Algebraic & Geometric Topology 19 (2019) 3601–3614

An upper bound on the LS category in presence of the fundamental group

ALEXANDER DRANISHNIKOV

We prove that
$$\mathrm{cat}_{\mathrm{LS}} X \le \tfrac{1}{2}\bigl(\mathrm{cd}(\pi_1(X)) + \dim X\bigr)$$
for every CW complex X, where $\mathrm{cd}(\pi_1(X))$ denotes the cohomological dimension of the fundamental group of X. We obtain this as a corollary of the inequality
$$\mathrm{cat}_{\mathrm{LS}} X \le \tfrac{1}{2}(\mathrm{cat}_{\mathrm{LS}}(u_X) + \dim X),$$
where $u_X\colon X \to B\pi_1(X)$ is a classifying map for the universal covering of X.

55M30

1 Introduction

The reduced *Lusternik–Schnirelmann category* (briefly LS category) $\mathrm{cat}_{\mathrm{LS}} X$ of a topological space X is the minimal number n such that there is an open cover $\{U_0, \dots, U_n\}$ of X by $n+1$ contractible sets in X. We note that the LS category is a homotopy invariant. The Lusternik–Schnirelmann category has many applications. Perhaps the most famous is the classical Lusternik–Schnirelmann theorem — see Cornea, Lupton, Oprea and Tanré [5] — which states that $\mathrm{cat}_{\mathrm{LS}} M$ gives a low bound for the number of critical points on a manifold M of any smooth not necessarily Morse function. This theorem was used by Lusternik and Schnirelmann in their solution of Poincaré's problem on the existence of three closed geodesics on a 2–sphere [14]. In modern time the LS category was used in the proof of the Arnold conjecture on symplectomorphisms; see Rudyak [17].

The LS category is a numerical homotopy invariant which is difficult to compute. Even to get a reasonable bound for $\mathrm{cat}_{\mathrm{LS}}$ very often is a serious problem. In this paper we discuss only upper bounds. For nice spaces, such as CW complexes, it is an easy observation that $\mathrm{cat}_{\mathrm{LS}} X \le \dim X$. In the 1940s Grossman [11] (and independently in the 1950s G W Whitehead [19]; see [5]) proved that, for simply connected CW complexes, $\mathrm{cat}_{\mathrm{LS}} X \le \frac{1}{2}\dim X$.

Published: 17 December 2019 DOI: 10.2140/agt.2019.19.3601

In the presence of the fundamental group the LS category can be equal to the dimension. In fact, $\mathrm{cat}_{\mathrm{LS}} X = \dim X$ if and only if X is essential in the sense of Gromov. This was proven for manifolds by Dranishnikov, Katz and Rudyak [13]. For general CW complexes we refer to Proposition 2.6 of this paper. We recall that an n–dimensional complex X is called *inessential* if a map $u_X\colon X \to B\pi_1(X)$ that classifies its universal cover can be deformed to the $(n-1)$–skeleton $(B\pi_1(X))^{(n-1)}$. Otherwise, it is called *essential*. Typical examples of essential CW complexes are aspherical manifolds.

Rudyak conjectured that in the case of a free fundamental group there should be a Grossman–Whitehead-type inequality, at least for closed manifolds. There were partial results towards Rudyak's conjecture by Dranishnikov, Katz and Rudyak [9] and Strom [18], until it was settled in Dranishnikov [6]. Later it was shown in Dranishnikov [7] (also see the followup by Oprea and Strom [15]) that the Grossman–Whitehead-type estimate holds for complexes with the fundamental group having small cohomological dimension. Namely, it was shown that $\mathrm{cat}_{\mathrm{LS}} X \le \mathrm{cd}(\pi_1(X)) + \frac{1}{2}\dim X$. Clearly, this upper bound is far from being optimal for fundamental groups with sufficiently large cohomological dimension. Indeed, for the product of an aspherical m–manifold M with the complex projective space we have $\mathrm{cat}_{\mathrm{LS}}(M \times \mathbb{C}P^n) = m + n$ but our upper bound is $m + \frac{1}{2}(m+2n) = \frac{3}{2}m + n$. Moreover, our bound fails to be useful for complexes with $\mathrm{cd}(\pi_1(X)) \ge \frac{1}{2}\dim X$. The desirable bound here is

$$\mathrm{cat}_{\mathrm{LS}} X \le \tfrac{1}{2}(\mathrm{cd}(\pi_1(X)) + \dim X).$$

Such an upper bound was proven in [9] for the systolic category, a differential geometry relative of the LS category. Nevertheless, for the classical LS category a similar estimate was missing until now.

In this paper we prove the desirable upper bound. We obtain such a bound as a corollary of the inequality

$$\mathrm{cat}_{\mathrm{LS}} X \le \tfrac{1}{2}(\mathrm{cat}_{\mathrm{LS}}(u_X) + \dim X),$$

where $u_X\colon X \to B\pi_1(X)$ is a classifying map for the universal covering of X. We note that this inequality gives a meaningful upper bound on the LS category for complexes with any fundamental group. Also we note that the new upper bound gives the optimal estimate for the above example $M \times \mathbb{C}P^n$, the product of an aspherical manifold and the complex projective space. Namely,

$$\mathrm{cat}_{\mathrm{LS}}(M \times \mathbb{C}P^n) \le \tfrac{1}{2}(m + (m+2n)) = m + n.$$

The author is thankful to the referee for valuable remarks.

