdera, editors), volume II, Eur. Math. Soc., Zürich (2006) 1083–1099 MR

[15] **P Ozsváth**, **Z Szabó**, *Holomorphic disks, link invariants and the multi-variable Alexander polynomial*, Algebr. Geom. Topol. 8 (2008) 615–692 MR

[16] **P Ozsváth**, **Z Szabó**, *Link Floer homology and the Thurston norm*, J. Amer. Math. Soc. 21 (2008) 671–709 MR

[17] **M Powell**, *The four-genus of a link, Levine–Tristram signatures and satellites*, J. Knot Theory Ramifications 26 (2017) art. id. 1740008 MR

[18] **J A Rasmussen**, *Floer homology and knot complements*, PhD thesis, Harvard University (2003) MR arXiv

[19] **J Rasmussen**, *Lens space surgeries and a conjecture of Goda and Teragaito*, Geom. Topol. 8 (2004) 1013–1031 MR

[20] **V G Turaev**, *Reidemeister torsion in knot theory*, Uspekhi Mat. Nauk 41 (1986) 97–147 MR In Russian; translated in Russian Math. Surveys 41 (1986) 119–182

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