Tight small Seifert fibered manifolds with $e_0 = -2$

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We provide the classification of tight contact structures on some small Seifert fibered manifolds. As an application of this classification, combined with work of Lekili, we obtain infinitely many counterexamples to a question of Honda, Kazez and Matić that asks whether a right-veering, nondestabilizable open book necessarily supports a tight contact structure.

1 Introduction

The classification of tight contact structures on a given closed, oriented 3–manifold is still one of the main and widely open problems of three-dimensional contact geometry. Deep work of Colin, Giroux and Honda in [1] proves that a closed, oriented, atoroidal 3–manifold has finitely many tight contact structures up to contact isotopy. Conversely, every closed, oriented, irreducible, toroidal 3–manifold has infinitely many tight contact structures up to contact isotopy. In particular, in the case of closed, oriented Seifert fibered manifolds, if the base orbifold has positive genus or if the number of singular fibers is greater than three — so, in both cases, the manifold contains a (vertical) incompressible torus — then there are infinitely many tight contact structures on the manifold up to contact isotopy. Finally, in [30], the existence question for tight contact structures on the remaining Seifert fibered manifolds was resolved by Lisca and Stipsicz, who proved that, except for a small infinite family, all others admit tight contact structures.

In this paper we are interested in the classification problem of tight contact structures on *small* Seifert fibered manifolds, that is, the Seifert fibered manifolds over S^2 with three singular fibers. Such manifolds, by using normalized Seifert invariants, are denoted by $M = M(e_0; r_1, r_2, r_3)$, where the *integer Euler number* $e_0(M) \in \mathbb{Z}$ is an invariant of the Seifert fibration once we require $r_i \in \mathbb{Q} \cap (0, 1)$. The manifold $M(e_0; r_1, r_2, r_3)$ is also described by the surgery diagram in Figure 1. The manifold $M(e_0; r_1, r_2, r_3)$ is an irreducible, atoroidal rational homology sphere, except possibly in the degenerate
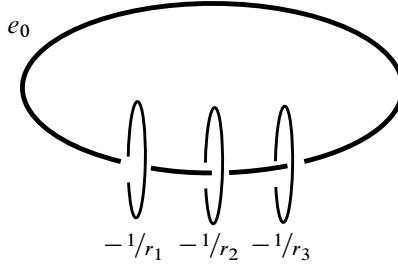
Figure 1: The manifold $M(e_0; r_1, r_2, r_3)$. Small unknots in the surgery diagram give rise to singular fibers of Seifert fibration.

situation $r_1 + r_2 + r_3 = -e_0$: in this case M contains a nonseparating, incompressible, horizontal surface, and is therefore a bundle over S^1 with periodic monodromy (see Hatcher [18, Proposition 1.11]). Note that the normalized Seifert invariants clearly satisfy $0 < r_1 + r_2 + r_3 < 3$, so M can be a surface bundle only when $e_0 = -2$ or -1. Our ability to classify tight contact structures on $M(e_0; r_1, r_2, r_3)$ depends crucially on the value of $r_1 + r_2 + r_3$.

The aforementioned result of Colin, Giroux and Honda says that unless the small Seifert fibered space is also a torus bundle over the circle, the number of tight contact structures it can admit, up to isotopy, is finite. Hence, if $e_0 \neq -1, -2$, then there are always finitely many isotopy classes of tight contact structures. The classification of tight contact structures on $M(e_0; r_1, r_2, r_3)$ for $e_0 \leq -3$ and $e_0 > 0$ was completed by Wu [37]. The case $e_0 = 0$ was completed by Ghiggini, Lisca and Stipsicz [10]. The cases $e_0 = -1, -2$ are incomplete; only some partial results are available, due to Ghiggini, Lisca and Stipsicz [11] in case of $e_0 = -1$ and Ghiggini [7] in case of $e_0 = -2$. The purpose of this paper is to study the classification of tight contact structures on $M(-2; r_1, r_2, r_3)$. The classification on such manifolds was initiated in [7], where Ghiggini among other things gave the classification of tight contact structures on those $M(-2; r_1, r_2, r_3)$ which are L–spaces. In this paper we generalize his result to some $M(-2; r_1, r_2, r_3)$ that are not L–spaces. First we set some notation.

For each of the rational numbers r_1, r_2 and r_3 in $(0, 1)$ we write

$$(1\text{-}1) \qquad -\frac{1}{r_i} = [a_0^i, a_1^i, \ldots, a_{n_i}^i] = a_0^i - \cfrac{1}{a_1^i - \cfrac{1}{\ddots - \cfrac{1}{a_{n_i}^i}}}$$

for some uniquely determined integers

$$a_0^i, a_1^i, \ldots, a_{n_i}^i \leq -2 \quad \text{for } i = 1, 2, 3.$$

Let $T(r_i)$ denote

$$T(r_i) = \left| \prod_{k=0}^{n_i} (a_k^i + 1) \right|.$$

Recall that we are using the normalized Seifert invariants, that is, $r_i \in \mathbb{Q} \cap (0, 1)$, and hence $0 < r_1 + r_2 + r_3 < 3$. We state our first result.

Theorem 1.1 *If*

(1) $r_1 + r_2 + r_3 \geq \frac{9}{4}$;

(2) $r_1 + r_2 + r_3 < 2$;

(3) $r_1 = \frac{1}{2}$, $r_2 = \frac{2}{3}$ and $r_3 = \frac{k}{k+1}$ for $k \geq 6$

holds, then the manifold $M(-2; r_1, r_2, r_3)$ admits exactly $T(r_1)T(r_2)T(r_3)$ isotopy classes of tight contact structures, all of which are Stein fillable. The explicit Stein filling can be described by Legendrian surgery on all possible Legendrian realizations of the link (after converting each $-1/r_i$–framed unknot component to a link of a_j^i–framed unknots with the help of equation (1-1) *in* Figure 3).

The third part of Theorem 1.1 indicates that the bounds on the sum of the Seifert invariants in the previous two are not necessarily sharp. On the other hand, our next result shows that the classification scheme for $M(-2; r_1, r_2, r_3)$ with $2 \leq r_1 + r_2 + r_3 < \frac{9}{4}$ is much more complicated.

Theorem 1.2 *The manifold $M\left(-2; \frac{1}{2}, \frac{2}{3}, \frac{5n+1}{6n+1}\right)$ for $n \geq 1$ admits exactly $\frac{1}{2}n(n+1)$ isotopy classes of tight contact structures, which are all homotopic and strongly fillable. On the other hand, at least n of them are Stein fillable, and when n is even at least $\frac{n}{2}$ of the remaining ones are not Stein fillable.*

Remark 1.3 As mentioned above the manifolds $M(-2; r_1, r_2, r_3)$ with $r_1 + r_2 + r_3 = 2$ are particularly interesting, as they also enjoy a (unique) surface bundle structure over the circle. One can easily determine the fiber genus from the Seifert invariants. For example, the manifolds $M\left(-2; \frac{1}{2}, \frac{3}{4}, \frac{3}{4}\right)$, $M\left(-2; \frac{1}{2}, \frac{2}{3}, \frac{5}{6}\right)$ and $M\left(-2; \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$ are the only torus bundles over the circle. Our finiteness result in Theorem 1.1 of course cannot include these sporadic cases. We only comment that the classification of tight contact structures on these manifolds is given independently by Honda in [20] and Giroux
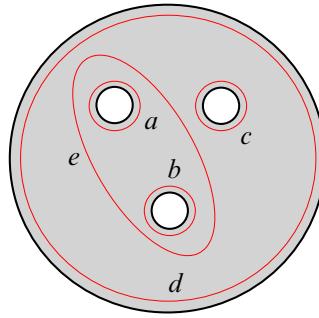
Figure 2: The open book (Σ, ϕ_k) on the Seifert fibered manifolds $M\left(-2; \frac{1}{2}, \frac{2}{3}, \frac{k}{k+1}\right)$.

in [16]. According to that classification each of these manifolds carries infinitely many tight contact structures, distinguished by their Giroux torsion. Among them there is a unique one with Giroux torsion zero; all others have positive Giroux torsion. A result of Gay in [4] proves that positive Giroux torsion is an obstruction to (strong) fillability (see Ghiggini, Honda and Van Horn-Morris [9]), so these manifolds can admit at most one Stein fillable tight structure. On the other hand one can easily see an explicit Stein fillable tight structure from Figure 1 on each of these manifolds. The remaining surface bundles are of higher genus and necessarily have periodic monodromy. In particular, they do carry finitely many tight contact structures. Unfortunately we do not have a technique yet to address the classification on such manifolds.

By using Theorem 1.1(3), we can extend work of Lekili in [26, Theorem 1.2] to provide an infinite family of examples of right-veering mapping classes on the four-holed sphere Σ each of which is nondestabilizable and yet supports an overtwisted contact structure. These examples then provide an infinite family of counterexamples for a conjecture of Honda, Kazez and Matić in [21]. See Lisca [27], Kazez and Roberts [25] and Ito and Kawamuro [22] for more such examples.

Corollary 1.4   *For each $k \geq 6$, there are open books (Σ, ϕ_k) on the Seifert fibered manifolds $M\left(-2; \frac{1}{2}, \frac{2}{3}, \frac{k}{k+1}\right)$, where the mapping classes $\phi_k = t_a^{k+1} t_b^2 t_c t_d t_e^{-2}$ are right-veering, cannot be destabilized and support overtwisted contact structures.* (See Figure 2.)

