Algebraic & Geometric Topology 19 (2019) 3615–3639

On the local homology of Artin groups of finite and affine type

Giovanni Paolini

We study the local homology of Artin groups using weighted discrete Morse theory. In all finite and affine cases, we are able to construct Morse matchings of a special type (we call them "precise matchings"). The existence of precise matchings implies that the homology has a squarefree torsion. This property was known for Artin groups of finite type, but not in general for Artin groups of affine type. We also use the constructed matchings to compute the local homology in all exceptional cases, correcting some results in the literature.

05E45, 20F36, 52C35

1 Introduction

We study the local homology of Artin groups with coefficients in the Laurent polynomial ring $R = \mathbb{Q}[q^{\pm 1}]$, where each standard generator acts as a multiplication by $-q$. This homology has been already thoroughly investigated for groups of finite type — see

Published: 17 December 2019 DOI: 10.2140/agt.2019.19.3615

Frenkel [14], De Concini, Procesi and Salvetti [11; 10], Callegaro [3], Salvetti [23] and Paolini and Salvetti [19] — also with integral coefficients — see Callegaro and Salvetti [8; 4] — and for some groups of affine type; see Callegaro, Moroni and Salvetti [5; 6; 7], Salvetti and Villa [24] and Paolini and Salvetti [19]. This work is meant to be a natural continuation of [19] and is based on the combinatorial techniques developed in [24; 19].

In [24] Salvetti and Villa introduced a new combinatorial method to study the homology of Artin groups, based on discrete Morse theory. They described the general framework, and made explicit computations for all exceptional affine groups. In [19] the author and Salvetti developed the theory further, and made explicit computations for the affine family $\widetilde{C}_n$ as well as for the (already known) families A_n, B_n and $\widetilde{A}_n$. A common ingredient emerged in all the cases considered there, namely *precise matchings*. It was shown that whenever an Artin group admits precise matchings, then its local homology has a squarefree torsion. For Artin groups of finite type, the absence of higher powers in the torsion is a consequence of the isomorphism with the (constant) homology of the corresponding Milnor fiber [3]. This geometric argument does not apply to Artin groups of infinite type, but precise matchings proved to be useful also beyond the finite case.

In the present work, we show that precise matchings exist for all Artin groups of finite and affine type. As we said, this implies the squarefreeness of the torsion in the local homology. The elegance of this conclusion seems to hint at some unknown deeper geometric reason. The main results, stated in Section 3, are the following:

Theorem 3.1 *Every Artin group of finite or affine type admits a φ–precise matching for each cyclotomic polynomial φ.*

Corollary 3.2 *Let G_W be an Artin group of finite or affine type. Then the local homology $H_*(X_W; R)$ has no φ^k–torsion for $k \geq 2$.*

We are able to use precise matchings to carry out explicit homology computations for all exceptional finite and affine cases. In particular we recover the results of [11; 24], with small corrections. The matchings we find for D_n, $\widetilde{B}_n$ and $\widetilde{D}_n$ are quite complicated, so we prefer to omit explicit homology computations for these cases (the homology for D_n and $\widetilde{B}_n$ was already computed in [11] and [7], respectively). The remaining finite and affine cases, namely A_n, B_n, $\widetilde{A}_n$ and $\widetilde{C}_n$, were already discussed in [19].

We also provide a software library which can be used to generate matchings for any finite or affine Artin group, check precision and compute the homology. Source code and instructions are available online [18].

This paper is structured as follows. In Section 2 we review the general combinatorial framework developed in [24; 19]. We introduce the local homology $H_*(X_W; R)$, which is the object of our study, together with algebraic complexes to compute it. We present weighted discrete Morse theory and precise matchings, and recall some useful results. In Section 3 we state and discuss the main results of this paper. Subsequent sections are devoted to the proof of the main theorem. In Section 4 we show that it is enough to construct precise matchings for irreducible Artin groups. In Section 5 we recall the computation of the weight of irreducible components of type A_n, B_n and D_n, which is used later. In Sections 6–10 we construct precise matchings for the families A_n, D_n, $\widetilde{B}_n$, $\widetilde{D}_n$ and $I_2(m)$. Finally, in Section 11 we deal with the exceptional cases.

2 Local homology of Artin groups via discrete Morse theory

In this section we are going to recall the general framework of [24; 19] for the computation of the local homology $H_*(X_W; R)$.

Let (W, S) be a Coxeter system on a finite generating set S, and let Γ be the corresponding Coxeter graph (with S as its vertex set). Denote by G_W the corresponding Artin group, with standard generating set $\Sigma = \{g_s \mid s \in S\}$. Define K_W as the (finite) simplicial complex over S given by

$$K_W = \{\sigma \subseteq S \mid \text{the parabolic subgroup } W_\sigma \text{ generated by } \sigma \text{ is finite}\}.$$

It is convenient to include the empty set $\varnothing$ in K_W. Let X_W be the quotient of the Salvetti complex of W by the action of W. This is a finite (nonregular) CW complex, with polyhedral cells indexed by K_W. It has G_W as its fundamental group, and it is conjectured to be a space of type $K(G_W, 1)$ [21; 22; 20]. This conjecture is known to be true for all groups of finite type [12] and for some families of groups of infinite type, including the affine groups of type $\widetilde{A}_n$, $\widetilde{B}_n$ and $\widetilde{C}_n$ [17; 7].

Consider the action of the Artin group G_W on the ring $R = \mathbb{Q}[q^{\pm 1}]$ given by

$$g_s \mapsto [\text{multiplication by } -q] \quad \text{for all } s \in S.$$

We are interested in studying the local homology $H_*(X_W; R)$, with coefficients in the local system defined by the above action of $G_W = \pi_1(X_W)$ on R. Whenever X_W is a $K(G_W, 1)$ space, this coincides with the group homology $H_*(G_W; R)$ with coefficients in the same representation.

The local homology $H_*(X_{\mathbf{W}}; R)$ is computed by the algebraic complex

$$C_k = \bigoplus_{\substack{\sigma \in K_{\mathbf{W}} \\ |\sigma| = k}} R e_\sigma$$

with boundary

$$\partial(e_\sigma) = \sum_{\tau \lhd \sigma} [\sigma : \tau] \frac{\mathbf{W}_\sigma(q)}{\mathbf{W}_\tau(q)} e_\tau,$$

where $\mathbf{W}_\sigma(q)$ is the Poincaré polynomial of the parabolic subgroup $\mathbf{W}_\sigma$ of $\mathbf{W}$.

Let $C^{\mathbf{0}}_*$ be the 1–shifted algebraic complex of free R–modules which computes the reduced simplicial homology of $K_{\mathbf{W}}$ with (constant) coefficients in R. Namely,

$$C^{\mathbf{0}}_k = \bigoplus_{\substack{\sigma \in K_{\mathbf{W}} \\ |\sigma| = k}} R e^{\mathbf{0}}_\sigma$$

with boundary

$$\partial^{\mathbf{0}}(e^{\mathbf{0}}_\sigma) = \sum_{\tau \lhd \sigma} [\sigma : \tau] e^{\mathbf{0}}_\tau.$$

Then we have an injective chain map $\Delta\colon C_* \to C^{\mathbf{0}}_*$ defined by

$$e_\sigma \mapsto \mathbf{W}_\sigma(q) e^{\mathbf{0}}_\sigma.$$

Therefore there is an exact sequence of complexes

$$0 \to C_* \xrightarrow{\Delta} C^{\mathbf{0}}_* \xrightarrow{\pi} L_* \to 0,$$

where

$$L_k = \bigoplus_{\substack{\sigma \in K_{\mathbf{W}} \\ |\sigma| = k}} \frac{R}{(\mathbf{W}_\sigma(q))} \bar{e}_\sigma,$$

with boundary induced by the boundary of $C^{\mathbf{0}}_*$. The associated long exact sequence in homology then allows to compute the homology of C_* in terms of the homology of $C^{\mathbf{0}}_*$ and of L_*:

$$\cdots \xrightarrow{\pi_*} H_{k+1}(L_*) \xrightarrow{\delta} H_k(C_*) \xrightarrow{\Delta_*} H_k(C^{\mathbf{0}}_*) \xrightarrow{\pi_*} H_k(L_*) \xrightarrow{\delta} H_{k-1}(C_*) \xrightarrow{\Delta_*} \cdots .$$

In this paper, we mostly focus on Artin groups of finite and affine type, for which $K_{\mathbf{W}}$ is either the full simplex (in the finite case) or its boundary (in the affine case). In the former case $H_*(C^{\mathbf{0}}_*)$ is trivial, and in the latter case the only nontrivial term is $H_{|S|-1}(C^{\mathbf{0}}_*) \cong R$. Therefore the challenging part consists in understanding the homology of the complex L_*, which encodes all the torsion.

