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On local tameness of certain graphs of groups

RITA GITIK

Let G be the fundamental group of a finite graph of groups with Noetherian edge groups and locally tame vertex groups. We prove that G is locally tame. It follows that if a finitely presented group H has a nontrivial JSJ–decomposition over the class of its VPC(k) subgroups for $k=1$ or $k=2$, and all the vertex groups in the decomposition are flexible, then H is locally tame.

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1 Introduction

Let H be a subgroup of a group G given by any presentation $G=\langle X \mid R\rangle$. Let K be the standard presentation 2–complex of G, ie K has one vertex, K has an edge, which is a loop, for every generator $x\in X$, and K has a 2–cell for every relator $r\in R$. The Cayley complex of G, denoted by $\mathrm{Cayley}_2(G)$, is the universal cover of K. Denote by $\mathrm{Cayley}_2(G,H)$ the cover of K corresponding to a subgroup H of G.

Definition 1 (see Gitik [3] and Mihalik [8]) A finitely generated subgroup H of a finitely presented group G is tame in G if, for any finite subcomplex C of $\mathrm{Cayley}_2(G,H)$ and for any component C_0 of $\mathrm{Cayley}_2(G,H)-C$, the group $\pi_1(C_0)$ is finitely generated.

Note that Definition 1 makes sense for any subgroup H and for any presentation of a group G, however it was shown in [8] that tameness of a finitely generated subgroup H is independent of a finite presentation of a group G. As we are interested in investigating a property of a group, rather than a property of a presentation, we will work with finitely presented groups, unless explicitly stated otherwise.

It is not known if there exists a finitely generated subgroup H of a finitely presented group G such that H is not tame in G. Moreover, it is not known if the trivial subgroup is tame in any finite presentation of a group. However, an infinitely generated subgroup might not be tame in a finitely presented group.

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Example 2 Let $F = \langle x, y \rangle$ be a free group of rank two, and let F' be its commutator subgroup. The complex $\mathrm{Cayley}_2(F, F')$ is one-dimensional and is homeomorphic to a lattice in the Euclidean plane, consisting of the horizontal lines $\{y = n \mid n \in \mathbb{Z}\}$ and the vertical lines $\{x = n \mid n \in \mathbb{Z}\}$. As the fundamental group of the complement of any finite connected subcomplex C in $\mathrm{Cayley}(F, F')$ is infinitely generated, F' is not tame in F.

The following definition was given in Gitik [4]:

Definition 3 A finitely presented group G is locally tame if all finitely generated subgroups of G are tame in G.

Note that Definition 3 makes sense for any presentation of a group G.

A finite subgroup can be tame in an infinite presentation of a group. Moreover, an infinite presentation of a group can be locally tame.

Example 4 Let $F = \langle X \rangle$ be an infinitely generated free group and let H be a finitely generated subgroup of F. Note that $\mathrm{Cayley}_2(F, H)$ is one-dimensional. It has a finite subcomplex, called the core, which carries H, and the complement of the core in $\mathrm{Cayley}_2(F, H)$ is a forest. Hence, for any finite subcomplex of $\mathrm{Cayley}_2(F, H)$, the fundamental group of any component of $\mathrm{Cayley}_2(F, H) - C$ is finitely generated. Therefore any free group is locally tame.

Definition 1 was motivated by the topology of 3–manifolds.

A manifold M is called a missing boundary manifold if it can be embedded in a compact manifold $\overline{M}$ such that $\overline{M} - M$ is a closed subset of the boundary of $\overline{M}$. Simon conjectured in [13] that if M_0 is a compact orientable irreducible 3–manifold, and M is the cover of M_0 corresponding to a finitely generated subgroup of $\pi_1(M_0)$, then M is a missing boundary manifold. Perelman's solution of Thurston's geometrization conjecture in 2003 implies that Simon's conjecture holds for all compact orientable irreducible 3–manifolds; see Bessières, Besson, Maillot, Boileau and Porti [1] and Kleiner and Lott [7].

Tucker proved in [14] that a noncompact orientable irreducible 3–manifold M is a missing boundary manifold if and only if for any compact submanifold C of M the fundamental group of any connected component of $M - C$ is finitely generated. It follows that $\pi_1(M)$ is tame in $\pi_1(M_0)$, so the fundamental groups of compact orientable irreducible 3–manifolds are locally tame.

Tameness of a subgroup is connected to other properties which have been studied for a long time.

