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The complement of a nIL graph with thirteen vertices is IL

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We show that for any simple nonoriented graph G with at least thirteen vertices either G or its complement is intrinsically linked.

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1 Introduction

Conway and Gordon [4] and Sachs [8] spearheaded the theory of spatial graphs by showing that every embedding of the complete graph on six vertices K_6 in $\mathbb{R}^3$ contains a nontrivial link. Such graphs are called intrinsically linked (IL), since the property is intrinsic to the graph and does not depend on the particular embedding. If not intrinsically linked, G is said to be *linklessly embeddable* (nIL). Conway and Gordon [4] also showed that K_7 is an *intrinsically knotted* graph, a graph whose every embedding in $\mathbb{R}^3$ contains a cycle which is a nontrivial knot. These results have generated a significant amount of work, including the classification of intrinsically linked graphs by Robertson, Seymour and Thomas [7]: a graph is intrinsically linked if and only if it contains one of the graphs in the Petersen family of graphs as a minor. For a graph G, *a minor of* G is any graph that can be obtained from G by a sequence of vertex deletions, edge deletions and edge contractions. An edge contraction means identifying its endpoints, deleting that edge, and deleting any double edges thus created. In this article, all graphs are nonoriented, without loops and without multiple edges.

The work in this article was motivated in part by the result of Battle, Harary and Kodoma [1] saying that for a graph with nine vertices, either the graph or its complement is nonplanar, and nine is minimal with this property. In particular, there exists a graph G with 8 vertices such that both G and its complement cG are planar. An example is given by the graph induced on vertices $v_1, \dots, v_8$ in Figure 1, left. Here we look at the similar question for intrinsic linkness.

We note that there are graphs with ten vertices such that both the graph and its complement are linklessly embeddable. We give an example in Figure 1. In this figure,

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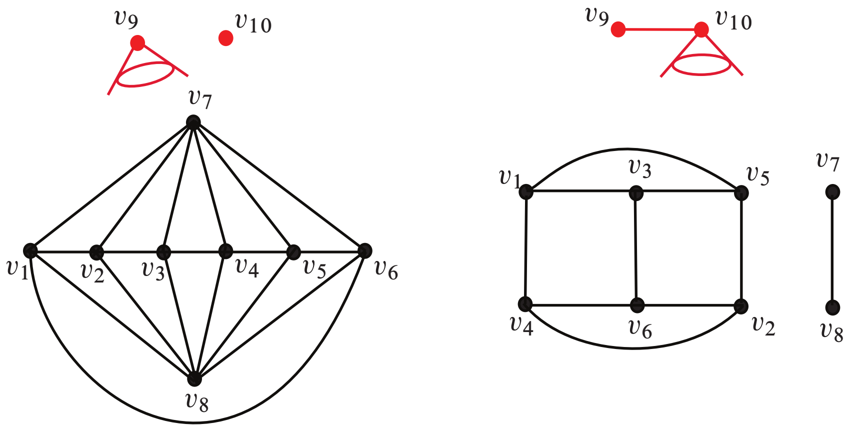

Figure 1: The graphs G (left) and cG (right) are both linklessly embeddable. The vertex v_9 of G is adjacent to all vertices $v_1, \dots, v_8$, and the vertex v_{10} of cG is adjacent to all vertices $v_1, \dots, v_9$.

the vertex v_9 of G is adjacent to all vertices $v_1, \dots, v_8$ of G (left) and the vertex v_{10} of cG is adjacent to all vertices $v_1, \dots, v_9$ of cG (right). In both G and cG, the subgraph induced by $v_1, v_2, \dots, v_7$ and v_8 is planar. To see why this is an example, we remember the result of Sachs that a graph is planar if and only if the cone over it is linklessly embeddable [8].