Poincaré polynomials of Coxeter groups are products of cyclotomic polynomials φ_d with $d \geq 2$. Thus the complex L_* decomposes as a direct sum of φ–primary components $(L_*)_\varphi$, where $\varphi = \varphi_d$ varies among the cyclotomic polynomials. Each component $(L_*)_\varphi$ takes the form

$$(L_k)_\varphi = \bigoplus_{\substack{\sigma \in K_W \\ |\sigma| = k}} \frac{R}{(\varphi^{v_\varphi(\sigma)})} \overline{e}_\sigma,$$

where $v_\varphi(\sigma)$ is the multiplicity of φ in the factorization of $W_\sigma(q)$. Again, the boundary in $(L_*)_\varphi$ is induced by the boundary of C^0_*. The homology of L_* decomposes accordingly, so our goal is to study $H_*((L_*)_\varphi)$ for each cyclotomic polynomial φ.

The main tool we are going to use is algebraic Morse theory for weighted complexes. We recall here the main points of the theory, referring to [13; 9; 1; 16; 24] for more details. Let G be the incidence graph of K_W, ie the graph having K_W as its vertex set and with a directed edge $\sigma \to \tau$ whenever $\tau \lhd \sigma$. We still call *simplices* the vertices of G, to avoid confusion with the vertices of K_W. A matching $\mathcal{M}$ on G is a set of edges of G such that each simplex $\sigma \in K_W$ is adjacent to at most one edge in $\mathcal{M}$. We say that σ is *critical* (with respect to the matching $\mathcal{M}$) if none of its adjacent edges is in $\mathcal{M}$. We also call an *alternating path* a sequence

$$(\tau_0 \lhd)\, \sigma_0 \rhd \tau_1 \lhd \sigma_1 \rhd \tau_2 \lhd \sigma_2 \rhd \cdots \rhd \tau_m \lhd \sigma_m \,(\rhd \tau_{m+1})$$

such that each pair $\tau_i \lhd \sigma_i$ belongs to $\mathcal{M}$ and no pair $\tau_i \lhd \sigma_{i-1}$ belongs to $\mathcal{M}$. An *alternating cycle* is a closed alternating path. The following are key definitions of the theory:

- $\mathcal{M}$ is *acyclic* if all alternating cycles are trivial.
- $\mathcal{M}$ is φ*–weighted* if $v_\varphi(\sigma) = v_\varphi(\tau)$ whenever $(\sigma \to \tau) \in \mathcal{M}$.

Notice that in general, by construction, one has $v_\varphi(\sigma) \geq v_\varphi(\tau)$ for $\tau \lhd \sigma$.

Theorem 2.1 [24, Theorem 2] *Fix a cyclotomic polynomial φ. Let $\mathcal{M}$ be an acyclic and φ–weighted matching on G. Then the homology of $(L_*)_\varphi$ is the same as the homology of the* ***Morse complex***

$$(L_*)^{\mathcal{M}}_\varphi = \bigoplus_{\sigma \text{ critical}} \frac{R}{(\varphi^{v_\varphi(\sigma)})} \overline{e}_\sigma$$

with boundary

$$\partial^{\mathcal{M}}(\overline{e}_\sigma) = \sum_{\substack{\tau \text{ critical} \\ |\tau| = |\sigma| - 1}} [\sigma : \tau]^{\mathcal{M}} \overline{e}_\tau,$$

where $[\sigma : \tau]^{\mathcal{M}} \in \mathbb{Z}$ *is given by the sum over all alternating paths*

$$\sigma \rhd \tau_1 \lhd \sigma_1 \rhd \tau_2 \lhd \sigma_2 \rhd \cdots \rhd \tau_m \lhd \sigma_m \rhd \tau$$

from σ *to* τ *of the quantity*

$$(-1)^m [\sigma : \tau_1][\sigma_1 : \tau_1][\sigma_1 : \tau_2][\sigma_2 : \tau_2] \cdots [\sigma_m : \tau_m][\sigma_m : \tau].$$

In [19] a special class of weighted matchings was introduced, namely precise matchings. We say that $\mathcal{M}$ is φ*–precise* (or simply *precise*) if $\mathcal{M}$ is acyclic and φ–weighted, and has the following additional property: $v_\varphi(\sigma) = v_\varphi(\tau) + 1$ whenever $[\sigma : \tau]^{\mathcal{M}} \neq 0$ (here σ and τ are critical simplices, so that $[\sigma : \tau]^{\mathcal{M}}$ is defined). This condition can be thought of as a rigid (weight-consistent) maximality condition. It appears to arise naturally in the study of the local homology of Artin groups, as shown in [19] and in the present work.

We refer to [19, Section 4] for a general introduction to precise matchings. Here we are only going to briefly recall what we need to study Artin groups. Let $C^0_*(\mathbb{Q})$ be the 1–shifted algebraic complex of free $\mathbb{Q}$–modules which computes the reduced simplicial homology of K_W with coefficients in $\mathbb{Q}$ (this is the same as C^0_*, but with $\mathbb{Q}$ instead of R in the definition). An acyclic matching $\mathcal{M}$ on G can be used to compute a Morse complex $C^0_*(\mathbb{Q})^{\mathcal{M}}$ of $C^0_*(\mathbb{Q})$ as well. Call $\delta^{\mathcal{M}}_*$ the boundary of the Morse complex $C^0_*(\mathbb{Q})^{\mathcal{M}}$.

Theorem 2.2 [19, Theorem 5.1] *Fix a cyclotomic polynomial* φ*. Let* $\mathcal{M}$ *be a* φ*–precise matching on* G*. Then the* φ*–torsion component of the local homology* $H_*(\boldsymbol{X_W}; R)$ *in each dimension, as an* R*–module, is given by*

$$H_m(\boldsymbol{X_W}; R)_\varphi \cong \left(\frac{R}{(\varphi)}\right)^{\oplus \operatorname{rk} \delta^{\mathcal{M}}_{m+1}}.$$

Theorem 2.3 [19, Theorem 5.1] *Suppose we have a* φ*–precise matching* $\mathcal{M}_\varphi$ *on* G *for every cyclotomic polynomial* φ*. Then the local homology of* $\boldsymbol{X_W}$ *in each dimension, as an* R*–module, is given by*

$$H_m(\boldsymbol{X_W}; R) \cong \left(\bigoplus_\varphi \left(\frac{R}{(\varphi)}\right)^{\oplus \operatorname{rk} \delta^{\mathcal{M}_\varphi}_{m+1}}\right) \oplus H_m(C^0_*).$$

In particular the term $H_*(C^0_*)$ *gives the free part of the homology, whereas the other direct summands give the torsion part.*

Corollary 2.4 [19, Corollary 5.2] *Suppose that G_W is an Artin group that admits a φ–precise matching for every cyclotomic polynomial φ. Then the homology $H_*(X_W; R)$ has no φ^k–torsion for $k \geq 2$.*

The formula of Theorem 2.3 simplifies further when G_W is of finite type (the free part disappears) or of affine type (the free part only appears in dimension $|S|-1$ and has rank 1).

3 The main theorem

As mentioned in the introduction, the main result of this paper is the following:

Theorem 3.1 *Every Artin group of finite or affine type admits a φ–precise matching for each cyclotomic polynomial φ.*

By Corollary 2.4, this has the following immediate consequence:

Corollary 3.2 *Let G_W be an Artin group of finite or affine type. Then the local homology $H_*(X_W; R)$ has no φ^k–torsion for $k \geq 2$.* □

For Artin groups of finite type, the local homology $H_*(X_W; R)$ coincides with the (constant) homology $H_*(F_W; \mathbb{Q})$ of the Milnor fiber of the associated hyperplane arrangement [3]. The q–multiplication on the homology of X_W corresponds to the action of the monodromy operator on the homology of F_W. The monodromy operator has a finite order N, thus the polynomial $q^N - 1$ must annihilate the homology. Therefore there can only be squarefree torsion.

