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Lusternik–Schnirelmann category of products with half-smashes

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We show that for a fixed space X and any sufficiently highly connected space A $(\mathrm{conn}(A) > \dim(X)$ is more than enough), the Lusternik–Schnirelmann category of products with X is remarkably stable with respect to changes in the second variable:

$$\mathrm{cat}(X \times A) = \mathrm{cat}(X \times (A \rtimes B)) \quad \text{for all spaces } B.$$

Taking $X = S^n$ leads to a closure property for the collections of spaces which do or do not satisfy the Ganea condition $\mathrm{cat}(S^n \times A) = 1 + \mathrm{cat}(A)$.

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1 Introduction

Lusternik–Schnirelmann (often abbreviated to L–S) category was introduced by Lusternik and Schnirelmann in the 1930s as part of an effort to quantify the way in which the topology of a manifold M forces smooth real-valued functions to have critical points. The behavior of Lusternik–Schnirelmann category with respect to products has been a continual puzzle. The fundamental product inequality

$$\mathrm{cat}(X \times Y) \leq \mathrm{cat}(X) + \mathrm{cat}(Y)$$

was discovered very early in the development of the theory and simple examples were soon found showing that the inequality could be strict. But all of these examples involved spaces having incompatible torsion in homology, and, in the following decades, no other examples were found. This led naturally to the suspicion that perhaps these torsion phenomena were the one and only way in which strict inequality could hold in the product formula. Ganea asked in 1970 [4] about a particularly attractive test case: does every space X satisfy the *Ganea condition*

$$\mathrm{cat}(X \times S^n) = \mathrm{cat}(X) + 1 \quad \text{for all } n \geq 1?$$

The conjecture (often referred to as the "Ganea conjecture", though it was not conjectured by him) that all spaces satisfy the Ganea condition was an important motivation for

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research into L–S category for more than twenty years until it was shown around 1990 by Hess [5] and Jessup [8] that it is true for simply connected rational spaces and by Iwase in 1997 [6] that there are simply connected finite CW complexes that are counterexamples.

Finding and recognizing spaces that do not satisfy the Ganea condition remains an interesting problem. For example, it is natural to ask: given X, for which values of n is $\mathrm{cat}(X \times S^n) = \mathrm{cat}(X)$? Iwase conjectured that if $\mathrm{cat}(X \times S^n) = \mathrm{cat}(X)$, then $\mathrm{cat}(X \times S^{n+t}) = \mathrm{cat}(X)$ for all $t \geq 0$. In [7], Iwase, Stanley and Strom proved this (for n sufficiently large, depending on the category, dimension and connectivity of X) as a special case of a theorem that says that if the connectivity of A is large enough (depending on the category, dimension and connectivity of X), then

$$\mathrm{cat}(X \times A) = \mathrm{cat}(X) \implies \mathrm{cat}(X \times (A \rtimes B)) < \mathrm{cat}(X) + \mathrm{cat}(A \rtimes B)$$

for all spaces B. Another direction of inquiry concerns the extent to which we can vary X while preserving the failure of the Ganea condition. Stanley, Scheerer and Tanré [9] introduced a stabilized variant of L–S category, called Qcat, and they conjectured that X satisfies the Ganea condition if and only if $\mathrm{cat}(X) = \mathrm{Qcat}(X)$. One implication of this Qcat conjecture (for finite complexes) follows from a theorem of Vandembroucq [10]: $\mathrm{Qcat}(X \times S^k) = \mathrm{Qcat}(X) + 1$ for every finite CW complex X and every $k \geq 1$. It is not hard to check that

$$\mathrm{Qcat}(X \rtimes B) = \mathrm{Qcat}(X) \quad \text{and} \quad \mathrm{cat}(X \rtimes B) = \mathrm{cat}(X)$$

for all spaces B. Thus it would follow from the Qcat conjecture that the property of satisfying the Ganea condition is stable under half-smash products: X satisfies the Ganea condition if and only if $X \rtimes B$ satisfies the Ganea condition for all spaces B.

Our purpose in this paper is to prove the following substantial generalization of the main theorem of [7]. As a corollary, we find that the stability of the Ganea condition under half-smash products does hold (subject to some connectivity conditions). This can be regarded as evidence for the Qcat conjecture of Scheerer, Stanley and Tanré.

