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On the Hikami–Inoue conjecture

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Given a braid presentation D of a hyperbolic knot, Hikami and Inoue consider a system of polynomial equations arising from a sequence of cluster mutations determined by D. They show that any solution gives rise to shape parameters and thus determines a boundary-parabolic $\mathrm{PSL}(2,\mathbb{C})$–representation of the knot group. They conjecture the existence of a solution corresponding to the geometric representation. Here we show that a boundary-parabolic representation ρ arises from a solution if and only if the length of D modulo 2 equals the obstruction to lifting ρ to a boundary-parabolic $\mathrm{SL}(2,\mathbb{C})$–representation (as an element in $\mathbb{Z}_2$). In particular, the Hikami–Inoue conjecture holds if and only if the length of D is odd. This can always be achieved by adding a kink to the braid if necessary. We also explicitly construct the solution corresponding to a boundary-parabolic representation given in the Wirtinger presentation of the knot group.

57M05, 57M25, 57M50, 57M60, 57N16; 13F60

1 Introduction

Let D be a braid of length n and width m. Hikami and Inoue [6] considered $n+1$ cluster variables $\boldsymbol{x}^1, \boldsymbol{x}^2, \dots, \boldsymbol{x}^{n+1}$, each of which consists of $3m+1$ variables, and related two consecutive cluster variables $\boldsymbol{x}^i$ and $\boldsymbol{x}^{i+1}$ $(1 \le i \le n)$ by an operator arising from cluster mutations. Precisely, if D has a braid group presentation $\sigma_{k_1}^{\epsilon_1}\sigma_{k_2}^{\epsilon_2}\cdots\sigma_{k_n}^{\epsilon_n}$, where σ_{k_i} denotes the standard generator of the m–braid group and $\epsilon_i = \pm 1$, then we have

$$\boldsymbol{x}^2 = R_{k_1}^{\epsilon_1}(\boldsymbol{x}^1), \quad \boldsymbol{x}^3 = R_{k_2}^{\epsilon_2}(\boldsymbol{x}^2), \quad \dots, \quad \boldsymbol{x}^{n+1} = R_{k_n}^{\epsilon_n}(\boldsymbol{x}^n),$$

where $R_k^{\pm}$ is the operator given by

$$R_k^{\pm}(x_1, \dots, x_{3m+1}) = (x_1, \dots, x_{3k-3}, R^{\pm}(x_{3k-2}, \dots, x_{3k+4}), x_{3k+5}, \dots, x_{3m+1}).$$

We refer to the equations (5) and (6) for the definition of $R^{\pm}$. See also [6, Section 2.2].

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Definition 1.1 The initial cluster variable $\boldsymbol{x}^1 \in \mathbb{C}^{3m+1}$ is called a *solution* if $\boldsymbol{x}^1 = \boldsymbol{x}^{n+1}$.

Recall that the space $S^3 \setminus (K \cup \{p, q\})$ admits a decomposition into ideal octahedra, where K is the knot represented by D and $p \neq q \in S^3$ are two points not in K. See for instance Dylan Thurston [9], Weeks [10] or Section 3.1. Dividing each ideal octahedron into ideal tetrahedra as in Figure 4 of [6], Hikami and Inoue proved that a nondegenerate solution (see Definition 3.2) determines the shape parameter of each ideal tetrahedron, so that these tetrahedra satisfy the gluing equations and completeness condition. In particular, we obtain a boundary-parabolic representation

$$\rho_{\boldsymbol{x}^1}\colon \pi_1(S^3 \setminus K) = \pi_1(S^3 \setminus (K \cup \{p, q\})) \to \mathrm{PSL}(2, \mathbb{C})$$

up to conjugation from a nondegenerate solution $\boldsymbol{x}^1$.

Conjecture 1.2 [6, Conjecture 3.2] Let D be a braid presentation of a hyperbolic knot K. Then there exists a nondegenerate solution $\boldsymbol{x}^1$ such that the induced representation $\rho_{\boldsymbol{x}^1}$ is geometric, i.e. discrete and faithful.

Remark 1.3 In this paper, we shall use a different subdivision of an octahedron from [6] (see Figure 3). A nondegenerate solution, implying the nondegeneracy of the ideal tetrahedra, thus requires a slightly different condition (see Definition 3.2) from [6]. Henceforth, by a nondegenerate solution we mean a solution that satisfies the condition in Definition 3.2. We stress that this change of an ideal triangulation is essential for the existence of a nondegenerate solution (see Remark 4.3).

The main purpose of this paper is to analyze the conjecture. In particular, we prove the following, which is a consequence of the more general Theorems 1.5 and 1.6 below:

Theorem 1.4 *Conjecture 1.2 holds if and only if the length of the braid is odd.*

Note that one can always make the braid length odd by adding a kink if necessary.

1.1 Main results

Let M be a compact 3–manifold with nonempty boundary and G be either $\mathrm{PSL}(2, \mathbb{C})$ or $\mathrm{SL}(2, \mathbb{C})$. Recall that a representation $\rho\colon \pi_1(M) \to G$ is *boundary-parabolic* if it maps peripheral subgroups to conjugates of the subgroup P of G consisting of upper-triangular matrices with ones on the diagonal. We shall sometimes call such a representation ρ a *(G, P)–representation*.

A representation $\pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ may or may not lift to $\mathrm{SL}(2,\mathbb{C})$ and the obstruction to lifting is a class in $H^2(M;\{\pm 1\})$. Also, a boundary-parabolic $\mathrm{PSL}(2,\mathbb{C})$–representation may lift to an $\mathrm{SL}(2,\mathbb{C})$–representation which is not boundary-parabolic. The obstruction to lifting a boundary-parabolic $\mathrm{PSL}(2,\mathbb{C})$–representation ρ to a boundary-parabolic $\mathrm{SL}(2,\mathbb{C})$–representation is a class in $H^2(M,\partial M;\{\pm 1\})$, called the *obstruction class of* ρ; see Garoufalidis, Goerner, Thurston and Zickert [4; 3]. Note that the image of this class in $H^2(M;\{\pm 1\})$ is the obstruction to lifting ρ to $\mathrm{SL}(2,\mathbb{C})$. If $M = S^3 \setminus \nu(K)$, where $\nu(K)$ denotes a small open regular neighborhood of a knot K, then we have $H^2(M,\partial M;\{\pm 1\}) \simeq \{\pm 1\}$. Therefore, the obstruction class of a boundary-parabolic $\mathrm{PSL}(2,\mathbb{C})$–representation $\rho\colon \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ can be viewed as an element of $\{\pm 1\}$.

Theorem 1.5 *Let D be a braid of a knot K* (not necessarily hyperbolic). *Then the obstruction class of $\rho_{\boldsymbol{x}^1}$ induced from a nondegenerate solution $\boldsymbol{x}^1$ is $(-1)^n$, where n is the length of D.*

The obstruction class of the geometric representation of a hyperbolic knot is nontrivial. This follows from the fact that any lift of the geometric representation maps a longitude to an element with trace -2 (see eg Calegari [1], Menal-Ferrer and Porti [7, Section 3.2] and also Proposition 2.2 below). Hence, Theorem 1.5 shows that having odd braid length is necessary for Conjecture 1.2 to hold. The fact that this is also sufficient follows from the result below, which is proved in Section 4.2:

Theorem 1.6 *Let D be a braid of a knot K* (not necessarily hyperbolic) *and let $\rho\colon \pi_1(S^3 \setminus K) \to \mathrm{PSL}(2,\mathbb{C})$ be a nontrivial boundary-parabolic representation. If the obstruction class of ρ is $(-1)^n$, where n is the length of D, then there exists a nondegenerate solution $\boldsymbol{x}^1$ such that the induced representation $\rho_{\boldsymbol{x}^1}$ coincides with ρ up to conjugation.*

We remark that the solution can be constructed explicitly when ρ is given using the Wirtinger presentation of the knot group. This uses techniques developed in Cho [2].

Organization of the paper

The paper is organized as follows. In Section 2, we recall the notion of Ptolemy coordinates with obstruction class. In Section 3, we give a short review on the Hikami–Inoue cluster variables and clarify the relation between these cluster variables and Ptolemy assignments by constructing a particular obstruction cocycle (Section 3.3). This gives a proof of Theorem 1.5. In Section 4, we prove Theorem 1.6 and present an explicit way to compute a solution when a boundary-parabolic representation is given in the Wirtinger presentation of the knot group.

