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Trisections, intersection forms and the Torelli group

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We apply mapping class group techniques and trisections to study intersection forms of smooth 4–manifolds. Johnson defined a well-known homomorphism from the Torelli group of a compact surface. Morita later showed that every homology 3–sphere can be obtained from the standard Heegaard decomposition of S^3 by regluing according to a map in the kernel of this homomorphism. We prove an analogous result for trisections of 4–manifolds. Specifically, if X and Y admit handle decompositions without 1– or 3–handles and have isomorphic intersection forms, then a trisection of Y can be obtained from a trisection of X by cutting and regluing by an element of the Johnson kernel. We also describe how invariants of homology 3–spheres can be applied, via this result, to obstruct intersection forms of smooth 4–manifolds. As an application, we use the Casson invariant to recover Rohlin's theorem on the signature of spin 4–manifolds.

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1 Introduction

Every closed, oriented 3–manifold Y admits a Heegaard decomposition into the union of two genus-g handlebodies. Such a decomposition can be obtained from the standard Heegaard splitting of S^3 by cutting and regluing by some element of the mapping class group $\mathcal{M}(\Sigma)$ of a closed, genus-g surface. When Y is an integral homology 3–sphere, the map can be chosen in the Torelli group $\mathcal{I}_g$, which consists of mapping classes that act trivially on homology. This connection has proven enormously useful in understanding both the mapping class group and integral homology 3–spheres.

Trisections of smooth 4–manifolds are analogous to Heegaard splittings in dimension 3. Specifically, a $(g;k_1,k_2,k_3)$*–trisection* $\mathcal{T}$ of a smooth 4–manifold X is a decomposition $X = Z_1 \cup Z_2 \cup Z_3$ such that

(1) the sector Z_λ is diffeomorphic to the 4–dimensional 1–handlebody $\natural_{k_\lambda} S^1 \times B^3$,

(2) each double intersection $H_\lambda = Z_{\lambda-1} \cap Z_\lambda$ is diffeomorphic to the 3–dimensional 1–handlebody $\natural_g S^1 \times B^2$,

(3) the triple intersection $\Sigma_g = Z_1 \cap Z_2 \cap Z_3$ is a closed, oriented surface of genus g.

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It follows immediately from the definition that $H_\lambda \cup_\Sigma H_{\lambda+1}$ is a genus-g Heegaard splitting of the boundary $\partial Z_\lambda = \#_{k_\lambda} S^1 \times S^2$. Conversely, take a genus-g handlebody H_g and three homeomorphisms $\{\phi_\lambda \colon \partial H_g \to \Sigma_g\}$. Suppose the union

$$Y_\lambda = H_g \cup_{\phi_\lambda \phi_{\lambda+1}^{-1}} (-H_g)$$

is homeomorphic to $\#_{k_\lambda} S^1 \times S^2$ for $\lambda = 1, 2, 3$. Then a result of Laudenbach and Poénaru [14] states that we can build a unique smooth 4–manifold, up to diffeomorphism, by gluing three handlebodies H_1, H_2 and H_3 to Σ_g via the homeomorphisms ϕ_1, ϕ_2 and ϕ_3 and then capping off with three 4–dimensional 1–handlebodies.

Given the analogy between Heegaard splittings and trisections, it is natural to ask:

(1) Do results about Heegaard splittings and 3–manifolds extend to 4–dimensions?

(2) Can we apply 3–dimensional techniques to answer questions in 4–manifold topology?

Of specific interest in this paper is the work of Birman and Craggs [1], Johnson [7; 8; 9], Casson, and Morita [16; 17] relating the Torelli group to invariants of integral homology 3–spheres. Birman and Craggs used the Rohlin invariant of homology 3–spheres to define a collection of homomorphisms from the Torelli group to $\mathbb{Z}/2\mathbb{Z}$. Johnson later completely classified these homomorphisms and subsequently used these maps, along with his namesake homomorphism, to determine the abelianization of the Torelli group. Casson lifted the Rohlin invariant and defined a famous $\mathbb{Z}$–valued invariant of homology 3–spheres. Morita then applied Johnson's work to reinterpret the Casson invariant in terms of the mapping class group. In particular, every integral homology 3–sphere can be obtained by taking the standard Heegaard decomposition of S^3, cutting along the Heegaard surface, and regluing by an element of $\mathcal{K}_g$, the kernel of the Johnson homomorphism. As Johnson showed, this subgroup is precisely the subgroup generated by separating twists. The change in the Casson invariant can then be computed using the surgery formula.

The main result of this paper is a 4–dimensional analogue of Morita's result:

Theorem 1.1 *Let X and Y be homeomorphic, closed smooth 4–manifolds. Suppose that X and Y admit $(g; k, 0, 0)$–trisections $\mathcal{T}_X$ and $\mathcal{T}_Y$, respectively. Then $\mathcal{T}_Y$ can be obtained from $\mathcal{T}_X$ by cutting and regluing by an element of the Johnson kernel $\mathcal{K}_g$.*

Note that the condition that X and Y admit $(g; k, 0, 0)$–trisections is equivalent to the condition that they are simply connected and admit perfect Morse functions (see Theorem 6.2).

1.1 Intersection forms

A second goal is this paper is to describe a way in which, via trisections, mapping class group techniques may be used to obstruct intersection forms. Let X be a 4–manifold. Multiplication in the cohomology ring $H^*(X;\mathbb{Z})$ determines a symmetric, bilinear pairing $Q\colon H^2(X,\mathbb{Z})\times H^2(X,\mathbb{Z})\to H^4(X,\mathbb{Z})\cong\mathbb{Z}$. When X is simply connected, it is a result of Wall that Q determines the homotopy type of X and a result of Freedman that Q determines the homeomorphism type of X. Every unimodular quadratic form over $\mathbb{Z}$ is the intersection form of a topological 4–manifold. However, this is not true for smooth 4–manifolds and determining which quadratic forms can be realized is an important open problem in 4–manifold theory.

There is an extra subtlety that distinguishes trisections from Heegaard splittings. It is not true that an arbitrary choice of gluing maps $\{\phi_\lambda\}$ determines a closed 4–manifold. The pairwise union $Y_\lambda = H_g \cup_{\phi_\lambda\phi_{\lambda+1}^{-1}} (-H_g)$ may not be homeomorphic to $\#_{k_\lambda} S^1\times S^2$, let alone have the same integral homology. This gluing data only determines a more general *Heegaard triple* and only when each Y_λ has the correct homeomorphism type does this construction describe a unique, closed 4–manifold. When Y_1 is homeomorphic to $\#_k S^1\times S^2$, and Y_2 and Y_3 are integral homology 3–spheres, we refer to the triple $\{\phi_\lambda\}$ as a *pseudotrisection*.

While this appears unfortunate, in fact it is an opportunity to show that some quadratic forms cannot be realized as the intersection form of any closed, smooth, oriented 4–manifold. Let $\{\phi_\lambda\}$ be a triple of gluing maps. Each map induces a map on homology $(\phi_\lambda^{-1})_*\colon H_1(\Sigma)\to H_1(H_g)$. The kernel of this map is a g–dimensional subspace L_λ that is Lagrangian with respect to the intersection form on $H_1(\Sigma)$. As shown by Feller, Klug, Schirmer and Zemke [4], the triple of Lagrangian subspaces determines the cohomology ring of the induced 4–manifold. Moreover, even if the triple $\{\phi_\lambda\}$ is only a pseudotrisection, we can formally compute the "intersection form" using the same algebraic formula.

We then have a strategy to obstruct Q as the intersection form of smooth 4–manifold. Take the set of all pseudotrisections $\{\phi_\lambda\}$ whose "intersection form" is Q. As in Theorem 1.1, any two pseudotrisections with the same "intersection form" are related by an element of the Johnson kernel. To exclude Q, it now suffices to show that after cutting and regluing by any element ρ in $\mathcal{K}_g$, at least one of the resulting 3–manifolds $Y_{\lambda,\rho}$ is always a nontrivial homology 3–sphere.

1.2 Rohlin's theorem

A standard fact in 4–manifold topology is Rohlin's theorem:

Theorem 1.2 [18] *Let X be a closed, spin 4–manifold. Then $\sigma(X) = 0 \mod 16$.*

There exist several alternate proofs of this result in the literature (see Freedman and Kirby [5], Kervaire and Milnor [10], Kirby [11], Kirby and Melvin [12] and Lawson and Michelsohn [15]). We present a new proof here using the Casson invariant λ of homology 3–spheres, whose reduction mod 2 is better known as the Rohlin invariant μ. A standard approach uses Theorem 1.2 to show that μ is well defined. However, using trisections we can reverse the logical causality and apply the mod 2 Casson invariant to exclude spin 4–manifolds whose signature is $8 \mod 16$.