Proof   Lekili proves that the mapping classes $\phi_k = t_a^{k+1} t_b^2 t_c t_d t_e^{-2}$ are right-veering and nondestabilizable. Moreover he proves [26, Proposition 3.2] that the contact invariants $c(\xi_{(\Sigma, \phi_k)})$ of the corresponding contact structures vanish. In particular, the contact structures $\xi_{(\Sigma, \phi_k)}$ cannot be Stein fillable. It is not difficult to check
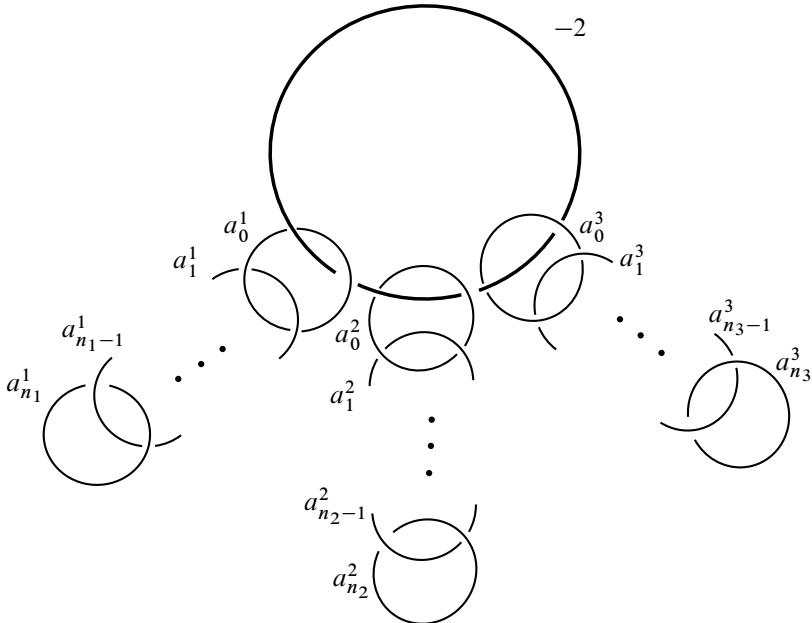
Figure 3: The manifold $M(-2; r_1, r_2, r_3)$, where $-1/r_i = [a_0^i, a_1^i, \ldots, a_{n_i}^i]$.

that the open book smoothly describes the manifold $M\left(-2; \frac{1}{2}, \frac{2}{3}, \frac{k}{k+1}\right)$ on which, by Theorem 1.1(3) above, for each $k \geq 6$ there is exactly one tight contact structure which is Stein fillable.   □

2   Preliminaries

In this section, assuming the reader is familiar with convex surface theory of Giroux [14], we will list some results regarding bypasses and their consequences due to Honda [19]. These results will be used again and again in the rest of the paper.

We first recall the twisting number, which is an invariant in the classification of the tight structures on Seifert fibered manifolds.

Let ξ be a tight structure on a Seifert fibered manifold and L be a Legendrian knot (smoothly) isotopic to a regular fiber of the Seifert fibration. Let ϕ denote this smooth isotopy. The twisting number $t(L, \phi)$ is defined as the difference between contact framing and the framing induced by the Seifert fibration. The maximal twisting number of the contact structure ξ is defined to be $t(\xi) = \max_\phi \min\{t(L, \phi), 0\}$. Wu [37, Theorem 1.3] proved that $t(\xi) < 0$ for any tight contact structure ξ on $M(-2; r_1, r_2, r_3)$. The same result was also obtained by Ghiggini [7, Corollary 4.6] and independently by Massot [31, Theorem B].

We now return to describe the effect of bypass attachment in terms of the Farey tessellation of the hyperbolic unit disk.

Theorem 2.1 (the Farey tessellation [19]) *Let T be a convex torus in standard form with $|\Gamma_T| = 2$, dividing slope s and ruling slope $r \neq s$. Let D be a bypass for T attached to the front of T along a ruling curve. Let T' be the torus obtained from T by attaching the bypass D. Then $|\Gamma_{T'}| = 2$ and the dividing slope s' of $\Gamma_{T'}$ is determined as follows: let $[r, s]$ be the arc on $\partial \mathbb{D}$ (where $\mathbb{D}$ is the disc model of the hyperbolic plane) running from r counterclockwise to s, then s' is the point in $[r, s]$ closest to r with an edge to s. If the bypass is attached to the back of T then the same algorithm works except one uses the interval $[s, r]$ on $\partial \mathbb{D}$.*

We now recall a standard way to find bypasses.

Theorem 2.2 (the imbalance principle [19]) *Suppose that Σ and Σ' are two disjoint convex surfaces and $\mathcal{A}$ is a convex annulus whose interior is disjoint from Σ and Σ' but its boundary is Legendrian with one component on each surface. If $|\Gamma_\Sigma \cdot \partial \mathcal{A}| > |\Gamma_{\Sigma'} \cdot \partial \mathcal{A}|$ then there will be a dividing curve on $\mathcal{A}$ that cuts a disk off of $\mathcal{A}$ that has part of its boundary on Σ, and hence a bypass for Σ on $\mathcal{A}$.*

Theorem 2.3 (the twist number lemma [19]) *Consider a Legendrian curve L with twisting number n, relative to a fixed framing and a standard tubular neighborhood V of L with boundary slope $\frac{1}{n}$. If there exists a bypass D which is attached along a Legendrian ruling curve of slope r, and $\frac{1}{r} \geq n + 1$, then there exists a Legendrian curve with larger twisting number isotopic (but not Legendrian isotopic) to L.*

Theorem 2.4 (the edge rounding lemma [19]) *Let Σ_1 and Σ_2 be convex surfaces with collared Legendrian boundary which intersect transversely inside the ambient contact manifold along a common boundary Legendrian curve. Assume the neighborhood of the common boundary Legendrian is locally isomorphic to the neighborhood $N_\epsilon = \{x^2 + y^2 \leq \epsilon\}$ of $M = \mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})$ with coordinates (x, y, z) and contact 1–form $\alpha = \sin(2\pi nz)\,dx + \cos(2\pi nz)\,dy$ for some $n \in \mathbb{Z}^+$, and that $\Sigma_1 \cap N_\epsilon = \{x = 0,\ 0 \leq y \leq \epsilon\}$ and $\Sigma_2 \cap N_\epsilon = \{y = 0,\ 0 \leq x \leq \epsilon\}$. If we join Σ_1 and Σ_2 along $x = y = 0$ and round the common edge so that the orientations of Σ_1 and Σ_2 are compatible and induce the same orientation after rounding, the resulting surface is convex, and the dividing curve $z = \frac{k}{2n}$ on Σ_1 will connect to the dividing curve $z = \frac{k}{2n} - \frac{1}{4n}$ on Σ_2, where $k = 0, \ldots, 2n - 1$.*

Theorem 2.4 says if the convex surfaces Σ_1 and Σ_2 are positioned so that Σ_2 is to the right (left) of Σ_1, then, after rounding the corner, the dividing curves of Σ_1 move up (down) to connect up the dividing curves of Σ_2 (see Figure 2 in [12]).

3 Proof of Theorem 1.1

We first provide some basic facts about continued fractions, and set up our framing convention.

3.1 Continued fractions

A simple fact for the Farey graph says that two points on $\partial \mathbb{D}$ correspond to an integral basis of $\mathbb{Z}^2$ if and only if there is an edge in the Farey tessellation connecting them. Using this fact and a simple induction argument we obtain the following lemma (see [37, Lemma 3.1]):

Lemma 3.1 *Suppose*

$$(3\text{-}1) \qquad -\frac{q_i}{p_i} = [a_0^i, a_1^i, \ldots, a_{n_i}^i] = a_0^i - \cfrac{1}{a_1^i - \cfrac{1}{\ddots - \cfrac{1}{a_{n_i}^i}}}$$

for some uniquely determined integers

$$a_0^i, a_1^i, \ldots, a_{n_i}^i \leq -2 \quad \text{for } i = 1, 2, 3.$$

Then:

(1) *For each* $i = 1, 2, 3$, *the numbers* $-v_i/u_i = [a_0^i, a_1^i, \ldots, a_{n_i-1}^i]$ *satisfy* $p_i \geq u_i > 0$, $q_i \geq v_i > 0$ *and* $p_i v_i - q_i u_i = 1$.

(2) $(p_i - q_i)/(v_i - u_i) = [a_{n_i}^i, a_{n_i-1}^i, \ldots, a_0^i + 1]$. $\square$

3.2 Framing

We now specify our framing convention. Let $M = M(p_1/q_1, p_2/q_2, p_3/q_3)$ be a Seifert fibered space over S^2 with three singular fibers where $p_i/q_i \in \mathbb{Q}$ are unnormalized Seifert invariants. For the surgery description of M, take $e_0 = 0$ and allow the surgery coefficients $-q_i/p_i$ be any rational numbers in Figure 1. Let $V_i \cong D^2 \times S^1$ denote a tubular neighborhood of the singular fibers F_i for $i = 1, 2, 3$, and choose identifications $\partial V_i \cong \mathbb{R}^2/\mathbb{Z}^2$ such that $(1, 0)^T$ is the direction of the meridian and $(0, 1)^T$ is the direction of a longitude. We have that $M \setminus V_1 \cup V_2 \cup V_3$ is diffeomorphic to $\Sigma \times S^1$, where Σ is a pair of pants, and choose an identification $-\partial(M \setminus V_i) \cong \mathbb{R}^2/\mathbb{Z}^2$ by setting $(0, 1)^T$ as the direction of the S^1 fiber and $(1, 0)^T$ as the direction given by $-\partial(\text{pt} \times \Sigma)$. We then obtain the Seifert fibered manifold M as $M \cong (\Sigma \times S^1) \cup_{(A_1 \cup A_2 \cup A_3)} (V_1 \cup V_2 \cup V_3)$, where the attaching maps $A_i : \partial V_i \to -\partial(\Sigma \times S^1)_i$ are given by

$$A_i = \begin{pmatrix} q_i & v_i \\ -p_i & -u_i \end{pmatrix} \quad \text{for some } u_i \text{ and } v_i \text{ such that } p_i v_i - q_i u_i = 1.$$

Proof of Theorem 1.1 We consider the manifold $M(-2; p_1/q_1, p_2/q_2, p_3/q_3)$ as in Figure 1, where $p_i/q_i \in \mathbb{Q} \cap (0, 1)$. For the rest of the paper we assume that $0 < p_1/q_1 \leq p_2/q_2 \leq p_3/q_3 < 1$. We first note that, by applying positive Rolfsen twists to the unknots in the surgery diagram for $M(-2; p_1/q_1, p_2/q_2, p_3/q_3)$ with framing $-q_2/p_2$ and $-q_3/p_3$, one can obtain that $M(-2; p_1/q_1, p_2/q_2, p_3/q_3) \cong M(p_1/q_1, -(q_2 - p_2)/q_2, -(q_3 - p_3)/q_3)$.

If $M = M(p_1/q_1, -(q_2 - p_2)/q_2, -(q_3 - p_3)/q_3)$ is endowed with a contact structure, we can isotope each singular fiber F_i to be Legendrian with twisting number $k_i < 0$, and take V_i to be its standard neighborhood with slope$(\Gamma_{\partial V_i}) = 1/k_i$ — in particular, with our framing convention above, the slope $1/k_i$ on ∂V_i corresponds to the vector $(k_i, 1)^T$. The attaching maps $A_i : \partial V_i \to -\partial(\Sigma \times S^1)_i$ are given by

$$A_1 = \begin{pmatrix} q_1 & v_1 \\ -p_1 & -u_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} q_2 & v_2 \\ q_2 - p_2 & v_2 - u_2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} q_3 & v_3 \\ q_3 - p_3 & v_3 - u_3 \end{pmatrix},$$

where $-q_i/p_i = [a_0^i, a_1^i, \ldots, a_{n_i}^i]$ and $-v_i/u_i = [a_0^i, a_1^i, \ldots, a_{n_i-1}^i]$ as in Lemma 3.1.