It is shown in Mihalik and Tschantz [9] that if the trivial subgroup is tame in G then $\pi_1^\infty(G)$ (the fundamental group at infinity of G) is pro-finitely generated. It is shown in [8] that if a finitely generated subgroup H is tame in G then $\pi_1^\infty(G, H)$ is pro-finitely generated.

It is shown in [9] that if the trivial subgroup is tame in G then G is QSF (quasi-simply-filtrated).

Recall that a group is called Noetherian or slender if all its subgroups are finitely generated. A group is polycyclic if it is Noetherian and solvable. For $n \geq 0$ a group G is VPC(n) (virtually polycyclic of length n) if it has $n+1$ subgroups, $G_0, \dots, G_n$, such that G_{i+1} is a normal subgroup of G_i for $0 \leq i \leq n-1$, the quotient groups G_i/G_{i+1} are isomorphic to $\mathbb{Z}$ for $0 \leq i \leq n-1$, G_n is the trivial subgroup, and G_0 has finite index in G.

Note that VPC(0) groups are finite, VPC(1) groups are finite extensions of $\mathbb{Z}$, and VPC(2) groups are finite extensions of an extension of $\mathbb{Z}$ by $\mathbb{Z}$. There are only two nonisomorphic extensions of $\mathbb{Z}$ by $\mathbb{Z}$, namely the fundamental group of a torus and the fundamental group of a Klein bottle.

It is unknown whether all finitely presented Noetherian groups are virtually polycyclic (Question 11.38 from the Kourovka notebook in Khukhro and Mazurov [6]), however there exist finitely generated Noetherian groups that are not virtually polycyclic, for example the Tarski monster.

The main result of this paper is the following theorem:

Theorem 5 *Let G be the fundamental group of a finite graph of groups with Noetherian edge groups such that the presentation of G given by that graph-of-groups structure is finite. If all the vertex groups of G are locally tame then G is locally tame.*

Theorem 5 has the following corollary:

Lemma 6 *The fundamental group of a surface is locally tame.*

Remark 7 The fundamental group of a compact two-dimensional orbifold is locally tame. Indeed, the fundamental group of a compact two-dimensional orbifold is a finite extension of the fundamental group of a surface, so the result follows from Lemma 6 and Remark 11.

Recall that a subgroup H is elliptic in a graph of groups G if H is contained in a conjugate of a vertex group. A vertex group K of a JSJ–decomposition of G which fails to be elliptic in some other JSJ–decomposition of G is called flexible; see Guirardel and Levitt [5, page 6]. For a background on JSJ–decompositions see Fujiwara and Papasoglu [2], Guirardel and Levitt [5] and Scott and Swarup [10].

Theorem 5 implies the following interesting result:

Lemma 8 *If a finitely presented group G has a nontrivial* JSJ*–decomposition over the class of its* VPC(k) *subgroups for $k = 1$ or $k = 2$, and all the vertex groups in the decomposition are flexible, then G is locally tame.*

Corollary to the proof of Lemma 8 *Let G be the fundamental group of a finite graph of groups which has all the vertex groups isomorphic to the product of $\mathbb{Z}$ with a finitely generated surface group, and all the edge groups isomorphic to either $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$. Then G is locally tame.*

Indeed, $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ are Noetherian. The proof of Lemma 8 demonstrates that a product of $\mathbb{Z}$ with a finitely generated surface group is locally tame, so the result follows from Theorem 5.

Remark 9 (see [5, Corollary 6.3]) Let G be a finitely presented group which has a JSJ–decomposition over the class of its VPC$(n+1)$ subgroups. Let K be a flexible vertex group of this decomposition. Then K is either VPC$(n+1)$ or K has a finite-index subgroup L such that L has a normal VPC(n) subgroup N with L/N the finitely generated fundamental group of a surface. Furthermore, if L/N is the fundamental group of a closed surface, then $K = G$.

Conjecture If a finitely presented group G has a nontrivial JSJ–decomposition over the class of its VPC$(n+1)$ subgroups for $n \geq 0$, and all the vertex groups in the decomposition are flexible, then G is locally tame.

2 Proof of Theorem 5

Let G be a group generated by a set X and let H be a subgroup of G. Let $\{Hg\}$ be the set of right cosets of H in G.