Theorem 1 *Let X be a connected CW complex, let A be a well-pointed path-connected space*[1] *and write*

$$m = \mathrm{cat}(X \times A), \quad \mathrm{conn}(X) = n - 1 \quad \textit{and} \quad \mathrm{conn}(A) = a - 1.$$

[1] A pointed space is *well-pointed* if the inclusion of the basepoint is an unpointed cofibration; $\mathrm{conn}(X)$ denotes the connectivity of X; see Section 2.1

If $\dim(X) < m \cdot \min\{n, a\} + a - 1$, *then*

$$\operatorname{cat}(X \times A) = \operatorname{cat}(X \times (A \rtimes B))$$

for every space B.

Let us consider the case $X = S^n$. If $A \not\simeq *$, then $m = \operatorname{cat}(S^n \times A) \geq 2$. It follows that Theorem 1 applies provided $n < 3a - 1$. Thus we have the following corollary, which provides supporting evidence for the Qcat conjecture described above:

Corollary 2 *If* A *is a well-pointed space and* $n \leq 3\,\operatorname{conn}(A) + 1$, *then*

$$\operatorname{cat}(S^n \times (A \rtimes B)) = \operatorname{cat}(S^n \times A)$$

for every space B.

Since Iwase's examples Q_p (for p prime) are simply connected and they satisfy $\operatorname{cat}(S^2 \times Q_p) = \operatorname{cat}(Q_p) = 2$, we obtain a vast collection of new examples: $Q_p \rtimes B$ does not satisfy the Ganea condition for any space B.

2 Preliminaries

2.1 Basics

We work in an unpointed convenient category $\mathcal{T}$ for doing homotopy theory, such as the category of compactly generated weak Hausdorff spaces, and the corresponding pointed category $\mathcal{T}_*$.

The *half-smash* product of the well-pointed space X and the unpointed space Y is the pointed space

$$X \rtimes Y = X \wedge Y_+.$$

This can also be described as the pointed space $(X \times Y)/(* \times Y)$. Since smash products preserve cofiber sequences, so do half-smash products.

2.1.1 Quotient maps A (continuous) map $q\colon X \to Y$ is called a *quotient map* if the continuity of a function $f\colon Y \to Z$ can be decided by composition with q; that is, f is continuous if and only if $f \circ q$ is continuous. The proofs of the following results concerning quotient maps are left as a pleasant exercise for the reader:

(1) A retraction is a quotient map.

(2) Pushouts of quotient maps are quotient maps.

2.1.2 Connectivity, fibers and cofibers A space X is *n–connected* if the unique map $X \to *$ is an n–equivalence or, equivalently, if $\pi_k(X) = 0$ for all $k \le n$. We write $\mathrm{conn}(X) = n$ to mean that X is n–connected, but not $(n{+}1)$–connected; similarly, we write $\mathrm{conn}(f) = n$ to indicate that f is an n–equivalence but not an $(n{+}1)$–equivalence. The following is well-known:

Proposition 3 *Let $f\colon X \to Y$, and let F_f be its homotopy fiber. Then*

(a) $\mathrm{conn}(f) = \mathrm{conn}(F_f) + 1$;

(b) *if $A \to X \xrightarrow{f} Y$ is a cofiber sequence, then $\mathrm{conn}(F_f) \ge \mathrm{conn}(A)$.*

2.2 Lusternik–Schnirelmann category and cone length

2.2.1 L–S category A *Lusternik–Schnirelmann cover* of *size n* for a topological space X is an open cover $X = U_0 \cup U_1 \cup \cdots \cup U_n$ such that each inclusion $U_i \hookrightarrow X$ is nullhomotopic. The *Lusternik–Schnirelmann category* of X is

$$\mathrm{cat}(X) = \inf\{n \mid X \text{ has an L–S cover with size at least } n\},$$

with the standard convention that $\inf(\varnothing) = \infty$. The reader is directed to the book [2] for any and all details about L–S category.

2.2.2 Cone length The way in which a CW complex is iteratively constructed by attaching cells can be generalized to the concept of a cone decomposition. Our approach here is slightly different from the usual one in that it uses infinitely long decompositions.