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2 Ptolemy varieties with obstruction class

Let M be an oriented compact 3–manifold with nonempty boundary. We fix an ideal triangulation $\mathcal{T}$ of the interior of M. This endows M with a decomposition into truncated tetrahedra whose triangular faces triangulate ∂M (see Figure 1). We denote by M^i or ∂M^i the set of the oriented i–cells (unoriented when $i=0$). We call an edge of ∂M a *short edge* and call an edge of M not in ∂M a *long edge*. For an oriented 1–cell e, we let $-e$ denote e with its opposite orientation.

2.1 Obstruction classes

For a group G the set $C^i(M;G)$ of all set maps from M^i to G forms a group with the operation naturally induced from G. We call $\sigma \in C^1(M;G)$ a *G–cocycle* if it satisfies

(1) $\sigma(e)\sigma(-e)=1$ for all $e \in M^1$;

(2) $\sigma(e_1)\sigma(e_2)\cdots\sigma(e_m)=1$ for each face f of M, where $e_1,\dots,e_m$ are the boundary edges of the face in the cyclic order determined by a choice of orientation of f.

The set $Z^1(M;G)$ of all G–cocycles admits a $C^0(M;G)$–action defined by

$$Z^1(M;G)\times C^0(M;G)\to Z^1(M;G), \quad (\sigma,\tau)\mapsto \sigma\bullet\tau,$$

where $\sigma\bullet\tau\colon M^1\to G$ is given by $(\sigma\bullet\tau)(e)=\tau(v)^{-1}\sigma(e)\tau(w)$ for $e\in M^1$, where v and w are the initial and terminal vertices of e, respectively. The following fact is well known (see eg [11; 8]).

Proposition 2.1 *The orbit space $H^1(M;G):=Z^1(M;G)/C^0(M;G)$ has a natural bijection with the set of all conjugacy classes of representations $\rho\colon \pi_1(M)\to G$.*

Note that if G is abelian, $H^1(M;G)$ is canonically isomorphic to the usual cellular cohomology group with the coefficient G.

Let G be either $\mathrm{SL}(2,\mathbb{C})$ or $\mathrm{PSL}(2,\mathbb{C})$ and P be the subgroup of G consisting of the upper-triangular matrices with ones in the diagonal. We let $C^i(M,\partial M;G,P)$ be the subset of $C^i(M;G)$ consisting of elements $\sigma\in C^i(M;G)$ satisfying $\sigma(x)\in P$ for all $x\in\partial M^i$, and let $Z^1(M,\partial M;G,P):=Z^1(M;G)\cap C^1(M,\partial M;G,P)$. An element of $Z^1(M,\partial M;G,P)$ is called a (G,P)*–cocycle*. One can easily check (see eg [11]) that every (G,P)–representation can be represented by a (G,P)–cocycle. In fact, $H^1(M,\partial M;G,P):=Z^1(M,\partial M;G,P)/C^0(M,\partial M;G,P)$ is in natural bijection with the set of (conjugacy classes of) so-called *decorated* (G,P)*–representations* (see eg [11; 4]), but we shall not need this here.

From the short exact sequence of groups $1\to\{\pm1\}\to\mathrm{SL}(2,\mathbb{C})\to\mathrm{PSL}(2,\mathbb{C})\to 1$, we obtain exact sequences (the standard proof of exactness still works in low degree even though the terms are only sets, not groups)

$$H^1(M;\mathrm{SL}(2,\mathbb{C}))\to H^1(M;\mathrm{PSL}(2,\mathbb{C}))\to H^2(M;\{\pm1\})$$

and

$$H^1(M,\partial M;\mathrm{SL}(2,\mathbb{C}),P)\to H^1(M,\partial M;\mathrm{PSL}(2,\mathbb{C}),P)\xrightarrow{\delta} H^2(M,\partial M;\{\pm1\}).$$

In particular, the latter sequence tells us that a $(\mathrm{PSL}(2,\mathbb{C}),P)$–representation ρ admits a $(\mathrm{SL}(2,\mathbb{C}),P)$–lifting if and only if $\delta(\rho)\in H^2(M,\partial M;\{\pm1\})$ vanishes, where ρ is viewed as a $(\mathrm{PSL}(2,\mathbb{C}),P)$–cocycle. The element $\delta(\rho)$ is called the *obstruction class* of ρ. Note that it does not depend on the choice of a $(\mathrm{PSL}(2,\mathbb{C}),P)$–cocycle representing ρ. Recall that we have the long exact sequence

$$H^1(M;\{\pm1\})\to H^1(\partial M;\{\pm1\})\to H^2(M,\partial M;\{\pm1\})\to H^2(M;\{\pm1\}).$$

It thus follows that if ρ lifts to $\mathrm{SL}(2,\mathbb{C})$ (eg if $H^2(M;\{\pm1\})=0$), then the obstruction class of ρ in $H^2(M,\partial M;\{\pm1\})$ can be viewed as an element of

$$\mathrm{Coker}\big(H^1(M;\{\pm1\})\to H^1(\partial M;\{\pm1\})\big).$$

In particular, if M is a knot exterior in S^3, the obstruction class of ρ is determined by the lift of the longitude. More precisely, the following holds:

Proposition 2.2 *Let $K\subset S^3$ be a knot with knot exterior M. Then the obstruction class of the* $(\mathrm{PSL}(2,\mathbb{C}),P)$*–representation ρ* (as an element of $H^2(M,\partial M;\{\pm1\})\simeq\{\pm1\}$) *coincides with half of* $\mathrm{tr}(\tilde\rho(\lambda))$, *where $\tilde\rho\colon\pi_1(M)\to\mathrm{SL}(2,\mathbb{C})$ is any lift of ρ and λ is the canonical longitude of K.*

Proof Considering any Wirtinger presentation of $\pi_1(M)$, it is easy to check that ρ has only two lifts, $\widetilde{\rho}_+$ and $\widetilde{\rho}_-\colon \pi_1(M) \to \mathrm{SL}(2,\mathbb{C})$, such that $\mathrm{tr}(\widetilde{\rho}_+(\mu)) = 2$ and $\mathrm{tr}(\widetilde{\rho}_-(\mu)) = -2$, respectively, where μ is a meridian of K. Since $\pi_1(\partial M)$ is an abelian group generated by μ and λ, ρ admits an $(\mathrm{SL}(2,\mathbb{C}), P)$–lifting if and only if $\mathrm{tr}(\widetilde{\rho}_+(\lambda)) = 2$. Therefore, by definition, the obstruction class of ρ coincides with half of $\mathrm{tr}(\widetilde{\rho}_+(\lambda))$. On the other hand, the canonical longitude λ is in the commutator subgroup of $\pi_1(M)$ and thus it should be expressed in Wirtinger generators of even length. Therefore, we have $\widetilde{\rho}_+(\lambda) = \widetilde{\rho}_-(\lambda)$. □

2.2 Ptolemy varieties

Recall that $\mathcal{T}$ is an ideal triangulation of the interior of a compact manifold M. We denote by $\mathcal{T}^1$ the set of the oriented 1–cells. We shall often identify each $e \in \mathcal{T}^1$ with a long edge of M in a natural way.

The third author with Garoufalidis and Thurston [4] (see also [11]) gave an efficient parametrization of $(\mathrm{PSL}(2,\mathbb{C}), P)$–representations with a given obstruction class. Precisely, for $\sigma \in Z^2(M, \partial M; \{\pm 1\})$ they defined the *Ptolemy variety* $P^\sigma(\mathcal{T})$ *with the obstruction cocycle* σ by the set of all set maps $c\colon \mathcal{T}^1 \to \mathbb{C} \setminus \{0\}$ satisfying $-c(e) = c(-e)$ for all $e \in \mathcal{T}^1$ and

$$\sigma_2 c(l_{02}) c(l_{13}) = \sigma_3 c(l_{03}) c(l_{12}) + \sigma_1 c(l_{01}) c(l_{23}) \tag{1}$$

for each ideal tetrahedron Δ (with vertices $\{0,1,2,3\}$) of $\mathcal{T}$, where l_{ij} is the oriented edge of Δ going from vertex i to vertex j and σ_i is the σ–value on the hexagonal face opposite to the vertex i. See Figure 1. We call an element $c \in P^\sigma(\mathcal{T})$ a *Ptolemy assignment*.

