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Incidence bicomodules, Möbius inversion and a Rota formula for infinity adjunctions

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In the same way decomposition spaces, also known as unital 2–Segal spaces, have incidence (co)algebras, and certain relative decomposition spaces have incidence (co)modules, we identify the structures that have incidence bi(co)modules: they are certain augmented double Segal spaces subject to some exactness conditions. We establish a Möbius inversion principle for (co)modules and a Rota formula for certain more involved structures called Möbius bicomodule configurations. The most important instance of the latter notion arises as mapping cylinders of infinity adjunctions, or more generally of adjunctions between Möbius decomposition spaces, in the spirit of Rota's original formula.

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Introduction

The theory of Möbius categories, developed by Leroux [21] generalises the theory for locally finite posets — see Rota [25] — and Cartier–Foata finite-decomposition monoids [8], which admit incidence (co)algebras and a Möbius inversion principle. It has recently been generalised to ∞–categories and decomposition spaces by Gálvez-Carrillo, Kock and Tonks [13; 14]. (Decomposition spaces are the same thing as the unital 2–Segal spaces of Dyckerhoff and Kapranov [10].)

An important tool for computing the Möbius function in the incidence algebra of a locally finite poset is the classical formula of Rota [25, Theorem 1], which compares the Möbius functions across a Galois connection. Rota's original work already gave many applications of this formula, and it also features prominently in standard textbooks such as Aigner [3] and Stanley [26]. The idea that one should not just work with posets individually, but rather exploit relationships between posets is of course a very modern idea, appealing to any mathematician with a categorical bias.

The original goal of the present work, thought to be a routine exercise, was to generalise this formula to ∞–adjunctions. It turned out a lot of machinery was required to do

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this in a satisfactory way, and developing this machinery ended up as a substantial contribution, warranting the present paper: they are general constructions in the theory of decomposition spaces/2–Segal spaces, concerning bicomodules, which are of interest not just in combinatorics but also in representation theory, in connection with Hall algebras.

Before stating and proving this formula for categories, let us recall some definitions.

Incidence coalgebras

Given a small category X, write X_0 for its set of objects and X_1 for its set of arrows. Let $\mathbb{Q}_{X_1}$ be the free vector space on X_1. We say a category X is *locally finite* if each morphism $f\colon x \to z$ in X admits only finitely many two-step factorisations $x \xrightarrow{g} y \xrightarrow{h} z$. This condition guarantees that the comultiplication on $\mathbb{Q}_{X_1}$, given by

$$\Delta\colon \mathbb{Q}_{X_1} \to \mathbb{Q}_{X_1} \otimes \mathbb{Q}_{X_1}, \quad f \mapsto \sum_{hg=f} g \otimes h,$$

is well defined. The counit $\delta\colon \mathbb{Q}_{X_1} \to \mathbb{Q}$ is given by $\delta(\mathrm{id}_x) = 1$, and $\delta(f) = 0$ else.

The *incidence algebra* $\mathcal{I}_X$ is the linear dual, $(\mathrm{Lin}(\mathbb{Q}_{X_1}, \mathbb{Q}), *, \delta)$ with the convolution product

$$(\alpha * \beta)(f) = \sum_{hg=f} \alpha(g)\beta(h),$$

where $\alpha, \beta \in \mathcal{I}_X$ and $f \in \mathbb{Q}_{X_1}$.

The *zeta function* $\zeta_X\colon \mathbb{Q}_{X_1} \to \mathbb{Q}$ is defined by $\zeta_X(f) = 1$ for all $f \in \mathbb{Q}_{X_1}$.

Define $\Phi_{\text{even}}\colon \mathbb{Q}_{X_1} \to \mathbb{Q}$ to be the number of even-length factorisations of a morphism, without identities, and $\Phi_{\text{odd}}\colon \mathbb{Q}_{X_1} \to \mathbb{Q}$ to be the number of odd-length factorisations, without identities. A category is *Möbius* [21] if it is locally finite and Φ_{even} and Φ_{odd} are finite.

Theorem (Content, Lemay and Leroux [9]) *If X is a Möbius category then the zeta function is invertible, and the inverse, called the Möbius function, is given by $\mu = \Phi_{\text{even}} - \Phi_{\text{odd}}$.*

Examples of Möbius categories are locally finite posets and monoids with the finite-decompositions property, and this theorem generalises similar theorems for these more specialised settings.

Rota formula for categories

A classical formula due to Rota [25] compares the Möbius functions of two posets related by a *Galois connection*. The following generalisation of Rota's formula to Möbius categories is both natural and straightforward (but seems not to have been made before):

Theorem (Rota formula for Möbius categories) *Let X and Y be Möbius categories and let $F\colon X \rightleftarrows Y :G$ be an adjunction, $F \dashv G$. Then, for all $x \in X$ and $y \in Y$,*

$$\sum_{\substack{x' \in X \\ f: x \to x' \\ Fx' = y}} \mu_X(f) = \sum_{\substack{y' \in Y \\ g: y' \to y \\ Gy' = x}} \mu_Y(g).$$

The reader is not expected to read the following elementary proof, but only notice that it looks like an associativity formula for a convolution product, except that the arrows live in different categories.

Proof of the Rota formula

$$\sum_{\substack{x' \in X \\ f: x \to x' \\ Fx' = y}} \mu_X(x \xrightarrow{f} x')$$

$$\stackrel{(1)}{=} \sum_{\substack{x' \in X \\ f: x \to x' \\ h: Fx' \to y}} \mu_X(x \xrightarrow{f} x') \delta_Y(Fx' \xrightarrow{h} y)$$

$$\stackrel{(2)}{=} \sum_{\substack{x' \in X \\ f: x \to x' \\ h: Fx' \to y}} \mu_X(x \xrightarrow{f} x') \Bigg(\sum_{\substack{y' \in Y \\ g: y' \to y \\ h': Fx' \to y' \\ h = gh'}} \zeta_Y(Fx' \xrightarrow{h'} y') \mu_Y(y' \xrightarrow{g} y) \Bigg)$$

$$\stackrel{(3)}{=} \sum_{\substack{x' \in X, y' \in Y \\ f: x \to x' \\ h: Fx' \to y \\ g: y' \to y \\ h': Fx' \to y' \\ h = gh'}} \mu_X(x \xrightarrow{f} x') \zeta_Y(Fx' \xrightarrow{h'} y') \mu_Y(y' \xrightarrow{g} y)$$

$$\overset{(4)}{=} \sum_{\substack{x'\in X, y'\in Y\\ g\colon y'\to y\\ k\colon x\to Gy'\\ f\colon x\to x'\\ k'\colon x'\to Gy'\\ k=k'f}} \mu_X(x \xrightarrow{f} x')\zeta_X(x' \xrightarrow{k'} Gy')\mu_Y(y' \xrightarrow{g} y) \qquad \text{(by adjunction)}$$

$$\overset{(5)}{=} \sum_{\substack{y'\in Y\\ g\colon y'\to y\\ k\colon x\to Gy'}} \Bigg(\sum_{\substack{x'\in X\\ f\colon x\to x'\\ k'\colon x'\to Gy'\\ k=k'f}} \mu_X(x \xrightarrow{f} x')\zeta_X(x' \xrightarrow{k'} Gy')\Bigg)\mu_Y(y' \xrightarrow{g} y)$$

$$\overset{(6)}{=} \sum_{\substack{y'\in Y\\ g\colon y'\to y\\ k\colon x\to Gy'}} \delta_X(x \xrightarrow{k} Gy')\mu_Y(y' \xrightarrow{g} y)$$

$$\overset{(7)}{=} \sum_{\substack{y'\in Y\\ g\colon y'\to y\\ Gy'=x}} \mu_Y(y' \xrightarrow{g} y). \qquad \square$$

In the main result of the present paper, Theorem 4.5.2, we write this formula as

$$\mu_X \star_l \delta_Y = \delta_X \star_r \mu_Y,$$

with the more conceptual proof

$$\mu_X \star_l \delta_Y \overset{(2)}{=} \mu_X \star_l (\zeta \star_r \mu_Y) \overset{(3\text{–}5)}{=} (\mu_X \star_l \zeta) \star_r \mu_Y \overset{(6)}{=} \delta_X \star_r \mu_Y,$$

referring to certain left and right convolution actions. The important insight here, which is due to Aguiar and Ferrer [2], is that the "mixed arrows" which appear in the middle factors (those of the form $Fx \to y$, which in the crucial step of the proof are reinterpreted as $x \to Gy$ by adjunction) belong to a *bimodule*: they are acted upon from the left by arrows in the category X and from the right by arrows in Y (see Example 3.2.3 for an explicit description). The long complicated sums in the proof are thus condensed into convolution actions from the left and right, denoted by $\star_l$ and $\star_r$. Aguiar and Ferrer [2] established a bimodule proof of the Rota formula in the setting of posets.

Two ingredients are necessary to make sense of the pleasing convolution proof above: one is to exhibit the data necessary to induce bicomodules and establish that adjunctions constitute an example. This already accounts for equalities (3–5) in the proof. The other is to establish a Möbius inversion principle for (co)modules (a notion which has not previously been considered in the literature, to the knowledge of the author), to account for the equalities (2) and (6).

In fact, as the notions and arguments become increasingly abstract and conceptual, it is natural to ask for further generalisation. In this work we take three considerable abstraction steps (beyond passing from posets to categories, which is already a fruitful step). First, we pass from categories and adjunctions to ∞–categories and ∞–adjunctions. Any ∞–adjunction defines a bicomodule in our sense. This step in itself is not so easy to justify from the viewpoint of combinatorics, but the homotopy content inherent in ∞–categories is important since already classical combinatorial structures have symmetries, and these can be handled more conveniently with groupoids than with sets, as advocated by Baez and Dolan [5], Gálvez-Carrillo, Kock and Tonks [11] and others. (This aspect will not be of importance in the present contribution, though.) Second, we pass from ∞–categories to decomposition spaces (also called 2–Segal spaces) and introduce a notion of adjunction for them. This step has an important combinatorial motivation, because many combinatorial coalgebras admit a natural realisation as incidence coalgebras of decomposition spaces which are not posets or categories. An important example is the coalgebra of all finite posets, which will serve as a running example in this paper. The final abstraction step consists in noticing that the abstract Rota formula works equally well for certain bicomodules which do *not* come from adjunctions or ∞–adjunctions. In fact the running example chosen to illustrate the theory is of this type: it is a certain bicomodule interpolating between the decomposition space of finite sets and the decomposition space of finite posets. The outcome is the formula $\mu(P) = (-1)^n$ for the Möbius function of a poset with n elements. This formula is well known (see for example Aguiar, Bergeron and Sottile [1]) but its derivation via a Rota formula is new and interesting, since the coalgebra of finite posets is not the incidence coalgebra of a locally finite poset or Möbius category.

Finally, a word should be said about the objective approach, an important aspect of the decomposition-space viewpoint on incidence algebras and Möbius inversion. The point here is to lift combinatorial identities to bijections of sets, and more generally equivalences of ∞–groupoids. Specifically, the Möbius inversion formula, which classically is an equation in vector spaces, is lifted to an equivalence between ∞–groupoids defining spans, whose homotopy cardinality are the linear maps in question (the introductions of [13; 11] contain further motivation). The present work inscribes itself in this tradition, seeking to find objective structures for bicomodules and the Rota formula. However, it must be admitted that the objective level is not fully achieved in this paper. On one hand, the Möbius formula obtained for comodules is not directly realised as the homotopy cardinality of an equivalence: it has been found necessary here to take homotopy cardinality a little bit earlier in the constructions, so that the final arguments take place at the vector-space level. This is due to the increased complexity compared to the plain Möbius inversion formula of [14], where an even–odd splitting could be found for the single decomposition space involved. In the present situation, *two* decomposition spaces are involved, and the even–odd splitting at the objective level could not be found. Furthermore, the objective analogue of bicomodules given here is not fully satisfactory from the homotopy viewpoint. While it is shown to induce bicomodules up to homotopy, the *coherence* of this up-to-homotopy structure is not established in this work, and would seem to require considerable further efforts, in the line of coherence proofs given by Gálvez-Carrillo, Kock and Tonks [13] and Penney [24]. Further discussion is included in the main text. The justification for not establishing coherence in the present contribution is that it is not necessary for the sake of taking cardinality, as required anyway in the final constructions for the Rota formula established.

Although the motivation and the statement of the theorem belongs to combinatorics, the setting for this work and the tools employed are from simplicial homotopy theory, in the style of [13; 14], working with ∞–groupoids, homotopy pullbacks, mapping spaces, fibrations and fibre sequences. One technical novelty compared to [13; 14] is that we exploit general simplicial maps between decomposition spaces, not just culf ones, and introduce the notion of adjunction between decomposition spaces. Another is that the notion of mapping cylinder is exploited systematically: on one hand locally to model the shapes needed to index the various configurations, and on the other hand globally, as infinity mapping cylinders.

Outline of the paper

We begin in Section 1 with a brief review of needed notions from the theory of ∞–categories, with an emphasis on decomposition spaces.

In Section 2, following Walde [27] and Young [28], we first explain how to obtain a comodule in the context of decomposition spaces.

Proposition 2.1.1 *If $f\colon C \to X$ is a culf map between two simplicial ∞–groupoids such that C is Segal and X is a decomposition space, then the span*

$$C_0 \xleftarrow{d_1} C_1 \xrightarrow{(f_1,d_0)} X_1 \times C_0$$

induces on the slice ∞–category $\mathcal{S}_{/C_0}$ the structure of a left $\mathcal{S}_{/X_1}$–comodule (at the π_0 level), and the span

$$C_0 \xleftarrow{d_0} C_1 \xrightarrow{(d_1,f_1)} C_0 \times X_1$$

induces on $\mathcal{S}_{/C_0}$ the structure of a right $\mathcal{S}_{/X_1}$–comodule (at the π_0 level).

