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A new approach to twisted K–theory of compact Lie groups

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We further explore the computation of the twisted K–theory and K–homology of compact simple Lie groups, previously studied by Hopkins, Moore, Maldacena–Moore–Seiberg, Braun and Douglas, with a focus on groups of rank 2. We give a new method of computation based on the Segal spectral sequence, which seems to us appreciably simpler than the methods used previously, at least in many key cases.

19L50; 55R20, 55T15, 57T10, 81T30

1 Introduction

This paper is an outgrowth of the paper [32] by Mathai and the author, where we started studying a new approach to the computation of the twisted K–theory of compact simple Lie groups. This problem was first studied by physicists (eg Moore [36], Maldacena, Moore and Seiberg [31], Fredenhagen and Schomerus [19], Bouwknegt, Dawson and Ridout [9], Braun [11], Braun and Schäfer-Nameki [12] and Gaberdiel and Gannon [22]) because of interest in the WZW (Wess–Zumino–Witten) model, which appears both in conformal field theory and as a string theory whose underlying spacetime manifold is a Lie group, usually compact and simple. In string theories in general, D–brane charges are expected to take their values in twisted K–theory of spacetime, so the study of WZW models led to the study of twisted K–theory of compact Lie groups. The calculation of twisted K–theory of Lie groups turned out to be sufficiently interesting that it was eventually taken up by mathematicians (Hopkins, unpublished but quoted in [31], and Douglas [16]).

Section 2 then revisits the topic of computing twisted K–theory $K^\bullet(G,h)$ for arbitrary choices of the twisting h. This has been the subject of an extensive literature, most notably [31; 36; 11; 16], and the results are rather complicated and hard to understand. However, this is an important problem because of the connection, discovered by physicists, between these twisted K–groups and fusion rings and representations of loop groups. We therefore present in Section 3 an easier way of computing these twisted K–groups for compact, simply connected, simple Lie groups of rank two. Theorems 8,

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14, 15 and 12 recover all the known results for rank-2 groups using our direct methods. Section 4 goes on to discuss nonsimply connected groups. While many of the results of this paper were previously known, to the best of my knowledge, Theorems 6, 9, 16 and 17 are new.

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2 Review and machinery

In this paper we will deal exclusively with complex periodic K–theory, which is 2–periodic by Bott periodicity. Given a topological space X (which for present purposes we can take to be compact) and a principal bundle P over X with fibers the projective unitary group $\mathrm{PU}(\mathcal{H}) = U(\mathcal{H})/\mathbb{T}$ of an infinite-dimensional separable Hilbert space $\mathcal{H}$, P defines a bundle of spectra over X with fibers the K–theory spectrum, and from this one can construct *twisted K–theory* of X in a standard way (see for example [40; 41; 6; 27; 5; 28] — this is only a small subset of the literature).

Since $\mathrm{PU}(\mathcal{H}) \simeq \mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$ space, bundles P as above are classified by classes $h \in H^3(X, \mathbb{Z})$, and we will denote the twisted K–theory or twisted K–homology of X by $K^\bullet(X, h)$ or $K_\bullet(X, h)$, even though, strictly speaking, h only determines these groups up to noncanonical isomorphism. (The noncanonicity will not be important in anything we do.)

Let G be a compact Lie group. In this section we restrict to the case where G is simple, connected and simply connected, which is the most studied case. Since G is then 2–connected with $\pi_3(G) \cong \mathbb{Z}$, we have $H^3(G, \mathbb{Z}) \cong \mathbb{Z}$. There is in fact a canonical isomorphism of $H^3(G, \mathbb{Z})$ with $\mathbb{Z}$ (ie a canonical choice of generator), due to the fact that $(X,Y,Z)\mapsto\langle X,[Y,Z]\rangle$, with $\langle\_\,,\_\rangle$ the Killing form, defines a canonical 3–form on the Lie algebra of G, and thus a preferred orientation on $H^3_{\mathrm{deR}}(G,\mathbb{R})$. In what follows we will mostly consider the case of twistings $h>0$ (when $H^3(G,\mathbb{Z})$ is identified with $\mathbb{Z}$). Changing the sign of h preserves the isomorphism types of $K^\bullet(X,h)$ and $K_\bullet(X,h)$, and when $h=0$, Hodgkin [26] proved that $K^\bullet(G)$ is an exterior algebra over $\mathbb{Z}$ with n generators, where $n=\operatorname{rank} G$. Thus taking $h\geq 1$ is no loss of generality.

For $G=\mathrm{SU}(2)=\mathrm{Sp}(1)$, the twisted K–theory $K^\bullet(G,h)$ for $h\neq 0$ was already computed in [40], with the result that it is 0 in even degree and $\mathbb{Z}/h$ in odd degree. The following result was proved in [11; 16]:

Theorem 1 [16, Theorem 1.1] *For G a simple, connected and simply connected compact Lie group,* $\operatorname{rank} G=n$*, and for twisting $h>0$, $K_\bullet(X,h)$ (even as a ring) is the tensor product of an exterior algebra over $\mathbb{Z}$ on $n-1$ odd-degree generators with a finite cyclic group of order $c(G,h)$ a divisor of h.*

As we will see, for the cases at least of $\mathrm{SU}(n+1)$, $\mathrm{Sp}(n)$ and G_2, this is not particularly difficult, and the hard part is to compute the numbers $c(G,h)$.

Incidentally, the distinction between $K_\bullet(X,h)$ and $K^\bullet(X,h)$ is not particularly important here. Since these twisted K–groups are the actual K–groups of a continuous trace C^*–algebra A over G (having h as Dixmier–Douady class), $K_\bullet(X,h)\cong K^{-\bullet}(A)$ and $K^{-\bullet}(X,h)\cong K_\bullet(A)$ are related by the universal coefficient theorem for type I C^*–algebras A [13], which says that there is a canonical exact sequence

$$0\to\operatorname{Ext}^1_{\mathbb{Z}}(K_{\bullet+1}(A),\mathbb{Z})\to K^\bullet(A)\to\operatorname{Hom}_{\mathbb{Z}}(K_\bullet(A),\mathbb{Z})\to 0.$$

Since A here has finitely generated K–theory and $K^\bullet(A)$ is torsion, $K_\bullet(A)$ has to be torsion, and so $K_\bullet(X,h)$ and $K^\bullet(X,h)$ agree except for a degree shift. Thus, for SU(2) and $h\neq 0$, $K_\bullet(G,h)$ is $\mathbb{Z}/h$ in *even* degree instead of odd degree, and in all other cases (again, with $h\neq 0$), $K_\bullet(G,h)$ and $K^\bullet(G,h)$ are actually noncanonically isomorphic.

In [11], a simple form for the numbers $c(G,h)$ was proposed, and was proven modulo a conjecture about the commutative algebra of Verlinde rings. (The conjecture is that Verlinde rings are the coordinate rings of complete intersection affine varieties.) This conjecture is known for $\mathrm{SU}(n+1)$, $\mathrm{Sp}(n)$ and G_2, but to the best of my knowledge it might still be open for the spin groups and the other exceptional groups (see eg

[10; 17; 3] for partial results). Thus the following should be regarded as a definitive theorem for $\mathrm{SU}(n+1)$, $\mathrm{Sp}(n)$ and G_2, but a "conditional theorem" in the other cases.

Theorem 2 ([11], but note comments above) *Assume the conjecture in the paragraph above, which is known at least in types A_n, C_n and G_2. For G a simple, connected and simply connected compact Lie group,* rank $G = n$*, and for twisting $h > 0$, the order $c(G,h)$ of the torsion in $K_\bullet(X,h)$ and $K^\bullet(X,h)$ is given by the formula $c(G,h) = h/\gcd(h, y(G))$, where the number $y(G)$ is given by the following table*:

G	$y(G)$
$A_n = \mathrm{SU}(n+1)$	$\mathrm{lcm}(1,2,\dots,n)$
$B_n = \mathrm{Spin}(2n+1)$	$\mathrm{lcm}(1,2,\dots,2n-1)$
$C_n = \mathrm{Sp}(n)$	$\mathrm{lcm}(1,2,\dots,n,1,3,\dots,2n-1)$
$D_n = \mathrm{Spin}(2n)$ $(n > 3)$	$\mathrm{lcm}(1,2,\dots,2n-3)$
G_2	60
F_4	27 720
E_6	27 720
E_7	12 252 240
E_8	2 329 089 562 800

Formulas were also given for $c(G,h)$ in [16, Theorem 1.2] for the classical groups and [16, page 797] for G_2, but they have a totally different form; for example,

$$c(\mathrm{SU}(n+1),h) = \gcd\left(\binom{h+i}{i} - 1 \;\middle|\; 1 \le i \le n\right)$$

and

$$c(\mathrm{Sp}(n),h) = \gcd\left(\sum_{-h\le j\le -1} \binom{2j+2(i-1)}{2(i-1)} \;\middle|\; 1 \le i \le n\right).$$

Appendix C in [31] proved that the Douglas and Braun formulas coincide in the case of $\mathrm{SU}(n+1)$. In Propositions 11 and 13, we will also see that the Douglas and Braun formulas coincide in the case of $\mathrm{Sp}(2)$ and G_2.

We now move on to the question of how to prove results like Theorems 1 and 2 in an easier way. Computation of $K^\bullet(\mathrm{SU}(n+1),h)$ was discussed in [19; 31; 36] using methods motivated by physics, based on a study of wrapping of branes in WZW theories. However, those papers don't quite give a mathematically rigorous proof, except in the simplest cases. More sophisticated methods for computing $K^\bullet(G,h)$ were used in [11; 16], but the techniques are decidedly not elementary. Braun [11] used the Hodgkin Künneth spectral sequence in equivariant K–theory together with the calculations of Freed, Hopkins and Teleman [20; 21],[1] while [16] used a Rothenberg–Steenrod spectral sequence and K–theory of loop spaces. So our purpose here is to give a more direct approach. We will need the Segal spectral sequence (from [42, Proposition 5.2]), though for our purposes it is easiest to reformulate it in homology instead of cohomology.

Theorem 3 *Let $F \xrightarrow{\iota} E \xrightarrow{\mathrm{pr}} B$ be a fiber bundle, say of CW complexes, and let $h \in H^3(E)$. Then there is a homological spectral sequence*

$$H_p(B, K_q(F, \iota^* h)) \Rightarrow K_\bullet(E, h).$$

Proof In the absence of the twist, this is precisely the homology dual of the spectral sequence of [42, Proposition 5.2], in the case where the cohomology theory used is complex K–theory. If $h = 0$, $E = B$ and $F = \mathrm{pt}$, this reduces to the usual Atiyah–Hirzebruch spectral sequence (AHSS) for K–homology. Similarly, if $E = B$ and $F = \mathrm{pt}$, but $h \neq 0$, this is the AHSS for twisted K–homology. To get the general case, we filter B by its skeleta. This induces a filtration of $K_\bullet(E, h)$ for which this is the induced spectral sequence (by Segal's proof). □

Remark 4 The spectral sequence of Theorem 3 will be strongly convergent if the ordinary homology of B is bounded. This will be the case if B is weakly equivalent to a finite-dimensional CW complex, and in particular covers all the cases considered in this paper.

As a simple application of Theorem 3, we can immediately prove the easiest part of Theorem 1. (However, this result is rather weak and we will want to improve on it.)

Theorem 5 *Let G be a simple, connected and simply connected compact Lie group. For any twisting $h > 0$, $K_\bullet(X, h)$ is a finite abelian group, and all elements have order a divisor of a power of h. In particular, if $h = 1$, then $K_\bullet(X, h)$ vanishes identically, and if $h = p^r$ is a prime power, then $K_\bullet(X, h)$ is a p–primary torsion group.*

Proof First observe that G contains a subgroup $H \cong \mathrm{SU}(2) \cong \mathrm{Sp}(1) \cong \mathrm{Spin}(3)$ such that the inclusion $H \hookrightarrow G$ is an isomorphism on π_j for $j \le 3$. Assuming this structural fact, the theorem follows immediately. Consider the fibration $H \to G \to G/H$. From Theorem 3, we get a spectral sequence converging to $K_\bullet(X, h)$,

[1] There is indirect physics input here since Freed, Hopkins and Teleman showed that the *equivariant* twisted K–theory is the same as the Verlinde ring of the associated WZW model.

with $E^2_{p,q} = H_p\big(G/H, K_q(\mathrm{SU}(2), h)\big)$. But $K_q(\mathrm{SU}(2), h)$ is nonzero only for q even, where it is $\mathbb{Z}/h$. Since E^2 is thus torsion with all elements of order dividing h, the same is true of E^∞. And even if there are nontrivial extensions involved in going from E^∞ to $K_\bullet(X, h)$, the result still follows.