Campbell, Mattman, Ottman, Pyzer, Rodrigues and Williams [2] showed that a graph on $n \geq 6$ vertices and at least $4n-9$ edges is intrinsically linked because it contains a K_6 minor. Since the complete graph on n vertices has $\binom{n}{2}$ edges and $\binom{n}{2} \geq 2(4n-9)$ for $n \geq 15$, for a graph G with $n \geq 15$ vertices, either G or its complement is intrinsically linked. We improve this result by proving the following:

Theorem 1 *If G is a graph with at least thirteen vertices, either G or its complement is intrinsically linked.*

2 Notation and background

We first introduce notation. The complete graph with n vertices is denoted by K_n. For a graph G with n vertices, cG represents the *complement of* G in K_n. If $\{v_1, v_2, \dots, v_k\}$ are vertices of G, then $\langle v_1, v_2, \dots, v_k \rangle_G$ denotes the subgraph of G induced on these vertices. If v is a vertex of G, $N(v)$ is the subgraph of G induced by v and the vertices adjacent to v in G. If v is a vertex of G, $G-v$ is the subgraph of G obtained by deleting the vertex v and all edges of G that v is incident to. The maximum degree of a vertex over all vertices of G is denoted by $\operatorname{maxdeg}(G)$. For a graph G, *the cone over* G is the graph obtained from G by adding one extra vertex and all edges between this vertex and vertices of G.

Colin de Verdière introduced the new graph parameter μ based on spectral properties of matrices associated to a graph [3]. He showed that μ is monotone under taking minors and that planarity of G is equivalent to $\mu(G) \leq 3$. Lovász and Schrijver [6] showed that linkless embeddability of G is equivalent to $\mu(G) \leq 4$. Kotlov, Lovász and Vempala [5] showed that, for a graph G with n vertices,

(1) if G is outerplanar, then $\mu(cG) \geq n-4$, and

(2) if G is planar, then $\mu(cG) \geq n-5$.

These results give the following:

Lemma 2 *Consider a graph G with $n \geq 10$ vertices. If G is planar, then its complement cG is IL.*

Proof By [5], if G is planar, then $\mu(cG) \geq n-5 \geq 5$. This means cG is intrinsically linked [6]. □

Lemma 3 *For a graph G with n vertices, with $n \geq 11$, if there exists a vertex of G whose degree is at least 10, either G or its complement is intrinsically linked.*

Proof Let v be a vertex of G with $\deg(v) = q \geq 10$. Since $q \geq 10$, $N(v)$ is a cone over a graph with minimum 10 vertices, $N(v)-v$. If $N(v)-v$ is not planar, then $N(v)$ is intrinsically linked [8] and thus G is intrinsically linked. If $N(v)-v$ is planar, then the complement of $N(v)-v$ in K_q is intrinsically linked by Lemma 2, and therefore cG is intrinsically linked. □

Observation 4 *If G' is a minor of G obtained through an edge contraction, then cG' is a subgraph of cG.*

This observation and Lemma 3 imply the following:

Lemma 5 *Assume G is a graph with at least 12 vertices. If an edge contraction in G creates a vertex of degree at least 10, then either G or cG is intrinsically linked.*

Lemma 6 *For $n \geq 12$, if $\operatorname{maxdeg}(G) \geq 9$ then G or cG is intrinsically linked.*

Proof If G contains a vertex of degree at least 10, Lemma 3 implies the conclusion. Assume that $a \in V(G)$ has degree 9 and denote by $v_1, v_2, \dots, v_9$ its neighbors in G. If there exists a neighbor of a, say v_1, connecting to at least two vertices not in $N(a)$, then, since these would not be neighbors of a, by contracting the edge av_1, we obtain a minor of G with at least 11 vertices and a newly created vertex $a = v_1$ of degree at least 10. Lemma 3 provides the intrinsic linkness conclusion.