2 Preliminaries

The proof of the new upper bound for $\mathrm{cat}_{\mathrm{LS}} X$ is based on a further modification of the Kolmogorov–Ostrand multiple cover technique [6]. That technique was extracted by Ostrand from the work of Kolmogorov on the 13^{th} Hilbert problem [16]. Also in this paper we make use of the following well-known fact:

Proposition 2.1 *Let $f\colon X \to Y$ be a homotopy domination. Then $\mathrm{cat}_{\mathrm{LS}} Y \le \mathrm{cat}_{\mathrm{LS}} X$.*

Proof Let $s\colon Y \to X$ be a left homotopy inverse to f, ie $f \circ s \sim 1_Y$. Let $U_0, \dots, U_k$ be an open cover of X by sets contractible in X. One can easily check that $s^{-1}(U_0), \dots, s^{-1}(U_k)$ is an open cover by sets contractible in Y. □

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a family of sets in a topological space X. The *multiplicity* of $\mathcal{U}$ (or the *order*) at a point $x \in X$, denoted by $\mathrm{Ord}_x \mathcal{U}$, is the number of elements of $\mathcal{U}$ that contain x. A family $\mathcal{U}$ is a cover of X if $\mathrm{Ord}_x \mathcal{U} \neq 0$ for all x.

Definition 2.2 A family $\mathcal{U}$ of subsets of X is called a *k–cover*, with $k \in \mathbb{N}$, if every subfamily of $\mathcal{U}$ that consists of k sets forms a cover of X.

The following is obvious (see [6]):

Proposition 2.3 *A family $\mathcal{U}$ that consists of m subsets of X is an $(n+1)$–cover of X if and only if $\mathrm{Ord}_x \mathcal{U} \ge m - n$ for all $x \in X$.*

Let K be a simplicial complex. By definition, the dual to the m–skeleton $K^{(m)}$ is a subcomplex $L = L(K, m)$ of the barycentric subdivision βK that consists of simplices of βK which do not intersect $K^{(m)}$. Note that βK is naturally embedded in the join product $K^{(n)} * L$. Then the following is obvious:

Proposition 2.4 *For any n–dimensional complex K the complement $K \setminus K^{(m)}$ to the m–skeleton is homotopy equivalent to an $(n-m-1)$–dimensional complex L.*

Proof The complex L is the dual to $K^{(m)}$. Clearly, $\dim L = n - m - 1$. The complement $K \setminus K^{(m)}$ can be deformed to L along the field of intervals defined by the embedding $\beta K \subset K^{(n)} * L$. □

Let $f\colon X \to Y$ be a continuous map. We recall that the LS category of f, $\operatorname{cat}_{\mathrm{LS}} f$, is the smallest number k such that X can be covered by $k+1$ open sets $U_0, \dots, U_k$ so that the restriction $f|_{U_i}\colon U_i \to Y$ of f to each of them is null-homotopic. Clearly,

$$\operatorname{cat}_{\mathrm{LS}} f \le \operatorname{cat}_{\mathrm{LS}} X, \operatorname{cat}_{\mathrm{LS}} Y.$$

We denote by $u_X\colon X \to B\pi$, $\pi = \pi_1(X)$, a map that classifies the universal covering $p\colon \widetilde{X} \to X$ of X. Thus, p is the pullback of the universal covering $q\colon E\pi \to B\pi$. Here $B\pi$ is any aspherical CW complex with the fundamental group π. Thus, any map $u\colon X \to B\pi$ that induces an isomorphism of the fundamental groups is a classifying map.

The following proposition is proven in [8, Proposition 4.3]:

Proposition 2.5 *A classifying map $u_X\colon X \to B\pi$ of the universal covering of a CW complex X can be deformed into the d–skeleton $B\pi^{(d)}$ if and only if $\operatorname{cat}_{\mathrm{LS}}(u_X) \le d$.*

The following proposition for closed manifolds was proven by Katz and Rudyak [13], although it was already known to Berstein in a different equivalent formulation [1].

Proposition 2.6 *For an n–dimensional CW complex X, $\operatorname{cat}_{\mathrm{LS}} X = n$ if and only if X is essential.*

Proof Suppose that X is essential. By Proposition 2.5 we obtain that $\operatorname{cat}_{\mathrm{LS}}(u_X) > n-1$. Thus, $\operatorname{cat}_{\mathrm{LS}} X \ge \operatorname{cat}_{\mathrm{LS}}(u_X) \ge n$ and, since $\dim X = n$, $\operatorname{cat}_{\mathrm{LS}} X = n$.

