Algebraic & Geometric Topology 20 (2020) 1041–1058

Ribbon distance and Khovanov homology

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We study a notion of distance between knots, defined in terms of the number of saddles in ribbon concordances connecting the knots. We construct a lower bound on this distance using the X–action on Lee's perturbation of Khovanov homology.

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1 Introduction

Ever since its inception, Khovanov homology [12], a categorification of the Jones polynomial, has attracted tremendous interest and has produced an entire new field of research. It has been generalized in several orthogonal directions (see Bar-Natan [3], Khovanov [13], Khovanov and Rozansky [15] and Lee [17]) and continues to generate intense activity. While the primary focus of the field has been categorification of various low-dimensional topological invariants — endowing them with new algebraic and higher categorical structure — it has also produced a small number of stunning applications in low-dimensional topology as a byproduct. Specifically, Lee's perturbation of Khovanov homology has been instrumental in producing several applications for knot cobordisms; the author's personal favorites are Rasmussen's proof of the Milnor conjecture [23] (bypassing the earlier gauge-theoretic proof by Kronheimer and Mrowka) and Piccirillo's proof that the Conway knot is not slice [22].

We define a notion of distance between knots, using the number of saddles in ribbon concordances connecting the knots. This distance is finite if and only if the knots are concordant, but it is hard to find examples of knots arbitrarily large finite distance apart. Using the X–action on Lee's perturbation of Khovanov homology, we construct a lower bound on this distance, which is the main result of this paper.

Theorem 1.1 *If d is the ribbon distance* (defined in Section 3) *between knots K and K', then*

$$(2X)^d \operatorname{Kh}_{\mathscr{L}}(K) \cong (2X)^d \operatorname{Kh}_{\mathscr{L}}(K'),$$

where $\operatorname{Kh}_{\mathscr{L}}$ is Lee's perturbation of Khovanov homology.

Published: 23 April 2020 DOI: 10.2140/agt.2020.20.1041

In particular, the distance of K from the unknot defines a notion of complexity for K, which produces lower bounds for many classical notions of knot complexities like band number, ribbon number and bridge index, and it itself has a lower bound coming from Khovanov homology. (Coincidentally, this lower bound agrees with the lower bound on unknotting number from Alishahi and Dowlin [1].)

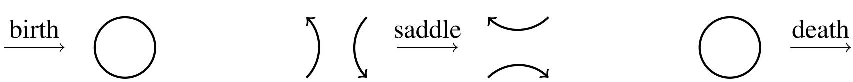
Corollary 1.2 *For any knot K, and over any field $\mathbb{F}$ with $2 \neq 0$, the extortion order* $\mathrm{xo}(K)$ (defined in Section 5) *is a lower bound for the ribbon distance of K from the unknot.*

Acknowledgements There has been a sudden abundance of short cute papers on applications of modern knot homology theories to knot cobordisms, and in particular ribbon concordances [24; 1; 19; 18]; the present paper is a result of the author's desire to jump on the bandsumwagon. Some of the ideas of this paper are recycled from the abovementioned papers, and therefore, he is grateful to their authors. He would also like to thank Brendan Owens for pointing out some lower bounds for the band number, Ciprian Manolescu for some computations, Robert Lipshitz for suggesting the wordplay in this paragraph, and the referee for several helpful comments.

The author was supported by NSF Grant DMS-1643401.

2 Knot cobordisms

A *cobordism* from a link $K_0 \subset \mathbb{R}^3 \times \{0\}$ to a link $K_1 \subset \mathbb{R}^3 \times \{1\}$ is a properly embedded oriented surface $F \subset \mathbb{R}^3 \times [0,1]$ with boundary the union of K_1 and the orientation-reversal of K_0.[1] Call the projection $\pi_t \colon \mathbb{R}^3 \times [0,1] \to [0,1]$ the time function, and assume $\pi_t|_F$ is Morse; its index 0, 1 and 2 critical points are called births, saddles and deaths. The cobordism may be viewed as a movie as time runs from 0 to 1. For regular values t of $\pi_t|_F$, $K_t := F \cap (\mathbb{R}^3 \times \{t\})$ is a link; K_t changes by isotopy with time, with the following local modifications occurring at births, saddles, and deaths:



[1] We are working with cobordisms in $\mathbb{R}^3 \times [0,1]$ as opposed to the more standard $S^3 \times [0,1]$ for a couple of reasons: naturality of Khovanov cobordism maps has only been established in $\mathbb{R}^3 \times [0,1]$ (even up to sign); and a subtle sign discrepancy for dotted cobordism maps can be resolved in $\mathbb{R}^3 \times [0,1]$.

We usually work with the projection $\pi_R\colon \mathbb{R}^2 \times \mathbb{R} \times [0,1] \to \mathbb{R}^2$, and represent each $K_t \subset \mathbb{R}^3 \times \{t\}$ by the link diagram $\pi_R(K_t)$. We then represent the cobordism as a *movie of link diagrams*; usually $\pi_R(K_t)$ changes by planar isotopy with time, with Reidemeister moves and the above moves happening at certain time instances (which by genericity we will assume to be distinct). Two such movies represent isotopic cobordisms (relative K_0 and K_1) if and only if they are related by a sequence of movie moves [5].

If F is diffeomorphic to a cylinder, then F is said to be a *concordance*[2] from the knot K_0 to the knot K_1. The concordance is said to be *ribbon* if there are no births [8]. The famous slice-ribbon conjecture states that every slice knot has a ribbon concordance to the unknot.

We will also be interested in *dotted cobordisms*, that is, cobordisms F decorated with finite number of dots in the interior. We can also represent them by movies of link diagrams, except now dots are present at certain instances. As before, by genericity, we will assume these instances are separate from the births, saddles, deaths, and the Reidemeister moves; moreover, at each such t, the link K_t contains exactly one dot and its projection to the link diagram $\pi_R(K_t)$ is away from the crossings.