By standard surgery theory techniques, every spin 4–manifold is spin-cobordant to a simply connected 4–manifold X that has indefinite intersection form and a handle decomposition without 1– and 3–handles (Lemma 6.1). Since X is spin, the second Stiefel–Whitney class $w_2(X)$ vanishes and so the intersection form is even. By the classification of unimodular, indefinite and even symmetric bilinear forms, the intersection form Q_X is isomorphic to $mE_8 \oplus nH$, where a negative value of m corresponds to summands of $-E_8$. Rohlin's theorem is then equivalent to the statement that m must be even. Thus, it suffices to prove the following theorem.

Theorem 1.3 *Suppose that X is a spin 4–manifold that admits a $(g; k, 0, 0)$–trisection. Then the intersection form of X has the form*

$$Q_X \cong 2mE_8 \oplus nH$$

for some integer m.

The first step in proving Theorem 1.3 is to find a triple $\{\phi_\lambda\}$ of maps that determines an $(8; 0, 0, 0)$–pseudotrisection with intersection form E_8 (Figure 1). The resulting 3–manifold Y_3 can be immediately identified as the Poincaré homology sphere, whose Casson invariant is -1. In particular, since Y_2 is homeomorphic to S^3, we see that

$$\mu(Y_2) + \mu(Y_3) = 1 \mod 2.$$

Then, we use the surgery formulas for the Casson invariant to prove that

$$\mu(Y_{2,\rho}) + \mu(Y_{3,\rho}) = 1 \mod 2$$

for all $\rho \in \mathcal{K}_8$. More generally, if we start with a $(g;k,0,0)$–pseudotrisection with intersection form $mE_8 \oplus nH$, we show that

$$\mu(Y_{2,\rho}) + \mu(Y_{3,\rho}) = m \mod 2$$

for any $\rho \in \mathcal{K}_g$. This suffices to prove Theorem 1.3 and therefore Rohlin's theorem.

It would be interesting to apply a stronger invariant of homology 3–spheres, in order to obstruct other intersection forms. A naive hope is that, since the Casson invariant counts SU(2) representations of the fundamental group and Donaldson invariants count SU(2) instantons, one might recover Donaldson's diagonalization theorem [2]. However, the full Casson invariant does not appear to add more information than its mod 2 version. For example, it does not obstruct the intersection forms $E_8 \oplus E_8$ or $E_8 \oplus \langle 1 \rangle$, which are definite but not diagonal and therefore excluded by Donaldson.

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2 Torelli group and the Johnson homomorphism

2.1 Torelli group

Let Σ_g be a closed, oriented surface of genus g and let $\Sigma_{g,1}$ denote a compact, oriented surface of genus g and one boundary component. Let $\mathcal{M}_g$ denote the mapping class group of Σ_g and let $\mathcal{I}_g$ denote the Torelli group of Σ_g. It is the subgroup of $\mathcal{M}_g$ consisting of classes of homeomorphisms of Σ_g to itself that induce the trivial homomorphism on homology. Let $\mathcal{M}_{g,1}$ be the mapping class group of $\Sigma_{g,1}$, consisting of isotopy classes of homeomorphisms of $\Sigma_{g,1}$ that fix the boundary pointwise. Let $\mathcal{I}_{g,1}$ denote the Torelli group of $\Sigma_{g,1}$. For technical reasons, it is often convenient to work with $\mathcal{I}_{g,1}$ instead of $\mathcal{I}_g$. Every element of $\mathcal{M}_{g,1}$ can be extended to an element of $\mathcal{M}_g$. The Torelli groups are related by the exact sequence

$$1 \to \pi_1(\mathrm{UT}\Sigma_g) \to \mathcal{I}_{g,1} \to \mathcal{I}_g \to 1,$$

where $\mathrm{UT}\Sigma_g$ denotes the unit tangent bundle of Σ_g. Let $\mathcal{K}_g$ and $\mathcal{K}_{g,1}$ be the subgroups of $\mathcal{I}_g$ and $\mathcal{I}_{g,1}$ generated by Dehn twists on separating curves.

The following is a basic fact in the theory of mapping class groups.

Proposition 2.1 *Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_g)$ be a collection of disjoint, simple closed curves representing linearly independent classes in $H_1(\Sigma)$. Let $\boldsymbol{\alpha}' = (\alpha_1', \dots, \alpha_g')$ be a second collection of disjoint, simple closed curves satisfying $[\alpha_i] = [\alpha_i']$ for all $i = 1, \dots, g$. There exists an element $\phi \in \mathcal{I}_g$ such that $\phi(\alpha_i) = \alpha_i'$ for all $i = 1, \dots, g$.*

Proof We sketch the proof; more details can be found in [3, Chapter 6.3.2]. The cut systems $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$ can be extended to geometric symplectic bases $\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ and $\{\boldsymbol{\alpha}', \boldsymbol{\beta}'\}$ such that $[\beta_i] = [\beta_i']$. Each complement $\Sigma \smallsetminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ and $\Sigma \smallsetminus (\boldsymbol{\alpha}' \cup \boldsymbol{\beta}')$ is a sphere with g boundary components. We can choose a homeomorphism between the spheres that identifies α_i with α_i' and β_i with β_i' and extends to a homeomorphism ϕ of Σ. The induced map on homology sends $[\alpha_i]$ to $[\alpha_i']$ and $[\beta_i]$ to $[\beta_i']$. In other words, ϕ acts trivially on homology and is therefore in the Torelli group. □

2.2 Johnson homomorphism

Let $\Gamma = \Gamma_0$ denote the fundamental group of $\Sigma_{g,1}$, with a basepoint chosen in the boundary, let $\Gamma_k = [\Gamma, \Gamma_{k-1}]$ denote the k^{th} term in the lower central series for Γ and let $N_k = \Gamma / \Gamma_k$ be the k^{th} nilpotent quotient. Note that $N_1 = \Gamma/[\Gamma, \Gamma] \cong H_1(\Sigma_{g,1}; \mathbb{Z})$. There is an exact sequence of groups

$$1 \to \mathcal{L}_{k+1} \to N_{k+1} \to N_k \to 1,$$

where $\mathcal{L}_{k+1}$ is the center of N_{k+1}. There is a natural action of $\mathcal{M}_{g,1}$ on Γ and the canonical morphism $\mathcal{M}_{g,1} \to \mathrm{Aut}(\Gamma)$ is injective. The subgroups Γ_k are characteristic and therefore preserved by any automorphism of Γ. Therefore, we obtain a map $\mathcal{M}_{g,1} \to \mathrm{Aut}(N_k)$ and denote its kernel by $\mathcal{M}_{g,1}(k)$. The Torelli group $\mathcal{I}_{g,1}$ is precisely $\mathcal{M}_{g,1}(1)$. There is a sequence of homomorphisms

$$\tau_k \colon \mathcal{M}_{g,1}(k) \to \mathrm{Hom}(N_1, \mathcal{L}_{k+1})$$

for $k \geq 1$. For $k = 1$, the homomorphism

$$\tau \colon \mathcal{I}_{g,1} \cong \mathcal{M}_{g,1}(1) \to \mathrm{Hom}(N_1, \mathcal{L}_2) \cong \Lambda^3 H_1(\Sigma)$$

is known as the *the* Johnson homomorphism.

Theorem 2.2 [8] *The kernel of τ is $\mathcal{K}_{g,1}$.*

Recall that a *bounding pair* consists of two disjoint, homologous curves d and d' and the corresponding bounding pair map $T_d T_{d'}^{-1}$ is an element of the Torelli group.

The genus of a bounding pair is the genus of the subsurface cut off by $d \cup d'$. A *k–chain* $(c_1, \dots, c_k)$ is a collection of simple closed curves such that $i(c_j, c_{j+1}) = 1$ and $i(c_j, c_k) = 0$ for $|j-k| > 1$. If k is odd, then the boundary of a neighborhood of $c_1 \cup \cdots \cup c_k$ is a bounding pair of genus $\frac{1}{2}(k-1)$. If k is even, then the boundary of a neighborhood of $c_1 \cup \cdots \cup c_k$ is a separating curve that cuts off a subsurface of genus $\frac{1}{2}k$.

Lemma 2.3 *Let (a, b, c) be a 3–chain and let $T_d T_{d'}^{-1}$ be the corresponding bounding pair map. Then*

$$\tau(T_d T_{d'}^{-1}) = [a] \wedge [b] \wedge [c].$$

To understand equivalences between Heegaard splittings and between trisections, we are interested in bounding pair maps that extend across a handlebody H_g.