The data needed to obtain a comodule is called a *comodule configuration*. In order to obtain a bicomodule structure, we first need an augmented bisimplicial ∞–groupoid Segal in each direction. We furthermore require this bisimplicial ∞–groupoid to be stable; see Section 2.3. This stability condition is a pullback condition on certain squares, and is a ∞–categorical reformulation of the notion of Bergner, Osorno, Ozornova, Rovelli and Scheimbauer [6], suitable for ∞–groupoids.

Theorem 2.4.1 *Let B be an augmented stable double Segal space, and such that the augmentation maps are culf. Suppose moreover $X := B_{\bullet,-1}$ and $Y := B_{-1,\bullet}$ are decomposition spaces. Then the spans*

$$B_{0,0} \xleftarrow{e_1} B_{1,0} \xrightarrow{(u,e_0)} X_1 \times B_{0,0}$$

and

$$B_{0,0} \xleftarrow{d_0} B_{0,1} \xrightarrow{(d_1,v)} B_{0,0} \times Y_1$$

induce on $\mathcal{S}_{/B_{0,0}}$ the structure of a bicomodule over $\mathcal{S}_{/X_1}$ and $\mathcal{S}_{/Y_1}$ (at the π_0 level).

An augmented bisimplicial ∞–groupoid satisfying the conditions of the theorem is called a *bicomodule configuration*. We describe as an example the bicomodule configuration of layered sets and posets, treated in more detail in Carlier [7].

In Section 3, we introduce the notion of correspondence of decomposition spaces: it is a decomposition space $\mathcal{M}$ with a map $\mathcal{M} \to \Delta^1$. We show that any correspondence of decomposition spaces gives rise to a bicomodule configuration. We then introduce the notion of *cartesian* and *cocartesian fibration* of decomposition spaces, adapting a homotopy-invariant definition for ∞–categories which can be found in Ayala and Francis [4]. They give rise to left and right pointed comodule configurations. We define an *adjunction* between decomposition spaces X and Y to be a simplicial map between decomposition spaces $p\colon \mathcal{M} \to \Delta^1$ which is both a cartesian and a cocartesian fibration, equipped with equivalences $X \simeq \mathcal{M}_{\{0\}}$ and $Y \simeq \mathcal{M}_{\{1\}}$. Adjunctions give rise to bicomodule configurations with two pointings.

In Section 4, we define left and right convolution actions $\star_l$ and $\star_r$ dual to the comodule structures. The following is a consequence of Theorem 2.4.1:

Corollary 4.3.1 *Given a bicomodule configuration, the left and right convolutions satisfy the associative law*

$$\alpha \star_l (\theta \star_r \beta) \simeq (\alpha \star_l \theta) \star_r \beta.$$

We then establish in Section 4.4 a Möbius inversion principle for complete comodules. Let $C \to Y$ be a right comodule configuration such that the simplicial ∞–groupoid C is augmented and with new bottom degeneracies $s_{-1}\colon C_{n-1} \to C_n$ which are sections to d_0. We say it is *complete* (Section 4.2) if the sections s_{-1} are monomorphisms.

For a complete decomposition space Y, let $\vec{Y}_n$ denote the full subgroupoid of simplices with all principal edges nondegenerate. The spans $Y_1 \xleftarrow{d_1^{n-1}} \vec{Y}_n \to 1$ define linear functors, the *Phi functors* $\Phi_n\colon \mathcal{S}_{/Y_1} \to \mathcal{S}$. We also put $\Phi_{\text{even}} := \sum_{n \text{ even}} \Phi_n$ and $\Phi_{\text{odd}} := \sum_{n \text{ odd}} \Phi_n$.

The zeta functor $\zeta^C\colon \mathcal{S}_{/C_0} \to \mathcal{S}$ is the linear functor defined by the span $C_0 \xleftarrow{=} C_0 \to 1$, and $\delta^R\colon \mathcal{S}_{/C_0} \to \mathcal{S}$ is the linear functor given by the span $C_0 \xleftarrow{s_{-1}} C_{-1} \to 1$. We define δ^L similarly for left comodule configurations.

Theorem (Theorems 4.4.4 and 4.4.5) *Given $C \to Y$ a complete right comodule configuration and $D \to X$ a complete left comodule configuration,*

$$\zeta^C \star_r \Phi^Y_{\text{even}} \simeq \delta^R + \zeta^C \star_r \Phi^Y_{\text{odd}}, \quad \Phi^X_{\text{even}} \star_l \zeta^D \simeq \delta^L + \Phi^X_{\text{odd}} \star_l \zeta^D.$$

In Section 4.5, we establish a Möbius inversion principle at the algebraic level. To this end, we need to impose some finiteness conditions in order to take homotopy cardinality. Define the *Möbius functions* as the homotopy cardinalities $|\mu^Y| := |\Phi^Y_{\text{even}}| - |\Phi^Y_{\text{odd}}|$ and $|\mu^X| := |\Phi^X_{\text{even}}| - |\Phi^X_{\text{odd}}|$.

Theorem 4.5.1 *Given $C \to Y$ a right Möbius comodule configuration and $D \to X$ a left Möbius comodule configuration,*

$$|\zeta^C| \star_r |\mu^Y| = |\delta^R|, \quad |\mu^X| \star_l |\zeta^D| = |\delta^L|.$$

Finally we can extend the Rota formula to bicomodules with Möbius inversion for both comodules, called *Möbius bicomodule configurations*. Combining Corollary 4.3.1 and Theorem 4.5.1, we obtain the main theorem of the present paper:

Theorem 4.5.2 *Given a Möbius bicomodule configuration* B *with* $X := B_{\bullet,-1}$ *and* $Y := B_{-1,\bullet}$, *we have*

$$|\mu^X| \star_l |\delta^R| = |\delta^L| \star_r |\mu^Y|,$$

where δ^R *is the linear functor given by the span*

$$B_{0,0} \leftarrow X_0 \rightarrow 1$$

and δ^L *is the linear functor given by the span*

$$B_{0,0} \leftarrow Y_0 \rightarrow 1.$$

The motivating example, treated in Section 4.6, shows that any (co)cartesian fibration $p\colon \mathcal{M} \rightarrow \Delta^1$ such that $\mathcal{M}$ is a complete decomposition space gives rise to a complete left (or right) comodule configuration:

Theorem 4.6.3 *Given an adjunction of decomposition spaces in the form of a bicartesian fibration* $p\colon \mathcal{M} \rightarrow \Delta^1$, *suppose moreover that* $\mathcal{M}$ *is a Möbius decomposition space. Then the bicomodule configuration extracted from this data is Möbius. In particular, we have the Rota formula for the adjunction* p,

$$|\mu^X| \star_l |\delta^R| = |\delta^L| \star_r |\mu^Y|.$$

When specialised to the case of a classical adjunction between 1–categories, this is the classical Rota formula from page 171.

Finally, the bicomodule configuration of layered sets and posets defined in Section 2 is Möbius, and we apply the generalised Rota formula to compute the Möbius function of the incidence algebra of the decomposition space of finite posets.

Theorem 4.7.3 [7, Theorem 3.4] *The Möbius function of the incidence algebra of the decomposition space* X *of finite posets is*

$$\mu(P) = \begin{cases} (-1)^n & \text{if } P \in X_1 \text{ is a discrete poset with } n \text{ elements,} \\ 0 & \text{else.} \end{cases}$$

It is shown in [7] how this result implies similar results for any directed restriction species or free operad.

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1 Preliminaries

We work in the ∞–category of ∞–groupoids, denoted by $\mathcal{S}$, following the notation of [13]. Our ∞–categories are quasi-categories; the theory of quasi-categories has been substantially developed by Joyal [18; 19] and Lurie [22]. An ∞–groupoid is an ∞–category in which all morphisms are invertible. They are precisely Kan complexes: simplicial sets in which every horn admits a filler (and not only the inner ones).

1.1 Pullbacks and fibres, slices and linear functors

The main tool used throughout this paper are pullbacks. We use the following standard lemma many times:

Lemma 1.1.1 [22, Lemma 4.4.2.1] *Given a prism diagram of ∞–groupoids*

$$\begin{array}{ccccc} X & \longrightarrow & X' & \longrightarrow & X'' \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ Y & \longrightarrow & Y' & \longrightarrow & Y'' \end{array}$$

in which the right-hand square is a pullback, the outer rectangle is a pullback if and only if the left-hand square is.

Given a map of ∞–groupoids $p\colon X \to S$ and an object $s \in S$, the *fibre* X_s of p over s is the pullback

$$\begin{array}{ccc} X_s & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ 1 & \xrightarrow[\ulcorner s \urcorner]{} & S \end{array}$$

A map of ∞–groupoids is a *monomorphism* when its fibres are (-1)–groupoids, that is, are either empty or contractible. If $f\colon X \to Y$ is a monomorphism, then there is a complement $Z := Y \setminus X$ such that $Y \simeq X + Z$; a monomorphism is essentially an equivalence from X onto some connected components of Y.

Recall that the objects of the slice ∞–category $\mathcal{S}_{/I}$ are maps of ∞–groupoids with codomain I. Pullback along a morphism $f\colon J \to I$ defines an functor $f^*\colon \mathcal{S}_{/I} \to \mathcal{S}_{/J}$. This functor is right adjoint to the functor $f_!\colon \mathcal{S}_{/J} \to \mathcal{S}_{/I}$ given by postcomposing with f. A span $I \xleftarrow{p} M \xrightarrow{q} J$ induces a functor between the slices by pullback and postcomposition,

$$\mathcal{S}_{/I} \xrightarrow{p^*} \mathcal{S}_{/M} \xrightarrow{q_!} \mathcal{S}_{/J}.$$

A functor is *linear* if it is homotopy equivalent to a functor induced by a span. The following Beck–Chevalley rule holds for ∞–groupoids: for any pullback square

$$\begin{array}{ccc} J & \xrightarrow{f} & I \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle q} \\ V & \xrightarrow[g]{} & U \end{array}$$

the functors $p_! f^*, g^* q_!\colon \mathcal{S}_{/I} \to \mathcal{S}_{/V}$ are naturally homotopy equivalent (see [16] for the technical details regarding coherence of these equivalences). By the Beck–Chevalley rule, the composition of two linear functors is linear. For an extended treatment of linear functors and homotopy linear algebra, we refer to [15].

1.2 Segal spaces and decomposition spaces

We consider the functor ∞–category

$$\mathrm{Fun}\,(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S})$$

whose objects are *simplicial ∞–groupoids*, that is, functors from the ∞–category $\mathbf{\Delta}^{\mathrm{op}}$ to the ∞–category $\mathcal{S}$.

A simplicial ∞–groupoid X is called a *Segal space* if the squares

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{d_0} & X_n \\ {\scriptstyle d_{n+1}}\downarrow & \lrcorner & \downarrow{\scriptstyle d_n} \\ X_n & \xrightarrow[d_0]{} & X_{n-1} \end{array}$$

are pullbacks for all $n > 0$.

The simplex category $\mathbf{\Delta}$ has an active–inert factorisation system. A morphism $[m] \to [n]$ is *active* (also called *generic*) if it preserves endpoints: $g(0) = 0$ and $g(m) = n$. A morphism is *inert* (also called *free*) if it is distance-preserving: $f(i+1) = f(i) + 1$ for $0 \leq i \leq m-1$. The active maps are generated by the codegeneracy maps and the inner coface maps, and the inert maps are generated by the outer coface maps $d^{\perp} := d^0$ and $d^{\top} := d^n$.

A *decomposition space* $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}$ is a simplicial ∞–groupoid such that the image of any pushout diagram in $\mathbf{\Delta}$ of an active map g along an inert map f is a pullback of ∞–groupoids. It is enough to check that the squares

$$
\begin{array}{ccc}
X_{n+1} & \xrightarrow{s_{k+1}} & X_{n+2} \\
{\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\
X_n & \xrightarrow[s_k]{} & X_{n+1}
\end{array}
\qquad\qquad
\begin{array}{ccc}
X_{n+2} & \xleftarrow{d_{k+2}} & X_{n+3} \\
\downarrow{\scriptstyle d_\perp} & \llcorner & \downarrow{\scriptstyle d_\perp} \\
X_{n+1} & \xleftarrow[d_{k+1}]{} & X_{n+2}
\end{array}
$$

$$
\begin{array}{ccc}
X_{n+1} & \xrightarrow{s_{k}} & X_{n+2} \\
{\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\
X_n & \xrightarrow[s_k]{} & X_{n+1}
\end{array}
\qquad\qquad
\begin{array}{ccc}
X_{n+2} & \xleftarrow{d_{k+1}} & X_{n+3} \\
\downarrow{\scriptstyle d_\top} & \llcorner & \downarrow{\scriptstyle d_\top} \\
X_{n+1} & \xleftarrow[d_{k+1}]{} & X_{n+2}
\end{array}
$$

are pullbacks for $0 \leq k \leq n$.

The notion of decomposition space was introduced by Gálvez-Carrillo, Kock and Tonks [13], and independently by Dyckerhoff and Kapranov [10] under the name unital 2–Segal space. It can be seen as an abstraction of posets. The equivalence of the two notions follows from the pullback formulation of 2–Segal spaces given in Proposition 2.3.2 of [10]. It is precisely the condition required to obtain a counital coassociative comultiplication on $\mathcal{S}_{/X_1}$; see also [24] for the exact role played by the decomposition-space condition. Since the motivation in the present paper comes from combinatorics, we follow the terminology of [13]; for a survey motivated by combinatorics, see [11].

Proposition 1.2.1 [10, Proposition 2.3.3; 13, Proposition 3.5] *Every Segal space is a decomposition space.*

There are plenty of examples of decomposition spaces which are not Segal, eg Schmitt's Hopf algebra of graphs, which is a running example in [11].