So we may assume that each vertex of $N(a)$ has at most one neighbor not in $N(a)$. Let $N := \langle v_1, v_2, \dots, v_9 \rangle_G$. Since $\langle a, N \rangle$ is a cone, unless G is intrinsically linked, N is planar. By [1], cN is nonplanar. In cG, a is adjacent to $v_{10}, v_{11}, \dots, v_{n-1}$, and each vertex $v_1, v_2, \dots, v_9$ connects to at least one vertex among $v_{10}, v_{11}, \dots, v_{n-1}$, since for $n \geq 12$, $\{v_{10}, v_{11}, \dots, v_{n-1}\}$ has at least 2 elements. By contracting all the edges av_{10}, av_{11}, …,av_{n-1} in cG, a cone over cN is obtained. Since cN is nonplanar, cG is intrinsically linked [8]. □

3 Graphs with thirteen vertices

Theorem 7 *Let G denote a graph with 13 vertices. Then either G or cG is intrinsically linked.*

Proof Let $k := \operatorname{maxdeg}(G)$. By Lemma 6, we need only consider $k \leq 8$. If $k \leq 5$, then G has at most 32 edges, and cG has at least 46 edges and is intrinsically linked [2]. We look at $k \in \{6, 7, 8\}$. We show that in each case, a sequence of edge contractions in either G or cG creates a vertex of degree 10 or a K_7 minor. Lemma 3 provides the intrinsic linkness conclusion. We note that, using symmetry, the assumption that $\operatorname{maxdeg}(G) = k$ adds that $\operatorname{maxdeg}(cG) = k$.

Case $k = 8$ Assume $\deg(v_1) = 8$. Let $v_2, v_3, \dots, v_9$ be the neighbors of v_1 in G; let $N := N(v_1)$ and $H := \langle v_{10}, v_{11}, v_{12}, v_{13} \rangle_G$. If any of the vertices of N has more than 2 neighbors in the set $\{v_{10}, v_{11}, v_{12}, v_{13}\}$, then a one-edge contraction creates a vertex of degree at least 10.

Assume that each of the vertices of N neighbors at most 2 vertices of H in G. Assume v_2v_{10} and v_2v_{11} are edges in G. If any of the v_i with $3 \leq i \leq 9$ neighbors both v_{12} and v_{13} in G, then contracting edges v_1v_2 and v_1v_i creates a degree 10 vertex. So each of the vertices $v_2, v_3, \dots, v_9$ misses at least one element of $\{v_{12}, v_{13}\}$ in G, which implies that contracting the edges v_1v_{12} and v_1v_{13} creates a degree 10 vertex neighboring $v_2, v_3, \dots, v_{11}$ in a minor of cG.

We may then assume each of the vertices $v_2, v_3, \dots, v_9$ neighbor at most one of the vertices of H in G, which means there are at most 8 edges between the vertices of $N := N(v_1)$ and those of H in G. This implies there are at least $36 - 8 = 28$ edges between cN and cH in cG. Since four of these edges connect v_1 to the vertices of cH, there are at least 24 edges between $\{v_2, \dots, v_9\}$ and cH in cG. Unless this lower bound is attained, the pigeonhole principle implies that one vertex of cH, say v_{10}, neighbors at least 7 vertices among $v_2, v_3, \dots, v_8$ and v_9. Contracting v_1v_{10} creates a degree ten vertex in a minor of cG. This implies that in cG each vertex of cH is adjacent to at most six vertices among $v_2, v_3, \dots, v_8$ and v_9, thus giving a maximum of $4 \cdot 6 = 24$ edges between cH and $cN \setminus v_1$ in cG. It follows that in G each of $v_2, v_3, \dots, v_8$ and v_9 has exactly one neighbor in H and each vertex of H neighbors exactly 2 of $v_2, v_3, \dots, v_8$ and v_9, thus partitioning this set into 4 blocks of 2 elements each. Contracting any two edges in the set $\{v_1v_{10}, v_1v_{11}, v_1v_{12}, v_1v_{13}\}$ creates a degree 10 vertex in cG.