Lemma 2.4 *Let d and d' be a bounding pair that bounds an annulus in a handlebody H_g. Then the bounding pair map $T_d T_{d'}^{-1}$ extends across the handlebody.*

Proof Cut H_g along the annulus and reglue via a Dehn twist. This gives a homeomorphism of the handlebody to itself that restricts to the bounding pair map on the boundary surface. □

Lemma 2.5 *Let (a, b, c) be a 3–chain and suppose that one of the three curves bounds in the handlebody H_g. Then the corresponding bounding pair map $T_d T_{d'}^{-1}$ extends across the handlebody.*

Proof Without loss of generality, assume that either a or b bound in the handlebody. Then the separating curve s, which is the boundary of a neighborhood of the 2–chain (a, b), bounds in the handlebody. Consequently, the curve d' is obtained from d by a band sum with this curve s. It is now clear that d and d' bound an annulus in the handlebody. Thus by Lemma 2.4, the bounding pair map $T_d T_{d'}^{-1}$ extends across the handlebody. □

3 Heegaard splittings and trisections

Throughout this section, let H_g denote a 3–dimensional, genus-g handlebody and let Σ_g be an abstract, closed surface of genus g. Let $\mathcal{M}(\partial H_g)$ denote the mapping class group of ∂H_g and let $\mathcal{M}(\Sigma_g)$ denote the mapping class group of Σ_g. Let $\mathcal{H}_g$

denote the subgroup of $\mathcal{M}(\partial H_g)$ consisting of classes that can be represented by a homeomorphism that extends across H_g.

A *cut system of disks* for H_g is a collection of g disjoint, properly embedded disks $\mathcal{D} = \{D_i\}$ in H_g whose complement is homeomorphic to B^3. The boundaries of the disks are a collection $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_g\}$ of disjoint, simple closed curves in ∂H_g that generate a g–dimensional subspace in $H_1(\partial H_g; \mathbb{Z})$. If $\phi \in \mathcal{H}_g$, then the cut system $\phi(\mathcal{D})$ can be connected to $\mathcal{D}$ by a sequence of handleslides. A *handleslide* of D_i over D_j consists of replacing D_i by D_i', which is obtained by joining D_i to a second copy of D_j by an embedded arc in ∂H_g.

A collection $\boldsymbol{\alpha}$ of g disjoint, simple closed curves in Σ_g whose complement has genus 0 is known as a *cut system of curves*. A homeomorphism $\phi\colon \partial H_g \to \Sigma_g$ sends the boundaries of a cut system of disks in H_g to a cut system of curves $\boldsymbol{\alpha}$.

We assume, once and for all, that we have fixed a cut system of disks $\mathcal{D}$ with boundary $\boldsymbol{\alpha}$ for H_g.

3.1 Heegaard splittings

Let Θ_g denote the set $\theta = \{\theta_1, \theta_2\}$, where each $\theta_\lambda\colon \partial H_g \to \Sigma$ is an isotopy class of homeomorphisms. Given any element θ, we can build a closed 3–manifold

$$Y_\theta = H_g \cup_{\theta_2^{-1}\theta_1} (-H_g).$$

This decomposition of Y_θ into two handlebodies is a *Heegaard decomposition*.

A collection of $2g$ simple closed curves $\{x_i, y_i\}$ on Σ is a *geometric symplectic basis* if the geometric intersection numbers satisfy

$$i(x_i, y_j) = \delta_{i,j}, \quad i(x_i, x_j) = 0 \quad \text{and} \quad i(y_i, y_j) = 0.$$

Definition 3.1 A Heegaard decomposition $S^3 = Y_\theta$, with $\theta = \{\theta_1, \theta_2\}$, is *standard* if $\{\theta_1(\alpha_i), \theta_2(\alpha_i)\}$ is a geometric symplectic basis for Σ_g.

Definition 3.2 A Heegaard decomposition $\#_k\, S^1 \times S^2 = Y_\theta$, with $\theta = \{\theta_1, \theta_2\}$, is *standard* if there exists a geometric symplectic basis $\{x_i, y_i\}$ such that

$$\theta_1(\alpha_i) = x_i \quad \text{and} \quad \theta_2(\alpha_i) = \begin{cases} x_i & \text{if } 1 \le i \le k, \\ y_i & \text{if } k+1 \le i \le g. \end{cases}$$

Many pairs $\{\theta_1, \theta_2\}$ specify the same Heegaard decomposition up to homeomorphism. Recall that $\mathcal{H}_g$ denotes the subgroup of $\mathcal{M}(\partial H_g)$ consisting of the classes of homeomorphisms that extend across the handlebody H_g. Then $\mathcal{H}_g \times \mathcal{H}_g$ acts on Θ_g on the right as

$$\{\theta_1, \theta_2\} \cdot (\mu_1, \mu_2) = \{\theta_1 \mu_1, \theta_2 \mu_2\},$$

and the mapping class group $\mathcal{M}(\Sigma)$ acts on Θ_g on the left as

$$\rho \cdot \{\theta_1, \theta_2\} = \{\rho\theta_1, \rho\theta_2\}.$$

Let $\mathcal{G}_2 = \mathcal{H}_g \times \mathcal{H}_g \times \mathcal{M}(\Sigma)$.

Lemma 3.3 *The $\mathcal{G}_2$–orbits of Θ_g are precisely the equivalence classes of Heegaard splittings.*

An important and well-known fact, due to Waldhausen, is that the standard Heegaard splitting of S^3 is essentially unique.

Theorem 3.4 [20] *Suppose that the pair $\theta = \{\theta_1, \theta_2\}$ determines a Heegaard splitting of S^3. Then*

$$\{\theta_1, \theta_2\} \sim \{\iota_1, \iota_2\},$$

where $\iota = \{\iota_1, \iota_2\}$ is standard.

By inductively applying Haken's lemma, a similar statement holds for connected sums of $S^1 \times S^2$.

Corollary 3.5 *Suppose that the pair $\theta = \{\theta_1, \theta_2\}$ determines a Heegaard splitting of $\#_k\, S^1 \times S^2$. Then*

$$\{\theta_1, \theta_2\} \sim \{\iota_1, \iota_2\},$$

where $\iota = \{\iota_1, \iota_2\}$ is standard.

The homology of Y_θ can be computed using the chain complex

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z}^g \xrightarrow{Q} \mathbb{Z}^g \xrightarrow{0} \mathbb{Z} \to 0.$$

All of the maps are zero except for Q. Let $\{x_i\}$ denote a basis for $\mathbb{Z}^g$ and $\{y_j\}$ denote a second basis. The linear map Q is defined by the formula

$$Qy_j = \sum_{i=1}^{g} \langle \theta_1(\alpha_i), \theta_2(\alpha_j) \rangle_\Sigma,$$

where $\langle\ ,\ \rangle_\Sigma$ denotes the intersection pairing on $H_1(\Sigma; \mathbb{Z})$.

Proposition 3.6 *Let $\theta = \{\theta_1, \theta_2\} \in \Theta_g$, and let Y_θ be the associated 3–manifold.*

(1) *Y_θ is an integral homology 3–sphere if and only if Q is unimodular.*

(2) *If $\rho \in \mathcal{I}_g$ and $\iota = \{\iota_1, \iota_2\}$ is a standard Heegaard decomposition of S^3, then the pair $\{\iota_1, \rho\iota_2\}$ determines an integral homology 3–sphere Y_ρ.*

(3) *If Y_θ is an integral homology sphere, then there exists some standard Heegaard decomposition $\iota = \{\iota_1, \iota_2\}$ of S^3 and an element $\rho \in \mathcal{I}_g$ such that*

$$\{\theta_1, \theta_2\} \sim \{\iota_1, \rho\iota_2\}.$$

Morita proved a stronger version of Proposition 3.6, which we state in the current formalism.

Theorem 3.7 [16] *If Y_θ is an integral homology sphere, then there exists some standard Heegaard decomposition $\iota = \{\iota_1, \iota_2\}$ of S^3 and an element $\rho \in \mathcal{K}_g$ such that*

$$\{\theta_1, \theta_2\} \sim \{\iota_1, \rho\iota_2\}.$$

3.2 Trisections of closed, smooth 4–manifolds

Recall that a $(g; k_1, k_2, k_3)$–trisection $\mathcal{T}_X$ of a closed, oriented, smooth 4–manifold is a decomposition $X = Z_1 \cup Z_2 \cup Z_3$, where each Z_λ is a handlebody $\natural_{k_\lambda} S^1 \times B^3$, each double intersection $H_\lambda = Z_{\lambda-1} \cap Z_\lambda$ is a genus-g handlebody, and $\Sigma = Z_1 \cap Z_2 \cap Z_3$ is a closed, oriented genus-g surface. We can identify each double intersection with a fixed, abstract H_g and the central surface with a fixed, abstract Σ_g. These identifications induce a triple of maps

$$\phi_\lambda \colon \partial H_g \to \Sigma_g$$

for $\lambda = 1, 2, 3$. Thus, every trisection $\mathcal{T}_X$ determines a triple $\phi = \{\phi_1, \phi_2, \phi_3\}$ of homeomorphisms. This triple uniquely describes X up to diffeomorphism. Furthermore, the pair $\{\phi_\lambda, \phi_{\lambda+1}\}$ determines a Heegaard splitting of $\#_{k_\lambda} S^1 \times S^2$.