When measured in $-\partial(M \setminus V_i)$, the slopes become

$$s_1 = \frac{-p_1 k_1 - u_1}{q_1 k_1 + v_1} = -\frac{p_1}{q_1} + \frac{1}{q_1(k_1 q_1 + v_1)},$$

$$s_2 = \frac{(q_2 - p_2)k_2 + (v_2 - u_2)}{q_2 k_2 + v_2},$$

$$s_3 = \frac{(q_3 - p_3)k_3 + (v_3 - u_3)}{q_3 k_3 + v_3}.$$

So, $k_1 < 0$ implies that $-1 = \lfloor -p_1/q_1 \rfloor < s_1 < -p_1/q_1$ and $k_i < 0$ implies that $0 = \lfloor (q_i - p_i)/q_i \rfloor \leq s_i < (q_i - p_i)/q_i$ for $i = 2, 3$.

In what follows we prove that by finding enough bypasses we can thicken the V_i to have boundary slopes $s_1 = -1$ and $s_2 = s_3 = 0$.

First note that after a small isotopy in the neighborhood V_i, we can make the ruling curves on $-\partial(M \setminus V_i)$ have infinite slope; in short these curves will be called vertical. Let $\mathcal{A}$ be an annulus with boundary being Legendrian vertical ruling curves along V_1 and V_2; such an annulus in short will be called vertical. Note that since $t(\xi) < 0$, we can make $\mathcal{A}$ convex. Moreover, $\partial \mathcal{A}$ intersects $\Gamma_{\partial(M \setminus V_1)}$ and $\Gamma_{\partial(M \setminus V_2)}$ exactly $2(q_1 k_1 + v_1)$ and $2(q_2 k_2 + v_2)$ times, respectively. There are two cases.

Case 1 If $q_1 k_1 + v_1 \neq q_2 k_2 + v_2$, the dividing set of $\mathcal{A}$, by the imbalance principle, has at least one boundary parallel arc, which bounds a bypass disk, on the ∂V_1 and/or ∂V_2 side. By attaching this bypass, we can increase the twisting numbers n_1 and n_2 incrementally by the twist number lemma. As long as we remain under Case 1, we can continue this process and increase twisting numbers n_1 and n_2 up to -1 because of our choice of ruling slopes and the twist number lemma.

Case 2 If $q_1 k_1 + v_1 = q_2 k_2 + v_2$ and the annulus $\mathcal{A}$ has no boundary parallel arcs, then we cut along $\mathcal{A}$ and round the corners by using the edge rounding lemma. To this end, observe first that a neighborhood of $M \setminus (V_1 \cup V_2 \cup \mathcal{A})$ is a solid torus with four edges. By rounding these edges we obtain a smooth torus $\partial(M \setminus V_1 \cup V_2 \cup \mathcal{A})$ that is isotopic to a neighborhood of F_3, and identify this torus with $\mathbb{R}^2/\mathbb{Z}^2$ in the same way as $\partial(M \setminus V_3)$. We can use the edge rounding lemma to compute the slope of the convex boundary: Each rounding between $-\partial(M \setminus V_1)$ or $-\partial(M \setminus V_2)$ and $\mathcal{A}$ changes the slope by an amount of $-1/2(2(q_1 k_1 + v_1))$ (here we use the assumption that $q_1 k_1 + v_1 = q_2 k_2 + v_2$). Since there are four edges to round, we compute the

slope on the $\partial(M \setminus V_1 \cup V_2 \cup \mathcal{A})$ as

$$
\begin{aligned}
s(\Gamma_{\partial(M \setminus V_1 \cup V_2 \cup \mathcal{A})}) &= -\frac{p_1 k_1 + u_1}{q_1 k_1 + v_1} + \frac{(q_2 - p_2)k_2 + v_2 - u_2}{q_2 k_2 + v_2} - \frac{1}{q_1 k_1 + v_1} \\
&= -\frac{p_1 k_1 + u_1}{q_1 k_1 + v_1} + \frac{(q_2 - p_2)(q_1 k_1 + v_1 - v_2)/q_2 + v_2 - u_2}{q_1 k_1 + v_1} - \frac{1}{q_1 k_1 + v_1} \\
&= \frac{(q_1 q_2 - q_1 p_2 - p_1 q_2)k_1 + v_1 q_2 - v_1 p_2 - u_1 q_2 - q_2 + 1}{q_1 q_2 k_1 + v_1 q_2} \\
&= \frac{\alpha}{\beta}.
\end{aligned}
$$

When measured in ∂V_3 we get

(3-2) $$ s_{k_1} = s(\Gamma_{\partial V_3}) = A_3^{-1}(-\beta, \alpha) = \frac{(Ak_1 + F)q_3}{(Ck_1 + D)v_3}, $$

where

$$
\begin{array}{ll}
A = \dfrac{p_1}{q_1} + \dfrac{p_2}{q_2} + \dfrac{p_3}{q_3} - 2, & C = 2 - \dfrac{p_1}{q_1} - \dfrac{p_2}{q_2} - \dfrac{u_3}{v_3}, \\[2ex]
F = \left(\dfrac{p_3}{q_3} + \dfrac{p_2}{q_2} - 2 \right) \dfrac{v_1}{q_1} + \dfrac{u_1 q_2 + q_2 - 1}{q_1 q_2}, & D = \left(2 - \dfrac{p_2}{q_2} - \dfrac{u_3}{v_3} \right) \dfrac{v_1}{q_1} - \dfrac{u_1 q_2 + q_2 - 1}{q_1 q_2}.
\end{array}
$$

Note that by our assumption in Theorem 1.1 we will consider only the following cases:

(1) $A \geq \frac{1}{4}$ (which implies $C < 0$).

(2) $A < 0$ (which implies $C > 0$).

(3) $p_1/q_1 = \frac{1}{2}$, $p_2/q_2 = \frac{2}{3}$ and $p_3/q_3 = k/(k+1)$ with $k \geq 6$.

Before analyzing the cases above in detail we make the following observation which will help us to bypass much of calculations (in particular for (2) and (3)).

3.3 Key shortcut

Suppose, for any $k_1 < 0$, we find a convex neighborhood of F_3 in V_3 with slope $(p_3 - q_3)/(v_3 - u_3)$. For notational ease we continue denoting this neighborhood by V_3. Now, the slope, when measured in $-\partial(M \setminus V_3)$, becomes $s_3 = 0$. In particular, there is a Legendrian curve L isotopic to a regular fiber with twisting -1, which is the maximal twisting of a tight contact structure ξ on M [7, Corollary 4.6]. We shall use this information to find thickenings of V_1 and V_2 such that their boundary slopes are $s_1 = -1$ and $s_2 = 0$. We first put a vertical annulus $\mathcal{A}$ between V_1 and V_3 such that $\partial \mathcal{A} = F_1 \cup L$. Note that $\Gamma_{\mathcal{A}}$ cannot have boundary parallel curves on the V_3 side. On the other hand, since $|q_1 k_1 + v_1| > 1$ whenever $k_1 < -1$, there are boundary parallel

curves in the dividing set of $\mathcal{A}$ on the V_1 side. By attaching bypasses along those boundary parallel arcs, we can increase the twisting number up to $k_1 = -1$. If we still have $|v_1 - q_1| > 1$, then there are more bypasses. We keep attaching these bypasses until $s_1 = -1$ or $|v_1 - q_1| = 1$. The latter possibility implies that $|p_1 - u_1| = 1$, as $0 < p_1 < q_1$, $0 < u_1 < v_1$ and $p_1 v_1 - u_1 q_1 = 1$. So, in either case we can thicken V_1 so that $s_1 = -1$. Similar arguments show that we can thicken V_2 so that $s_2 = 0$. Hence we conclude that $M(p_1/q_1, (q_2 - p_2)/q_2, (q_3 - p_3)/q_3) = \Sigma \times S^1 \cup_{A_1 \cup A_2 \cup A_3} (V_1 \cup V_2 \cup V_3)$ with $s_1 = -1$ and $s_2 = s_3 = 0$ measuring in $-\partial(M \setminus V_i)$ for $i = 1, 2, 3$. We will now count the total possible number of tight contact structures, up to isotopy, on M from each of the pieces. By [20, Lemma 5.1(3b)] there is a unique tight contact structure on $\Sigma \times S^1$ relative to the prescribed boundary data above. By Lemma 3.1(2) the slope of $\Gamma_{\partial V_1}$ is

$$\begin{pmatrix} -u_1 & -v_1 \\ p_1 & q_1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{p_1 - q_1}{v_1 - u_1} = [a_{n_1}^1, a_{n_1-1}^1, \ldots, a_1^1, a_0^1 + 1].$$

Thus, by the classification of tight contact structures on solid tori [19, Theorem 4.16], there are exactly $|(a_0^1 + 1)(a_1^1 + 1) \cdots (a_{n_1+1}^1)| = T(r_1)$ tight contact structures on V_1. Similarly on V_2 and V_3, as

$$\begin{pmatrix} v_i - u_i & -v_i \\ p_i - q_i & q_i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{p_i - q_i}{v_i - u_i} = [a_{n_i}^i, a_{n_i-1}^i, \ldots, a_1^i, a_0^i + 1] \quad \text{for } i = 2, 3,$$

there are exactly $T(r_2)$ and $T(r_3)$ tight contact structures, respectively. Hence, under the assumption that there is a neighborhood V_3 of F_3 with slope $(p_3 - q_3)/(v_3 - u_3)$, we obtain an upper bound on the number of tight contact structure on M. Finally, this upper bound is achieved by counting all distinct Stein fillable structures on M [28, Theorem 1.2] which are obtained by all possible realizations of the smooth diagram in Figure 3 by Legendrian surgeries.