A *cone decomposition* of $f\colon X \to Y$ is a homotopy-commutative diagram $\mathcal{D}$ of the form

$$(\mathcal{D})\qquad \begin{tikzcd}
A_0 \arrow[d] & A_1 \arrow[d] & & A_{n-1} \arrow[d] & A_n \arrow[d] & & \\
X_0 \arrow[r] \arrow[d, "\simeq"'] & X_1 \arrow[r] & \cdots \arrow[r] & X_{n-1} \arrow[r] & X_n \arrow[r] & \cdots \arrow[r] & X_\infty \arrow[d, "\simeq"] \\
X \arrow[rrrrrr, "f"] & & & & & & Y
\end{tikzcd}$$

in which each sequence $A_i \to X_i \to X_{i+1}$ is a mapping cone sequence (that is, $X_n = X_{n-1} \cup CA_{n-1}$ for each n and each inclusion $X_{n-1} \to X_n$ is the natural inclusion, which is a cofibration) and X_∞ is the categorical (and hence homotopy) colimit of the middle row.

A cone decomposition $\mathcal{D}$ has *length* at most n if the maps $X_m \to X_{m+1}$ are homotopy equivalences for all $m \geq n$; we write $\ell(\mathcal{D}) \leq n$ to indicate this.[2] The *cone length* of a well-pointed space X is defined to be

$$\mathrm{cl}(X) = \inf\{\ell(\mathcal{D}) \mid \mathcal{D} \text{ is a cone decomposition of } * \to X\}.$$

Since the functor $\mathbf{?} \rtimes B$ commutes with homotopy colimits, we have

$$\mathrm{cl}(A \rtimes B) \leq \mathrm{cl}(A) \quad \text{for any space } A.$$

It is well-known that $\mathrm{cat}(X) \leq \mathrm{cl}(X) \leq \mathrm{cat}(X) + 1$ for any space X, and since A is a retract of $A \rtimes B$, we easily obtain the formulas

$$\mathrm{cat}(A \rtimes B) = \mathrm{cat}(A) \quad \text{and} \quad \mathrm{cat}(X \times A) \leq \mathrm{cat}(X \times (A \rtimes B))$$

for any spaces A, B and X.

2.3 Ganea fibrations

The *Ganea construction* takes a fiber sequence $F \to E \xrightarrow{p} B$, constructs the obvious map $E/F \to B$, and converts it to a new fibration $G(p)\colon G(E) \to B$ with fiber $G(F)$. Ganea proved in [3] that $G(F) \simeq F * \Omega(B)$.

If we begin with the path fibration $\mathcal{P}(X) \to X$ and repeat the Ganea construction, we obtain a diagram of the form

$$(\mathcal{D}) \qquad \begin{tikzcd} \Omega(X)^{*1} \arrow[d] & & \Omega(X)^{*n} \arrow[d] & \Omega(X)^{*n+1} \arrow[d] & & \\ G_0(X) \arrow[r] \arrow[d, "p_0"'] & \cdots \arrow[r] & G_{n-1}(X) \arrow[r] \arrow[d, "p_{n-1}"'] & G_n(X) \arrow[r] \arrow[d, "p_n"] & \cdots \arrow[r] & G_\infty(X) \arrow[d] \\ X \arrow[r, equal] & \cdots \arrow[r, equal] & X \arrow[r, equal] & X \arrow[r, equal] & \cdots \arrow[r, equal] & X \end{tikzcd}$$

in which each column is a fiber sequence and $G_\infty(X)$ is the homotopy colimit of the middle row. Since an n–fold join is at least $(n-1)$–connected, the induced map $G_\infty(X) \to X$ is a homotopy equivalence.

Theorem 4 (Ganea [3]) $\mathrm{cat}(X) \leq n$ *if and only if* p_n *has a section.*

[2] A homotopy equivalence $f\colon X \xrightarrow{\simeq} Y$ has a cone decomposition of length 0 in which $A_n = *$ and $X_n = X$ for all n.