1.3 Incidence coalgebras and culf functors

For any decomposition space X, we get an incidence coalgebra [10; 13]. The span $X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1$ defines a linear functor, the *comultiplication*

$$\Delta\colon \mathcal{S}_{/X_1} \to \mathcal{S}_{/X_1\times X_1}, \quad (T \xrightarrow{t} X_1) \mapsto (d_2, d_0)_! \circ d_1^*(t).$$

The span $X_1 \xleftarrow{s_0} X_0 \xrightarrow{z} 1$ defines a linear functor, the *counit*

$$\delta\colon \mathcal{S}_{/X_1} \to \mathcal{S}, \quad (T \xrightarrow{t} X_1) \mapsto z_! \circ s_0^*(t).$$

The up-to-coherent-homotopy coassociativity follows from the decomposition-space axioms; see [13, Sections 5 and 7] or [24, Section 4.3] for a proof. We obtain a coalgebra $(\mathcal{S}_{/X_1}, \Delta, \delta)$, called the *incidence coalgebra*.

The ∞–category $\mathcal{S}_{/I}$ plays the role of the vector space with basis I. The presheaf category $\mathcal{S}^I$ can be considered the linear dual of the slice category $\mathcal{S}_{/I}$ (see [15] for the precise statements and proofs). A span $I \leftarrow M \to J$ defines both a linear functor $\mathcal{S}_{/I} \to \mathcal{S}_{/J}$ and the dual linear functor $\mathcal{S}^J \to \mathcal{S}^I$. If X is a decomposition space, the coalgebra structure on $\mathcal{S}_{/X_1}$ therefore induces an algebra structure on $\mathcal{S}^{X_1}$. In details, the *convolution product* of two linear functors $F, G\colon \mathcal{S}_{/X_1} \to \mathcal{S}$, given by the spans $X_1 \leftarrow M \to 1$ and $X_1 \leftarrow N \to 1$, is the composite of their tensor product $F \otimes G$ with the comultiplication,

$$F * G\colon \mathcal{S}_{/X_1} \xrightarrow{\Delta} \mathcal{S}_{/X_1} \otimes \mathcal{S}_{/X_1} \xrightarrow{F\otimes G} \mathcal{S} \otimes \mathcal{S} \simeq \mathcal{S},$$

where the tensor product $F \otimes G$ is given by the span $X_1 \times X_1 \leftarrow M \times N \to 1$. The neutral element for convolution is

$$\delta\colon \mathcal{S}_{/X_1} \to \mathcal{S}$$

defined by the span $X_1 \xleftarrow{s_0} X_0 \to 1$.

A map $f\colon X \to Y$ of simplicial spaces is *cartesian* on an arrow $[n] \to [k]$ in $\mathbf{\Delta}$ if the naturality square for F with respect to this arrow is a pullback. It is called a *right fibration* if it is cartesian on $d_\bot$ and on all active maps, and is called a *left fibration* if it is cartesian on $d_\top$ and on all active maps.

A simplicial map $f\colon X \to Y$ is *conservative* if it is cartesian with respect to codegeneracy maps

$$\begin{array}{ccc} X_n & \xrightarrow{s_i} & X_{n+1} \\ {\scriptstyle f_n}\downarrow & \lrcorner & \downarrow{\scriptstyle f_{n+1}} \\ Y_n & \xrightarrow[s_i]{} & Y_{n+1} \end{array} \qquad \text{for } 0 \le i \le n.$$

It is *ulf* (unique lifting of factorisations) if it is cartesian with respect to inner coface maps

$$\begin{array}{ccc} X_{n+1} & \xleftarrow{d_{i+1}} & X_{n+2} \\ {\scriptstyle f_{n+1}}\downarrow & \lrcorner & \downarrow{\scriptstyle f_{n+2}} \\ Y_{n+1} & \xleftarrow{d_{i+1}} & Y_{n+2} \end{array} \qquad \text{for } 0 \le i \le n.$$

We write *culf* for conservative and ulf, that is, cartesian on all active maps. The culf functors induce coalgebra homomorphisms between the incidence algebras. They play an essential role in [13; 14] as a natural notion of morphism between decomposition spaces, but the present paper deals also with general simplicial maps.

Layered finite posets and layered finite sets We refer to [12] for the following material. An *n–layering* of a finite poset P is a monotone map $l\colon P \to \underline{n}$, where $\underline{n} = \{1, \dots, n\}$ are the objects of the skeleton of the category of finite ordered sets (possibly empty) and monotone maps. The fibres $P_i = l^{-1}(i)$ for $i \in \underline{n}$ are called layers, and can be empty. The objects of the groupoid $\boldsymbol{C}_n$ of n–layered finite posets are monotone maps $l\colon P \to \underline{n}$ and the morphisms are triangles

$$\begin{array}{ccc} P & \xrightarrow{\simeq} & P' \\ & \searrow \quad \swarrow & \\ & \underline{n} & \end{array}$$

where $P \to P'$ is a monotone bijection. They assemble into a simplicial groupoid $\boldsymbol{C}$. The face maps are given by joining layers, or deleting an outer layer for the top and bottom face maps. The degeneracy maps are given by inserting empty layers.

Proposition 1.3.1 [12, Proposition 6.12, Lemma 6.13] *The simplicial groupoid $\boldsymbol{C}$ of layered finite posets is a decomposition space* (*but not a Segal space*), *and is complete*, *locally finite*, *locally discrete and of locally finite length.*

The incidence coalgebra of $\boldsymbol{C}$ has comultiplication given by the span

$$\boldsymbol{C}_1 \xleftarrow{d_1} \boldsymbol{C}_2 \xrightarrow{(d_2,d_0)} \boldsymbol{C}_1 \times \boldsymbol{C}_1,$$

where d_1 joins the two layers, and d_2 and d_0 return the two layers. The comultiplication of a poset is thus obtained by summing over admissible cuts (a 2–layering of the poset) and taking tensor product of the two layers.

Similarly, let $\boldsymbol{I}_n$ denote the groupoid of all layerings of finite sets. Again these groupoids assemble into a simplicial groupoid, denoted by $\boldsymbol{I}$.

Proposition 1.3.2 [12, Proposition 4.3, Lemma 4.4] *The simplicial groupoid $\boldsymbol{I}$ is a Segal space, and hence a decomposition space, which is complete, locally finite, locally discrete and of locally finite length.*

The simplicial groupoid $\boldsymbol{C}$ is the decomposition space corresponding to the terminal directed restriction species, finite posets and convex maps, while $\boldsymbol{I}$ is the decomposition space corresponding to the terminal restriction species, finite sets and injections. The incidence coalgebra of $\boldsymbol{I}$ is the binomial coalgebra [12, Section 2.4] with well-known Möbius function $(-1)^n$ for a set with n elements.

2 Bicomodules

2.1 Comodules

The theory of modules in the context of decomposition spaces has been developed by Walde [27], and independently by Young [28], both in the context of Hall algebras. They call them relative 2–Segal spaces. Here we give a conceptual way to reformulate their definitions using linear functors.

Given a map between two simplicial ∞–groupoids $f\colon C \to X$, the span $C_0 \xleftarrow{d_1} C_1 \xrightarrow{(f_1,d_0)} X_1 \times C_0$ defines a linear functor $\gamma_l\colon \mathcal{S}_{/C_0} \to \mathcal{S}_{/X_1} \otimes \mathcal{S}_{/C_0}$, and the span $C_0 \xleftarrow{d_0} C_1 \xrightarrow{(d_1,f_1)} C_0 \times X_1$ defines a linear functor $\gamma_r\colon \mathcal{S}_{/C_0} \to \mathcal{S}_{/C_0} \otimes \mathcal{S}_{/X_1}$.

Proposition 2.1.1 *Let $f\colon C \to X$ be a map between two simplicial ∞–groupoids. Suppose moreover that C is Segal, X is a decomposition space and the map $f\colon C \to X$ is culf; then the span*

$$C_0 \xleftarrow{d_1} C_1 \xrightarrow{(f_1,d_0)} X_1 \times C_0$$

induces on the slice ∞–category $\mathcal{S}_{/C_0}$ the structure of a left $\mathcal{S}_{/X_1}$–comodule (at the π_0 level), and the span

$$C_0 \xleftarrow{d_0} C_1 \xrightarrow{(d_1,f_1)} C_0 \times X_1$$

induces on $\mathcal{S}_{/C_0}$ the structure of a right $\mathcal{S}_{/X_1}$–comodule (at the π_0 level).

The data needed to obtain a comodule is called a *comodule configuration*, that is, a culf map from a Segal space to a decomposition space.

Remark 2.1.2 The relevance of the Segal condition on C and the culf condition on f can be explained individually as follows. It is standard that for a category C, the coalgebra of arrows C_1 coacts on C_0; the coaction (from the right) is given by $b \mapsto \sum_{f : a \to b} a \otimes f$. Coassociativity of this coaction is equivalent to the Segal condition. Now a culf map $C \to X$ defines a coalgebra homomorphism, and. in this way, also X_1 coacts on C_0, by "corestriction of coscalars".

Remark 2.1.3 The proposition is stated only at the π_0–level. This means that we establish only the comodule structure up to homotopy, but do not establish the *coherence* of this up-to-homotopy structure. A stronger result, a partial coherence result, is given by [27; 28], who establish the coherence at the 1–truncated level (rather than the 0–truncated level established here). It is most likely that full coherence can be established by exploiting the techniques employed by [13; 24]. While only a small bit of the axioms are used to establish the proposition as stated, the full decomposition-space axioms and the culf condition are expected to be required for the fully coherent result, and this is why these conditions have been included in the definition of comodule configuration.

Proof We want to prove that the map γ_l is a left $\mathcal{S}_{/X_1}$–coaction. The desired diagram, commutative up to homotopy,

$$\begin{array}{ccc} \mathcal{S}_{/C_0} & \xrightarrow{\gamma_l} & \mathcal{S}_{/X_1 \times C_0} \\ {\scriptstyle \gamma_l}\downarrow & & \downarrow{\scriptstyle \mathrm{Id} \otimes \gamma_l} \\ \mathcal{S}_{/X_1 \times C_0} & \xrightarrow[\Delta \otimes \mathrm{Id}]{} & \mathcal{S}_{/X_1 \times X_1 \times C_0} \end{array}$$

is induced by the solid spans in the diagram

$$\begin{array}{ccccc}
C_0 & \xleftarrow{\ d_1\ } & C_1 & \xrightarrow{\ (f_1,d_0)\ } & X_1 \times C_0 \\
{\scriptstyle d_1}\uparrow & & \uparrow{\scriptstyle d_2} & \lrcorner & \uparrow{\scriptstyle \mathrm{Id}\otimes d_1} \\
C_1 & \xleftarrow[\ d_1\ ]{} & C_2 & \xrightarrow[\ (d_2 f_1,d_0)\ ]{} & X_1 \times C_1 \\
{\scriptstyle (f_1,d_0)}\downarrow & \llcorner & \downarrow{\scriptstyle (f_1,d_0 d_0)} & & \downarrow{\scriptstyle \mathrm{Id}\otimes(f_1,d_0)} \\
X_1 \times C_0 & \xleftarrow[\ d_1\otimes\mathrm{Id}\ ]{} & X_2 \times C_0 & \xrightarrow[\ (d_2,d_0)\otimes\mathrm{Id}\ ]{} & X_1 \times X_1 \times C_0
\end{array}$$

The coassociativity (at the π_0 level) will follow from Beck–Chevalley equivalences if we have the two pullbacks indicated in the diagram. The upper right-hand square is a pullback if and only if its composite with the second projection is a pullback. This composite outer square is a pullback because C satisfies the Segal condition. Similarly, the lower left-hand square is a pullback if its composite with the first projection is a pullback. This composite outer square is a pullback because $f\colon C \to X$ is culf. $\square$

Example 2.1.4 (Décalage [17]) Given a simplicial space X, the *lower décalage* $\mathrm{Dec}_\perp(X)$ is the simplicial space obtained by deleting X_0, all d_0 face maps and s_0 degeneracy maps. The original d_0 maps induce a simplicial map $d_\perp\colon \mathrm{Dec}_\perp(X) \to X$, called the décalage map. Similarly, the *upper décalage* $\mathrm{Dec}_\top(X)$ is the simplicial space obtained by deleting X_0, all last face maps $d_\top$ and last degeneracy maps $s_\top$. The original $d_\top$ maps induce a simplicial map $d_\top\colon \mathrm{Dec}_\top X \to X$.

It is well known that $\mathrm{Dec}_\perp(X)$ is a Segal space and the décalage map is culf (see [13, Proposition 4.9]). Hence we have a comodule configuration. The resulting comodule is the incidence coalgebra of X as a (right) comodule over itself.

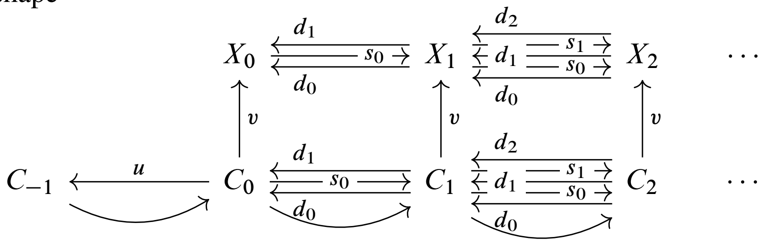
For categories, given a functor $f\colon C \to D$, define the *mapping cylinder* (or *collage* in [20]) M_f to be the category where objects are either objects of C or objects of D and

$$\mathrm{Hom}_{M_f}(x,y) = \begin{cases} \mathrm{Hom}_C(x,y) & \text{if } x,y \in C, \\ \mathrm{Hom}_D(x,y) & \text{if } x,y \in D, \\ \mathrm{Hom}_D(f(x),y) & \text{if } x \in C, y \in D, \\ \varnothing & \text{else.} \end{cases}$$

There exists a unique $p\colon M_f \to \Delta^1$ such that $p^{-1}(0) = C$ and $p^{-1}(1) = D$. This is moreover a cocartesian fibration, the cocartesian lift for $x \in C$ being given by

$\mathrm{Id}_{f(x)} \in \mathrm{Map}_{M_f}(x, f(x))$. The shape of a comodule configuration is that of $(M_{\mathrm{id}})^{\mathrm{op}}$, where M_{id} is the mapping cylinder of the identity of $\boldsymbol{\Delta}$. In other words, a comodule configuration is a functor from $(M_{\mathrm{id}})^{\mathrm{op}}$ to $\mathcal{S}$ (satisfying certain conditions).