A Ptolemy assignment $c \in P^\sigma(\mathcal{T})$ corresponds to a $(\mathrm{PSL}(2,\mathbb{C}), P)$–cocycle, Φ_c, such that $\delta(\Phi_c) = [\sigma] \in H^2(M, \partial M; \{\pm 1\})$. It thus *induces* a $(\mathrm{PSL}(2,\mathbb{C}), P)$–

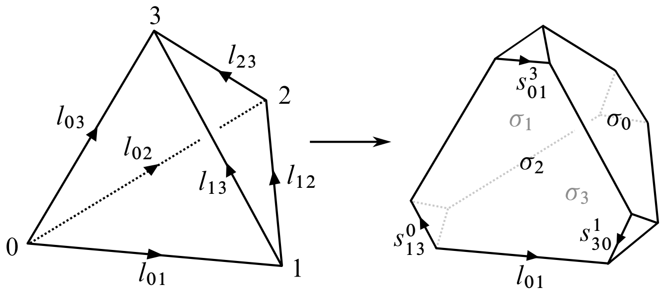

Figure 1: A truncated tetrahedron.

representation $\rho_c\colon \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ up to conjugation whose obstruction class is $[\sigma]$. Note that Φ_c is explicitly expressed by $c \in P^{\sigma}(\mathcal{T})$ as

$$\Phi_c(l_{ij}) = \pm\begin{pmatrix} 0 & -c(l_{ij})^{-1} \\ c(l_{ij}) & 0 \end{pmatrix}, \quad \Phi_c(s_{ij}^k) = \pm\begin{pmatrix} 1 & -\sigma_l \dfrac{c(l_{ji})}{c(l_{ik})c(l_{kj})} \\ 0 & 1 \end{pmatrix}$$

for Figure 1. Here $\{i,j,k,l\} = \{0,1,2,3\}$ and $s_{ij}^k \in \partial M^1$ is the edge contained in the face $[i,j,k]$ and parallel to l_{ij} (see Figure 1). The cocycle condition (2) is then automatically satisfied for the hexagonal faces, and the Ptolemy relation (1) ensures that it is also satisfied for the triangular faces. We refer to [4, Section 9] for details.

Now let us consider the map

$$d\colon Z^1(\partial M;\{\pm 1\}) \to Z^2(M,\partial M;\{\pm 1\}), \quad \epsilon \mapsto d(\epsilon), \tag{2}$$

defined by $d(\epsilon)$–value on a face of M by multiplying ϵ–values of all edges of ∂M that are contained in the face. We note that it induces the usual map $H^1(\partial M;\{\pm 1\}) \to H^2(M,\partial M;\{\pm 1\})$.

Proposition 2.3 *Let $\epsilon \in Z^1(\partial M;\{\pm 1\})$. Then any $(\mathrm{PSL}(2,\mathbb{C}), P)$–representation ρ_c induced from $c \in P^{d(\epsilon)}(\mathcal{T})$ admits a lift $\widetilde{\rho}_c\colon \pi_1(M) \to \mathrm{SL}(2,\mathbb{C})$ such that*

$$\widetilde{\rho}_c(\gamma) = \begin{pmatrix} \overline{\epsilon}(\gamma) & * \\ 0 & \overline{\epsilon}(\gamma) \end{pmatrix}$$

for all $\gamma \in \pi_1(\partial M)$ up to conjugation, where $\overline{\epsilon}\colon \pi_1(\partial M) \to \{\pm 1\}$ is the homomorphism induced from ϵ.

Proof We may choose a lift $\widetilde{\Phi}_c \in C^1(M;\mathrm{SL}(2,\mathbb{C}))$ of Φ_c such that

$$\widetilde{\Phi}_c(l) = \begin{pmatrix} 0 & -c(l)^{-1} \\ c(l) & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{\Phi}_c(s) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

for all $l \in M^1 \setminus \partial M^1$ and $s \in \partial M^1$. One can check that $\widetilde{\Phi}_c$ satisfies the cocycle condition for every triangular face of M (but may not for all faces). Let $\widetilde{\epsilon} \in C^1(M;\{\pm 1\})$ be the trivial extension of ϵ, i.e. $\widetilde{\epsilon}(e) := \epsilon(e)$ if $e \in \partial M^1$ and $\widetilde{\epsilon}(e) := 1$ otherwise. Then, by definition, $\widetilde{\epsilon} \cdot \widetilde{\Phi}_c\colon M^1 \to \mathrm{SL}(2,\mathbb{C})$ is a cocycle satisfying

$$(\widetilde{\epsilon} \cdot \widetilde{\Phi}_c)(e) = \begin{pmatrix} \epsilon(e) & * \\ 0 & \epsilon(e) \end{pmatrix}$$

for all $e \in \partial M$. The proposition follows by letting $\widetilde{\rho}_c\colon \pi_1(M) \to \mathrm{SL}(2,\mathbb{C})$ be a representation induced from $\widetilde{\epsilon} \cdot \widetilde{\Phi}_c$. □

Combining Propositions 2.2 and 2.3, we obtain the following:

Theorem 2.4 *Let $K \subset S^3$ be a knot and $M = S^3 \setminus \nu(K)$. Then any $(\mathrm{PSL}(2,\mathbb{C}), P)$–representation ρ_c induced from $c \in P^{d(\epsilon)}(\mathcal{T})$ has the obstruction class $\overline{\epsilon}(\lambda) \in \{\pm 1\} \simeq H^2(M, \partial M; \{\pm 1\})$, where $\overline{\epsilon}\colon \pi_1(\partial M) \to \{\pm 1\}$ is the homomorphism induced from $\epsilon \in Z^1(\partial M; \{\pm 1\})$ and λ is the canonical longitude of K.*

3 The Hikami–Inoue cluster variables

3.1 The octahedral decomposition of a knot complement with two points removed

Let $K \subset S^3$ be a knot and let $\nu(K \cup \{p, q\})$ denote a tubular neighborhood of the union of K with two points $p \neq q \in S^3$ not in K. Whenever we choose a knot diagram of K, we have a decomposition of the space $M = S^3 \setminus \nu(K \cup \{p, q\})$ into blocks each of which is a cube with two cylinders (whose core is the knot) removed. See Figure 2. Note that M is a 3–manifold with three boundary components (two spheres and a torus) whose interior is homeomorphic to $S^3 \setminus (K \cup \{p, q\})$. Now consider two quadrilaterals Q_1 and Q_2 in each block as in Figure 2 and collapse them horizontally so that their vertical edges are respectively identified. We call the resulting object a *pinched block*.

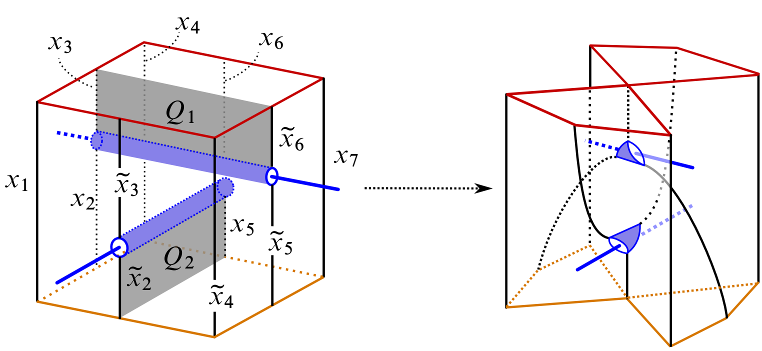

Figure 2: A pinched block.

On the other hand, a pinched block can also be obtained from a truncated octahedron by identifying two pairs of edges as in Figure 3 (right). Therefore, one can obtain M by gluing truncated octahedra, and it thus follows that the interior of M can be decomposed into ideal octahedra (one per crossing). We denote by $\mathcal{O}$ this octahedral decomposition of $S^3 \setminus (K \cup \{p, q\})$. It is due to Thurston [9] (see also [10]).

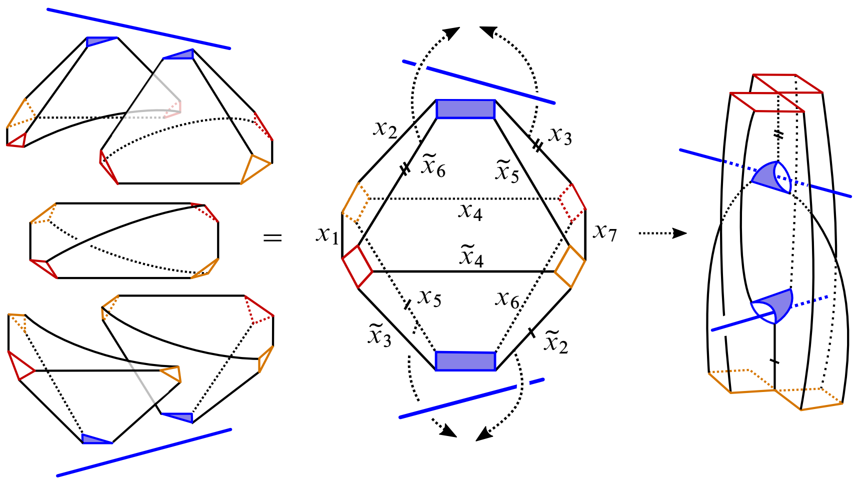

Figure 3: A truncated octahedron.