The implication in the other direction can be derived from the proof of Theorem 4.4 in [8]. Here we give the sketch of the proof. Let $u_X\colon X \to B\pi^{(n-1)}$ be a classifying map. To prove the inequality $\operatorname{cat}_{\mathrm{LS}} X \le n-1$ it suffices to show that the Ganea–Schwarz fibration $p^X_{n-1}\colon G_{n-1}(X) \to X$ admits a section. Since the fiber of the Ganea–Schwarz fibration $p^{B\pi}_{n-1}$ is $(n-2)$–connected, it admits a section over $B\pi^{(n-1)}$ and, hence, the map u_X admits a lift $f\colon X \to G_{n-1}(B\pi)$. Then the map p' in the pullback diagram

$$\begin{tikzcd}
G_{n-1}(X) \arrow[r, "q"] & Z \arrow[r, "u'_X"] \arrow[d, "p'"'] & G_{n-1}(B\pi) \arrow[d, "p^{B\pi}_{n-1}"'] \\
& X \arrow[r, "u_X"] & B\pi
\end{tikzcd}$$

admits a section $s\colon X \to Z$. Here $p^X_{n-1} = p' \circ q$. Since X is n–dimensional, to show that s has a lift with respect to q it suffices to prove that the homotopy fiber F of the map q is $(n-1)$–connected. Note that the homotopy exact sequence of the fibration

$$F \to (p^X_{n-1})^{-1}(x_0) \xrightarrow{u'} (p^{B\pi}_{n-1})^{-1}(y_0),$$

where u' is the restriction of $u'_X \circ q$ to the fiber $(p^X_{n-1})^{-1}(x_0)$ coincides with the homotopy exact sequence of the fibration

$$F \to *_n\Omega(X) \xrightarrow{*\Omega(u_X)} *_n\Omega(B\pi)$$

obtained from the loop map $\Omega(u_X)$ turned into a fibration by taking the iterated join product. Since $\pi_0(\Omega u_X) = 0$, we obtain $\pi_i(*_n\Omega u_X) = 0$ for $i \le n$ (see [8, Proposition 2.4] or [10, Proposition 3.3]) and hence $\pi_i(F) = 0$ for $i \le n-1$. □

3 Multiple covers of polyhedra

For a point $x \in X$ in a CW complex X, by $d(x)$ we denote the dimension of the open cell e containing x. We call a subset $A \subset X$ in a CW complex X *r–deformable* if A can be deformed in X to the r–skeleton $X^{(r)}$. A deformation $H\colon A \times I \to X$ to the 0–skeleton $X^{(0)}$ is called *monotone* if $d(H(x,t))$ is a monotonically decreasing function of t for all $x \in A$.

Proposition 3.1 *Let X be a connected simplicial complex of dimension $\le (r+1)N-1$. Then for any $m \ge N$ there exists an open cover $\mathcal{U} = \{U_1, \dots, U_m\}$ of X by r–deformable sets such that $\mathrm{Ord}_x\,\mathcal{U} \ge m-k+1$ for every $k \le N$ and all $x \in X^{((r+1)k-1)}$. Equivalently, the restriction of $\mathcal{U}$ to the $((r+1)k-1)$–skeleton is a k–cover.*

Moreover, for $r = 0$ we may assume that each set U_i is monotone r–deformable.

Proof It suffices to prove the proposition for complexes with $\dim X = (r+1)N-1$. We do it by induction on n. For $N = 1$ the statement is obvious. Suppose that it holds true for $N-1 \ge 1$. We prove it for N by induction on m. First we establish the base of induction by proving the statement for $m = N$. By the external induction applied to $X^{((r+1)(N-1)-1)}$ with $m = N-1$ there is an open cover $\mathcal{U} = \{U_1, \dots, U_{N-1}\}$ of $X^{((r+1)(N-1)-1)}$ such that each U_i is r–deformable and $\mathrm{Ord}_x\,\mathcal{U} \ge (N-1)-k+1 = N-k$ for all $x \in X^{((r+1)k-1)}$. We can enlarge each U_i to an r–deformable open in X set $U'_i \subset X$.

Let $G = \bigcup_{i=1}^{N-1} U'_i$. Since the complement $X \setminus X^{((r+1)(N-1)-1)}$ is homotopy equivalent to an r–dimensional complex (see Proposition 2.4), $Z_0 = X \setminus G$ is r–deformable.

Since Z_0 is closed, we can find an open enlargement W_0 to an r–deformable set whose closure does not intersect $X^{((r+1)(N-1)-1)}$. Thus, the cover $\{U'_1,\dots,U'_{N-1},W_0\}$ satisfies the condition of the proposition for $k=N$.

Consider the set

$$Z_1=\{x\in X^{((r+1)(N-1)-1)}\mid \operatorname{Ord}_x\mathcal{U}=1\}.$$

Clearly, Z_1 is closed. By the induction assumption Z_1 does not intersect the skeleton $X^{((r+1)(N-2)-1)}$. Since the complement,

$$X^{((r+1)(N-1)-1)}\setminus X^{((r+1)(N-2)-1)}$$

is homotopy equivalent to an r–dimensional complex, Z_1 is r–deformable in the skeleton $X^{((r+1)(N-1)-1)}$. Let W_1 be an enlargement of Z_1 to an open r–deformable in X set such that the closure $\overline{W_1}$ does not intersect $\overline{W_0}\cup X^{((r+1)(N-2)-1)}$. Note that the cover $\{U'_1,\dots,U'_{N-1},W_0\cup W_1\}$ satisfies the condition of the proposition with $k=N$ and $k=N-1$.