Proof of (1) Let ξ be a tight contact structure on M with maximal twisting number $t(\xi) = -c < 0$ with $c > 0$. We claim that, under the assumption

$$A = \frac{p_1}{q_1} + \frac{p_2}{q_2} + \frac{p_3}{q_3} - 2 \geq \frac{1}{4},$$

this twisting number is actually -1. We prove this by putting together some results of Ghiggini from [7]. In [7, Proposition 2.1(1)], it was proved that if ξ is a tight structure with $t(\xi) = -c$, then for $i = 1, 2, 3$ there exist neighborhoods U_i of the singular fibers F_i such that $M \setminus (U_1 \cup U_2 \cup U_3)$ has minimal convex boundary with slopes $s(-\partial(M \setminus U_i)) = d_i/c$ satisfying $(d_i, c) = 1$ and $d_i/c < -p_i/q_i$. Moreover,

it was shown in [7, Proposition 2.6] that the integers d_1, d_2, d_3 and $c > 0$ satisfy $d_1 + d_2 + d_3 = -2c - 1$. Using these we obtain that $(-2c - 1)/c = (d_1 + d_2 + d_3)/c < -(p_1/q_1 + p_2/q_2 + p_3/q_3)$, and hence $p_1/q_1 + p_2/q_2 + p_3/q_3 - 2 < 1/c$. Now, combining this last inequality with the assumption that $p_1/q_1 + p_2/q_2 + p_3/q_3 - 2 \geq \frac{1}{4}$, we obtain that $c < 4$. Finally, it was shown in [7, Corollary 2.7] that if $t(\xi) < -1$ for a tight structure ξ on M, then it must be that $t(\xi) \leq -4$. Therefore, in our case, we obtain that $c = 1$ and that $t(\xi) = -1$. In particular, any tight structure ξ on $M(-2; p_1/q_1, p_2/q_2, p_3/q_3)$ with $p_1/q_1 + p_2/q_2 + p_3/q_3 - 2 \geq \frac{1}{4}$ has $t(\xi) = -1$. This information together with the reasoning in Section 3.3 finishes the proof of Theorem 1.1 for the case (1). $\qquad\square$

Proof of (2) We claim that under the assumption in the case (2), the slope s_{k_1} in (3-2) is less than or equal to Aq_3/Cv_3 for all $k_1 < 0$. To see this, we first show that s_{k_1} as a rational function in the variable k_1 has nonpositive first derivative (indeed it is negative except when $q_1 = q_2 = 2$). To see this, we calculate that $s'_{k_1} = (AD - FC)q_3/(Ck_1 + D)^2 v_3$ and $AD - FC = q_1 - q_2 - q_1 q_2$ (recall by definition — see (1-1) — that $q_1 > 1$ and $q_2 > 1$), and hence $s'_{k_1} < 0$ except when $q_1 = q_2 = 2$. When $q_1 = q_2 = 2$, one can directly check that $s_{k_1} = (p_3 - q_3)/(v_3 - u_3) = Aq_3/Cv_3$. In all other cases, we have that s'_{k_1} is negative. Moreover, we show that under the assumption in (2) the vertical asymptote $-D/C$ of s_{k_1} is always greater than -1. To prove this last claim, first observe that, by the definitions, and the fact that $p_1 v_1 - q_1 u_1 = 1$, we can write $D = (C + p_1/q_1)v_1/q_1 - (u_1 q_2 + q_2 - 1)/q_1 q_2 = Cv_1/q_1 + 1/q_1^2 - 1/q_1 + 1/q_1 q_2$. Substituting this in $-D/C$ we get that $-D/C = -v_1/q_1 + (q_1 q_2 - q_1 - q_2)/Cq_1^2 q_2$, which is obviously greater than -1 when C is positive, as $0 < v_1 \leq q_1$ and $q_1 q_2 - q_1 - q_2 > 0$. This, together with the fact that $s'_{k_1} < 0$, proves that $s_{k_1} \leq Aq_3/Cv_3$ for all $k_1 < 0$. Moreover,

$$\frac{Aq_3}{Cv_3} = \frac{p_3 - (2 - p_1/q_1 - p_2/q_2)q_3}{(2 - p_1/q_1 - p_2/q_2)v_3 - u_3} \leq \frac{p_3 - q_3}{v_3 - u_3}$$

if $p_1/q_1 + p_2/q_2 \leq 1$. So, by using this last inequality and Theorem 4.16 of [19] we find that there is a neighborhood V_3 with $s(\Gamma_{\partial V_3}) = (p_3 - q_3)/(v_3 - u_3)$. In particular, by arguments in Section 3.3 above we prove Theorem 1.1 for the case (2) when $p_1/q_1 + p_2/q_2 \leq 1$.

We want to remark that by [29, Theorem 1.1] and [7, Theorem 1.2], we know that $M(-2; p_1/q_1, p_2/q_2, p_3/q_3)$ is an L–space if and only if there are no relatively prime integers $m > a$ such that $1 - p_1/q_1 < a/m$, $1 - p_2/q_2 < (m - a)/m$ and

$1 - p_3/q_3 < 1/m$. From those inequalities, we easily conclude that the condition that $p_1/q_1 + p_2/q_2 \leq 1$ implies that $M(-2; p_1/q_1, p_2/q_2, p_3/q_3)$ is an L–space. In particular our result reproves the classification of tight structures on such manifolds, which was first obtained by Ghiggini in [7]. But the criteria above further tells us that for $M(-2; p_1/q_1, p_2/q_2, p_3/q_3)$ to be an L–space, it is necessary that $p_1/q_1 + p_2/q_2 + p_3/q_3 < 2$, and sufficient that $p_1/q_1 + p_2/q_2 + p_3/q_3 \leq \frac{3}{2}$. In particular, the condition that $p_1/q_1 + p_2/q_2 \leq 1$ does not characterize all manifolds $M(-2; p_1/q_1, p_2/q_2, p_3/q_3)$ which are L–spaces. It is worth noting that, at the moment, we do not have a closed formula in terms of the Seifert invariants to determine exactly which $M(-2; p_1/q_1, p_2/q_2, p_3/q_3)$ are L–spaces when $\frac{3}{2} < p_1/q_1 + p_2/q_2 + p_3/q_3 < 2$ (see [24] for more on this problem and its complexity). So it is left to analyze the case (2) when $p_1/q_1 + p_2/q_2 > 1$. We first give an explicit demonstration of the classification of tight contact structures on a particular family of small Seifert fibered spaces of interest which are not L–spaces: $M_r = M\left(-2; \frac{1}{2}, \frac{2}{3}, r\right)$, where $r = p/q \in \mathbb{Q} \cap \left[\frac{4}{5}, \frac{5}{6}\right)$ and $-q/p = [a_1, a_2, \ldots, a_m]$ with $a_i \leq -2$ for $i = 1, 2, \ldots, m$.

Lemma 3.2 *On M_r, up to isotopy, there are exactly $|(a_1 + 1)(a_2 + 1) \cdots (a_m + 1)|$ tight contact structures, which are all Stein fillable.*

We note that Lemma 3.2 in particular says that on $M_n = M\left(-2; \frac{1}{2}, \frac{2}{3}, (5n-1)/(6n-1)\right)$ with $n \geq 2$ (note $-(6n-1)/(5n-1) = [-2, -2, -2, -2, -3, -2, \ldots, -2]$, where -3 is followed by $n-2$ -2's) there are, up to isotopy, exactly two tight contact structures, which are both Stein fillable. It is interesting to compare this infinite family with its orientation-reversal one, $-M_n = M\left(-1; \frac{1}{2}, \frac{1}{3}, n/(6n-1)\right)$, which has been very instrumental in our understanding of 3–dimensional contact geometry, by work of Etnyre and Honda [3], Lisca and Matić [28] and Ghiggini [5]. Finally, recent work of Ghiggini and Van Horn-Morris [13], shows that on $-M_n$ there are, up to isotopy, exactly $\frac{1}{2}n(n-1)$ tight contact structures, which are all strongly fillable but at least one of them is not Stein fillable.

Proof Let ξ be a tight contact structure on $M_r = M\left(-2; \frac{1}{2}, \frac{2}{3}, \frac{p}{q}\right) \cong M\left(\frac{1}{2}, -\frac{1}{3}, -\frac{q-p}{q}\right)$. As above we can make singular fibers F_i Legendrian with the twisting numbers $k_i < 0$, then take their standard neighborhoods V_i with slope $s(\Gamma_{V_i}) = 1/k_i$.

When measured in $-\partial(M \setminus V_i)$, the slopes are

$$s_1 = -\frac{k_1}{2k_1 + 1}, \quad s_2 = \frac{k_2 + 1}{3k_2 + 2}, \quad s_3 = \frac{(q-p)k_3 + v - u}{qk_3 + v}.$$

Let $\mathcal{A}$ be a vertical annulus between V_1 and V_2. We first consider the case when $2k_1 + 1 \neq 3k_2 + 2$; then the imbalance principle applies: The dividing set of $\mathcal{A}$ has at least one boundary parallel arc, which bounds a bypass disk, on the V_1 and/or V_2 side. By attaching this bypass we can increase the twisting number k_1 and/or k_2 by one because of our choice of ruling slopes (which on ∂V_2 and ∂V_3 are -2 and 3, respectively) and the twist number lemma. As long as $2k_1 + 1 \neq 3k_2 + 2$, we can continue this process of increasing twisting numbers up to $k_1 = -1$ and $k_2 = 0$. Next suppose $2k_1 + 1 = 3k_2 + 2$ and $\mathcal{A}$ has no boundary parallel arcs. After cutting and rounding $\mathcal{A}$ between V_1 and V_2 we get a neighborhood V_3 of F_3 such that its slope in coordinates of ∂V_3, by formula (3-2), becomes

$$s_{k_1} = s(\Gamma_{\partial V_3}) = \frac{(6p - 5q)k_1 + 3p - 2q}{(5v - 6u)k_1 + 2v - 3u}.$$