3.2 The Hikami–Inoue cluster variables

The edges of an ideal octahedron (as in Figure 3) correspond to vertical edges of a block as in Figure 2 (left). We label these edges by $x_1, \dots, x_7, \widetilde{x}_1, \dots, \widetilde{x}_7$ as in Figure 4 with the obvious relations $x_1 = \widetilde{x}_1$ and $x_7 = \widetilde{x}_7$. As indicated in Figure 4 (left) we shall regard the edges x_i as being above a crossing, and the edges $\widetilde{x}_i$ as being below the crossing.

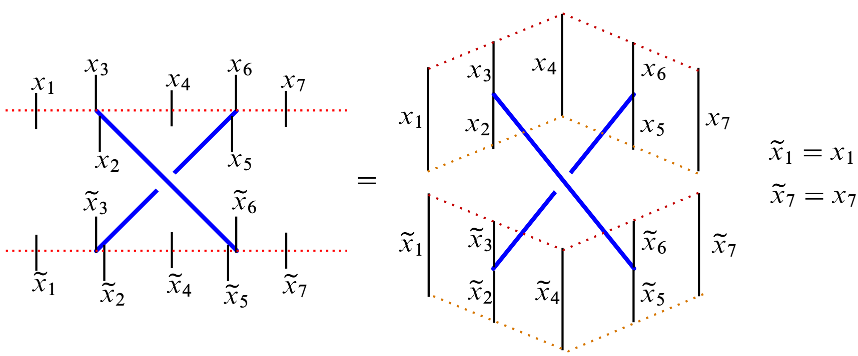

Figure 4: Edges of an octahedron at a crossing.

Assigning a complex-valued variable to each of the edges $x_1, \dots, x_7, \widetilde{x}_1, \dots, \widetilde{x}_7$ with the same label as the edge itself, Hikami and Inoue considered the equation

$$(\widetilde{x}_1, \dots, \widetilde{x}_7) = R^{\pm}(x_1, \dots, x_7)$$

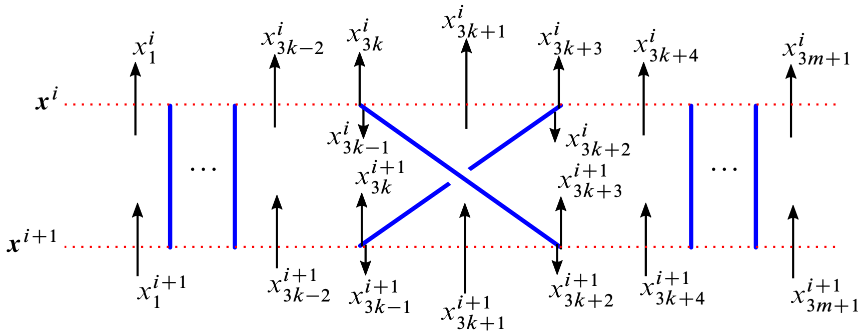

Figure 5: Edges of $\mathcal{O}$ around the i^{th} level of a braid.

(see [5, Section 2.2]), where $R^{\pm}$ is a certain operator defined by rational polynomial equations. As we shall see in Section 3.3, these equations are equivalent to Ptolemy relations for a particular obstruction cocycle.

Now suppose that the knot diagram D is given by a braid with presentation $\sigma_{k_1}^{\epsilon_1}\cdots\sigma_{k_n}^{\epsilon_n}$. Here σ_{k_i} denotes the standard generator of the m–braid group and $\epsilon_i \in \{\pm 1\}$. Similar to the edge-labeling described in the previous paragraph, we label the oriented edges of the octahedral decomposition $\mathcal{O}$ as follows:

(1) Draw $n+1$ imaginary horizontal lines on the braid D so that there is only one crossing between two consecutive lines (see Figures 5 and 12).

(2) As in Figure 4 (left), whenever a horizontal line meets D there are two corresponding edges, and whenever a horizontal line meets a region of (the closure of) D, there is one corresponding edge. Since each of the horizontal lines meets the braid m times and the regions $m+1$ times, it corresponds to $3m+1$ edges of $\mathcal{O}$.

(3) For the i^{th} horizontal line we orient the corresponding edges and denote them by $x^i_1,\dots,x^i_{3m+1}$ as in Figure 5, and let $\boldsymbol{x}^i=(x^i_1,\dots,x^i_{3m+1})$.

Note that there are many overlapped labelings; for instance, in Figure 5 we have $x^i_j = x^{i+1}_j$ for $j=1,\dots,3k-2$ and $j=3k+4,\dots,3m+1$.

We again assign a complex-valued variable to each oriented edge of $\mathcal{O}$ and denote the variable by the same as the edge itself. Hikami and Inoue [5] related consecutive cluster variables $\boldsymbol{x}^i$ and $\boldsymbol{x}^{i+1}$ $(1 \le i \le n)$ by the equation

$$\boldsymbol{x}^{i+1} = R^{\epsilon_i}_{k_i}(\boldsymbol{x}^i),$$

where the operator $R_k^\pm$ is defined by

$$R_k^\pm(x_1,\dots,x_{3m+1})=(x_1,\dots,x_{3k-3},R^\pm(x_{3k-2},\dots,x_{3k+4}),x_{3k+5},\dots,x_{3m+1}).$$

Note that $R_k^\pm$ only affects the variables above and below the crossing.

Recall that an initial cluster variable $\boldsymbol{x}^1\in\mathbb{C}^{3m+1}$ is called a solution if $\boldsymbol{x}^1=\boldsymbol{x}^{n+1}$. Whenever we have a solution $\boldsymbol{x}^1\in\mathbb{C}^{3m+1}$, we define the set map

$$c_{\boldsymbol{x}^1}\colon \mathcal{O}^1\to\mathbb{C}$$

by assigning the variable x_j^i to the oriented edge of $\mathcal{O}$ labeled by the same name. The fact that this assignment respects the face identifications in $\mathcal{O}$ follows directly from the definitions of $R^\pm$ and $R_k^\pm$.

3.3 The obstruction cocycle

Let $\mathcal{T}$ be the ideal triangulation of $S^3\setminus(K\cup\{p,q\})$ obtained by decomposing each octahedron of $\mathcal{O}$ into five ideal tetrahedra as in Figure 3 (left). As explained earlier, this induces a triangulation of the boundary of $M=S^3\setminus\nu(K\cup\{p,q\})$.

We define a cocycle $\epsilon_D\in Z^1(\partial M;\{\pm1\})$ on ∂M by assigning signs to the short edges of the truncated tetrahedra. Note that each short edge either lies in the top/bottom of a truncated octahedron, or on one of the sides. We shall call the edges *top/bottom*

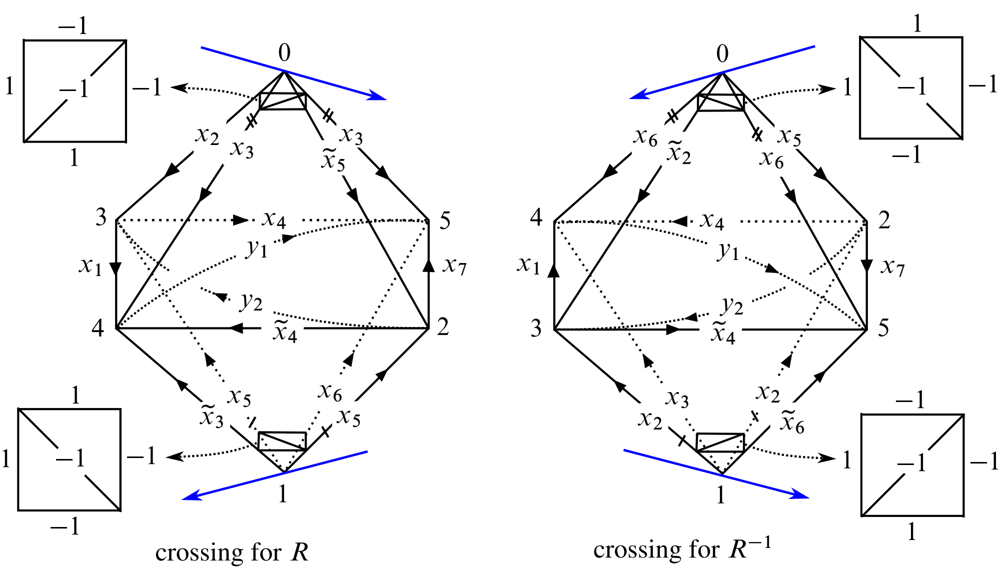

Figure 6: An ideal octahedron at a crossing.