It still remains to verify the structural statement. For the classical groups, $\mathrm{SU}(2)$ sits in $\mathrm{SU}(n)$, $\mathrm{Spin}(3)$ sits in $\mathrm{Spin}(n)$, and $\mathrm{Sp}(1)$ sits in $\mathrm{Sp}(n)$ for all relevant values of n. The fact that these inclusions are isomorphisms on π_3 is standard, and follows from the classical fibrations

$$\mathrm{SU}(n) \to \mathrm{SU}(n+1) \to S^{2n+1},$$
$$\mathrm{Sp}(n) \to \mathrm{Sp}(n+1) \to S^{4n+3},$$
$$\mathrm{Spin}(n) \to \mathrm{Spin}(n+1) \to S^n,$$

together with the facts that $\mathrm{SU}(2)$ and G are both 2–connected. In the case of G_2, there is a fibration $\mathrm{SU}(2) \to G_2 \to V_{7,2}$ [8, Lemme 17.1]. In the case of F_4, there is a fibration $\mathrm{Spin}(9) \to F_4 \to \mathbb{OP}^2$ [7]. For the E–series we can use the fibration $F_4 \to E_6 \to E_6/F_4$ along with what we know about F_4, then use the inclusions $E_6 \hookrightarrow E_7 \hookrightarrow E_8$. □

In order to apply Theorem 3 more precisely, in some cases we will need an explicit description of some of the differentials. Thus the following theorem is useful. It applies with basically the same proof to other exceptional homology theories, though we won't need these here.

Theorem 6 *In the situation of Theorem 3, suppose that ι^* is an isomorphism (or even just an injection) on H^3 (so that the twisting class on E can be identified with the restricted twisting class on F), the differentials $d^2, \dots, d^{r-1}$ leave $E^2_{r,0} = H_r(B, K_0(F, \iota^* h))$ unchanged, and one has a class x in this group which comes from a class $\alpha \in \pi_r(B)$ under the composite*

$$\pi_r(B) \xrightarrow{\text{Hurewicz}} H_r(B, K_0(F, \iota^* h)).$$

Then $d^r(x) \in E^r_{0,r-1}$, a quotient of $K_{r-1}(F, \iota^* h)$, is the image of α under the composite

$$\pi_r(B) \xrightarrow{\partial} \pi_{r-1}(F) \xrightarrow{\text{Hurewicz}} K_{r-1}(F, \iota^* h),$$

where the first map is the boundary map in the long exact sequence of the fibration $F \xrightarrow{\iota} E \xrightarrow{\mathrm{pr}} B$. (The Hurewicz map in twisted homology is easy to understand as follows, at least if $r \geq 5$: $\iota^* h$ defines a principal $K(\mathbb{Z}, 2)$–bundle $P_{\iota^* h}$ over F, and the pullback of this bundle to the total space $P_{\iota^* h}$ is trivial, so there is a natural map $K_\bullet(P_{\iota^* h}) \to K_\bullet(F, \iota^* h)$ [29, page 536]; the Hurewicz map is the composite

$$\pi_{r-1}(F) \cong \pi_{r-1}(P_{\iota^* h}) \xrightarrow{\wedge 1} \pi_{r-1}(P_{\iota^* h} \wedge \boldsymbol{K}) = K_{r-1}(P_{\iota^* h}) \to K_{r-1}(F, \iota^* h),$$

where $\pi_{r-1}(F) \cong \pi_{r-1}(P_{\iota^* h})$ if $r > 4$, by the long exact sequence of the fibration $K(\mathbb{Z}, 2) \to P_{\iota^* h} \to F$.)

Proof Since the class x by assumption was not changed under the earlier differentials, and since the twisting comes entirely from the fiber, we can, without loss of generality, reduce to the case where B is a sphere S^r and thus $E = (\mathbb{R}^r \times F) \cup F$, where $\mathbb{R}^r \times F$ corresponds to pr^{-1} of the open r–cell in the base. In this case the spectral sequence comes directly from the long exact sequence

$$(1) \quad \cdots \to K_r(F, \iota^* h) \xrightarrow{\iota_*} K_r(E, h) \to K_r(E, F, h) \\ \cong K_r(E \smallsetminus F, h) \cong K_0(F, \iota^* h) \xrightarrow{\partial} K_{r-1}(F, \iota^* h) \to \cdots .$$

Note here that $H_r(B, K_0(F, \iota^* h))$ can be identified with the term $K_0(F, \iota^* h)$ in (1). So the differential d^r is the boundary map in (1), and we use commutativity of the diagram

$$\begin{array}{ccc} \pi_r(B) & \xrightarrow{\partial} & \pi_{r-1}(F) \\ \downarrow{\scriptstyle \text{Hurewicz}} & & \downarrow{\scriptstyle \text{Hurewicz}} \\ H_r(B, K_0(F, \iota^* h)) & \xrightarrow{\partial} & K_{r-1}(F, \iota^* h) \end{array}$$

a consequence of naturality of the Hurewicz homomorphism. □

Another useful result for us will be the "universal coefficient theorem" of Khorami [29].

Theorem 7 (Khorami [29]) *Let X be a space (say, a compact CW-complex), let $h \in H^3(X, h)$ and let P_h be the associated principal bundle with structure group $\mathrm{PU}(\mathcal{H}) \simeq \mathbb{CP}^\infty$. Then $K_\bullet(X, h) \cong K_\bullet(P_h) \otimes_R \mathbb{Z}$, where $R = K_0(\mathbb{CP}^\infty)$ is a ring under Pontrjagin product acting on $K_\bullet(P_h)$ via the principal $\mathbb{CP}^\infty$–bundle structure on P_h and on $\mathbb{Z}$ via the ring homomorphism $R \to \mathbb{Z}$ sending $\beta_j \mapsto 1$ if $j = 0$ or 1 and $\beta_j \mapsto 0$ if $j > 1$. Here R is the free $\mathbb{Z}$–module on generators $1 = \beta_0, \beta_1, \ldots,$ where β_j is dual to $(\gamma - 1)^j$, with γ the Hopf line bundle in $K(\mathbb{CP}^\infty)$.*

In fact, Khorami mentions at the end of his paper that he suspects that his theorem can be used to recover Theorem 1, though he gives no details except in the case

$G = \mathrm{SU}(2)$, where he points out that for P_h as in Theorem 7, $K_\bullet(P_h) \cong R/(h\beta_1)$ and thus $K_\bullet(\mathrm{SU}(2), h) \cong R/(h\beta_1) \otimes_R \mathbb{Z} \cong \mathbb{Z}/h$.

3 Twisted K–theory of rank-two simple Lie groups

3.1 The case of SU(3)

To explain how we use these tools, we start with the simplest nontrivial case, namely $G = \mathrm{SU}(3)$, which was first treated in [31; 36]. We recall the result:

Theorem 8 *Let h be a positive integer, viewed as a twisting class for* SU(3). *Then (in both even and odd degree) $K_\bullet(\mathrm{SU}(3), h) \cong \mathbb{Z}/h$ if h is odd and $K_\bullet(\mathrm{SU}(3), h) \cong \mathbb{Z}/(h/2)$ if h is even.*

Proof We use the standard fibration

$$\mathrm{SU}(2) = S^3 \xrightarrow{\iota} \mathrm{SU}(3) \xrightarrow{\mathrm{pr}} S^5.$$

Here ι^* is an isomorphism on H^3, and we already know that $K_\bullet(\mathrm{SU}(2), h)$ is $\mathbb{Z}/h$ in even degree, 0 in odd degree. So apply Theorems 3 and 6. The picture of the spectral sequence is given in Figure 1. To compute the differential d^5, we use Theorem 6, along with the exact homotopy sequence

$$\pi_5(\mathrm{SU}(2)) \to \pi_5(\mathrm{SU}(3)) \to \pi_5(S^5) \xrightarrow{\partial} \pi_4(\mathrm{SU}(2)) \to \pi_4(\mathrm{SU}(3)).$$

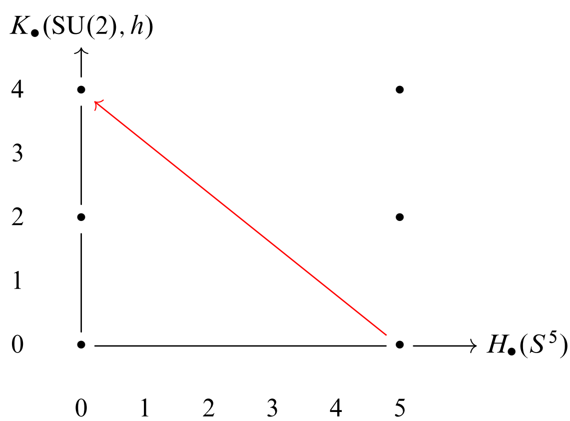

Figure 1: The Segal SS for twisted K–homology of SU(3). Heavy dots indicate copies of $\mathbb{Z}/h$. The diagonal arrow shows the differential d^5.

Here it is classical that $\pi_5(\mathrm{SU}(2)) \cong \pi_4(\mathrm{SU}(2)) \cong \mathbb{Z}/2$, and $\pi_5(\mathrm{SU}(3)) \cong \mathbb{Z}$ and $\pi_4(\mathrm{SU}(3)) = 0$ by [35]. Thus the boundary map ∂ in this sequence has kernel of index 2. Now we need to understand the Hurewicz maps

$$\begin{gathered} \pi_5(S^5) \to H_5(S^5, K_0(\mathrm{SU}(2), h)), \\ \pi_4(\mathrm{SU}(2)) \to K_4(\mathrm{SU}(2), h) \cong K_0(\mathrm{SU}(2), h). \end{gathered}$$

The generator of $\pi_5(S^5)$ is suspended from the generator of $\pi_0(S^0)$, just as we have $H_5(S^5, K_0(\mathrm{SU}(2), h)) \cong K_0(\mathrm{SU}(2), h)$ via suspension, so the generator 1 of $\pi_5(S^5)$ goes to the generator 1 of the cyclic group $\mathbb{Z}/h$. To finish the proof, we need the following Theorem 9. Thus we see that d^5 in the Segal spectral sequence has kernel of order 2 if h is even and trivial kernel if h is odd, and the result follows. □

Theorem 9 *Let $h \in \mathbb{Z}$, $h \neq 0$. Then the Hurewicz map $\pi_4(S^3) \to K_4(S^3, h) \cong \mathbb{Z}/h$ is nonzero if and only if h is even.*

Proof Since $\pi_4(S^3) \cong \mathbb{Z}/2$, obviously the Hurewicz map is 0 if h is odd, since then there is no 2–torsion in $K_4(S^3, h)$. So assume h is even. The Hurewicz map in twisted K–homology is a bit more mysterious than the usual Hurewicz map in K–homology, but we can apply Theorem 7 to help clarify things. Let P_h be the principal $\mathbb{CP}^\infty$–bundle over S^3 classified by the nonzero integer $h \in \mathbb{Z} \cong H^3(S^3, \mathbb{Z})$. The Serre spectral sequence for the fibration $\mathbb{CP}^\infty \to P_h \to S^3$ has only two columns, so in cohomology the only differential is d_3, which sends the generator u of $H^2(\mathbb{CP}^\infty)$ to h times the usual generator y of $H^3(S^3)$. Since d_3 is a derivation, $d_3(u^n) = nhu^{n-1}y$, and the homology differential is similar, but just points in the opposite direction. Hence $H_{2n}(P_h) \cong \mathbb{Z}/(nh)$ for $n \geq 1$, and $H_{\mathrm{odd}}(P_h)$ vanishes. Thus the AHSS for P_h collapses (as does the ku–AHSS for P_h, a fact we will use later), and Khorami computed that $K_0(P_h) \cong R/(h\beta_1)$ as an R–module, where $R = K_0(\mathbb{CP}^\infty)$ (with multiplication defined by Pontrjagin product), and $K_0(S^3, h) \cong \mathbb{Z}/h$ is gotten from this by tensoring with $\mathbb{Z}$ (viewed as an R–module under $\beta_1 \mapsto 1$, $\beta_j \mapsto 0$ for $j > 1$).

Note that a map $S^3 \xrightarrow{h} S^3$ of degree h pulls the $\mathbb{CP}^\infty$–bundle P_1 over S^3 back to the $\mathbb{CP}^\infty$–bundle P_h over S^3. So we get a pullback square

$$\begin{array}{ccccc} \mathbb{CP}^\infty & \longrightarrow & P_h & \longrightarrow & S^3 \\ \Big\| & & \Big\downarrow f_h & & \Big\downarrow h \\ \mathbb{CP}^\infty & \longrightarrow & P_1 & \longrightarrow & S^3 \end{array} \tag{2}$$

and the map $f_h\colon P_h \to P_1$ induces a map of R–modules $R/(h\beta_1) \to R/(\beta_1)$ on K–homology. Comparison of the Serre spectral sequences also shows that $(f_h)_*$ is surjective on integral homology. From the long exact homotopy sequences associated to the two rows, we also see that $\pi_j(P_h) \cong \mathbb{Z}/h$ for $j = 2$ and $\cong \pi_j(S^3)$ for $j \geq 4$, and that the map $f_h\colon P_h \to P_1$ induces multiplication by h on π_j for $j \geq 4$.