Lemma 2.1 *Assume F and F' are dotted cobordisms (in generic position) with the same underlying surface and the same number of dots on each component, but differing only in the placement of the dots. Then the movies for F and F' are related by a sequence of the following movie moves:*

(1) ***Far commutation*** *We may switch the order of the following operations:*
   - (a) *adding a dot, and then adding another dot;*
   - (b) *adding a dot, and then performing a birth;*
   - (c) *adding a dot, and then performing a saddle;*
   - (d) *performing a death, and then adding a dot;*
   - (e) *adding a dot, and then performing a Reidemeister move far away.*

(2) ***Moving dots on link diagrams*** *If we are adding a dot on one side of a crossing on a link diagram $\pi_R(K_t)$, then we can instead add it on the other side of the crossing.*

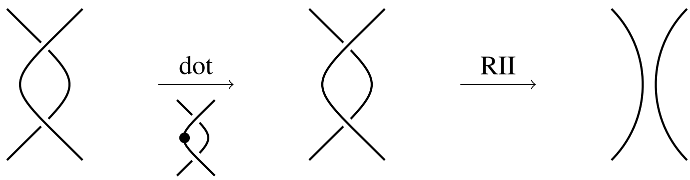
[2] In old literature, the word "cobordism" was used instead of "concordance".

Proof We can use the second movie move to move dots freely on K_t for each t. To move dots in the time direction, we use the first movie move, which allows us to move dots past each other, and also past births, deaths, saddles, and Reidemeister moves; all the possibilities are listed, except the following:

(a) Perform a birth, and then add a dot on the newborn unknot component. In this case, it is impossible to switch the order.

(b) Add a dot to a small unknot component, and then perform a death on that component. This is simply the time-reversal of the previous case.

(c) Add a dot on some strand of the knot diagram, and then perform a Reidemeister move that involves that strand; see below for an example with Reidemeister II move.

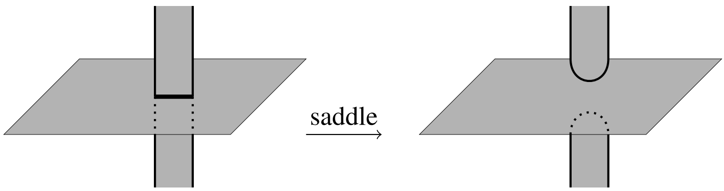


However, in this case, we may move the dot on the link diagram (using the second movie move) away from the strands involved in the Reidemeister move, and then use far commutation with the Reidemeister move (using the first movie move) to change the temporal order of the dot addition and the Reidemeister move. □

3 Ribbon complexities

There are certain notions of complexities that we can associate to ribbon concordances. If K is a ribbon knot — that is, if K has a ribbon concordance to the unknot U — then we can define the *band number* $b(K)$ to be the smallest number of saddles in a ribbon concordance $K \to U$; this is also the smallest number of bands if we write K as a band sum of an unlink. This number is usually called the *ribbon-fusion number* and has lower bounds coming from the Jones polynomial [11]; more classically, twice this number is bounded below by $\mathrm{gn}(H_1(\Sigma_K))$ — the smallest number of generators for the first homology of the double branched cover [21]. In turn, for any knot K, $1+b(K\#m(K))$ itself gives a lower bound for the bridge index of K (here $m(K)$ denotes the orientation-reversal of the mirror of K) [11].

Each ribbon knot bounds a ribbon disk in $\mathbb{R}^3$, that is, an immersed disk with only ribbon singularities (as shown with the thick line in the leftmost figure below). So we may define the *ribbon number* $r(K)$ to be smallest number of ribbon singularities for ribbon disks bounded by K. We may perform saddles near each ribbon singularity (as shown below) to convert K to an unlink, so $r(K)$ is bounded below by $b(K)$. A nice argument shows that the knot genus $g(K)$ also provides a lower bound for $r(K)$ [7].



We may also define a notion of distance on knots coming from ribbon concordances. For any two knots K and K', define the *ribbon distance* $d(K, K')$ to be the smallest k such that there is a sequence of knots $K = K_0, K_1, \ldots, K_{n-1}, K_n = K'$ from K to K' and a ribbon concordance (in some direction) between every consecutive pair K_i and K_{i+1} with at most k saddles. The following properties are immediate:

(1) $d(K, K') < \infty$ if and only if K and K' are concordant. (For the slightly nonobvious direction, note that if K and K' are concordant, then there is some K'' with ribbon concordances to both K and K'; see [8].)

(2) $d(K, K') = 0$ if and only if K and K' are isotopic.

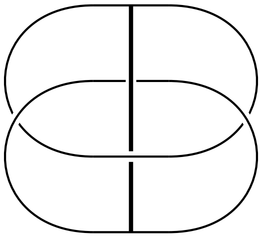
(3) $d(K, K') = d(K', K)$.

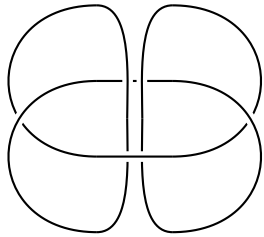
(4) $d(K, K'') \leq \max\{d(K, K'), d(K', K'')\}$, and hence d satisfies the triangle inequality.

This notion of distance complements the more standard notion of *cobordism distance*, which is defined to be the smallest genus of a cobordism between the two knots. (Cobordism distance between any two knots is finite, and is zero if and only if the knots are concordant.)

For any slice knot K, its distance from the unknot, $d(K, U)$, therefore provides yet another notion of complexity. It is clear from the definitions that $d(K, U) \leq b(K)$.