The fact that the same conclusion holds for Artin groups of affine type is surprising, and might be due to some deeper geometric reasons which we still do not know.

The proof of Theorem 3.1 is split throughout the rest of this paper. In Section 4 we show that it is enough to construct precise matchings in the irreducible finite and affine cases. Case A_n was done in [19]. However, we study it again in Section 6 as we need it for D_n, $\widetilde{B}_n$ and $\widetilde{D}_n$. Cases B_n, $\widetilde{A}_n$ and $\widetilde{C}_n$ were done as well in [19], so we do not treat them again here. Case D_n is considered in Section 7, case $\widetilde{B}_n$ in Section 8, case $\widetilde{D}_n$ in Section 9, and case $I_2(m)$ in Section 10. In all remaining exceptional cases, we construct precise matchings via a computer program, as discussed in Section 11.

We provide a software library to construct precise matchings for any given finite or affine Artin group, following [19] and the present paper. Source code and instructions can be found online [18]. This library can be used to check precision and compute the homology.

Remark 3.3 Not all Artin groups admit precise matchings for every cyclotomic polynomial. For example, consider the Coxeter system (W, S) defined by the Coxeter matrix

$$\begin{pmatrix} 1 & 3 & 3 & 2 & \infty & 4 \\ 3 & 1 & 3 & 4 & 2 & \infty \\ 3 & 3 & 1 & \infty & 4 & 2 \\ 2 & 4 & \infty & 1 & \infty & \infty \\ \infty & 2 & 4 & \infty & 1 & \infty \\ 4 & \infty & 2 & \infty & \infty & 1 \end{pmatrix}.$$

The simplicial complex K_W consists of: three 2–simplices ($\{1,2,4\}$, $\{2,3,5\}$ and $\{1,3,6\}$), all having φ_2–weight equal to 3; six 1–simplices with φ_2–weight equal to 2 ($\{1,4\}$, $\{2,4\}$, $\{2,5\}$, $\{3,5\}$, $\{1,6\}$ and $\{3,6\}$), and three 1–simplices with φ_2–weight equal to 1 ($\{1,2\}$, $\{2,3\}$ and $\{1,3\}$); six 0–simplices ($\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$ and $\{6\}$), all having φ_2–weight equal to 1; one empty simplex, with φ_2–weight equal to 0. A φ_2–weighted matching can only contain edges between simplices of weight 1. Since the three 1–simplices of weight 1 form a cycle, at least one of them (say $\{1,2\}$) is critical. Then the incidence number between $\{1,2,4\}$ and $\{1,2\}$ is nonzero, and their φ_2–weights differ by 2. Therefore the matching cannot be φ_2–precise.

We introduce here some notation that will be used later. Given a simplex $\sigma \in K_W$, denote by $\Gamma(\sigma)$ the subgraph of Γ induced by σ. We will sometimes speak about the connected components of $\Gamma(\sigma)$, which will be denoted by $\Gamma_i(\sigma)$ for some index i. Also, given a vertex $v \in S$, define

$$\sigma \veebar v = \begin{cases} \sigma \cup \{v\} & \text{if } v \notin \sigma, \\ \sigma \setminus \{v\} & \text{if } v \in \sigma. \end{cases}$$

4 Reduction to the irreducible cases

Let (W_1, S_1) and (W_2, S_2) be Coxeter systems, and consider the product Coxeter system $(W_1 \times W_2, S_1 \sqcup S_2)$. Suppose that the Artin groups $\boldsymbol{G_{W_1}}$ and $\boldsymbol{G_{W_2}}$ admit φ–precise matchings $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. In the following lemma we construct a φ–precise matching for $\boldsymbol{G_{W_1}} \times \boldsymbol{G_{W_2}} = \boldsymbol{G_{W_1 \times W_2}}$:

Lemma 4.1 *Fix a cyclotomic polynomial φ. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be φ–precise matchings on $\boldsymbol{G}_{\boldsymbol{W}_1}$ and $\boldsymbol{G}_{\boldsymbol{W}_2}$, respectively. Then $\boldsymbol{G}_{\boldsymbol{W}_1\times \boldsymbol{W}_2}$ also admits a φ–precise matching.*

Proof First notice that the simplicial complex $K_{\boldsymbol{W}_1\times\boldsymbol{W}_2}$ consists of the simplicial join $K_{\boldsymbol{W}_1} * K_{\boldsymbol{W}_2}$ of the two simplicial complexes $K_{\boldsymbol{W}_1}$ and $K_{\boldsymbol{W}_2}$:

$$K_{\boldsymbol{W}_1\times\boldsymbol{W}_2} = K_{\boldsymbol{W}_1} * K_{\boldsymbol{W}_2} = \{\sigma_1 \sqcup \sigma_2 \mid \sigma_1 \in K_{\boldsymbol{W}_1},\ \sigma_2 \in K_{\boldsymbol{W}_2}\}.$$

The weights behave well with respect to this decomposition:

$$v_\varphi(\sigma_1 \sqcup \sigma_2) = v_\varphi(\sigma_1) + v_\varphi(\sigma_2).$$

Construct a matching $\mathcal{M}$ on $K_{\boldsymbol{W}_1\times\boldsymbol{W}_2}$ as follows:

$$\begin{aligned}\mathcal{M} = \{&\sigma_1 \sqcup \sigma_2 \to \sigma_1 \sqcup \tau_2 \mid (\sigma_2 \to \tau_2) \in \mathcal{M}_2\}\\ &\cup \{\sigma_1 \sqcup \sigma_2 \to \tau_1 \sqcup \sigma_2 \mid (\sigma_1 \to \tau_1) \in \mathcal{M}_2 \text{ and } \sigma_2 \text{ is critical in } K_{\boldsymbol{W}_2}\}.\end{aligned}$$

The critical simplices $\sigma_1 \sqcup \sigma_2$ of $K_{\boldsymbol{W}_1\times\boldsymbol{W}_2}$ are those for which σ_1 is critical in $K_{\boldsymbol{W}_1}$ and σ_2 is critical in $K_{\boldsymbol{W}_2}$.

Any alternating path in $K_{\boldsymbol{W}_1\times\boldsymbol{W}_2}$ projects onto an alternating path in $K_{\boldsymbol{W}_2}$ via the map $\sigma_1 \sqcup \sigma_2 \mapsto \sigma_2$ (provided that multiple consecutive occurrences of the same simplex are replaced by a single occurrence). This is because an edge of the form $\sigma_1 \sqcup \sigma_2 \to \sigma_1 \sqcup \tau_2$ is in $\mathcal{M}$ if and only if $\sigma_2 \to \tau_2$ is in $\mathcal{M}_2$. The same statement is not true for $K_{\boldsymbol{W}_1}$, but we still have a weaker property which will be useful later: the projection of an alternating path to $K_{\boldsymbol{W}_1}$ cannot have two consecutive edges both traversed "upwards" (ie increasing dimension). This is because if an edge of the form $\sigma_1 \sqcup \sigma_2 \to \tau_1 \sqcup \sigma_2$ is in $\mathcal{M}$, then $\sigma_1 \to \tau_1$ is in $\mathcal{M}_1$.

Let us prove that $\mathcal{M}$ is acyclic. Consider an alternating cycle c in $K_{\boldsymbol{W}_1\times\boldsymbol{W}_2}$. Its projection onto $K_{\boldsymbol{W}_2}$ gives an alternating cycle, which must be trivial because $\mathcal{M}_2$ is acyclic. Therefore c takes the form, for some fixed simplex $\sigma_2 \in K_{\boldsymbol{W}_2}$,

$$\sigma_{1,0} \sqcup \sigma_2 \rhd \tau_{1,1} \sqcup \sigma_2 \lhd \sigma_{1,1} \sqcup \sigma_2 \rhd \cdots \rhd \tau_{1,m} \sqcup \sigma_2 \lhd \sigma_{1,0} \sqcup \sigma_2.$$

If σ_2 is critical in $K_{\boldsymbol{W}_2}$, then also the projection of c onto $K_{\boldsymbol{W}_1}$ is an alternating cycle. By acyclicity of $\mathcal{M}_1$ such a projection must be the trivial cycle, so also c is trivial. On the other hand, if σ_2 is not critical, then none of the edges $\sigma_{1,i} \sqcup \sigma_2 \to \tau_{1,i} \sqcup \sigma_2$ is in $\mathcal{M}$, thus c must be trivial as well.