Now note that P_1 is the homotopy fiber of the canonical map $S^3 \to K(\mathbb{Z}, 3)$ inducing an isomorphism on π_3, and thus $P_1 \to S^3 \to K(\mathbb{Z}, 3)$ is the beginning of the Postnikov tower of S^3. Thus P_1 is 3–connected and $\pi_j(P_1) \cong \pi_j(S^3)$ for $j \geq 4$. So, by the Hurewicz theorem, the Hurewicz map $\pi_4(P_1) \to H_4(P_1) \cong \mathbb{Z}/2$ is an isomorphism.

We can also consider the diagram

$$\begin{array}{ccccc} & & P_1 & & \\ & \overset{i_h}{\swarrow} & \downarrow & & \\ P_h & \longrightarrow & S^3 & \xrightarrow{\;h\;} & K(\mathbb{Z}, 3) \end{array}$$

where the downward solid arrow is the bundle projection of the $\mathbb{CP}^\infty$–bundle P_1 over S^3. From the exact homotopy sequence

$$[P_1, P_h] \to [P_1, S^3] \to [P_1, K(\mathbb{Z}, 3)],$$

we see that we get a lifting $i_h\colon P_1 \to P_h$, which is the first stage of the Postnikov fibration $P_1 \xrightarrow{i_h} P_h \to K(\mathbb{Z}/h, 2)$ for P_h. Unlike the map f_h in the other direction, i_h is not a map of $\mathbb{CP}^\infty$–bundles.

Putting everything together, we see that the Hurewicz map $\pi_j(S^3) \to K_j(S^3, h)$ (for $j \geq 4$ even) is the composite

$$\pi_j(S^3) \cong \pi_j(P_1) \xrightarrow[\cong]{(i_h)_*} \pi_j(P_h) \to K_j(P_h) = R/(h\beta_1) \to \mathbb{Z}/h.$$

Now $K_{\text{even}}(P_1)$ and $K_{\text{even}}(P_h)$ have skeletal filtrations $F_0 = \mathbb{Z} \subset F_1 \subset F_2 \subset \cdots$, where F_j is generated (additively) by the images of $\beta_0, \ldots, \beta_j$, and since the AHSS for K–homology of P_1 collapses, we have maps $F_j \to H_{2j}$ identifying F_j/F_{j-1} with H_{2j}. The image of π_4 under the Hurewicz map must lie in F_2 (just on dimensional grounds). Thus, since the Hurewicz map $\pi_4(P_1) \to H_4(P_1)$ is an isomorphism, the Hurewicz map in K–homology for P_1 maps $\pi_4(P_1) \cong \mathbb{Z}/2$ onto the cyclic group generated by β_2, of order 2 in $R/(\beta_1)$, that maps onto $H_4(P_1)$. (One can compute that $2\beta_2 = \beta_1^2 - \beta_1$ lies in the ideal generated by β_1.)

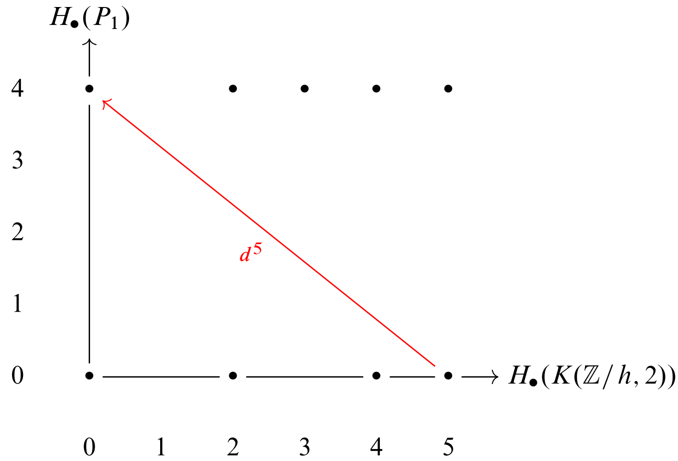

Figure 2: The Serre SS for the first Postnikov fibration of P_h.

The Hurewicz map in ordinary homology $\pi_4(P_h) \to H_4(P_h)$ can be identified with the edge homomorphism $(i_h)_*\colon H_4(P_1) \to H_4(P_h)$ associated to the Serre spectral sequence for $P_1 \xrightarrow{i_h} P_h \to K(\mathbb{Z}/h,2)$, which is shown schematically in Figure 2. The integral homology of $K(\mathbb{Z}/h,2)$ is a bit complicated, but we only need its 2–primary part in low degree. When $h = 2$, Serre showed that $H^\bullet(K(\mathbb{Z}/h,2),\mathbb{F}_2)$ is a polynomial ring on generators $\iota, \mathrm{Sq}^1\iota, \mathrm{Sq}^2\mathrm{Sq}^1\iota, \dots$, where ι is the canonical generator in degree 2 [24, page 500]. Thus the $\mathbb{F}_2$–Betti numbers of $K(\mathbb{Z}/2,2)$ are $1,0,1,1,1,2,\dots$. In particular we can see from this that rank $H_4(K(\mathbb{Z}/2,2);\mathbb{Z}) = 1$.

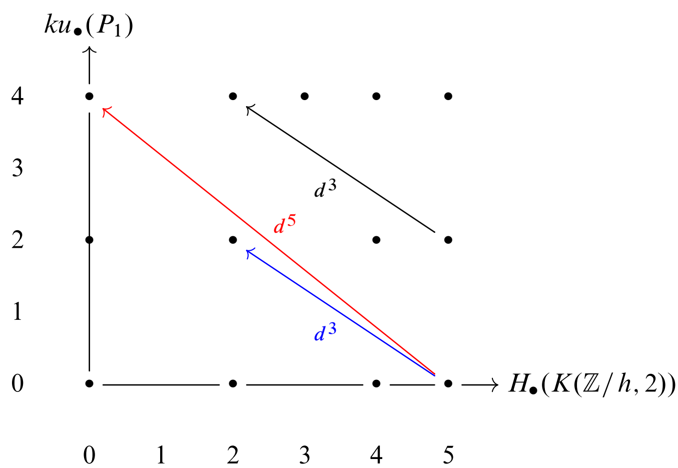

Figure 3: The Segal SS in ku for the first Postnikov fibration of P_h.

The complete calculation of $H_4(K(\mathbb{Z}/h,2);\mathbb{Z})$ may be found in [18, Theorem 21.1] and in [14; 15] (which even computes the integral homology in arbitrary degree, at least in principle), and it turns out that $H_4(K(\mathbb{Z}/h,2);\mathbb{Z}) \cong \Gamma_4(\mathbb{Z}/h)$, where Γ_4 is the functor defined in [45], and for h even, this is a cyclic group of order $2h$, while $H_5(K(\mathbb{Z}/h,2);\mathbb{Z}) \cong \mathbb{Z}/2$ [18, Theorem 22.1].

But recall that $H_4(P_h;\mathbb{Z}) \cong \mathbb{Z}/(2h)$. Thus the red arrow in Figure 2 has to be an isomorphism and the edge homomorphism $(i_h)_*\colon H_4(P_1) \to H_4(P_h)$, which is the Hurewicz map, vanishes. One way of thinking about this is that we can view the Hurewicz map as being about the embedding of an S^4 in P_h via the generator η of $\pi_4(P_h)$. This sphere doesn't bound a disk (if it did, the homotopy class of η would be trivial), but it does bound a homology chain, and even an oriented manifold (in this low dimension, oriented bordism is almost the same as homology). The question for us now is: does it bound a Spin^c manifold? This determines the Hurewicz map in K–homology since (when we localize everything at 2) the low-degree summand of $M\text{Spin}^c$ is ku (connective K–theory) [38, Section 8; 43, Chapter XI] and the K–homology class of this 4–sphere, which is the image of the Hurewicz map, comes from its ku–homology class.

So let's reconsider Figure 2 redone in ku–homology, which is Figure 3.

Since $ku_\bullet(P_h)$ is all concentrated in even degree, everything in odd degree must cancel. We have $\widetilde{ku}_2(P_1)=0$, $\widetilde{ku}_4(P_1) \cong H_4(P_1) \cong \mathbb{Z}/2$, $\widetilde{ku}_2(P_h) \cong H_2(P_h) \cong \mathbb{Z}/h$ and $\widetilde{ku}_4(P_h)$ is an extension of $H_4(P_h) \cong \mathbb{Z}/(2h)$ by $H_2(P_h) \cong \mathbb{Z}/h$. The Hurewicz map $\pi_4(P_h) \to K_{\text{even}}(P_h)$ now comes from the edge homomorphism $\widetilde{ku}_4(P_1) \to \widetilde{ku}_4(P_h)$, and so the relevant question is which of the two arrows (d^3 and d^5) starting at the position $(5,0)$ in Figure 3 is nonzero.

To answer this question we can consider the map $P_h \to K(\mathbb{Z}/h,2)$, which induces a morphism of spectral sequences from the spectral sequence of Figure 3 to the Atiyah–Hirzebruch spectral sequence for computing $ku_\bullet(K(\mathbb{Z}/h,2))$. By [4; 46, Theorem 2], $\widetilde{K}_\bullet(K(\mathbb{Z}/h,2))$ vanishes. Looking then at the AHSS for $ku_\bullet(K(\mathbb{Z}/h,2))$, we see that $d^3\colon H_5 \to H_2$ is nonzero, since the dual differential for computing ku–cohomology,

$$d_3 = \text{Sq}^3\colon H^3(K(\mathbb{Z}/h,2);\mathbb{Z}) \to H^6(K(\mathbb{Z}/h,2);\mathbb{Z}),$$

is nonzero, and thus the blue arrow in Figure 3 is nontrivial. This implies that the map $\widetilde{ku}_4(P_1) \to \widetilde{ku}_4(P_h)$ is nontrivial, and the image must go to $\left(\frac{h}{2}\right)\beta_1$ in $\widetilde{K}_{\text{even}}(P_h)$. So, under the map $K_0(P_h) \to K_0(S^3,h)$, it maps to $\frac{h}{2}$ in $\mathbb{Z}/h$, which is nontrivial. This gives the desired result. □

3.2 The Braun–Douglas theorem for Sp(2) ≅ Spin(5)

We begin by showing that the Douglas and Braun formulas coincide in the case of $\mathrm{Sp}(2)\cong\mathrm{Spin}(5)$. Here it is convenient to use the following standard definition:

Definition 10 Fix a prime p, and for x a positive integer, let $\nu_p(x)$ be the number of times that p divides x. In other words, $\nu_p(x)$ is defined by the property that $x=p^{\nu_p(x)}x'$, where $\gcd(x',p)=1$. Thus $x=\prod_{p\text{ prime}}p^{\nu_p(x)}$.

Proposition 11 *Let h be a positive integer, let*

$$h'=\frac{h}{\gcd(h,\mathrm{lcm}(1,2,3))}=\frac{h}{\gcd(h,6)},$$

which is the Braun formula for $c(G_2,h)$, and let

$$h''=\gcd\Bigl(h,2\binom{h}{3}+\binom{h}{2}\Bigr),$$

which is Douglas's formula for $c(G_2,h)$ in [16, Theorem 1.2]. *Then $h'=h''$.*

Proof It will suffice to show that $\nu_p(h')=\nu_p(h'')$ for all primes p. Clearly $\nu_p(h')=\nu_p(h'')=0$ if p does not divide h. So assume p divides h, and we'll consider in turn the cases of $p=2,3$ and $p>3$. If $p=2$ and h is even, then $\nu_2(h')=\nu_2(h)-1$, while $2\binom{h}{3}=\frac{1}{3}h(h-1)(h-2)$ is divisible by 8 and $\binom{h}{2}=\frac{h}{2}(h-1)$ has the same divisibility by 2 as $\frac{h}{2}$. Thus $\nu_2(h'')=\nu_2(h)-1=\nu_2(h')$. If $p=3$ and $h\equiv 0 \pmod 3$, then $\nu_3(h')=\nu_3(h)-1$, while $\nu_3\bigl(\binom{h}{2}\bigr)=\nu_3(h)$ and

$$\nu_3\Bigl(2\binom{h}{3}\Bigr)=\nu_3\bigl(\tfrac{1}{3}h(h-1)(h-2)\bigr)=\nu_3(h)-1.$$

Thus $\nu_3(h'')=\nu_3(h)-1$. Finally, if $p>3$ and p divides h, then $\nu_p(h)=\nu_p\bigl(\binom{h}{2}\bigr)=\nu_p\bigl(2\binom{h}{3}\bigr)$ and so $\nu_p(h')=\nu_p(h'')=\nu_p(h)$. □

Theorem 12 (Braun–Douglas theorem for $\mathrm{Sp}(2)$) *Let h be a positive integer and let $G=\mathrm{Sp}(2)\cong\mathrm{Spin}(5)$. Then, in any degree, $K_\bullet(G,h)$ is cyclic of order $h/\gcd(h,6)$.*

Proof We argue as in Theorem 8, using the usual fibration $\mathrm{Sp}(1)\to G\to S^7$, where $\mathrm{Sp}(1)\cong\mathrm{SU}(2)\cong S^3$. The inclusion of $\mathrm{Sp}(1)$ into $\mathrm{Sp}(2)$ induces an isomorphism on H^3, and the groups $K_j(\mathrm{Sp}(1),h)$ are cyclic of order h for j even, zero for j odd.