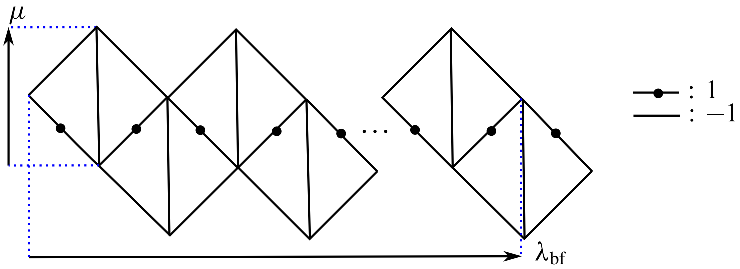

Figure 7: Configuration of ϵ_D on the boundary torus.

edges or *side edges* accordingly. We assign signs to the top/bottom edges as indicated in Figure 6 and assign $+1$ to all of the side edges. This is clearly a cocycle, which respects the face pairings and thus gives rise to a cocycle in $\epsilon_D \in Z^1(\partial M;\{\pm 1\})$, as desired. We stress that ϵ_D depends on the decomposition of M, in particular the choice of a braid presentation D of K.

The cocycle ϵ_D is illustrated in Figure 7, where μ and λ_{bf} denote a meridian and the blackboard-framed longitude of K, respectively. In particular, ϵ_D induces the homomorphism $\overline{\epsilon}_D\colon \pi_1(\nu(K)) \to \{\pm 1\}$ that maps μ to -1 and λ_{bf} to 1.

3.4 Proof of Theorem 1.5

Let us consider an octahedron of $\mathcal{O}$. We index the vertices by $\{0,\dots,5\}$ and denote the oriented edges as in Figure 6. We compute the Ptolemy relation (1) for each of the ideal tetrahedra with the obstruction cocycle $\sigma_D := d(\epsilon_D) \in Z^2(M,\partial M;\{\pm 1\})$. Recall that the map d is given in the equation (2). For example, the tetrahedron with vertices $\{0,3,4,5\}$ in Figure 6 (left) gives

$$\begin{aligned}\sigma_4 c(l_{04})c(l_{35}) = \sigma_5 c(l_{05})c(l_{34}) + \sigma_3 c(l_{03})c(l_{45}) &\iff (-1)x_3x_4 = x_3x_1 + (-1)x_2y_1\\ &\iff \quad x_2y_1 = x_3x_4 + x_1x_3.\end{aligned}$$

Similarly, we obtain the following from the other ideal tetrahedra:

Figure 6 (left)		Figure 6 (right)	
$\{0,3,4,5\}$	$x_2y_1 = x_3x_4 + x_1x_3$	$\{0,2,4,5\}$	$y_1x_5 = x_4x_6 + x_6x_7$
$\{1,2,3,5\}$	$x_6y_2 = x_5x_7 + x_4x_5$	$\{1,2,3,4\}$	$x_3y_2 = x_1x_2 + x_2x_4$
$\{2,3,4,5\}$	$x_4\widetilde{x}_4 = x_1x_7 + y_1y_2$	$\{2,3,4,5\}$	$x_4\widetilde{x}_4 = y_1y_2 + x_1x_7$
$\{0,2,4,5\}$	$\widetilde{x}_5y_1 = x_3\widetilde{x}_4 + x_3x_7$	$\{0,3,4,5\}$	$\widetilde{x}_2y_1 = x_6\widetilde{x}_4 + x_1x_6$
$\{1,2,3,4\}$	$\widetilde{x}_3y_2 = x_5\widetilde{x}_4 + x_1x_5$	$\{1,2,3,5\}$	$\widetilde{x}_6y_2 = x_2x_7 + x_2\widetilde{x}_4$

Considering $x_1, \dots, x_7$ as given variables, we obtain

$$(y_1, y_2) = \left(\frac{x_3(x_1+x_4)}{x_2}, \frac{x_5(x_4+x_7)}{x_6}\right), \tag{3}$$

$$(\tilde{x}_3, \tilde{x}_4, \tilde{x}_5) = \begin{pmatrix} \dfrac{x_1x_3x_5+x_3x_4x_5+x_1x_2x_6}{x_2x_4} \\ \dfrac{x_1x_3x_4x_5+x_3x_4^2x_5+x_1x_3x_5x_7+x_3x_4x_5x_7+x_1x_2x_6x_7}{x_2x_4x_6} \\ \dfrac{x_3x_4x_5+x_3x_5x_7+x_2x_6x_7}{x_4x_6} \end{pmatrix}^T$$

for Figure 6 (left) and

$$(y_1, y_2) = \left(\frac{x_6(x_4+x_7)}{x_5}, \frac{x_2(x_1+x_4)}{x_3}\right), \tag{4}$$

$$(\tilde{x}_2, \tilde{x}_4, \tilde{x}_6) = \begin{pmatrix} \dfrac{x_1x_3x_5+x_1x_2x_6+x_2x_4x_6}{x_3x_4} \\ \dfrac{x_1x_2x_4x_6+x_2x_4^2x_6+x_1x_3x_5x_7+x_1x_2x_6x_7+x_2x_4x_6x_7}{x_3x_4x_5} \\ \dfrac{x_2x_4x_6+x_3x_5x_7+x_2x_6x_7}{x_4x_5} \end{pmatrix}^T$$

for Figure 6 (right). Letting $\tilde{x}_1 := x_1$, $\tilde{x}_2 := x_5$, $\tilde{x}_6 := x_3$ and $\tilde{x}_7 := x_7$ for Figure 6 (left) and $\tilde{x}_1 := x_1$, $\tilde{x}_3 := x_6$, $\tilde{x}_5 := x_2$ and $\tilde{x}_7 := x_7$ for Figure 6 (right) (note that these equations come from Figures 2 and 4), we obtain

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x}_5 \\ \tilde{x}_6 \\ \tilde{x}_7 \end{pmatrix}^T = \begin{pmatrix} x_1 \\ x_5 \\ \dfrac{x_1x_3x_5+x_3x_4x_5+x_1x_2x_6}{x_2x_4} \\ \dfrac{x_1x_3x_4x_5+x_3x_4^2x_5+x_1x_3x_5x_7+x_3x_4x_5x_7+x_1x_2x_6x_7}{x_2x_4x_6} \\ \dfrac{x_3x_4x_5+x_3x_5x_7+x_2x_6x_7}{x_4x_6} \\ x_3 \\ x_7 \end{pmatrix}^T = R \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}^T \tag{5}$$

for Figure 6 (left) and

$$(6)\qquad \begin{pmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \widetilde{x}_3 \\ \widetilde{x}_4 \\ \widetilde{x}_5 \\ \widetilde{x}_6 \\ \widetilde{x}_7 \end{pmatrix}^T = \begin{pmatrix} x_1 \\ \dfrac{x_1x_3x_5+x_1x_2x_6+x_2x_4x_6}{x_3x_4} \\ x_6 \\ \dfrac{x_1x_2x_4x_6+x_2x_4^2x_6+x_1x_3x_5x_7+x_1x_2x_6x_7+x_2x_4x_6x_7}{x_3x_4x_5} \\ x_2 \\ \dfrac{x_3x_5x_7+x_2x_4x_6+x_2x_6x_7}{x_4x_5} \\ x_7 \end{pmatrix}^T = R^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}^T$$

for Figure 6 (right). The equations (5) and (6) exactly coincide with the definition of $R^{\pm}$ in [6]. See [6, Equation (2.12)].

Remark 3.1 Subdividing the octahedron into five tetrahedra as in Figure 6 corresponds to taking the form of the R–operator given in the equation (2.14) instead of (2.9) in [6].

Now let D be a braid of length n and width m. Let $c_{\boldsymbol{x}^1}\colon \mathcal{O}^1 \to \mathbb{C}$ be the set map induced from a solution $\boldsymbol{x}^1 \in \mathbb{C}^{3m+1}$ as in Section 3.2. Recall that $\mathcal{T}$ has two additional edges per crossing compared to $\mathcal{O}$. We extend the set map to $c_{\boldsymbol{x}^1}\colon \mathcal{T}^1 \to \mathbb{C}$ by defining the values on the added edges using the equations (3) and (4). We say that a solution $\boldsymbol{x}^1$ is *nondegenerate* if

$$c_{\boldsymbol{x}^1}(e) \neq 0$$

for all $e \in \mathcal{T}^1$. One can easily check from the equations (3) and (4) that this is equivalent to the following:

Definition 3.2 A solution $\boldsymbol{x}^1$ is said to be *nondegenerate* if every cluster variable $\boldsymbol{x}^i = (x^i_1, \dots, x^i_{3m+1})$ satisfies $x^i_j \neq 0$ for all $1 \leq j \leq 3m+1$ and $x^i_{3j-2} \neq -x^i_{3j+1}$ for all $1 \leq j \leq m$.