By construction, and by additivity of the weight function v_φ, the matching $\mathcal{M}$ is φ–weighted.

Finally, suppose that $[\sigma_1 \sqcup \sigma_2 : \tau_1 \sqcup \tau_2]^{\mathcal{M}} \neq 0$, where $\sigma_1 \sqcup \sigma_2$ and $\tau_1 \sqcup \tau_2$ are critical simplices of $K_{\boldsymbol{W}_1 \times \boldsymbol{W}_2}$ with $\dim(\sigma_1 \sqcup \sigma_2) = \dim(\tau_1 \sqcup \tau_2) + 1$. Let $\mathcal{P} \neq \varnothing$ be the set of alternating paths from $\sigma_1 \sqcup \sigma_2$ to $\tau_1 \sqcup \tau_2$. Given any path $p \in \mathcal{P}$, its projection onto $K_{\boldsymbol{W}_2}$ is an alternating path from σ_2 to τ_2.

(1) Suppose $\sigma_2 = \tau_2$. Then the projected paths are trivial in $K_{\boldsymbol{W}_2}$, so $\mathcal{P}$ is in bijection with the set of alternating paths from σ_1 to τ_1 in $K_{\boldsymbol{W}_1}$. Therefore $[\sigma_1 : \tau_1]^{\mathcal{M}_1} = \pm[\sigma_1 \sqcup \sigma_2 : \tau_1 \sqcup \tau_2]^{\mathcal{M}} \neq 0$. Since $\mathcal{M}_1$ is φ–precise, we conclude that $v_\varphi(\sigma_1) = v_\varphi(\tau_1) + 1$ and so $v_\varphi(\sigma_1 \sqcup \sigma_2) = v_\varphi(\tau_1 \sqcup \sigma_2) + 1$.

(2) Suppose $\sigma_2 \neq \tau_2$. The projection of any $p \in \mathcal{P}$ onto $K_{\boldsymbol{W}_2}$ is a nontrivial alternating path, and $\mathcal{P}$ is nonempty, so $\dim(\sigma_2) = \dim(\tau_2) + 1$. Then $\dim(\sigma_1) = \dim(\tau_1)$. For any alternating path $p \in \mathcal{P}$, consider now its projection q onto $K_{\boldsymbol{W}_1}$. We want to prove that q is a trivial path (thus in particular $\sigma_1 = \tau_1$). Suppose by contradiction that q is nontrivial. Then, since σ_1 and τ_1 have the same dimension, one of the following three possibilities must occur:

- The path q begins with an upward edge $\sigma_1 \lhd \rho$. Then $(\rho \to \sigma_1) \in \mathcal{M}_1$, which is not possible because σ_1 is critical.
- The path q ends with an upward edge $\rho \lhd \tau_1$. Then $(\tau_1 \to \rho) \in \mathcal{M}_1$, which is not possible because τ_1 is critical.
- The path q begins and ends with a downward edge, so it must have two consecutive upward edges somewhere in the middle. This is also not possible by previous considerations.

We proved that the projection on $K_{\boldsymbol{W}_1}$ of any alternating path $p \in \mathcal{P}$ is trivial, and thus in particular $\sigma_1 = \tau_1$ (because $\mathcal{P}$ is nonempty). Then $\mathcal{P}$ is in bijection with the set of alternating paths from σ_2 to τ_2 in $K_{\boldsymbol{W}_2}$. We conclude as in case (1). □

In view of this lemma, from now on we only consider irreducible Coxeter systems.

5 Weight of irreducible components

In order to compute the weight $v_\varphi(\sigma)$ of a simplex $\sigma \in K_{\boldsymbol{W}}$, one needs to know the Poincaré polynomial of the parabolic subgroup $\boldsymbol{W}_\sigma$ of $\boldsymbol{W}$. Let $\Gamma_1(\sigma), \dots, \Gamma_m(\sigma)$ be

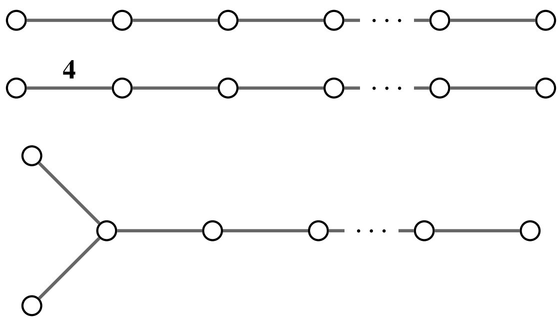

Figure 1: Coxeter graphs of type A_n, B_n and D_n. All these graphs have n vertices.

the connected components of the subgraph $\Gamma(\sigma) \subseteq \Gamma$ induced by σ. Then the Poincaré polynomial of W_σ splits as a product of the Poincaré polynomials of irreducible components of finite type:

$$W_\sigma(q) = \prod_{i=1}^{m} W_{\Gamma_i(\sigma)}(q), \quad \text{and therefore} \quad v_\varphi(\sigma) = \sum_{i=1}^{m} v_\varphi(\Gamma_i(\sigma)).$$

In this section we derive formulas for the φ–weight of an irreducible component of type A_n, B_n and D_n (see Figure 1).

Components of type A_n

The exponents of a Coxeter group W_{A_n} of type A_n are $1, 2, \dots, n$. Then its Poincaré polynomial is $W_{A_n}(q) = [n+1]_q!$. If φ_d is the d^{th} cyclotomic polynomial (for $d \geq 2$), the φ_d–weight is then

$$\omega_{\varphi_d}(A_n) = \left\lfloor \frac{n+1}{d} \right\rfloor.$$

Components of type B_n

In this case the exponents are $1, 3, \dots, 2n-3, 2n-1$, and the Poincaré polynomial is $W_{B_n}(q) = [2n]_q!!$. The φ_d–weight (for $d \geq 2$) is given by

$$\omega_{\varphi_d}(B_n) = \begin{cases} \lfloor \frac{n}{d} \rfloor & \text{if } d \text{ is odd,} \\ \lfloor \frac{n}{d/2} \rfloor & \text{if } d \text{ is even.} \end{cases}$$

Components of type D_n

Here the exponents are $1, 3, \dots, 2n-3, n-1$, and the Poincaré polynomial is $W_{D_n}(q) = [2n-2]_q!! \cdot [n]_q$. The φ_d–weight (for $d \geq 2$) is given by

$$\omega_{\varphi_d}(D_n) = \begin{cases} \lfloor \frac{n}{d} \rfloor & \text{if } d \text{ is odd,} \\ \lfloor \frac{n-1}{d/2} \rfloor & \text{if } d \text{ is even and } d \nmid n, \\ \frac{n}{d/2} & \text{if } d \text{ is even and } d \mid n. \end{cases}$$

6 Case A_n revisited

The construction of a precise matching for the case A_n was thoroughly discussed in [19]. However, in order to describe precise matchings for the cases D_n, $\widetilde{B}_n$ and $\widetilde{D}_n$, we need a slightly more general construction.

Throughout this section, let (W_{A_n}, S) be a Coxeter system of type A_n with generating set $S = \{1, 2, \dots, n\}$, and let $K_n^A = K_{W_{A_n}}$. See Figure 2 for the corresponding Coxeter graph.

Figure 2: A Coxeter graph of type A_n.

For integers $f, g \geq 0$, define $K_{n,f,g}^A \subseteq K_n$ by

$$K_{n,f,g}^A = \bigl\{\sigma \in K_n \mid \{1, 2, \dots, f\} \subseteq \sigma \text{ and } \{n-g+1, n-g+2, \dots, n\} \subseteq \sigma\bigr\}.$$

In other words, $K_{n,f,g}^A$ is the subset of K_n^A consisting of the simplices which contain the first f vertices and the last g vertices. In general, $K_{n,f,g}^A$ is not a subcomplex of K_n^A. For any $d \geq 2$, we are going to recursively construct a φ_d–weighted acyclic matching on $K_{n,f,g}^A$. This matching coincides with the one of [19, Section 5.1] when $g = 0$ and $f \leq d-1$. See also Table 1 for an example.

In what follows, the notation "$n \equiv a, \dots, b \pmod{d}$" means that n is congruent modulo d to some integer in the closed interval $[a, b]$.