Example 3.1 Let K_1 be the connected sum of the positive and the negative trefoil, and let K_n be the connected sum of n copies of K_1. We have $r(K_1) = g(K_1) = \mathrm{gn}(H_1(\Sigma_{K_1})) = 2$ and $b(K_1) = d(K_1, U) = 1$; K_1 can be obtained by adding a band (shown by the thick line below) to the 2–component unlink, which intersects the natural disks bounded by the unlink in 2 ribbon singularities:





We get $r(K_n) \geq g(K_n) = ng(K_1) = 2n$. The ribbon number is subadditive under connected sum, so $r(K_n) \leq nr(K_1) = 2n$; therefore, $r(K_n) = 2n$. We also get $b(K_n) \geq \mathrm{gn}(H_1(\Sigma_{K_n}))/2 = \mathrm{gn}(\oplus^n H_1(\Sigma_{K_1}))/2 = n$. The band number is also subadditive under connected sum, so $b(K_n) \leq nb(K_1) = n$; therefore, $b(K_n) = n$. Finally $d(K_n, U) = 1$ since we have a sequence of knots $K_n, K_{n-1}, \ldots, K_0 = U$, and a single-saddle ribbon concordance $K_{i+1} \to K_i$ for all i, obtained by connected summing K_i with the single-saddle ribbon concordance $K_1 \to U$.

It is unclear if $d(K, U)$ can be arbitrarily large (while staying finite). In this paper, we will give an example of a knot with $d(K, U) = 2$ (Example 6.1), and indeed one with $d(K, U) > 2$ (Example 6.2). It is reasonable to guess that the techniques of this paper, but using knot Floer homology instead of Khovanov homology, might produce examples of knots with larger values of $d(K, U)$.[3]

4 Khovanov homology

Fix a ground ring R and consider the 2–dimensional Frobenius algebra

$$V = R[T][X]/\{X^2 = T\}$$

over $R[T]$ with comultiplication $V \to V \otimes_{R[T]} V$ given by

$$1 \mapsto 1 \otimes X + X \otimes 1, \quad X \mapsto X \otimes X + T 1 \otimes 1$$

and counit $V \to R[T]$ given by $1 \mapsto 0$, $X \mapsto 1$. This produces a Khovanov-style link homology theory [12] for any link K by applying it to the Kauffman cube of resolutions

[3] And indeed, using knot Floer homology instead of Khovanov homology, within one month of the first appearance of this paper, Juhász, Miller and Zemke [10] constructed knots with arbitrarily large values of $d(K, U)$.

of its link diagram. The resulting theory is usually called the *Lee perturbation of Khovanov homology* [17], and we will denote it $\mathrm{Kh}_{\mathscr{L}}(K)$. It is a bigraded homology theory over $R[T]$ with R in bigrading $(0,0)$ and T in bigrading $(0,-4)$.

A dotted cobordism $F\colon K_0 \to K_1$ (in generic position) with $\delta(F)$ dots induces a map

$$\mathrm{Kh}_{\mathscr{L}}(F)\colon \mathrm{Kh}_{\mathscr{L}}(K_0) \to \mathrm{Kh}_{\mathscr{L}}(K_1)$$

of $R[T]$–modules of bigrading $(0, \chi(F) - 2\delta(F))$ [14; 3; 9; 18], defined as follows. The movie presentation for F is a sequence of planar isotopy, Reidemeister moves, births, saddles, deaths, and dot additions. Except for dot addition, each of the other moves induces a map on $\mathrm{Kh}_{\mathscr{L}}$ using the Frobenius algebra V. The dot addition map is defined slightly differently. We present a careful definition below that avoids a sign issue.

An *elementary dotted cobordism* from $K \to K$ is a product cobordism decorated with a single dot. Consider (the projection of) the dot on the oriented link diagram $\pi_{\boldsymbol{R}}(K)$. Checkerboard color the complement of the link diagram in $\mathbb{R}^2$ so that the unbounded region is colored white. If the arc in the link diagram containing the dot is oriented as the boundary of a black region, define the sign of the dot to be $(+1)$, otherwise, define it to be (-1). Then define the dotted cobordism map $\mathrm{Kh}_{\mathscr{L}}(K) \to \mathrm{Kh}_{\mathscr{L}}(K)$ to be the map merging a small unknot labeled X near the dot, times the sign of the dot.

It is well-known that two isotopic (rel boundary) undotted knot cobordisms induce the same map $\mathrm{Kh}_{\mathscr{L}}(K_0) \to \mathrm{Kh}_{\mathscr{L}}(K_1)$, up to an overall sign.[4] We have a similar variant for dotted cobordisms.

Lemma 4.1 *Assume F and F' are dotted cobordisms (in generic position) with the same underlying surface and the same number of dots on each component, but differing only in the placement of the dots. Then they induce the same map on $\mathrm{Kh}_{\mathscr{L}}$, including the sign.*

Proof We merely have to check that the map is unchanged under the movie moves listed in Lemma 2.1. The first movie move (far commutation) is clear. For the second movie move (moving the dot past a crossing), we may check directly that on the Khovanov chain complex level, the map associated to merging a small unknot labeled X to some strand is homotopic to the *negative* of the map associated to merging a small unknot labeled X to the corresponding strand on the opposite side of a crossing; see [2].

[4] This sign issue can also be resolved [6], but we will not need to.