The coset graph of G with respect to H, denoted by $\mathrm{Cayley}(G, H)$, is the oriented graph whose vertices are the cosets $\{Hg\}$, the set of edges is $\{Hg\} \times (X \cup X^{-1})$, and for $x \in X \cup X^{-1}$ the edge (Hg, x) begins at the vertex Hg and ends at the vertex Hgx. Denote the Cayley graph of G by $\mathrm{Cayley}(G)$. Note that $\mathrm{Cayley}(G, H)$ is the quotient of $\mathrm{Cayley}(G)$ by left multiplication by H. Also note that the 1–skeleton of $\mathrm{Cayley}_2(G)$ is $\mathrm{Cayley}(G)$, and the 1–skeleton of $\mathrm{Cayley}_2(G, H)$ is $\mathrm{Cayley}(G, H)$.

Let G be generated by a disjoint union of sets X_i for $1 \le i \le n$. We call a connected subcomplex of $\mathrm{Cayley}_2(G, H)$ an X_i–component if all the edges in the 1–skeleton of that subcomplex have the form (Hg, x) with $x \in X_i \cup X_i^{-1}$.

Proof of Theorem 5 Let G be the fundamental group of a finite graph of groups with vertex groups V_i for $1 \le i \le n$ and edge groups E_j for $1 \le j \le m$. As the presentation of G given by that graph-of-groups structure is finite and all the edge groups are Noetherian, hence finitely generated, Lemma 10 implies that all the vertex groups are finitely presented. Let the vertex group V_i be generated by a finite set X_i such that the sets X_i and X_k are disjoint for $i \neq k$.

The group G acts on the Bass–Serre tree T of the graph-of-groups decomposition of G; see [12], Chapter 5. The vertices of the tree T are the G–cosets of the vertex groups V_i and the edges of T are the G–cosets of the edge groups E_j. The stabilizer of a vertex gV_i of T is the subgroup gV_ig^{-1} of G and the stabilizer of an edge gE_j of T is the subgroup gE_jg^{-1} of G.

Consider a finitely generated subgroup H of G. It acts on the tree T. The H–stabilizer of a vertex gV_i of T is $gV_ig^{-1} \cap H$ and the H–stabilizer of an edge gE_j of T is $gE_jg^{-1} \cap H$. Hence H is the fundamental group of a (possibly infinite) graph of groups which has the vertex groups of the form $gV_ig^{-1} \cap H$ and the edge groups of the form $gE_jg^{-1} \cap H$; see [11, pages 157 and 162]. As the edge groups of G are Noetherian, the edge groups of H are also Noetherian, hence they are finitely generated. As H is finitely generated, Lemma 10 implies that the vertex groups of H are finitely generated.

Note that as the presentation of G is given by the graph-of-groups structure, any 2–cell in $\mathrm{Cayley}_2(G)$ belongs to either a maximal X_i–component of $\mathrm{Cayley}_2(G)$ or to the union of two such components. Hence a maximal X_i–component of $\mathrm{Cayley}_2(G)$ is simply connected because it contains all the 2–cells of $\mathrm{Cayley}_2(G)$ with boundaries in that maximal X_i–component.

All maximal X_i–components of $\mathrm{Cayley}_2(G, H)$ have fundamental groups which are the vertex groups of H. Indeed, let Y_i be the maximal X_i–component of $\mathrm{Cayley}_2(G)$ containing 1_G, the basepoint of $\mathrm{Cayley}_2(G)$. The set X_i generates the vertex group V_i and the vertices of $\mathrm{Cayley}_2(G)$ are the elements of G. For $v \in V_i$ consider the vertices g and gv of gY_i. There exists $h \in H$ such that $hg = gv$ if and only if $gvg^{-1} = h \in H$. So the fundamental group of the image of gY_i under the quotient map of $\mathrm{Cayley}_2(G)$ to $\mathrm{Cayley}_2(G, H)$ is isomorphic to $gV_ig^{-1} \cap H$, which is a finitely generated vertex group of H.

Therefore the maximal X_i–components of $\mathrm{Cayley}_2(G, H)$ are homeomorphic to $\mathrm{Cayley}_2(V_i, U_i)$, with U_i a finitely generated subgroup of V_i.

As H is finitely generated, there exists a finite connected subcomplex $(K, H \cdot 1)$ of $\mathrm{Cayley}_2(G, H)$ such that the inclusion map of $(K, H \cdot 1)$ in $\mathrm{Cayley}_2(G, H)$ induces an epimorphism of $\pi_1(K, H \cdot 1)$ with $\pi_1(\mathrm{Cayley}_2(G, H), H \cdot 1) = H$.