Matching 6.1 (φ_d–matching on $K_{n,f,g}^A$) (a) If $f + g \geq n$ then $K_{n,f,g}^A$ has size at most 1, and the matching is empty. In the subsequent cases, assume $f + g < n$.

simplices		$v_\varphi(\sigma)$
 →		2
 →		2
 →		1
	(critical)	2
	(critical)	1

Table 1: Matching 6.1 on $K^A_{7,1,3}$ for $d = 3$.

(b) If $f \geq d$, then $K^A_{n,f,g} \cong K^A_{n-d,\,f-d,\,g}$ via removal of the first d vertices. Define the matching recursively, as in $K^A_{n-d,\,f-d,\,g}$. In the subsequent cases, assume $f \leq d-1$.

(c) Case $n \geq d+g$.

(c1) If $\{1,\dots,d-1\} \subseteq \sigma$, then match σ with $\sigma \veebar d$ (here the vertex d exists and can be removed, because $n \geq d+g$). Notice that for $f = d-1$ this is always the case, thus in the subsequent cases we can assume $f \leq d-2$.

(c2) Otherwise, if $f+1 \in \sigma$ then match σ with $\sigma \setminus \{f+1\}$.

(c3) Otherwise, if $\{f+2,\dots,d-1\} \not\subseteq \sigma$ then match σ with $\sigma \cup \{f+1\}$.

(c4) We are left with the simplices σ such that $\{1,\dots,f,f+2,\dots,d-1\} \subseteq \sigma$ and $f+1 \notin \sigma$. If we ignore the vertices $1,\dots,f+1$ we are left with the simplices on the vertex set $\{f+2,\dots,n\}$ which contain $f+2,\dots,d-1$; relabeling the vertices, these are the same as the simplices on the vertex set $\{1,\dots,n-f-1\}$ which contain $1,\dots,d-2-f$. Then construct the matching recursively as in $K^A_{n-f-1,\,d-2-f,\,g}$.

(d) Case $n < d+g$ (in particular, $f \leq d-2$).

(d1) If $n \equiv -1,0,1,\dots,f \pmod{d}$ and σ is either

$$\{1,\dots,n\} \quad \text{or} \quad \{1,\dots,f,f+2,\dots,n\},$$

then σ is critical.

(d2) Otherwise, match σ with $\sigma \veebar f+1$.

	case	# critical	$\lvert\sigma\rvert - v_\varphi(\sigma)$
	$f > n$ or $g > n$	0	-
	$f, g \leq n$ and $f+g \geq n$	1	$n - \lfloor \frac{n+1}{d} \rfloor$
$f+g<n$	$n \equiv \max(d-1,\ f_0+g_0+1), \dots,$ $\min(f_0+d-1, g_0+d-1) \pmod d$	2	$n - \lfloor \frac{n-f}{d} \rfloor - \lfloor \frac{n-g}{d} \rfloor - 1$
$f+g<n$	$n \equiv \max(f_0, g_0), \dots,$ $\min(f_0+g_0, d-2) \pmod d$	2	$n - \lfloor \frac{n-f}{d} \rfloor - \lfloor \frac{n-g}{d} \rfloor$
$f+g<n$	otherwise	0	-

Table 2: Critical simplices of Matching 6.1. Here $f_0, g_0 \in \{0, \dots, d-1\}$ are defined as $f \mod d$, and $g \mod d$, respectively.

The following lemma can be proved by induction on n, similarly to [19, Lemma 5.6 and Theorem 5.7]. We omit its proof, as well as subsequent proofs of the same type.

Lemma 6.2 *Matching 6.1 is acyclic and φ_d–weighted. Its critical simplices are given by Table 2. In particular, the matching is φ_d–precise for $f = g = 0$.* □

Remark 6.3
- The conditions in Table 2 are symmetric in f and g, even if the definition of Matching 6.1 is not.
- The two intervals of Table 2 in the case $f + g < n$ are always disjoint.
- For $g = 0$, Matching 6.1 coincides with the matching defined in [19, Section 5.1]. Table 2 simplifies a lot in this case, as both intervals contain only one element ($d-1$ and f_0, respectively). See [19, Table 2].
- If $f \equiv -1 \pmod d$ or $g \equiv -1 \pmod d$, then the two intervals of Table 2 are empty, thus there is at most one critical simplex.
- If $\sigma \to \tau$ is in the matching, then $\sigma = \tau \cup \{v\}$ with $v \equiv 0$ or $v \equiv f+1 \pmod d$. This can be easily checked by induction.

7 Case D_n

In [19] precise matchings were constructed for A_n and B_n, but the third infinite family of groups of finite type, namely D_n, was left out (see Figure 3).

For $n \geq 4$ let (W_{D_n}, S) be a Coxeter system of type D_n, with generating set $S = \{1, 2, \dots, n\}$, and let $K_n^D = K_{W_{D_n}}$. We are going to construct a φ_d–precise matching

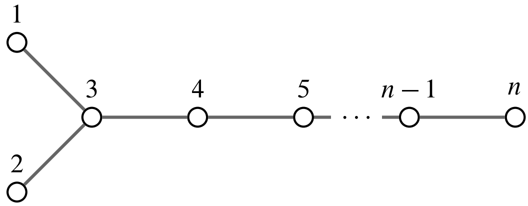

Figure 3: A Coxeter graph of type D_n.

on K_n^D. We actually split the definition according to the parity of d, and for d even we construct a matching on each

$$K_{n,g}^D = \{\sigma \in K_n^D \mid \{n-g+1, n-g+2, \dots, n\} \subseteq \sigma\}$$

for $0 \le g \le n-1$. We will need this construction for $\widetilde{B}_n$ and $\widetilde{D}_n$.

Matching 7.1 (φ_d–matching on K_n^D for d odd) (a) If $1 \in \sigma$ then match σ with $\sigma \veebar 2$.

(b) Otherwise, relabel the vertices $\{2, \dots, n\}$ as $\{1, \dots, n-1\}$ and construct the matching as in K_{n-1}^A.

Matching 7.2 (φ_d–matching on $K_{n,g}^D$ for d even) (a) If $2 \notin \sigma$, relabel the vertices $\{1, 3, 4, \dots, n\}$ as $\{1, \dots, n-1\}$ and construct the matching as in $K_{n-1,0,g}^A$.

(b) Otherwise, if $d = 2$ and $\{1, 2, 3, 4\} \not\subseteq \sigma$, proceed as follows:

(b1) If $\{1, 2, 4\} \subseteq \sigma$, match σ with $\sigma \veebar 5$ if possible (ie if $n - g \ge 5$); otherwise σ is critical.

(b2) Otherwise, match σ with $\sigma \veebar 3$ if possible (ie if $n - g \ge 3$); otherwise σ is critical.

(c) Otherwise, if $d \ge 4$ and $3 \notin \sigma$, match σ with $\sigma \veebar 1$.

(d) Otherwise, if $d = 4$ and $4 \notin \sigma$ (recall that at this point $\{2, 3\} \subseteq \sigma$), ignore vertex 1, relabel vertices $\{5, \dots, n\}$ as $\{1, \dots, n-4\}$, and construct the matching as in $K_{n-4,0,g}^A$.

(e) Otherwise, if $d \ge 6$ and $4 \notin \sigma$, match σ with $\sigma \veebar 1$.

(f) Otherwise, if $d \ge 4$ and $1 \notin \sigma$, proceed as follows. Recall that at this point $\{2, 3, 4\} \subseteq \sigma$.

(f1) If $\left\{2, \dots, \frac{d}{2} + 1\right\} \subseteq \sigma$, relabel the vertices $\{2, \dots, n\}$ as $\{1, \dots, n-1\}$ and construct the matching as in $K_{n-1,\max(d/2,3),g}^A$.

(f2) Otherwise, match σ with $\sigma \cup \{1\}$.

case	# critical	$\lvert\sigma\rvert - v_\varphi(\sigma)$
$n \equiv 0 \pmod{d}$	2	$n - 2\frac{n}{d}$
$n \equiv 1 \pmod{d}$	2	$n - 2\frac{n-1}{d} - 1$
otherwise	0	-

Table 3: Critical simplices of Matching 7.1 (case D_n, d odd).