Let $\boldsymbol{\Delta}_{\mathrm{bot}}$ be the simplex category of finite linear orders with a specified bottom element, and bottom-preserving monotone maps. Consider the mapping cylinder M_j of the functor $j\colon \boldsymbol{\Delta} \to \boldsymbol{\Delta}_{\mathrm{bot}}$ freely adding a bottom element. Presheaves on M_j are diagrams of the shape



This is the shape of what we call a *right pointed comodule configuration*: it is a comodule configuration $C \to X$ such that the Segal space C is augmented, and with new bottom sections $s_{-1}\colon C_{n-1} \to C_n$. The importance of the pointing (the extra bottom degeneracy maps) is that it makes possible to formulate the notion of completeness and the condition locally finite length — see Section 4.5 below — and it guarantees the existence of a filtration on the associated comodule (see [14, Section 6] for a similar argument), which is of independent interest.

Example 2.1.5 The comodule configuration obtained from the lower décalage of a decomposition space X is also right pointed, the augmentation map is given by $d_1\colon X_1 \to X_0$, and the extra bottom sections by s_0.

2.2 Augmented bisimplicial infinity-groupoids

We shall establish conditions under which left and right comodule structures define a bicomodule. The main objects of interest are augmented bisimplicial ∞–groupoids subject to conditions, which are formulated in terms of pullbacks. We consider the functor ∞–category

$$\mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}} \times \boldsymbol{\Delta}^{\mathrm{op}}, \mathcal{S})$$

whose objects are *bisimplicial ∞–groupoids*, that is, functors from the ∞–category $\boldsymbol{\Delta}^{\mathrm{op}} \times \boldsymbol{\Delta}^{\mathrm{op}}$ to the ∞–category $\mathcal{S}$.

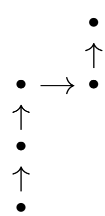
A *double Segal space* is a bisimplicial ∞–groupoid satisfying the Segal condition for each restriction $\boldsymbol{\Delta}^{\mathrm{op}} \times \{[n]\} \to \mathcal{S}$ (the columns) and $\{[n]\} \times \boldsymbol{\Delta}^{\mathrm{op}} \to \mathcal{S}$ (the rows).

Let $\mathbf{\Delta}_+$ be the augmented simplex category of all finite ordinals and order-preserving maps. An *augmented* bisimplicial ∞–groupoid B has in addition ∞–groupoids $B_{i,-1}$ and $B_{-1,i}$ of (-1)–simplices. We consider the functor ∞–category

$$\mathrm{Fun}\,(\mathbf{\Delta}_+^{\mathrm{op}} \times \mathbf{\Delta}_+^{\mathrm{op}} \setminus \{(-1,-1)\}, \mathcal{S})$$

whose objects are *augmented bisimplicial ∞–groupoids.*

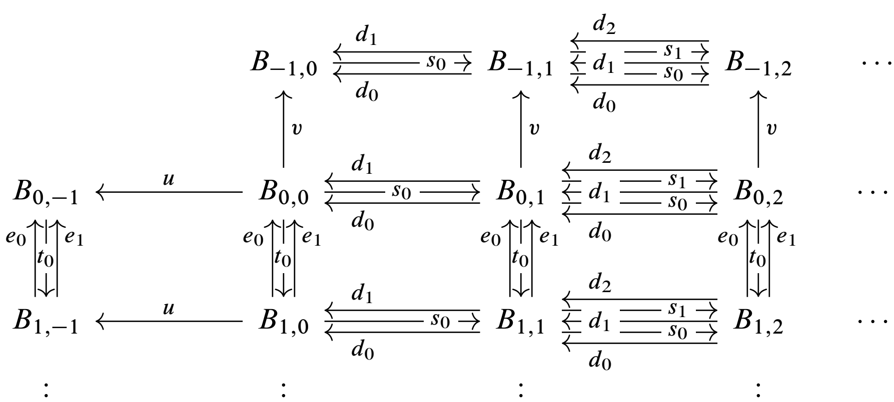
Remark 2.2.1 The shape of an augmented bisimplicial ∞–groupoid is $(\mathbf{\Delta}_{/\Delta^1})^{\mathrm{op}}$. We denote by $[i,j]$ the object given by the map $\Delta^{i+1+j} \to \Delta^1$ sending the $i+1$ first vertices to 0 and the others to 1. We allow i or j to be equal to -1 but not both. Maps $[i,j] \to [k,l]$ are given by the inclusions respecting the horizontal map. For example, the object $[2,1]$ can be drawn as



where the horizontal maps lie over the map in Δ^1.

We can draw $[i,j]$ as a column of $i+1$ black dots followed by $j+1$ white dots. Maps send black dots to black dots and white dots to white dots, without crossing.

We use the following notation for an augmented bisimplicial ∞–groupoid. We denote by $d_k\colon B_{i,j} \to B_{i,j-1}$ and $e_l\colon B_{i,j} \to B_{i-1,j}$ the face maps, and $s_k\colon B_{i,j-1} \to B_{i,j}$ and $t_l\colon B_{i-1,j} \to B_{i,j}$ the degeneracy maps; u and v are the augmentation maps:



An augmented double Segal space satisfies that rows and columns are Segal. If we suppose that augmentations are culf and $B_{\bullet,-1}$ and $B_{-1,\bullet}$ are decomposition spaces, we can apply Proposition 2.1.1 to obtain comodules: the span

$$B_{0,0} \xleftarrow{e_1} B_{1,0} \xrightarrow{(u,e_0)} B_{1,-1} \times B_{0,0}$$

induces on $\mathcal{S}_{/B_{0,0}}$ the structure of a left comodule over $\mathcal{S}_{/B_{1,-1}}$, and the span

$$B_{0,0} \xleftarrow{d_0} B_{0,1} \xrightarrow{(d_1,v)} B_{0,0} \times B_{-1,1}$$

induces on $\mathcal{S}_{/B_{0,0}}$ the structure of a right comodule over $\mathcal{S}_{/B_{-1,1}}$.

2.3 Stability

We say a bisimplicial ∞–groupoid is *stable* if the squares

$$\begin{array}{ccc} B_{i-1,j-1} & \xleftarrow{d_k} & B_{i-1,j} \\ {\scriptstyle e_l}\uparrow & \ulcorner & \uparrow{\scriptstyle e_l} \\ B_{i,j-1} & \xleftarrow[d_k]{} & B_{i,j} \end{array} \qquad \begin{array}{ccc} B_{i-1,j-1} & \xrightarrow{s_k} & B_{i-1,j} \\ {\scriptstyle e_l}\uparrow & \urcorner & \uparrow{\scriptstyle e_l} \\ B_{i,j-1} & \xrightarrow[s_k]{} & B_{i,j} \end{array}$$

$$\begin{array}{ccc} B_{i-1,j-1} & \xleftarrow{d_k} & B_{i-1,j} \\ \downarrow{\scriptstyle t_l} & \llcorner & \downarrow{\scriptstyle t_l} \\ B_{i,j-1} & \xleftarrow[d_k]{} & B_{i,j} \end{array} \qquad \begin{array}{ccc} B_{i-1,j-1} & \xrightarrow{s_k} & B_{i-1,j} \\ \downarrow{\scriptstyle t_l} & \lrcorner & \downarrow{\scriptstyle t_l} \\ B_{i,j-1} & \xrightarrow[s_k]{} & B_{i,j} \end{array}$$

are pullbacks for all values of the indices except for $d_\perp$ along $e_\top$ and $d_\top$ along $e_\perp$.

Remark 2.3.1 A bisimplicial ∞–groupoid is stable if it satisfies all the following properties:

- $s_k\colon B_{i,j-1} \to B_{i,j}$ is a cartesian natural transformation for all $0 \le k \le j-1$.
- d_k for $k \ne \top, \perp$ is a cartesian natural transformation.
- $d_\top$ is a left fibration.
- $d_\perp$ is a right fibration.

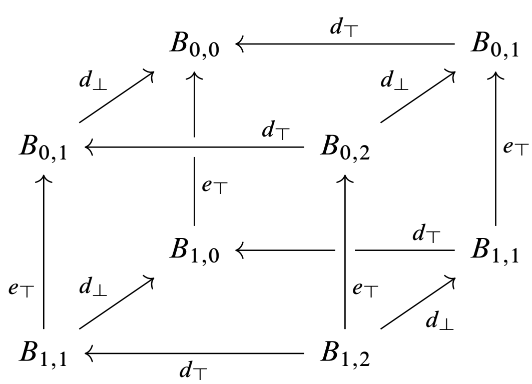
Remark 2.3.2 Bergner, Osorno, Ozornova, Rovelli, and Scheimbauer introduced the notion of stable double category (bisimplicial *set*) in [6]: they define a double category to be stable if every square is uniquely determined by its span of source morphisms and, independently, by its cospan of target morphisms. The present definition is a categorical reformulation of their notion suitable for ∞–groupoids. The motivation for the terminology is the following example. Let C be a stable ∞–category (see [23, Chapter 1]). Define a double Segal space B where $B_{0,0}$ is the ∞–groupoid of objects of C, where $B_{0,1}$ is the ∞–groupoid of arrows of C (as in the fat nerve) and $B_{1,1}$ is the ∞–groupoid of pullback squares (equivalently, pushout squares). More generally, $B_{m,n}$ is the ∞–groupoid of $(\Delta^m \times \Delta^n)$–diagrams in C for which all the rectangles are pullbacks (and hence pushouts). This is a stable bisimplicial ∞–groupoid (which of course is a double Segal space). This is almost by definition: since we only took pullback and pushout squares, they are determined by their sources by pushout or their targets by pullback, in the sense of our definition.

Lemma 2.3.3 *Let B be a double Segal space. If we have the two pullbacks*

$$\begin{array}{ccc} B_{0,0} & \xleftarrow{d_0} & B_{0,1} \\ {\scriptstyle e_0}\uparrow & \lrcorner & \uparrow{\scriptstyle e_0} \\ B_{1,0} & \xleftarrow[d_0]{} & B_{1,1} \end{array} \qquad \begin{array}{ccc} B_{0,0} & \xleftarrow{d_1} & B_{0,1} \\ {\scriptstyle e_1}\uparrow & \lrcorner & \uparrow{\scriptstyle e_1} \\ B_{1,0} & \xleftarrow[d_1]{} & B_{1,1} \end{array}$$

then the double Segal space is stable.

Proof First, the second pullback implies that every square with top face maps $d_\top\colon B_{i,j+1} \to B_{i,j}$ is a pullback. Indeed, in the cube



the top and bottom squares are pullbacks because every row is Segal, and the back square is a pullback by hypothesis. Thus the rectangle consisting of bottom and back is a pullback because bottom and back squares are; front is a pullback because top and rectangle are. By induction, supposing the squares

$$\begin{array}{ccc} B_{i-1,j-1} & \xleftarrow{d_\top} & B_{i-1,j} \\ {\scriptstyle e_\top}\uparrow & \lrcorner & \uparrow{\scriptstyle e_\top} \\ B_{i,j-1} & \xleftarrow[d_\top]{} & B_{i,j} \end{array}$$

are pullbacks, we can form cubes with the top and bottom faces pullbacks thanks to the Segal condition, and the back square is a pullback by hypothesis. This proves that every square involving top face maps are pullbacks. Starting with the first pullback, we prove in the same way that every square involving bottom face maps are pullbacks.

Now we want to prove that the following square is a pullback for $0 < k < i$:

$$\begin{array}{ccc} B_{i-1,j-1} & \xleftarrow{d_\top} & B_{i-1,j} \\ {\scriptstyle e_k}\uparrow & & \uparrow{\scriptstyle e_k} \\ B_{i,j-1} & \xleftarrow[d_\top]{} & B_{i,j} \end{array}$$

If $k = i - r$, we postcompose vertically with $(e_\top)^r$:

$$\begin{array}{ccc} B_{k-1,j-1} & \xleftarrow{d_\top} & B_{k-1,j} \\ {\scriptstyle (e_\top)^r}\uparrow & & \uparrow{\scriptstyle (e_\top)^r} \\ B_{i-1,j-1} & \xleftarrow[d_\top]{} & B_{i-1,j} \\ {\scriptstyle e_k}\uparrow & & \uparrow{\scriptstyle e_k} \\ B_{i,j-1} & \xleftarrow[d_\top]{} & B_{i,j} \end{array}$$

Then the vertical composite is equivalent to $e_\top \circ (e_\top)^r$ (by face map identities), so both the rectangle and the upper square are pullbacks by assumption, and therefore, by Lemma 1.1.1, the lower square is a pullback too, as required.

We can do the same proof with bottom face maps. We can also replace $d_\top$ in the new previous pullback squares and obtain the remaining pullbacks involving the face maps.

For squares with face and degeneracy maps, we use the following strategy: in the diagram

$$\begin{array}{ccccc} B_{00} & \xrightarrow{s_0} & B_{01} & \xrightarrow{d_\bot} & B_{00} \\ {\scriptstyle e_\bot}\uparrow & & \uparrow{\scriptstyle e_\bot} & & \uparrow{\scriptstyle e_\bot} \\ B_{10} & \xrightarrow[s_0]{} & B_{11} & \xrightarrow[d_\bot]{} & B_{10} \end{array}$$

the map s_0 is a section of d_1, then the long edge is an identity. The right-hand square is a pullback (it is one of the two pullback in the hypothesis). Thus the left-hand square is a pullback. We can proceed in the same way for the other degeneracy maps.