This is the place that it is not necessarily true that $s_{k_1} \leq (p - q)/(v - u)$ for all $k_1 < 0$. That is, it is not immediate that there is a thickening of V_3 such that its slope when measured with respect to $-\partial(M_r \setminus V_3)$ is $s_3 = 0$. But since $p/q \in \mathbb{Q} \cap \left[\frac{4}{5}, \frac{5}{6}\right)$, $p \geq u > 0$, $q \geq v > 0$ and $q - p \geq v - u$, we still have that

$$\frac{(6p - 5q)k_1 + 3p - 2q}{(5v - 6u)k_1 + 2v - 3u} < \frac{6p - 5q}{5v - 6u}$$

for all $k_1 < 0$. So, by [19, Theorem 4.16], we can find a convex neighborhood $V_3' \subset V_3$ of the singular fiber F_3 such that $s(\Gamma_{\partial V_3'}) = (6p - 5q)/(5v - 6u)$. This slope becomes $\frac{1}{6}$ when measured in coordinates of $-\partial(M_r \setminus V_3')$. We now take a vertical annulus between V_1 and V_3'. Note that as long as $n_1 \leq -4$ we have $|2n_1 + 1| > 6$ and hence the convex annulus will have a bypass on the V_1 side. By the twist number lemma attaching this bypass will increase the twisting n_1 to -3. Similarly we can increase the twisting n_2 to -2. So, slopes in the coordinates of $-\partial(M \setminus V_i)$ become $s_1 = -\frac{3}{5}$, $s_2 = \frac{1}{4}$ and $s_3 = \frac{1}{6}$. Observe that yet another vertical annulus $\mathcal{A}$ between V_1 and V_2 will result in bypasses which allow us to thicken V_1 and V_2 so that their boundary slopes become $s_1 = -1$ and $s_2 = 0$. At this point we can assume that the vertical annulus $\mathcal{A}$ has no boundary parallel arcs in its dividing set because $t(\xi) < 0$ for any tight structure on M_r. We cut along $\mathcal{A}$ and round the corners of $(M \setminus V_1 \cup V_2 \cup \mathcal{A})$. Now $\partial(M \setminus V_1 \cup V_2 \cup \mathcal{A})$ is smoothly isotopic to $\partial(M \setminus V_3)$ and has slope 0, by the edge rounding lemma. Since the solid tori V_1 and V_2 have boundary slopes $A_1^{-1}(-1, 1) = -1$ and $A_2^{-1}(1, 0) = \infty$, respectively, they are standard neighborhoods of Legendrian (singular) fibers. So each carries a unique tight contact structure. On the other hand, since $A_3^{-1}(1, 0) = (p - q)/(v - u)$ and, by the second part

of Lemma 3.1, $(p-q)/(v-u) = [a_m, a_{m-1}, \ldots, a_2, a_1+1]$, we conclude by using the classification of tight contact structures on solid tori [19, Theorem 2.3] that V_3 admits exactly $|(a_m+1)(a_{m-1}+1)\cdots(a_2+1)(a_1+1)|$ tight contact structures. Therefore, as explained in Section 3.3, M_r admits exactly $|(a_m+1)(a_{m-1}+1)\cdots(a_2+1)(a_1+1)|$ tight contact structures, up to isotopy, which are all Stein fillable. $\qquad\square$

We now return to the general case. Note that $s_{k_1} < Aq_3/Cv_3$ for every $k_1 < 0$, so there is a convex neighborhood V_3' of F_3 such that $s(\Gamma_{\partial V_3'}) = Aq_3/Cv_3$. Now the slope, when measured in $-\partial(M \setminus V_3')$, becomes

$$s_3 = \begin{pmatrix} q_3 & v_3 \\ q_3 - p_3 & v_3 - u_3 \end{pmatrix} \begin{pmatrix} Cv_3 \\ Aq_3 \end{pmatrix} = \frac{p_1}{q_1} + \frac{p_2}{q_2} - 1 > 0.$$

Step I Assume first that $q_1 = q_2 = q$; we then have $s_1 = (-p_1k_1 - u_1)/(qk_1 + v_1)$, $s_2 = ((q - p_2)k_2 + (v_2 - u_2))/(qk_2 + v_2)$ and $s_3 = (p_1 + p_2 - q)/q$. Put a vertical annulus between V_1 and V_3'. One can easily see that $|qk_1 + v| > q$ whenever $k_1 < -1$. So by the imbalance principle there are bypasses on the V_1 side. By the twist number lemma from [19] we can increase the twisting number k_1 up to -1. Similarly we can increase the twisting number k_2 up to -1. So, with this thickening at hand the slopes become $s_1 = (p_1 - u_1)/(v_1 - q)$ and $s_2 = (p_2 - q + v_2 - u_2)/(v_2 - q)$. Now a vertical annulus between V_1 and V_2 will have boundary parallel arcs in its dividing set, and hence bypasses, if $|v_1 - q| \neq |v_2 - q|$. Attaching those bypasses we either obtain that $s_1 = -1$ and $s_2 = 0$ or $v_1 = v_2$. If the former happens then we are done by Section 3.3. So we can assume without loss of generality that $v_1 = v_2 = v$. This time a vertical annulus between V_1 and V_3', as $|v - q| < q$, by the imbalance principle, must yield bypasses on the V_3' side. Attaching those bypasses and tracing their effect via the Farey tessellation we eventually get first that $s_3 = 1/q'$ and then a further bypass — which is still available as $0 < (p_1 + p_2 - q)/q < s_2$ — gives that $s_3 = 0$. Now, using Section 3.3, we obtain the desired classification.

Step II Assume now that $q_1 \neq q_2$. So, $s_3 = (p_1q_2 + p_2q_1 - q_1q_2)/q_1q_2 > 0$. We now put a vertical annulus between V_1 and V_3'. Since $|q_1k_1 + v_1| > q_1q_2$ whenever $k_1 < -q_2$, by the imbalance principle there are bypasses on the V_1 side which, by the twist number lemma from [19] increase the twisting number k_1 up to $-q_2$. Similarly we can increase the twisting number k_2 up to $-q_1$ and the slopes become $s_1 = (p_1q_2 + u_1)/(q_1q_2 - v_1)$ and $s_2 = (q_1q_2 - q_1p_2 + u_2 - v_2)/(q_1q_2 - v_2)$. If $v_1 \neq v_2$, then $|v_1 - q_1q_2| \neq |v_2 - q_1q_2|$. So a vertical annulus between V_1 and V_2 will have bypasses. Once again by the twist number lemma we increase the twisting numbers

k_1 and k_2 up to $-q_2 + l$ and $-q_1 + k$ for $1 \le l \le q_2 - 1$ and $1 \le k \le q_1 - 1$. As in Step I above, from this we either obtain that $s_1 = -1$ and $s_2 = 0$, and are done with the proof or $v_1 = v_2 = v$. In the latter case we have $|v - q_1 q_2| = |v - q_1 q_2| \ne q_1 q_2$. Hence a vertical annulus between V_1 (or V_2) and V_3' will produce bypasses on the V_3' side that changes its slopes, again referring to the Farey tessellation, to a sequence of fractions ending at $1/q'$ for some q'. Since $|v - q_1 q_2| \ne q'$ is still the case, we get one more bypass, and conclude that $s_3 = 0$. By using Section 3.3, the desired classification is obtained. This completes the proof of (2).   □

Proof of (3)   We substitute $p_1/q_1 = \frac{1}{2}$, $p_2/q_2 = \frac{2}{3}$ and $p_3/q_3 = k/(k+1)$ to obtain that the slopes s_i for $i = 1, 2, 3$, measured with respect to $-\partial(M \setminus V_i)$, are $s_1 = -n_1/(2n_1 + 1)$, $s_2 = (n_2 + 1)/(3n_2 + 2)$ and $s_3 = (n_3 + 1)/((k+1)n_3 + k)$. Moreover, the new boundary slope, after cutting and rounding a standard vertical annulus between V_1 and V_2, for ∂V_3, by formula (3-2), is

$$s_{n_1} = -\frac{(k-5)n_1 + k - 2}{(k-6)n_1 + k - 3}.$$

Recall we assumed that $k \ge 6$. There are several cases to consider. If $k \ge 8$, then it is easy to see that $s_{n_1} \le -1$ for all $n_1 < 0$. In particular, there exists a neighborhood $V_3' \subset V_3$ with boundary slope $s(\Gamma_{\partial V_3'}) = -1$. When measured with respect to $-\partial(M \setminus V_3')$, the slope becomes 0. Now, by the argument in Section 3.3 (note $(p_3 - q_3)/(v_3 - u_3) = -1$), we finish the proof. When $k = 6$ or 7, one needs more care. For example it is not true that we can find a convex neighborhood of F_3 with the boundary slope -1 for all $n_1 < 0$. In some sense this is the primary reason that we cannot extend our classification to include all the values of the Seifert invariants p_i/q_i. Indeed, in the next section we will exhibit an infinite family of examples for which the classification is very different than what we have in Theorem 1.1. Nevertheless, for $k = 6, 7$, it is easy to check that, similar to the arguments as in Lemma 3.2, the classification can be obtained as claimed (indeed $k = 6$ corresponds to $n = 1$ in Theorem 1.2).   □

This concludes the proof of Theorem 1.1.   □

4   Proof of Theorem 1.2

Let M_n denote the manifold $M\left(-2; \frac{1}{2}, \frac{2}{3}, (5n+1)/(6n+1)\right)$. The proof starts by explaining why M_n has at most $\frac{1}{2}n(n+1)$ tight contact structures. This first part

is a classic convex surface theory argument as in [3; 37; 7; 10; 11; 13]. The second part is devoted to detecting the claimed number of tight contact structures, which makes an essential use of the contact invariants in Heegaard Floer homology with twisted coefficients. We refer the reader to [13, Section 3] for basics of Heegaard Floer homology with twisted coefficients. We still assume the terminology from the previous section.

4.1 Upper bound

We prove that M_n admits at most $\frac{1}{2}n(n+1)$ tight contact structures, up to isotopy, for any $n \geq 1$. The manifold M_n can also be described as $M\left(\frac{1}{2}, -\frac{1}{3}, -n/(6n+1)\right)$. In particular, we can decompose the manifold M_n as

$$M_n \cong (\Sigma \times S^1) \cup_{(A_1 \cup A_2 \cup A_3)} (V_1 \cup V_2 \cup V_3),$$

where the attaching maps $A_i = \partial V_i \to -\partial(\Sigma \times S^1)_i$ are given by

$$A_1 = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 6n+1 & 6 \\ n & 1 \end{pmatrix}.$$

Lemma 4.1 *The manifold M_n admits at most $\frac{1}{2}n(n+1)$ tight contact structures up to isotopy for any $n \geq 1$.*

Proof We would like to first determine the maximal twisting number of tight contact structures on M_n. It was proven first by Wu in [37] that any tight contact structure ξ on small Seifert fibered spaces with $e_0 = -2$ has negative maximal twisting number. In particular, this fact applies to each tight contact structure on M_n.

Claim 4.2 *Let ξ be a tight contact structure on M_n. Then the maximal twisting number is $\mathrm{tw}(\xi) = -6k - 1$ for some k with $0 \leq k \leq n-1$.*

Proof As practiced in the previous sections, we want to use a vertical annulus between the standard neighborhoods V_1 or V_2 and Legendrian regular fiber L with maximal twisting number to produce bypasses on the V_1 or V_2 side, and hence normalize/thicken these neighborhoods as much as we can. Then, based on their slopes we determine the twisting number of potential tight contact structures that permit those slopes.