2.3.1 Cone length and category of products If well-pointed spaces X and Y have cone decompositions, then we can derive a cone decomposition for their product $X\times Y$ by setting

$$(X\times Y)_m = \bigcup_{i+j=m} X_i \times X_j.$$

It is well-known that there are cofiber sequences

$$\Bigl(\bigvee_{i+j=m+1} A_i * B_j\Bigr) \to (X\times Y)_m \hookrightarrow (X\times Y)_{m+1}$$

($U * V$ denotes the *join* of U and V; it is homotopy equivalent to $\Sigma(U\wedge V)$). Since $X_\infty \times Y_\infty = \bigcup_{n=0}^{\infty}(X\times Y)_m$, $X_\infty\times Y_\infty$ is the (homotopy) colimit of the resulting telescope diagram, so this sequence is indeed a cone decomposition (see [1] for details). It follows from this construction that

$$\mathrm{cl}(X\times Y)\le \mathrm{cl}(X)+\mathrm{cl}(Y)$$

for any two spaces X and Y.

We can apply these observations to the cone decompositions provided by the Ganea construction. For given spaces X and Y, we can form

$$\widetilde{G}_m(X,Y) = \bigcup_{i+j=m} G_i(X)\times G_j(Y) \subseteq G_m(X)\times G_m(Y),$$

which is the m^{th} step in a cone decomposition of $G_m(X)\times G_m(Y)$. Crucially for us, the composition

$$\widetilde{G}_m(X,Y) \xhookrightarrow{u} G_m(X)\times G_m(Y) \xrightarrow{p_m\times p_m} X\times Y,$$

which we'll denote by $\widetilde{p}_m$, has the same category-detecting property that $G_m(X\times Y)$ has.

Proposition 5 [6; 9] *The following are equivalent*:

(1) $\mathrm{cat}(X\times Y)\le m$.

(2) $\widetilde{p}_m\colon \widetilde{G}_m(X,Y)\to X\times Y$ *has a section up to homotopy.*

3 Two lemmas

Before embarking on our proof of Theorem 1, we establish two technical results. First, we show that the inequality hypothesis of Theorem 1 implies lower bounds on the connectivities of two maps. Secondly, we estimate the L–S category of a certain kind homotopy pushout.

Lemma 6 *Under the conditions of Theorem 1,*

(a) $\dim(X) < \mathrm{conn}(u\colon \widetilde{G}_m(X,A) \hookrightarrow G_m(X) \times G_m(A))$, *and*

(b) $\dim(X) < \mathrm{conn}(p_m\colon G_m(A) \to A)$.

Proof If $a \geq n$ and $\dim(X) < mn + a - 1$, then also $\dim(X) < (m+1)a - 1$, so we have this latter relation in either case. To ease the notational burden, we write

$$X_i = G_i(X) \quad \text{and} \quad A_j = G_j(A).$$

To prove (a) we observe that the inclusion $\widetilde{G}_m(X \times A) \hookrightarrow X_m \times A_m$ is the composition of a great many maps $\tau_{i,j}$, indexed by the integers i and j with $m < i + j \leq 2m$ and, crucially, with $i, j \geq 1$. These maps $\tau_{i,j}$ sit in (homotopy) pushout squares of the form

$$\begin{tikzcd}
(X_{i-1} \times A_j) \cup (X_i \times A_{j-1}) \arrow[r, "t_{i,j}"] \arrow[d] \arrow[dr, phantom, "\text{(H)PO}"] & X_i \times A_j \arrow[d] \\
Y \arrow[r, "\tau_{i,j}"] & Z
\end{tikzcd}$$

Therefore,

$$\mathrm{conn}(u) \geq \min\{\mathrm{conn}(\tau_{i,j}) \mid m < i + j \leq 2m\}.$$

To estimate the connectivity of u, we find a uniform lower bound for

$$\mathrm{conn}(t_{i,j}) \leq \mathrm{conn}(\tau_{i,j}).$$

By construction, there are cofiber sequences

$$(\Omega X)^{*i} \to X_{i-1} \to X_i \quad \text{and} \quad (\Omega A)^{*j} \to A_{j-1} \to A_j$$

and hence cofiber sequences

$$(\Omega X)^{*i} * (\Omega A)^{*j} \to (X_{i-1} \times A_j) \cup (X_i \times A_{j-1}) \xrightarrow{t_{i,j}} X_i \times A_j.$$