Case $k = 7$ Assume $\deg(v_1) = 7$. Notice that this implies that the minimal degree in cG is 5. Let $v_2, v_3, \dots, v_8$ be the neighbors of v_1 in G. If any of these vertices has more than 3 neighbors in the set $\{v_9, v_{10}, v_{11}, v_{12}, v_{13}\}$ then a one-edge contraction creates a vertex of degree at least 10. Without loss of generality, assume v_2 neighbors v_9, v_{10} and v_{11} in G. If another v_i with $3 \le i \le 8$ neighbors both v_{12} and v_{13} in G, contracting the edges v_1v_2 and v_1v_i creates a vertex of degree 10 vertex in a 11–vertex minor of G. But this implies that contracting the edges v_1v_{12} and v_1v_{13} in cG creates a degree ten vertex.

It follows that there are at most 14 edges with one endpoint in $N := N(v_1)$ and the other endpoint in $H := \langle v_9, v_{10}, v_{11}, v_{12}, v_{13}\rangle_G$ in G. This implies that there are $40 - 5 - 14 = 21$ edges between cH and $\{v_2, v_3, \dots, v_8\}$ in cG. By the pigeonhole principle, there is at least one vertex in cH, say v_9, which connects with at least 5 vertices in $\{v_2, v_3, \dots, v_8\}$. If it connects with 6 of them, then contracting the edge v_1v_9 creates a degree 10 vertex in a minor of cG. So we may assume v_9 misses v_2 and v_3 in cG. Since $k = 7$ for both G and cG, v_9 neighbors at most one other vertex of cH.

If $\deg_{cG}(v_9) = 6$, contracting the edges v_1v_2 and v_2v_9 in G creates a degree 10 vertex in a minor of G. If $\deg_{cG}(v_9) = 7$ and v_9 neighbors v_{10} in cG, there is at least one element of $\{v_2, \dots, v_8\}$ which misses both v_9 and v_{10} in cG, since otherwise one might contract v_1v_9 and v_1v_{10} to create a degree 10 vertex in a minor of cG. This vertex must be one of the v_2 or v_3, so we assume it is v_2. Contracting v_1v_2 and v_2v_9 creates a degree 10 vertex in a minor of G. □

Case $k = 6$ Every vertex of G (and cG) has degree 6 and G has 39 edges. The graph G is said to be 6–regular. Denote by $v_2, v_3, \dots, v_7$ the neighbors of v_1 in G. Let $N := N(v_1) = \langle v_2, v_3, \dots, v_7 \rangle_G$. Let $H := \langle v_8, v_9, \dots, v_{13} \rangle_G$. If any of the vertices of N neighbors more than 4 of the vertices of H, then a one-edge contraction creates a degree 10 vertex in a minor of G.

Assume there exists a neighbor of v_1 in G, say v_2, which neighbors 4 vertices of H, say v_8, v_9, v_{10}, v_{11}. If, for any $3 \le i \le 7$, v_i neighbors both v_{12} and v_{13}, contracting v_1v_2 and v_1v_i produces a degree 10 vertex in a minor of G. It follows that each of $v_2, v_3, \dots, v_6$ and v_7 misses at least one of the v_{12} and v_{13} in G, hence contracting v_1v_{12} and v_1v_{13} in cG produces a degree ten vertex.