Let $\mathrm{tw}(\xi) = -t < 0$, and L be a Legendrian regular fiber with $\mathrm{tw}(L, \xi) = -t$. After some small isotopy, we can arrange the decomposition above so that the neighborhoods

V_i of the singular fibers F_i and L do not intersect. The slopes of the standard neighborhoods are $-1/m_i$. When measured in $-\partial(M_n \setminus V_i)$ they become

$$s_1 = -\frac{m_1}{2m_1 + 1}, \quad s_2 = \frac{m_2 + 1}{3m_2 + 2}, \quad s_3 = \frac{nm_3 + 1}{(6n + 1)m_3 + 6}.$$

Let $\mathcal{A}$ be a vertical annulus whose boundary is a vertical curve on the V_1 side and the Legendrian regular fiber L. Then the dividing set of $\mathcal{A}$ will have boundary parallel arcs (hence bypasses) on the $-\partial(M \setminus V_1)$ side whenever $2m_1 + 1 < -t$. Those bypasses will potentially increase the twisting number m_1. More precisely, as the ruling slope on ∂V_1 is -2, by the twist number lemma from [19] and the choice of L, we can increase m_1 incrementally up to either $2m_1 + 1 = -t$ or $m_1 = -1$. Similarly, m_2 can be increased up to either $3m_1 + 2 = -t$ or $m_2 = -1$. In particular, there is a nonnegative integer k satisfying $m_1 = -3k - 1$, $m_2 = -2k - 1$ and $t = 6k + 1$. Let $\mathcal{A}$ be a vertical annulus between V_1 and V_2. Note that the dividing set of $\mathcal{A}$ cannot have any boundary parallel arcs because of the maximality of $-t$. We can cut along $\mathcal{A}$ and round the corners to get a smooth manifold $M_n \setminus V_1 \cup V_2 \cup \mathcal{A}$ such that $\partial(M_n \setminus V_1 \cup V_2 \cup \mathcal{A})$ is smoothly isotopic to $\partial(M \setminus V_3)$. Moreover, by the edge rounding lemma from [19] we compute its slope as

$$s(\Gamma_{\partial(M_n \setminus V_1 \cup V_2 \cup \mathcal{A})}) = -\frac{m_1}{2m_1 + 1} + \frac{m_2 + 1}{3m_2 + 2} - \frac{1}{2m_1 + 1} = -\frac{k}{6k + 1}.$$

When measured in ∂V_3,

$$s(\Gamma_{\partial V_3}) = A_3^{-1}(6k + 1, k) = -n + k.$$

We show that $k \geq n$ is impossible. To this end, suppose $k \geq n$, which then implies $t = 6k + 1 \geq 6n + 1$ and that there is a neighborhood V_3' of F_3 such that $s(\Gamma_{\partial V_3'}) = \infty$. When measured with respect to $-\partial(M_n \setminus V_3')$ this slope becomes $\frac{1}{6}$, contradicting $t \geq 6n + 1$ whenever $n \geq 1$. Indeed, in this case a Legendrian ruling curve fiber on $-\partial(M_n \setminus V_3')$ will have twisting -6. Therefore, we obtain that $\mathrm{tw}(\xi) = -6k - 1$ for some k with $0 \leq k \leq n - 1$, proving the claim. $\qquad\square$

Now we can decompose M_n as $(M_n \setminus V_3') \cup V_3'$, where $(M_n \setminus V_3')$ is made of V_1, V_2 and a neighborhood of the annulus $\mathcal{A}$. Since V_1 and V_2 are the standard neighborhoods for each k, each carries a unique tight contact structure. Moreover the dividing set of $\mathcal{A}$ uniquely determines a tight contact structure in the neighborhood of $\mathcal{A}$. Therefore, there is a unique tight contact structure on $M_n \setminus V_3'$ relative to its boundary $\partial(M_n \setminus V_3') \cong \partial V_3'$. On the other hand, V_3' has boundary slope $-n + k$, and, by [19, Theorem 4.16], it

carries exactly $n-k$ tight contact structures relative to its boundary. Since $0 \leq k \leq n-1$, we get that the total number of tight contact structures on M_n is at most $\frac{1}{2}n(n+1)$. □

4.2　Lower bound

We take M_∞ to be the 3–manifold obtained by smooth 0–surgery along the left-handed trefoil knot; see Figure 4. This is the small Seifert fibered manifold $M\left(-2; \frac{1}{2}, \frac{2}{3}, \frac{5}{6}\right)$, which also has the structure of a torus bundle over the circle as

$$M_\infty = T^2 \times [0, 1]/\sim_A,$$

where $(x, 1) = (Ax, 0)$ and A is the diffeomorphism of T^2 given by

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Recall that the homotopy class of an oriented tangent 2–plane field ξ on a 3–manifold M with $c_1(\xi)$ torsion is determined by two invariants [17, Theorem 4.16]: the spin^c–structure $\mathfrak{s}_\xi$ induced by ξ and an invariant[1] $\theta(\xi) \in \mathbb{Z}$ defined by

$$\theta(\xi) = c_1^2(X, J) - 3\sigma(X) - 2\chi(X),$$

for any almost-complex 4–manifold (X, J) with $\partial X = M$ and such that ξ is the field of complex tangencies $TM \cap J(TM)$. Here $\chi(X)$ is the Euler characteristic and $\sigma(X)$ is the signature. If $H_1(M)$ has no 2–torsion (this is true for example for the manifold M_∞), then $c_1(\xi)$ does determine the $\mathfrak{s}_\xi$.

The classification of tight contact structures on M_∞, up to isotopy/homotopy, and their fillability are well understood. The following theorem summarizes them:

Theorem 4.3　M_∞ *admits an infinite family of vertical tight contact structures $\{\xi_i\}_{i=0}^\infty$ such that*:

(1) [20, Theorem 0.1; 15]　*They have the Giroux torsion $Tor(\xi_i) = i$, and hence are pairwise nonisotopic. Moreover, any other tight structure on M_∞ is isotopic to ξ_i for some i.*

(2) [20, Theorem 0.1; 15]　*All are homotopic with $\theta(\xi_i) = 4$ (note that $c_1(\xi_i) = 0$ for all $i \geq 0$).*

(3) [20, Theorem 0.1; 15]　*All are weakly fillable.*

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[1]This is related to $d_3(\xi)$ in some other references. We will stick with Gompf's notation. The two are related by $4d_3(\xi) = \theta(\xi)$.
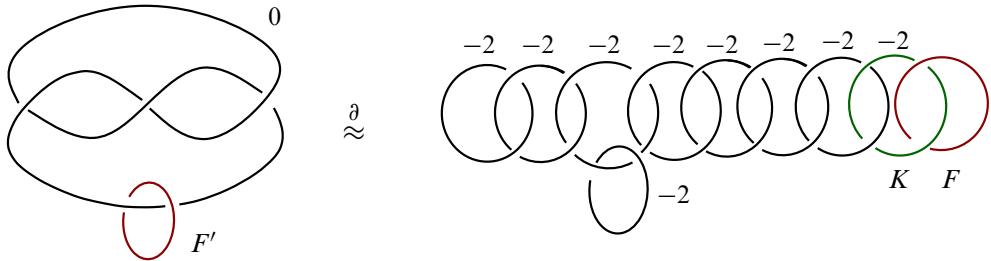
Figure 4

(4) [4] *The contact structure ξ_0 is Stein (and hence strongly) fillable, while
for $i > 0$, ξ_i is not strongly fillable.*

(5) [9, Theorem 1] *The contact invariants $c(\xi_i)$ have degree $-\frac{1}{4}\theta(\xi_i) - \frac{1}{2}$. They,
with untwisted coefficients, are all zero for $i > 0$, while with twisted coefficients
all are nonzero and pairwise different for $i \geq 0$.*

Since the contact structures ξ_i are all vertical, the knot

$$F' = \mathbf{0} \times [0, 1]/\sim_A$$

is tangent to ξ_i for all i. If we use the Seifert fibration structure on M_∞, ie the surgery
description in Figure 4, then the knot F' is topologically isotopic to a meridian of
the left-handed trefoil. The manifold M_∞ can also be described as the boundary of
E_9 plumbing, from which we obtain its unique Stein fillable contact structure. If F
denotes the image of F' under this isotopy and F' has a framing, say f, then F has
framing $f + 1$.

Lemma 4.4 [6, Lemma 3.5] *There exists a framing on F such that $\mathrm{tw}(F, \xi_i) = -i-1$.
Moreover, smooth $(-n-1)$–surgery along F gives M_n.*

Let $F_{i,j}$ denote the Legendrian knot obtained from F by applying $n-i-1$ stabilizations
such that the rotation number $\mathrm{r}(F_{i,j}, \xi_i) = j$, where $0 \leq i \leq n-1$ and $|j| \leq n-i-1$
with $j \equiv n+1-i \pmod 2$. Finally, let $\xi_{i,j}^n$ denote tight contact structures on M_n
obtained by Legendrian surgery along $F_{i,j}$ in (M_∞, ξ_i). Altogether we have $\frac{1}{2}n(n+1)$
tight structures on M_n, and for which we claim:

Lemma 4.5 *For each pair (i, j) where $0 \leq i \leq n-1$ and $|j| \leq n-i-1$ and
$j \equiv n+1-i \pmod 2$, the contact structures $\xi_{i,j}^n$ as described above are pairwise
nonisotopic. More precisely, we show that:*
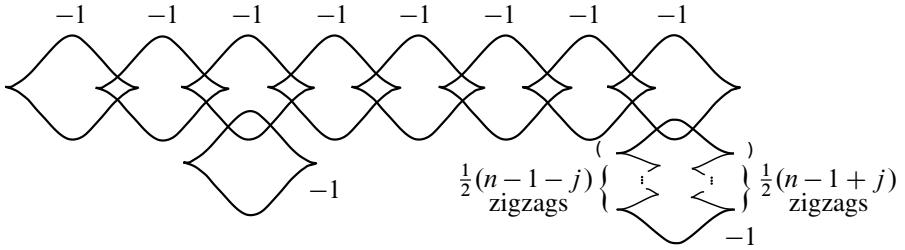
Figure 5: Stein fillable tight contact structures $\xi^n_{0,j}$ on M_n.

(1) If $-(n-i-1) \leq j_1, j_2 \leq n-i-1$ and $j_m \equiv n+1-i \pmod 2$ for $m = 1,2$ and $j_1 \neq j_2$, then ξ^n_{i,j_1} and ξ^n_{i,j_2} are not isotopic.