Therefore we have the same map on homology for dot addition on either side of a crossing. □

The main advantage of using dotted cobordisms is the famous *neck-cutting relation*. We will need it in the following two forms:

Lemma 4.2 *Assume the link diagram $\pi_R(K)$ for K contains a small unknot U. Then, up to an overall sign, the identity map $\mathrm{Kh}_{\mathscr{L}}(K) \to \mathrm{Kh}_{\mathscr{L}}(K)$ is the sum of the following two maps:*

(1) *Add a dot to U, perform a death on U, and perform a rebirth for U.*

(2) *Perform a death on U, perform a rebirth for U, and add a dot to U.*

In terms of movies,

$$\pm\mathrm{Kh}_{\mathscr{L}}(\bigcirc \xrightarrow{\text{Id}} \bigcirc) = \mathrm{Kh}_{\mathscr{L}}(\bigcirc \xrightarrow{\text{dot}} \bigcirc \xrightarrow{\text{death}} \quad \xrightarrow{\text{birth}} \bigcirc) + \mathrm{Kh}_{\mathscr{L}}(\bigcirc \xrightarrow{\text{death}} \quad \xrightarrow{\text{birth}} \bigcirc \xrightarrow{\text{dot}} \bigcirc).$$

Proof If the dot addition maps are given by merging small unknots labeled X, then it is easy to check that the above equation holds (without the sign) on the nose at the Khovanov chain complex level. However, the actual dot addition map has an extra sign given by the sign of the dot. But the unknot U before death and the unknot U after birth are oriented in the same way, so the two dots have the same sign, and consequently, the above equation holds up to an overall sign. □

Lemma 4.3 *Assume $F\colon K \to K$ is a cobordism obtained by performing an elementary saddle on the link diagram $\pi_R(K)$ for K, followed by performing the saddle in reverse. Then, up to an overall sign, the map $\mathrm{Kh}_{\mathscr{L}}(K) \to \mathrm{Kh}_{\mathscr{L}}(K)$ is the sum of the following two maps:*

(1) *Add a dot to one of the two strands in $\pi_R(K)$ involved in the saddle.*

(2) *Add a dot to the other strand in $\pi_R(K)$ involved in the saddle.*

In terms of movies,

$$\pm\mathrm{Kh}_{\mathscr{L}}(\,)\,(\xrightarrow{\text{saddle}} \asymp \xrightarrow{\text{saddle}}\,)\,(\,) = \mathrm{Kh}_{\mathscr{L}}(\,)\,(\xrightarrow[\bullet)\,(]{\text{dot}}\,)\,(\,) + \mathrm{Kh}_{\mathscr{L}}(\,)\,(\xrightarrow[)\,(\bullet]{\text{dot}}\,)\,(\,).$$

Proof The proof is very similar to the previous proof. If the dot addition maps are given by merging small unknots labeled X, then the equation holds (without the sign) at the Khovanov chain complex level. However, since the saddle is an oriented saddle, the two dots on the two strands have the same sign, and consequently, the above equation holds up to an overall sign. □

5 X–action on Khovanov homology

If we fix a component of K, then the map $\mathrm{Kh}_{\mathscr{L}}(K) \to \mathrm{Kh}_{\mathscr{L}}(K)$ associated to the elementary dotted cobordism $K \to K$ that has a single dot on the chosen component is denoted by X (since it comes from merging an unknot labeled X), and is often called the *X–action* on $\mathrm{Kh}_{\mathscr{L}}(K)$. It is clear from the Frobenius algebra V that $X^2 = T$. This makes $\mathrm{Kh}_{\mathscr{L}}(K)$ a module over $R[T, X]/\{X^2{=}T\} = R[X]$ (although the module structure depends on chosen link component).

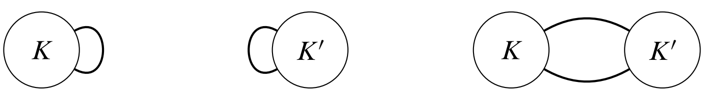
Khovanov homology of connected sums has a nice expression using the X–actions.

Lemma 5.1 *Let K and K' be links with chosen components, and let $K \# K'$ be the link obtained by connected summing the chosen components. Then*

$$\mathrm{Kh}_{\mathscr{L}}(K \# K') \cong \Sigma^{0,-1} \operatorname{Tor}_{R[X]}(\mathrm{Kh}_{\mathscr{L}}(K), \mathrm{Kh}_{\mathscr{L}}(K'))$$

as bigraded $R[X]$–modules, with the X–action on the right-hand side induced from the X–action on either $\mathrm{Kh}_{\mathscr{L}}(K)$ or $\mathrm{Kh}_{\mathscr{L}}(K')$. Here $\Sigma^{a,b}$ denotes an upward bigrading shift by (a, b), that is, tensoring with a single R in bigrading (a, b), and $\operatorname{Tor}_{R[X]}(A, A')$ is the total homology of the derived tensor product $A \otimes^{L}_{R[X]} A'$ of two bigraded modules A, A' over $R[X]$.

Proof The argument entirely follows Khovanov's argument for his original invariant (which is the specialization $X^2 = T = 0$) [13], so we skip some details. Consider the following link diagrams for K, K', and $K \# K'$, so that the induced diagram for $K \amalg K'$ differs from the diagram of $K \# K'$ locally by an elementary saddle:



Let $C\mathrm{Kh}_{\mathscr{L}}$ be the Khovanov chain complexes associated to these diagrams. They become modules over $R[X]$ by the X–action at the strands that are shown in the above diagram. (For $K \# K'$ either strand works.) By construction, these complexes are free over $R[T]$, but indeed, they are free over $R[X]$ as well. Therefore, it is enough to construct an isomorphism of chain complexes over $R[X]$,

$$C\mathrm{Kh}_{\mathscr{L}}(K \# K') \cong \Sigma^{0,-1} C\mathrm{Kh}_{\mathscr{L}}(K) \otimes_{R[X]} C\mathrm{Kh}_{\mathscr{L}}(K').$$

Consider the saddle map

$$C\mathrm{Kh}_{\mathscr{L}}(K) \otimes_{R[T]} C\mathrm{Kh}_{\mathscr{L}}(K') \cong C\mathrm{Kh}_{\mathscr{L}}(K \amalg K') \to \Sigma^{0,1} C\mathrm{Kh}_{\mathscr{L}}(K \# K'),$$

and it is easy to check that it factors through $C\mathrm{Kh}_{\mathscr{L}}(K)\otimes_{R[X]} C\mathrm{Kh}_{\mathscr{L}}(K')$. So all that remains is to check that this $R[X]$–module chain map

$$C\mathrm{Kh}_{\mathscr{L}}(K)\otimes_{R[X]} C\mathrm{Kh}_{\mathscr{L}}(K') \to \Sigma^{0,1} C\mathrm{Kh}_{\mathscr{L}}(K\,\#\,K')$$

is an isomorphism on the chain groups.