We just need to compute the differential in the Segal spectral sequence

$$d^7\colon H_7(S^7, K_0(\mathrm{Sp}(1), h)) \to K_6(\mathrm{Sp}(1), h).$$

As explained in Theorem 6, this differential is related to the boundary map ∂ in the long exact homotopy sequence

$$\pi_7(\mathrm{Sp}(1)) \to \pi_7(\mathrm{Sp}(2)) \to \pi_7(S^7) \xrightarrow{\partial} \pi_6(\mathrm{Sp}(1)) \to \pi_6(\mathrm{Sp}(2)).$$

Here $\pi_6(S^3) \cong \mathbb{Z}/12$, $\pi_7(S^3) \cong \mathbb{Z}/2$, $\pi_7(\mathrm{Sp}(2)) \cong \mathbb{Z}$ and $\pi_6(\mathrm{Sp}(2)) = 0$. (See for example [24, page 339] for the homotopy groups of S^3 and [35] for the homotopy groups of Sp(2).) From this exact sequence, ∂ is surjective onto $\pi_6(S^3) \cong \mathbb{Z}/12$. Away from the primes 2 and 3, ∂ vanishes and so does the differential in the spectral sequence for $K_\bullet(G, h)$. So we only need to analyze what happens at the primes 2 and 3. This involves understanding the Hurewicz homomorphism $\pi_6(S^3) \to K_6(S^3, h)$ or $\pi_6(P_h) \to K_6(P_h)$, where P_h is the principal $\mathbb{CP}^\infty$–bundle over SU(2) associated to the twist h as in the proof of Theorem 9.

We can proceed as we did there. If h is divisible by neither 2 nor 3, then obviously the Hurewicz homomorphism is zero. If h is divisible by 3 and we localize at 3, then everything is largely as in the proof of Theorem 9, and we keep the notation used there. The fiber P_1 of the Postnikov fibration $P_1 \xrightarrow{h} P_h \to K(\mathbb{Z}/h, 2)$ is 3–locally 5–connected, so by the mod-$\mathcal{C}$ Hurewicz theorem (with $\mathcal{C}$ the Serre class of prime-to-3 torsion groups), the 3–torsion subgroup $\mathbb{Z}/3$ of $\pi_6(S^3) \cong \pi_6(P_1) \cong \pi_6(P_h)$ maps isomorphically to $H_6(P_1) \cong \mathbb{Z}/3$. We have $H_6(P_h) \cong \mathbb{Z}/(3h)$ and $H_6(K(\mathbb{Z}/h, 2)) \cong \mathbb{Z}/(3h)$ by [18, Theorem 21.1], so the differential $d^7\colon H_7(K(\mathbb{Z}/h, 2)) \to H_6(P_1)$ in the analogue of Figure 2 must be nonzero and the Hurewicz homomorphism in ordinary homology, which can be identified with the map $(i_h)_*\colon H_6(P_1) \to H_6(P_h)$, vanishes. To study the corresponding map in ku–homology, we compare the diagram analogous to Figure 3 with the AHSS for $ku_\bullet(K(\mathbb{Z}/h, 2))$. $H^\bullet(K(\mathbb{Z}/h, 2); \mathbb{F}_p)$ has generators ι_2, $\beta_r\iota_2$, $P^1\beta_r\iota_2$, etc (β_r the r^{th}–power Bockstein, 3^r the biggest power of 3 dividing h) and the first nontrivial differential in the AHSS for computing $K^\bullet(K(\mathbb{Z}/3^r, 2))$ from $H^\bullet(K(\mathbb{Z}/3^r, 2); \mathbb{Z})$ is (up to a nonzero constant) $d_5 = \beta_r P^1\colon \beta_r\iota_2 \mapsto \beta_r P^1 \beta_r \iota_2$. This is dual to a nonzero differential d^5 with target $H_7(K(\mathbb{Z}/3^r, 2); \mathbb{Z})$, and so the Hurewicz map $ku_6(P_1) \to ku_6(P_h)$ will be nonzero, just as in the proof of Theorem 9.

The hardest step is the 2–local calculation in the case where h is even, which involves the 2–local part of the Hurewicz map $\pi_6(P_h) \to ku_6(P_h)$ for h even. We defer this calculation to Theorem 16. □

3.3 The Braun–Douglas theorem for G_2

The following result and its proof are partially modeled on Appendix C in [31], and proves that the Douglas and Braun formulas coincide in the case of G_2.

Proposition 13 *Let h be a positive integer, let*

$$h' = \frac{h}{\gcd(h, 60)},$$

and let

$$h'' = \gcd\Bigl(h, \binom{h+2}{2} - 1, \tfrac{1}{120}(h+1)(h+2)(2h+3)(3h+4)(3h+5) - 1\Bigr).$$

Note that h' is Braun's formula for $c(G_2, h)$ and h'' is Douglas's formula for $c(G_2, h)$. Then $h' = h''$.

Proof We again use Definition 10. It will suffice to show that $\nu_p(h') = \nu_p(h'')$ for all primes p.

First consider $p = 2$. If h is odd, then $\nu_2(h) = \nu_2(h') = \nu_2(h'') = 0$. If $\nu_2(h) = 1$, then, since $\nu_2(\gcd(h, 60)) = 1$, $\nu_2(h') = 0$. Consider h''. We have $h \equiv 2 \pmod 4$, so $h+2 \equiv 0 \pmod 4$ and $\binom{h+2}{2}$ is even, hence $\binom{h+2}{2} - 1$ is odd. Thus $\nu_2(h'') = 0 = \nu_2(h')$ in this case. If $\nu_2(h) \geq 2$, then since $\nu_2(60) = 2$, $\nu_2(h') = \nu_2(h) - 2$. But $\binom{h+2}{2} - 1 = \frac{1}{2}((h+2)(h+1) - 2) = \frac{1}{2}h(h+3)$. Thus if $\nu_2(h) \geq 2$, $\nu_2(\gcd(h, \binom{h+2}{2} - 1)) = \nu_2(h) - 1$. On the other hand,

$$\begin{aligned}\tfrac{1}{120}&(h+1)(h+2)(2h+3)(3h+4)(3h+5) - 1 \\ &= \tfrac{1}{120}(18h^5 + 135h^4 + 400h^3 + 585h^2 + 422h + 120 - 120) \\ &= \tfrac{1}{120}h(18h^4 + 135h^3 + 400h^2 + 585h + 422).\end{aligned}$$

The denominator is $2^3 \cdot 15$ and, since $h \equiv 0 \pmod 4$ and $422 \equiv 2 \pmod 4$, ν_2 of the numerator is $\nu_2(h) + 1$. Thus ν_2 of this fraction is $\nu_2(h) + 1 - 3 = \nu_2(h) - 2 = \nu_2(h')$. So again $\nu_2(h') = \nu_2(h'')$.

Next, consider $p = 3$. If $\nu_3(h) = 0$, then clearly $\nu_3(h') = \nu_3(h'') = 0$. If $\nu_3(h) \geq 1$, then $\nu_3(\gcd(h, 60)) = 1$, so $\nu_3(h') = \nu_3(h) - 1$. On the other hand, $\nu_3\bigl(\frac{1}{2}h(h+3)\bigr) > \nu_3(h)$, so taking the gcd with $\frac{1}{2}h(h+3)$ doesn't change $\nu_3(h)$. With regard to $\frac{1}{120}h(18h^4 + 135h^3 + 400h^2 + 585h + 422)$, if h is divisible by 3, then ν_3 of the numerator is the same as for h (since $422 \equiv 2 \pmod 3$), while $\nu_3(120) = 1$, so ν_3 of the fraction, as well as $\nu_3(h'')$, is $\nu_3(h) - 1$, which agrees with $\nu_3(h')$.

Consider now $p=5$. If $\nu_5(h)=0$, then clearly $\nu_5(h')=\nu_5(h'')=0$. If $\nu_5(h)\geq 1$, then $\nu_5(\gcd(h,60))=1$, so $\nu_5(h')=\nu_5(h)-1$. For h divisible by 5, $h+3\equiv 3 \pmod 5$, so $\nu_5\bigl(\frac{1}{2}h(h+3)\bigr)=\nu_5(h)$, and taking the gcd with $\frac{1}{2}h(h+3)$ doesn't change $\nu_5(h)$. Again, for h divisible by 5, $18h^4+135h^3+400h^2+585h+422\equiv 2 \pmod 5$, while $\nu_5(120)=1$, so ν_5 of the big fraction, as well as $\nu_5(h'')$, is $\nu_5(h)-1$, which agrees with $\nu_5(h')$.

Finally, suppose $p\geq 7$. Then 2, 60 and 120 are all relatively prime to p. If $\nu_p(h)=0$, then clearly $\nu_p(h')=\nu_p(h'')=0$. If $\nu_p(h)\geq 1$, then $\nu_p(\gcd(h,60))=0$, so $\nu_p(h')=\nu_p(h)$. On the other hand, if $\nu_p(h)\geq 1$, then

$$\nu_p\bigl(\tfrac{1}{2}h(h+3)\bigr)=\nu_p(h) \quad\text{and}\quad \nu_p\bigl(\tfrac{1}{120}h(18h^4+135h^3+400h^2+585h+422)\bigr)\geq \nu_p(h),$$

so $\nu_p(h'')=\nu_p(h)=\nu_p(h')$. This concludes the proof. □

We now want to give an elementary but rigorous proof of the Braun–Douglas theorem for G_2. We start with analysis of the odd torsion. For convenience in what follows, if x is a positive integer, let x_{odd} denote the maximal odd factor of x. Of course, $x_{\mathrm{odd}}=\prod_{p\text{ prime}\geq 3} p^{\nu_p(x)}$.

Theorem 14 *Let h be a positive integer, viewed as a twisting class on G_2. Then $K_\bullet(G_2,h)$ is a finite torsion group in all degrees. Its odd torsion (in any degree) is cyclic of order*

$$c(G_2,h)_{\mathrm{odd}}=h_{\mathrm{odd}}/\gcd(h_{\mathrm{odd}},15).$$

Proof We use the fibration [8, Lemme 17.1]

$$\mathrm{SU}(2)\to G_2\to V_{7,2},$$

and get from Theorem 3 a spectral sequence

$$E^2_{p,q}=H_p(V_{7,2},K_q(\mathrm{SU}(2),h))\Rightarrow K_\bullet(G_2,h).$$

(The restriction map $H^3(G_2)\to H^3(\mathrm{SU}(2))$ is an isomorphism.) Here $K_q(\mathrm{SU}(2),h)\cong\mathbb{Z}/h$ for q even and is 0 for q even. Since E^2 is torsion, so is $K_\bullet(G_2,h)$. The Stiefel manifold $V_{7,2}$ is 11–dimensional and has only one nontrivial homology group below the top dimension, namely a $\mathbb{Z}/2$ in dimension 5 (see for example [24, Section 3.D]), so, after inverting 2, $V_{7,2}$ becomes homotopy equivalent to S^{11} by the Hurewicz theorem modulo the Serre class of 2–primary torsion groups. Thus, from the point of view of odd torsion, we are in the situation of Theorem 6 with a unique differential d^{11}. We have the long exact homotopy sequence

$$\pi_{11}(G_2) \to \pi_{11}(V_{7,2}) \to \pi_{10}(\mathrm{SU}(2)) \to \pi_{10}(G_2),$$

and $\pi_{10}(\mathrm{SU}(2)) \cong \mathbb{Z}/15$, $\pi_{10}(G_2) = 0$ [34]. Thus the boundary map $\pi_{11}(V_{7,2}) \to \pi_{10}(\mathrm{SU}(2))$ has kernel of order 15, and the theorem follows from Theorem 6, exactly as in the proof of Theorem 8.

Let's first deal with the 5–primary torsion. If $\gcd(h, 5) = 1$, then $K_\bullet(G_2, h)$ can't have 5–primary torsion, by Theorem 5. So assume h is divisible by 5 and localize everything at 5. Once again, let's use the notation of Theorem 9. The first 5–primary torsion in the homotopy groups and homology groups of P_1 occurs in degree 10. So the Hurewicz map $\pi_{10}(S^3) \cong \pi_{10}(P_1) \to H_{10}(P_1) \cong \mathbb{Z}/5$ is a 5–local isomorphism, as is the Hurewicz map to $\widetilde{ku}_{10}(P_1)$. Just as in the proof of Theorem 9, we need to show that the map $\widetilde{ku}_{10}(P_1) \to \widetilde{ku}_{10}(P_h)$ is injective on the 5–torsion. And, again, we do this by comparing the Segal spectral sequence

$$H_p(K(\mathbb{Z}/h, 2), ku_q(P_1)) \Rightarrow ku_\bullet(P_h)$$

with the AHSS for computing $ku_\bullet(K(\mathbb{Z}/h, 2))$. (Here everything is localized at the prime 5.) This is exactly like the 3–primary calculation in Theorem 12.