Next we consider

$$Z_2=\{x\in X^{((r+1)(N-2)-1)}\mid \operatorname{Ord}_x\mathcal{U}=2\}$$

and similarly define an open set W_2 and so on up to W_{N-1}. By the construction each set W_i is r–deformable and the closures $\overline{W_i}$ are disjoint. Therefore, the union $U'_N=W_0\cup\dots\cup W_{N-1}$ is r–contractible. Then the cover $U'_0,\dots,U'_N$ satisfies all the conditions of the proposition for all $k\le N$.

The proof of the inductive step is very similar to the above. Assume that the statement of the proposition holds for N and $m-1\ge N$. We prove it for N and m. Let $\mathcal{U}=\{U_1,\dots,U_{m-1}\}$ be an open cover of X by r–deformable sets such that for any $k\le N$ the restriction of $\mathcal{U}$ to $X^{((r+1)k-1)}$ is a k–cover. Thus, $\operatorname{Ord}_x\mathcal{U}\ge(m-1)-N+1=m-N$ for all x. Let

$$Z_0=\{x\in X\mid \operatorname{Ord}_x\mathcal{U}=m-N\}.$$

By the induction assumption, $Z_0\cap X^{((r+1)(N-1)-1)}=\varnothing$. Thus, Z_0 is r–deformable in X. We consider an open r–deformable neighborhood W_0 of Z_0 for which $\overline{W_0}\cap X^{(r+1)(N-1)-1}=\varnothing$.

Next we consider the closed set

$$Z_1=\{x\in X^{((r+1)(N-1)-1)}\mid \operatorname{Ord}_x\mathcal{U}=m-N+1\}.$$

By the induction assumption, Z_1 does not intersect $X^{((r+1)(N-2)-1)}$. As above, we define an r–deformable set W_1 with

$$\overline{W}_1 \cap (\overline{W}_0 \cup X^{((r+1)(N-2)-1)}) = \varnothing$$

and so on. We define $U_m = W_0 \cup \cdots \cup W_{N-1}$. Then the condition of the proposition is satisfied for all k with $\mathcal{U}' = \{U_1, \dots, U_{m-1}, U_m\}$.

Now we revise our proof for $r = 0$ in order to verify the extra condition of the proposition. Note that $\dim X \le N-1$ in this case. In the proof of the base of induction on m the enlargements U_i' can be chosen monotone deformable to U_i. Hence, each U_i' is monotone 0–deformable. Since W_0 lives in the complement to the $(N-2)$–skeleton, it is monotone 0–deformable. The set W_1 can be chosen monotone deformable to the monotone 0–deformable set $W_1 \cap X^{(N-2)} \subset X^{(N-2)} \setminus X^{(N-3)}$. Thus, W_1 is monotone 0–deformable and so on. As the result we obtain that the set $U_N' = W_0 \cup \cdots \cup W_{N-1}$ is monotone 0–deformable. In the proof of inductive step the same argument shows that the set $U_m = W_0 \cup \cdots \cup W_{N-1}$ is monotone 0–deformable. □

3.1 Borel construction

Let a group π act on spaces X and E with the projections onto the orbit spaces $q_X\colon X \to X/\pi$ and $q_E\colon E \to E/\pi = B$. Let $q_{X\times E}\colon X \times E \to X \times_\pi E = (X \times E)/\pi$ denote the projection onto the orbit space of the diagonal action of π on $X \times E$. Then there is a commutative diagram, called the *Borel construction* [2],

$$\begin{tikzcd}
X \arrow[d, "q_X"'] & X \times E \arrow[l, "\mathrm{pr}_X"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, "q"'] & E \arrow[d, "q_E"'] \\
X/\pi & X \times_\pi E \arrow[l, "p_E"'] \arrow[r, "p_X"] & B
\end{tikzcd}$$

If π is discrete and the actions are free and proper, then all projections in the diagram are locally trivial bundles with the structure group π. Then the fiber of p_X is homeomorphic to X and the fiber of p_E is homeomorphic to E. For any invariant subset $Q \subset X$ the map p_X defines the pair of bundles $p_X\colon (X \times_\pi E, Q \times_\pi E) \to B$ with the stratified fiber (X, Q) and the structure group π.

If X/π and B are CW complexes for proper free actions of the discrete group π, their CW structures define a natural CW structure on $X \times_\pi E$ as follows: First, X and E, being covering spaces, inherit CW structures from X/π and B, respectively. Since the diagonal action of π on $X \times E$ preserves the product CW complex structure on $X \times E$ and takes cells to cells homeomorphically, the orbit space $X \times_\pi E$ receives the induced CW complex structure.