The chain groups $C\mathrm{Kh}_{\mathscr{L}}$ decompose as direct sums of chain groups of various resolutions of the link diagrams, so it is enough to check that the above map is an isomorphism at each resolution of K and K' — that is, it is enough to check the case when K and K' are planar unlinks, which is trivial to check. □

Khovanov homology of the mirror can also be determined.

Lemma 5.2 *Let K be a link with a chosen component, and let $m(K)$ be the orientation reversal of its mirror with the same chosen component. Then*

$$\mathrm{Kh}_{\mathscr{L}}(m(K)) \cong \Sigma^{0,2}\,\mathrm{Ext}_{R[X]}(\mathrm{Kh}_{\mathscr{L}}(K), R[X])$$

as bigraded $R[X]$–modules. Here, $\Sigma^{a,b}$ denotes a bigrading shift as before, and $\mathrm{Ext}_{R[X]}(A, R[X])$ is the total homology of the derived dual $\mathrm{RHom}_{R[X]}(A, R[X])$ of a bigraded module A over $R[X]$, but with the bigradings negated.

Proof This argument is also very close to Khovanov's argument for his original invariant (the specialization $X^2 = T = 0$) [12], so once again, we will skip some details. Fix a pointed link diagram K, and the mirrored diagram $m(K)$.

For any pointed link diagram L, the Khovanov generators are pairs (ρ,λ), where ρ is some complete resolution of L (obtained by resolving each crossing by the 0–resolution or the 1–resolution), and λ is a labeling of the components of ρ by $\{1, X\}$. Let $G_1(L)$ (respectively, $G_X(L)$) be the set of Khovanov generators that label the pointed circle in any resolution by 1 (respectively, X). The whole chain complex $C\mathrm{Kh}_{\mathscr{L}}(L)$ is then freely generated over $R[X]$ by $G_1(L)$, while the subcomplex $XC\mathrm{Kh}_{\mathscr{L}}(L)$ (which is isomorphic to $\Sigma^{0,-2}C\mathrm{Kh}_{\mathscr{L}}(L)$) is freely generated over $R[X]$ by $G_X(L)$.

Being free, it is enough to show that the dual complex $\mathrm{Hom}_{R[X]}(C\mathrm{Kh}_{\mathscr{L}}(K), R[X])$, after negating the bigradings, is isomorphic to $\Sigma^{0,-2}C\mathrm{Kh}_{\mathscr{L}}(m(K)) \cong XC\mathrm{Kh}_{\mathscr{L}}(m(K))$. The dual complex is also free, with generators given by the dual generators: namely, for any generator $(\rho,\lambda)\in G_1(K)$, its dual generator $(\rho,\lambda)^*$ is the linear map that sends (ρ,λ) to $1\in R[X]$, and the rest of the generators to 0.

For any generator $(\rho,\lambda)\in G_1(K)$, consider the generator $(\rho,\overline{\lambda})\in G_X(m(K))$ where ρ is the exact same resolution (that is, each crossing in $m(K)$ is resolved as a 1–resolution if and only if the corresponding crossing in K is resolved as a 0–resolution) and $\overline{\lambda}$ is the labeling that labels each component of ρ by 1 if and only if λ labels that component by X. The map $(\rho,\lambda)^*\mapsto(\rho,\overline{\lambda})$ then induces the required isomorphism

$$\mathrm{Hom}_{R[X]}(C\mathrm{Kh}_{\mathscr{L}}(K),R[X])\to XC\mathrm{Kh}_{\mathscr{L}}(m(K))$$

of chain complexes over $R[X]$. □

If K is a knot and R is a field $\mathbb{F}$ with $2\neq 0$, then the module $\mathrm{Kh}_{\mathscr{L}}(K)$ over $\mathbb{F}[X]$ takes a particularly simple form. It decomposes (noncanonically) as $\Sigma^{0,s(K)+1}\mathbb{F}[X]\oplus\mathscr{T}(K)$, where $s(K)$ is Rasmussen's s–invariant, and $\mathscr{T}(K)$ is the (canonical) subgroup of $\mathrm{Kh}_{\mathscr{L}}(K)$ consisting of the X–torsion elements,

$$\mathscr{T}(K)=\{a\in\mathrm{Kh}_{\mathscr{L}}(K)\mid X^n a=0\text{ for some }n\},$$

which we will call the *extortion group* of K.

The smallest n such that $X^n\mathscr{T}(K)=0$ is called the *extortion order*, and denoted by $\mathrm{xo}(K)$. This was denoted by $\mathfrak{u}_X(K)$ in [1], who used it to provide a lower bound on the unknotting number. This number $\mathrm{xo}(K)$ can very well depend on the ground field $\mathbb{F}$, but since it is usually less than 3 for small knots, we do not have actual examples where this extortion order is different over different fields; consequently, we have chosen our notation $\mathrm{xo}(K)$ not to reflect this possible dependence on the field $\mathbb{F}$. The extortion order $\mathrm{xo}(K)$ is related to the Lee spectral sequence (coming from the filtered chain complex for $\mathrm{Kh}_{\mathscr{L}}(K)$ with filtration given by powers of T) as follows. If the Lee spectral sequence collapses at the E_k page, then $\mathrm{xo}(K)\in\{2k-3,2k-2\}$.