The result for 3–primary torsion follows from Theorem 16 below. □

Theorem 15 *Let h be a positive integer, viewed as a twisting class on G_2. Then $K_\bullet(G_2, h)$ is a finite torsion group in all degrees. Its 2–primary torsion (in any degree) is cyclic of order*

$$c(G_2, h)_{2\text{–primary}} = 2^{\max(0, \nu_2(h)-2)}.$$

In other words,

$$\nu_2(c(G_2, h)) = \begin{cases} 0, & \nu_2(h) \le 2, \\ \nu_2(h) - 2, & \nu_2(h) > 2. \end{cases}$$

Proof First suppose that $\nu_2(h) \le 1$. This time we use the fibration

$$\mathrm{SU}(3) \to G_2 \to S^6,$$

coming from the action of G on the unit sphere of the imaginary octonians. We can apply Theorem 8 together with Theorems 3 and 6. The inclusion $\mathrm{SU}(3) \hookrightarrow G_2$ induces an isomorphism on H^3, and $K_\bullet(\mathrm{SU}(3), h)$ has no 2–torsion, and that proves the

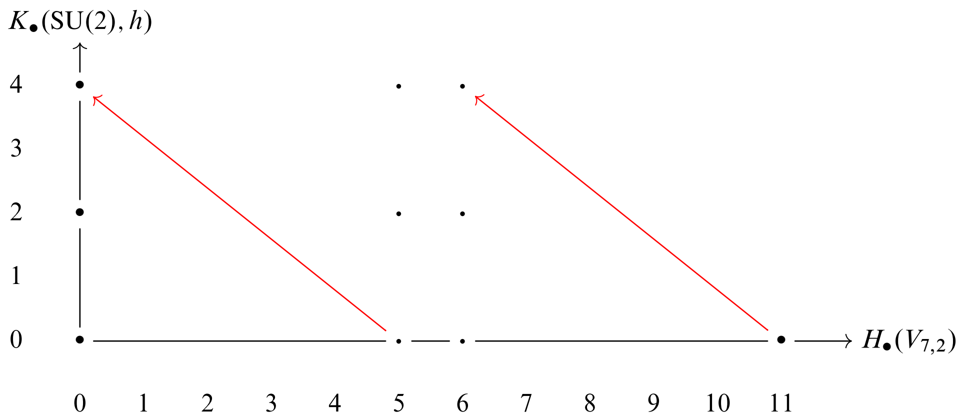

Figure 4: The Segal SS for 2–primary twisted K–homology of G_2 when $\nu_2(h) \geq 1$. Heavy dots indicate copies of $\mathbb{Z}/2^{\nu_2(h)}$. Light dots indicate copies of $\mathbb{Z}/2$. The diagonal red arrow shows the differential d^5.

theorem in this case. Note that in the case $\nu_2(h) = 1$, we see that the picture for the Segal spectral sequence attached to

$$\mathrm{SU}(2) \to G_2 \to V_{7,2}$$

has to look like Figure 4, with the red arrows isomorphisms, so that everything cancels out.

If $\nu_2(h) \geq 2$, then, in Figure 4, the red d^5 arrows will still be nonzero for the same reasons as before,[2] but this time the copies of $\mathbb{Z}/2^{\nu_2(h)}$ are reduced to $\mathbb{Z}/2^{\nu_2(h)-1}$ at the E^6 stage. At this point the dots in the H_5 and H_6 columns have disappeared and the spectral sequence now looks like one for a fibration with S^{11} as the base and with $\mathbb{Z}/2^{\nu_2(h)-1}$ in even degrees in the twisted K–homology of the fiber. There is still the differential

$$d^{11}\colon E^{11}_{11,j} \to E^{11}_{0,j+10}$$

to be reckoned with. We claim that this differential has image isomorphic to $\mathbb{Z}/2$, which will give the desired result.

This comes about in the following way. If G_2 really fits into a fibration $S^3 \to G_2 \to S^{11}$ and we were looking at the associated 2–local Segal spectral sequence for computing

[2]The map $E^5_{5,0} \to E^5_{0,4}$ is determined by Theorem 6 and the calculation of the Hurewicz map in twisted K–homology from Theorem 9. The other map $E^5_{11,0} \to E^5_{6,4}$ is linked to this one by the module action of $K_\bullet(G_2)$, which is an exterior algebra over $\mathbb{Z}$, on $K_\bullet(G_2,h)$.

$K_\bullet(G_2, h)$, then d^{11} would vanish as a consequence of Theorem 6, since $\pi_{10}(S^3)$ has no 2–torsion. But in our situation, the groups $\mathbb{Z}/2^{\nu_2(h)-1}$ in $E^{11}_{11,2k}$ and in $E^{11}_{0,2k+10}$ are really different. The former arose as *kernels* of d^5, a map $\mathbb{Z}/2^{\nu_2(h)} \to \mathbb{Z}/2$, and the latter as *cokernels* of a map $\mathbb{Z}/2 \to \mathbb{Z}/2^{\nu_2(h)}$ (see Figure 4 again). By [26], $K_\bullet(G_2) \cong \bigwedge(x_3, x_{11})$, an exterior algebra on two odd generators, and the module action of $x_{11} \in K_\bullet(G_2)$ on $K_\bullet(G_2, h)$ and on the spectral sequence sets up an isomorphism $E^2_{0,2k} \to E^2_{11,2k}$ which has to pass to an isomorphism $K_{2k}(G_2, h) \to K_{2k+11}(G_2, h)$. This can only happen if $E^2_{0,2k}$ and $E^2_{11,2k}$ are each quotients of an index-2 subgroup of $K_{2k}(S^3, 2^{\nu_2(h)}) \cong \mathbb{Z}/2^{\nu_2(h)}$ by a subgroup of order 2, and we end up with a 2–torsion subgroup of order $2^{\nu_2(h)-2}$.

An alternative way to prove the theorem when $\nu_2(h) \geq 2$ is to use the Segal spectral sequence for the fibration $\mathrm{SU}(3) \to G_2 \to S^6$ to compute the order of $KU_\bullet(G_2, h)$, along with the previous argument with the other spectral sequence, to deduce that the group in each degree is cyclic. For example, suppose $h = 4$ and consider the differential $H_6(S^6, K_0(\mathrm{SU}(3), h)) \to H_0(S^6, K_5(\mathrm{SU}(3), h))$. Both groups here are cyclic of order 2, and the generator of the domain is the image of the Hurewicz map from $\pi_6(S^6)$. Since $\pi_5(\mathrm{SU}(3)) \cong \mathbb{Z}$ [35] and $\pi_5(G_2) = 0$ [34], the boundary map $\partial\colon \pi_6(S^6) \to \pi_5(\mathrm{SU}(3))$ is an isomorphism and we just need to determine what happens to the generator of $\pi_5(\mathrm{SU}(3))$ under the Hurewicz map to $K_5(\mathrm{SU}(3), h)$. But the Hurewicz map $\pi_5(\mathrm{SU}(3)) \to H_5(\mathrm{SU}(3))$ has image which is of index 2 [44, (4.3)], so the image of $\pi_5(\mathrm{SU}(3))$ in $K_5(\mathrm{SU}(3), h)$ corresponds exactly to the kernel of

$$d^5\colon H_5\big(S^5, K_0(\mathrm{SU}(2), 4)\big) \to H_0\big(S^5, K_4(\mathrm{SU}(2), 4)\big)$$

in the Segal spectral sequence for $K_\bullet(\mathrm{SU}(3), 4)$ from the fibration $S^3 \to \mathrm{SU}(3) \to S^5$, and this is the entire group $K_5(\mathrm{SU}(3), h) \cong \mathbb{Z}/2$. Thus, in the Segal spectral sequence, the differential

$$H_6\big(S^6, K_0(\mathrm{SU}(3), h)\big) \to H_0\big(S^6, K_5(\mathrm{SU}(3), h)\big)$$

is an isomorphism. The differential with the other parity can also be seen to be an isomorphism, and in this way one can show that $K_\bullet(G_2, 4) = 0$. □

3.4 Analysis of Hurewicz maps

To complete all the theorems from Section 3, we need the following technical result:

Theorem 16 *Let h be a positive integer and let P_h be the principal $\mathbb{CP}^\infty$–bundle over S^3 classified by a map of degree h from S^3 to $K(\mathbb{Z}, 3)$, as in the proof of*

Theorem 9. Then the Hurewicz maps $\pi_j(S^3) \cong \pi_j(P_h) \to ku_j(P_h)$ *are injective on the*

(1) 2*–torsion* $\mathbb{Z}/2$ *when* $j = 4$ *and* h *is even* (this case was in Theorem 9),

(2) 2*–primary torsion* $\mathbb{Z}/4$ *when* $j = 6$ *and* h *is divisible by* 4 *— if* $h \equiv 2 \pmod 4$, *the map is nonzero* (these cases were needed for Theorem 12),

(3) 3*–torsion* $\mathbb{Z}/3$ *when* $j = 10$ *and* 3 *divides* h (this case was needed for Theorem 14),

(4) 5*–torsion* $\mathbb{Z}/5$ *when* $j = 10$ *and* 5 *divides* h (this case was also needed for Theorem 14) .

In all cases, the image maps injectively under the quotient map $K_0(P_h) \to K_0(S^3, h)$.

Before starting the proof, let's explain a bit about strategy. We have lumped all of these results together and included cases (1) and (4) (even though we already did those by another method) to illustrate a common method of attack using either the Adams–Novikov spectral sequence (ANSS) or the classical Adams spectral sequence (ASS). For this we localize at the appropriate prime (2, 3 or 5). We will use the fact that the unstable Hurewicz homomorphism $\pi_j(P_h) \to ku_j(P_h)$ factors through the stable Hurewicz homomorphism $\pi_j^s(P_h) \to ku_j(P_h)$, and the latter can be computed by comparing $\pi_\bullet^s$ and $ku_\bullet$ with the help of the ANSS or the ASS.

Proof We localize at the appropriate prime. It suffices to look at the associated Brown–Peterson homology $\mathrm{BP}_\bullet$, since $MU_\bullet$ splits as a wedge of shifted copies of BP and $K_\bullet$ can be recovered from $MU_\bullet$ by the Conner–Floyd isomorphism $MU_\bullet(P_h) \otimes_{MU_\bullet} K_\bullet \cong K_\bullet(P_h)$. The ANSS has the form (see [39; 23] or [2, Part III, Section 15])

$$\mathrm{Ext}^s_{\mathrm{BP}_\bullet\mathrm{BP}}(\Sigma^t\,\mathrm{BP}_\bullet, \widetilde{\mathrm{BP}}_\bullet(P_h)) \Rightarrow \{S^{t-s}, P_h\},$$

and the edge homomorphism

$$\{S^t, P_h\} \to \mathrm{Hom}_{\mathrm{BP}_\bullet\mathrm{BP}}(\Sigma^t\,\mathrm{BP}_\bullet, \widetilde{\mathrm{BP}}_\bullet(P_h))$$

is the stable Hurewicz map in BP–homology. One can compare this with study of the classical Adams spectral sequence (ASS)

$$\mathrm{Ext}^s_{\mathcal{A}_*}(\Sigma^t\,\mathbb{F}_p, \widetilde{H}_\bullet(P_h;\mathbb{F}_p)) \Rightarrow \{S^{t-s}, P_h\},$$

for which the edge homomorphism

$$\{S^t, P_h\} \to \mathrm{Hom}_{\mathcal{A}_*}(\Sigma^t\, \mathbb{F}_p, \widetilde{H}_\bullet(P_h; \mathbb{F}_p))$$

is the stable Hurewicz map in ordinary mod- p homology. (We have deliberately ignored a few localization and completion issues which don't cause problems in our case. Here $\mathcal{A}_*$ is the dual Steenrod algebra at the prime p.)