There remains the case of squares involving only degeneracy:

$$\begin{array}{ccc} B_{ij} & \xrightarrow{s_0} & B_{i,j+1} \\ {\scriptstyle t_k}\downarrow & & \downarrow{\scriptstyle t_k} \\ B_{i+1,j} & \xrightarrow[s_0]{} & B_{i+1,j+1} \end{array}$$

We again glue on the right a square with face map such that the long edge is an identity and use once again Lemma 1.1.1. □

2.4 Bicomodules

Theorem 2.4.1 *Let B be an augmented stable double Segal space, and such that the augmentation maps are culf. Suppose moreover $X := B_{\bullet,-1}$ and $Y := B_{-1,\bullet}$ are decomposition spaces. Then the spans*

$$B_{0,0} \xleftarrow{e_1} B_{1,0} \xrightarrow{(u,e_0)} X_1 \times B_{0,0}$$

and

$$B_{0,0} \xleftarrow{d_0} B_{0,1} \xrightarrow{(d_1,v)} B_{0,0} \times Y_1$$

induce on $\mathcal{S}_{/B_{0,0}}$ the structure of a bicomodule over $\mathcal{S}_{/X_1}$ and $\mathcal{S}_{/Y_1}$ (at the π_0 level).

A bisimplicial ∞–groupoid satisfying the conditions of the theorem is called a *bicomodule configuration*.

Remark 2.4.2 In analogy with Remark 2.1.2, the notion of bicomodule configuration can be broken up into steps. First, for any double Segal space B, since the zeroth column $B_{\bullet,0}$ is a Segal space, $\mathcal{S}_{/B_{0,0}}$ is a left comodule over $\mathcal{S}_{/B_{1,0}}$, and similarly $\mathcal{S}_{/B_{0,0}}$ is a right comodule over $\mathcal{S}_{/B_{0,1}}$. It is now the stability of B that expresses the bicomodule condition. From here, a culf augmentation on the left to a decomposition space X induces a coalgebra homomorphism, and a culf augmentation on the right to a decomposition space Y induces another coalgebra homomorphism, and coextension of coscalars along these coalgebra homomorphisms makes $\mathcal{S}_{/B_{0,0}}$ an $\mathcal{S}_{/X_1}$–$\mathcal{S}_{/Y_1}$–bicomodule. This viewpoint might well be useful in the proof of full coherence.

Remark 2.4.3 As for Proposition 2.1.1, the theorem is stated at the π_0–level. It is most likely that the full coherence can be established using the techniques employed in [13; 24]. It is expected that all the stability pullbacks are required for the fully coherent result. For the present purposes, we are going to take homotopy cardinality anyway, and for that, coherence is not essential.

Proof The left and right comodule structures were established in Proposition 2.1.1. The desired homotopy coherent diagram

$$\begin{array}{ccc}
\mathcal{S}_{/\boldsymbol{B}_{0,0}} & \xrightarrow{\gamma_l} & \mathcal{S}_{/\boldsymbol{B}_{1,-1}\times \boldsymbol{B}_{0,0}} \\
{\scriptstyle \gamma_r}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}\otimes\gamma_r} \\
\mathcal{S}_{/\boldsymbol{B}_{0,0}\times \boldsymbol{B}_{-1,1}} & \xrightarrow[\gamma_l\otimes\mathrm{Id}]{} & \mathcal{S}_{/\boldsymbol{B}_{1,-1}\times \boldsymbol{B}_{0,0}\times \boldsymbol{B}_{-1,1}}
\end{array}$$

is induced by the solid spans in the diagram

$$\begin{array}{ccccc}
B_{0,0} & \xleftarrow{e_1} & B_{1,0} & \xrightarrow{(u,e_0)} & B_{1,-1}\times B_{0,0} \\
{\scriptstyle d_0}\uparrow & & {\scriptstyle d_0}\uparrow & \lrcorner & \uparrow{\scriptstyle \mathrm{Id}\otimes d_0} \\
B_{0,1} & \xleftarrow[e_1]{} & B_{1,1} & \xrightarrow[(ud_0,e_0)]{} & B_{1,-1}\times B_{0,1} \\
{\scriptstyle (d_1,v)}\downarrow & \llcorner & \downarrow{\scriptstyle (d_1,ve_1)} & & \downarrow{\scriptstyle \mathrm{Id}\otimes(d_1,v)} \\
B_{0,0}\times B_{-1,1} & \xleftarrow[e_1\otimes\mathrm{Id}]{} & B_{1,0}\times B_{-1,1} & \xrightarrow[(u,e_0)\otimes\mathrm{Id}]{} & B_{1,-1}\times B_{0,0}\times B_{-1,1}
\end{array}$$

The homotopy commutativity of the squares follows once again from the new augmentation simplicial identities. The upper-right hand square is a pullback if and only if its composite with the second projection is a pullback and, similarly, the lower-left hand square is a pullback if and only if its composite with the first projection is a pullback. These composite outer squares are pullbacks due to the stability condition. □

Example 2.4.4 In analogy with Example 2.1.4, given a decomposition space X, let B be the total décalage of X. (Its zeroth column is $\mathrm{Dec}_\top(X)$ and its zeroth row is $\mathrm{Dec}_\perp(X)$.) With its natural augmentation maps, this becomes a bicomodule configuration, realising the coalgebra of X as a bicomodule over itself.

2.5 Augmented double Segal space of layered sets and posets

The following example is developed in [7], where details of the general construction and proofs can be found.

Suppose $\boldsymbol{I}$ is the decomposition space of layered finite sets and $\boldsymbol{C}$ is the decomposition space of layered finite posets. We obtain a bisimplicial groupoid $\boldsymbol{B}$ with the following explicit description: the groupoid $\boldsymbol{B}_{i,j}$ consists of pairs of layerings $(S \to \underline{i},\, P \to \underline{j+1})$ where S is a finite set, and P is a finite poset. For example, $\boldsymbol{B}_{0,0}$ is the groupoid of 1–layered finite posets. The horizontal face maps (taking place only on the $(j+1)$–layered finite poset part) are given by

- $d_k \colon \boldsymbol{B}_{i,j} \to \boldsymbol{B}_{i,j-1}$ joins the layers $(k+1)$ and $(k+2)$ of the poset for all $j > 0$ and $0 \leq k \leq j-1$;
- $d_\top = d_j \colon \boldsymbol{B}_{i,j} \to \boldsymbol{B}_{i,j-1}$ deletes the last layer.

Horizontal degeneracy maps are given by inserting empty layers: $s_k \colon \boldsymbol{B}_{i,j} \to \boldsymbol{B}_{i,j+1}$ inserts an empty $(k+2)^{\text{nd}}$ layer in the poset for all $j \geq 0$ and $0 \leq k \leq j$.

The vertical face maps are given by

- $e_\bot = e_0 \colon \boldsymbol{B}_{i,j} \to \boldsymbol{B}_{i-1,j}$ deletes the first layer of the set for all $i > 0$;
- $e_k \colon \boldsymbol{B}_{i,j} \to \boldsymbol{B}_{i-1,j}$ joins the layers k and $k+1$ of the set for all $0 < k < i$;
- $e_\top = e_i \colon \boldsymbol{B}_{i,j} \to \boldsymbol{B}_{i-1,j}$ joins the last layer of the set and the first layer of the poset into the first layer of the poset.

Vertical degeneracy maps are given by inserting empty layers: $t_k \colon \boldsymbol{B}_{i,j} \to \boldsymbol{B}_{i+1,j}$ inserts an empty $(k+1)^{\text{st}}$ layer to the set for all $0 \leq k \leq i$. The augmentation maps are $u \colon \boldsymbol{B}_{i,0} \to \boldsymbol{I}_i$, deleting the whole 1–layered poset, and $v \colon \boldsymbol{B}_{0,j} \to \boldsymbol{C}_j$, deleting the first layer of the poset. It should be noted that the row $\boldsymbol{B}_{0,\bullet}$ is the lower décalage of $\boldsymbol{C}$, that v is the décalage map given by the original d_0, and that u is the augmentation map that décalage always have.

Proposition 2.5.1 [7, Proposition 2.14] *With augmentations maps u and v, the bisimplicial groupoid $\boldsymbol{B}$ is a bicomodule configuration.*

3 Correspondences, fibrations and adjunctions

3.1 Decomposition-space correspondences

A *correspondence* is by definition a decomposition space $\mathcal{M}$ with a map to the 1–simplex Δ^1. We consider the slice ∞–category $\boldsymbol{Cat}_{\infty/\Delta^1}$. It contains in particular $\boldsymbol{\Delta}_{/\Delta^1}$, whose objects are $[i,j]$; see Remark 2.2.1. There is now a natural notion of nerve in this context. Given a correspondence $p\colon \mathcal{M} \to \Delta^1$, the relative nerve $N_{\Delta^1}\colon \boldsymbol{Cat}_{\infty/\Delta^1} \to \mathrm{Fun}((\boldsymbol{\Delta}_{/\Delta^1})^{\mathrm{op}}, \mathcal{S})$ of p is the augmented bisimplicial ∞–groupoid given by $B_{i,j} := N_{\Delta^1}(p)_{i,j} = \mathrm{Map}_{/\Delta^1}([i,j], p)$, where $[i,j]$ is given by the map $\Delta^{i+1+j} \to \Delta^1$ sending the $i+1$ first vertices to 0 and the others to 1. It is allowed for i or j to be equal to -1 but not both.

From the nerve definition, the following square is a standard mapping-space fibre sequence for slices:

$$\begin{array}{ccc} B_{i,j} & \longrightarrow & \mathrm{Map}(\Delta^{i+1+j}, \mathcal{M}) \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \mathrm{post}\, p} \\ 1 & \xrightarrow[\ulcorner [i,j] \urcorner]{} & \mathrm{Map}(\Delta^{i+1+j}, \Delta^1) \end{array}$$

Proposition 3.1.1 *Given a decomposition-space correspondence* $p\colon \mathcal{M} \to \Delta^1$, *the bisimplicial* ∞*–groupoid* B *described above enjoys the following properties*:

(1) *it is Segal in both directions*;

(2) *it is stable*;

(3) *it is augmented*;

(4) *these augmentations are culf.*

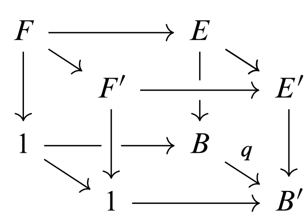
To prove these properties, we will use the following lemmas:

Lemma 3.1.2 *Given a diagram*

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle q} \\ F' & \longrightarrow & E' & \longrightarrow & B' \end{array}$$

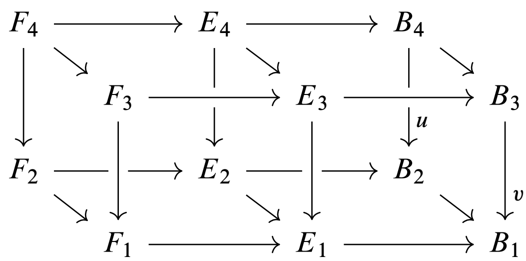
such that top and bottom are two fibre sequences, if q *is an equivalence, then the left-hand square is a pullback.*

Proof In the following cube, the front and back squares are pullbacks by assumption; the bottom one is since q is an equivalence:



The rectangle consisting of the back square and the bottom square is a pullback by Lemma 1.1.1, since both squares are pullbacks. Thus the rectangle consisting of the top square and the front one is. Applying one more time Lemma 1.1.1, we conclude the top square is a pullback. □

Lemma 3.1.3 *Given a diagram*



such that horizontal maps form fibre sequences, if the vertical middle square (involving E_i *for* $1 \leq i \leq 4$*) is a pullback and if* u *and* v *are equivalences, then the left vertical square is a pullback.*

Proof By Lemma 3.1.2, since u and v are equivalences, the front and back squares of the left cube of the diagram are pullbacks. We conclude by applying Lemma 1.1.1 twice, as in the proof of Lemma 3.1.2. □

Proof of Proposition 3.1.1 (1) Segal in both directions means, for any i, the squares

$$\begin{array}{ccc} B_{n+1,i} & \xrightarrow{e_0} & B_{n,i} \\ {\scriptstyle e_{n+1}}\downarrow & \lrcorner & \downarrow{\scriptstyle e_n} \\ B_{n,i} & \xrightarrow[e_0]{} & B_{n-1,i} \end{array} \qquad \begin{array}{ccc} B_{i,n+1} & \xrightarrow{d_0} & B_{i,n} \\ {\scriptstyle d_{n+1}}\downarrow & \lrcorner & \downarrow{\scriptstyle d_n} \\ B_{i,n} & \xrightarrow[d_0]{} & B_{i,n-1} \end{array}$$

are pullbacks.

The ∞–groupoids $B_{n,i}$ for $n \geq 0$ are also given by the fibre sequences

$$\begin{array}{ccc} B_{n,i} & \longrightarrow & \mathrm{Map}(\Delta^{n+1+i}, \mathcal{M}) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle R_{n+1+i}} \\ 1 & \xrightarrow[\ulcorner \mathrm{id} \urcorner]{} & \mathrm{Map}(\Delta^1, \Delta^1) \end{array}$$

where the right-hand map R_{n+1+j} sends $\sigma \in \mathrm{Map}(\Delta^{n+1+j}, \mathcal{M})$ to $p \circ \sigma \circ \rho_{n+1}$, where $\rho_{n+1}\colon \Delta^1 \to \Delta^{n+1+j}$ maps the arrow in Δ^1 to the $(n{+}1)^{\mathrm{st}}$ edge of Δ^{n+1+j}, that is, $\rho_{n+1} = (d^{\perp})^{n}(d^{\top})^{j}$. Indeed, in the diagram

$$
\begin{tikzcd}
B_{i,j} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathrm{Map}(\Delta^{i+1+j}, \mathcal{M}) \arrow[d, "\text{post } p"] \\
1 \arrow[r, "\ulcorner \beta_{i,j} \urcorner"'] \arrow[d] & \mathrm{Map}(\Delta^{i+1+j}, \Delta^{1}) \arrow[d, "\text{pre } \rho_{i+1}"] \\
1 \arrow[r, "\ulcorner \mathrm{id} \urcorner"'] & \mathrm{Map}(\Delta^{1}, \Delta^{1})
\end{tikzcd}
$$

the bottom square is a pullback, because the fibre of the right bottom map is contractible, thus the whole rectangle is a pullback by Lemma 1.1.1.