(g) Otherwise, proceed as follows. Recall that at this point $\{1,2,3,4\} \subseteq \sigma$. Let $k \geq 4$ be the size of the connected component $\Gamma_1(\sigma)$ of the vertex 1, in the subgraph $\Gamma(\sigma) \subseteq \Gamma$ induced by σ. Write $k = q\frac{d}{2} + r$, where

$$\begin{cases} 0 < r < \frac{d}{2} & \text{if } k \not\equiv 0 \left(\operatorname{mod} \frac{d}{2}\right), \\ r \in \left\{0, \frac{d}{2}\right\} \text{ and } q \text{ even} & \text{if } k \equiv 0 \left(\operatorname{mod} \frac{d}{2}\right). \end{cases}$$

Define a vertex v by

$$v = \begin{cases} q\frac{d}{2} + 1 & \text{if } q \text{ is even}, \\ q\frac{d}{2} + 2 & \text{if } q \text{ is odd}. \end{cases}$$

It can be checked that $v = 1$ or $v \geq 5$. The idea now is that most of the times $\sigma \veebar v$ has the same φ_d–weight as σ. Unfortunately there are some exceptions, so we still have to examine a few subcases.

(g1) Suppose $v \in \sigma$. If $v \leq n - g$, match σ with $\sigma \setminus \{v\}$. Otherwise σ is critical.

(g2) Suppose $v \notin \sigma$. Match σ with $\sigma \cup \{v\}$, unless one of the following occurs:

(g2.1) $v > n$ (ie the vertex v doesn't exist in S). Then σ is critical.

case	origin	$\lvert\sigma\rvert - v_\varphi(\sigma)$
$n \equiv 0 \pmod{d}$	(a), (b) for $n = 4$ and $d = 2$, (d) for $d = 4$, (f) for $d \geq 6$ or $n = d = 4$, (g2.1)–(g2.3)	$n - 2\frac{n}{d}$
$n \equiv 1 \pmod{d}$	(a)	$n - 2\frac{n-1}{d} - 1$
$n \equiv \frac{d}{2} + 1 \pmod{d}$ for $d \geq 4$	(d) for $d = 4$, (f) for $d \geq 6$, (g2.1)–(g2.3)	$n - 2\frac{n-1}{d}$
otherwise	-	-

Table 4: Critical simplices of Matching 7.2 (case D_n, d even) for $g = 0$. In the second column we indicate in which parts of Matching 7.2 the critical simplices arise.

(g2.2) q is even, and $\left\{q\frac{d}{2}+2,\dots,(q+1)\frac{d}{2}+1\right\} \subseteq \sigma$. In this case the connected components $\Gamma_1(\sigma)$ and $\Gamma_1(\sigma \cup \{v\})$ have a different φ_d–weight. Then ignore the vertices up to $q\frac{d}{2}+1$, relabel the vertices $\left\{q\frac{d}{2}+2,\dots,n\right\}$ as $\left\{1,\dots,n-q\frac{d}{2}-1\right\}$ and construct the matching as in $K^A_{n-qd/2-1,\,d/2,\,g}$.

(g2.3) q is odd, and $\left\{q\frac{d}{2}+3,\dots,(q+1)\frac{d}{2}\right\} \subseteq \sigma$. Similarly to case (g2.2), relabel the vertices and construct the matching as in $K^A_{n-qd/2-2,\,d/2-2,\,g}$.

Lemma 7.3 *Matchings 7.1 and 7.2 are acyclic and φ_d–weighted. Critical simplices for these matchings on K^D_n are given by Tables 3 and 4. In particular, both matchings on K^D_n are φ_d–precise.* □

8 Case $\widetilde{B}_n$

Consider now, for $n \geq 3$, an affine Coxeter system $(\boldsymbol{W}_{\widetilde{B}_n}, S)$ of type $\widetilde{B}_n$ (see Figure 4). Throughout this section, let $K_n = K_{\boldsymbol{W}_{\widetilde{B}_n}}$. We are going to describe a φ_d–precise matching on K_n. For d odd the matching is very simple, and has exactly one critical simplex. For d even the situation is more complicated.

Matching 8.1 (φ_d–matching on $K_n = K_{\boldsymbol{W}_{\widetilde{B}_n}}$ for d odd) For $\sigma \neq \{1,2,\dots,n\}$, match σ with $\sigma \veebar 0$. Then $\{1,2,\dots,n\}$ is the only critical simplex.

Matching 8.2 (φ_d–matching on $K_n = K_{\boldsymbol{W}_{\widetilde{B}_n}}$ for d even) For $\sigma \in K_n$, let k be the size of the connected component $\Gamma_n(\sigma)$ of the vertex n, in the subgraph $\Gamma(\sigma) \subseteq \Gamma$ induced by σ. Let $k = q\frac{d}{2}+r$, with $0 \leq r < \frac{d}{2}$.

(a) If $r \geq 1$, match σ with $\sigma \veebar \left(n-q\frac{d}{2}\right)$, unless $\sigma = \{0,2,3,\dots,n\}$ and $r = 1$ (in this case σ is critical).

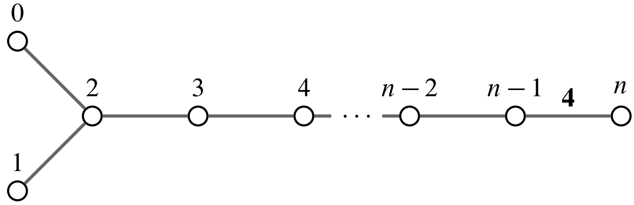

Figure 4: A Coxeter graph of type $\widetilde{B}_n$.

(b) If $r=0$ and $\left\{n-(q+1)\frac{d}{2}+1,\dots,n-q\frac{d}{2}-1\right\}\subseteq\sigma$, ignore vertices $\geq n-q\frac{d}{2}$, relabel vertices $\left\{0,1,\dots,n-q\frac{d}{2}-1\right\}$ as $\left\{1,2,\dots,n-q\frac{d}{2}\right\}$, and construct the matching as in $K^D_{n-qd/2,\,d/2-1}$.

(c) If $r=0$ and $\left\{n-(q+1)\frac{d}{2}+1,\dots,n-q\frac{d}{2}-1\right\}\not\subseteq\sigma$, proceed as follows:

  (c1) If $|\sigma|=n$ (ie σ is either $\{0,2,3,\dots,n\}$ or $\{1,2,3,\dots,n\}$), then σ is critical.

  (c2) If $n=(q+1)\frac{d}{2}$ and $\sigma=\left\{0,2,3,\dots,n-q\frac{d}{2}-1,n-q\frac{d}{2}+1,\dots,n\right\}$, then σ is critical.

  (c3) Otherwise, match σ with $\sigma\veebar\left(n-q\frac{d}{2}\right)$.

Lemma 8.3 *Matchings 8.1 and 8.2 are acyclic and φ_d–weighted. For d odd, Matching 8.1 has exactly one critical simplex σ which satisfies $|\sigma|-v_{\varphi_d}(\sigma)=n-\left\lfloor\frac{n}{d}\right\rfloor$. For d even, all critical simplices σ of Matching 8.2 satisfy $|\sigma|-v_{\varphi_d}(\sigma)=n-\left\lfloor\frac{n}{d/2}\right\rfloor$. In particular, both matchings are φ_d–precise.* □

9 Case $\widetilde{D}_n$

In this section we consider a Coxeter system $(W_{\widetilde{D}_n},S)$ of type $\widetilde{D}_n$ for $n\geq 4$ (see Figure 5). Throughout this section, let $K_n=K_{W_{\widetilde{D}_n}}$. We are going to describe a φ_d–precise matching on K_n. Again, this will be easier for d odd and quite involved for d even.

Matching 9.1 (φ_d–matching on $K_n=K_{W_{\widetilde{D}_n}}$ for d odd) (a) If $\sigma=\{1,2,\dots,n\}$, then σ is critical.

(b) If $1\notin\sigma$, relabel vertices $\{n,n-1,\dots,3,2,0\}$ as $\{1,2,\dots,n\}$ and generate the matching as in K^D_n.

(c) In all remaining cases, match σ with $\sigma\veebar 0$.