Proof of case (4) Let's start with case (4), taking $p = 5$, starting with ordinary homology. We have $H^\bullet(P_h; \mathbb{F}_5) \cong \mathbb{F}_5[u] \otimes \bigwedge(y)$, where u is in degree 2 and y is in degree 3. The Bockstein $\beta_{\nu_5(h)}$ sends u to y, and $P^j(u^j) = u^{5j}$. In particular, there is no nonzero $\mathcal{A}$–module map $H^\bullet(P_h; \mathbb{F}_5) \to \Sigma^{10}\, \mathbb{F}_5$, since any such map would send u to 0 and $P^1 u$ to something nonzero, and thus the Hurewicz map in $\mathbb{F}_p$–homology (which would be dual to this map in cohomology) has to be 0. Of course, we could also observe this from the fact that $H_{10}(P_h; \mathbb{Z}) \cong \mathbb{Z}/(5h)$, which has 5–primary subgroup cyclic of order $5^{\nu_5(h)+1} \ge 25$, and since the 5–primary subgroup of $\pi_{10}(P_h) \cong \pi_{10}(S^3) \cong \mathbb{Z}/15$ is cyclic of order 5, the image of the Hurewicz map has to reduce mod 5 to 0. But in fact the *integral* Hurewicz map in degree 10 vanishes for any $\nu_5(h) \ge 1$; one way to see this is to use the 5–local Serre spectral sequence of the fibration $P_1 \to P_h \to K(\mathbb{Z}/h, 2)$ as in Figure 2. In the range of dimensions we're interested in, $H^\bullet(K(\mathbb{Z}/5^{\nu_5(h)}, 2); \mathbb{F}_5)$ agrees with

$$\mathbb{F}_5[\iota_2, \beta_{\nu_5(h)} P^1 \beta_{\nu_5(h)}\iota_2] \otimes \textstyle\bigwedge(\beta_{\nu_5(h)}\iota_2, P^1\beta_{\nu_5(h)}\iota_2).$$

The generators here have degrees 2, 12, 3 and 11, respectively. Via the calculation of the Bockstein spectral sequence in [33, Theorem 10.4],

$$\begin{aligned} H^{11}(K(\mathbb{Z}/5^{\nu_5(h)}, 2); \mathbb{Z}) &\cong H_{10}(K(\mathbb{Z}/5^{\nu_5(h)}, 2); \mathbb{Z}) \\ &\cong (\mathbb{Z}/5^{\nu_5(h)})(\iota_2)^4(\beta_{\nu_5(h)}\iota_2) \oplus (\mathbb{Z}/5) P^1 \beta_{\nu_5(h)}\iota_2. \end{aligned}$$

The k–invariant of the 5–local Postnikov approximation

$$K(\mathbb{Z}/5, 10) \to X_1 \to K(\mathbb{Z}/5^{\nu_5(h)}, 2)$$

to P_h can be identified with the image under d_{11} of the canonical generator of $H^{10}(K(\mathbb{Z}/5, 10); \mathbb{F}_5)$ in the Serre spectral sequence for this Postnikov approximant, and has to be nonzero, since otherwise the 5–primary torsion in $H_{10}(P_h; \mathbb{Z})$ would be $\mathbb{Z}/5 \oplus H_{10}(K(\mathbb{Z}/5^{\nu_5(h)}, 2)) \cong (\mathbb{Z}/5)^2 \oplus (\mathbb{Z}/5^{\nu_5(h)})$, not $\mathbb{Z}/5^{\nu_5(h)+1}$, so the k–invariant can be seen to be a nonzero multiple of $P^1(\beta_{\nu_5(h)}\iota_2)$, and the Hurewicz map (which corresponds to the image of $H_{10}(K(\mathbb{Z}/5,10))$ under the edge homomorphism) has to vanish.

On the other hand, consider the ANSS. The generators of $\mathrm{BP}_\bullet$ are v_1 in degree $2(p-1)=8$, v_2 in degree $2(p^2-1)=48$, etc. Since these are all in even degree and the homology of P_h is also all in even degree, the AHSS for $\mathrm{BP}_\bullet$ collapses and $\mathrm{BP}_{\mathrm{odd}}(P_h)$ vanishes identically. Since we've already seen that the Hurewicz map $\pi_{10}(P_h)\to H_{10}(P_h)$ vanishes, the image of the Hurewicz map $\pi_{10}(P_h)\to \mathrm{BP}_\bullet(P_h)$ has to map to 0 in $E^\infty_{10,0}\cong H_{10}(P_h)$ and thus has to lie in $E^\infty_{2,8}\cong(\mathbb{Z}/5^{\nu_5(h)})v_1$ (here the indexing of E^∞ corresponds to the AHSS for $\mathrm{BP}_\bullet$). Note that $\mathrm{BP}_\bullet(P_h)$ does not necessarily split as a direct sum of $\mathrm{BP}_\bullet\mathrm{BP}$–comodules corresponding to the summands of E^∞ for the AHSS, but it has a filtration for which this is the associated graded $\mathrm{BP}_\bullet\mathrm{BP}$–comodule.

Note that $\mathrm{BP}_\bullet/(hj)=\mathrm{BP}_\bullet/p^{\nu_p(h)+\nu_p(j)}$, so we get a spectral sequence converging to the E^2–term of the ANSS for which E^1 is a sum of copies of

$$\mathrm{Ext}^{s,t-2j}_{\mathrm{BP}_\bullet\mathrm{BP}}(\mathrm{BP}_\bullet,\mathrm{BP}_\bullet/p^{\nu_p(h)+\nu_p(j)}),\quad 2j\le t.$$

Under the map $\mathrm{BP}_\bullet(P_h)\to K_0(S^3,h)$, 1 and v_1 map to 1 and the other generators v_j for $j\ge 2$ map to 0. So we are particularly interested in what happens in low topological degree ($t-s=10$ for case (4) of the theorem, other values no larger than this for the other cases) and with regard to v_1.

For our case at hand with $p=5$, where $\nu_5(h)=1$ for simplicity, a diagram of the Ext groups may be found in [39, Figure 4.4.16]. In low degrees [39, Theorem 4.4.15], $\mathrm{Ext}^{\bullet,\bullet}_{\mathrm{BP}_\bullet\mathrm{BP}}(\mathrm{BP}_\bullet,\mathrm{BP}_\bullet/p)$ is a polynomial algebra on v_1 (which has bidegree $s=0$, $t=8$) tensored with an exterior algebra on $h_{1,0}$ (which has bidegree $s=1$, $t=8$). We see that not very much can contribute, except for

$$\mathrm{Ext}^0_{\mathrm{BP}_\bullet\mathrm{BP}}(\Sigma^{10}\,\mathrm{BP}_\bullet,\Sigma^2\,\mathrm{BP}_\bullet/5)\cong\mathbb{F}_5v_1,$$

corresponding to $E^\infty_{2,8}$ in the AHSS (which is where we expected the Hurewicz homomorphism to land). This can't be killed by a differential, so the Hurewicz map is nonzero, and since $v_1\mapsto 1$, this maps to an element of order 5 in $K_0(S^3,h)$.

Proof of case (3) The other cases of the theorem are treated in a similar fashion. Let's next deal with the other odd torsion case, (3), with $p=3$ and again degree 10. We'll take $\nu_3(h)=1$ (again for simplicity — when $\nu_3(h)$ is larger, things are similar but the bookkeeping is more complicated). Again, the Hurewicz map $\pi_{10}(P_h)\to H_{10}(P_h;\mathbb{F}_3)$ vanishes since if there were a map $f\colon S^{10}\to P_h$ which were nonzero on homology with

$\mathbb{F}_3$ coefficients, the dual map on cohomology would send the generator $u \in H^2(P_h;\mathbb{F}_3)$ to 0, and thus would have to kill u^5, which is the generator in degree 10. So once again we look at the ANSS to study the Hurewicz map in BP homology. This time, v_1 is in degree $2\cdot(3-1)=4$, v_2 in degree $2\cdot(3^2-1)=16$, etc, so the Hurewicz map in BP homology will have target in a $\mathrm{BP}_\bullet\mathrm{BP}$–subcomodule M of $\mathrm{BP}_\bullet(P_h)$ which is an extension

$$0 \to \Sigma^2\,\mathrm{BP}_\bullet/3 \to M \to \Sigma^6\,\mathrm{BP}_\bullet/9 \to 0,$$

where the subobject comes from $H_2(P_h)\cong\mathbb{Z}/3$ and the quotient comes from $H_6(P_h)\cong\mathbb{Z}/9$. This extension is nontrivial since, in cohomology, P^1 is nonzero from $H^2(P_h;\mathbb{F}_3)$ to $H^6(P_h;\mathbb{F}_3)$. We get a long exact sequence of Ext groups (all over $\mathrm{BP}_\bullet\mathrm{BP}$, which we omit for conciseness)

$$0 \to \mathrm{Ext}^{0,10}(\mathrm{BP}_\bullet, \Sigma^2\,\mathrm{BP}_\bullet/3) \to \mathrm{Ext}^{0,10}(\mathrm{BP}_\bullet, M) \to \mathrm{Ext}^{0,10}(\mathrm{BP}_\bullet, \Sigma^6\,\mathrm{BP}_\bullet/9)$$
$$\to \mathrm{Ext}^{1,10}(\mathrm{BP}_\bullet, \Sigma^2\,\mathrm{BP}_\bullet/3) \to \cdots.$$

Here v_1^2 gives a nonvanishing contribution to $\mathrm{Ext}^{0,10}(\mathrm{BP}_\bullet, M)$ which can't be killed under any differential of the ANSS. The upshot of all of this is that the Hurewicz map in BP–homology is nonzero $\pi_{10}(P_h)\to\mathrm{BP}_{10}(P_h)$, and that, under the map to $K_0(S^3,h)$, this goes to nonzero 3–torsion.

Proof of case (1) Now let's consider cases (1) and (2), which involve the prime $p=2$. First consider case (1), which is relatively easy; we want to compute the Hurewicz map in BP in degree 4 for P_h, h even, using the ANSS. This time the generators are v_1 in degree 2, v_2 in degree 6, etc, and $\mathrm{Ext}^0_{\mathrm{BP}_\bullet\mathrm{BP}}(\Sigma^4\,BP_\bullet, \widetilde{\mathrm{BP}}_\bullet(P_h))$ potentially has contributions from

$$\mathrm{Ext}^{0,2}(\mathrm{BP}_\bullet, \mathrm{BP}_\bullet/2^{\nu_2(h)}) \quad \text{and} \quad \mathrm{Ext}^{0,0}(\mathrm{BP}_\bullet, \mathrm{BP}_\bullet/2^{\nu_2(h)+1}).$$

Since the Hurewicz map vanishes in ordinary homology, the composite $\pi_4(P_h)\to\mathrm{BP}_4(P_h)\to H_4(P_h)$ (the last map being the edge homomorphism of the AHSS) has to vanish, so we are only interested in the first term. Say that $\nu_2(h)=1$; then the picture of $\mathrm{Ext}^{s,t}(\mathrm{BP}_\bullet,\mathrm{BP}_\bullet/2)$ is shown in [39, Figure 4.4.32]. Our candidate for the image of the Hurewicz map is $v_1\in\mathrm{Ext}^{0,2}$; this is a permanent cycle, as one can see from the picture, so the Hurewicz map is nonzero. And v_1 reduces to 1 in $K_0(S^3,h)$.

Proof of case (2) Finally we have the case (2) in topological degree 6. First take $\nu_2(h)=1$; then $H^\bullet(P_h;\mathbb{F}_2)=\mathbb{F}_2[u]\otimes\bigwedge\mathrm{Sq}^1 u$, with the polynomial generator u in degree 2. The ordinary Hurewicz map has to vanish, since there is no nonzero ring homomorphism $H^\bullet(P_h;\mathbb{F}_2) \to H^\bullet(S^6,\mathbb{F}_2)$. So candidates for the BP Hurewicz map have to live in a $\mathrm{BP}_\bullet\mathrm{BP}$–subcomodule M of $\mathrm{BP}_\bullet(P_h)$ which is an extension

$$0 \to \Sigma^2\,\mathrm{BP}_\bullet/2 \to M \to \Sigma^4\,\mathrm{BP}_\bullet/4 \to 0,$$

where the subobject comes from $H_2(P_h)\cong\mathbb{Z}/2$ and the quotient comes from $H_4(P_h)\cong\mathbb{Z}/4$. Once again the contribution of $v_1^2\in\mathrm{Ext}^{0,4}$ to $\mathrm{Ext}^{0,6}(\mathrm{BP}_\bullet,\mathrm{BP}_\bullet(M))$ is a permanent cycle mapping nontrivially to $K_0(S^3,h)$.