The only knot with $\mathrm{xo}(K)=0$ (that is, $\mathscr{T}(K)=0$) is the unknot [16]. All other Kh–thin knots have $\mathrm{xo}(K)=1$; 8_{19} is the first knot with $\mathrm{xo}(K)=2$. Since the Lee spectral sequence collapses at the E_2 page for small knots, it is hard to find examples of knots with $\mathrm{xo}(K)>2$; the first example of a knot with $\mathrm{xo}(K)>2$ was constructed in [19].

The extortion order can be computed from the Mathematica package KnotTheory [4] using the function UniversalKh, the standard reference for which seems to be "Scott's slides" [20]. UniversalKh works over $\mathbb{Q}$ and returns a free resolution of $\mathrm{Kh}_{\mathscr{L}}(K)$ over $\mathbb{Q}[X]$. Each term t^aq^bKhE contributes a tower $\mathbb{Q}[X]\langle p\rangle$ with the generator p in bigrading (a,b), and each term t^aq^bKhC[n] contributes a two-step complex

$\mathbb{Q}[X]\langle p,q\rangle$ with generators p and q in bigradings $(a-1,b-2n)$ and (a,b), and differential $p\mapsto X^nq$. Therefore, the extortion group $\mathscr{T}(K)$ over $\mathbb{Q}$ is isomorphic to the homology of complexes coming from the KhC[n] terms and the extortion order $\mathrm{xo}(K)$ over $\mathbb{Q}$ is the largest n such that KhC[n] appears.

The extortion groups and extortion orders behave nicely under connected sums.

Lemma 5.3 *Consider knots K and K' and their connected sum $K\#K'$. Then, over any field $\mathbb{F}$ with $2\neq 0$,*

$$\mathscr{T}(K\#K')\cong\Sigma^{0,s(K')}\mathscr{T}(K)\oplus\Sigma^{0,s(K)}\mathscr{T}(K')\oplus\Sigma^{0,-1}\operatorname{Tor}_{\mathbb{F}[X]}(\mathscr{T}(K),\mathscr{T}(K'))$$

and

$$\mathrm{xo}(K\#K')=\max\{\mathrm{xo}(K),\mathrm{xo}(K')\}.$$

Proof The first statement is immediate from Lemma 5.1 and the isomorphism

$$\mathrm{Kh}_{\mathscr{L}}(L)\cong\Sigma^{0,s(L)+1}\mathbb{F}[X]\oplus\mathscr{T}(L)$$

for all knots L.

For the second statement, we immediately get

$$\mathrm{xo}(K\#K')\geq\max\{\mathrm{xo}(K),\mathrm{xo}(K')\}$$

from the first two summands in the decomposition of $\mathscr{T}(K\#K')$. So it is enough to prove that the extortion order of the summand $\operatorname{Tor}_{\mathbb{F}[X]}(\mathscr{T}(K),\mathscr{T}(K'))$ equals the minimum of the extortion orders of $\mathscr{T}(K)$ and $\mathscr{T}(K')$.

Consider free resolutions $\widetilde{\mathscr{T}}(K)$ and $\widetilde{\mathscr{T}}(K')$ of the extortion groups over $\mathbb{F}[X]$. By the classification of finitely generated modules over PIDs, they decompose into a direct sum of 2–step complexes $\mathbb{F}[X]\langle p,q\rangle$, with the differential given by $p\mapsto\alpha(X)q$, where $\alpha(X)$ is some power of some irreducible homogeneous polynomial in X. Since X has nonzero bigrading, the only possibilities are $\alpha(X)=X^n$. Each such summand contributes $\mathbb{F}[X]\langle q\rangle/\{X^nq=0\}$ in homology, so the extortion orders are the maximum values of n that appear in such a decomposition.

If $n\geq m$, then by a simple change of basis, the tensor product of the 2–step complexes $\mathbb{F}[X]\xrightarrow{X^n}\mathbb{F}[X]$ and $\mathbb{F}[X]\xrightarrow{X^m}\mathbb{F}[X]$ decomposes as

$$(\mathbb{F}[X]\xrightarrow{X^m}\mathbb{F}[X])\oplus(\mathbb{F}[X]\xrightarrow{X^m}\mathbb{F}[X]),$$

and, hence, the extortion order of the summand $\operatorname{Tor}_{\mathbb{F}[X]}(\mathscr{T}(K),\mathscr{T}(K'))$ is equal to $\min\{\mathscr{T}(K),\mathscr{T}(K')\}$. □

Similarly, these invariants are well-behaved under mirroring as well.

Lemma 5.4 *Consider a knot K and the orientation reversal of its mirror, $m(K)$. Then, over any field $\mathbb{F}$ with $2 \neq 0$,*

$$\mathscr{T}(m(K)) \cong \Sigma^{0,2}\operatorname{Ext}_{\mathbb{F}[X]}(\mathscr{T}(K), \mathbb{F}[X]) \quad \textit{and} \quad \mathrm{xo}(m(K)) = \mathrm{xo}(K).$$

Proof The first statement is immediate from Lemma 5.2 and the isomorphism

$$\mathrm{Kh}_{\mathscr{L}}(L) \cong \Sigma^{0,s(L)+1}\mathbb{F}[X] \oplus \mathscr{T}(L)$$

for all knots L.

For the second, as before, consider a free resolution $\widetilde{\mathscr{T}}(K)$ of $\mathscr{T}(K)$ over $\mathbb{F}[X]$. Once again, it decomposes into a direct sum of 2–step complexes $\mathbb{F}[X]\langle p, q\rangle$, with the differential given by $p \mapsto X^n q$. The dual of such a summand over $\mathbb{F}[X]$ is the 2–step complex $\mathbb{F}[X]\langle p^*, q^*\rangle$, with the differential given by $q^* \mapsto X^n p^*$. Since extortion order is the maximum such n that appears, the claim follows. □

6 Main theorem

This section is devoted to the proof of the main theorems from Section 1.