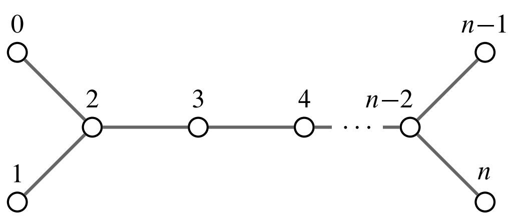

Figure 5: A Coxeter graph of type $\widetilde{D}_n$.

Matching 9.2 (φ_d–matching on $K_n = K_{W_{\widetilde{D}_n}}$ for d even) For $n=4$ and $d \leq 6$, we construct the matching separately as follows:

- Case $n=4$, $d=2$. If $|\sigma|=1$ and $2 \notin \sigma$, or $|\sigma|=2$, or $|\sigma|=3$ and $2 \in \sigma$, then match σ with $\sigma \veebar 2$. Otherwise, σ is critical.
- Case $n=4$, $d=4$. If $2 \notin \sigma$ or $\sigma \cap \{1,3,4\} = \varnothing$, then match σ with $\sigma \veebar 0$. Otherwise, σ is critical.
- Case $n=4$, $d=6$. Match σ with $\sigma \veebar 0$, except in the following two cases: $2 \in \sigma$, $0 \notin \sigma$ and $|\sigma| \geq 3$, or $\{0,2\} \subseteq \sigma$ and $|\sigma|=4$.

In the remaining cases ($n \geq 5$ or $d \geq 8$), the matching is constructed as follows:

(a) If $1 \notin \sigma$, relabel vertices $\{n, n-1, \dots, 3, 2, 0\}$ as $\{1, 2, \dots, n\}$ and construct the matching as in K_n^D.

(b) Otherwise, if $d=2$ and $\{0,1,2,3\} \not\subseteq \sigma$, proceed as follows:

(b1) If $\{0,1,3\} \subseteq \sigma$, match σ with $\sigma \veebar 4$ if $\{5,6,\dots,n\} \not\subseteq \sigma$; otherwise σ is critical.

(b2) Otherwise, if $\{1,3,4,\dots,n\} \not\subseteq \sigma$ then match σ with $\sigma \veebar 2$; otherwise σ is critical.

(c) Otherwise, if $d \geq 4$ and $0 \notin \sigma$, proceed as follows:

(c1) If $\left\{1, 2, \dots, \frac{d}{2}\right\} \subseteq \sigma$, relabel vertices $\{n, n-1, \dots, 2, 1\}$ as $\{1, 2, \dots, n\}$ and construct the matching as in $K_{n,d/2}^D$.

(c2) Otherwise, if $n = \frac{d}{2}+1$ and $\sigma = \{1, 2, \dots, n-2, n\}$, then σ is critical.

(c3) Otherwise, if $\sigma = \{1, 2, \dots, n\}$, then σ is critical.

(c4) Otherwise, match σ with $\sigma \cup \{0\}$.

(d) Otherwise, if $d \geq 4$ and $2 \notin \sigma$, match σ with $\sigma \veebar 0$.

(e) Otherwise, if $d=4$ and $3 \notin \sigma$, ignore vertices 0, 1 and 2, relabel vertices $\{n, n-1, \dots, 4\}$ as $\{1, 2, \dots, n-3\}$ and construct the matching as in K_{n-3}^D.

(f) Otherwise, if $d \geq 6$ and $3 \notin \sigma$, match σ with $\sigma \veebar 0$.

(g) Otherwise, proceed as follows. Recall that at this point $\{0,1,2,3\} \subseteq \sigma$. Let $k \geq 4$ be the size of the leftmost connected component $\Gamma_0(\sigma)$ of the subgraph $\Gamma(\sigma) \subseteq \Gamma$ induced by σ. Notice that $\{0, 1, \dots, k-1\} \subseteq \sigma$, unless $k=n$ and $\sigma = \{0, 1, \dots, n-2, n\}$. Similarly to Matching 7.2, write $k = q\frac{d}{2} + r$, where

$$\begin{cases} 0 < r < \frac{d}{2} & \text{if } k \not\equiv 0 \left(\operatorname{mod} \frac{d}{2}\right), \\ r \in \left\{0, \frac{d}{2}\right\} \text{ and } q \text{ even} & \text{if } k \equiv 0 \left(\operatorname{mod} \frac{d}{2}\right). \end{cases}$$

case	# critical	$\lvert\sigma\rvert - v_\varphi(\sigma)$
$n \equiv 0 \pmod{d}$	3	$n - \frac{n}{d}$ (once), $n - 2\frac{n}{d}$ (twice)
$n \equiv 1 \pmod{d}$	3	$n - \frac{n-1}{d}$ (once), $n - 2\frac{n-1}{d} - 1$ (twice)
otherwise	1	$n - \lfloor \frac{n}{d} \rfloor$

Table 5: Critical simplices of Matching 9.1 (case $\widetilde{D}_n$, d odd).

Define a vertex v by

$$v = \begin{cases} q\frac{d}{2} & \text{if } q \text{ is even,} \\ q\frac{d}{2}+1 & \text{if } q \text{ is odd.} \end{cases}$$

(g1) If $d = 4$, q odd and $r = 1$, proceed as follows:

(g1.1) If $k \le n-2$, ignore vertices $0, \dots, k$, relabel vertices $\{n, n-1, \dots, k+1\}$ as $\{1, 2, \dots, n-k\}$, and construct the matching as in K^D_{n-k}.

(g1.2) Otherwise, σ is critical.

(g2) Otherwise, if $\sigma = \{0, 1, \dots, n-2, n\}$ and $v \ge n-1$, then σ is critical.

(g3) Otherwise, if $v \in \sigma$, match σ with $\sigma \veebar v$.

(g4) Otherwise, if $v > n$, then σ is critical.

(g5) Otherwise, proceed as follows. Let c be the size of the (possibly empty) connected component $C = \Gamma_{v+1}(\sigma)$ of the vertex $v+1$ in the subgraph $\Gamma(\sigma) \subseteq \Gamma$ induced by σ. Let

$$\ell = \begin{cases} \frac{d}{2} & \text{if } q \text{ even,} \\ \frac{d}{2}-2 & \text{if } q \text{ odd.} \end{cases}$$

(g5.1) If $\{n-1, n\} \subseteq C$, then σ is critical.

(g5.2) Otherwise, if $c < \ell$, match σ with $\sigma \cup \{v\}$.

(g5.3) Otherwise, if $c = \ell$, $n-1 \notin C$ and $n \in C$, then σ is critical.

(g5.4) Otherwise, ignore vertices $0, 1, \dots, v-1$, relabel $\{n, n-1, \dots, v+1\}$ as $\{1, 2, \dots, n-v\}$ and construct the matching as in $K^D_{n-v,\ell}$.

Lemma 9.3 *Matchings 9.1 and 9.2 are φ_d–precise. In addition, critical simplices of Matching 9.1 are as in Table 5.* □

Table 6 shows the local homology in the case $\widetilde{D}_n$ for $n \le 9$, computed using the software library [19]. We employ the notation $\{d\} = R/(\varphi_d)$, as in [11; 10].

	$\widetilde{D}_4$	$\widetilde{D}_5$	$\widetilde{D}_6$	$\widetilde{D}_7$	$\widetilde{D}_8$	$\widetilde{D}_9$
H_0	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$
H_1	$\{3\}$	0	0	0	0	0
H_2	$\{4\}^3$	$\{4\}$	$\{3\}$	$\{3\}$	0	0
H_3	$m_{\widetilde{D}_4}$	$\{5\}$	$\{5\}$	0	0	$\{3\}$
H_4	R	$m_{\widetilde{D}_5}$	$\{4\} \oplus \{6\}^3$	$\{4\}^3 \oplus \{6\}$	$\{4\}^3$	$\{4\}$
H_5		R	$m_{\widetilde{D}_6}$	$\{4\}^4 \oplus \{7\}$	$\{4\}^2 \oplus \{7\}$	0
H_6			R	$m_{\widetilde{D}_7}$	$\{4\}^2 \oplus \{6\} \oplus \{8\}^3$	$\{8\}$
H_7				R	$m_{\widetilde{D}_8}$	$\{6\} \oplus \{9\}$
H_8					R	$m_{\widetilde{D}_9}$
H_9						R

$$m_{\widetilde{D}_4} = \{2\}^4 \oplus \{4\}^3 \oplus \{6\}^3$$
$$m_{\widetilde{D}_5} = \{2\}^2 \oplus \{4\} \oplus \{6\} \oplus \{8\}^3$$
$$m_{\widetilde{D}_6} = \{2\}^5 \oplus \{4\}^2 \oplus \{6\}^3 \oplus \{8\} \oplus \{10\}^3$$
$$m_{\widetilde{D}_7} = \{2\}^3 \oplus \{4\}^5 \oplus \{6\} \oplus \{8\} \oplus \{10\} \oplus \{12\}^3$$
$$m_{\widetilde{D}_8} = \{2\}^6 \oplus \{4\}^4 \oplus \{6\}^2 \oplus \{8\}^3 \oplus \{10\} \oplus \{12\} \oplus \{14\}^3$$
$$m_{\widetilde{D}_9} = \{2\}^4 \oplus \{4\}^2 \oplus \{6\}^2 \oplus \{8\} \oplus \{10\} \oplus \{12\} \oplus \{14\} \oplus \{16\}^3$$

Table 6: Homology in the case $\widetilde{D}_n$ for $n \leq 9$.