Using the Lemma 3.1.2, we only have to check that the front square in the cube

$$
\begin{tikzcd}
B_{n+1,i} \arrow[rr] \arrow[dr] \arrow[dd] & & B_{n+1,i} \arrow[dr] \arrow[dd] & \\
& \mathrm{Map}(\Delta^{n+1+1+i}, \mathcal{M}) \arrow[rr] \arrow[dd] & & \mathrm{Map}(\Delta^{n+1+i}, \mathcal{M}) \arrow[dd] \\
B_{n,i} \arrow[rr] \arrow[dr] & & B_{n-1,i} \arrow[dr] & \\
& \mathrm{Map}(\Delta^{n+1+i}, \mathcal{M}) \arrow[rr] & & \mathrm{Map}(\Delta^{n+i}, \mathcal{M})
\end{tikzcd}
$$

is a pullback, and apply Lemma 1.1.1. But the squares

$$
\begin{tikzcd}
\mathcal{M}_{n+1+1+i} \arrow[r, "d_{\perp}"] \arrow[d, "d_{n+1}"'] & \mathcal{M}_{n+1+i} \arrow[d, "d_{n}"] \\
\mathcal{M}_{n+1+i} \arrow[r, "d_{\perp}"'] & \mathcal{M}_{n+i}
\end{tikzcd}
$$

are pullbacks because $\mathcal{M}$ is a decomposition space, $d_{\perp}$ is an inert map, and d_{n+1} and d_{n} are always inner coface maps, thus active maps. For the remaining squares, we use that the squares

$$
\begin{tikzcd}
\mathcal{M}_{i+1+n+1} \arrow[r, "d_{i+1}"] \arrow[d, "d_{\top}"'] & \mathcal{M}_{i+1+n} \arrow[d, "d_{\top}"] \\
\mathcal{M}_{i+1+n} \arrow[r, "d_{i+1}"'] & \mathcal{M}_{i+1+n-1}
\end{tikzcd}
$$

are also pullbacks because $\mathcal{M}$ is a decomposition space.

(2) To establish the stability condition, by Lemma 2.3.3 it is enough to prove that the two following squares are pullbacks:

$$
\begin{tikzcd}
B_{0,0} & B_{0,1} \arrow[l, "d_0"'] \\
B_{1,0} \arrow[u, "e_0"] & B_{1,1} \arrow[l, "d_0"] \arrow[u, "e_0"'] \arrow[ul, phantom, "\ulcorner" very near start]
\end{tikzcd}
\qquad
\begin{tikzcd}
B_{0,0} & B_{0,1} \arrow[l, "d_1"'] \\
B_{1,0} \arrow[u, "e_1"] & B_{1,1} \arrow[l, "d_1"] \arrow[u, "e_1"'] \arrow[ul, phantom, "\ulcorner" very near start]
\end{tikzcd}
$$

We can prove this with the same strategy used above. The decomposition-space axioms used here are that the following squares are pullbacks:

$$
\begin{tikzcd}
\mathcal{M}_0 \arrow[r, "s_0"] & \mathcal{M}_1 \\
\mathcal{M}_1 \arrow[u, "d_\bot"] \arrow[r, "s_1"'] \arrow[ur, phantom, "\urcorner" very near start] & \mathcal{M}_2 \arrow[u, "d_\bot"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{M}_0 \arrow[r, "s_0"] & \mathcal{M}_1 \\
\mathcal{M}_1 \arrow[u, "d_\top"] \arrow[r, "s_0"'] \arrow[ur, phantom, "\urcorner" very near start] & \mathcal{M}_2 \arrow[u, "d_\top"']
\end{tikzcd}
$$

(3)–(4) The augmentations $Y_j := B_{-1,j}$ and $X_i := B_{i,-1}$ are also given by the fibre sequences

$$
\begin{tikzcd}
Y_j \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \operatorname{Map}(\Delta^j, \mathcal{M}) \arrow[d, "S"] \\
1 \arrow[r, "\ulcorner d_0 \urcorner"'] & \operatorname{Map}(\Delta^0, \Delta^1)
\end{tikzcd}
\qquad
\begin{tikzcd}
X_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \operatorname{Map}(\Delta^i, \mathcal{M}) \arrow[d, "T"] \\
1 \arrow[r, "\ulcorner d_1 \urcorner"'] & \operatorname{Map}(\Delta^0, \Delta^1)
\end{tikzcd}
$$

where the map S sends σ to $p \circ \sigma \circ (d^\top)^j$ and the map T sends σ to $p \circ \sigma \circ (d^\bot)^i$. Since the squares

$$
\begin{tikzcd}
\operatorname{Map}(\Delta^{i+1+j}, \mathcal{M}) \arrow[r, "(d_\bot)^{i+1}"] \arrow[d, "R_{i+1+j}"'] & \operatorname{Map}(\Delta^j, \mathcal{M}) \arrow[d, "S"] \\
\operatorname{Map}(\Delta^1, \Delta^1) \arrow[r, "d_\bot"'] & \operatorname{Map}(\Delta^0, \Delta^1)
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{Map}(\Delta^{i+1+j}, \mathcal{M}) \arrow[r, "(d_\top)^{j+1}"] \arrow[d, "R_{i+1+j}"'] & \operatorname{Map}(\Delta^i, \mathcal{M}) \arrow[d, "T"] \\
\operatorname{Map}(\Delta^1, \Delta^1) \arrow[r, "d_\top"] & \operatorname{Map}(\Delta^0, \Delta^1)
\end{tikzcd}
$$

commute, it is enough to define maps $B_{i,j} \to Y_j$ and $B_{i,j} \to X_i$.

The augmentation maps are culf: we need to prove that the back squares of the following cubes are pullbacks:

$$
\begin{tikzcd}[column sep=small]
Y_0 \arrow[rr, "s_0"] \arrow[dr] & & Y_1 \arrow[dr] & \\
& \operatorname{Map}(\Delta^0, \mathcal{M}) \arrow[rr, "s_0"] & & \operatorname{Map}(\Delta^1, \mathcal{M}) \\
B_{0,0} \arrow[uu, "d_0"] \arrow[rr, "s_0"] \arrow[dr] & & B_{0,1} \arrow[uu, "d_0"'] \arrow[dr] & \\
& \operatorname{Map}(\Delta^1, \mathcal{M}) \arrow[uu, "d_0"] \arrow[rr, "s_1"] & & \operatorname{Map}(\Delta^2, \mathcal{M}) \arrow[uu, "d_0"']
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
Y_1 \arrow[dr] & & Y_2 \arrow[ll, "d_1"'] \arrow[dr] & \\
& \operatorname{Map}(\Delta^1, \mathcal{M}) & & \operatorname{Map}(\Delta^2, \mathcal{M}) \arrow[ll, "d_1"'] \\
B_{0,1} \arrow[uu, "d_0"] \arrow[dr] & & B_{0,2} \arrow[ll, "d_1"'] \arrow[uu, "d_0"'] \arrow[dr] & \\
& \operatorname{Map}(\Delta^2, \mathcal{M}) \arrow[uu, "d_0"] & & \operatorname{Map}(\Delta^3, \mathcal{M}) \arrow[ll, "d_2"'] \arrow[uu, "d_0"']
\end{tikzcd}
$$

We can apply Lemma 3.1.3 since the front square is a pullback because $\mathcal{M}$ is a decomposition space. □

To summarise, given a decomposition-space correspondence $p\colon \mathcal{M} \to \Delta^1$, we get a bicomodule configuration and then $\mathcal{S}_{/B_{0,0}}$ is a bicomodule by Theorem 2.4.1.

3.2 Cocartesian and cartesian fibrations of decomposition spaces

Ayala and Francis [4] formulate a homotopy-invariant definition of cartesian and cocartesian fibrations so it can be equally well formulated in any model for ∞–categories. We adapt here those definitions to decomposition spaces.

Let $p\colon X \to Y$ be a simplicial map between decomposition spaces. A morphism $\Delta^1 \xrightarrow{\langle s \xrightarrow{a} t\rangle} X$ is *p–cocartesian* if the diagram of coslices of decomposition spaces

$$\begin{array}{ccc} {}^{a/}X & \longrightarrow & {}^{s/}X \\ \downarrow & & \downarrow \\ {}^{pa/}Y & \longrightarrow & {}^{ps/}Y \end{array}$$

is a pullback, where the coslice ${}^{s/}X$ is given by pullback of the lower décalage $\mathrm{Dec}_\perp(X)$,

$$\begin{array}{ccc} ({}^{s/}X)_n & \longrightarrow & \mathrm{Dec}_\perp(X)_n \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (d_\top)^{n+1}} \\ 1 & \xrightarrow[\ulcorner s \urcorner]{} & X_0 \end{array}$$

and similarly the coslice ${}^{a/}X$ is given by pullback of the $\mathrm{Dec}_\perp(\mathrm{Dec}_\perp(X))$,

$$\begin{array}{ccc} ({}^{a/}X)_n & \longrightarrow & \mathrm{Dec}_\perp(\mathrm{Dec}_\perp(X))_n \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (d_\top)^{n+1}} \\ 1 & \xrightarrow[\ulcorner a \urcorner]{} & X_1 \end{array}$$

and the functor ${}^{a/}X \to {}^{s/}X$ is given by $\mathrm{Dec}_\perp(d_\perp)$, where the simplicial map

$$d_\perp\colon \mathrm{Dec}_\perp(X) \to X$$

is given by the original d_0.

The functor $p\colon X \to Y$ is a *cocartesian fibration* if any diagram of solid arrows

$$\begin{array}{ccc} \Delta^0 & \longrightarrow & X \\ {\scriptstyle d_1}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ \Delta^1 & \longrightarrow & Y \end{array}$$

admits a p–cocartesian diagonal filler.

Similarly, a morphism $\Delta^1 \xrightarrow{\langle s \xrightarrow{a} t\rangle} X$ is *p–cartesian* if the diagram of slice decomposition spaces

$$\begin{array}{ccc} X_{/a} & \longrightarrow & X_{/t} \\ \big\downarrow & & \big\downarrow \\ Y_{/pa} & \longrightarrow & Y_{/pt} \end{array}$$

is a pullback, where the slice $X_{/t}$ is given by pullback of the upper décalage $\mathrm{Dec}_\top(X)$, the slice $X_{/a}$ is given by pullback of $\mathrm{Dec}_\top(\mathrm{Dec}_\top(X))$, and the functor $X_{/a} \to X_{/t}$ is given by $\mathrm{Dec}_\top(d_\top) = d_{\top-1}$, where $d_\top\colon \mathrm{Dec}_\top(X) \to X$ is given by the original $d_\top$.

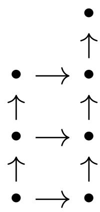
The functor $p\colon X \to Y$ is a *cartesian fibration* if any diagram of solid arrows

$$\begin{array}{ccc} \Delta^0 & \longrightarrow & X \\ {\scriptstyle d_0}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ \Delta^1 & \longrightarrow & Y \end{array}$$

admits a p–cartesian diagonal filler.

Bisimplex category with diagonal maps We define $\overline{[i,j]}\colon M_{\phi_{i,j}} \to \Delta^1$ to be the canonical projection from the mapping cylinder of $\phi_{i,j} := (d^\top)^j\colon \Delta^i \to \Delta^{i+j}$; it is a cocartesian fibration. They assemble into a category, denoted by $\overline{\mathbf{\Delta}}_{/\Delta^1}$, of shape like $\mathbf{\Delta}_{/\Delta^1}$, but with extra diagonal maps $d\colon \overline{[i-1,j]} \to \overline{[i,j-1]}$ given by inclusion. These satisfy new simplicial identities $\sigma_k d = d\sigma_{k+1}$ for $0 \le k \le j$, where σ_k are degeneracy maps "on j" (horizontal) and d are diagonal maps, and, with face maps, $d\delta_{k+1} = \delta_k d$ for $0 \le k \le j$, where δ_k are horizontal face maps. Similarly for degeneracy maps τ_k "on i" (vertical), $\tau_k d = d\tau_k$ for $0 \le k < i$ and $d\epsilon_k = \epsilon_k d$ for

$0 \le k < i$, where ϵ_k are vertical face maps. For example, we can draw $\overline{[2,1]}$ as

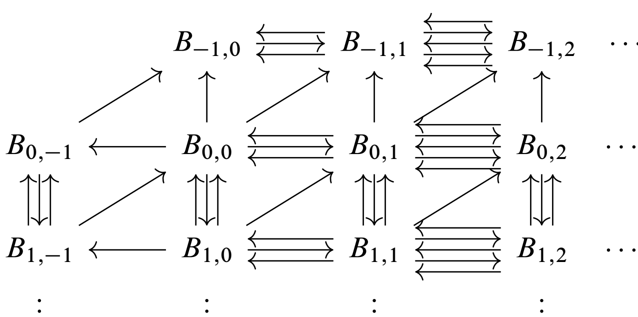


where the horizontal maps lie over the map in Δ^1. It is like a cocartesian version of the earlier drawing of Remark 2.2.1.

Remark 3.2.1 We can draw $\overline{[i,j]}$ as a column of $i+1$ black dots followed by $j+1$ white dots. Whereas arrows in $\mathbf{\Delta}_{/\Delta^1}$ send black dots to black dots and white dots to white dots (without crossing), in $\overline{\mathbf{\Delta}}_{/\Delta^1}$ we allow moreover to map white dots to black dots.

There is a natural notion of nerve in the context of cocartesian fibrations over Δ^1: given a cocartesian fibration $p\colon \mathcal{M} \to \Delta^1$ between decomposition spaces, define the cocartesian nerve $N_{\text{cocart}}\colon \boldsymbol{Cat}^{\text{cocart}}_{\infty/\Delta^1} \to \text{Fun}((\overline{\mathbf{\Delta}}_{/\Delta^1})^{\text{op}}, \mathcal{S})$ by $N_{\text{cocart}}(p)_{\overline{i,j}} := \text{Map}^{\text{cocart}}_{/\Delta^1}(\overline{[i,j]}, \mathcal{M})$, the mapping space preserving cocartesian arrows.