The previous computation in this section tells us that the set map $c_{\boldsymbol{x}^1}\colon \mathcal{T}^1 \to \mathbb{C} \setminus \{0\}$ induced from a nondegenerate solution $\boldsymbol{x}^1$ is a point of the Ptolemy variety $P^{\sigma_D}(\mathcal{T})$ with the obstruction cocycle $\sigma_D \in Z^2(M, \partial M; \{\pm 1\})$. We have thus proven:

Proposition 3.3 *A nondegenerate solution $\boldsymbol{x}^1$ induces a boundary parabolic representation*

$$\rho_{\boldsymbol{x}^1}\colon \pi_1(S^3 \setminus K) = \pi_1(M) \to \mathrm{PSL}(2, \mathbb{C})$$

(*up to conjugation*) *whose obstruction class is* $[\sigma_D] \in H^2(M, \partial M; \{\pm 1\})$.

Remark 3.4 A Ptolemy assignment (with obstruction class) determines the shape (or cross-ratio parameters) of the ideal tetrahedra so that they fulfill Thurston's gluing equations. We refer to [4, Section 12] for details. This implies that the proof of the second part of Theorem 3.1 in [6] is unnecessary.

Proposition 3.5 *Let D be a braid of length n representing a knot. Then $[\sigma_D]$ is $(-1)^n$ under the isomorphism $H^2(M, \partial M; \{\pm 1\}) \simeq \{\pm 1\}$.*

Proof It suffices to show that $\overline{\epsilon}_D(\lambda) = (-1)^n$, where λ is the canonical longitude and $\overline{\epsilon}_D$ denotes the homomorphism induced from $\epsilon_D \in Z^1(\partial M; \{\pm 1\})$. Recall Section 3.3 that we have $\overline{\epsilon}_D(\mu) = -1$ and $\overline{\epsilon}_D(\lambda_{\mathrm{bf}}) = 1$ for the meridian μ and blackboard-framed longitude λ_{bf}. We thus obtain

$$\overline{\epsilon}_D(\lambda) = \overline{\epsilon}_D(\lambda_{\mathrm{bf}})\overline{\epsilon}_D(\mu)^{-w(D)} = \overline{\epsilon}_D(\lambda_{\mathrm{bf}})\overline{\epsilon}_D(\mu)^{-n} = (-1)^n.$$

Here $w(D)$ denotes the writhe of the closure of D, which is congruent to the length n modulo 2. □

4 The existence of a nondegenerate solution

Let $\widetilde{M}$ be the universal cover of $M = S^3 \setminus \nu(K \cup \{p, q\})$ and $\widehat{\widetilde{M}}$ be the space obtained from $\widetilde{M}$ by collapsing each boundary component to a point. We denote by $I(\widetilde{M})$ the set of these points. Note that $\pi_1(M)$ acts on $I(\widetilde{M})$.

Definition 4.1 For a $(\mathrm{PSL}(2,\mathbb{C}), P)$–representation $\rho\colon \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$, a *decoration* $\mathcal{D}\colon I(\widetilde{M}) \to \mathrm{PSL}(2,\mathbb{C})/P$ is a ρ–equivalent assignment, i.e. $\mathcal{D}(\gamma \cdot v) = \rho(\gamma)\mathcal{D}(v)$ for all $\gamma \in \pi_1(M)$ and $v \in I(\widetilde{M})$.

Recall that $\mathrm{PSL}(2,\mathbb{C})/P$ denotes the (left) P–coset space, where P is the subgroup of $\mathrm{PSL}(2,\mathbb{C})$ consisting of upper-triangular matrices with ones on the diagonal. We may identify a P–coset gP with a vector $g\binom{1}{0}$ which is well defined up to sign. In particular, by $\det(gP, hP)$ we mean $\det\big(g\binom{1}{0}, h\binom{1}{0}\big) \in \mathbb{C}/\{\pm 1\}$.

We now fix a braid presentation D of a knot K and let $\mathcal{T}$ be the ideal triangulation of $S^3 \setminus (K \cup \{p, q\})$ given as in Section 3. For any decoration $\mathcal{D}$ we define an assignment $c\colon \mathcal{T}^1 \to \mathbb{C}/\{\pm 1\}$ by

$$c(e) = \det(\mathcal{D}(v_1), \mathcal{D}(v_2))$$

for $e \in \mathcal{T}^1$, where v_1 and $v_2 \in I(\widetilde{M})$ are endpoints of a lift of e. Note that $c(e)$ does not depend on the choice of a lift of e, since $\mathcal{D}$ is ρ–equivariant.

Proposition 4.2 *Given a nontrivial* $(\mathrm{PSL}(2,\mathbb{C}), P)$*–representation* $\rho\colon \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$*, there exists a decoration* $\mathcal{D}$ *such that the induced assignment* c *satisfies* $c(e) \neq 0$ *for all* $e \in \mathcal{T}^1$*.*

The proof of Proposition 4.2 (see Section 4.1 for details) relies on the following basic facts: (i) every edge of $\mathcal{T}$ is connected to either p or q; (ii) a decoration on the lifts of p and q can be chosen freely and independently (respecting ρ–equivalence only). The observation that (i) and (ii) implies Proposition 4.2 was first pointed out to the authors by Seonhwa Kim. We also note that there are edges connecting p (or q) to itself and this is the reason why we can not detect the trivial representation. Namely, these edges become generators in the Wirtinger presentation (see Figure 9 (left)) and thus the image of the generators under ρ must be nontrivial.

Remark 4.3 The ideal triangulation used in [6] does not satisfy fact (i) above. The edge x_c in Figure 4 of [6] joins an ideal vertex corresponding to the knot to itself. In this case, Proposition 4.2 may not hold, whenever the closure of D has a kink. Therefore, it is essential to use another subdivision of an octahedron (for instance, as in Figure 6) so that fact (i) holds.

Proposition 4.2 implies the existence of a nondegenerate solution desired as in Theorem 1.6. More precisely, the following holds:

Theorem 4.4 *Let* $\sigma_D \in Z^2(M, \partial M; \{\pm 1\})$ *be the cocycle given as in Section 3. If a nontrivial* $(\mathrm{PSL}(2,\mathbb{C}), P)$*–representation* ρ *has the obstruction class* $[\sigma_D] \in H^2(M, \partial M; \{\pm 1\})$*, then there exists a point* $c \in P^{\sigma_D}(\mathcal{T})$ *such that* ρ_c *coincides with* ρ *up to conjugation.*

Proof Let $\mathcal{D}$ be a decoration as in Proposition 4.2. Whenever one chooses a sign of each $c(e)$, it is known that $c\colon \mathcal{T}^1 \to \mathbb{C} \setminus \{0\}$ is a point of $P^{\sigma}(\mathcal{T})$ for some $\sigma \in Z^2(M, \partial M; \{\pm 1\})$ such that $\rho_c = \rho$ up to conjugation. In particular, the obstruction class of ρ is $[\sigma] \in H^2(M, \partial M; \{\pm 1\})$. Then the theorem follows from the fact that if σ_0 and $\sigma_1 \in Z^2(M, \partial M; \{\pm 1\})$ satisfy $[\sigma_0] = [\sigma_1]$, then two varieties $P^{\sigma_0}(\mathcal{T})$ and $P^{\sigma_1}(\mathcal{T})$ are canonically isomorphic. □

As we computed in Section 3.4, the class $[\sigma_D]$ viewed as an element of $\{\pm 1\}$ coincides with $(-1)^n$, where n is the length of D. We therefore obtain Theorem 1.6 as a consequence of Theorem 4.4.

4.1 Proof of Proposition 4.2

We first consider edges, say $e_1, \dots, e_m$, of $\mathcal{T}$ that join p and q. We orient these edges from q to p. We choose a lift $\widetilde{e}_j$ of each e_j so that their terminal points agree as in Figure 8. We denote by $\widetilde{p}$ the terminal point and by $\widetilde{q}_j$ the initial point of $\widetilde{e}_j$. From ρ–equivariance of $\mathcal{D}$, we have

$$\mathcal{D}(\widetilde{q}_j) = \rho(g)\mathcal{D}(\widetilde{q}_k)$$

for some $g \in \pi_1(M)$. From elementary covering theory one can check that if $e_j \cup e_k$ wraps an arc of K as in Figure 8, then the loop g should be the Wirtinger generator corresponding to the arc. Note that $c(e_k) \neq 0$ if and only if $\det(\mathcal{D}(\widetilde{p}), \mathcal{D}(\widetilde{q}_k)) \neq 0$.