The triple ϕ depends on choices. For any choices $\mu_\lambda \in \mathcal{H}_g$ and $\rho \in \mathcal{M}(\Sigma_g)$, the triple $\{\rho\phi_1\mu_1, \rho\phi_2\mu_2, \rho\phi_3\mu_3\}$ determines a diffeomorphic 4–manifold. Let Φ_g denote the set of triples $\phi = \{\phi_1, \phi_2, \phi_3\}$ where each $\phi_\lambda \colon \partial H_g \to \Sigma$ is an isotopy class of homeomorphisms. The group $\mathcal{H}_g \times \mathcal{H}_g \times \mathcal{H}_g$ acts on Φ_g on the right by the rule

$$\{\phi_1, \phi_2, \phi_3\} \cdot (\mu_1, \mu_2, \mu_3) = \{\phi_1\mu_1, \phi_2\mu_2, \phi_3\mu_3\},$$

and the group $\mathcal{M}(\Sigma)$ acts on Φ_g on the left by the rule

$$\rho \cdot \{\phi_1, \phi_2, \phi_3\} = \{\rho\phi_1, \rho\phi_2, \rho\phi_3\}.$$

Let $\mathcal{G}_3 = \mathcal{H}_g \times \mathcal{H}_g \times \mathcal{H}_g \times \mathcal{M}(\Sigma)$. We can combine the above actions into a single $\mathcal{G}_3$ action on Φ_g, and each diffeomorphism class of trisections corresponds with a unique $\mathcal{G}_3$–orbit. We refer to the action of $\mathcal{M}(\Sigma)$ as *global reparametrization* and the action of $\mathcal{H}_g \times \mathcal{H}_g \times \mathcal{H}_g$ as a *handlebody diffeomorphism*.

Remark 3.8 If $\phi = \{\phi_1, \phi_2, \phi_3\}$ is a triple arising from a trisection of a closed 4–manifold, then by Theorem 3.4 and Corollary 3.5, each triple $\{\phi_\lambda, \phi_{\lambda+1}\}$ is $\mathcal{G}_2$–equivalent to a standard $\{\iota_1, \iota_2\}$. However, it is not true in general that ϕ is $\mathcal{G}_3$–equivalent to some $\{\iota_1, \iota_2, \iota_3\}$ where every pair $\{\iota_\lambda, \iota_{\lambda+1}\}$ is standard. It is known that only connected sums of $\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2$ and $S^2 \times S^2$ admit this property.

It is often convenient to work just in $\mathcal{M}(\Sigma_g)$ and interpret $\mathcal{H}_g$ as a subgroup of $\mathcal{M}(\Sigma_g)$. Let $\phi = \{\phi_\lambda\}$ be a fixed triple. Define subgroups of $\mathcal{M}(\Sigma)$

$$\begin{aligned}\mathcal{A}_\phi &:= \{\phi_1 \mu \phi_1^{-1} : \mu \in \mathcal{H}_g\},\\ \mathcal{B}_\phi &:= \{\phi_2 \mu \phi_2^{-1} : \mu \in \mathcal{H}_g\},\\ \mathcal{C}_\phi &:= \{\phi_3 \mu \phi_3^{-1} : \mu \in \mathcal{H}_g\}.\end{aligned}$$

Consequently, the orbits of the action of $\mathcal{H}_g \times \{1\} \times \{1\}$ on the right are precisely the orbits of the action of $\mathcal{A}$ on the right. Similarly for the orbits of $\mathcal{B}$ and $\mathcal{C}$. Furthermore, let $\mathcal{AB}_\phi = \mathcal{A}_\phi \cap \mathcal{B}_\phi$.

Lemma 3.9 *Fix a triple $\phi = \{\phi_1, \phi_2, \phi_3\}$. If $\rho \in \mathcal{AB}_\phi$, then*

$$\{\phi_1, \phi_2, \phi_3\} \sim \{\phi_1, \phi_2, \rho\phi_3\}.$$

Proof Since $\rho \in \mathcal{AB}_\phi = \mathcal{A}_\phi \cap \mathcal{B}_\phi$, there exist $a, b \in \mathcal{H}_g$ such that

$$\phi_1 a = \rho\phi_1 \quad \text{and} \quad \phi_2 b = \rho\phi_2.$$

Thus

$$\{\phi_1, \phi_2, \phi_3\} \sim \{\rho\phi_1, \rho\phi_2, \rho\phi_3\} \sim \{\phi_1 a, \phi_2 b, \rho\phi_3\} \sim \{\phi_1, \phi_2, \rho\phi_3\}. \qquad \square$$

3.3 Heegaard diagrams

For many purposes, such as computing the algebraic topology of X, it is often easier and sufficient to encode $\phi_i\colon \partial H_g \to \Sigma_g$ by its action on a cut system.

Given a collection of maps $\{\phi_\lambda\}$ that determine a trisection of a 4–manifold X, we obtain a *trisection diagram* $(\Sigma, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3)$, where $\alpha_\lambda = \phi_\lambda(\boldsymbol{\alpha})$. To agree with

conventions in Heegaard Floer theory, we often denote the trisection diagram as $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ instead. A collection of the form $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$, where each of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ is a cut system of curves, is known as a *Heegaard triple*. Conversely, given a Heegaard triple $(\Sigma, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3)$, we can choose a (nonunique) triple of homeomorphisms $\{\phi_\lambda \colon \partial H_g \to \Sigma_g\}$ such that $\boldsymbol{\alpha}_\lambda$ is the image under ϕ_λ of a fixed cut system of curves.

Lemma 3.10 *A Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ determines a unique $\mathcal{G}_3$–orbit in Φ_g.*

The cohomology ring of X can be computed from the Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ alone. In particular, the cut systems $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ respectively determine g–dimensional subspaces L_α, L_β and L_γ in $H_1(\Sigma; \mathbb{Z})$. As detailed in [4], the cohomology ring $H^*(X; \mathbb{Z})$ of X is determined purely by the triple L_α, L_β, L_γ, up to a symplectic automorphism of $H_1(\Sigma; \mathbb{Z})$.

Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ be a trisection diagram for a $(g; 0, k, 0)$–trisection. Let ${}_\alpha Q_\beta$ be the $(g \times g)$–matrix of intersection numbers $\langle \alpha_i, \beta_j \rangle_\Sigma$ and define ${}_\beta Q_\gamma$ and ${}_\gamma Q_\alpha$ similarly. Then the intersection form Q_X is given in matrix form by the following formula [4, Theorem 4.3]:

$$Q_X = (-1) \cdot_\gamma Q_\beta \cdot ({}_\alpha Q_\beta)^{-1} \cdot_\gamma Q_\alpha. \tag{1}$$

Theorem 3.11 [4] *Let $\mathcal{T}_X$ be a $(g; 0, k, 0)$–trisection of X. Then $\mathcal{T}_X$ admits a diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ such that*:

(1) *$(\Sigma, \alpha, \boldsymbol{\beta})$ is a standard Heegaard diagram for S^3.*

(2) *In $H_1(\Sigma)$, we have*

$$[\gamma_i] = -[\alpha_i] - \sum_{j=1}^{g} \widetilde{Q}_{i,j} [\beta_j],$$

where Q is the intersection form of X and $\widetilde{Q} = Q \oplus \langle 0 \rangle^k$.

Combining the above theorem with Proposition 2.1, we obtain:

Corollary 3.12 *Let X and Y be closed, smooth, oriented 4–manifolds such that $Q_X \cong Q_Y$. Suppose X and Y admit $(g; 0, k, 0)$–trisections $\mathcal{T}_X$ and $\mathcal{T}_Y$. Then there exists a diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ for $\mathcal{T}_X$ and an element $\rho \in \mathcal{I}_g$ such that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \rho(\boldsymbol{\gamma}))$ is a diagram for $\mathcal{T}_Y$.*

3.4 Pseudotrisections

We introduce the more general notion of a *pseudotrisection*. Recall from the previous subsection that a trisection of a closed, smooth, oriented 4–manifold can be encoded by a triple of maps $\{\phi_\lambda\colon \partial H_g \to \Sigma_g\}$ or a Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$. In an honest trisection, the 3–manifolds obtained as the union of any pair of handlebodies must be homeomorphic to S^3 or $\#_k\, S^1 \times S^2$ for some $k \geq 1$. In a pseudotrisection, we relax that condition and allow two of the 3–manifolds to merely have the same $\mathbb{Z}$–homology as one of those manifolds.