(2) The contact structures $\xi^n_{i_1,j_1}$ and $\xi^n_{i_2,j_2}$ are not isotopic whenever $i_1 \neq i_2$ and $-(n-i_m-1) \leq j_m \leq n-i_m-1$ for $m = 1,2$.

Note that since Legendrian surgery preserves fillability, we obtain that all $\xi^n_{i,j}$ are weakly fillable and hence tight. Among them, each $\xi^n_{0,j}$ — see Figure 5 — is Stein fillable (as ξ_0 is Stein fillable) and we explicitly calculate that $\theta(\xi^n_{0,j}) = 2$. All other tight structures $\xi^n_{i,j}$ for $i > 0$ can be made at least strongly fillable because M_n is an integral homology sphere. By work of Lisca and Matić [28], we have that for any $n \geq 1$, the tight contact structures $\xi^n_{0,j}$ are pairwise nonisotopic. Moreover, by work of Plamenevskaya [36, Section 3] the contact invariants $c(\xi^n_{0,j}) \in \widehat{\mathrm{HF}}_{(-1)}(-M_n)$ are linearly independent over $\mathbb{Z}$. Since $\widehat{\mathrm{HF}}_{(-1)}(-M_n) \cong \mathbb{Z}^n$ [32, Section 8.1], we obtain that $\widehat{\mathrm{HF}}_{(-1)}(-M_n) \cong \langle c(\xi^n_{0,-n+1}), c(\xi^n_{0,-n+3}), \ldots, c(\xi^n_{0,n-1}) \rangle$.

Moreover, by the second part of Lemma 4.5, and its proof below, we will be able to conclude that indeed the contact structures $\xi^n_{i,j}$ all are homotopic with $\theta = 2$.

Proof of Lemma 4.5 We prove the first part of the lemma in Step I and the second part in Step II.

Step I This will follow from applying a result of Wu [38, Theorem 1.4(1)]. The actual result of Wu is much stronger, but here we state a simplified version.

Theorem 4.6 (Wu) *Let (M, ξ) be a tight contact 3–manifold and L a Legendrian knot in it. Fix integers s and p_r for $r = 1, 2$ such that $0 \leq p_r \leq s$. Let L_r be the Legendrian knot constructed from L by p_r positive stabilizations and $s - p_r$ negative stabilizations. Then the Legendrian surgeries on L_r give two contact structures ξ_1*

and ξ_2 on the same ambient 3–manifold M'. Assume ξ_1 and ξ_2 are isotopic. If (M, ξ) is weakly filled by a symplectic 4–manifold (W, ω) and L is nullhomologous in W, then $p_1 = p_2$.

In our case (M, ξ) will be taken to be the contact manifold (M_∞, ξ_i), where $i \geq 0$ is fixed. The Legendrian link L will be taken to be the Legendrian knot $F \subset (M_\infty, \xi_i)$, and hence $L_r = F_{i,j}$. Now note that the parameter j in our description means $\mathrm{r}(F_{i,j}) = j$, where $|j| \leq n - i - 1$ and $j \equiv n + 1 - i \pmod 2$. When n and i are fixed, the parameters j and p, where p as in Theorem 4.6 stands for the number of positive stabilizations on F and ranges in $0 \leq p \leq n - i - 1$, determine one another. Indeed, given p, the corresponding j value will be $j = -(n - i - 1) + 2p$.

Next we want to show that (M_∞, ξ_i) for all $i \geq 0$ has a simply connected weak symplectic filling (W, ω). In [2, Propositions 15 and 16] a weak symplectic filling (W, ω) for (M_∞, ξ_i) for all $i \geq 0$ is constructed (and it is interesting to note that weak fillings constructed by Ding and Geiges for (M, ξ_i) are really all the same symplectic manifold, which fills them all simultaneously). Moreover, in the same reference it was proven, by an explicit construction, that there is a Lefschetz fibration $W \to D^2$ with generic torus fibers and monodromy $A \in \mathrm{SL}_2(\mathbb{Z})$ that determines the torus bundle on the boundary. We show that this W is simply connected. To this end, first observe that the monodromy $A \in \mathrm{SL}_2(\mathbb{Z})$ of M_∞ satisfies $A = (ab)^5$, where a and b are a generating set for $\mathrm{SL}_2(\mathbb{Z})$ given as

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

From this we obtain that the Lefschetz fibration $W \to D^2$ has ten singular fibers, and the vanishing cycles a and b corresponding to these singular points induce a $\mathbb{Z}$–basis for $\pi_1(T^2)$. Because of this latter observation and the short exact sequence $\pi_1(T^2) \to \pi_1(W) \to \pi_1(D^2) = 0$ induced by $T^2 \hookrightarrow W \to D^2$, we obtain that $\pi_1(W) = 0$. Another, and possibly shorter, way to see that W is simply connected is that W is actually diffeomorphic to the complement of a cusp fiber in the elliptic surface $E(1)$ (this is by the fact that $(ab)^6 = 1$ and that ab represents a cusp fiber), and this latter is known to be simply connected (eg [12, Exercise 7.3.21(b)]). Now Theorem 4.6 applies to prove that if $-(n - i - 1) \leq j_1, j_2 \leq n - i - 1$ with $j_m \equiv n + 1 - i \pmod 2$ for $m = 1, 2$ and $j_1 \neq j_2$, then ξ^n_{i,j_1} and ξ^n_{i,j_2} are not isotopic.

Step II As will be evident below, for our purpose here we are interested in the Ozsváth–Szabó contact classes with twisted coefficients, and their behavior under

Legendrian surgery. Let $(Y, \mathfrak{t})$ be a closed, connected, oriented spin^c 3–manifold; Ozsváth and Szabó in [33; 35] defined the Heegaard Floer homology with twisted coefficients $\widehat{\mathrm{HF}}(Y, \mathfrak{t}; M)$, whose coefficients form a module M over $\mathbb{Z}[H^1(Y)]$. If this module is $M = \mathbb{Z}$ with the trivial action of $H^1(Y)$, then one obtains the usual Heegaard Floer homology $\widehat{\mathrm{HF}}(Y, \mathfrak{t})$. Moreover, to a cobordism V from Y_0 to Y_1 they associated a morphism between $\widehat{\mathrm{HF}}(Y_0, \mathfrak{t})$ and $\widehat{\mathrm{HF}}(Y_1, \mathfrak{t})$. One needs to pay extra attention in the twisted case, as the groups are usually modules over different rings. Regardless, in [35, Theorem 3.8] it was proven that the cobordism V with a spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(V)$ induces an anti-$\mathbb{Z}[H^1(Y_0)]$–linear map

$$F_{V,\mathfrak{s}} \colon \widehat{\mathrm{HF}}(Y_0, \mathfrak{t}; M) \to \widehat{\mathrm{HF}}(Y_1, \mathfrak{t}; M(V))$$

which is well defined up to sign. Here $M(V)$ is the coefficient module induced from M by the cobordism. The equivalence class of such a map is denoted by $[F_{W,\mathfrak{s}}]$. We refer to [33; 35; 23; 13] for more details. Moreover, to a contact structure ξ on Y, one associates a class $c(\xi; M) \in \widehat{\mathrm{HF}}(Y, \mathfrak{t}_\xi; M)$, where $\mathfrak{t}_\xi$ is the spin^c structure induced by ξ. This element is well defined up to sign and multiplication by invertible elements of $\mathbb{Z}[H^1(Y)]$, and $[c(\xi, M)]$ will denote its equivalence class. In [34], Ozsváth and Szabó proved that $[c(\xi, M)]$ is an isotopy invariant of ξ, and that if $c_1(\xi)$ is a torsion class (which is the case for ξ_i on M_∞ for all $i \geq 0$), then $[c(\xi, M)]$ is a set of homogeneous elements of degree $-\frac{1}{4}\theta(\xi) - \frac{1}{2}$. We refer to [13, Section 3] for further and helpful details relevant to contact classes with twisted coefficients, for example where the following theorem [13, Theorem 3.6], due to Ghiggini, is stated:

Theorem 4.7 (Ghiggini) *Let (Y_0, ξ_0) and (Y_1, ξ_1) be two contact manifolds, and (V, J) be the Stein cobordism induced by a Legendrian surgery on a Legendrian link in Y_0. If $\mathfrak{t}$ is the canonical spin^c –structure on V for the complex structure J, then*

$$[F_{V,\mathfrak{s}}(c(\xi_1, M))] = \begin{cases} [c(\xi_0, M(V))] & \text{if } \mathfrak{s} = \mathfrak{t}, \\ 0 & \text{if } \mathfrak{s} \neq \mathfrak{t}. \end{cases}$$

We want to emphasize that for our purpose here M will be assumed to be $\mathbb{F}$ and $M(V)$ is $\mathbb{F}[H^1(M_\infty)]$

Recall the contact structures $\xi_{i,j}^n$ are obtained by Legendrian surgery along the Legendrian knot $F_{i,j} \subset (M_\infty, \xi_i)$. Let V be the corresponding cobordism from (M_∞, ξ_i) to $(M_n, \xi_{i,j}^n)$. Applying Theorem 4.7, we obtain that

$$(4\text{-}1) \qquad\qquad [F_V(c(\xi_{i,j}^n))] = [c(\xi_i)].$$

We note that the map F_V induced by the various Legendrian surgeries are all associated to the same smooth manifold V, ie F_W does not depend on i. Now it is proven in [8, Theorem 2] that the equivalence classes of contact invariants $[c(\xi_i)]$ with twisted coefficients all are nonzero and pairwise different for $i \geq 0$. Therefore we obtain that $\xi^n_{i_1,j_1}$ and $\xi^n_{i_2,j_2}$ are not isotopic whenever $i_1 \neq i_2$ and $-(n-i_m-1) \leq j_m \leq n-i_m-1$ with $j_m \equiv n+1-i \pmod 2$ for $m = 1, 2$. We would like to emphasize again that the use of contact classes here with twisted coefficients is crucial as the contact classes $c(\xi_i)$ with untwisted coefficients are all zero for $i > 0$ [9, Theorem 1].