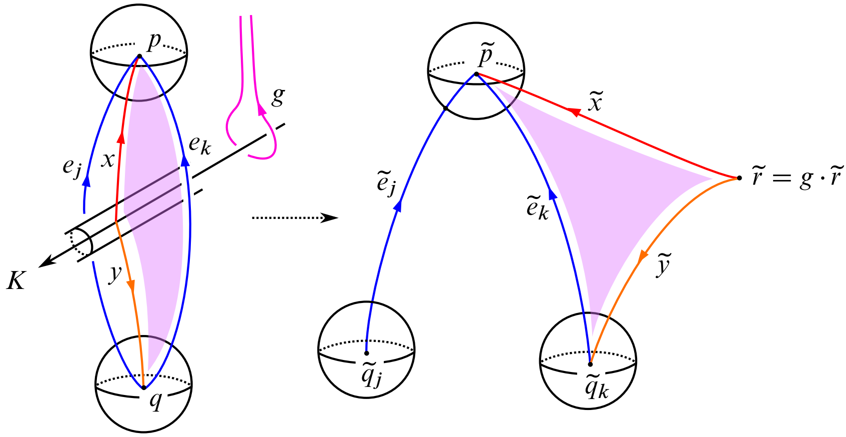

Figure 8: Local configuration of a lift.

We then consider edges of $\mathcal{T}$ that are connected to the knot K, for example edges x and y as in Figure 8. We consider an ideal triangle in $S^3 \setminus (K \cup \{p, q\})$ with edges x, y, e_k as in Figure 8, and its lift so that p corresponds to the point $\widetilde{p}$. We denote the edges of the lift by $\widetilde{x}$, $\widetilde{y}$, $\widetilde{e}_k$. Since the terminal point, $\widetilde{r}$, of $\widetilde{x}$ (or $\widetilde{y}$) is fixed by the Wirtinger generator g, we obtain

$$\mathcal{D}(\widetilde{r}) = \mathcal{D}(g \cdot \widetilde{r}) = \rho(g)\mathcal{D}(\widetilde{r}).$$

Since $\operatorname{tr}(\rho(g)) = \pm 2$ and $\rho(g) \neq \mathrm{Id}$ — otherwise ρ should be a trivial representation — $\rho(g)$ has a unique eigenvector up to scaling. It thus follows that $c(x) = \det(\mathcal{D}(\widetilde{p}), \mathcal{D}(\widetilde{r})) \neq 0$ if and only if $\mathcal{D}(\widetilde{p})$ is not an eigenvector of $\rho(g)$. Similarly, $c(y) \neq 0$ if and only if $\mathcal{D}(\widetilde{q}_k)$ is not an eigenvector of $\rho(g)$.

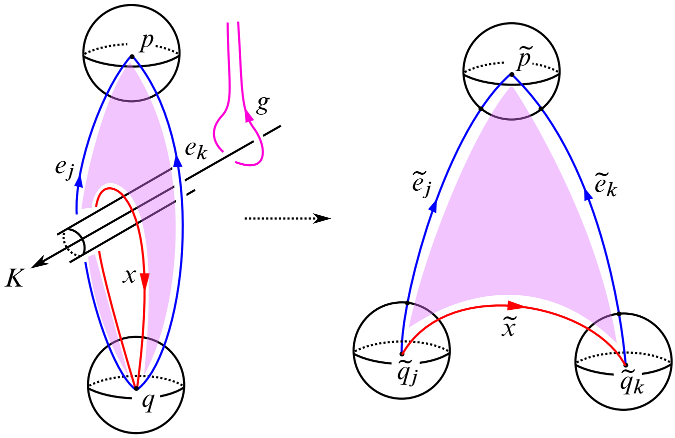

Figure 9: Local configuration of a lift.

We finally consider edges of $\mathcal{T}$ joining q (or p) to itself, for example an edge x as in Figure 9. We consider an ideal triangle in $S^3 \setminus (K \cup \{p, q\})$ with edges e_j, e_k, x as in Figure 9, and its lift so that p corresponds to the point $\widetilde{p}$. We denote the edges of the lift by $\widetilde{e}_j$, $\widetilde{e}_k$, $\widetilde{x}$. It directly follows that $c(x) \neq 0$ if and only if

$$\det(\mathcal{D}(\widetilde{q}_j), \mathcal{D}(\widetilde{q}_k)) = \det(\rho(g)\mathcal{D}(\widetilde{q}_k), \mathcal{D}(\widetilde{q}_k)) \neq 0.$$

Again, this is equivalent to the condition that $\mathcal{D}(\widetilde{q}_k)$ is not an eigenvector of $\rho(g)$.

Let us sum up the conditions. To be precise, we enumerate the Wirtinger generators by $g_1, \dots, g_l$. Our desired decoration as in Proposition 4.2 should satisfy (i) $\det(\mathcal{D}(\widetilde{p}), \mathcal{D}(\widetilde{q}_j)) \neq 0$; (ii) $\mathcal{D}(\widetilde{p})$ is not an eigenvector of $\rho(g_i)$; (iii) $\mathcal{D}(\widetilde{q}_j)$ is not an eigenvector of $\rho(g_i)$ for all $1 \leq j \leq m$ and $1 \leq i \leq l$. Since we can choose $\mathcal{D}(\widetilde{p})$ and one of the $\mathcal{D}(\widetilde{q}_j)$ freely, such a decoration exists.

4.2 Explicit computation from a representation

Let D be a braid presentation of a knot K and let $\rho\colon \pi_1(S^3 \setminus K) \to \mathrm{PSL}(2, \mathbb{C})$ be a nontrivial $(\mathrm{PSL}(2, \mathbb{C}), P)$–representation whose obstruction class is $(-1)^n$, where n is the length of D. We devote this subsection to presenting an explicit formula for computing a solution.

Let $\widetilde{\rho}$ be an $\mathrm{SL}(2, \mathbb{C})$–lift of ρ satisfying

$$\widetilde{\rho}(\mu) = \begin{pmatrix} -1 & * \\ 0 & -1 \end{pmatrix} \neq -\mathrm{Id} \quad \text{and} \quad \widetilde{\rho}(\lambda) = \begin{pmatrix} (-1)^n & * \\ 0 & (-1)^n \end{pmatrix}$$

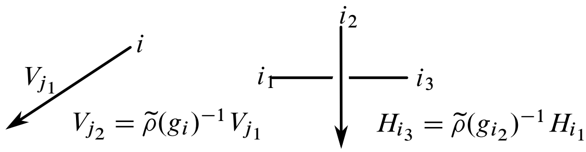

Figure 10: Rules for region- and arc-colorings.

(recall Proposition 2.2). We index the regions of the closure of D by $1 \leq j \leq n+2$ and the arcs by $1 \leq i \leq n$. We then assign a nonzero column vector V_j to the j^{th} region so that these vectors satisfy

$$V_{j_2} = \widetilde{\rho}(g_i)^{-1} V_{j_1} \tag{7}$$

for Figure 10 (left), where m_i is the Wirtinger generator corresponding to the i^{th} arc. The region-colorings are well determined whenever an initial vector is chosen arbitrarily. Note that V_j corresponds to $\mathcal{D}(\widetilde{q}_j)$ in Section 4.1.

We also assign a nonzero column vector H_i to the i^{th} arc so that these vectors satisfy $\widetilde{\rho}(g_i) H_i = -H_i$ for $1 \leq i \leq m$ (recall that the eigenvalue of $\widetilde{\rho}(g_i)$ is -1) and

$$H_{i_3} = \widetilde{\rho}(g_{i_2})^{-1} H_{i_1} \tag{8}$$

for Figure 10 (right). We remark that the fact that the eigenvalue of $\widetilde{\rho}(\lambda_{\text{bf}})$ is 1 (equivalently, the eigenvalue of $\widetilde{\rho}(\lambda)$ is $(-1)^n$) is required here.

Recall that the octahedral decomposition $\mathcal{O}$ has $3n+2$ edges; (i) n of them, called *over-edges*, stand above the knot; (ii) another n of them, called *under-edges*, stand below the knot; (iii) the last $n+2$ of them, called *regional edges*, stand on the regions. See Figure 11. We choose an additional nonzero column vector W (which corresponds to $\mathcal{D}(\widetilde{p})$ in Section 4.1) and define the set map $c\colon \mathcal{O}^1 \to \mathbb{C}$ as follows:

(i) $c(e) = \det(H_i, W)$ if e is the over-edge standing over the i^{th} arc;

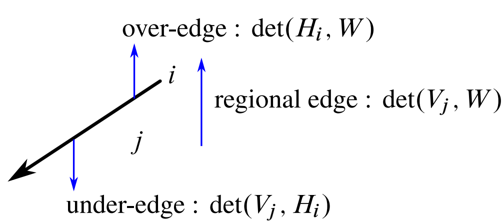

Figure 11: Edges of $\mathcal{O}$ with their c–values.