Definition 3.13 Suppose that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ is a diagram that defines a $(g; 0, k, 0)$–pseudotrisection $p\mathcal{T}$. The *intersection form* $Q_{p\mathcal{T}}$ of the pseudotrisection is the quadratic form determined by the formula in (1).

For each unimodular intersection form Q, we can build a pseudotrisection with intersection form Q as follows:

Proposition 3.14 *Let Q be a unimodular, symmetric bilinear form of rank g. Then there exists a $(g+k; k, 0, 0)$–pseudotrisection $(\Sigma, \boldsymbol{\alpha}_Q, \boldsymbol{\beta}_Q, \boldsymbol{\gamma}_Q)$ such that*

(1) *the pair $\{\alpha, \beta\}$ is standard,*

(2) *the pair $\{\beta, \gamma\}$ is standard, and*

(3) $[\gamma_i] = -[\alpha_i] - \sum_{j=1}^{g} \widetilde{Q}_{i,j}[\beta_j]$,

where $\widetilde{Q} = Q \oplus \langle 0\rangle^k$.

Proof Start with the standard $(g, 0)$–trisection $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}')$ of $\#_g\, \mathbb{CP}^2$. The standard $(1, 0)$–trisection of $\mathbb{CP}^2$ is given by the diagram $(\Sigma, \alpha, \beta, \gamma)$ where α and β intersect geometrically once and γ represents the class $-[\alpha] - [\beta]$. Taking the connected sum of g copies gives the standard $(g, 0)$–trisection of $\#_g\, \mathbb{CP}^2$.

Let b_i and b_j be curves homotopic to β_i and β_j, respectively, and disjoint from $\boldsymbol{\beta}$. In addition, let $b_{i,j}$ be a band sum of b_i and b_j disjoint from $\boldsymbol{\beta}$. Now set

$$\phi = \left(\prod_{i=1}^{g} T_{b_i}^{Q_{i,i}-1} e\right)\left(\prod_{1\leq i<j\leq g} (T_{b_{i,j}} T_{b_i}^{-1} T_{b_j}^{-1})^{Q_{i,j}}\right),$$

and let $\boldsymbol{\gamma} = \phi(\boldsymbol{\gamma}')$. We can assume that each Dehn twist is along a curve disjoint from a fixed collection of curves $\{\beta_1, \ldots, \beta_g\}$. Thus, the geometric intersection number

between $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ never changes. In particular, $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ still form a geometric symplectic basis for the surface. The induced map ϕ_* on $H_1(\Sigma_g)$ satisfies

$$\phi_*([\alpha_i]+[\beta_i]) = [\alpha_i] + \sum_{j=1}^{g} Q_{i,j}[\beta_j].$$

Therefore, $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ gives a $(g;0)$–pseudotrisection with intersection form Q.

To obtain a $(g+k;k,0,0)$–pseudotrisection, we can stabilize k times by connected sum with the standard $(1;1,0,0)$–trisection of S^4. A diagram for this trisection is $(T^2, \alpha, \alpha, \beta)$, where (T^2, α, β) is a standard Heegaard diagram for the genus-1 splitting of S^3. □

We refer to the pseudotrisection $\mathcal{T}_Q = (\Sigma, \boldsymbol{\alpha}_Q, \boldsymbol{\beta}_Q, \boldsymbol{\gamma}_Q)$ constructed in Proposition 3.14 as the *standard pseudotrisection* for Q.

Proposition 3.15 *Let Q be an intersection form.*

(1) *Let $\mathcal{T} = (\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\gamma}')$ be a $(g;k,0,0)$–pseudotrisection with intersection form Q. There exists an equivalent diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ for $\mathcal{T}$ such that $\{[\alpha_i],[\beta_j]\}$ is a symplectic basis for $H_1(\Sigma)$ and*

$$[\gamma_i] = -[\alpha_i] - \sum_{j=1}^{g} \widetilde{Q}_{i,j}[\beta_j],$$

where $\widetilde{Q} = Q \oplus \langle 0\rangle^k$.

(2) *Let $\mathcal{T} = (\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\gamma}')$ be a $(g;k,0,0)$–pseudotrisection diagram satisfying the conclusions of part (1). Then there exists some $\phi \in \mathcal{I}_g$ such that $\mathcal{T}_{Q,\phi} := (\Sigma, \boldsymbol{\alpha}_Q, \boldsymbol{\beta}_Q, \phi(\boldsymbol{\gamma}_Q))$ is equivalent to $\mathcal{T}$.*

(3) *Let*

$$\mathcal{T}_1 = (\Sigma_g, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \boldsymbol{\gamma}_1) \quad \text{and} \quad \mathcal{T}_2 = (\Sigma_g, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, \boldsymbol{\gamma}_2)$$

be $(g+k;k,0,0)$–pseudotrisections that satisfy the conclusions of part (1) and have the same intersection form Q. Then there exists some $\phi \in \mathcal{I}_g$ such that $\phi(\boldsymbol{\gamma}_1) = \boldsymbol{\gamma}_2$.

Proof The proof of (1) is essentially the same as the proof of Theorem 3.11, which is a restatement of [4, Theorem 4.4]. The only difference is that we cannot assume $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is geometrically standard. However, the proof only requires that $\{[\alpha_i],[\beta_i]\}$

form a symplectic basis for $H_1(\Sigma)$. This follows since the 3–manifold constructed from the diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is an integral homology 3–sphere.

Part (2) follows from part (1) by Proposition 2.1.

To prove (3), note that by (2) we can assume $\mathcal{T}_1$ has a diagram $(\Sigma, \boldsymbol{\alpha}_Q, \boldsymbol{\beta}_Q, \phi_1(\boldsymbol{\gamma}_Q))$ and $\mathcal{T}_2$ has a diagram $(\Sigma, \boldsymbol{\alpha}_Q, \boldsymbol{\beta}_Q, \phi_2(\boldsymbol{\gamma}_Q))$. Now set $\phi = \phi_2 \circ \phi_1^{-1}$. □

4 Trisections and Torelli group

The main result of this section is an analogue of Morita's result (Theorem 3.7) for trisections of closed, smooth 4–manifolds.

Theorem 4.1 *Let Q be a unimodular, symmetric bilinear form of rank n over $\mathbb{Z}$. Let X be a closed, smooth 4–manifold with intersection form Q and which admits a $(g;k,0,0)$–trisection $\mathcal{T}_X$. Let $(\Sigma, \boldsymbol{\alpha}_Q, \boldsymbol{\beta}_Q, \boldsymbol{\gamma}_Q)$ be the pseudotrisection from* Proposition 3.14. *Then there exists a map $\phi \in \mathcal{K}_g$ such that $(\boldsymbol{\alpha}_Q, \boldsymbol{\beta}_Q, \phi(\boldsymbol{\gamma}_Q))$ is a trisection diagram for the trisection $\mathcal{T}_X$.*

Fix an intersection form Q and let $\mathcal{T}_Q$ be the standard $(g;k,0,0)$–pseudotrisection with intersection form Q. For concreteness, let $\phi = \{\phi_1, \phi_2, \phi_3\}$ be a triple of maps $\phi_\lambda \colon \partial H_g \to \Sigma_g$ encoding this pseudotrisection. We then obtain subgroups $\mathcal{A}_\phi, \mathcal{B}_\phi, \mathcal{C}_\phi \subset \mathcal{M}(\Sigma_g)$ of homeomorphisms that extend across the α–, β– and γ–handlebodies, respectively. From now on, we suppress the subscript ϕ, although it is important that these groups depend on ϕ.

Let $\mathcal{TA}$ be the intersection of $\mathcal{A}$ with the Torelli group $\mathcal{I}_g$. Define $\mathcal{TB}$, $\mathcal{TC}$ and $\mathcal{TAB}$ similarly, where $\mathcal{AB}$ denotes the intersection of $\mathcal{A}$ and $\mathcal{B}$. Finally, denote the Johnson homomorphism by $\tau \colon \mathcal{I}_{g,1} \to \Lambda^3 H_1(\Sigma_g)$.