Lemma 3.2 *Let $\widetilde{X}$ be the universal covering of an n–dimensional simplicial complex X with fundamental group $\pi = \pi_1(X)$. Suppose that the universal covering admits a classifying map $u\colon X \to B$ to a d–dimensional simplicial complex, $\pi_1(B) = \pi$. Let E be the universal covering of B. Then, for the n–skeleton,*

$$\mathrm{cat}_{\mathrm{LS}}(\widetilde{X} \times_\pi E)^{(n)} \le \tfrac{1}{2}(d+n),$$

where the CW complex structure on $\widetilde{X} \times_\pi E$ is defined by the simplicial complex structures on X and B.

Proof Let $K = \widetilde{X} \times_\pi E$. Since $(\widetilde{X} \times E)^{(n)} = \bigcup_j \widetilde{X}^{(n-j)} \times E^{(j)}$, we have

$$K^{(n)} = \bigcup_{j=0}^{d} \widetilde{X}^{(n-j)} \times_\pi E^{(j)}.$$

We show that $\mathrm{cat}_{\mathrm{LS}} K^{(n)} \le d + \lfloor \frac{1}{2}(n-d) \rfloor = \lfloor \frac{1}{2}(d+n) \rfloor$.

Let $m = \lfloor \frac{1}{2}(d+n) \rfloor + 1$. We apply Proposition 3.1 to B with $r = 0$ to obtain an open cover $\mathcal{U} = \{U_1, \dots, U_m\}$ by monotone 0–deformable in B sets with $\mathrm{Ord}_x \mathcal{U} \ge m - j$ for $x \in B^{(j)}$. We note that we apply Proposition 3.1 here with $r = 0$ and $N = d+1$. Thus, we need to be sure that $m \ge d+1$, which is satisfied since $d \le n$. The substitution $i = k-1$ helps to see the inequality $\mathrm{Ord}_x \mathcal{U} \ge m - i$ for $x \in B^{(j)}$.

Since $m > \frac{1}{2}(d+n)$, we have $2m-1 > d+n-1$ and, hence, $2m-1 \ge n = \dim X$. Hence we can apply Proposition 3.1 with $N = m$ and $r = 1$ to get an open cover $\mathcal{V} = \{V_1, \dots, V_m\}$ of X by 1–deformable in X sets such that the restriction of $\mathcal{V}$ to $X^{(2j-1)}$ is a j–cover for $j = 1, \dots, k$, where k is the smallest integer satisfying the inequality $n \le 2k-1$.

For every $i \le m$ we define

$$W_i = p_E^{-1}(V_i) \cap p_{\widetilde{X}}^{-1}(U_i).$$

We claim that the collection of sets $\{W_1, \dots, W_m\}$ covers $K^{(n)}$. Let $x \in \widetilde{X}^{(n-j)} \times_\pi E^{(j)}$. Then the point $p_{\widetilde{X}}(x) \in B^{(j)}$ is covered by at least $m-j$ sets $U_{k_1}, \dots, U_{k_{m-j}} \in \mathcal{U}$. Since $\mathcal{V}$ restricted to $X^{(2(m-j)-1)}$ is an $(m-j)$–cover, the sets $V_{k_1}, \dots, V_{k_{m-j}}$ cover $X^{(2(m-j)-1)}$. Note that $2(m-j)-1 \ge d+n+2-2j-1 \ge n-j$. Therefore, the point $p_E(x) \in X^{(n-j)}$ is covered by V_{k_s} for some $s \in \{1, \dots, m-j\}$. Hence, $x \in W_{k_s}$.

We note that $W_i = Q_i \times_\pi P_i \subset \widetilde{X} \times_\pi E$, where $P_i = q_B^{-1}(U_i)$ and $Q_i = q_X^{-1}(V_i)$. Thus, its intersection with $K^{(n)}$ can be written as

$$W_i(n) = W_i \cap K^{(n)} = \bigcup_j^d Q_i(n-j) \times_\pi P_i(j),$$

where $P_i(k) = P_i \cap E^{(k)}$ and $Q_i(\ell) = Q_i \cap \widetilde{X}^{(\ell)}$.

To complete the proof we show that each set $W_i(n)$ is contractible in $K^{(n)}$. We consider a monotone deformation $h_t\colon U_i \to B$ of U_i to $B^{(0)}$. Let $\widetilde{h}_t\colon P_i \to E$ be the lifting of h_t. Thus, $\widetilde{h}_t$ is a π–equivariant deformation of P_i to $E^{(0)}$. Then $1_{\widetilde{X}} \times h_t\colon \widetilde{X} \times P_i \to \widetilde{X} \times E$ is a π–equivariant deformation and, hence, it defines a deformation of the orbit space $\bar{h}_t\colon \widetilde{X} \times_\pi P_i \to K$ which is a lift of h_t with respect to $p_{\widetilde{X}}$. Since each skeleton $\widetilde{X}^{(i)}$ is π–invariant, the deformation $\bar{h}_t$ preserves the filtration of the fibers $\widetilde{X}$ of the bundle $p_{\widetilde{X}}$ by the skeleta. For the same reason, $\bar{h}_t$ moves the set $Q_i(n-j) \times_\pi P_i$ within $Q_i(n-j) \times_\pi B$. Since h_t is monotone, $\bar{h}_t$ moves $Q_i(n-j) \times_\pi P^{(j)}$ within $Q_i(n-j) \times_\pi B^{(j)} \subset K^{(n)}$ for all j. Thus, $\bar{h}_t$ deforms $W_i(n)$ within $K^{(n)}$ to the set

$$Q_i \times_\pi E^{(0)} \subset \widetilde{X} \times_\pi E^{(0)} = p_{\widetilde{X}}^{-1}(B^{(0)}) \cong \coprod_{b \in B^{(0)}} \widetilde{X}.$$