Proof of Theorem 1.1 Since the ribbon distance is defined using a sequence of ribbon concordances, it is enough to do the case when there is a ribbon concordance $K \xrightarrow{F} K'$ with at most d saddles. After isotopy, we assume the movie of the cobordism F has the following form:

(1) First we perform some Reidemeister moves and planar isotopy on K. Since we are free to choose the link diagram for K, we actually do not need this move.

(2) Then we perform d elementary (planar) saddles, one at a time.

(3) Then we perform further Reidemeister moves and planar isotopy.

(4) Then we perform d elementary (planar) deaths, again one at a time.

(5) Then we again perform some Reidemeister moves and planar isotopy to end at K'. Once again, since we are free to choose the link diagram for K', we do not need this move.

So the ribbon concordance $K \xrightarrow{F} K'$ decomposes as $K \xrightarrow{F_2} \widetilde{K} \xrightarrow{F_3} K' \amalg U^d \xrightarrow{F_4} K'$, where the piece F_i comes from item (i) above, and U^d denotes the d–component planar unlink.

Let $K' \xrightarrow{\bar{F}} K$ be the cobordism viewed in reverse, which decomposes as

$$K' \xrightarrow{\bar{F}_4} K' \amalg U^d \xrightarrow{\bar{F}_3} \widetilde{K} \xrightarrow{\bar{F}_2} K.$$

Let $K \xrightarrow{W} K$ be the cobordism

$$K \xrightarrow{F_2} \widetilde{K} \xrightarrow{F_3} K' \amalg U^d \xrightarrow{\bar{F}_3} \widetilde{K} \xrightarrow{\bar{F}_2} K.$$

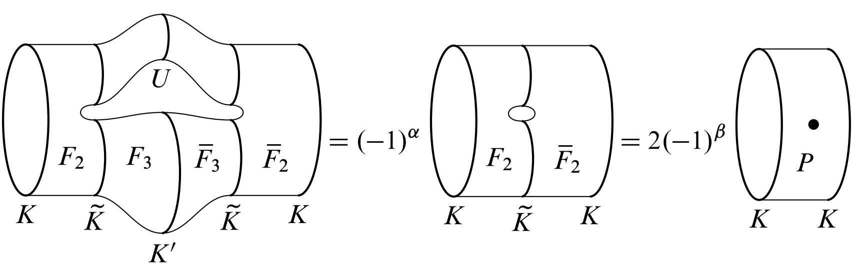
We will prove Theorem 1.1 by computing the image of the map $\mathrm{Kh}_{\mathscr{L}}(W)$ in two different ways, corresponding to the two sides of the equation in the statement of the theorem.

Method 1 The cobordism $\widetilde{K} \xrightarrow{F_3} K' \amalg U^d \xrightarrow{\bar{F}_3} \widetilde{K}$ is isotopic (rel boundary) to the identity cobordism $\widetilde{K} \to \widetilde{K}$ since the cobordism F_3 corresponds to a link isotopy in $\mathbb{R}^3$, and $\bar{F}_3$ is the same isotopy performed in reverse. Therefore, the image of $\mathrm{Kh}_{\mathscr{L}}(W)$ is same as the image of the map associated to the cobordism

$$K \xrightarrow{F_2} \widetilde{K} \xrightarrow{\bar{F}_2} K.$$

This cobordism performs d planar saddles, and then performs them in reverse. So repeated application of Lemma 4.3 tells us that the map $\mathrm{Kh}_{\mathscr{L}}(\bar{F}_2) \circ \mathrm{Kh}_{\mathscr{L}}(F_2)$ associated to this cobordism, up to an overall sign, is 2^d times the map associated to the dotted cobordism $K \xrightarrow{P} K$, where P is the product cobordism decorated with d dots. (Note that, since P is connected, by Lemma 4.1, the map $\mathrm{Kh}_{\mathscr{L}}(P)$ is independent of the placement of the d dots on P.) By definition, the image of $\mathrm{Kh}_{\mathscr{L}}(P)$ is $X^d \mathrm{Kh}_{\mathscr{L}}(K)$; therefore, the image of the original cobordism map is $(2X)^d \mathrm{Kh}_{\mathscr{L}}(K)$.

Schematically (with $d = 1$),



Method 2 Insert the identity cobordism $K' \amalg U^d \to K' \amalg U^d$ into W to get an isotopic cobordism

$$K \xrightarrow{F_2} \widetilde{K} \xrightarrow{F_3} K' \amalg U^d \xrightarrow{\mathrm{Id}} K' \amalg U^d \xrightarrow{\bar{F}_3} \widetilde{K} \xrightarrow{\bar{F}_2} K.$$

The cobordism $K' \amalg U^d \xrightarrow{\mathrm{Id}} K' \amalg U^d$ has d necks coming from U^d, so repeated application of Lemma 4.2 tells us that the map $\mathrm{Id}\colon \mathrm{Kh}_{\mathscr{L}}(K' \amalg U^d) \to \mathrm{Kh}_{\mathscr{L}}(K' \amalg U^d)$, up to an overall sign, is the sum of 2^d maps associated to the following 2^d dotted cobordisms with d dots: each has the same underlying surface

$$K' \amalg U^d \xrightarrow{F_4} K' \xrightarrow{\bar{F}_4} K' \amalg U^d,$$

which has d death–birth pairs; and the 2^d dotted cobordisms are obtained by distributing d dots in 2^d different ways so that each death–birth pair has exactly one dot.