Finally, since ξ_i are all homotopic with $\theta(\xi_i) = 4$, and hence their contact classes have degree $-\frac{1}{2}\theta(\xi_i) - \frac{1}{2} = -\frac{3}{2}$ (see Theorem 4.3(2)–(3)), and since (4-1) holds, we obtain via the degree-shift formula in Heegaard Floer homology [32] that $\xi^n_{i,j}$ are all homotopic with $\theta(\xi^n_{i,j}) = 2$.   □

4.3 Strongly fillable but not Stein fillable structures

Lemma 4.8 *The manifold M_n for $n \geq 1$ even admits at least $\frac{1}{2}n$ tight contact structures which are strongly fillable but not Stein fillable.*

Proof We claim the tight contact structures $\xi^n_{i,0}$ are not Stein fillable for $0 < i \leq n-1$ with $i \equiv n-1 \pmod 2$. The proof we present here is very parallel to beautiful work of Ghiggini in [5]. Let $\mathcal{J}$ denote the conjugation map in Heegaard Floer homology $\widehat{\mathrm{HF}}$ with coefficients in $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. We recall that, by work of Plamenevskaya [36, Section 3] the contact invariants $c(\xi^n_{0,j}) \in \widehat{\mathrm{HF}}_{(-1)}(-M_n)$ are linearly independent over $\mathbb{F}$ and, since $\widehat{\mathrm{HF}}_{(-1)}(-M_n) \cong \mathbb{F}^n$ [32, Section 8.1], we obtain that

$$\widehat{\mathrm{HF}}_{(-1)}(-M_n) \cong \langle c(\xi^n_{0,-n+1}), c(\xi^n_{0,-n+3}), \ldots, c(\xi^n_{0,n-1}) \rangle.$$

Moreover, it is easy to see from the surgery description in Figure 5 that the contact structure $\overline{\xi^n_{0,j}} = \mathcal{J}(\xi^n_{0,j})$ is isotopic to $\xi^n_{0,-j}$, and $\xi^n_{i,0}$ is isotopic to its conjugate for each $0 \leq i \leq n-1$ with $i \equiv n-1 \pmod 2$ (see also [6, Lemma 3.8]). Therefore, since $c(\xi^n_{i,0})$ are fixed points of the action $\mathcal{J}$ on $\widehat{\mathrm{HF}}_{(-1)}(-M_n)$, we conclude that $c(\xi^n_{i,0})$ is a linear combination of elements of the form $c(\xi^n_{0,j}) + c(\xi^n_{0,-j})$ for $j = -n+1, -n+3, \ldots, n-1$ (and we note $j = 0$ is not possible as n is assumed to be even).

All in place, we are ready to finish the proof. Suppose (X_i, J_i) is a Stein filling of $(M_n, \xi^n_{i,0})$. If $\mathfrak{t}_i$ denotes the canonical Spin^c structure of (X_i, J_i), then $(X_i, -J_i)$ is a Stein filling of $(M_n, \overline{\xi^n_{i,0}})$ and $\overline{\mathfrak{t}}_i$ is the canonical Spin^c structure of $(X_i, -J_i)$. Now, we puncture the Stein filling (X_i, J_i) and view it as a Stein cobordism from $-M_n$

to S^3. Because $\xi_{i,0}^n \cong \overline{\xi_{i,0}^n}$, we get that $F_{X_i,\mathfrak{t}_i}^+(c(\xi_{i,0}^n)) = F_{X_i,\overline{\mathfrak{t}_i}}^+(c(\xi_{i,0}^n)) = c(\xi_{\text{std}}) \neq 0$. In particular, by [36, Theorem 4] we conclude that $\mathfrak{t}_i$ is isomorphic to its conjugate $\overline{\mathfrak{t}_i}$ (see also [5, Theorem 4.1]) and $F_{X_i,\mathfrak{t}_i}^+(c(\xi_{i,0}^n))$ is a generator of $\text{HF}^+(S^3) \cong \mathbb{F}$. On the other hand, by using the fact that the conjugation homomorphism $\mathcal{J}$ has trivial action on $\text{HF}^+(S^3)$ and the observations above, we conclude that the map $F_{X_i,\mathfrak{t}_i}^+$ evaluates on each $c(\xi_{0,j}^n) + c(\xi_{0,-j}^n)$ as

$$F_{X_i,\mathfrak{t}_i}^+(c(\xi_{0,j}^n) + c(\xi_{0,-j}^n)) = 2F_{X_i,\mathfrak{t}_i}^+(c(\xi_{0,j}^n)) = 0.$$

In particular, we end up with the contradiction that $F_{X_i,\mathfrak{t}_i}^+(c(\xi_{i,0}^n)) = 0$. Thus, $(M_n, \xi_{i,0}^n)$ when n is even cannot be Stein fillable for any $0 < i \leq n-1$ with $i \equiv n-1 \pmod{2}$. This counts to the total of $\frac{1}{2}n$ strongly fillable tight structures on M_n that are not Stein fillable. $\square$

References

[1] **V Colin**, **E Giroux**, **K Honda**, *Finitude homotopique et isotopique des structures de contact tendues*, Publ. Math. Inst. Hautes Études Sci. 109 (2009) 245–293 MR

[2] **F Ding**, **H Geiges**, *Symplectic fillability of tight contact structures on torus bundles*, Algebr. Geom. Topol. 1 (2001) 153–172 MR

[3] **J B Etnyre**, **K Honda**, *On the nonexistence of tight contact structures*, Ann. of Math. 153 (2001) 749–766 MR

[4] **D T Gay**, *Four-dimensional symplectic cobordisms containing three-handles*, Geom. Topol. 10 (2006) 1749–1759 MR

[5] **P Ghiggini**, *Strongly fillable contact 3–manifolds without Stein fillings*, Geom. Topol. 9 (2005) 1677–1687 MR

[6] **P Ghiggini**, *Ozsváth–Szabó invariants and fillability of contact structures*, Math. Z. 253 (2006) 159–175 MR

[7] **P Ghiggini**, *On tight contact structures with negative maximal twisting number on small Seifert manifolds*, Algebr. Geom. Topol. 8 (2008) 381–396 MR

[8] **P Ghiggini**, **K Honda**, *Giroux torsion and twisted coefficients* (2008) arXiv

[9] **P Ghiggini**, **K Honda**, **J Van Horn-Morris**, *The vanishing of the contact invariant in the presence of torsion*, preprint (2007) arXiv

[10] **P Ghiggini**, **P Lisca**, **A I Stipsicz**, *Classification of tight contact structures on small Seifert 3–manifolds with $e_0 \geq 0$*, Proc. Amer. Math. Soc. 134 (2006) 909–916 MR

[11] **P Ghiggini**, **P Lisca**, **A I Stipsicz**, *Tight contact structures on some small Seifert fibered 3–manifolds*, Amer. J. Math. 129 (2007) 1403–1447 MR

[12] **P Ghiggini**, **S Schönenberger**, *On the classification of tight contact structures*, from "Topology and geometry of manifolds" (G Matić, C McCrory, editors), Proc. Sympos. Pure Math. 71, Amer. Math. Soc., Providence, RI (2003) 121–151 MR

[13] **P Ghiggini**, **J Van Horn-Morris**, *Tight contact structures on the Brieskorn spheres $-\Sigma(2, 3, 6n - 1)$ and contact invariants*, J. Reine Angew. Math. 718 (2016) 1–24 MR

[14] **E Giroux**, *Convexité en topologie de contact*, Comment. Math. Helv. 66 (1991) 637–677 MR

[15] **E Giroux**, *Une infinité de structures de contact tendues sur une infinité de variétés*, Invent. Math. 135 (1999) 789–802 MR

[16] **E Giroux**, *Structures de contact en dimension trois et bifurcations des feuilletages de surfaces*, Invent. Math. 141 (2000) 615–689 MR

[17] **R E Gompf**, *Handlebody construction of Stein surfaces*, Ann. of Math. 148 (1998) 619–693 MR

[18] **A Hatcher**, *Notes on basic of 3–manifold topology*, book project (2007) Available at http://pi.math.cornell.edu/~hatcher/3M/3Mdownloads.html

[19] **K Honda**, *On the classification of tight contact structures, I*, Geom. Topol. 4 (2000) 309–368 MR Correction in *Factoring nonrotative $T^2 \times I$ layers*, Geom. Topol. 5 (2001) 925–938

[20] **K Honda**, *On the classification of tight contact structures, II*, J. Differential Geom. 55 (2000) 83–143 MR

[21] **K Honda**, **W H Kazez**, **G Matić**, *Right-veering diffeomorphisms of compact surfaces with boundary*, Invent. Math. 169 (2007) 427–449 MR

[22] **T Ito**, **K Kawamuro**, *Visualizing overtwisted discs in open books*, Publ. Res. Inst. Math. Sci. 50 (2014) 169–180 MR

[23] **S Jabuka**, **T E Mark**, *Product formulae for Ozsváth–Szabó 4–manifold invariants*, Geom. Topol. 12 (2008) 1557–1651 MR

[24] **M Jankins**, **W D Neumann**, *Rotation numbers of products of circle homeomorphisms*, Math. Ann. 271 (1985) 381–400 MR

[25] **W H Kazez**, **R Roberts**, *Fractional Dehn twists in knot theory and contact topology*, Algebr. Geom. Topol. 13 (2013) 3603–3637 MR

[26] **Y Lekili**, *Planar open books with four binding components*, Algebr. Geom. Topol. 11 (2011) 909–928 MR

[27] **P Lisca**, *On overtwisted, right-veering open books*, Pacific J. Math. 257 (2012) 219–225 MR

[28] **P Lisca**, **G Matić**, *Tight contact structures and Seiberg–Witten invariants*, Invent. Math. 129 (1997) 509–525 MR

[29] **P Lisca**, **A I Stipsicz**, *Ozsváth–Szabó invariants and tight contact 3–manifolds, III*, J. Symplectic Geom. 5 (2007) 357–384 MR

[30] **P Lisca**, **A I Stipsicz**, *On the existence of tight contact structures on Seifert fibered 3–manifolds*, Duke Math. J. 148 (2009) 175–209 MR

[31] **P Massot**, *Geodesible contact structures on 3–manifolds*, Geom. Topol. 12 (2008) 1729–1776 MR

[32] **P Ozsváth**, **Z Szabó**, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math. 173 (2003) 179–261 MR

[33] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. 159 (2004) 1159–1245 MR

[34] **P Ozsváth**, **Z Szabó**, *Heegaard Floer homology and contact structures*, Duke Math. J. 129 (2005) 39–61 MR

[35] **P Ozsváth**, **Z Szabó**, *Holomorphic triangles and invariants for smooth four-manifolds*, Adv. Math. 202 (2006) 326–400 MR

[36] **O Plamenevskaya**, *Contact structures with distinct Heegaard Floer invariants*, Math. Res. Lett. 11 (2004) 547–561 MR

[37] **H Wu**, *Legendrian vertical circles in small Seifert spaces*, Commun. Contemp. Math. 8 (2006) 219–246 MR

[38] **H Wu**, *On Legendrian surgeries*, Math. Res. Lett. 14 (2007) 513–530 MR

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