Since V_i is 1–deformable in X, so is Q_i in $\widetilde{X}$. Since $\widetilde{X}$ is simply connected, Q_i is contractible in $\widetilde{X}$. Thus, we obtain that the set

$$Q_i \times_\pi E^{(0)} \cong \coprod_{b \in B^{(0)}} Q_i \subset \coprod_{b \in B^{(0)}} \widetilde{X}$$

is 0–deformable in $\widetilde{X} \times_\pi E^{(0)} \subset K^{(n)}$. Therefore, $W_i(n)$ is 0–deformable in $K^{(n)}$. Since K is connected, $W_i(n)$ is contractible in $K^{(n)}$.

Thus, $\operatorname{cat}_{\mathrm{LS}} K^{(n)} \le m-1 = \lfloor \frac{1}{2}(d+n) \rfloor \le \frac{1}{2}(d+n)$. □

4 Main result

Theorem 4.1 *For every simplicial complex X there is the inequality*

$$\operatorname{cat}_{\mathrm{LS}} X \le \tfrac{1}{2}(\operatorname{cat}_{\mathrm{LS}}(u_X) + \dim X),$$

where $u_X\colon X \to B\pi$ is a classifying map for the universal cover of X.

Proof Let $\dim X = n$ and $\mathrm{cat}_{\mathrm{LS}}(u_X) = d$. In the proof we use the notation $B = B\pi$, $B^d = B\pi^{(d)}$ and $E = E\pi$, $E^d = E\pi^{(d)}$. By Proposition 2.5 we may assume that the map u_X lands in B^d. Consider the diagram generated by the Borel construction,

$$\begin{tikzcd} X & \widetilde{X} \times_\pi E \arrow[l, "p_E"'] \arrow[r, "p_{\widetilde{X}}"] & B \\ X \arrow[u, equal] & \widetilde{X} \times_\pi E^d \arrow[l, "p_{E^d}"'] \arrow[r, "p_{\widetilde{X}}|"] \arrow[u, "\subset"] & B^d \arrow[u, "\subset"] \end{tikzcd}$$

Since E is contractible, the map p_E is a homotopy equivalence. Let g be its homotopy inverse. Applying the homotopy lifting property we may assume that g is a section of p_E. Then the map $p_{\widetilde{X}}$ is homotopic to $p_{\widetilde{X}} \circ g \circ p_E$. Note that the map $p_{\widetilde{X}} \circ g \colon X \to B$ is a classifying map for X. Thus, it is homotopic to the map $u_X \colon X \to B$, whose image is in B^d. Therefore, $p_{\widetilde{X}} \colon \widetilde{X} \times_\pi E \to B$ is homotopic to a map with image in B^d. Let $p_t \colon \widetilde{X} \times_\pi E \to B$ be such a homotopy. Thus, $p_0 = p_{\widetilde{X}}$ and $p_1(\widetilde{X} \times_\pi E) \subset B^d$. Let $\bar{p}_t \colon \widetilde{X} \times_\pi E \to \widetilde{X} \times_\pi E$ be the lift of p_t with $\bar{p}_0 = \mathrm{id}$. Then $\bar{p}_1(\widetilde{X} \times_\pi E) \subset \widetilde{X} \times_\pi E^d$.

First, we note that $s = \bar{p}_1 \circ g \colon X \to \widetilde{X} \times_\pi E^d$ is a homotopy section of p_{E^d}. Indeed, the homotopy $h_t = p_E \circ \bar{p}_t \circ g \colon X \to X$ joins $h_0 = p_E \circ \bar{p}_0 \circ g = p_E \circ g = 1_X$ with $h_1 = p_E \circ \bar{p}_1 \circ g = p_{E^d} \circ \bar{p}_1 \circ g = p_{E^d} \circ s$.

We may assume that B is a simplicial complex. Let $K = \widetilde{X} \times_\pi E^d$. We consider the CW complex structure on K defined by the simplicial complex structures on X and B. Next we show that the restriction $(p_{E^d})|_{K^{(n)}} \colon K^{(n)} \to X$ is a homotopy domination. Since $\dim X = n$, there is a homotopy $s_t \colon X \to K$ with $s_0 = s$ and $s_1(X) \subset K^{(n)}$. Then the homotopy $q_t = p_{E^d} \circ s_t \colon X \to X$ joins $q_0 = p_{E^d} \circ s \sim 1_X$ with $q_1 = p_{E^d} \circ s_1 = (p_{E^d})|_{K^{(n)}} \circ s_1$.