(ii) $c(e) = \det(V_j, H_i)$ if e is the under-edge standing below the i^{th} arc whose left-side region is indexed by j;

(iii) $c(e) = \det(V_j, W)$ if e is the regional edge corresponding to the j^{th} region.

Here we oriented the edge e as in Figure 11.

We again extend the above set map to $c\colon \mathcal{T}^1 \to \mathbb{C}$ by using the equations (3) and (4). As we showed in Section 4.1, for a generic choice of W and the V_j, we have $c(e) \neq 0$ for all $e \in \mathcal{T}^1$.

Example 4.5 (The 4_1 knot with a kink) Let us consider a braid of the knot 4_1 as in Figure 12. The geometric representation ρ lifts to an $\mathrm{SL}(2,\mathbb{C})$–representation $\tilde{\rho}$ such that

$$\tilde{\rho}(g_1) = \tilde{\rho}(g_2) = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \quad \tilde{\rho}(g_3) = \begin{pmatrix} -1 & 0 \\ -\lambda & -1 \end{pmatrix},$$

$$\tilde{\rho}(g_4) = \begin{pmatrix} -1-\lambda & \lambda \\ -\lambda & -1+\lambda \end{pmatrix}, \quad \tilde{\rho}(m_4) = \begin{pmatrix} -2 & \lambda \\ -1+\lambda & 0 \end{pmatrix},$$

where $\lambda^2 - \lambda + 1 = 0$.

We enumerate the arcs and regions of the closure of the braid as in Figure 12. Choosing the vector $H_1 = \binom{1}{0}$, the equation (8) gives

$$H_2 = \tilde{\rho}(g_2)^{-1} H_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad H_3 = \tilde{\rho}(g_5)^{-1} H_2 = \begin{pmatrix} 0 \\ -1+\lambda \end{pmatrix},$$

$$H_4 = \tilde{\rho}(g_2) H_3 = \begin{pmatrix} 1-\lambda \\ 1-\lambda \end{pmatrix}, \quad H_5 = \tilde{\rho}(g_3)^{-1} H_4 = \begin{pmatrix} -1+\lambda \\ \lambda \end{pmatrix}.$$

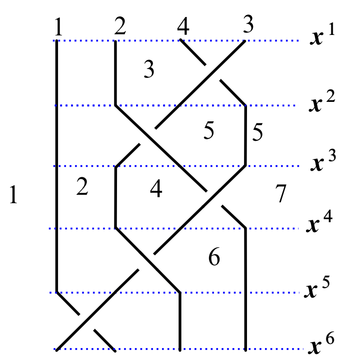

Figure 12: A braid presentation of the 4_1 knot.

Similarly, letting the vector $V_1 = \binom{\alpha}{\beta}$ for some $\alpha, \beta \in \mathbb{C}$, the equation (7) gives

$$V_2 = \widetilde{\rho}(g_1)^{-1} V_1 = \begin{pmatrix} -\alpha+\beta \\ -\beta \end{pmatrix}, \qquad V_3 = \widetilde{\rho}(g_2)^{-1} V_2 = \begin{pmatrix} \alpha-2\beta \\ \beta \end{pmatrix},$$

$$V_4 = \widetilde{\rho}(g_4)^{-1} V_2 = \begin{pmatrix} \alpha(1-\lambda)+\beta(-1+2\lambda) \\ -\alpha\lambda+\beta(1+2\lambda) \end{pmatrix}, \qquad V_5 = \widetilde{\rho}(g_3)^{-1} V_3 = \begin{pmatrix} -\alpha+2\beta \\ \alpha\lambda-\beta(1+2\lambda) \end{pmatrix},$$

$$V_6 = \widetilde{\rho}(g_5)^{-1} V_4 = \begin{pmatrix} \alpha(-1+\lambda)+\beta(2-3\lambda) \\ \alpha\lambda-\beta(1+3\lambda) \end{pmatrix}, \qquad V_7 = \widetilde{\rho}(g_5)^{-1} V_5 = \begin{pmatrix} \alpha(1-\lambda)+\beta(-2+3\lambda) \\ -\alpha(1+\lambda)+2\beta(2+\lambda) \end{pmatrix}.$$

Then, finally, letting the vector $W = \binom{\gamma}{1}$ for some $\gamma \in \mathbb{C}$, we obtain the cluster variables $\boldsymbol{x}^1, \dots, \boldsymbol{x}^5$ as follows. We note that a generic choice for α, β, and γ gives a nondegenerate solution. Here we abbreviate $\det(\cdot,\cdot)$ by $|\cdot,\cdot|$:

$$\boldsymbol{x}^1 = \begin{pmatrix} |V_1,W| \\ |V_2,H_1| \\ |H_1,W| \\ |V_2,W| \\ |V_3,H_2| \\ |H_2,W| \\ |V_3,W| \\ |V_6,H_4| \\ |H_4,W| \\ |V_6,W| \\ |V_7,H_3| \\ |H_3,W| \\ |V_7,W| \end{pmatrix}^T, \quad \boldsymbol{x}^2 = \begin{pmatrix} |V_1,W| \\ |V_2,H_1| \\ |H_1,W| \\ |V_2,W| \\ |V_3,H_2| \\ |H_2,W| \\ |V_3,W| \\ |V_5,H_3| \\ |H_3,W| \\ |V_5,W| \\ |V_7,H_5| \\ |H_5,W| \\ |V_7,W| \end{pmatrix}^T, \quad \boldsymbol{x}^3 = \begin{pmatrix} |V_1,W| \\ |V_2,H_1| \\ |H_1,W| \\ |V_2,W| \\ |V_4,H_4| \\ |H_4,W| \\ |V_4,W| \\ |V_5,H_2| \\ |H_2,W| \\ |V_5,W| \\ |V_7,H_5| \\ |H_5,W| \\ |V_7,W| \end{pmatrix}^T,$$

$$\boldsymbol{x}^4 = \begin{pmatrix} |V_1,W| \\ |V_2,H_1| \\ |H_1,W| \\ |V_2,W| \\ |V_4,H_4| \\ |H_4,W| \\ |V_4,W| \\ |V_6,H_5| \\ |H_5,W| \\ |V_6,W| \\ |V_7,H_3| \\ |H_3,W| \\ |V_7,W| \end{pmatrix}^T, \quad \boldsymbol{x}^5 = \begin{pmatrix} |V_1,W| \\ |V_2,H_1| \\ |H_1,W| \\ |V_2,W| \\ |V_3,H_1| \\ |H_1,W| \\ |V_3,W| \\ |V_6,H_4| \\ |H_4,W| \\ |V_6,W| \\ |V_7,H_3| \\ |H_3,W| \\ |V_7,W| \end{pmatrix}^T,$$

where

$$|V_1, W| = \alpha - \beta\gamma, \quad |V_2, H_1| = \beta, \quad |H_1, W| = 1, \quad |V_2, W| = -\alpha + \beta\gamma + \beta,$$
$$|V_3, H_2| = \beta, \quad |H_2, W| = -1, \quad |V_3, W| = \alpha - \beta(\gamma + 2), \quad |V_6, H_4| = (\lambda - 1)(\alpha - 3\beta),$$
$$|H_4, W| = (\gamma - 1)(\lambda - 1), \quad |V_6, W| = \alpha(-\gamma\lambda + \lambda - 1) + \beta(3(\gamma - 1)\lambda + \gamma + 2),$$
$$|V_7, H_3| = \alpha\lambda - \beta(2\lambda + 1), \quad |H_3, W| = \gamma - \gamma\lambda,$$
$$|V_7, W| = \alpha((\gamma - 1)\lambda + \gamma + 1) - \beta(2\gamma(\lambda + 2) - 3\lambda + 2), \quad |V_5, H_3| = \lambda^2(-(\alpha - 2\beta)),$$
$$|V_5, W| = \beta(2\gamma\lambda + \gamma + 2) - \alpha(\gamma\lambda + 1), \quad |V_7, H_5| = (\lambda - 1)(\alpha - 3\beta),$$
$$|H_5, W| = -\gamma\lambda + \lambda - 1, \quad |V_4, H_4| = (\lambda - 1)(-(\alpha - 2\beta)), \quad |H_4, W| = (\gamma - 1)(\lambda - 1),$$
$$|V_4, W| = (\gamma - 1)\lambda(\alpha - 2\beta) + \alpha - \beta(\gamma + 1), \quad |V_5, H_2| = \alpha\lambda - \beta(2\lambda + 1),$$
$$|H_2, W| = -1, \quad |V_6, H_5| = -\beta, \quad |V_3, H_1| = -\beta.$$

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