Proposition 4.2 *Suppose that Q is a unimodular, symmetric bilinear form, and let $(\Sigma, \boldsymbol{\alpha}_Q, \boldsymbol{\beta}_Q, \boldsymbol{\gamma}_Q)$ be the standard $(g;k,0,0)$–pseudotrisection for Q. Then*

$$\tau(\mathcal{TAB}) + \tau(\mathcal{TC}) = \Lambda^3 H_1(\Sigma).$$

Proof We just need to find explicit bounding pairs that map onto a basis for $\Lambda^3 H_1(\Sigma_g)$. To do this, we decompose $\Lambda^3 H_1(S_g)$ into subspaces and then check that each subspace is in $\tau(\mathcal{TAB}) + \tau(\mathcal{TC})$.

We can assume that we have a geometric symplectic basis $\{x_i, y_i\}$ for the surface Σ_g such that $\{x_1, \dots, x_g\}$ bound in H_α, such that $\{y_1, \dots, y_{g-k}, x_{g-k+1}, \dots, x_g\}$ bound in H_β and such that some curves $\{z_1, \dots, z_g\}$ bound in H_γ, where

$$[z_i] = \begin{cases} [x_i] + Q[y_i] & \text{for } 1 \le i \le g-k, \\ [y_i] & \text{for } g-k+1 \le i \le g. \end{cases}$$

We partition the basis curves into two sets. Let $\{x_i, y_i\}$ for $1 \le i \le g-k$ be the *left* generators and $\{x_l, y_l\}$ for $g-k+1 \le l \le g$ be the *right* generators. Let W_X^1 be the subspace spanned by elements of the form $x_i \wedge a \wedge b$, where x_i, a and b are left generators; let W_Y^1 be the subspace spanned by elements of the form $y_i \wedge a \wedge b$, where y_i, a and b are left generators; let $W_{XY}^1 = W_X^1 \cap W_Y^1$ be their intersection and $W^1 = W_X^1 + W_Y^1$ their sum. Define W_X^2, W_Y^2, W_{XY}^2 and W^2 similarly using only right generators. Finally, define W_X, W_Y, W_{XY} and W in the same manner but allowing elements of any combination between left and right.

First, it is a consequence of Morita's original argument that W^2 is in the image $\tau(\mathcal{TAB}) + \tau(\mathcal{TC})$. Specifically, the following elements lie in $\tau(\mathcal{TAB}) \cap \tau(\mathcal{TC})$:

$$\begin{gathered} [x_i] \wedge [y_i] \wedge ([x_i] + [x_j]), \qquad [y_i] \wedge [x_i] \wedge ([y_i] + [y_j]), \\ [x_i] \wedge ([y_i] + [x_j]) \wedge ([y_j] + [y_k]), \quad [y_i] \wedge ([x_i] + [y_j]) \wedge ([x_j] + [x_k]). \end{gathered}$$

In addition, the elements

$$[x_i] \wedge ([y_i] + [x_j]) \wedge ([y_j] + [x_k])$$

lie in $\tau(\mathcal{TAB})$, and the elements

$$[y_i] \wedge ([x_i] + [y_j]) \wedge ([x_j] + [y_k])$$

lie in $\tau(\mathcal{TC})$. These elements suffice to span W^2.

Next, let $1 \le i, j, l \le g-k$. The following elements are each the image of a 3–chain, at least one of which bounds in H_α and at least one of which bounds in H_β. Thus, by Lemma 2.5, they lie in the image $\tau(\mathcal{AB})$:

$$\begin{gathered} [x_i] \wedge [y_i] \wedge ([x_i] + [x_j]), \qquad [y_i] \wedge [x_i] \wedge ([y_i] + [y_j]), \\ ([x_i] + [x_l]) \wedge [y_i] \wedge ([x_i] + [x_j]), \quad ([y_i] + [y_l]) \wedge [x_i] \wedge ([y_i] + [y_j]). \end{gathered}$$

Thus $W_{XY}^1 \subset \tau(\mathcal{TAB})$.

Next, since $\{y_i, z_i\}$ are geometrically standard, we can find 3–chains that live in $\mathcal{C}$ mapping to

$$[y_i] \wedge ([x_i] + Q[y_i]) \wedge ([y_i] + [y_l]) = [y_i] \wedge Q[y_i] \wedge [y_j] \mod W_{XY}$$

and

$$([y_i] + [y_l]) \wedge ([x_i] + Q[y_i]) \wedge ([y_i] + [y_l]) \\ = ([y_i] + [y_l]) \wedge Q[y_i] \wedge ([y_i] + [y_l]) \mod W_{XY}.$$

Because Q is unimodular, we can express any $[y_m]$ as a $\mathbb{Z}$–linear combination of the $\{Q[y_i]\}$, and thus $W_Y^1 \subset \tau(\mathcal{AB}) + \tau(\mathcal{C})$. Finally, it again follows from Morita's original proof that

$$[z_i] \wedge [z_j] \wedge [z_i] = [x_i] \wedge [x_j] \wedge [x_k] \mod W_Y$$

is in $\tau(\mathcal{C})$. Consequently, $W^2 \subset \tau(\mathcal{TAB}) + \tau(\mathcal{TC})$.

Finally, we need to ensure all mixed elements lie in the image. From now on, we will let $1 \le i, j \le g-k$ and $g-k+1 \le l, m \le g$.

By an argument similar to that above, we can obtain the elements

$$\begin{array}{lll} [x_i] \wedge [y_i] \wedge [x_l], & ([x_i] + [x_j]) \wedge [y_i] \wedge [x_l], & [x_i] \wedge ([y_i] + [y_j]) \wedge [x_l], \\ [x_i] \wedge [y_i] \wedge [y_l], & ([x_i] + [x_j]) \wedge [y_i] \wedge [y_l], & [x_i] \wedge ([y_i] + [y_j]) \wedge [y_l], \\ [x_l] \wedge [y_l] \wedge [x_i], & ([x_l] + [x_m]) \wedge [y_l] \wedge [x_i], & [x_l] \wedge ([y_l] + [y_m]) \wedge [x_i], \\ [x_l] \wedge [y_l] \wedge [y_i], & ([x_l] + [x_m]) \wedge [y_l] \wedge [y_i], & [x_l] \wedge ([y_l] + [y_m]) \wedge [y_i], \end{array}$$

in the image $\tau(\mathcal{TAB})$. This is enough to completely span W_{XY}.

Moreover, the elements

$$[x_i] \wedge [y_i + x_l] \wedge [x_j] = [x_i] \wedge [x_l] \wedge [x_j] \mod W_{XY},$$
$$[x_i] \wedge [y_i + x_l] \wedge [x_m] = [x_i] \wedge [x_l] \wedge [x_m] \mod W_{XY}$$

also lie in $\tau(\mathcal{TAB})$ and so we can span W_X. And finally, the elements

$$[y_i] \wedge ([z_i] + [z_l]) \wedge [y_j] = [y_i] \wedge ([y_i] + [y_l]) \wedge [y_j] \mod W_X,$$
$$[y_i] \wedge ([z_i] + [z_l]) \wedge [y_m] = [y_i] \wedge ([y_i] + [y_l]) \wedge [y_m] \mod W_X$$

are in the image $\tau(\mathcal{TC})$. We have thus constructed a basis for $\Lambda^3 H_1$ that lies in $\tau(\mathcal{TAB}) + \tau(\mathcal{TC})$. □

Theorem 4.3 *Let X be a closed, oriented, smooth 4–manifold X that admits a $(g;k,0,0)$–trisection. Then X admits a trisection diagram*

$$\mathcal{T}_{Q,\phi} = (\Sigma, \boldsymbol{\alpha}_Q, \boldsymbol{\beta}_Q, \phi(\boldsymbol{\gamma}_Q)),$$

where Q is the intersection form of X, where $\mathcal{T}_Q = (\Sigma, \boldsymbol{\alpha}_Q, \boldsymbol{\beta}_Q, \boldsymbol{\gamma}_Q)$ is the standard pseudotrisection for Q constructed in Proposition 3.14, and where $\phi \in \mathcal{K}_g$.

Proof Just as in the proof of Corollary 3.12, we can apply Proposition 2.1 to find a trisection diagram $(\Sigma, \boldsymbol{\alpha}_Q, \boldsymbol{\beta}_Q, \psi(\boldsymbol{\gamma}_Q))$ for X, where ψ is an element of the Torelli group $\mathcal{I}_g$. By Proposition 4.2, we can find some elements $a \in \mathcal{TAB}$ and $c \in \mathcal{TC}$ such that

$$\tau(\psi) = \tau(a) + \tau(c).$$

Hence, $\tau(a^{-1}\psi c^{-1}) = -\tau(a) + \tau(\phi) - \tau(c) = 0$. Thus, $\phi = a^{-1}\psi c^{-1} \in \ker(\tau) = \mathcal{K}_g$. It now follows from Lemma 3.9 that the trisection diagrams

$$(\Sigma, \boldsymbol{\alpha}_Q, \boldsymbol{\beta}_Q, \phi(\boldsymbol{\gamma}_Q)) \quad \text{and} \quad (\Sigma, \boldsymbol{\alpha}_Q, \boldsymbol{\beta}_Q, \psi(\boldsymbol{\gamma}_Q))$$

are equivalent. □

5 Casson invariant and linking forms

5.1 Casson invariant

We refer the reader to [19] for an introduction to the Casson invariant and its relationship to the Alexander polynomial and Arf invariant.