Proposition 3 tells us that

$$\begin{aligned}
\mathrm{conn}(t_{i,j}) &= \mathrm{conn}(F_{t_{i,j}}) + 1 \\
&\geq \mathrm{conn}((\Omega X)^{*i} * (\Omega A)^{*j}) + 1 \\
&= ((in + ja) - 2) + 1 \\
&= (in + ja) - 1.
\end{aligned}$$

The minimum value for $(in + ja) - 1$ must occur at some values of $i, j \geq 1$ for which $i + j$ takes its least possible sum, namely $m + 1$. In the case $a \geq n$, the value of

$(in+ja)-1$ is the one with $i=m$ and $j=1$; thus, if $a\geq n$, we have

$$\operatorname{conn}(t_{i,j}) = mn + a - 1 > \dim(X).$$

On the other hand, if $a<n$, the $t_{i,j}$ with smallest value is the one with $j=m$ and $i=1$, and so, in this case,

$$\operatorname{conn}(t_{i,j}) = n + ma - 1 > (m+1)a - 1 > \dim(X).$$

Together, these estimates show

$$\begin{aligned}\operatorname{conn}(u) &\geq \min\{\operatorname{conn}(\tau_{i,j}) \mid m < i+j \leq 2m\}\\ &\geq \min\{\operatorname{conn}(t_{i,j}) \mid m < i+j \leq 2m\}\\ &> \dim(X).\end{aligned}$$

This proves (a).

We complete the proof of Lemma 6 by analyzing the connectivity of the Ganea fibration $p_m\colon G_m(A)\to A$. Since its fiber is $(\Omega(A))^{*(m+1)}$, we have

$$\begin{aligned}\operatorname{conn}(p_m) &= \operatorname{conn}(F_m) + 1\\ &= \operatorname{conn}(\Omega(A)^{*(m+1)}) + 1\\ &\geq ((m+1)a - 2) + 1\\ &= (m+1)a - 1\\ &> \dim(X)\end{aligned}$$

by Proposition 3, regardless of the relationship between a and n. □

Lemma 7 *Let*

$$X_0 \to X_1 \to \cdots \to X_m \to \cdots \quad \textit{and} \quad A_0 \to A_1 \to \cdots \to A_m \to \cdots$$

be cone decompositions. Then the homotopy pushout of the diagram

$$X_m \times A_0 \times CB \leftarrow X_m \times A_0 \times B \to \Bigl(\bigcup_{i+j=m} X_i \times A_j\Bigr) \times B$$

has cone length, hence L–S category, at most m.

Proof We begin by identifying the homotopy type of the homotopy pushout, which we denote by P. The diagram defining P maps to the diagram

$$X_m \xleftarrow{\mathrm{pr}_{X_m}} X_m \times A_0 \times B \to \Bigl(\bigcup_{i+j=m} X_i \times A_j\Bigr) \times B$$

by a pointwise homotopy equivalence (since $A_0 \simeq CB \simeq *$). Since both diagrams involve at least one cofibration, the categorical pushouts are also homotopy pushouts, and the induced map $P \to Q$ of categorical pushouts is a homotopy equivalence. Therefore it suffices to work out the cone length of Q.

We identify Q up to homeomorphism by studying the diagram

$$\begin{tikzcd}
X_m \times A_0 \times B \arrow[r] \arrow[d, "\mathrm{pr}_{X_m}"'] \arrow[dr, phantom, "\text{(H)PO}"] & \left(\bigcup_{i+j=m} X_i \times A_j\right) \times B \arrow[d, "r"] \arrow[ddr, bend left, "s"] & \\
X_m \arrow[r] \arrow[drr, "t"'] & Q \arrow[dr, "\xi"] & \\
& & \bigcup_{i+j=m} X_i \times (A_j \rtimes B)
\end{tikzcd}$$

in which s is the obvious quotient map and t is the obvious inclusion. Since pr_{X_m} has a section, it is a quotient map, and, since the square is a categorical pushout square,

$$r\colon \left(\bigcup_{i+j=m} X_i \times A_j\right) \times B \to Q$$

is also a quotient map. On the point-set level, the effect of the map s is to identify points of the form $(x, *, b)$ with $(x, *, *)$ for $x \in X_m$ and $b \in B$; as precisely the same identifications are made by the map r, we see that ξ is a bijection. Finally, since r and s are both quotient maps, it follows that ξ is a homeomorphism.