Similarly to the previous Section 3.1, we get a bicomodule configuration and $\mathcal{S}_{/B_{0,0}}$ is a bicomodule over $\mathcal{S}_{/X_1}$ and $\mathcal{S}_{/Y_1}$. We have here moreover diagonal maps $B_{i,j-1} \to B_{i-1,j}$ and new sections $s_{-1}\colon B_{i,j-1} \to B_{i,j}$ for $i \ge 0$ given by the composition with a diagonal map. That is, $\mathcal{S}_{/B_{0,0}}$ is pointed as a right comodule over $\mathcal{S}_{/Y_1}$:



We now adapt Lurie's definition of adjunction of ∞–categories [22] to decomposition spaces.

An *adjunction* between decomposition spaces X and Y is a simplicial map between decomposition spaces $p\colon \mathcal{M} \to \Delta^1$ which is both a cartesian and a cocartesian fibration together with equivalences $X \simeq \mathcal{M}_{\{0\}}$ and $Y \simeq \mathcal{M}_{\{1\}}$.

Proposition 3.2.2 *Given an adjunction $p\colon \mathcal{M} \to \Delta^1$, the bisimplicial ∞–groupoid B described above is a bicomodule configuration. Moreover, $\mathcal{S}_{/B_{0,0}}$ is pointed as a right comodule over $\mathcal{S}_{/Y_1}$, and as a left comodule over $\mathcal{S}_{/X_1}$.*

Proof By Proposition 3.1.1, the bisimplicial ∞–groupoid B is a bicomodule configuration. The pointings are given by the cartesian and cocartesian conditions. □

Adjunctions between ∞–categories are examples of adjunctions between decomposition spaces.

Example 3.2.3 To illustrate these abstract concepts, let us spell out the bicomodule configuration associated to an ordinary adjunction of 1–categories $F\colon X \rightleftarrows Y :G$. The ∞–groupoid $B_{0,0}$ is now just the set of arrows $Fx \to y$, which by adjunction correspond to arrows $x \to Gy$, and $B_{1,0}$ is the set of composable pair $Fx \to Fx' \to y$ (which is the same as $x \to x' \to Gy$). In general $B_{i,j}$ is the set of chains of composable arrows $Fx_0 \to \cdots \to Fx_i \to y_j \to \cdots \to y_0$. These chains can be drawn as in the figure in Remark 2.2.1, where the horizontal arrow is a "mixed" arrow $Fx \to y$. This drawing can be filled as in the figure on page 200, and thus giving a right pointing by the rearrangement

$$\begin{array}{ccc} Fx' \longrightarrow y & & y \\ \uparrow & \longmapsto & \uparrow \\ Fx & & Fx \longrightarrow Fx' \end{array}$$

Similarly the functor $G\colon Y \to X$ induces a left pointing on the equivalent (by adjunction) augmented simplicial set of chains $x_0 \to \cdots \to x_i \to Gy_j \to \cdots \to Gy_0$.

4 Möbius inversion for comodules and a Rota formula

4.1 Finiteness and cardinality

An ∞–groupoid X is *locally finite* if at each basepoint x the homotopy groups $\pi_i(X,x)$ are finite for $i \geq 1$ and are trivial for i sufficiently large. It is called *finite* if furthermore it has only finitely many components. We denote by $\mathcal{F}$ (following the notation of [14]) the ∞–category of finite ∞–groupoids. A map is *finite* if each fibre is finite. A pullback of any homotopy finite map is again finite. A span $I \xleftarrow{p} M \xrightarrow{q} J$ and the corresponding linear functor $\mathcal{S}_{/I} \to \mathcal{S}_{/J}$ are *finite* if the map p is finite.

A decomposition space X is *locally finite* if X_1 is locally finite and both s_0 and d_1 are finite maps [14, Section 7.4].

Proposition 4.1.1 [15, Proposition 4.3] *Let I, J and M be locally finite ∞–groupoids and $I \xleftarrow{p} M \xrightarrow{q} J$ a finite span. Then the induced finite linear functor $\mathcal{S}_{/I} \to \mathcal{S}_{/J}$ restricts to $\mathcal{F}_{/I} \to \mathcal{F}_{/J}$.*

The cardinality [5] of a finite ∞–groupoid X is the alternating product of cardinalities of the homotopy groups,

$$|X| = \sum_{x \in \pi_0(X)} \prod_{k=1}^{\infty} |\pi_k(X, x)|^{(-1)^k}.$$

For a locally finite ∞–groupoid S, there is a notion of cardinality $|-|\colon \mathcal{F}_{/S} \to \mathbb{Q}_{\pi_0 S}$ sending a basis element $\ulcorner s \urcorner$ to the basis element $\delta_s = |\ulcorner s \urcorner|$.

For any locally finite decomposition space X, we can take the cardinality of the linear functors $\delta\colon \mathcal{F}_{/X_1} \to \mathcal{F}$ and $\Delta\colon \mathcal{F}_{/X_1} \to \mathcal{F}_{/X_1 \times X_1}$ to obtain a coalgebra structure

$$\mathbb{Q}_{\pi_0 X_1} \xrightarrow{|\delta|} \mathbb{Q}, \quad \mathbb{Q}_{\pi_0 X_1} \xrightarrow{|\Delta|} \mathbb{Q}_{\pi_0 X_1} \otimes \mathbb{Q}_{\pi_0 X_1},$$

called the *numerical incidence coalgebra* of X; see [14, Section 7.7].

4.2 Completeness and Möbius condition

A decomposition space is called *complete* if $s_0\colon X_0 \to X_1$ is a monomorphism [14, Section 2]. Since s_0 is a monomorphism, we can identify X_0 with an ∞–subgroupoid of X_1. We denote by X_a its complement: $X_1 \simeq X_0 + X_a$. More generally, recall the word notation from [14]: consider the alphabet with three letters $\{0, 1, a\}$; 0 indicates degenerate edges $s_0(x) \in X_1$, a denotes edges specified to be nondegenerate, and 1 denotes unspecified edges. For w a word of length n in this alphabet, define

$$X^w = \prod_{i \in w} X_i \subset (X_1)^n.$$

This inclusion is full since $X_a \subset X_1$ is full by completeness.

Denote by X_w the ∞–groupoid of n–simplices whose principal edges have the types indicated in the word w, that is, the full subgroupoid of X_n given by the pullback

$$\begin{array}{ccc} X_w & \longrightarrow & X_n \\ \big\downarrow & \lrcorner & \big\downarrow \\ X^w & \longrightarrow & (X_1)^n \end{array}$$

We define $\vec{X}_n = X_{a\ldots a} \subset X_n$ to be the full subgroupoid of simplices with all principal edges nondegenerate. It is the complement of the union of the essential images of the degeneracy maps $s_i\colon X_{n-1} \to X_n$, that is,

$$\vec{X}_n = X_n \setminus \bigcup_{i=0}^{n-1} \mathrm{Im}(s_i).$$

By definition, $\vec{X}_0 = X_0$.

For a complete decomposition space, the spans $X_1 \xleftarrow{d_1^{n-1}} \vec{X}_n \to 1$ define linear functors, the *Phi functors*

$$\Phi_n\colon \mathcal{S}_{/X_1} \to \mathcal{S}.$$

We also put $\Phi_{\text{even}} := \sum_{n \text{ even}} \Phi_n$ and $\Phi_{\text{odd}} := \sum_{n \text{ odd}} \Phi_n$.

The incidence algebra of a decomposition space contains the *zeta functor*

$$\zeta\colon \mathcal{S}_{/X_1} \to \mathcal{S}$$

given by the span $X_1 \xleftarrow{=} X_1 \to 1$.

Theorem 4.2.1 [14, Theorem 3.8] *For a complete decomposition space, the following Möbius inversion holds:*

$$\begin{array}{c} \zeta * \Phi_{\text{even}} \simeq \delta + \zeta * \Phi_{\text{odd}} \\ \wr\vert \\ \Phi_{\text{even}} * \zeta \simeq \delta + \Phi_{\text{odd}} * \zeta. \end{array}$$

This is however not enough to allow the Möbius inversion formula to descend to the vector space level. A complete decomposition space X is of *locally finite length* [14] if every edge $f \in X_1$ has finite length, that is, the fibres $(\vec{X}_n)_f$ of $d_1^{(n)}\colon \vec{X}_n \to X_1$ over f are empty for n sufficiently large.

A *Möbius decomposition space* [14] is a decomposition space which is locally finite and of locally finite length; the fibre $(\vec{X}_n)_f$ is finite (eventually empty). It follows that the map

$$\sum_n d_1^{n-1}\colon \sum_n \vec{X}_n \to X_1$$

is finite; by Proposition 4.1.1, the Phi functors descend to

$$\Phi_n\colon \mathcal{F}_{/X_1} \to \mathcal{F}$$

and we can take cardinality to obtain functions $|\Phi_n|\colon \mathbb{Q}_{\pi_0 X_1} \to \mathbb{Q}$.

Finally, we can take cardinality of the abstract Möbius inversion formula of Theorem 4.2.1; see [14] for a complete exposition.

Theorem 4.2.2 [14, Theorem 8.9] *If X is a Möbius decomposition space, then the cardinality of the zeta functor, $|\zeta|\colon \mathbb{Q}_{\pi_0 X_1} \to \mathbb{Q}$, is convolution invertible with inverse $|\mu| := |\Phi_{\text{even}}| - |\Phi_{\text{odd}}|$:*

$$|\zeta| * |\mu| = |\delta| = |\mu| * |\zeta|.$$

4.3 Right and left convolutions

We introduce left and right convolution actions as dual to the comodule structures. Explicitly, given a right comodule configuration $C \to Y$, we get a right comodule $\mathcal{S}_{/C_0}$ over $\mathcal{S}_{/Y_1}$. The *right convolution* $\theta \star_r \beta$ of the two functors $\theta\colon \mathcal{S}_{/C_0} \to \mathcal{S}$ and $\beta\colon \mathcal{S}_{/Y_1} \to \mathcal{S}$, given by the spans $C_0 \leftarrow M \to 1$ and $Y_1 \leftarrow N \to 1$, is the composite of $\theta \otimes \beta$ with the right coaction γ_r,

$$\theta \star_r \beta\colon \mathcal{S}_{/C_0} \xrightarrow{\gamma_r} \mathcal{S}_{/C_0} \otimes \mathcal{S}_{/Y_1} \xrightarrow{\theta\otimes\beta} \mathcal{S},$$

where the tensor product $\theta \otimes \beta$ is given by the span $C_0 \times Y_1 \leftarrow M \times N \to 1$.

Similarly, given a left comodule configuration, we can define the *left convolution* $\alpha \star_l \theta$ of $\alpha\colon \mathcal{S}_{/X_1} \to \mathcal{S}$ and $\theta\colon \mathcal{S}_{/C_0} \to \mathcal{S}$,

$$\alpha \star_l \theta\colon \mathcal{S}_{/C_0} \xrightarrow{\gamma_l} \mathcal{S}_{/X_1} \otimes \mathcal{S}_{/C_0} \xrightarrow{\alpha\otimes\theta} \mathcal{S}.$$

If we have a bicomodule configuration, then the following associativity formula expresses the compatibility of coactions from Theorem 2.4.1:

Corollary 4.3.1 *Given a bicomodule configuration, the convolutions defined above satisfy*

$$\alpha \star_l (\theta \star_r \beta) \simeq (\alpha \star_l \theta) \star_r \beta.$$

4.4 Möbius inversion for (co)modules

Let $C \to Y$ be a comodule configuration. The zeta functor

$$\zeta^C \colon \mathcal{S}_{/C_0} \to \mathcal{S}$$

is the linear functor defined by the span

$$C_0 \xleftarrow{=} C_0 \to 1.$$

Let $C \to Y$ be a right pointed comodule configuration. The augmented simplicial ∞–groupoid C is an object of the functor ∞–category

$$\mathrm{Fun}\,(\boldsymbol{\Delta}_{\mathrm{bot}}^{\mathrm{op}}, \mathcal{S}),$$

where $\boldsymbol{\Delta}_{\mathrm{bot}}$ is the simplex category of finite linear orders with a specified bottom element, and with monotone maps preserving the bottom element. The forgetful functor $\boldsymbol{\Delta}_{\mathrm{bot}} \to \boldsymbol{\Delta}$ is right adjoint to the functor $j \colon \boldsymbol{\Delta} \to \boldsymbol{\Delta}_{\mathrm{bot}}$ adding a bottom element.

Remark 4.4.1 In the situation where Y is Segal and $C = \mathrm{Dec}_\perp Y$, we can take C_{-1} to be Y_0, with d_0 as augmentation map. By [22, Lemma 6.1.3.16], this is a colimit diagram.

A right pointed comodule configuration $f \colon C \to Y$ is *complete* if the new degeneracies $s_{-1} \colon C_{n-1} \to C_n$ are monomorphisms. Since s_{-1} is a monomorphism, we can identify C_{-1} with a ∞–subgroupoid of C_0. We denote by C_b its complement: $C_0 = C_{-1} + C_b$. Denote by C_{vw} the ∞–groupoid of simplices whose principal edges have the type indicated in the word vw, where $v \in \{-1, 0, b\}$ and w is a word in the alphabet $\{0, 1, a\}$, that is, the full ∞–subgroupoid of C_n given by the pullback

$$\begin{array}{ccc} C_{vw} & \longrightarrow & C_n \\ \big\downarrow & \lrcorner & \big\downarrow \\ C_v \times Y^w & \longrightarrow & C_0 \times (Y_1)^n \end{array}$$

where $n = |w| \geq 0$. The principal edges of the ∞–groupoid C_n consist of an element in C_0 given by $(d_\top)^n$, and n edges in Y_1, the principal edges of the image of C_n by f. In this situation, we define $\vec{C}_n = C_{ba\ldots a} \subset C_n$ to be the full subgroupoid of simplices with all principal edges nondegenerate. It is given by the pullback diagram

$$\begin{array}{ccc} C_{ba\dots a} & \longrightarrow & C_n \\ \big\downarrow & \lrcorner & \big\downarrow \\ C_b \times Y^{a\dots a} & \longrightarrow & C_0 \times Y_1^n \end{array}$$

Define

$$\delta^R\colon \mathcal{S}_{/C_0} \to \mathcal{S}$$

to be the linear functor given by the span

$$C_0 \xleftarrow{s_{-1}} C_{-1} \to 1$$

and define the *right Phi functors*

$$\Phi_n^R\colon \mathcal{S}_{/C_0} \to \mathcal{S}$$

to be the linear functors given by the spans

$$C_0 \leftarrow \vec{C}_n \to 1.$$

If $n=-1$, then $\vec{C}_{-1} = C_{-1}$ (by convention) and Φ_{-1}^R is the linear functor δ^R.