If K is a knot in a homology 3–sphere Y, let $Y + (1/m)K$ denote the integral homology 3–sphere obtained by Dehn surgery on K with slope $1/m$.

Let $\lambda(Y) \in \mathbb{Z}$ denote the Casson invariant of an integral homology 3–sphere Y. It has the following properties:

(1) $\lambda(S^3) = 0$.

(2) For any Y, any knot K in Y and any integer $m \in \mathbb{Z}$, the difference

$$\lambda'(K) := \lambda\left(Y + \frac{1}{m+1}K\right) - \lambda\left(Y + \frac{1}{m}K\right)$$

is independent of m.

(3) If $K \cup L$ is a boundary link in Y, then

$$\begin{aligned}\lambda''(K,L) := \lambda\left(Y + \frac{1}{m+1}K + \frac{1}{n+1}L\right) - \lambda\left(Y + \frac{1}{m}K + \frac{1}{n+1}L\right) \\ -\lambda\left(Y + \frac{1}{m+1}K + \frac{1}{n}L\right) + \lambda\left(Y + \frac{1}{m}K + \frac{1}{n}L\right) \\ = 0.\end{aligned}$$

(4) $\lambda'(T(2,3)) = 1$.

(5) $\lambda'(K) = \frac{1}{2}\Delta''_K(1)$.

(6) $\lambda(-Y) = -\lambda(Y)$.

(7) $\lambda(Y_1 \# Y_2) = \lambda(Y_1) + \lambda(Y_2)$.

Moreover, we have that

$$\tfrac{1}{2}\Delta''_K(1) = \mathrm{Arf}(K) \mod 2.$$

5.2 Linking form

The Arf invariant and Casson invariant of knots can be computed in terms of a linking form on the homology of a Seifert surface.

Let Y be an oriented 3–manifold and Σ an oriented, embedded surface in Y. The linking form is a map

$$l\colon H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}$$

defined in terms of the pairwise linking of curves on Σ. Specifically, let a and b be simple closed curves in Σ. Then

$$l([a],[b]) = \mathrm{lk}_Y(a, b^+),$$

where b^+ denotes a pushoff of b in the positive normal direction to Σ. The linking pairing is well defined on homology and bilinear and satisfies the symmetry relation

$$l([b],[a]) = l([a],[b]) + \langle [a],[b]\rangle_\Sigma. \tag{2}$$

Reducing mod 2, we obtain a map $q\colon H_1(\Sigma;\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ defined by setting

$$q(a) = l(a,a).$$

It is a *quadratic enhancement* of the intersection pairing $\langle\ ,\ \rangle_\Sigma$, meaning that it satisfies the relation

$$q(x+y) = q(x) + q(y) + \langle x, y\rangle_\Sigma \mod 2. \tag{3}$$

Now, suppose that Σ is a Seifert surface for a knot $K \subset Y$, where Y is an integral homology 3–sphere. Let $\{a_i, b_i\}$ be a geometric symplectic basis for Σ. The Casson invariant $\lambda'(K)$ of K can be computed in terms of the linking form on Σ according to the formula

$$\lambda'(K) = \sum_{i=1}^{g} (l(a_i, a_i)l(b_i, b_i) - l(a_i, b_i)l(a_i, b_i)) + 2 \sum_{1 \le i < j \le g} (l(a_i, a_j)l(b_i, b_j) - l(a_i, b_j)l(a_j, b_i)). \tag{4}$$

The Arf invariant can be computing using the simpler formula

$$\mathrm{Arf}(K) = \sum_{i=1}^{g} q(a_i) \cdot q(b_i) \mod 2. \tag{5}$$

5.3 Linking forms on the central surface

Let Σ be the central surface of a pseudotrisection. This surface embeds in each 3–manifold Y_i as a Heegaard surface. Each embedding determines a linking form l_i on $H_1(\Sigma)$ and a quadratic enhancement q_i. In general, these linking forms are distinct. However, with respect to a particular basis, we can describe l_2 and l_3 in terms of the quadratic form of the pseudotrisection. In addition, let c be a separating simple closed curve in Σ. Then, via the embedding of Σ in Y_i, it determines a knot K_i. In general, the Arf invariant, Alexander polynomial and Casson invariant of K_i depend on i. However, these also can be described in terms of the quadratic form of the pseudotrisection.

Throughout this subsection, let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ be a pseudotrisection with quadratic form Q. Let $Y_2 = H_\beta \cup -H_\gamma$ and $Y_3 = H_\gamma \cup -H_\alpha$.

First, we describe the difference between the quadratic enhancements q_2 and q_3.

Lemma 5.1 *Suppose that Q is even. Then $q_2 = q_3$.*

Proof By (3), it is enough to check on the symplectic basis $\{[z_i], [x_i]\}$. Furthermore, we assume that we have performed handleslides so that the diagram satisfies the conclusions of Proposition 3.15.

First, the mod 2 linking form q_3 in Y_3 vanishes on each element of the basis $\{[x_i], [z_i]\}$ since we can choose representatives that bound disks in one of the two handlebodies H_γ and H_α. By the same argument, the mod 2 linking form q_2 in Y_2 vanishes on the

basis $\{[y_i],[z_i]\}$ since each can be represented by the boundary of a compressing disk. To check that the mod 2 linking forms agree, we need to check that $q_2([x_i])$ always vanishes. By (3), we have

$$q_2([x_i]) = q_2(-[z_i] + Q[y_i]) = q_2([z_i]) + q_2(Q[y_i]) + \langle [z_i], Q[y_i]\rangle_\Sigma = Q_{i,i} \mod 2.$$

But since Q is an even intersection form, every diagonal element $Q_{i,i}$ is even and vanishes mod 2. □

As a corollary, we see that the Arf invariants of K_2 and K_3 agree when the intersection form is even.

Lemma 5.2 *Suppose Q is even. Let c be a separating curve in Σ, and let K_2 and K_3 be the knots obtained as the image of c in Y_2 and Y_3. Then* $\operatorname{Arf}(K_2) = \operatorname{Arf}(K_3)$.

Proof Since c is separating on Σ, it cuts off a subsurface Σ'. The images of this surface in Y_2 and Y_3 are Seifert surfaces for K_2 and K_3, respectively. Choose a symplectic basis $\{a_i, b_i\}$ for $H_1(\Sigma')$. By (5) and Lemma 5.1,

$$\operatorname{Arf}(K_2) = \sum q_2(a_i)q_2(b_i) = \sum q_3(a_i)q_3(b_i) = \operatorname{Arf}(K_3). \qquad \square$$

Now we describe the linking forms l_2 and l_3 in terms of the quadratic form of the pseudotrisection. Recall that we have a symplectic basis $\{x_i, y_i\}$ for $H_1(\Sigma, \mathbb{Z})$ such that $\{x_i\}$ is a basis for L_α and $\{y_i\}$ is a basis for L_β. To simplify notation, let $Qy_i = \sum_{j=1}^n Q_{i,j}\, y_j$. Then $z_i = -x_i - Qy_i$ is a basis for L_γ.