An alternative method Before we deal with higher p–primary torsion, we should mention another approach to our theorem using the classical ASS, which is discussed in this context in [2, Part III, Section 16]. To avoid unnecessary repetitions, we go into detail only with $p=2$ and cases (1) and (2) of the theorem. Following Adams's notation, let $\mathcal{B}$ be the subalgebra of the mod-2 Steenrod algebra $\mathcal{A}$ generated by Sq^1 and $Q_1=\mathrm{Sq}^1\,\mathrm{Sq}^2+\mathrm{Sq}^2\,\mathrm{Sq}^1$. This is an exterior algebra on generators of degrees 1 and 3, so it has total dimension 4. By a change-of-rings argument, Adams [2, Part III, Proposition 16.1] proves that the ASS for $\widetilde{ku}_\bullet(X)$ has E_2 term which simplifies to $\mathrm{Ext}^{s,t}_{\mathcal{B}_*}(\mathbb{F}_2,\widetilde{H}_\bullet(X;\mathbb{F}_2))$. We can study the Hurewicz map $\pi^s_\bullet(X)\to ku_\bullet(X)$ by comparing this ASS with the one with E_2 terms $\mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathbb{F}_2,\widetilde{H}_\bullet(X;\mathbb{F}_2))$ converging to $\pi^s_\bullet(X)$. The natural map $\mathrm{Ext}^{s,t}_{\mathcal{A}_*}\to\mathrm{Ext}^{s,t}_{\mathcal{B}_*}$ comes from the forgetful functor from $\mathcal{A}_*$–comodules to $\mathcal{B}_*$–comodules. The advantage of this approach, applied to $X=$ the suspension spectrum of P_h, is that we know $H^\bullet(P_h;\mathbb{F}_2)$ quite explicitly as a module over $\mathcal{A}$ (and in particular over $\mathcal{B}$). Indeed, if h is even, in the Serre spectral sequence for computing $H^\bullet(P_h;\mathbb{F}_2)$ from $\mathbb{CP}^\infty\to P_h\to S^3$, the only differential d_3 vanishes, and so $H^\bullet(P_h;\mathbb{F}_2)=\mathbb{F}_2[u]\otimes\bigwedge(y)$, where u is in degree 2 and y is in degree 3. Since $H^2(P_h;\mathbb{Z})=0$ and $H^3(P_h;\mathbb{Z})\cong\mathbb{Z}/h$, if $\nu_2(h)=1$, then $\mathrm{Sq}^1\,u=y$, whereas if $\nu_2(h)>1$, then $\mathrm{Sq}^1\,u=0$ and $\beta_{\nu_2(h)}(u)=y$. In both cases we have $\mathrm{Sq}^2\,u=u^2$, $\mathrm{Sq}^j\,y=0$ for $j\ge1$. (The last identity follows from the fact that y is pulled back from $H^3(S^3;\mathbb{F}_2)$, on which $\mathcal{A}$ acts trivially.) The rest of the action of the Steenrod algebra can be determined from the Cartan relations. For the sake of definiteness, let's take $\nu_2(h)=1$. Note that the inclusion $\mathbb{CP}^\infty\hookrightarrow P_h$ induces an isomorphism of $\mathbb{F}_2[u]\subset H^\bullet(P_h;\mathbb{F}_2)$ onto $H^\bullet(\mathbb{CP}^\infty;\mathbb{F}_2)$. So, if $\mathcal{A}^{\mathrm{even}}$ is the subalgebra of $\mathcal{A}$ generated by the Sq^{2^j} for $j\ge1$, and $a\in\mathcal{A}^{\mathrm{even}}$, then $a(u)$ must be a linear combination of primitive elements of $\mathbb{F}_2[u]$, ie of the elements u^{2^j} for $j\ge0$, by the same argument as found in [1, pages 19–21].

Most of the work in computing the Ext and stable homotopy groups was done by Liulevicius [30] and Mosher [37]. Let M be the left $\mathcal{A}$–module $\widetilde{H}^\bullet(\mathbb{CP}^\infty;\mathbb{F}_2)$, and let N be the left $\mathcal{A}$–module $\widetilde{H}^\bullet(Y;\mathbb{F}_2)$, where Y is the result of attaching a 3–cell to $\mathbb{CP}^\infty$ via a map $S^2 \xrightarrow{h} \mathbb{CP}^1 \subset \mathbb{CP}^\infty$ of degree h. Y can be identified with a subcomplex of P_h and the cofiber of the inclusion $Y \to P_h$ can be identified with $\Sigma^3\,\mathbb{CP}^\infty$. So we have exact sequences of $\mathcal{A}$–modules

(3a) $$0 \to \Sigma^3\,\mathbb{F}_2 \to N \to M \to 0,$$

(3b) $$0 \to \Sigma^3\,M \to \widetilde{H}^\bullet(P_h;\mathbb{F}_2) \to N \to 0.$$

These extensions are nontrivial since we have the relations $\mathrm{Sq}^1(u^{2j+1}) = u^{2j}y$, and so there are classes $v \in \mathrm{Ext}^{1,3}_{\mathcal{A}}(M,\mathbb{F}_2)$, $w \in \mathrm{Ext}^{1,3}_{\mathcal{A}}(N,M)$, associated to (3a) and (3b), respectively.

In low dimensions, $\mathrm{Ext}^{s,t}_{\mathcal{A}}(M,\mathbb{F}_2)$ was computed in [30], and there is only one Adams differential in this range. There is a unique nonzero element in $\mathrm{Ext}^{1,3}_{\mathcal{A}}(M,\mathbb{F}_2)$, so that is v, and the connecting map in the long exact sequence coming from (3a) is Yoneda product with v by [39, Theorem 2.3.4]. From knowledge of $\mathrm{Ext}^{s,t}_{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2)$ [39, Theorem 3.2.11] and of $\mathrm{Ext}^{s,t}_{\mathcal{A}}(M,\mathbb{F}_2)$ [30, Proposition II.3] in low dimensions along with the long exact sequence, we get the diagram of the long exact sequence for $\mathrm{Ext}^{s,t}_{\mathcal{A}}(N,\mathbb{F}_2)$ shown in Figure 5. Here $\mathrm{Ext}^{s,t}_{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2)$ is depicted at the left, $\mathrm{Ext}^{s,t}_{\mathcal{A}}(M,\mathbb{F}_2)$ at the right, and red dots indicate elements paired under the connecting map (ie under product with v).

From this picture we can read off the stable homotopy groups of Y, since there are no Adams differentials in the dimension range we're interested in. So for example $\pi^s_4(Y) \cong \mathbb{Z} \oplus \mathbb{Z}/2$, with the $\mathbb{Z}$ coming from $\mathbb{CP}^\infty$ (the $t-s=4$ column on the right in Figure 5) and the $\mathbb{Z}/2$ coming from S^3 (the dot at $t-s=1$, $s=1$ on the left in Figure 5 — remember that we shift up in dimension by 3). Similarly, $\pi^s_6(Y) \cong \mathbb{Z}\oplus\mathbb{Z}/4$, with the $\mathbb{Z}$ coming from $\mathbb{CP}^\infty$ (the $t-s=6$ column on the right in Figure 5) and the $\mathbb{Z}/4$ coming from S^3 (the dots at $t-s=3$, $s=2$ and 3 on the left in Figure 5).

To compute $\mathrm{Ext}^{s,t}_{\mathcal{A}}(\widetilde{H}^\bullet(P_h;\mathbb{F}_2),\mathbb{F}_2)$, we need one more exact sequence coming from (3b). Since all the reduced homology of P_h is torsion, so is $\pi^s_\bullet(P_h)$, and the connecting map $\mathrm{Ext}^{s,t}_{\mathcal{A}}(M,\mathbb{F}_2) \to \mathrm{Ext}^{s+1,t+3}_{\mathcal{A}}(N,\mathbb{F}_2)$ kills off the $\mathbb{Z}$ summands. The Hurewicz maps we are interested in come from the composites $\pi_j(S^3) \to \pi^s_j(Y) \to \pi^s_j(P_h)$ with $j=4$ and $j=6$, so they come from the $\mathbb{Z}/2$ in $\pi^s_4(Y)$ and the $\mathbb{Z}/4$ in $\pi^s_6(Y)$ coming from the dots on the left in Figure 5 in bidegrees $(s=1, t=2)$, resp., $(s=2, t=5)$ and $(s=3, t=6)$. In both the cases $j=4$ and 6, the torsion summand in $\pi^s_j(Y)$ cannot be killed by $\pi^s_{j+1}(\Sigma^3\,\mathbb{CP}^\infty)$.

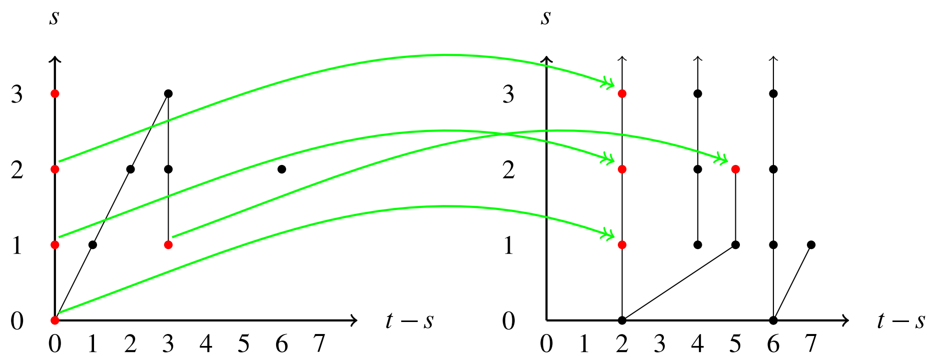

Figure 5: The groups $\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ (left) and $\mathrm{Ext}_{\mathcal{A}}^{s,t}(M, \mathbb{F}_2)$ (right, following Liulevicius) in low dimensions. Red dots indicate elements which cancel under the connecting map (green arrows) in the long exact sequence for $\mathrm{Ext}_{\mathcal{A}}^{s,t}(N, \mathbb{F}_2)$ (for $\nu_2(h) = 1$).

Next let's compute the $\mathcal{B}$–module structure on $H^\bullet(X)$, needed for the right side of Figure 6. When $\nu_2(h) > 1$, Sq^1 vanishes identically on $H^\bullet(P_h; \mathbb{F}_2)$, and so the $\mathcal{B}$–module structure is trivial and

$$\mathrm{Ext}_{\mathcal{B}_*}^{s,t}(\mathbb{F}_2, \widetilde{H}_\bullet(X; \mathbb{F}_2)) \cong \widetilde{H}_t(P_h; \mathbb{F}_2) \otimes \mathrm{Ext}_{\mathcal{B}_*}^{s,*}(\mathbb{F}_2, \mathbb{F}_2).$$

When $\nu_2(h) = 1$, then $\mathrm{Sq}^1(u^j) = j u^{j-1} y$ and $\mathrm{Sq}^1(u^j y) = 0$, while by an induction using the Cartan formula, we have $\mathrm{Sq}^2(u^j) = u^{j+1}$ for j odd, 0 for j even, and $\mathrm{Sq}^2(u^j y) = u^{j+1} y$ for j odd, 0 for j even. Thus

$$Q_1(u^j) = (\mathrm{Sq}^1 \mathrm{Sq}^2 + \mathrm{Sq}^2 \mathrm{Sq}^1)(u^j) = \mathrm{Sq}^1(j u^{j+1}) + \mathrm{Sq}^2(j u^{j-1} y) = 0$$

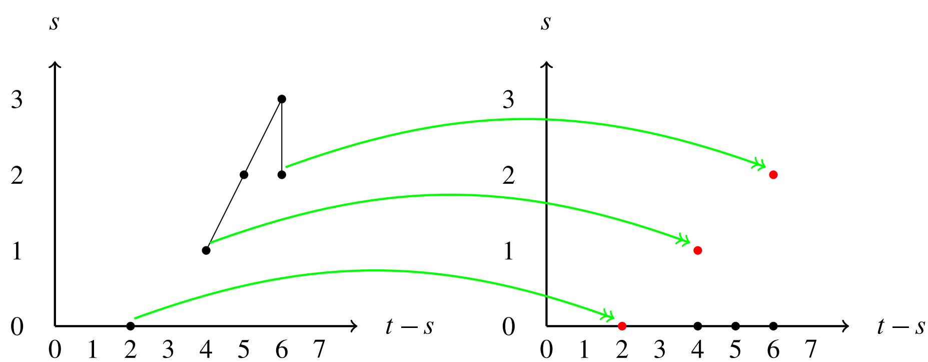

Figure 6: Comparing the 2–local Adams spectral sequences for computing $\pi_\bullet^s(P_h)$ and $ku_\bullet(P_h)$ for $h = 2k$, k odd. Red dots indicate the contribution from $\mathrm{Ext}_{\mathcal{C}_*}^{s,2+t}(\mathbb{F}_2, \mathbb{F}_2)$. Some other contributions on the right are omitted.

in all cases, and similarly $Q_1(u^j y) = 0$ in all cases. Let $\mathcal{C} = \bigwedge(Q_1)$ be the subalgebra of $\mathcal{B}$ generated by Q_1. Then we've seen that for j odd, u^j and $u^{j-1}y$ span a $\mathcal{B}$–module M_j on which Sq^1 acts cyclically and $\mathcal{C}$ acts trivially. So this module is $\mathcal{B}\otimes_{\mathcal{C}}\mathbb{F}_2$ and again by change of rings, $\mathrm{Ext}^{s,2j+t}_{\mathcal{B}_*}(\mathbb{F}_2, M_j) \cong \mathrm{Ext}^{s,2j+t}_{\mathcal{C}_*}(\mathbb{F}_2,\mathbb{F}_2)$. However, when j is even, u^j and $u^{j-1}y$ each span a trivial one-dimensional $\mathcal{B}$–module. Thus, for $\nu_2(h) = 1$,

$$\mathrm{Ext}^{s,t}_{\mathcal{B}_*}(\mathbb{F}_2, \widetilde{H}_\bullet(P_h;\mathbb{F}_2)) \cong \bigoplus_{j \text{ odd}} \mathrm{Ext}^{s,2j+t}_{\mathcal{C}_*}(\mathbb{F}_2,\mathbb{F}_2) \oplus \bigoplus_{j \text{ even}} \mathrm{Ext}^{s,2j+t}_{\mathcal{B}_*}(\mathbb{F}_2,\mathbb{F}_2) \oplus \bigoplus_{j \text{ even}} \mathrm{Ext}^{s,2j+1+t}_{\mathcal{B}_*}(\mathbb{F}_2,\mathbb{F}_2).$$

Note that a simple calculation gives $\mathrm{Ext}^{s,t}_{\mathcal{C}_*}(\mathbb{F}_2,\mathbb{F}_2) \cong \mathbb{F}_2$ for all $s \ge 0$ and $t = 3s$ (0 for other values of t) and $\mathrm{Ext}^{s,t}_{\mathcal{B}_*}(\mathbb{F}_2,\mathbb{F}_2)$ is a sum of copies of $\mathbb{F}_2$, one for each $s_1, s_2 \ge 0$ and $s = s_1 + s_2$, $t = 3s_1 + s_2$ (the formulas for t come from the fact that Sq^1 raises topological degree by 1 and Q_1 raises topological degree by 3). Increasing s_2 corresponds to multiplying by $h_0 \in \mathrm{Ext}^{1,1}$. (This is also all in [39, Theorem 3.1.16].) Thus, in case (1) with $\nu_2(h) = 1$, we get in the E_2 of the ASS for $\widetilde{ku}_\bullet(P_h)$ copies of $\mathbb{F}_2$ in bidegrees

$$(s,t) = (s, 2+3s), (s_1+s_2, 4+3s_1+s_2), (s_1+s_2, 5+3s_1+s_2), \text{ etc.}$$

These are shown on the right side of Figure 6. Note that the terms coming from homology in degrees $2j$ and $2j+1$ correspond to the image of $\mathbb{Z}\beta_j \subset K_0(P_h)$ (in Khorami's notation in [29]). Since β_j maps to 0 in $K_0(S^3, h)$ for $j \ge 2$, we are really only interested in the terms with $j = 1$, which are indicated by red dots in Figure 6. The nontriviality of the green arrows in Figure 6 (which is easy to check purely algebraically) immediately gives another proof of cases (1) and (2) when $\nu_2(h) = 1$.