Lemma 4.4.2 *For every word w in the alphabet $\{0,1,a\}$, the following square is a pullback:*

$$\begin{array}{ccc} C_{0w} & \longrightarrow & C_1 \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle (d_\top, f)} \\ C_0 \times Y_w & \longrightarrow & C_0 \times Y_1 \end{array}$$

Proof Let $n=|w|$. The square is the top rectangle of the diagram

$$\begin{array}{ccccc} C_{0w} & \longrightarrow & C_n & \longrightarrow & C_1 \\ \big\downarrow & & \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle (d_\top, f)} \\ C_0 \times Y_w & \longrightarrow & C_0 \times Y_n & \longrightarrow & C_0 \times Y_1 \\ \big\downarrow & \lrcorner & \big\downarrow & & \\ C_0 \times Y^w & \longrightarrow & C_0 \times Y_1^n & & \end{array}$$

The bottom square and left-hand rectangle are pullbacks by definition of Y_w and C_{0w}, hence the top left-hand square is a pullback. The right-hand square is a pullback because the augmentation map $C \to Y$ is culf. Hence the top rectangle, which is the desired square, is a pullback. □

Given a complete decomposition space Y, we denote by $\Phi_n^Y\colon \mathcal{S}_{/Y_1} \to \mathcal{S}$ the usual Phi functors; see Section 4.1 above.

Proposition 4.4.3 *The right Phi functors satisfy*

$$\zeta^C \star_r \Phi_n^Y \simeq \Phi_{n-1}^R + \Phi_n^R.$$

Proof Compute the convolution action $\zeta^C \star_r \Phi_n^Y$ by Lemma 4.4.2 as

$$\begin{array}{ccccc} C_0 & \nwarrow & & & \\ \uparrow & & & & \\ C_1 & \longleftarrow & C_{0a\dots a} & & \\ \downarrow & \llcorner & \downarrow & \searrow & \\ C_0 \times Y_1 & \longleftarrow & C_0 \times \vec{Y}_n & \longrightarrow & 1 \end{array}$$

But $C_{0a\dots a} \simeq C_{-1a\dots a} + C_{ba\dots a} \simeq \vec{C}_{n-1} + \vec{C}_n$. This is an equivalence of ∞–groupoids over C_0 and the resulting span is $\Phi_{n-1}^R + \Phi_n^R$. □

Write

$$\Phi_{\mathrm{even}}^Y := \sum_{n \text{ even}} \Phi_n^Y, \quad \Phi_{\mathrm{odd}}^Y := \sum_{n \text{ odd}} \Phi_n^Y.$$

The previous proposition implies the following Möbius inversion formula:

Theorem 4.4.4 *Given $C \to Y$ a complete right pointed comodule configuration,*

$$\zeta^C \star_r \Phi_{\mathrm{even}}^Y \simeq \delta^R + \zeta^C \star_r \Phi_{\mathrm{odd}}^Y.$$

Proof The two linear functors are equivalent to the sum of the right Phi functors:

$$\zeta^C \star_r \Phi_{\mathrm{even}}^Y \simeq \Phi_{-1}^R + \Phi_0^R + \Phi_1^R + \cdots \simeq \delta^R + \zeta^C \star_r \Phi_{\mathrm{odd}}^Y.$$ □

We can also define a *left pointed* comodule configuration $D \to X$, with new top sections instead of bottom: we consider instead the mapping cylinder of $\mathbf{\Delta} \to \mathbf{\Delta}_{\mathrm{top}}$, where $\mathbf{\Delta}_{\mathrm{top}}$ is the simplex category of finite linear orders with a specified top element, and with monotone maps preserving the top element. A left pointed comodule configuration is *complete* if the new degeneracies $t_{\top+1}\colon D_{n-1} \to D_n$ are monomorphisms. Similarly, we define the *left Phi functors* and δ^L using $t_{\top+1}$ and $e_\top$ and we obtain the following formula:

Theorem 4.4.5 *Given $D \to X$ a complete left pointed comodule configuration,*

$$\Phi_{\mathrm{even}}^X \star_l \zeta^D \simeq \delta^L + \Phi_{\mathrm{odd}}^X \star_l \zeta^D.$$

4.5 Möbius bicomodule configurations and the Rota formula

In order to take homotopy cardinality to recover the usual Möbius inversions, we need to impose some finiteness conditions. We adapt the approach of [14] summarised in Sections 4.1 and 4.2 above.

A *right Möbius comodule configuration* is a complete right pointed comodule configuration $C \to Y$ such that the decomposition space Y is Möbius and the augmented comodule is Möbius, that is:

- C is locally finite: the ∞–groupoid C_0 is locally finite and both s_{-1} and d_0 are finite maps.
- C is of locally finite length: every edge has a finite length, that is for all $a \in C_0$, the fibres of $d_0^{(n)}\colon \vec{C}_n \to C_0$ over a are empty for n sufficiently large.

Under these conditions, the Phi functors descend to

$$\Phi_n^R\colon \mathcal{F}_{/C_0} \to \mathcal{F}$$

and we can now take the cardinality of the "Möbius formulas" (Theorems 4.4.4 and 4.4.5).

Similarly, we define a *left Möbius comodule configuration* to be a complete left pointed comodule configuration $D \to X$ such that the decomposition space X is Möbius and the augmented comodule is Möbius, using $t_{\top+1}$ and $d_\top$.

Theorem 4.5.1 *Given $C \to Y$ a right Möbius comodule configuration and $D \to X$ a left Möbius comodule configuration,*

$$|\zeta^C| \star_r |\mu^Y| = |\delta^R|, \quad |\mu^X| \star_l |\zeta^D| = |\delta^L|,$$

where $|\mu^Y| := |\Phi^Y_{\text{even}}| - |\Phi^Y_{\text{odd}}|$ and $|\mu^X| := |\Phi^X_{\text{even}}| - |\Phi^X_{\text{odd}}|$.

A *Möbius bicomodule configuration* is a bicomodule configuration with two pointings such that both left and right comodule configurations are Möbius. It hence has extra degeneracy maps in both directions: extra bottom degeneracy maps in the horizontal direction and extra top degeneracy maps in the vertical direction.

Note that given a Möbius bicomodule configuration B, the zeta functors are defined only on the ∞–groupoid $B_{0,0}$ and then are the same for the two comodules. In both cases it is given by the span

$$B_{0,0} \xleftarrow{=} B_{0,0} \to 1.$$

Theorem 4.5.2 *Given a Möbius bicomodule configuration B with $X := B_{\bullet,-1}$ and $Y := B_{-1,\bullet}$, we have*

$$|\mu^X| \star_l |\delta^R| = |\delta^L| \star_r |\mu^Y|,$$

where δ^R is the linear functor given by the span

$$B_{0,0} \leftarrow X_0 \rightarrow 1$$

and δ^L is the linear functor given by the span

$$B_{0,0} \leftarrow Y_0 \rightarrow 1.$$

Proof Using the Möbius formulas at the algebraic level from Theorem 4.5.1, and the associativity of the convolution actions from Corollary 4.3.1, we compute

$$|\mu^X| \star_l |\delta^R| = |\mu^X| \star_l (|\zeta| \star_r |\mu^Y|) = (|\mu^X| \star_l |\zeta|) \star_r |\mu^Y| = |\delta^L| \star_r |\mu^Y|. \qquad \square$$

4.6 Möbius bicomodule configurations from adjunctions of Möbius decomposition spaces

We saw in Section 3.2 that given a cocartesian fibration $p\colon \mathcal{M} \to \Delta^1$ between decomposition spaces, we obtain a right comodule configuration B, with diagonal maps $B_{i,j-1} \to B_{i-1,j}$ and new sections $s_{-1}\colon B_{i,j-1} \to B_{i,j}$ for $i \geq 0$ given by the composition with a diagonal map.

Lemma 4.6.1 *Given a cocartesian fibration $p\colon \mathcal{M} \to \Delta^1$ between decomposition spaces, suppose moreover that $\mathcal{M}$ is complete. Then the associated right pointed comodule configuration is complete.*

Proof The new sections will be monomorphisms if the following square is a pullback:

$$\begin{array}{ccc} B_{i,j-1} & \xrightarrow{\text{id}} & B_{i,j-1} \\ {\scriptstyle \text{id}}\downarrow & & \downarrow{\scriptstyle s_{-1}} \\ B_{i,j-1} & \xrightarrow[s_{-1}]{} & B_{i,j} \end{array}$$

By assumption, $\mathcal{M}$ is a complete decomposition space, hence all degeneracy maps are monomorphisms, and we can apply Lemma 3.1.3, to obtain the desired pullbacks. $\square$

Instantiating the general definitions from Section 4.4, the zeta functor

$$\zeta\colon \mathcal{S}_{/B_{0,0}} \to \mathcal{S}$$

is given by the span

$$B_{0,0} \xleftarrow{=} B_{0,0} \to 1,$$

and the functor

$$\delta^R\colon \mathcal{S}_{/B_{0,0}} \to \mathcal{S}$$

is defined by the span

$$B_{0,0} \xleftarrow{s_{-1}} B_{0,-1} \to 1.$$

The right comodule configuration being complete, we get a Möbius inversion formula (Theorem 4.4.4),

$$\zeta \star_r \Phi^Y_{\mathrm{even}} \simeq \delta^R + \zeta \star_r \Phi^Y_{\mathrm{odd}},$$

where $Y := B_{-1,\bullet}$.

Similarly, given a cartesian fibration $p\colon \mathcal{M} \to \Delta^1$ between decomposition spaces, we obtain a left pointed comodule configuration.

Lemma 4.6.2 *Given a cartesian fibration $p\colon \mathcal{M} \to \Delta^1$ between decomposition spaces, suppose moreover that $\mathcal{M}$ is complete. Then the left pointed comodule configuration is complete.*

The functor

$$\delta^L\colon \mathcal{S}_{/B_{0,0}} \to \mathcal{S}$$

is given by the span

$$B_{0,0} \leftarrow B_{-1,0} \to 1.$$

This leads to the Möbius inversion formula

$$\Phi^X_{\mathrm{even}} \star_l \zeta \simeq \delta^L + \Phi^X_{\mathrm{odd}} \star_l \zeta.$$

Given an adjunction between decomposition spaces, that is, a simplicial map $\mathcal{M} \to \Delta^1$ which is both cartesian and cocartesian, and suppose that $\mathcal{M}$ is complete; then we just obtained two Möbius inversion formulas.

Theorem 4.6.3 *Given an adjunction of decomposition spaces in the form of a bicartesian fibration $p\colon \mathcal{M} \to \Delta^1$, suppose moreover that $\mathcal{M}$ is a Möbius decomposition space. Then the bicomodule configuration extracted from this data is Möbius. In particular, we have the Rota formula for the adjunction p,*

$$|\mu^X| \star_l |\delta^R| = |\delta^L| \star_r |\mu^Y|.$$

Proof First observe that $B_{0,0}$, and in fact all $B_{i,j}$, are locally finite since they are given by pullback (see page 194) of locally finite spaces. Second, note that $e_\top\colon B_{i+1,j} \to B_{i,j}$ is induced in the same way from the face map $d_i\colon \mathcal{M}_{i+2+j} \to \mathcal{M}_{i+1+j}$, which is an inner face map, and is therefore finite since $\mathcal{M}$ is Möbius. Similarly, $d_0\colon B_{i,j+1} \to B_{i,j}$ is obtained from $d_{i+1}\colon \mathcal{M}_{i+2+j} \to \mathcal{M}_{i+1+j}$ which is also an inner face map. Finally, the fibres of $e_\top^{(n)}$ are empty for n sufficiently large because the fibres of $d_{i-n+1} \circ \cdots \circ d_i\colon \mathcal{M}_{i+2+j} \to \mathcal{M}_{i+2-n+j}$ are empty for n sufficiently large since $\mathcal{M}$ is Möbius. Similarly, the fibres of $d_0^{(n)}$ are empty for n sufficiently large. □

4.7 The Möbius function of the incidence algebra of the decomposition space of finite posets

We come back to the bicomodule configuration $\boldsymbol{B}$ of layered sets and posets given in Section 2.5.

Proposition 4.7.1 [7, Proposition 3.3] *The bicomodule configuration $\boldsymbol{B}$ is Möbius.*

Remark 4.7.2 This Möbius bicomodule configuration does not come from an adjunction of decomposition spaces.

We can now apply the generalised Rota formula of Theorem 4.5.2 to compute the Möbius function of the bialgebra of finite posets. The result is known [1], but its derivation from the generalised Rota formula is new and interesting.

Theorem 4.7.3 [7, Theorem 3.4] *The Möbius function of the incidence algebra of the decomposition space $\boldsymbol{C}$ of finite posets is*

$$\mu(P) = \begin{cases} (-1)^n & \text{if } P \in \boldsymbol{C}_1 \text{ is a discrete poset with } n \text{ elements,} \\ 0 & \text{else.} \end{cases}$$

This result can be extended to the incidence algebra of any directed restriction species, including rooted forests, and free operads [7].

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