Proposition 5.3 *The linking form l_2 satisfies*

$$\begin{aligned} &l_2(z_i, z_j) = 0, \qquad l_2(z_i, x_j) = -Q_{i,j}, \quad l_2(x_i, z_j) = 0,\\ &l_2(x_i, x_j) = Q_{i,j}, \quad l_2(x_i, y_j) = -\delta_{i,j}, \quad l_2(y_i, y_j) = 0. \end{aligned}$$

Proof For any $A \in H_1(\Sigma; \mathbb{Z})$, we have that

$$l_2(y_i, A) = 0 \quad \text{and} \quad l_2(A, z_j) = 0.$$

Consequently, we easily see that

$$l_2(z_i, z_j) = l_2(x_i, z_j) = 0.$$

Using the relation in (2), we then see that

$$l_2(z_i, x_j) = l_2(x_j, z_i) + \langle x_j, z_j\rangle = 0 + \langle x_j, -x_i - Qy_i\rangle = \langle x_j, -Q_{i,j}\, y_j\rangle = -Q_{i,j}.$$

Furthermore, we have that $z_i + x_i = -Qy_i$. Consequently,

$$l_2(z_i + x_i, z_j + x_j) = 0.$$

Using the bilinearity relation, we obtain

$$l_2(z_i, z_j) + l_2(z_i, x_j) + l_2(x_i, z_j) + l_2(x_i, x_j) = 0.$$

Using the above formulas, we see that

$$l_2(x_i, x_j) = -l_2(z_i, x_j) = Q_{i,j}.$$

Next, we have

$$-Q_{i,j} = l_2(z_i, x_j) = l_2(-x_i, x_j) + l_2(-Qy_i, x_j) = -Q_{i,j} + l_2(Qy_i, x_j),$$

so $l_2(Qy_i, x_j) = 0$ for all i and j. This implies that $l_2(y_i, x_j) = 0$ and consequently that $l_2(x_j, y_i) = \langle y_i, x_j \rangle = \delta_{i,j}$. □

Proposition 5.4 *The linking form l_3 satisfies*

$$\begin{aligned} &l_3(z_i, z_j) = 0, \quad l_3(z_i, x_j) = 0, \qquad l_3(x_i, z_j) = Q_{i,j}, \\ &l_3(x_i, x_j) = 0, \quad l_3(x_i, y_j) = -\delta_{i,j}, \quad l_3(y_i, y_j) = Q^{-1}_{i,j}. \end{aligned}$$

Proof As in the proof of Proposition 5.3, we have that

$$l_3(z_i, A) = l_3(A, x_j) = 0$$

for any $A \in H_1(\Sigma; \mathbb{Z})$. We then easily obtain

$$l_3(z_i, z_j) = l_3(z_i, x_j) = l_3(x_i, x_j) = 0.$$

It then follows that

$$l_3(x_i, z_j) = \langle z_j, x_i \rangle = Q_{i,j}.$$

Now,

$$Q_{i,j} = l_3(x_i, z_j) = l_3(x_i, -x_j) + l_3(x_i, -Qy_j) = l_3(x_i, -Qy_j),$$

which implies that $l_3(x_i, y_j) = -\delta_{i,j}$. Finally,

$$\begin{aligned} l_3(z_i, z_j) &= l_3(-x_i, -x_j) + l_3(-x_i, -Qy_j) + l_3(-Qy_i, -x_j) + l_3(-Qy_i, -Qy_j) \\ &= 0 - Q_{i,j} + 0 + l_3(Qy_i, Qy_j) \\ &= 0. \end{aligned}$$

Therefore, $l_3(Qy_i, Qy_j) = Q_{i,j}$. Because Q is the matrix for the linking form in the basis $\{Qy_i\}$, we see that Q^{-1} is the matrix for the linking form in the basis $\{y_i\}$. □

6 Rohlin's theorem

Lemma 6.1 *Let X be a closed, spin 4–manifold. Then X is spin-cobordant to a 4–manifold X' such that*

(1) *X' has a handle decomposition without 1– or 3–handles, and*

(2) *the intersection form of X' is even and indefinite.*

In particular, X' is spin and satisfies $\sigma(X') = \sigma(X)$.

Proof Following the standard surgery theory trick, we can attach 5–dimensional 2–handles to kill the 1–handles of X. Moreover, we can extend the spin structure across these 2–handles. Turning the manifold upside down, we can also kill all of the 3–handles. This results in some spin X'' that has no 1– or 3–handles. By attaching one more 2–handle along a contractible loop, we get a spin cobordism to $X'' \# S^2 \times S^2$, which has an even, indefinite intersection form. □

The connection between handle decompositions and trisections is given by the following result.

Theorem 6.2 [6] *Let X be a closed, oriented 4–manifold that admits a handle decomposition with a single 0–handle, k_1 many 1–handles, n many 2–handles, and k_3 many 3–handles. Then X admits a $(g; k_1, g-n, k_2)$–trisection for some g.*

Corollary 6.3 *Suppose that a closed, smooth oriented 4–manifold X admits a handle decomposition without 1– or 3–handles. Then*

(1) *X admits a $(g; 0, k, 0)$–trisection for some g and k, and*

(2) *X admits a $(g; k, 0, 0)$–trisection for some g and k.*

Proof The first statement follows directly from Theorem 6.2, while the second follows from the first by cyclically permuting the sectors of the trisection. □

A pseudotrisection with intersection form E_8 is guaranteed by Proposition 3.14. An example of such a pseudotrisection (obtained by trial and error, not the method of the proof of Proposition 3.14) is given in Figure 1.

Proposition 6.4 *Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ be the Heegaard triple in Figure 1. Then $Y_{\alpha\beta} \cong Y_{\gamma\alpha} \cong S^3$, and $Y_{\beta\gamma}$ is the Poincaré homology sphere.*

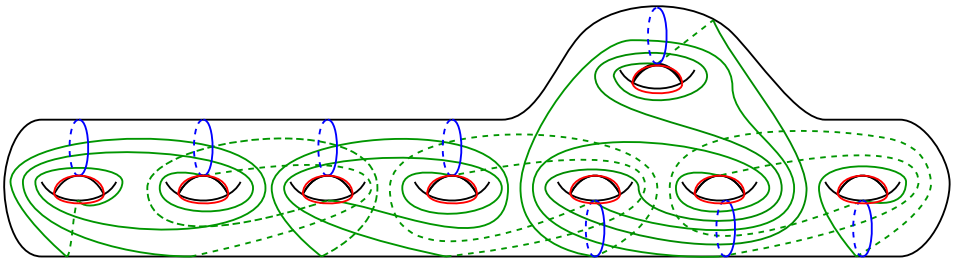

Figure 1: A pseudotrisection with intersection form E_8.

Proof Each α_i intersects a unique β–curve in a single point and a unique γ curve in a single point. This implies that the pairs (Σ, α, β) and (Σ, γ, α) are the standard Heegaard diagram for S^3.

Viewing Σ as a Heegaard surface in $Y_{\alpha\beta}$, the surface framing of each γ curve is $+2$ and the curves $\{\gamma_1, \dots, \gamma_8\}$ are a collection of unknots that link according to the E_8 Dynkin diagram. It is well known (eg [13]) that the result of surgery on this framed link results in the Poincaré homology sphere. □

We now have sufficient tools to prove Rohlin's theorem:

Proof of Theorem 1.3 Fix $m, n, k \geq 0$ and set $g = 8m + 2n + k$. Then we can obtain a $(g; k, 0, 0)$–pseudotrisection $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ with intersection form $Q = mE_8 \oplus nH$ by taking a connected sum of m copies of the counterfeit E_8, n copies of the standard trisection of $S^2 \times S^2$, and k copies of the standard $(1; 1, 0, 0)$–trisection of S^4.

Let $\phi \in \mathcal{K}_g$ be any element of the Johnson kernel and let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \phi(\boldsymbol{\gamma}))$ be the resulting pseudotrisection. We obtain 3–manifolds $Y_{2,\phi}$ and $Y_{3,\phi}$ as the union of handlebodies. Because $\phi \in \mathcal{K}_g$, it is the product of r separating twists. Consequently, $Y_{2,\phi}$ is obtained from Y_2 by surgery on a boundary link $L_2 = l_{2,1} \cup \cdots \cup l_{2,r}$, and $Y_{3,\phi}$ is obtained from Y_3 by surgery on a boundary link $L_3 = l_{3,1} \cup \cdots \cup l_{3,r}$. The surgery formula for the Casson invariant implies that

$$\mu(Y_{2,\phi}) - \mu(Y_2) = \sum_{i=1}^{r} \mathrm{Arf}(l_{2,i}) \quad \text{and} \quad \mu(Y_{3,\phi}) - \mu(Y_3) = \sum_{i=1}^{r} \mathrm{Arf}(l_{3,i}).$$

But by Lemma 5.2, we have that $\mathrm{Arf}(l_{2,i}) = \mathrm{Arf}(k_{3,i})$ for all $i = 1, \dots, r$. Thus,

$$\mu(Y_{2,\phi}) + \mu(Y_{3,\phi}) = \mu(Y_2) + \mu(Y_3) = m \mod 2.$$

Now, suppose that X is any closed, smooth, oriented 4–manifold with intersection Q and an $(8m+2n+k;k,0,0)$–trisection. By Theorem 4.1, we can assume there is some element $\phi \in \mathcal{K}_g$ such that $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, \phi(\boldsymbol{\gamma}))$ is a trisection diagram for X. Moreover, since this is an honest trisection decomposition, we have that $Y_{2,\phi} \cong Y_{3,\phi} \cong S^3$. Therefore

$$m = \mu(Y_{2,\phi}) + \mu(Y_{3,\phi}) = 0 \mod 2. \qquad \square$$

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