The underlying composed cobordism

$$(K \xrightarrow{F_2} \widetilde{K} \xrightarrow{F_3} K' \amalg U^d \xrightarrow{F_4} K' \xrightarrow{\bar{F}_4} K' \amalg U^d \xrightarrow{\bar{F}_3} \widetilde{K} \xrightarrow{\bar{F}_2} K) = K \xrightarrow{F} K' \xrightarrow{\bar{F}} K$$

is connected, so by Lemma 4.1, the 2^d dotted cobordism maps all induce the same map, which is the map $\mathrm{Kh}_{\mathscr{L}}(\bar{F}) \circ \mathrm{Kh}_{\mathscr{L}}(Q) \circ \mathrm{Kh}_{\mathscr{L}}(F)$ corresponding to the dotted cobordism

$$K \xrightarrow{F} K' \xrightarrow{Q} K' \xrightarrow{\bar{F}} K,$$

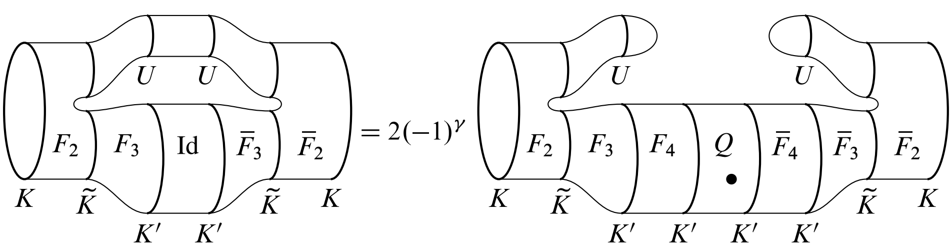
where $K' \xrightarrow{Q} K'$ is the product cobordism decorated with d dots.

Levine and Zemke have shown [18] that the map

$$\mathrm{Kh}_{\mathscr{L}}(F) \circ \mathrm{Kh}_{\mathscr{L}}(\bar{F})\colon \mathrm{Kh}_{\mathscr{L}}(K') \to \mathrm{Kh}_{\mathscr{L}}(K')$$

is $\pm\,\mathrm{Id}$,[5] and, therefore, the map $\mathrm{Kh}_{\mathscr{L}}(F)$ is surjective and the map $\mathrm{Kh}_{\mathscr{L}}(\bar{F})$ is injective. Consequently, the image of the map $\mathrm{Kh}_{\mathscr{L}}(\bar{F}) \circ \mathrm{Kh}_{\mathscr{L}}(Q) \circ \mathrm{Kh}_{\mathscr{L}}(F)$ is isomorphic to the image of $\mathrm{Kh}_{\mathscr{L}}(Q)$, which is $X^d\mathrm{Kh}_{\mathscr{L}}(K')$. Therefore, the image of the original cobordism map is isomorphic to $(2X)^d\mathrm{Kh}_{\mathscr{L}}(K')$.

Schematically,



□

[5] Actually, they proved it for Khovanov's specialization $X^2 = T = 0$, but the proof works in this more general case.

Proof of Corollary 1.2 Let d be the ribbon distance of the knot K to the unknot U. We know $\mathrm{Kh}_{\mathscr{L}}(U) \cong \Sigma^{0,1}\mathbb{F}[X]$ and, since we are working over a field with $2 \neq 0$,

$$\mathrm{Kh}_{\mathscr{L}}(K) \cong \Sigma^{0,s(K)+1}\mathbb{F}[X] \oplus \mathscr{T}(K).$$

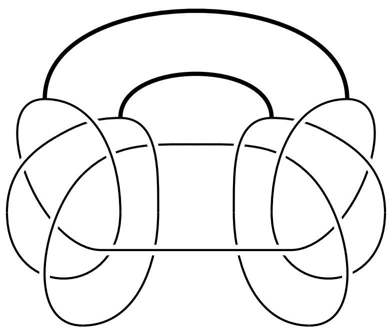
(If $d < \infty$, then K is slice, and hence $s(K) = 0$ [23], but we will not need this fact; indeed, this fact will follow from the proof.)

By Theorem 1.1, and since $2 \neq 0$,

$$X^d\mathrm{Kh}_{\mathscr{L}}(K) \cong \Sigma^{0,s(K)+1-2d}\mathbb{F}[X] \oplus X^d\mathscr{T}(K) \cong X^d\mathrm{Kh}_{\mathscr{L}}(U) \cong \Sigma^{0,1-2d}\mathbb{F}[X],$$

and therefore, $X^d\mathscr{T}(K) = 0$, and, hence, $d \geq \mathrm{xo}(K)$. □

Example 6.1 Let $K = 8_{19} \# m(8_{19})$. Using the function UniversalKh from the Mathematica package KnotTheory, we get $\mathrm{xo}(8_{19}) = 2$. By Lemmas 5.3 and 5.4, $\mathrm{xo}(K) = 2$, and hence, by Corollary 1.2, the distance of K from the unknot is at least 2. Indeed, adding untwisted (blackboard-framed) bands along the thick lines in the following knot diagram for K converts it to a 3–component unlink, so $d(K,U) = 2$:



Example 6.2 Let K_M be the knot from [19], and let $K = K_M \# m(K_M)$. Since the Lee spectral sequence for K_M collapses at the E_3 page, we know $\mathrm{xo}(K_M) \in \{3,4\}$. Indeed, using the function UniversalKh, we can conclude $\mathrm{xo}(K_M) = 3$. (Alternatively, $\mathrm{xo}(K_M)$ is a lower bound for the unknotting number [1], and K_M can clearly be unknotted with three crossing changes.) Once again, by Lemmas 5.3 and 5.4, $\mathrm{xo}(K) = 3$, and hence, by Corollary 1.2, the distance of K from the unknot is at least 3.

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Received: 4 June 2019 Revised: 25 July 2019