To prove the lemma, it suffices to show that $\mathrm{cl}(Q) \leq m$. Truncate the given cone decompositions to obtain

$$X_0 \to \cdots \to X_m \xrightarrow{\mathrm{id}} X_m \xrightarrow{\mathrm{id}} \cdots \quad \text{and} \quad A_0 \to \cdots \to A_m \xrightarrow{\mathrm{id}} A_m \xrightarrow{\mathrm{id}} \cdots .$$

Next half-smash the second decomposition with B to obtain the cone decomposition

$$A_0 \rtimes B \to A_1 \rtimes B \to \cdots \to A_m \rtimes B \xrightarrow{\mathrm{id}} A_m \rtimes B \xrightarrow{\mathrm{id}} \cdots$$

for $A_m \rtimes B$, also of length m. Now we can recognize that $Q = (X_m \times (A_m \rtimes B))_m$, the m^{th} step in the cone decomposition that the product $X_m \times (A_m \rtimes B)$ inherits from the given cone decompositions of X_m and $A_m \rtimes B$ (described in Section 2.2). Thus $\mathrm{cl}(P) = \mathrm{cl}(Q) \leq m$. $\square$

4 Proof of Theorem 1

Since $\mathrm{cat}(X \times A) \le \mathrm{cat}(X \times (A \rtimes B))$ for all spaces A, B and X, it suffices to prove the reverse inequality

$$\mathrm{cat}(X \times (A \rtimes B)) \le \mathrm{cat}(X \times A)$$

for spaces satisfying the hypotheses of Theorem 1. We will accomplish this by showing that $X \times (A \rtimes B)$ is a retract of a certain space P with cone length at most $m = \mathrm{cat}(X \times A)$.

Since $\mathrm{cat}(X \times A) = m$, Proposition 5 tells us that the composition

$$\widetilde{G}_m(X, A) \xrightarrow{u} G_m(X) \times G_m(A) \xrightarrow{p_m \times p_m} X \times A$$

has a homotopy section, $\sigma\colon X \times A \to \widetilde{G}_m(X, A)$. We use σ to define

$$\sigma' = \mathrm{pr}_{G_m(X)} \circ u \circ \sigma \circ \mathrm{in}_X,$$

so that the solid-arrow part of the diagram

$$\begin{tikzcd}
X \arrow[r, "\sigma'"] \arrow[d, "\mathrm{in}_X"'] & G_m(X) \arrow[r, equal] \arrow[d, dotted, "k"] & G_m(X) \arrow[r, "p_m"] & X \\
X \times A \arrow[r, "\sigma"] \arrow[rrr, bend right, "v"'] & \widetilde{G}_m(X, A) \arrow[r, "u"] & G_m(X) \times G_m(A) \arrow[r, "p_m \times p_m"] \arrow[u, "\mathrm{pr}_{G_m(X)}"'] & X \times A \arrow[u, "\mathrm{pr}_X"']
\end{tikzcd}$$

commutes on the nose. Since σ is a homotopy section of $(p_m \times p_m) \circ u$, the map v is homotopic to $\mathrm{id}_{X \times A}$. Then

$$p_m \circ \sigma' = \mathrm{pr}_X \circ i \circ \mathrm{in}_X \simeq \mathrm{pr}_X \circ \mathrm{id}_{X \times A} \circ \mathrm{in}_X = \mathrm{id}_X,$$

so σ' is a homotopy section for $p_m\colon G_m(X) \to X$.

The key point (this is where the connectivity hypothesis comes in) is to prove that the diagram above remains commutative (up to homotopy) when the dotted arrow k (which is the inclusion of the first coordinate) is added. Thus we must compare the two maps $k \circ \sigma'$ and $\sigma \circ \mathrm{in}_X$.