Therefore, by Proposition 2.1, $\mathrm{cat}_{\mathrm{LS}} X \le \mathrm{cat}_{\mathrm{LS}} K^{(n)}$. Lemma 3.2 implies

$$\mathrm{cat}_{\mathrm{LS}} X \le \tfrac{1}{2}(d + n). \qquad \square$$

Corollary 4.2 *For any CW complex X,*

$$\mathrm{cat}_{\mathrm{LS}} X \le \tfrac{1}{2}\big(\mathrm{cd}(\pi_1(X)) + \dim X\big).$$

Proof We note that every CW complex is homotopy equivalent to a simplicial complex of the same dimension. By the Eilenberg–Ganea theorem, $\pi = \pi_1(X)$ has a classifying complex $B\pi$ of dimension equal to $\mathrm{cd}(\pi)$ whenever $\mathrm{cd}(\pi) \neq 2$ (see [4]). Thus, if $\mathrm{cd}(\pi) \neq 2$, the result immediately follows from Theorem 4.1.

In the case when $\mathrm{cd}(\pi)=2$ one can find a classifying complex $B\pi$ of dimension three [4]. Then obstruction theory implies that there is a map $r\colon B\pi\to B\pi^{(2)}$ which is the identity on the 1–skeleton. It is easy to check that r induces an isomorphism of the fundamental groups: obviously it is surjective, and the kernel of $r_*\colon \pi_1(B)\to \pi_1(B\pi^{(2)})$ is trivial. In particular, its composition with a classifying map $r\circ u_X\colon X\to B\pi^{(2)}$ is a classifying map and we can apply Theorem 4.1 to it. $\square$

Theorem 4.3 *For any locally trivial bundle $p\colon E\to B$ with a simply connected fiber F and an aspherical base B,*

$$\mathrm{cat}_{\mathrm{LS}}\, E\le \dim B+\tfrac{1}{2}\dim F.$$

Proof By Corollary 4.2,

$$\begin{aligned}\mathrm{cat}_{\mathrm{LS}}\, E&\le \tfrac{1}{2}\big(\mathrm{cd}(\pi_1(E))+\dim E\big)=\tfrac{1}{2}\big(\mathrm{cd}(\pi_1(B))+\dim B+\dim F\big)\\&\le \tfrac{1}{2}(2\dim B+\dim F)=\dim B+\tfrac{1}{2}\dim F.\end{aligned}$$ $\square$

When B is an aspherical manifold we obtain an upper bound

$$\mathrm{cat}_{\mathrm{LS}}\, E\le \mathrm{cat}_{\mathrm{LS}}\, B+\tfrac{1}{2}\dim F.$$

Therefore, for every aspherical n–manifold M the LS category of the total manifold of an S^3–fibration $f\colon N\to M$ is at most $n+1$. For principal S^3–bundles the same estimate was obtained in [12]. For nonprincipal S^3–bundles the old upper bound was only $n+2$, just in view of the fact that N is inessential. A concrete example would be the total space N of the pullback of the nonprincipal S^3–bundle (we refer to [3] for the proof of nonprincipality) $h\colon \mathrm{SO}(5)\times_{\mathrm{SO}(4)}S^3\to S^4$ via an essential map of a 4–torus $g\colon T^4\to S^4$. I don't see how to get our estimate $\mathrm{cat}_{\mathrm{LS}}\, N\le 5$ by any other means.

In the case when additionally $\mathrm{cat}_{\mathrm{LS}}\, F=\frac{1}{2}\dim F$, like for $F=\mathbb{C}P^n$, we have a Hurewicz-type formula for $\mathrm{cat}_{\mathrm{LS}}$,

$$\mathrm{cat}_{\mathrm{LS}}\, E\le \mathrm{cat}_{\mathrm{LS}}\, B+\mathrm{cat}_{\mathrm{LS}}\, F.$$

We recall that for general fibrations the Hurewicz-type formula does not hold. The best-known estimate for general locally trivial bundles is

$$\mathrm{cat}_{\mathrm{LS}}\, E\le (\mathrm{cat}_{\mathrm{LS}}\, B+1)(\mathrm{cat}_{\mathrm{LS}}\, F+1)-1;$$

see [5]. Note that fibrations with the fiber $\mathbb{C}P^n$ can be produced by projectivization of the spherical bundles of complex vector bundles.

4.1 r–Connected universal cover

We recall a classical result that for an r–connected, n–dimensional complex X,

$$\mathrm{cat_{LS}}\, X \le \frac{n}{r+1}.$$

If $X = B \times Y$ with r–connected Y, we have

$$\begin{aligned}\mathrm{cat_{LS}}\, X \le \mathrm{cat_{LS}}\, B + \frac{\dim Y}{r+1} &= \mathrm{cat_{LS}}\, B + \frac{n - \dim B}{r+1}\\ \le \mathrm{cat_{LS}}\, B + \frac{n - \mathrm{cat_{LS}}\, B}{r+1} &= \frac{r\, \mathrm{cat_{LS}}\, B + n}{r+1}.\end{aligned}$$

Below we obtain a similar estimate for general X.