Proof of cases with $\nu_2(h) > 1$ Finally, we consider cases (1) and (2) when $\nu_2(h) > 1$, say for definiteness $\nu_2(h) = 2$. Then the $\mathcal{A}$–module extensions in (3a)–(3b) now split, and Figure 6 is modified as follows. On the left-hand side, since $\pi_2(X) = \mathbb{Z}/4$ (after localizing at 2), the columns with $t - s = 2, 3$ are modified as in [39, Example 2.1.19], with the addition of a differential. This will not matter for us since we only care about topological degrees 4 and up. The other change is that $H_2(P_h) \cong \mathbb{Z}/4$, and the $\mathcal{B}$–submodule of $H^\bullet(X;\mathbb{F}_2)$ generated by u and y is now trivial. That changes the picture on the right as shown in Figure 7. The differentials on the right are determined by the facts that $H_2(P_h) = \mathbb{Z}/h$ and that the AHSS for *ku* collapses at E^2. This picture proves the remaining cases of the theorem with $p = 2$. Cases (3) and (4), with

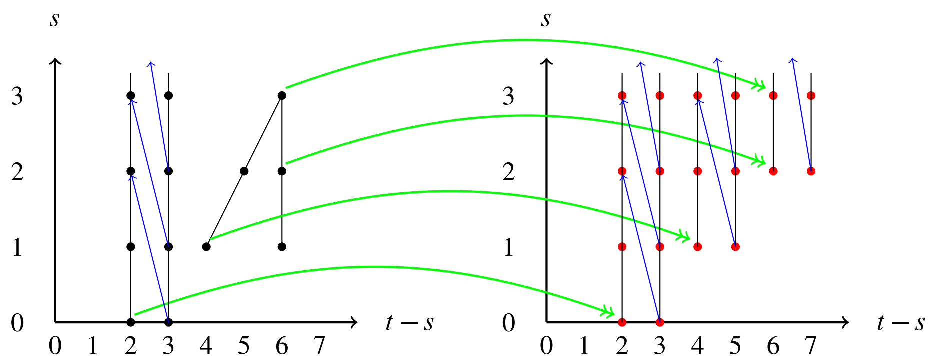

Figure 7: Comparing the 2–local Adams spectral sequences for computing $\pi^s_\bullet(P_h)$ and $ku_\bullet(P_h)$ for $h = 4k$, k odd. Red dots indicate the contribution from $\mathrm{Ext}^{s,2+t}_{\mathcal{B}_*}(\mathbb{F}_2, \mathbb{F}_2) \oplus \mathrm{Ext}^{s,3+t}_{\mathcal{B}_*}(\mathbb{F}_2, \mathbb{F}_2)$. Other contributions on the right are omitted. Blue arrows show d_2.

$p = 3$ or 5, can also be handled by the same methods as cases (1) and (2). The picture analogous to Figure 6 for $p = 3$, $\nu_3(h) = 1$, and case (3) appears as Figure 8. (At an odd prime p, $\mathcal{C}$ becomes the exterior algebra on Q_1, which is of degree $2p-1$.) □

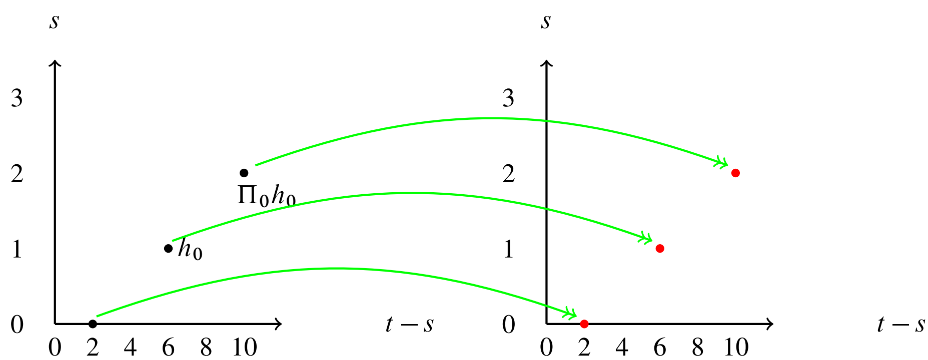

Figure 8: Comparing the 3–local Adams spectral sequences for computing $\pi^s_\bullet(P_h)$ and $ku_\bullet(P_h)$ for $h = 3k$, $\gcd(3, k) = 1$. Red dots indicate the contribution from $\mathrm{Ext}^{s,2+t}_{\mathcal{C}_*}(\mathbb{F}_3, \mathbb{F}_3)$. Other contributions on the right are omitted.

4 The nonsimply connected cases

Similar techniques can also be used to compute twisted K–theory for the nonsimply connected simple rank-2 groups. There are two of these: PSU(3), with fundamental group $\mathbb{Z}/3$, and $\mathrm{PSp}(2) \cong \mathrm{SO}(5)$, with fundamental group $\mathbb{Z}/2$. The case of $\mathrm{PSU}(3)$ was studied in [32, Theorem 19 and Remark 20], so we consider here the case of $\mathrm{PSp}(2)$. Note first of all that the covering map $\mathrm{Sp}(2) \xrightarrow{\pi} \mathrm{PSp}(2)$ induces an isomorphism on H^3 by [32, Theorem 1], and that $\mathrm{PSp}(2)$ fits into a fibration

$$S^3 = \mathrm{Sp}(1) \to \mathrm{PSp}(2) \to \mathbb{RP}^7, \tag{4}$$

which replaces the fibration $\mathrm{Sp}(1) \to \mathrm{Sp}(2) \to S^7$ used in the proof of Theorem 12. We have transfer and pushforward maps

$$\pi^*\colon K_\bullet(\mathrm{PSp}(2), h) \to K_\bullet(\mathrm{Sp}(2), h), \quad \pi_*\colon K_\bullet(\mathrm{Sp}(2), h) \to K_\bullet(\mathrm{PSp}(2), h),$$

and $\pi_* \circ \pi^*$ is multiplication by 2. Since $K_\bullet(\mathrm{Sp}(2), h)$ is cyclic in both even and odd degree, this implies that, when we localize at an odd prime p, $K_\bullet(\mathrm{PSp}(2), h)_{(p)} \cong K_\bullet(\mathrm{Sp}(2), h)_{(p)}$. If $p = 3$, this is a cyclic group of order $3^{\max(0, \nu_3(h)-1)}$, and if $p \geq 5$, this is a cyclic group of order $p^{\nu_p(h)}$. The only issue is therefore what happens with 2–primary torsion. Recall from Theorem 12 that $K_\bullet(\mathrm{Sp}(2), h)_{(2)}$ is a cyclic group of order $2^{\max(0, \nu_2(h)-1)}$. We have, by Theorem 3, from (4) a Segal spectral sequence

$$H_p(\mathbb{RP}^7, K_q(S^3, h)) \Rightarrow K_\bullet(\mathrm{PSp}(2), h). \tag{5}$$

If h is odd, this gives 0 after localizing at 2. So assume that $h = 2k$ with k odd. After localizing at 2, the left side of (5) becomes $H_p(\mathbb{RP}^7, \mathbb{F}_2)$ for q even and 0 for q odd. The transfer argument shows that multiplication by 2 on $K_\bullet(\mathrm{PSp}(2), h)_{(2)}$ factors through $K_\bullet(\mathrm{Sp}(2), h)_{(2)} = 0$, so all 2–primary torsion is of order 2.

Now, if h is even, it is 0 mod 2, so we have natural maps

$$K_0(S^3, h) \xrightarrow{\text{reduce mod } 2} K_0(S^3, h; \mathbb{F}_2) \cong K_0(S^3; \mathbb{F}_2),$$
$$K_\bullet(\mathrm{PSp}(2), h) \xrightarrow{\text{reduce mod } 2} K_\bullet(\mathrm{PSp}(2), h; \mathbb{F}_2) \cong K_\bullet(\mathrm{PSp}(2); \mathbb{F}_2),$$

the first of which is an isomorphism. So we get a map of spectral sequences

$$\begin{array}{ccc} H_p(\mathbb{RP}^7, K_q(S^3, h))_{(2)} & \Longrightarrow & K_\bullet(\mathrm{PSp}(2), h)_{(2)} \\ \big\downarrow & & \big\downarrow \\ H_p(\mathbb{RP}^7, K_q(S^3; \mathbb{F}_2)) & \Longrightarrow & K_\bullet(\mathrm{PSp}(2); \mathbb{F}_2) \end{array} \tag{6}$$

The K–theory of $\mathrm{PSp}(2) \cong \mathrm{SO}(5)$ was computed in [25, Satz 5.15]; as an abelian group it is $\mathbb{Z}^2 \oplus \mathbb{Z}/4$ in both even and odd degree. Hence, in the Segal spectral sequence $H_p(\mathbb{RP}^7, K_q(S^3)) \Rightarrow K_\bullet(\mathrm{PSp}(2))$, which has a $\mathbb{Z}/2$ in E^2 in bidegrees

$(2j-1,k)$ for $j=1,2,3$, there is room for only one differential. In fact, from the description of the torsion in $K^\bullet$ in [25], the torsion in K^0 is generated by the pullback of the generator of $\widetilde{K}^0(\mathbb{RP}^7)$, and the generator of the torsion in K^1 is generated by the product of this class with an odd generator λ_1 of a torsion-free exterior algebra, which is precisely the canonical representation $\mathrm{PSp}(2)\cong\mathrm{SO}(5)\to U(5)$ viewed as a class in K^1. So this determines the differentials in the Segal spectral sequence in K–cohomology; there must be differentials killing off $H^6(\mathbb{RP}^7,K^0(S^3))$ and $H^6(\mathbb{RP}^7,K^1(S^3))$. From the universal coefficient theorem, $K_\bullet(\mathrm{PSp}(2);\mathbb{F}_2)\cong\mathbb{F}_2^4$ in both even and odd degree. (We get a group of rank 3 from reducing the integral K–homology mod 2, and pick up another $\mathbb{F}_2$ from the Tor term.) If we compare the bottom spectral sequence in (6) with the one for integral K–homology and with the one for twisted K–homology of SO(4) (in which there are no differentials at all), we see that the only nonzero differentials are $d^2\colon E^2_{p+2,q}\to E^2_{p,q+1}$ with $p=4$ or 5. Now go back to the commuting diagram (6). There cannot be a nonzero differential in the upper spectral sequence, since it would imply existence of a forbidden differential in the lower sequence. So the spectral sequence for $K_\bullet(\mathrm{PSp}(2),h)_{(2)}$ collapses, and since all torsion is of order 2, we conclude that the 2–primary torsion in $K_\bullet(\mathrm{PSp}(2),h)_{(2)}$ is $(\mathbb{Z}/2)^4$ in both even and odd degree. Putting everything together, we see that we have proved the following:

Theorem 17 *Suppose that h is either odd or $2 \bmod 4$. Then $K_\bullet(\mathrm{PSp}(2),h)$ is finite, and is the same in both even and odd degree. The odd torsion in $K_\bullet(\mathrm{PSp}(2),h)$ is cyclic of order $h_{\mathrm{odd}}/\gcd(h,3)$. The 2–primary torsion vanishes if h is odd, and if h is $2 \bmod 4$, it is $(\mathbb{Z}/2)^4$ in each degree.*

Cases where h is divisible by a higher power of 2 can be handled similarly, though the results are more complicated.

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