Lemma 6(a) ensures that $\dim(X) < \mathrm{conn}(u)$, so it suffices to show that

$$u \circ (k \circ \sigma') = u \circ (\sigma \circ \mathrm{in}_X).$$

Since the target of these maps is a product, it suffices to study the compositions with the two projections and show that

$$\mathrm{pr}_{G_m(X)} \circ (u \circ (k \circ \sigma')) = \mathrm{pr}_{G_m(X)} \circ (u \circ (\sigma \circ \mathrm{in}_X))$$

and

$$\mathrm{pr}_{G_m(A)} \circ (u \circ (k \circ \sigma')) = \mathrm{pr}_{G_m(X)} \circ (u \circ (\sigma \circ \mathrm{in}_X)).$$

For the projection on $G_m(X)$, we have

$$\mathrm{pr}_{G_m(X)} \circ (u \circ (k \circ \sigma')) = \mathrm{id}_{G_m(X)} \circ \sigma' = \sigma' = \mathrm{pr}_{G_m(X)} \circ (u \circ (\sigma \circ \mathrm{in}_X)).$$

For the projection on $G_m(A)$, we compute

$$\mathrm{pr}_{G_m(A)} \circ (u \circ (k \circ \sigma')) = (\mathrm{pr}_{G_m(A)} \circ \mathrm{in}_{G_m(X)}) \circ \sigma' = *.$$

The calculation

$$\begin{aligned} p_m \circ (\mathrm{pr}_{G_m(A)} \circ (u \circ (\sigma \circ \mathrm{in}_X))) &= \mathrm{pr}_A \circ ((p_m \times p_m) \circ (u \circ (\sigma \circ \mathrm{in}_X))) \\ &\simeq \mathrm{pr}_A \circ \mathrm{id}_{X \times A} \circ \mathrm{in}_X \\ &\simeq * \end{aligned}$$

implies $\mathrm{pr}_{G_m(a)} \circ (u \circ (\sigma \circ \mathrm{in}_X)) \simeq *$ as well, because of Lemma 6(b). It follows then that $k \circ \sigma' \simeq \sigma \circ \mathrm{in}_X$, as required.

Since A is well-pointed, the inclusion $\mathrm{in}_X \colon X \to X \times A$ is a cofibration. Therefore we may find a map $s \colon X \times A \to G_m(X)$ that is homotopic to σ and which makes the square

$$\begin{tikzcd} X \arrow[r, "\mathrm{in}_X"] \arrow[d, "\sigma'"'] & X \times A \arrow[d, "s"] \\ G_m(X) \arrow[r, "k"] & \widetilde{G}_m(X, A) \end{tikzcd}$$

commute on the nose. This diagram becomes, after applying the functor $\mathbf{?} \times B$, the upper right square in the strictly commutative diagram

$$\begin{tikzcd} X \times CB \arrow[d, "\sigma' \times \mathrm{id}_{CB}"'] & X \times B \arrow[l] \arrow[r, "\mathrm{in}_X \times \mathrm{id}_B"] \arrow[d, "\sigma' \times \mathrm{id}_B"] & (X \times A) \times B \arrow[d, "s \times \mathrm{id}_B"] \\ G_m(X) \times \{*\} \times CB \arrow[d, "p_m \times \mathrm{id}_{CB}"'] & G_m(X) \times \{*\} \times B \arrow[l] \arrow[r, "k \times \mathrm{id}_B"] \arrow[d, "p_m \times \mathrm{id}_B"] & \widetilde{G}_m(X, A) \times B \arrow[d, "\widetilde{p}_m \times \mathrm{id}_B"] \\ X \times CB & X \times B \arrow[l] \arrow[r, "\mathrm{in}_{X \times B}"] & (X \times A) \times B \end{tikzcd}$$

in which each of the three vertical composites is a homotopy equivalence and all of the horizontal maps are cofibrations. The diagram induces maps of (homotopy) pushouts

$$X \times ((A \times B) \cup CB) \to P \to X \times ((A \times B) \cup CB)$$

in which the composition is a homotopy equivalence.

Writing $X_i = G_i(X)$ and $A_j = G_j(A)$ and recalling that $A_0 = X_0 = *$, we see that P is a space of the kind studied in Lemma 7. It follows that $\mathrm{cl}(P) \le m$ and hence

$$\mathrm{cat}(X \times (A \rtimes B)) = \mathrm{cat}(X \times ((A \times B) \cup CB)) \le \mathrm{cl}(P) \le m.$$

This completes the proof of Theorem 1. $\square$

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