In the proof of the main result we applied our technical proposition (Proposition 3.1) with $r = 0$ and $r = 1$. Using Proposition 3.1 with $r = 0$ and arbitrary $r > 0$ brings the following:

Lemma 4.4 *Suppose that $\widetilde{X}$ is the universal covering of an n–dimensional simplicial complex X where the fundamental group $\pi = \pi_1(X)$ is r–connected. Assume that $\widetilde{X}$ admits a classifying map to an d–dimensional complex B with $\pi_1(B) = \pi$. Let E be the universal covering of B. Then*

$$\mathrm{cat_{LS}}(\widetilde{X} \times_\pi E)^{(n)} \le \frac{rd+n}{r+1}.$$

This lemma brings the following generalization of Theorem 4.1:

Theorem 4.5 *For every simplicial complex X with r–connected universal cover $\widetilde{X}$, there is the inequality*

$$\mathrm{cat_{LS}}\, X \le \frac{r\, \mathrm{cat_{LS}}(u_X) + \dim X}{r+1},$$

where $u_X \colon X \to B\pi$ is a classifying map for the universal cover of X.

Corollary 4.6 *For any CW complex X with r–connected universal covering $\widetilde{X}$,*

$$\mathrm{cat_{LS}}\, X \le \frac{r\, \mathrm{cd}(\pi_1(X)) + \dim X}{r+1}.$$

References

[1] **I Berstein**, *On the Lusternik–Schnirelmann category of Grassmannians*, Math. Proc. Cambridge Philos. Soc. 79 (1976) 129–134 MR

[2] **A Borel**, *Seminar on transformation groups*, Annals of Mathematics Studies 46, Princeton Univ. Press (1960) MR

[3] **P Bouwknegt**, **J Evslin**, **V Mathai**, *Spherical T–duality, II: An infinity of spherical T–duals for non-principal* $\mathrm{SU}(2)$*–bundles*, J. Geom. Phys. 92 (2015) 46–54 MR

[4] **K S Brown**, *Cohomology of groups*, Graduate Texts in Mathematics 87, Springer (1982) MR

[5] **O Cornea**, **G Lupton**, **J Oprea**, **D Tanré**, *Lusternik–Schnirelmann category*, Mathematical Surveys and Monographs 103, Amer. Math. Soc., Providence, RI (2003) MR

[6] **A N Dranishnikov**, *On the Lusternik–Schnirelmann category of spaces with 2–dimensional fundamental group*, Proc. Amer. Math. Soc. 137 (2009) 1489–1497 MR

[7] **A Dranishnikov**, *The Lusternik–Schnirelmann category and the fundamental group*, Algebr. Geom. Topol. 10 (2010) 917–924 MR

[8] **A N Dranishnikov**, *The LS category of the product of lens spaces*, Algebr. Geom. Topol. 15 (2015) 2983–3008 MR

[9] **A N Dranishnikov**, **M G Katz**, **Y B Rudyak**, *Cohomological dimension, self-linking, and systolic geometry*, Israel J. Math. 184 (2011) 437–453 MR

[10] **A Dranishnikov**, **R Sadykov**, *On the LS-category and topological complexity of a connected sum*, Proc. Amer. Math. Soc. 147 (2019) 2235–2244 MR

[11] **D P Grossman**, *An estimation of the category of Lusternik–Shnirelman*, C. R. (Doklady) Acad. Sci. URSS 54 (1946) 109–112 MR

[12] **N Iwase**, **M Mimura**, **T Nishimoto**, *Lusternik–Schnirelmann category of non-simply connected compact simple Lie groups*, Topology Appl. 150 (2005) 111–123 MR

[13] **M G Katz**, **Y B Rudyak**, *Lusternik–Schnirelmann category and systolic category of low-dimensional manifolds*, Comm. Pure Appl. Math. 59 (2006) 1433–1456 MR

[14] **L Lusternik**, **L Schnirelmann**, *Sur le probleme de trois geodesiquesfermees sur les surfaces de genus* 0, C. R. Acad. Sci. Paris Sér. I Math. 189 (1929) 269–271

[15] **J Oprea**, **J Strom**, *Lusternik–Schnirelmann category, complements of skeleta and a theorem of Dranishnikov*, Algebr. Geom. Topol. 10 (2010) 1165–1186 MR

[16] **P A Ostrand**, *Dimension of metric spaces and Hilbert's problem* 13, Bull. Amer. Math. Soc. 71 (1965) 619–622 MR

[17] **Y B Rudyak**, *On analytical applications of stable homotopy (the Arnold conjecture, critical points)*, Math. Z. 230 (1999) 659–672 MR

[18] **J Strom**, *Lusternik–Schnirelmann category of spaces with free fundamental group*, Algebr. Geom. Topol. 7 (2007) 1805–1808 MR

[19] **G W Whitehead**, *The homology suspension*, from "Colloque de topologie algébrique", Georges Thone, Liège (1957) 89–95 MR

Department of Mathematics, University of Florida
Gainesville, FL, United States

dranish@math.ufl.edu

Received: 12 June 2018 Revised: 18 February 2019