10 Case $I_2(m)$

In this section we consider the case $I_2(m)$ for $m \geq 5$ (see Figure 6). The Poincaré polynomial is given by $W_{I_2(m)} = [2]_q[m]_q$. Then the φ_d–weight is

$$\omega_{\varphi_d}(I_2(m)) = \begin{cases} 2 & \text{if } d = 2 \text{ and } m \text{ even,} \\ 1 & \text{if } d = 2 \text{ and } m \text{ odd,} \\ 1 & \text{if } d \geq 3 \text{ and } d \mid m, \\ 0 & \text{if } d \geq 3 \text{ and } d \nmid m. \end{cases}$$

Figure 6: A Coxeter graph of type $I_2(m)$.

In this case φ_d–precise matchings are easy to construct by hand. As a straightforward consequence we also obtain the homology groups $H_*(\mathbf{X_W}; R)$.

Matching 10.1 (φ_d–matching on $K_{\mathbf{W}_{I_2(m)}}$)

- If $d = 2$ and m is even, every simplex is critical. Critical simplices are then $\{1,2\}$ (size 2, weight 2), $\{1\}$, $\{2\}$ (size 1, weight 1) and $\varnothing$ (size 0, weight 0). By Theorem 2.3, the homology groups are $H_0(\mathbf{X_W}; R)_{\varphi_2} \cong R/(\varphi_2)$ and $H_1(\mathbf{X_W}; R)_{\varphi_2} \cong R/(\varphi_2)$.
- If $d = 2$ and m is odd, match $\{1,2\}$ with $\{1\}$ (both simplices have weight 1). The critical simplices are $\{2\}$ (size 1, weight 1) and $\varnothing$ (size 0, weight 0). The homology groups are $H_0(\mathbf{X_W}; R)_{\varphi_2} \cong R/(\varphi_2)$ and $H_1(\mathbf{X_W}; R)_{\varphi_2} \cong 0$.
- If $d \geq 3$ and $d \mid m$, match $\{2\}$ with $\varnothing$ (both simplices have weight 0). The critical simplices are $\{1,2\}$ (size 2, weight 1) and $\{1\}$ (size 1, weight 0). The homology groups are $H_0(\mathbf{X_W}; R)_{\varphi_d} \cong 0$ and $H_1(\mathbf{X_W}; R)_{\varphi_d} \cong R/(\varphi_d)$.
- If $d \geq 3$ and $d \nmid m$, match $\{1,2\}$ with $\{1\}$ and $\{2\}$ with $\varnothing$ (all simplices have weight 0). There are no critical simplices and all homology groups are trivial.

	H_3	H_4	F_4	E_6	E_7	E_8
H_0	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$
H_1	0	0	$\{2\}$	0	0	0
H_2	m_{H_3}	0	$\{2\} \oplus \{3\} \oplus \{6\}$	0	0	0
H_3		m_{H_4}	m_{F_4}	0	0	0
H_4				$\{6\} \oplus \{8\}$	$\{6\}$	$\{4\}$
H_5				m_{E_6}	$\{6\}$	0
H_6					m_{E_7}	$\{8\} \oplus \{12\}$
H_7						m_{E_8}

$m_{H_3} = \{2\} \oplus \{6\} \oplus \{10\}$

$m_{H_4} = \{2\} \oplus \{3\} \oplus \{4\} \oplus \{5\} \oplus \{6\} \oplus \{10\} \oplus \{12\} \oplus \{15\} \oplus \{20\} \oplus \{30\}$

$m_{F_4} = \{2\} \oplus \{3\} \oplus \{4\} \oplus \{6\} \oplus \{8\} \oplus \{12\}$

$m_{E_6} = \{3\} \oplus \{6\} \oplus \{9\} \oplus \{12\}$

$m_{E_7} = \{2\} \oplus \{6\} \oplus \{14\} \oplus \{18\}$

$m_{E_8} = \{2\} \oplus \{3\} \oplus \{4\} \oplus \{5\} \oplus \{6\} \oplus \{8\} \oplus \{10\} \oplus \{12\} \oplus \{15\} \oplus \{20\} \oplus \{24\} \oplus \{30\}$

Table 7: Homology in the exceptional finite cases (see Section 11).

To summarize, the local homology is given by

$$H_0(X_W; R) \cong \frac{R}{(\varphi_2)}, \quad H_1(X_W; R) \cong \bigoplus_{\substack{d \mid m \\ d \geq 2}} \frac{R}{(\varphi_d)}.$$

This result corrects the one given in [11], where proper divisors of m were not taken into account.

11 Exceptional cases

In all exceptional finite and affine cases (see for example [2, Appendix A1] for a classification), we constructed precise matchings by means of a computer program. The explicit description of these matchings, together with proof of precision and homology computations, can be obtained through the software library available online [18].

The matchings were obtained using a variant of the algorithm described in [15]. We initially left out the constraint of precision; instead, we looked for acyclic and φ_d–weighted matchings with a minimal number of critical simplices. The core idea was to

	$\tilde{I}_1$	$\tilde{G}_2$	$\tilde{F}_4$	$\tilde{E}_6$	$\tilde{E}_7$	$\tilde{E}_8$
H_0	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$
H_1	R	$\{2\} \oplus \{3\}$	$\{2\}$	0	0	0
H_2		R	$\{2\} \oplus \{3\}$	0	0	0
H_3			$m_{\tilde{F}_4}$	$\{3\}$	$\{3\}$	0
H_4			R	$\{5\} \oplus \{8\}$	0	$\{4\}$
H_5				$m_{\tilde{E}_6}$	0	0
H_6				R	$m_{\tilde{E}_7}$	$\{5\} \oplus \{8\}$
H_7					R	$m_{\tilde{E}_8}$
H_8						R

$$m_{\tilde{F}_4} = \{2\}^2 \oplus \{3\} \oplus \{4\} \oplus \{8\}$$
$$m_{\tilde{E}_6} = \{2\} \oplus \{3\}^3 \oplus \{6\}^2 \oplus \{9\}^2 \oplus \{12\}^2$$
$$m_{\tilde{E}_7} = \{2\}^3 \oplus \{3\} \oplus \{4\} \oplus \{6\} \oplus \{8\} \oplus \{10\} \oplus \{14\} \oplus \{18\}$$
$$m_{\tilde{E}_8} = \{2\}^2 \oplus \{3\} \oplus \{4\} \oplus \{5\} \oplus \{8\} \oplus \{9\} \oplus \{14\}$$

Table 8: Homology in the exceptional affine cases.

rewrite this as a linear optimization problem, in terms of boolean variables $x_e \in \{0,1\}$ that indicate if an edge e appears in the matching. The matchings obtained by solving this optimization problem always turned out to be φ_d–precise. This is a further indication that precise matchings emerge naturally in the study of the local homology of Artin groups.

The homology groups can be computed using Theorem 2.3. They are described in Tables 7 and 8, where again we employ the notation $\{d\} = R/(\varphi_d)$. We recover the results of [11] (for the finite cases) and [24] (for the affine cases), except for minor corrections in the cases E_8 and $\widetilde{E}_8$.

Acknowledgements I would like to thank my supervisor Mario Salvetti, for his strong support and help throughout my PhD.

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Scuola Normale Superiore
Pisa, Italy

`giovanni.paolini@sns.it`

Received: 2 July 2018 Revised: 14 January 2019