Let C be a compact subcomplex of $\mathrm{Cayley}_2(G, H)$. Note that C has nonempty intersection with only finitely many maximal X_i–components of $\mathrm{Cayley}_2(G, H)$. The complex K can be enlarged to contain C. It can be enlarged more, so that it consists of finitely many maximal X_i–components of $\mathrm{Cayley}_2(G, H)$ which have nontrivial intersection with K and the 2–cells with boundaries in the union of those X_i–components. Denote the enlarged complex by L. The complex K is finite but the complex L might be infinite. By construction, $L - C$ has a finite number of connected components. Note that even though the inclusion of $\pi_1(K, H \cdot 1)$ into $\pi_1(\mathrm{Cayley}_2(G, H), H \cdot 1)$ is an epimorphism, the inclusion of $\pi_1(L, H \cdot 1)$ into $\pi_1(\mathrm{Cayley}_2(G, H), H \cdot 1)$ is an isomorphism, because $\pi_1(L, H \cdot 1)$ is the fundamental group of a subgraph of groups; see [11, Proposition 3.6].

As the vertex groups V_i are locally tame, the fundamental group of each component of the complement of C in any maximal X_i–component is finitely generated, so each component of $L - C$ is a finite union of 2–complexes with finitely generated fundamental groups. Note that if a locally finite complex M is the union of two connected subcomplexes A and B such that $\pi_1(A)$ and $\pi_1(B)$ are finitely generated and $A \cap B$ is connected, then $\pi_1(M)$ is finitely generated. Note also that if a locally finite complex M has finitely generated fundamental group and contains disjoint isomorphic subcomplexes A and B, then the fundamental group of the complex obtained by identifying A and B is finitely generated. It follows that the fundamental group of each component of $L - C$ is finitely generated.

By construction, $\mathrm{Cayley}_2(G,H) - C = (\mathrm{Cayley}_2(G,H) - L) \cup (L - C)$. Let W be a connected component of the closure of $\mathrm{Cayley}_2(G,H) - L$. Then $W \cap L$ is connected and $\pi_1(W \cap L)$ is isomorphic to $\pi_1(W)$ because L carries the fundamental group of $\mathrm{Cayley}_2(G,H)$. So, for each component L_i of $L - C$ which intersects W nontrivially, $\pi_1(L_i \cap W) = \pi_1(W)$. Let W^i be the (possibly infinite) union of all components of $\mathrm{Cayley}_2(G,H) - L$ which have nontrivial intersection with L_i. Then $\pi_1(W^i \cup L_i) = \pi_1(L_i)$, which is finitely generated. Hence the fundamental group of each component of $\mathrm{Cayley}_2(G,H) - C$ is finitely generated, proving Theorem 5. □

Lemma 10 *Let G be the fundamental group of a possibly infinite graph of groups.*

(1) *If the presentation of G given by that graph-of-groups structure is finitely generated and all the edge groups of G are finitely generated, then all the vertex groups of G are finitely generated.*

(2) *If the presentation of G given by that graph-of-groups structure is finite and all the edge groups of G are finitely generated, then all the vertex groups of G are finitely presented.*

Proof We give a proof in a special case, when G is a product of A and B amalgamated over a finitely generated group C.

(1) If G is finitely generated then A and B are also finitely generated. Indeed, assume without loss of generality that A is not finitely generated. Then $A = \bigcup_{i=1}^{\infty} A_i$ with $A_i \subsetneq A_{i+1}$, and all A_i are finitely generated. As C is finitely generated, there exists $j > 0$ such that $C \subset A_j$. Consider a sequence $G_i = A_i *_C B$ for $i \geq j$. By construction, $G_i \subsetneq G_{i+1}$ and $\bigcup G_i = G$. Hence G should be infinitely generated, contradicting the assumptions. Therefore both A and B are finitely generated.