Consequently, there are at most 18 edges joining N to H. Since the sum of the degrees in G of the vertices of H is 36, H has at least 9 edges. Similarly, it follows that N has at least $(42-18)/2 = 12$ edges. Looking in the complement, we notice that $v_1 \cup cH$ has at most $6+6 = 12$ edges and that it constitutes $N(v_1)$ in cG. By symmetry, it follows that $|N| = 12$, $|H| = 9$ and $|L| = 18$, where L denotes the set of edges joining N and H. The only distribution that respects the 6–regularity of both G and its complement is one in which each element of either $N - v_1$ or H is incident to exactly three edges of L. This implies the subgraph $\langle v_2, \dots, v_7 \rangle_G$ has six edges, and each vertex $v_2, \dots, v_7$ has degree 2 in $\langle v_2, \dots, v_7 \rangle_G$. Then $\langle v_2, \dots, v_7 \rangle_G$ is either two disjoint triangles or a full 6–cycle. If $\langle v_2, \dots, v_7 \rangle_G$ is two disjoint cycles, $\langle v_2, \dots, v_7 \rangle_{cG}$ is a $K_{3,3}$. Then cG contains a $K_{3,3,1}$ minor given by contracting all edges of cG which have v_1 as their endpoint. So it can assumed that $v_2v_3v_4v_5v_6v_7v_2$ is a cycle in G. By symmetry, $v_8, \dots, v_{12}$ and v_{13} are joined by all edges in the K_6 they determine except the cycle $v_8v_9v_{10}v_{11}v_{12}v_{13}v_8$. Notice that this fully describes the adjacency relations within the neighborhood of v_1 and those of the set of nonneighbors of v_1, and the adjacency relations are the same for all vertices in G and cG: for a vertex v, every neighbor of v shares two common neighbors with v, and any nonneighbor of v has three common neighbors with v. It follows that G and cG are both strongly regular graphs with parameters $(13, 6, 2, 3)$, thus they are isomorphic to the Paley graph on 13 vertices. This graph contains a K_7 minor and is thus intrinsically linked. The K_7 minor can be obtained by contracting the following edges of G, highlighted in Figure 2: v_2v_9, v_3v_{10}, v_4v_{11}, v_5v_{12}, v_6v_{13}, v_7v_8.

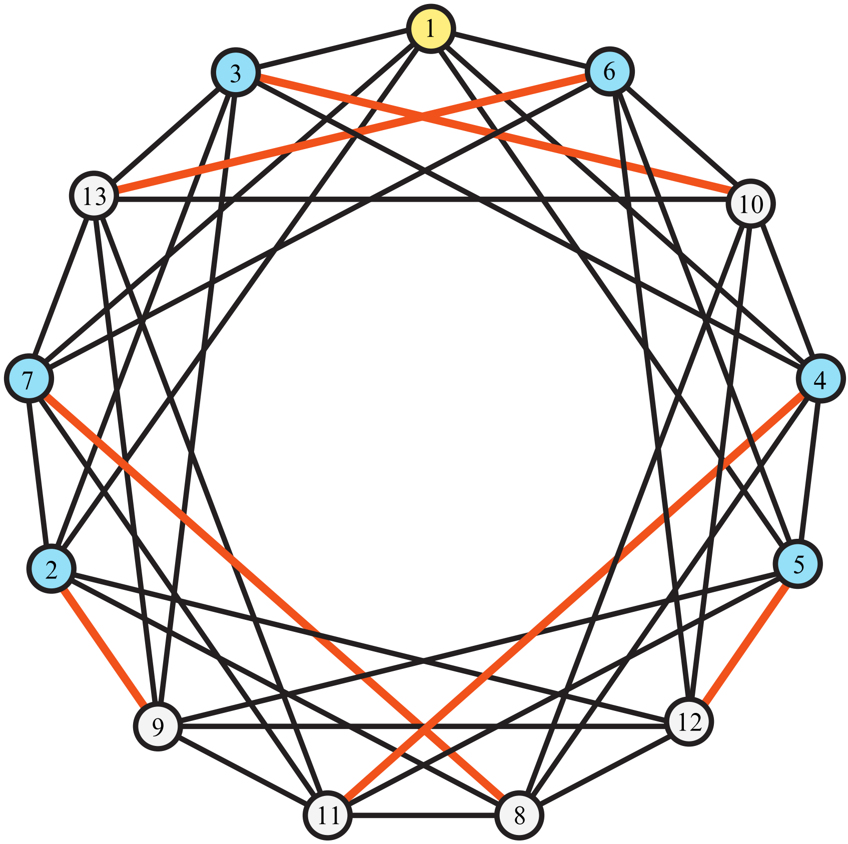

Figure 2: Paley graph with thirteen vertices. Contracting the highlighted edges yields a K_7 minor.

Remark 8 It is still an open question whether the complement of a linklessly embeddable graph with 11 or 12 vertices is necessarily intrinsically linked.

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