(2) Assume that G is finitely presented. We claim that in this case both A and B are finitely presented. Indeed, assume that A is infinitely presented. We have proved already that A is finitely generated, so $A = \langle a_1, \dots, a_n \mid r_1, r_2, \dots, r_k, \dots \rangle$. Let $B = \langle b_1, \dots, b_m \mid s_1, \dots, s_p, \dots \rangle$ and let C be generated by a finite set $c_1, \dots, c_l$. Let f_A and f_B be the inclusions of C in A and B, respectively, defining the amalgamated free product $G = A *_C B$. Then, as G is finitely presented, there exists $N > 0$ such that

$$G = \big\langle a_1, \dots, a_n, b_1, \dots, b_m \mid r_1, r_2, \dots, r_N, s_1, \dots, s_p, \dots, \\ f_A(c_1) = f_B(c_1), \dots, f_A(c_l) = f_B(c_l) \big\rangle.$$

Consider a sequence of groups $A_i = \langle a_1, \dots, a_n \mid r_1, r_2, \dots, r_i \rangle$. There is a surjection from A_i to A_{i+1} for every $i \geq 1$ and each A_i maps onto A. Let G_i denote the group obtained from A_i and B by adding the gluing relations in C, even though C may not be a subgroup of A_i. In particular, consider the map G_N to G, where N is as defined above. That map is an isomorphism, hence the map of A_N to A is also an isomorphism, so the group A is finitely presented.

The proof of the general case is obtained by collapsing all the edges in the graph-of-groups structure of G which are not incident to A. This yields a new graph-of-groups structure for G in which every edge is incident to A. Now we can apply the above arguments using all the vertices in the new graph of groups at once to construct the groups G_i. □

3 Proofs of Lemmas 6 and 8

Remark 11 The following result was proved in [4]. Let K_0 be a finite-index subgroup of a finitely presented group K. A finitely generated subgroup H of K is tame in K if and only if $H \cap K_0$ is tame in K_0.

It follows that virtually locally tame groups are locally tame.

Proof of Lemma 6 Example 4 shows that any free group is locally tame, so the fundamental group of a nonclosed surface is tame.

It is shown in [4] that finitely generated abelian groups are locally tame, so the fundamental group of a torus is locally tame.

Note that the fundamental group of a closed orientable surface of genus greater than one can be written as a product of two free groups amalgamated over a cyclic subgroup. Hence Theorem 5 implies that the fundamental groups of closed orientable surfaces of genus greater than one are locally tame.

As any nonorientable closed surface is double-covered by an orientable closed surface, Remark 11 implies that the fundamental groups of nonorientable closed surfaces are locally tame. □

Proof of Lemma 8 Consider, first, the case when a finitely presented group G has a nontrivial JSJ–decomposition over the class of its VPC(1) subgroups and all the vertex groups in the decomposition are flexible. Note that VPC(1) groups are Noetherian.

The flexible vertex groups in such a JSJ–decomposition are either VPC(1) or virtually a finitely generated fundamental group of surfaces; see [5, Corollary 6.3]. Furthermore, if a vertex group M in that decomposition is virtually the fundamental group of a closed surface, then $G = M$.

Hence, Remark 11 and Lemma 6 imply that the group G satisfies the conditions of Theorem 5, therefore it is locally tame.

Next, consider the case when a finitely presented group G has a nontrivial JSJ–decomposition over the class of its VPC(2) subgroups and all the vertex groups in the decomposition are flexible. Note that VPC(2) subgroups are Noetherian.

The flexible vertex groups in such JSJ–decomposition are either VPC(2) or virtually cyclic by a finitely generated surface group; see [5, Corollary 6.3]. Furthermore, if a flexible vertex group K in that decomposition is virtually cyclic by a closed surface group, then $G = K$.

If a group L is cyclic by a finitely generated surface group then there exists a surface M and a normal cyclic subgroup N of L such that the sequence

$$1 \to N \to L \to \pi_1(M) \to 1$$

is exact and L is the fundamental group of a bundle X over M with fiber $\mathbb{S}^1$.

If H is a finitely generated subgroup of L then either $H \cap N = \{1\}$ or $H \cap N$ is isomorphic to $\mathbb{Z}$. Let K be the image of H in $\pi_1(M)$. Note that K is finitely generated. Let M_K be the cover of M with fundamental group K. Then H is the fundamental group of a bundle X_H over M_K with fiber either $\mathbb{S}^1$ if $H \cap N = Z$ or $\mathbb{R}$ if $H \cap N = \{1\}$. As K is finitely generated, M_K is a missing boundary surface. It follows that, in either case, X_H is a missing boundary 3–manifold, so L is locally tame.

If a group L is VPC(2) then it is virtually either the fundamental group of a torus or the fundamental group of a Klein bottle, hence Remark 11 and Lemma 6 imply that L is locally tame.

Therefore, Remark 11 implies that the group G satisfies the conditions of Theorem 5, so it is locally tame. □

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Department of Mathematics, University of Michigan
Ann Arbor, MI, United States

ritagtk@umich.edu

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